Implementing two-qubit gates along paths on the Schmidt sphere

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Abstract

Qubits (quantum bits) are what runs quantum computers, like a bit in classical computers. Quantum gates are used to operate on qubits in order to change their states. As such they are what "programmes" a quantum computer. An unfortunate side effect of quantum physics is that coupling a quantum system (like our qubits) to an outside environment will lead to a certain loss of information. Reducing this decoherence effect is thus vital for the function of a quantum computer. Geometric quantum computation is a method for creating error robust quantum gates by using so called geometric phases which are solely reliant on the geometry of the evolution of the system.

The purpose of this project has been to develop physical schemes of geometric entangling two-qubit gates along the Schmidt sphere, a geometric construct appearing in two-qubit systems. Essentially the overall aim has been to develop new schemes for implementing robust entangling quantum gates solely by means of interactions intrinsic to the computational systems.

In order to create this gate four mutually orthogonal states were defined which together spanned the two-qubit state space. Two of the states were given time dependent variables containing a total of two angles, which were used to parameterize the Schmidt sphere. By designing an evolution for these angles that traced out a cyclical evolution along geodesic lines a quantum gate with exclusively geometric phases could be created. This gate was dubbed the "Schmidt gate" and could be shown to be entangling by analyzing a change in the concurrence of a two qubit system. Two Hamiltonians were also defined which when acted upon the predefined system of states would give rise to the aforementioned evolution on the Schmidt sphere.

The project was successful in creating an entangling quantum gate which could be shown by looking at difference in the concurrence of the input and output state of a two-qubit system passing through the gate.

Sammanfattning

Qubits (kvantbitar) är vad som driver en kvantdator, likt det bits gör i en klassisk dator. Kvantgrindar används för att verka på qubits för ändra på deras tillstånd. På så vis är det dessa kvantgrindar som programmerar"en kvantdator. En olycklig konsekvens av kvantfysiken är att när ett kvantsystem (t.ex. våra qubits) kopplas till en utomstående miljö så försvinner en viss del av information i systemet. Att minska på denna dekoherens effekt är däför väldigt viktigt för att en kvantdator ska fungera. Geometrisk kvantberäkning (eng. Geometric quantum computation) är en metod för att skapa felskyddade kvantgrindar genom att använda så kallade geometriska faser som endast beror på geometrin av systemets utveckling.

Målet med detta projekt har varit att utveckla fysiska system av geometriskt sammanflätande tvåqubitgrindar längs med Schmidtsfären, vilket är en geometrisk konstruktion som uppstår i tvåkubit-system. För att uppnå detta definierades fyra ortogonala tillstånd där två gavs tidsberoende paramaterar som ritade ut kurva på Schmidtsfären när de utvecklades. Från detta så kunde en kvantgrind skapas (döpt Schmidtgrinden) där endast geometriska faser plockades upp. Två Hamiltonianer definierades som då skulle ge upphov till den kurvan som tillstånden bildade på Schmidtsfären.

Projektet lyckades med att skapa en sammanflätande kvantgrind vilket kunde visas genomat att jämföra concurrence mellan input- och outputtillståndet hos ett tvåkubitsystem som passerat genom grinden.

Contents

1 Introduction

In a classical computer logic gates are either models or some physical electronic device that interacts with some binary input (the ones and zeroes) and returns an output. They are the building blocks of digital circuits that run classical computers. Similar to the logic gates, quantum computers use quantum logic gates (or simply quantum gates) to operate on quantum bits (qubits) changing their states. But quantum computers follow entirely different rules compared to the boolean logic of the classical computer. Constructing quantum gates requires an entirely different methodology having to adhere to the laws of quantum physic. One major difference being the use of entanglement between qubits, which allows quantum computers to utilize things such as quantum teleportation. An advantage quantum computers have over classical ones is the fact that a quantum computer can perform exponentially faster computation as the number of qubits grow. With just a few dozen qubits you could theoretically have as much computing power as a modern computer consisting of several billions of bits [3].

An issue with creating practical quantum computers with many different qubits working at once is maintaining the so called quantum coherence inside of the system. An unfortunate side effect of quantum physics is that coupling a quantum system to an outside environment will lead to a certain loss of information. Reducing this decoherence effect is thus vital for the function of a quantum computer. One can do so by creating quantum error correcting codes, but it is also possible to design quantum gates which are inherently more fault-tolerant[6].

There are many different ways to construct quantum gates but one strategy for creating error robust quantum gates is by using geometric quantum computation (GQC). GQC is the idea to use geometric phases in order to implement quantum gates. These geometric phases depend solely on the geometry of the evolution of the system. GQC has been used in several different experimental platforms, such as superconducting qubits [2].

The purpose of this project has been to develop physical schemes of geometric entangling two-qubit gates on the Schmidt sphere, a geometric construct appearing in pure two-qubit systems. In this way, entanglement is used as a pure resource for implementing a novel form of geometric gates. To achieve this pulsed realizations of different types of qubit-qubit interacting Hamiltonians were applied on a system of mutually orthogonal quantum states (the Hamiltonians being what traces out a curve on the sphere), which are typically on the form:

$$H_{\mathrm{I}} = \mathbf{S}_a \mathbb{J} \mathbf{S}_b. \tag{1}$$

$\mathbf{S}_{a,b}$ are the spin variables of the two particles. J is a coupling constant governing the strength of the interaction between the qubits which takes the following shape:

$$\mathbb{J} = \begin{pmatrix} J_{xx} & J_{xy} & 0 \\ J_{yx} & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{pmatrix}. \tag{2}$$

The two key objectives of the project have been to identify pulsed sequences of interactions to generate geodesic polygons on the Schmidt sphere with no change of Schmidt vectors and to implement different types of geometric gates controlled by piecewise variation of the coupling parameters $J_{xx}, J_{xy}, \ldots, J_{zz}$.

