Studies on Reverse Engineering of Constant Frequency Geometric Quantum Gates

Dissertation in partial fulfillment of the requirements for the degree of

Master of Theoretical Physics majoring in Quantum Field Theory and String Theory

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Abstract

Geometric Quantum Computation (GQC) is one of the effective methods to realize quantum computation [1]. By using geometric phases, it shows robustness to certain errors. This thesis aims to develop a new systematic technique for implementing GQC, especially in non-adiabatic systems. First, we examine the nature of the geometric phase with differential geometry. Then, we give a general theoretical method to realize a given quantum gate with a geometric phase through reverse engineering. We examine the method in the constant frequency quantum system with 2 or 3 energy levels. Finally, we analyze the non-Abelian case where some of the frequencies are degenerate.

Abstrakt

Geometrisk kvantberäkning (GQC) är en av de effektiva metoderna för att förverkliga kvantberäkningar[1]. Genom att använda geometriska faser visar den sig vara robust mot vissa fel. Syftet med denna avhandling är att utveckla en ny systematisk teknik för att genomföra GQC, särskilt i icke-adiabatiska system. Först undersöker vi den geometriska fasens natur. med hjälp av differentiell geometri. Därefter ger vi en allmän teoretisk metod för att realisera en given kvantgrind med en geometrisk fas genom reverse engineering. Vi undersöker metoden i kvantsystemet med konstant frekvens och med 2 eller 3 energinivåer. Slutligen analyserar vi det icke-Abeliska fallet där vissa av frekvenserna är degenererade.

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1 Introduction

The idea of quantum computing is to use some special phenomena of quantum systems such as interference and entanglement, to perform computation. Although most of today's quantum computing experiments are still simulations on classical computers, quantum computation shows great potential and is worthy of research since traditional approaches on improving computer technology are beginning to run up against practical limitations on equipment's size. More important, quantum computation provides means for efficient solution of some computationally hard problems.

It is easy to begin the construction of quantum computing from the analogy of classical computing. Using quantum gates instead of traditional gates, one needs to build a circuit with inputs, outputs, and logic gates to carry out a given computation. However, this circuit-like quantum computation has an important problem to solve. It depends on the ability to perform a universal set of quantum gate operations on a set of qubits; it needs the quantum gates in practice to be resilient to certain kinds of errors. At the same time, the quantum gates are too fragile to keep unaffected by the environment. A possible solution is to use Geometric Quantum Computation (GQC) [2], also termed as Holonomic Quantum computation(HQC) [3]. It allows building more robust quantum gates in a universal set by using geometric phases [4], which are also called 'Berry phases' [5].

The basic idea of GQC is to realize a given set of quantum gates by transport of state vectors around loops [3]. For a quantum system with parameter dependent Hamiltonian, one can consider quantum gates as unitary operators. The geometric phase is the difference between the beginning and the end of the loop. Therefore, we can also say that GQC is the idea to geometric phase to perform the computing process. For computation on a single qubit, we can also imagine the process as a rotation of vector on Bloch sphere.

This work presents a more detailed explanation of how this process works in theory and develops a new technique to realize quantum computing with GQC. In Chapter 2, we focus on understanding GQC from a perspective of differential geometry. Then we study the possible forms of GQC evolution, as arising in adiabatic or non-adiabatic as well as, Abelian or non-Abelian. Among these situations, we use a non-adiabatic system as our example to develop a new technique of performing given quantum gates using GQC with reverse engineering, which is the central part of chapter 3. We apply our new method to two simple systems and implicitly modify these systems to perform our expected function. This chapter deduces each step to build a specific quantum gate with a geometric phase.

2 Theory of quantum geometric phase

2.1 Mathematical preparation

The idea of geometry quantum computation (GQC) is based on a specific quantum phenomenon, the geometric phase [6]. Therefore we first need to understand the derivation and property of geometric phase both in physics and mathematical perspective. Then we can find detailed methods of applying geometric phase in quantum computing

The origin of geometric phase is differential geometry features of the state space corresponding to a given quantum system. Hence it is necessary to begin at learning some knowledge of differential geometry, which will be quite useful in describing and reconsidering the process of quantum computing in the geometric language.

2.1.1 Topology, differentiable manifold and diffeomorphism

To understand the more complicated concept, we first need to get familiar with some basic research objects in differential geometry.
The most fundamental object is **topology** [7]. In a short word, a topology, or a topological space, is an open set, denoted as X, and can be considered as a collection of open subsets of X, denoted as τ, $\tau = \{U_i | i \in I\}$. τ should meet the following requirements:

- The empty set $\varnothing$ and universal set X are in τ.
- The intersection of any subsets in τ is still in τ.
- The union of any subsets in τ is still in τ.

We can build a map for two given topologies.One may see that $\mathbb{R}^m$ is a topological space. Hence specially, we can select one open set U_i in a general topology U and define a map f from U_i to another open set U_i' in $\mathbb{R}^m$. If $f : U_i \longrightarrow U_i'$ is continuous and has an inverse $f^{-1} : U_i' \longrightarrow U_i$, the map f is called a **homeomorphsim** [7].

Then we can define the differentiable manifold. Consider a topological space M. It has a family of pairs between open sets U_i and homeomorphsim ϕ_i, like $\{(U_i, \phi_i)\}$. If given two open sets U_i and U_j with $U_i \cap U_j \neq \varnothing$, we can build a map ψ_{ij} from $\phi_j(U_i \cap U_j)$ to $\phi_i(U_i \cap U_j)$ and $\psi_{ij} = \phi_i \circ \phi_j^{-1}$.

If this map ψ_{ij} is inifinitely differential, then the topology M is a **differentiable manifold**. The homeomorphsim ϕ_i, ϕ_j are also called **coordinate functions**.

As a result, if we have a map $f : M \longrightarrow N$ from m-dimensional manifold M to another manifold N with n-dimensions, we can represent the map in the space R^m and R^n with coordinate functions. Assuming the coordinate function of M is ψ and the coordinate function of N is φ, then we can rewrite the map f as:

$$f \longrightarrow \varphi \circ f \circ \psi^{-1}. \tag{2.1}$$

Denote the new map as χ

$$\chi = \varphi \circ f \circ \psi^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n \tag{2.2}$$

and assume it has the inverse function χ^{-1}:

$$\chi^{-1} = \psi \circ f^{-1} \circ \varphi^{-1}. \tag{2.3}$$

If both χ and χ^{-1} are infinitely differentiable, the map f is a **diffeomorphism**.

2.1.2 Connection and holonomy

From the definition of manifold we know that a manifold is locally like $\mathbb{R}^n$. Hence we can also define a **vector** with the differentiable structure of the manifold.

Given a manifold M, the vector is a tangent vector to a curve in M. Then we need to select a curve $c(t)$ and a function $f : M \longrightarrow \mathbb{R}$, Furthermore, we define the vector as a directional derivative of $f(c(t))$ the tangent vector at $t = 0$ we can write the tangent vector clearly as:

$$\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}|_{t=0}. \tag{2.4}$$

With the coordinate function ϕ of M, we can continue reducing it into

$$\frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}|_{t=0} = X^\mu(\frac{\partial f}{\partial x^\mu}) = X[f] \tag{2.5}$$

where:

$$X = X^\mu(\frac{\partial}{\partial x^\mu}), (X^\mu = \frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}) \tag{2.6}$$

If $V[M]$ is smooth to any point of M, it is a **vector field**, denoted as $\mathscr{X}[M]$.

However, a manifold can also carry a more complex structure if it admits a specific metric tensor. For the manifold as tensor field, the vector field cannot directly arise without an extra structure called **connection**.The connection gives that how tensors are transported along a curve [7]. Therefore it also allows us to compare vectors in different points of a manifold. For a manifold M and its vector field X, a connection is a map ∇

$$\nabla : \mathscr{X}[M] \times \mathscr{X}[M] \longrightarrow \mathscr{X}[M]. \tag{2.7}$$

Now we can use the connection to define a set of special transformations. Consider a m-dimensional manifold (M, g) with a Riemann metric tensor g and a connection ∇. We pick a point p in (M, g), and draw a set of closed loops at p:

$$\{c(t)|0 \leq t \geq 1, c(0) = c(1) = p\}.. \tag{2.8}$$

If we take a vector X at p, from the definition of vector, we know that the vector is in the tangent space T_pM at p. Then we perform a parallel transportation to X along the loop $c(t)$, which means moving the vector without rotation. We will return to p and get a new vector X_c also in the tangent space T_pM. From the definition of connection, that it gives how a tensor moves along a curve, we know that we can use the connection and the loop to describe our operation. Actually we induce a map:

$$P_c : T_pM \longrightarrow T_pM. \tag{2.9}$$

The map is a series of transformation along closed loops. It meets the requirements of group. It is called a **holonomy group** [7] and denoted as $H(p)$.

2.1.3 Fiber bundle

We now introduce the concept of fiber bundle to complete our mathematical preparation of describing the process of GQC. A fiber bundle consist of the following elements [7]:

- A differentiable manifold E called the **total space**.
- A differentiable manifold M called the **base space**.
- A differentiable manifold F called the **fiber**.
- A surjection $\pi : E \longrightarrow M$ called the **projection**. The inverse image of a given point p, that $\pi^{-1}(p) = F_p \cong F$ is called the fiber at p.
- A Lie group G,which acts on the left of F, called the **structure group**.
- A map ϕ_i called **trivialization**. Given a set of open covering $\{(U_i)\}$ of M with a diffeomorphism ϕ_i: $U_i \times F \longrightarrow \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$.
- A map t_{ij} called the **transition function**: If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the new map $\phi_{i,p}$ is another diffeomorphism, which works as $F \longrightarrow F_p$. We define $t_{ij} = \phi_{i,p}^{-1} \circ \phi_{j,p}$

and require that When $U_i \cap Uj \neq \emptyset$, $t_{ij} = \phi_{i,p}^{-1} \circ \phi_{j,p} : F \longrightarrow F$ be an element of G. Then we build a smooth map $t_{ij} : U_i \cap U_j \longrightarrow G$ which relates ϕ_i and ϕ_j by:

$$\phi_i(p, f) = \phi_i(p, t_{ij}(p)f). \tag{2.10}$$

In general, the role of fiber bundles is it provide a tool to makes a topology look like a direct product of two topological space. With the term of fiber bundles, we can naturally describe the map from Hilbert space to state space, which is important in understanding the geometric phase.

2.2 Derivation of geometric phase

The fundamental concept of quantum computing is the qubit, built upon an analogous concept of classical computation. In physics, a qubit means a two-level quantum system. Performing a 'computation' on a qubit is naturally performing a unitary operator to the state.

The state is also a vector in state space, a topological space. Recalling the formerly defined holonomy group, we may wonder whether there is a kind of operator in the holonomy group that performs parallel transportation of the state vector. Moreover, we also wonder what will happen after the state transports along a loop.

2.2.1 Fibers of a quantum mechanic system

Given a quantum system with Hamiltonian $H(\vec{\lambda}_t)$. The parameter vector $\vec{\lambda}_t$ is time-dependent. Consider a state $|\psi\rangle$ of the system . Assume $|\psi\rangle$ is in the nth eigenstate of this system at $t = 0$, which means

$$|\psi(0)\rangle = |n, \vec{\lambda}_t(0)\rangle \tag{2.11}$$

Denote the state vector at later time t as $|\psi(t)\rangle$. Obviously, $|\psi(t)\rangle$ belongs to an (N+1)-dimensional complex vector space with the null vector subtracted, that is

$$|\psi(t)\rangle \in C^{N+1} - \{0\}. \tag{2.12}$$

We can decompose $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \{Z_i(t)\}, i = 1,2,3...N. \tag{2.13}$$

Each Z_i is a complex parameter. The normalization condition

$$\Sigma_{i=0}^{N} \overline{Z}_i(t) Z_i(t) = 1 \tag{2.14}$$

defines a sphere S^{2N+1}. Therefore a normalized state will be further limited into the space of sphere S^{2N+1}.

However the state vector $|\psi(t)\rangle$ is not enough to represent a physical state. The physical state is a ray. The ray is a one-dimensional subspace of the state space. Its components are linked by $U(1)$ group. If $|\psi\rangle' = e^{i\theta}|\psi\rangle$, the two state vectors are in the same ray or the same subspace.The subspace is a equivalence class, which we denoted as $S^{2N+1}/U(1)$. All these equivalence classes form the space of physical states:

$$P(C^{N+1}) = \frac{S^{2N+1}}{U(1)} = \frac{C^{N+1} - \{0\}}{C - \{0\}} \tag{2.15}$$

We can also use N-dimensional projective space CP^N [8] to describe the space of physical states.

Recalling our definition of fiber bundle before, we can represent the parallel transportation process in the terms of differential geometry. The total space E here is the space of normalizable state vectors $C^{N+1} - \{0\}$. The base space M is the space of physical states $P(C^{N+1})$. The fiber is a set of all normalized vectors from the same ray. The projection ϕ is the association of the normalized vector $|\psi(t)\rangle$ to the operator $|\psi(t)\rangle\langle\psi(t)|$ [9]. The structure group acting on the fiber is $U(1)$.

The time evolution of the state vector produces a path in the total space $C^{N+1} - \{0\}$ [9]. With the fiber bundles we can first build the corresponding path in the base space $P(C^{N+1})$ and map it into the total space. Under parallel transportation of a physical state, around a closed loop, the inverse projection π^{-1} gives rise to a open path in total space. This means the final state vector has an overall phase difference from the initial one. Part of the overall phase is independent of time it takes to traverse the loop and relies on the geometric structure of the fiber bundle.

2.2.2 The generation of geometric phase

Given a closed curve $|\varphi(x)\rangle$ in base space M, in differential geometry, $|\varphi(x)\rangle$ provides a continuous map from a patch U in base space into the fibers above. If we change the curve $|\varphi\rangle$ to a different curve $|\varphi(x)'\rangle$, it can be seen as changing the local section to a new one. We can describe this process the with structure group $U(1)$

$$|\psi(x)'\rangle = e^{i\theta(x)}|\psi(x)\rangle, \tag{2.16}$$

where $\theta(x)$ is a real function of coordinates X^μ on M [9].