Essentially the overall aim has been to develop new schemes for implementing robust entangling quantum gates solely by means of interactions intrinsic to the computational systems.

2 Background

2.1 Single qubit systems and the Bloch sphere

A single qubit is a quantum system that can be described by two normalized and orthogonal quantum states often denoted as $|0\rangle$ and $|1\rangle$. Unlike a regular bit which can only exist as a Boolean state (either 0 or 1) a qubit can exist as an arbitrary superposition of $|0\rangle$ and $|1\rangle$, which can be written on the form $\alpha|0\rangle + \beta|1\rangle$. This state can also be written by a geometric representation:

$$|\psi\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle), \tag{3}$$

where $e^{i\gamma}$ represents a global phase factor which has no observable effect. θ and ϕ parametrize a unit sphere referred to as the Bloch sphere, where each value of θ and ϕ define a point on this sphere (see figure (1)). The Bloch sphere works as a geometric representation of a single qubit system and is often used as a visualisation tool [5].

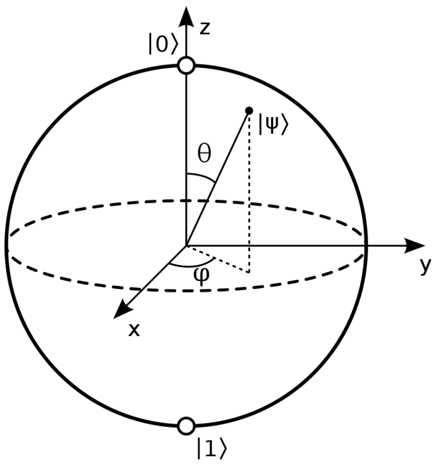

Figure 1: Image of the Bloch sphere. It can be seen that the angles $\theta = 0$ and $\theta = \pi$ corresponds to the states $|0\rangle$ and $\langle 1|$ respectively.

2.2 Two-qubit/bipartite systems and the Schmidt sphere

More qubits can be put together in order create higher order systems consisting of 2^n computational basis states, which can also exist as superpositions of these computational basis states. A two-qubit system thus has four computational basis states $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ each with an associated amplitude such that its state vector takes the form [5]:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \tag{4}$$

For multiple qubit systems there is unfortunately no simple generalization of the Bloch sphere, but there is a way to visualize bipartite systems in a similar way by using the so called Schmidt sphere [4].

In order to understand the Schmidt sphere we first look at Schmidt decomposition and how it can be used on bipartite systems. Schmidt's theorem states that any pure state $|\psi\rangle$ consisting of a composite system (which in the case of this project are bipartite states), there exist two orthonormal states for each subsystem such that the $|\psi\rangle$ can be written as a superposition of the products of these orthonormal states. The dimension of this decomposed state is set by the dimension of the lowest subsystem [5].

For the case of two qubits a and b this takes the form:

$$|\psi_{ab}\rangle = \cos\frac{\alpha}{2}\,|\mathbf{n}_a\rangle\,|\mathbf{n}_b\rangle + e^{i\beta}\sin\frac{\alpha}{2}\,|{-\mathbf{n}_a}\rangle\,|{-\mathbf{n}_b}\rangle\,, \tag{5}$$

where α and β are polar angles which are used to parametrize the "Schmidt sphere" similar to that of the Bloch sphere. $|\pm\mathbf{n}_k\rangle$, $k \in a, b$ are mutually orthogonal states [4].

2.3 Phases

"Phase" is a term often used in quantum mechanics with many different meanings. For example in equation (3) $e^{-i\phi}$ is called a global phase since it is physically indistinguishable from the same state without this phase factor. If M is a measurement operator, then the probability of a measurement is $\langle\psi|\,M^\dagger M\,|\psi\rangle$, which stays the same over a global phase [5]. In GQC the most important phases are the dynamical and the geometric phases, which we get by analyzing the Schrödinger equation [2].

The Schrödinger equation governs the time evolution of a system, where the Hamiltonian H contains the information of its evolution.

$$i\hbar\frac{d}{dt}\,|\psi(t)\rangle = H(t)\,|\psi(t)\rangle\,. \tag{6}$$

If some system undergoes a cyclic evolution i.e. an evolution which will bring it back to its starting state after some time τ, $|\psi\rangle$ representing the system will trace out a path C (e.g. on the Bloch or Schmidt sphere). The starting and final state will look the same but can be differentiated by some phase $e^{i\phi}$. Actually for any point $|\psi(t)\rangle$ on the curve we can choose a state $\left|\tilde{\psi}(t)\right\rangle$ so that $\left|\tilde{\psi}(0)\right\rangle = |\psi(\tau)\rangle$, from which we can then write:

$$|\psi(t)\rangle = e^{if(t)}\left|\tilde{\psi}(t)\right\rangle. \tag{7}$$

The change in the phase which is associated with the cyclic evolution on C will thus be given by $\phi = f(\tau) - f(0)$. It is said that the system has 'picked up' a phase ϕ after passing one evolution. The total phase ϕ can be calculated by rewriting the Schrödinger equation (6) by inserting (7) and integrating over the period τ. This gives the following expression:

$$\phi = -\frac{1}{\hbar}\int_0^T \langle\psi(t)|\,H\,|\psi(t)\rangle\,dt + i\int_0^T \left\langle\tilde{\psi}(t)\right|\frac{d}{dt}\left|\tilde{\psi}(t)\right\rangle dt. \tag{8}$$