The closed curve will also map a closed path into total space E, which denoted as $|\varphi(t)\rangle$. Then the corresponding transforming process in E is

$$|\varphi'(t)\rangle = e^{i\theta(t)} |\varphi(t)\rangle . \tag{2.17}$$

To keep the closure we need to regain:

$$|\varphi(T)\rangle = |\varphi(0)\rangle , \tag{2.18}$$

$$\theta(T) = \theta(0) + 2\pi n. \tag{2.19}$$

We called this type of transformation **gauge transformation**.

Then we want to find what will happen if we do a 'horizontal' lift to a closed curve in M. We denote the horizontal lifted path in total space E as $|\Phi(t)\rangle$, which is an open curve but comes from the closed loop $|\varphi(t)\rangle$

$$|\Phi(t)\rangle = e^{if(t)} |\varphi(t)\rangle . \tag{2.20}$$

It implies that:

$$|\Phi(T)\rangle = e^{if(T)-if(0)} |\Phi(0)\rangle . \tag{2.21}$$

As the definition, the tangent vectors of the lift should be 'horizontal'.To find the requirements of horizontal movement, we decompose the tangent vectors field $|\dot{\Phi}(t)\rangle$ of it into vertical and horizontal parts:

$$|\dot{\Phi}(t)\rangle = \langle \Phi(t)|\dot{\Phi}(t)\rangle |\Phi(t)\rangle + |h_\Phi(t)\rangle . \tag{2.22}$$

The vertical component is $\langle \Phi(t)|\dot{\Phi}(t)\rangle |\Phi(t)\rangle$ and the horizontal component is $|h_\Phi(t)\rangle$. This decomposition is independent with the selection of fiber element. To make the vertical part vanish

$$\langle \Phi(t)|\dot{\Phi}(t)\rangle = 0. \tag{2.23}$$

Apply the expression of $|\Phi(t)\rangle$ in terms of $|\varphi(t)\rangle$ yields:

$$\langle \varphi(t)e^{-if(t)}|i\dot{f}(t)e^{if(t)}|\varphi(t)\rangle + \langle \varphi(t)e^{-if(t)}|e^{if(t)}|\dot{\varphi}(t)\rangle = 0, \tag{2.24}$$

which implies

$$\dot{f}(t) = i\,\langle \varphi(t)|\dot{\varphi}(t)\rangle . \tag{2.25}$$

Integrate both sides from $t = 0$ to $t = T$, and denote $\beta = f(T) - f(0)$, we find

$$\beta = i\int_0^T \langle \varphi(t)|\dot{\varphi}(t)\rangle \,\mathrm{d}t. \tag{2.26}$$

To calculate $|\dot{\varphi}(t)\rangle$, we decompose the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ along vertical and horizontal direction:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\theta}\frac{\partial}{\partial\theta} + \dot{X}^{\mu}\frac{\partial}{\partial X^{\mu}} \tag{2.27}$$

and substitute $\frac{\mathrm{d}}{\mathrm{d}t}|\varphi(t)\rangle$ into (2.25)

$$\beta = i\int_0^T \mathrm{d}t\left[\dot{\theta}\left\langle \varphi(t)\middle|\frac{\partial}{\partial\theta}\middle|\varphi(t)\right\rangle + \dot{X}^{\mu}\left\langle \varphi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\varphi(t)\right\rangle\right]. \tag{2.28}$$

An infinitesimal $U(1)$ transformation on $\varphi(t)$ implies

$$|\varphi(t)\rangle_{\theta_0+\delta\theta} = |\varphi(t)\rangle_{\theta_0} + i\delta\theta\,|\varphi(t)\rangle_{\theta_0}. \tag{2.29}$$

By comparing with its Taylor expansion

$$|\varphi(t)\rangle_{\theta_0+\delta\theta} = |\varphi(t)\rangle_{\theta_0} + \delta\theta\frac{\partial|\varphi(t)\rangle}{\partial\theta}\bigg|_{\theta_0} \tag{2.30}$$

we find for an arbitrary θ_0 that

$$\frac{\partial|\varphi(t)\rangle}{\partial\theta}\bigg|_{\theta_0} = i\,|\varphi(t)\rangle_{\theta_0}. \tag{2.31}$$

Therefore also with $\langle\varphi(t)|\varphi(t)\rangle = 1$ we can simplify (2.27)

$$\beta = i\int_0^T \mathrm{d}t\left[\dot{\theta} + \dot{X}^{\mu}\left\langle \varphi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\varphi(t)\right\rangle\right] \tag{2.32}$$

Let us define

$$A_{\mu} = i\left\langle \varphi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\varphi(t)\right\rangle \tag{2.33}$$

and

$$\tilde{A} = A_{\mu}\mathrm{d}X^{\mu} \tag{2.34}$$

so that the second part of the integral of the right-hand side of Eq.(2.32) becomes

$$i\int_0^T \mathrm{d}t\dot{X}^{\mu}\left\langle \varphi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\varphi(t)\right\rangle = \oint_c \tilde{A}. \tag{2.35}$$

From Eq.(2.19), we know the first part of Eq.(2.32) equals

$$\int_0^T \dot{\theta}\mathrm{d}t = 2\pi n \tag{2.36}$$

thus implying

$$\beta = 2\pi n + \oint_c \tilde{A}. \tag{2.37}$$

For the horizontal lifting transformation $e^{i\beta}$, we have thus found

$$\exp[i\beta] = \exp[2\pi n + \oint_c \tilde{A}] = \exp[\oint_c \tilde{A}] \tag{2.38}$$

Hence, we can consider the phase β as

$$\beta = \oint_c \tilde{A}. \tag{2.39}$$

The phase β is right the **standard geometric phase** we want to find.

2.3 Properties of geometric phase

The geometric phase has some special physics features that become significant advantages in applying it in quantum computation from its mathematical property. Its essential characteristics mainly include gauge invariance, and it is decided by geometric structures, which gives the quantum computation using geometric phase good performance in robustness.

2.3.1 Gauge invariance

To prove the gauge invariance of the geometric phase, we consider two different lifts

$$|\Phi(t)\rangle = e^{if_1(t)} |\varphi(t)\rangle \tag{2.40}$$

$$|\widetilde{\Phi}(t)\rangle = e^{if_2(t)} |\varphi(t)\rangle . \tag{2.41}$$

For the period T, their corresponding overall phase are

$$\alpha_1 = f_1(T) - f_1(0) \tag{2.42}$$

and

$$\alpha_2 = f_2(T) - f_2(0). \tag{2.43}$$

These two lifts are connected by a gauge transformation $V(t)$

$$|\widetilde{\Phi}(t)\rangle = V(t) |\Phi(t)\rangle . \tag{2.44}$$

From the Schrödinger equation, we know

$$i\frac{\partial}{\partial t} |\widetilde{\Phi}(t)\rangle = \widetilde{H}(t) |\widetilde{\Phi}(t)\rangle . \tag{2.45}$$

Use (2.41), the Schrödinger equation becomes

$$-\dot{f}_2(t)e^{if_2(t)} |\varphi(t)\rangle + ie^{if_2(t)} |\dot{\varphi}(t)\rangle = \widetilde{H}(t) |\widetilde{\Phi}(t)\rangle . \tag{2.46}$$

Perform $\langle\widetilde{\Phi}(t)| = e^{-if_2}\langle\varphi(t)|$ on both sides

$$-\dot{f}_2(t) + i\langle\varphi(t)|\dot{\varphi}(t)\rangle = \langle\widetilde{\Phi}(t)|\widetilde{H}(t)|\widetilde{\Phi}(t)\rangle. \tag{2.47}$$

Rearrange

$$\dot{f}_2(t) = i\langle\varphi(t)|\dot{\varphi}(t)\rangle - \langle\widetilde{\Phi}(t)|\widetilde{H}(t)|\widetilde{\Phi}(t)\rangle. \tag{2.48}$$

Integrate bothe sides,then we get the representation of α_2 [10]:

$$\alpha_2 = \int_0^T f(t)\mathrm{d}t = \int_0^T \mathrm{d}t[i\langle\varphi(t)|\dot{\varphi}(t)\rangle - \langle\widetilde{\Phi}(t)|\widetilde{H}(t)|\widetilde{\Phi}(t)\rangle]. \tag{2.49}$$

Same for α_1

$$\alpha_1 = \int_0^T f(t)\mathrm{d}t = \int_0^T \mathrm{d}t[i\langle\varphi(t)|\dot{\varphi}(t)\rangle - \langle\Phi(t)|H(t)|\Phi(t)\rangle]. \tag{2.50}$$

Comparing α_1 and α_2, we find that the first part of the integrals are same. Recalling (2.25), the first part of each integral is the geometric phase β we have derived. Hence, the geometric phase keeps unchanged under the gauge transformation. It also means the geometric phase is system independent, which gives its more robustness.

2.3.2 Geometric interpretation

The mathematical explanation to the time-independence of geometric phase is that it is decided by the geometric structure of the fiber bundles for a given quantum system. Actually the geometric phase equals the integral of connection form along a specific curve.

A general tangent vector is produced by the operator $\frac{\mathrm{d}}{\mathrm{d}t}$. In the usual way, the operator can be decomposed into two components [9]

$$\frac{\mathrm{d}}{\mathrm{d}t} = \alpha\frac{\partial}{\partial\theta} + B^\mu D_\mu, \tag{2.51}$$

where D_μ is called the covariant derivative. We can define it in the base space CP^N [9]

$$D_\mu = \frac{\partial}{\partial X^\mu} + \Gamma_\mu\frac{\partial}{\partial\theta} \quad X \in CP^N \tag{2.52}$$

where $\mathbf{\Gamma} = \Gamma_\mu \mathrm{d}X^\mu$ is the connection of the base space.