The total phase ϕ can be decomposed into two phases. The first one consisting of the left component is the dynamical phase which is clearly time dependent since it contains the Hamiltonian. The second phase is the geometric phase consisting of the right part of the equation which only depends on the path the system travels along, which for a single qubit system could be its path along the Bloch sphere and for a two-qubit system its path along the Schmidt sphere. The dynamical phase is completely independent of the Hamiltonian as well as the rate at which the system evolves [2].

2.4 Geometric quantum computation

In order to create quantum gates solely relying on geometric phases the dynamical phase needs to be eliminated. Eliminating the dynamical phase would mean having the total phase be equal to the geometric phase ($\phi = \phi_G$), thus making it independent of time. There are several different methods for achieving this with one such method being creating a cyclical system that only travels along geodesic lines. A geodesic line is simply the shortest path between two points on a sphere. Creating a cyclical system will mean connecting two or more geodesic lines thus creating so called geodesic polygons [6].

The time evolution operator $U(t)$ is a unitary operator which as its name suggests describes the time evolution of a system. Using the operator at time τ on some state going through a cyclic evolution as mentioned above we can see from (7) that we get the following expression [5][2]:

$$U(\tau)\,|\psi(0)\rangle = \left|\tilde{\psi}(\tau)\right\rangle \rightarrow \tag{9}$$

$$U(\tau) = e^{-i\phi_G}\,|\psi(0)\rangle\,\langle\psi(0)|\,. \tag{10}$$

This type of time evolution operator can be seen as equivalent to a quantum gate, since it takes an input state and spits out an output state which will have changed the parameters of the system.

What defines GQC is in general a few criteria. The first is based around having a complete basis of orthonormal (ON) vectors that span the state space, where the amount of ON basis needed to span the state space is related to the number of basis states, $N = 2^n$ where N is the number of vectors needed for n qubits. The second criteria is based around each vector only picking up geometric phases after its evolution. A geometric quantum gate U at time τ for an n-qubit system would thus take the form [2]:

$$U(\tau) = \sum_{k=1}^{N} e^{i\phi_k}\,|\psi_k\rangle\,\langle\psi_k|\,, \tag{11}$$

which is nontrivial unless all ϕ_k are equal up to some integer multiplication of 2π.

2.5 Entanglement and concurrence

Entanglement is a characteristic property in quantum systems in which two or more states cannot be described separately from one another. I.e that the state vector describing the system can no longer be separated into a product of the subsystem states. Even if these states were to be separated physically they still would not be independent from one another. Thus measuring one subsystem gives information about the other system, but also gives the ability to manipulate the properties of the other system [8].

The level of which two qubits are entangled can be measured by its so called concurrence. Working for both pure and mixed states. Concurrence is defined by the formula [7]:

$$C = \left| \left\langle \psi \middle| \tilde{\psi} \right\rangle \right|, \tag{12}$$

where $\left| \tilde{\psi} \right\rangle$ is a "spin flip" transformation of $|\psi\rangle$ defined by:

$$\left| \tilde{\psi} \right\rangle = Y \otimes Y \left| \psi^* \right\rangle, \tag{13}$$

where Y is the Pauli Y operator and $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ expressed in fixed basis states. This transformation essentially takes the state of a single e.g. $|0\rangle$ and flips it to $|1\rangle$. In order to apply this to a two-qubit system the operation is applied on each individual qubit. A measurement like concurrence does not exist for systems containing more qubits and thus this simple analytical expression is a unique property of two-qubit systems. Concurrence goes between 0 and 1 where $C = 0$ describes a completely unentangled system and $C = 1$ describes one that is completely entangled. For a completely entangled bit the term e-bit (entanglement bit) is sometimes used. For systems with $C < 1$ the concurrence gives a measurement of the amount of an e-bit that system is. In something like quantum teleportation C can be seen as a measurement of how well this teleportation can be done, where $C = 1$ will represent a perfect teleportation. As C shrinks the teleportation becomes more and more randomized and at C = 0 the state achieved after the teleportation will have lost all its relation to the state it was supposed to teleport [7].

For a two-qubit system like the one described in equation (4) the concurrence will take the shape of the following equation:

$$C = 2|ad - bc|. \tag{14}$$

2.6 Spin-spin coupling

In order to prepare entangled states there needs to be an interaction between between the qubits. The strength of this coupling is often described using the coupling parameter J. J can be written in terms of a $3x3$ matrix like in (2), where each term $J_{i,j}, (i, j \in x, y, z)$ refers a certain type of coupling. For example J_{zz} is equal to terms in front of $Z \otimes Z$, J_{xy} for $X \otimes Y$ and so on. In general the word coupling is used in quantum mechanics when the evolution on some part of the system is dependent on some other quantity [5].

In multiple qubit operations it is common to write the operations using the Pauli operators $(\mathbb{1}, X, Y, Z)$, which generalize computation basis measurements to give the parity between the different qubits. Operators on Pauli form often include terms such as $Z \otimes \mathbb{1}, Y \otimes X$ and so on, which all have meanings for how the system is entangled. Each Pauli operator can be coupled to a spin operator S such that:

$$S_x = \frac{1}{2}X, S_y = \frac{1}{2}Y, S_z = \frac{1}{2}Z. \tag{15}$$

Remember that $\hbar = 1$.

The simplest interaction between qubits comes from the so called Ising model, which is a spin-spin coupling between two spin 1/2 particles on the form:

$$H = \mathbb{J}\mathbf{S}_{1,z} \otimes \mathbf{S}_{2,z}, \tag{16}$$

where $S_{i,j} i, j \in 1, 2$ refers to the spin operators of the two qubits. From the expression it can be seen that this type of Hamiltonian corresponds to J_{zz} [8]. A more general interaction comes from the Heisenberg model which takes on the form:

$$H = \frac{2\lambda}{\hbar}\bar{\mathbf{S}}_1 \cdot \bar{\mathbf{S}}_2, \tag{17}$$

where $S_{1,2}$ represent the spin operators belonging to the the spin pair and λ is the strength of the interaction. Dzyaloshinskii-Moriya (DM) also known an antisymmetric exchange is another type of spin-spin coupling:

$$H = D_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_j). \tag{18}$$

This type of coupling thus gives rise to cross terms in $\mathbb{J}$, such as J_{yx}, J_{xy} and so on [1].

The last coupling we will bring up is the XY-model, which is described by:

$$H = (1 + r)S_x \otimes S_x + (1 - r)S_y \otimes S_y. \tag{19}$$

Inserting the Pauli operators into the equation it is quite clear that these types of terms corresponds to J_{xx} and J_{yy}. Single qubit terms are also capable of appearing in a Hamiltonian affecting a two-qubit system. One such example is the Zeeman term which represents the qubits being places in an external magnetic field going in the z-direction. Such a term could be on the form [6]:

$$H = -\mu_z B. \tag{20}$$

3 Method

3.1 Defining states

The main method used in this project is based on so called reverse engineering, which works by first defining the states one wishes to end up with as well deciding the type of evolution one wishes to see and then in the end using the Schrödinger equation in order to find a Hamiltonian that gives the sought after evolution. Since the goal is to design two-qubit gates by means of GQC we start the analysis by defining four different mutually orthogonal states (one for each basis state) . Two of them being entangled systems and two being uncoupled. The four states were chosen to be on the form:

$$|\psi_1(t)\rangle = f\,|n,m\rangle + g\,|-n,-m\rangle\,, \tag{21}$$
$$|\psi_2(t)\rangle = -g^*\,|n,m\rangle + f^*\,|-n,-m\rangle\,, \tag{22}$$
$$|\psi_3(t)\rangle = |n,-m\rangle\,, \tag{23}$$
$$|\psi_4(t)\rangle = |-n,m\rangle\,. \tag{24}$$

The basis states $|i,j\rangle_{[i,j]=[\pm n,\pm m]}$ represent the Schmidt vectors, which can consist of the basis states of a two-qubit system. Actually n and m can be set as specific states without loss in generality, thus we immediately set $n, m = 0$ and $-n, -m = 1$. The coefficients f, g and their complex conjugate counterparts are set as time dependent variables containing the angles α and β which will parameterize the Schmidt sphere. Meanwhile the Schmidt vectors where are to be constant in time, thus the time evolution of the system only relies on the change of the coefficients in front of them. This in turn makes $|\psi\rangle_3$ and $|\psi_4\rangle$ constants in time.

3.2 Evolving the system and computing the parameters

The parameters f and g are set to be on a geometric form in order to parameterize the Schmidt sphere, where f is purely real since one can always choose to pull out an overall global phase out of the system. Since f is purely real it follows that $f = f^*$. f and g are thus chosen to be:

$$f = \cos\frac{\alpha}{2}, \tag{25}$$

$$g = \sin\frac{\alpha}{2}e^{-i\beta}. \tag{26}$$

After defining the states and the parameters the next step is defining the evolution of the system is such a way that it evolves purely on the Schmidt sphere. The system chosen travels only along geodesics in order to remove the previously mentioned dynamical phase (See figure (2)).

The system is chosen to start at a maximally entangled state, which is $\alpha = \frac{\pi}{2}$ and $\beta = 0$ (one can check that the amplitude of the Schmidt vectors are equal for these values). As a matter of fact any other point on the Schmidt sphere can be chosen as a starting point apart from the north pole, i.e $\alpha = 0$ and β some arbitrary angle between 0 and 2π. It can be shown using concurrence that this type of gate would be incapable of entanglement (see Appendix).