By applying it to a vector $|\phi(t)\rangle$ in total space E, we find the corresponding to tangent vector

$$|\dot{\phi}(t)\rangle = \alpha\frac{\partial}{\partial\theta}|\phi(t)\rangle + B^\mu D_\mu|\phi(t)\rangle, \tag{2.53}$$

where B^{μ} is a coefficient, representing the component in horizontal bias. The tangent vector itself can be decomposed along vertical and horizontal direction

$$|\dot{\phi}(t)\rangle = \langle\phi(t)|\dot{\phi}(t)\rangle\,|\phi(t)\rangle + |h_{\phi}(t)\rangle\,. \tag{2.54}$$

The vertical part $|\phi(t)\rangle$ and horizontal part $|h_{\phi}(t)\rangle$ satisfy the orthogonality relation

$$\langle\phi(t)|h_{\phi}(t)\rangle = 0. \tag{2.55}$$

By comparing with Eq.(2.53) we find

$$\begin{aligned}\alpha\frac{\partial}{\partial\theta}|\phi(t)\rangle &= \langle\phi(t)\rangle\,\dot{\phi}(t)\,|\phi(t)\rangle\,,\\ B^{\mu}D_{\mu}\,|\phi(t)\rangle &= |h_{\phi}(t)\rangle\,.\end{aligned} \tag{2.56}$$

By using the orthogonality Eq.(2.55), we can obtain

$$\langle\phi(t)|h_{\phi}(t)\rangle = \langle\phi(t)|B^{\mu}D_{\mu}|h_{\phi}(t)\rangle = 0. \tag{2.57}$$

Since Eq.(2.57) holds, for any B^{μ}, it follows that

$$\langle\phi(t)|D_{\mu}|\phi(t)\rangle = 0. \tag{2.58}$$

By substituting Eq.(2.52) into Eq.(2.58), implies

$$\begin{aligned}\left\langle\phi(t)\middle|\frac{\partial}{\partial X^{\mu}}|\phi(t)\right\rangle + \left\langle\phi(t)\middle|\Gamma_{\mu}\frac{\partial}{\partial\theta}\middle|\phi(t)\right\rangle &= 0,\\ \Gamma_{\mu}\left\langle\phi(t)\middle|\frac{\partial}{\partial\theta}\middle|\phi(t)\right\rangle &= -\left\langle\phi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\phi(t)\right\rangle.\end{aligned} \tag{2.59}$$

Substitute Eq.(2.31) into the left side of (2.59)

$$\Gamma_{\mu}\left\langle\varphi(t)\middle|\frac{\partial}{\partial\theta}\middle|\phi(t)\right\rangle = i\Gamma_{\mu}\,\langle\phi(t)|\phi(t)\rangle\,. \tag{2.60}$$

Assume the vector state is normalized, which means $\langle\phi(t)|\phi(t)\rangle = 1$, thus

$$\Gamma_{\mu} = i\left\langle\phi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\phi(t)\right\rangle. \tag{2.61}$$

The connection form $\mathbf{\Gamma}$ becomes

$$\mathbf{\Gamma} = \Gamma_{\mu}\mathrm{d}X^{\mu} = i\left\langle\phi(t)\middle|\frac{\partial}{\partial X^{\mu}}\middle|\phi(t)\right\rangle\mathrm{d}X^{\mu}. \tag{2.62}$$

Recalling (2.33), (2.34) and (2.39), we find that A_{μ} equals to Γ_{μ}. Therefore the expression of geometric phase meets the integral of the connection form along a closed curve. Then we give the independence of run time a geometric description. The connection form is only decided by the geometric structure of a manifold and the geometric structure will not change along the evolution of quantum system. So as the geometric phase.

3 A method to implement quantum gates with geometric phase

To build the interconnected system of a given quantum state, we utilize the idea of reverse engineering. In short, reverse engineering analyzes the existing system to find its core characteristics and then reproduce a new system. Therefore in designing a quantum computation system, we assume a system satisfies the requirements of performing the function of the given geometric phase, and we analyze its features, in theory, to find how to build a new one.

3.1 Introduction

The goal of our work is to design a system with a specific geometric phase, which generally reflects in the form of the evolution operator. The process can be divided into two parts.

First we transfer the requirements of parallel transportation into the limitations of gauge transformation operators. As we see in the Eq.(2.49) and Eq.(2.50), the overall phase includes two parts. While the geometric phase is invariant under the gauge transformation, the second part, called dynamical phase will change. It will show below that how we can use the gauge transformation to eliminate the dynamical phase, which add more limitation to the gauge transformation operator by an additional equation. This equation will be called parallel transportation, for it also makes the state vector do a parallel transformation.

The second part is to define Hamiltonian with known evolution operators. In this process, we choose to use reverse engineering, which allows us to calculate the corresponding Hamiltonian from the given evolution operator. With this Hamiltonian we obtained from reverse engineering, it becomes possible to build a system in experiments to realize the specific geometric phase and then perform the corresponding quantum gate.
We will elaborate our method with an example, a non-adiabatic system.