The evolution is then done in two steps (see figure (2)), using two different Hamiltonians. The first which takes the system from the starting point along the equator past the y-axis and to the negative x-axis and the second which

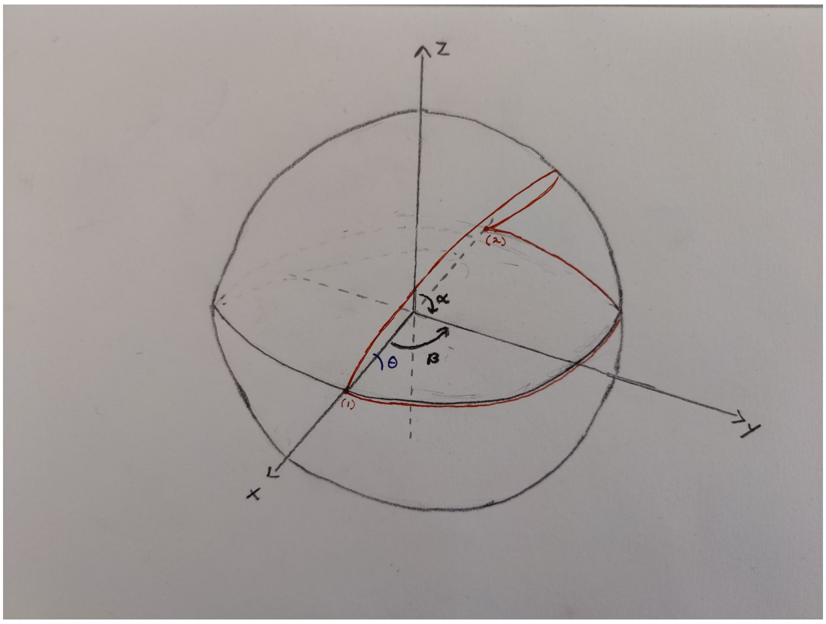

Figure 2: Illustration of the Schmidt sphere along with two curves along its surface. θ represents the angle which the second curve is lifted from the xy-plane. The solid angle of the curve Ω is defined as $\Omega = 2\theta$.

takes the system along a curve that is lifted from the xy plane by an angle θ from the negative x-axis back to the starting point.

In the case of the first Hamiltonian it can be seen that $\alpha(t)$ remains constant at $\frac{\pi}{2}$, which means that $\beta(t)$ can be chosen as some arbitrary function that takes β from 0 to π within some given period. For the second Hamiltonian α and β will depend on one another since they both evolve in time. By projecting down the curve onto the xy-plane it can be seen that it forms an elliptical shape as seen in figure (3), thus giving the following relation between x and y:

$$y = \cos\theta\sqrt{1 - x^2}. \tag{27}$$

By rewriting the equation into spherical coordinates ($x = \sin\alpha\cos\beta$, $y = \sin\alpha\sin\beta$) and solving for $\sin\alpha$ we get the following equation:

$$\alpha = \arcsin\left(\left(\frac{\sin^2\beta}{\cos^2\theta} + \cos^2\beta\right)^{-1/2} \right). \tag{28}$$

Similar to before β can then be chosen to be some function of t that takes β from π to 0 over some period.

3.3 The Schmidt gate

The two states $|\psi_{1,2}\rangle$ have both undergone identical cyclical evolutions on the Schmidt sphere with the same spatial angle defining the curve, which means

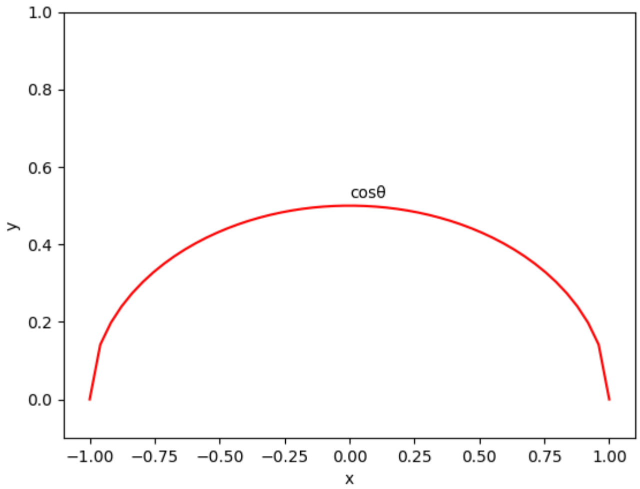

Figure 3: Graph showing the projection of the curve from figure 2 projected down on the xy-plane for some value θ. Based of of equation (27).

their solid angle Ω is also the same. The geometric phase of a two-qubit system traveling along a geodesic curve on the Schmidt sphere will have a geometric phase given by $\phi_G = \theta = \frac{1}{2}\Omega$[4]. The geometric phase picked up by both these states will thus be identical, their terms only being differentiated by being complex conjugates of one another. $|\psi_{3,4}\rangle$ also exist on their own Schmidt spheres, but since they undergo no evolution they will not pick up any phase. The quantum gate generated by this evolution we dub the "Schmidt gate". It is described by the four states $|\psi_{1,2,3,4}\rangle$ which are fixed throughout the evolution as well as the geometric phases picked up by $|\psi_{1,2}\rangle$. Based on (11) and the arguments above the Schmidt gate will take the form of:

$$U(t) = e^{-i\Omega/2} |\psi_1\rangle \langle\psi_1| + |01\rangle \langle 01| + |10\rangle \langle 10| + e^{i\Omega/2} |\psi_2\rangle \langle\psi_2| . \tag{29}$$

3.4 The Schmidt gate and entanglement

When applying the Schmidt gate (29) with a generic two qubit state vector as our input state (see equation (4)) we will end up with the following output state:

$$\begin{aligned} U|\psi\rangle = \quad & (a(e^{-i\Omega/2}\cos^2\tfrac{\alpha}{2} + e^{i\Omega/2}\sin^2\tfrac{\alpha}{2}) + \tfrac{d\sin\alpha e^{i\beta}}{2}(e^{-i\Omega/2} - e^{i\Omega/2}))\,|00\rangle \\ & (\tfrac{a\sin\alpha e^{-i\beta}}{2}(e^{-i\Omega/2} - e^{i\Omega/2}) + d(e^{-i\Omega/2}\sin^2\tfrac{\alpha}{2} + e^{i\Omega/2}\cos^2\tfrac{\alpha}{2}))\,|11\rangle \\ & +b\,|01\rangle + c\,|10\rangle . \end{aligned} \tag{30}$$

The new amplitudes a, d will thus have (hopefully) changed after passing through the gate. a', d' corresponds to the terms in front of $|00\rangle$ and $|11\rangle$ respectively as seen in the following equations:

$$a' = a(e^{-i\Omega/2}\cos^2\tfrac{\alpha}{2} + e^{i\Omega/2}\sin^2\tfrac{\alpha}{2}) + \tfrac{d\sin\alpha e^{i\beta}}{2}(e^{-i\Omega/2} - e^{i\Omega/2}),$$
$$d' = \tfrac{a\sin\alpha e^{-i\beta}}{2}(e^{-i\Omega/2} - e^{i\Omega/2}) + d(e^{-i\Omega/2}\sin^2\tfrac{\alpha}{2} + e^{i\Omega/2}\cos^2\tfrac{\alpha}{2}). \quad (31)$$

The concurrence of the input state is just equation (14), meanwhile the concurrence of the output state will be given a similar expression but with a, d replaced with a', d'

$$C' = 2|a'd' - bc|. \quad (32)$$

In order to see if the entanglement of the system has changed we look for a difference in concurrence between the input and output states. Which will only be the case if $a'd' \neq ad$. Inserting the expressions for a' and d' into $a'd'$ we get:

$$a'd' = \frac{\sin\alpha}{2}(a^2e^{-i\beta} + d^2e^{i\beta})((e^{-i\Omega} - 1)\sin^2\frac{\alpha}{2} + (1 - e^{i\Omega})\cos^2\frac{\alpha}{2}) \quad (33)$$
$$+ ad(1 + \frac{\sin^2\alpha}{2}(e^{i\Omega} + e^{-i\Omega} + 1)). \quad (34)$$

It can be seen that using the starting point of $\alpha_0 = 0$ or $\alpha_0 = \pi$ gives $C = C'$. Such a gate would thus give no change in the entanglement. These starting values is the same as starting at a product state of $|\psi_1\rangle$ and $|\psi_2\rangle$. It is also important to note that starting anywhere other than the poles will give a change in the concurrence thus making U an entangling quantum gate.

3.5 Finding the Hamiltonians

Now that we have created quantum gate all that is left is to find a Hamiltonian which would give rise to this gate. From the four states defined in the begining of this section the general form of the Hamiltonian can be calculated by using the Schrödinger equation (6). Since $|\psi_3\rangle$ and $|\psi_4\rangle$ are completely unchanged in time the Hamiltonian will not affect these states. The Schrödinger equation will thus only be written for $|\psi_1\rangle$ and $|\psi_2\rangle$. This gives the following equations:

$$i\begin{bmatrix}\dot{f}\\ \dot{g}\end{bmatrix} = \begin{bmatrix}h_{11}f & h_{12}g\\ h_{12}^*f & h_{22}g\end{bmatrix}, \quad (35)$$

$$i\begin{bmatrix}-\dot{g}^*\\ \dot{f}^*\end{bmatrix} = \begin{bmatrix}-h_{11}g^* & h_{12}f^*\\ -h_{12}^*g^* & h_{22}f^*\end{bmatrix}. \quad (36)$$

where $h_{i,j}$ are the components of the Hamiltonian. Solving this equation for the components gives the Hamiltonian:

$$H = i\begin{bmatrix}\dot{f}f^* + \dot{g}^*g & g^*\dot{f} - \dot{g}^*f\\ \dot{g}f^* - \dot{f}^*g & \dot{g}g^* + \dot{f}^*f\end{bmatrix}. \quad (37)$$

It can be seen that this Hamiltonian is trace-less, since $h_{11} + h_{22} = 0$. The Hamiltonian can be rewritten in terms tensor products of the Pauli operators giving the following expression:

$$H = \frac{1}{4}((h_{11} - h_{22})(Z \otimes \mathbb{1} + \mathbb{1} \otimes Z) + i(h_{12} - h_{12}^*)(X \otimes Y + Y \otimes X) \\ +(h_{12} + h_{12}^*)(X \otimes X - Y \otimes Y)). \quad (38)$$

These Pauli operators can later be coupled to the spin operators. Components containing $\mathbb{1} \otimes \mathbb{1}$ and $Z \otimes Z$ disappear due to containing the sum of h_{11} and h_{22}.

Next we rewrite (37) and (38) by inserting the expressions for f and g as well as their time derivatives, which gives the following expressions:

$$H = \begin{bmatrix} -\dot{\beta}\sin^2\frac{\alpha}{2} & \frac{e}{2}^{i\beta}(\dot{\beta}\sin\alpha - i\dot{\alpha}) \\ \frac{e}{2}^{-i\beta}(i\dot{\alpha} + \dot{\beta}\sin\alpha) & \dot{\beta}\sin^2\frac{\alpha}{2} \end{bmatrix}, \quad (39)$$

$$H = \tfrac{1}{4}(-2\dot{\beta}\sin^2\tfrac{\alpha}{2}(Z \otimes \mathbb{1} + \mathbb{1} \otimes Z) + (\dot{\alpha}\cos\beta - \dot{\beta}\sin\alpha\sin\beta)(X \otimes Y + Y \otimes X) + \\ (\dot{\beta}\sin\alpha\cos\beta + \dot{\alpha}\sin\beta)(X \otimes X - Y \otimes Y)). \quad (40)$$