3.2 Parallel transportation equation

We have calculated the overall phase α for the system $H(t)$ in section 2.3.1:

$$\alpha = \int_0^T f(t)\mathrm{d}t = \int_0^T \mathrm{d}t[i\langle\varphi(t)|\dot{\varphi}(t)\rangle - \langle\Phi(t)|H(t)|\Phi(t)\rangle]. \tag{3.1}$$

We hope to keep the first part, the geometric phase and annihilate the second part, the dynamical phase [11]. The dynamical phase also equals to the vertical component of the tangent vector [9]. Thus we have

$$\langle\Phi(t)|\dot{\Phi}(t)\rangle = -\langle\Phi(t)|H(t)|\Phi(t)\rangle = 0. \tag{3.2}$$

This equation is also called the parallel transport equation, for in the geometric interpretation, it describes a parallel transformation of a vector from a tangent vector space to another[7].

Consider a computational basis $\{|k(0)\rangle, k = 1,2,...,d\}$ [12], then we can write $|\Phi(t)\rangle$ as [13]:

$$|\Phi(t)\rangle = \sum_k |\Phi_k(t)\rangle. \tag{3.3}$$

For each k, we require

$$\langle\Phi_k(t)|\dot{\Phi}_k(t)\rangle = 0. \tag{3.4}$$

Then Eq.(3.2) will hold. To keep

$$H(t)|\Phi_k(t)\rangle = E_k|\Phi_k(t)\rangle, \tag{3.5}$$

we define [14]:

$$|\Phi_k(t)\rangle = V(t)|k(0)\rangle. \tag{3.6}$$

$V(t)$ is a unitary operator. Substitute it into Eq.(3.4), we find

$$i\langle k(0)|V^\dagger(t)\dot{V}(t)|k(0)\rangle = 0. \tag{3.7}$$

3.3 Reverse engineering method for a non-adiabatic system

Consider a state vector $|\Phi(t)\rangle$ in a non-adiabatic system [14]:

$$|\Phi(t)\rangle = U(t)|\Phi(0)\rangle. \tag{3.8}$$

$U(t)$ is the time evolution operator, and it is unitary. $|\Phi(t)\rangle$ satisfies the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H(t)|\Phi(t)\rangle \tag{3.9}$$

Substitute Eq.(3.8) into Eq.(3.9)

$$i\frac{\partial}{\partial t}U(t)\left|\Phi(0)\right\rangle = H(t)U(t)\left|\Phi(0)\right\rangle. \tag{3.10}$$

Act $U^{\dagger}(t)$ from the right for the both sides of the equation, we find

$$i\frac{\partial}{\partial t}U(t)U^{\dagger}(t)\left|\Phi(0)\right\rangle = H(t)\left|\Phi(0)\right\rangle, \tag{3.11}$$

so that

$$H(t) = i\dot{U}(t)U^{\dagger}(t). \tag{3.12}$$

Eq.(3.12) gives the relation between the unitary operator and Hamiltonian. For the non-adiabatic system $U(t)$ takes this form:

$$U(t) = V(\vec{\lambda}_t)D(\vec{\omega}t)V^{\dagger}(\vec{\lambda}_t). \tag{3.13}$$

where $\mathrm{diag}[D(\vec{\omega}(t)] = [e^{-i\omega_1 t}, e^{-i\omega_2 t}, ... e^{-i\omega_d t}]$, with ω_d is the frequency of the system. Hence, after we define $V(t)$ with the limitations from Eq.(3.7), we can calculate the Hamiltonian we need.

However, there is an inequivalence in this process. To derive $U(t)$ from $H(t)$, suppose:

$$H(t) = \sum_k \omega_k V(t)\left|k(0)\right\rangle\left\langle k(0)\right| V^{\dagger}(t). \tag{3.14}$$

Then [14]:

$$U(t) = Te^{-i\int_0^t H(s)\mathrm{d}s}. \tag{3.15}$$

It is different from Eq.(3.13).

4 Examples

We select two non-abadiatic quantum systems with different number of energy levels to demonstrate and examine the method introduced in Chapter 3. One is the 2-level non-abadiatic system and the other is 3-level.

4.1 The geometric phase gate system with 2 energy levels

First we write down the parallel transport equation of the non-abadiatic 2-level system, with $V(\vec{\lambda}_t)$ as the evolution operator for the state vector

$$\langle \psi(t)|\dot{\psi}(t)\rangle = \langle k|\frac{\partial V}{\partial \lambda_i} \cdot \dot{\lambda}_i|k\rangle = 0. \tag{4.1}$$

We need to calculate the Hermitian adjoint of $V(\vec{\lambda}_t)$ and the derivative of $V(\vec{\lambda}_t)$. For the system has two energy levels, $V(t)$ is an SU(2) element. To calculate its Hermitian adjoint and derivative, we have to represent it clearly with some parameters. We introduce unit quaternion representation of $SU(2)$.