The terms in front of the tensor products for XX,YY,YX,XY can be assigned to the coupling parameter J (2), where $J_{xy} = \dot{\alpha}\cos\beta - \dot{\beta}\sin\alpha\sin\beta$ and so on for the other terms. Expressing J as a 4x4 matrix it takes the shape:

$$\mathbb{J} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix}. \quad (41)$$

Inserting the coefficients from (40) into (41) gives the expression:

$$\mathbb{J} = \frac{1}{4}\begin{bmatrix} \dot{\beta}\sin\alpha\cos\beta + \dot{\alpha}\sin\beta & \dot{\alpha}\cos\beta - \dot{\beta}\sin\alpha\sin\beta \\ \dot{\alpha}\cos\beta - \dot{\beta}\sin\alpha\sin\beta & -\dot{\beta}\sin\alpha\cos\beta + \dot{\alpha}\sin\beta \end{bmatrix}. \quad (42)$$

In the end two Hamiltonians will be generated. One for each curve segment. They are both based of the same general expression (40) but they will differ by how the parameters α and β are defined. In the first Hamiltonian α is a constant $\alpha_1 = \frac{\pi}{2}$ and in the second α_2 is equal to equation (28). Both $\beta_{\{1,2\}}$ can as explained above be set to some arbitrary function that that take β from its starting position to its final one and are thus left as is in the results.

Inserting $\alpha_{\{1,2\}}$ and $\beta_{\{1,2\}}$ into equations (25) and (26) and calculating their time derivatives as well as g^* and its time derivative and inserting them into (38) finally gives the explicit form of the Hamiltonians as:

$$H_1 = \frac{\dot{\beta}_1}{4}(-(Z \otimes \mathbb{1} + \mathbb{1} \otimes Z) - \sin(\beta_1)(X \otimes Y + Y \otimes X) + \cos(\beta_1)(X \otimes X - Y \otimes Y)), \quad (43)$$

$$H_2 = \tfrac{1}{4}(-2\dot{\beta}\sin^2(\tfrac{1}{2}arcsin((\tfrac{\sin^2\beta}{\cos^2\theta} + \cos^2\beta)^{-1/2}))(Z \otimes \mathbb{1} + \mathbb{1} \otimes Z) \\ -(\frac{tan^2\theta\cos^2\beta\sin\beta}{(\frac{\sin^2\beta}{\cos^2\theta}+\cos^2\beta)^{3/2})\sqrt{1-(\frac{\sin^2\beta}{\cos^2\theta}+\cos^2\beta)^{-1})}} + \dot{\beta}\sin\beta(\tfrac{\sin^2\beta}{\cos^2\theta} + \cos^2\beta)^{-1/2})(X \otimes Y + Y \otimes X) + \\ (\dot{\beta}\cos\beta(\tfrac{\sin^2\beta}{\cos^2\theta} + \cos^2\beta)^{-1/2}) - \frac{tan^2\theta\cos\beta\sin^2\beta}{(\frac{\sin^2\beta}{\cos^2\theta}+\cos^2\beta)^{3/2})\sqrt{1-(\frac{\sin^2\beta}{\cos^2\theta}+\cos^2\beta)^{-1})}})(X \otimes X - Y \otimes Y)) \quad (44)$$

4 Discussion and outlook

One of the first things seen in the Hamiltonian (40) are the tensor product terms. The first term relates to a Zeeman term (see section 2.6), since it only goes in the z direction. The consequences of this term are quite big for the initial assumptions made in this project. This shows that the form that the Hamiltonian (1) was assumed to take was insufficient. The Zeeman term shows clearly that one-qubit terms are also necessary to create this system.

The term containing $XX - YY$ is similar to the XY model described in 2.6 with an $r = 0$, however the minus sign in between is quite unconventional (I will continue to call i the "XY term" in the rest of the discussion). Interestingly this term can be transformed into an XY coupling by rotating one of the qubits with the angle π around the x-axis (see Appendix). Actually the same transformation also affects the symmetric term turning it into the anti-symmetric DM (Dzyaloshinskii-Moriya) coupling.

Another thing to note are the terms that do not appear in the Hamiltonian. Due to the fact the Hamiltonian is trace-less the terms containing $\mathbb{1} \otimes \mathbb{1}$ and $Z \otimes Z$ disappeared. This meant that $J_{zz} = 0$ which represented the Heisenberg coupling. As a matter of fact neither the Ising or DM couplings where found in the Hamiltonian. Considering that a rotation of one of the qubits gives rise to a DM term, maybe some other choice of Schmidt basis might have led to both of their appearance.

There is quite a clear difference between the first and second Hamiltonian. While H_1 has a rather clear and simple expression H_2 is significantly longer and more complex. This difference comes rather clearly from the fact that in H_2 there is a trigonometric relation between α and β (28). An interesting fact about H_1 (43) is that the part containing the Zeeman term has no dependence on β, depending only on the value of $\dot{\beta}$. If $\beta(t)$ is set to be some linear function of t then this would mean that the strength of this term is constant throughout this curve. The parts containing the symmetric exchange and XY term both have a trigonometric dependency on β. By looking at this dependency it can be seen that at the starting and end time there will be no symmetric exchange, while on the contrary the XY term will be at a maximum. The opposite happens halfway through the curve at $\beta = \frac{\pi}{2}$ where the symmetric exchange instead reaches a maximum and the XY term disappears.

Due to the complexity of H_2 analyzing its properties is slightly harder, but there are still certain conclusion that can be made. The angle θ influences each coupling seen in equation (44). It can be seen that this Hamiltonian will not work for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ since certain parts will go of to infinity as θ approaches these values. These values for θ represent paths along the poles of the Schmidt sphere. In order to create such a curve one might have to create what is essentially three curves where one stops when it reaches the north pole. Similar to H_1 the XY and symmetric terms also get reduced to 0 at the same values of β. The Zeeman term is also nonzero for the whole path, however the strength of the coupling varies over the curve.

One of the more surprising findings of the project has been the effect that different starting values have on the concurrence. The fact that starting at one of pole of the Schmidt sphere gives a gate incapable of entangling two qubits was definitely non-trivial. This also shows a clear limitation of the Schmidt gate, however the actual effect this would have when designing specific gates meant for computation would have to be analyzed.

In the analysis two of the four states ($|\psi_3\rangle, |\psi_4\rangle$ where set to be constant. Their existence in the analysis where only really justified by $|01\rangle, |10\rangle$ needing to be a part of a two qubit gate. Other than that they remained unused. These states could have been set up in a similar way to $|\psi_1\rangle$ and $|\psi_2\rangle$ with their own dynamic parameters like g and f. One way to write this would have been:

$$|\psi_3\rangle = \eta |01\rangle + \kappa |10\rangle, \tag{45}$$

$$|\psi_4\rangle = -\kappa^* |01\rangle + \eta^* |10\rangle. \tag{46}$$

This type of system would have given rise to an entirely different Hamiltonian giving the need to analyse more parameters, as well as giving new relations when writing out the Hamiltonian in Pauli operator form. Giving these states their own cyclical evolution along the Schmidt sphere along with $|\psi_1\rangle$ and $|\psi_2\rangle$ could give rise to an entirely different form of quantum gate. A way to expand this project could be to apply the method developed in this project on a system which now includes the states (45) and (46).

There are several different ways to create geodesic polygons on a sphere. The evolution chosen for this project was to create a so called "orange slice" (Named so due to the shape of the curve, see figure (2)). One could instead create for example a spherical triangle. In order to create such an evolution three Hamiltonians would need to be defined, one for each side of the triangle. The general form of the Hamiltonian would remain the same in this scenario since it only depends on how the states are defined. In a future project one could analyse to see if this type of curve leads to some interesting type of entanglement or if the outcome is similar to the one in this project.

5 Conclusion

Through the use of reverse engineering a geodesic curve on the Schmidt sphere could be designed and two Hamiltonians were able to be found that through piece wise implementation would give rise to this evolution. Connections to spin-spin couplings potentially capable of being implemented in a real world experiment where found in the Hamiltonians in the form of Zeeman, XY and symmetric terms. The Hamiltonians found in turn created a geometric gate dubbed the Schmidt gate capable of entangling a two qubit state as shown by the change in concurrence of a two-qubit state passing through the gate.

Some surprising findings where made, such as starting the system from the north pole of the Schmidt sphere created a gate incapable of entanglement. Other finding was the lack of any J_{zz} term in the coupling of the Hamiltonian and the inclusion of single cubit terms in the coupling in the form of the Zeeman terms.

The Schmidt sphere proved to be both a powerful analytical tool in terms of being able to create this type of evolution in the first place, but also a power full visualisation tool. Both due to how it allowed for a clear mapping of the evolution of the system, but it also gave the possibility to clearly map the relation between the parameters α and β which was done by utilising the projection of geodesic curve on a sphere.

In the end the method developed in this project shows a clear success in creating geometric entangling quantum gates. The main theoretical way to expand upon this project would probably be to try and implement new gates by creating a system where all basis states go through an evolution on the Schmidt sphere. The inherent fault tolerance of creating gates using GQC would make it interesting to see if this type of geometric gate could be implemented experimentally or eventually even be implemented in a real quantum computer.

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Images

Picture of the Bloch sphere:

[9] By Smite-Meister - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=5829358

Appendix

Rotating one of the qubits around the x-axis with the angle π on the XY terms in the Hamiltonian responds to the transformation:

$$
\begin{gathered}
(iX) \otimes \mathbb{1}(X \otimes X - Y \otimes Y)(-iX) \otimes \mathbb{1} = \\
(iXX(-iX)) \otimes X - (iXY(-iX)) \otimes Y = \\
(XXX) \otimes X - (XYX)Y.
\end{gathered} \tag{47}
$$

By writing out X and Y in matrix form it can be shown that:

$$XXX = X, \tag{48}$$

$$XYX = -Y. \tag{49}$$

Inserting these into the above transformation we can see that it becomes the following:

$$XX + YY, \tag{50}$$

which is precisely what the standard XY-model looks like.

Applying the same transformation on the symmetric exchange gives the following:

$$
\begin{gathered}
(iX) \otimes \mathbb{1}(X \otimes Y + Y \otimes X)(-iX) \otimes \mathbb{1} = \\
(iXX(-iX)) \otimes Y + (iXY(-iX)) \otimes X = \\
XY - YX.
\end{gathered} \tag{51}
$$

This is the exact the expression for DM. This transformation thus seems to change the our couplings into a similar but slightly different type of coupling.