Given a unit quaternion

$$q = x_1\hat{1} + x_2\hat{i} + x_3\hat{j} + x_4\hat{k} \tag{4.2}$$

where

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{4.3}$$

It can be mapped to a complex matrix M

$$M(q) = \begin{bmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{bmatrix} \tag{4.4}$$

which is an $SU(2)$ element. This map is an isomorphism[15].Therefore we can let

$$\vec{\lambda}_t = [a,b,c,d]. \tag{4.5}$$

a,b,c,d are functions of t. We can just denote the quaternion corresponding to $\vec{\lambda}_t$ as λ. Then

$$V(\lambda) = \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}. \tag{4.6}$$

where $\lambda = a\hat{1} + b\hat{i} + c\hat{j} + d\hat{k}$.

The Hermitian adjoint of $V(\vec{\lambda}_t)$ is $V(\bar{\lambda})$ where $\bar{\lambda}$ is the conjugate of λ, that

$$\bar{\lambda} = a\hat{1} - b\hat{i} - c\hat{j} - d\hat{k}. \tag{4.7}$$

So the $V^{\dagger}(\lambda)$ is

$$V^{\dagger}(\lambda) = \begin{bmatrix} a - bi & -c - di \\ c - di & a + bi \end{bmatrix} \tag{4.8}$$

Then we calculate the derivative. Derive $V(\lambda_t)$ on t

$$\dot{V}(\lambda) = \Sigma_{j=1}^{4} \frac{\partial V(\lambda_j)}{\partial \lambda_j} \frac{\mathrm{d}\lambda_j}{\mathrm{d}t}. \tag{4.9}$$

Calculate each component of the derivative of $V(\vec{\lambda}_t)$

$$\dot{V}(\lambda_1) = \begin{bmatrix} \dot{a} & 0 \\ 0 & \dot{a} \end{bmatrix} \tag{4.10}$$

$$\dot{V}(\lambda_2) = \begin{bmatrix} i\dot{b} & 0 \\ 0 & -i\dot{b} \end{bmatrix} \tag{4.11}$$

$$\dot{V}(\lambda_3) = \begin{bmatrix} 0 & \dot{c} \\ -\dot{c} & 0 \end{bmatrix} \tag{4.12}$$

$$\dot{V}(\lambda_4) = \begin{bmatrix} 0 & i\dot{d} \\ i\dot{d} & 0 \end{bmatrix}. \tag{4.13}$$

where

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}, \quad y \in \{a, b, c, d\}. \tag{4.14}$$

Finally the derivative of $V(\lambda)$ is

$$\dot{V}(\lambda) = \begin{bmatrix} \dot{a} + i\dot{b} & \dot{c} + i\dot{d} \\ -\dot{c} + i\dot{d} & \dot{a} - i\dot{b} \end{bmatrix}. \tag{4.15}$$

Left multiply $V^{\dagger}(\lambda)$, we get a new matrix. We denote the new matrix as $G(\lambda)$ for convenience

$$G(\lambda) = \begin{bmatrix} a - bi & -c - di \\ c - di & a + bi \end{bmatrix} \begin{bmatrix} \dot{a} + i\dot{b} & \dot{c} + i\dot{d} \\ -\dot{c} + i\dot{d} & \dot{a} - i\dot{b} \end{bmatrix} \tag{4.16}$$

$G(\lambda)$ equals to

$$\begin{bmatrix} (a - ib)(\dot{a} + i\dot{b}) + (-c - id)(i\dot{d} - \dot{c}) & (-c - id)(\dot{a} - i\dot{b}) + (a - ib)(\dot{c} + i\dot{d}) \\ (c - id)(\dot{a} + i\dot{b}) + (a + bi)(i\dot{d} - \dot{c}) & (a + bi)(\dot{a} - i\dot{b}) + (c - id)(\dot{c} + i\dot{d}) \end{bmatrix}. \tag{4.17}$$

The meaning of $G(\lambda) = 0$ is that the diagonal elements of $G(\lambda)$ are zero. So when it acts a vector, the vector will vanish. The diagonal elements of the matrix are

$$(a - ib)(\dot{a} + i\dot{b}) + (-c - id)(-\dot{c} + i\dot{d}) \tag{4.18}$$

and

$$(a + ib)(\dot{a} - i\dot{b}) + (c - id)(\dot{c} + i\dot{d}). \tag{4.19}$$

Then we obtain

$$-ib\dot{a} + a\dot{a} + ia\dot{b} + b\dot{b} + id\dot{c} + c\dot{c} - ic\dot{d} + d\dot{d} = 0 \tag{4.20}$$

$$ib\dot{a} + a\dot{a} - ia\dot{b} + b\dot{b} - id\dot{c} + c\dot{c} + ic\dot{d} + d\dot{d} = 0 \tag{4.21}$$

From (4.3) we know

$$a^2 + b^2 + c^2 + d^2 = 1. \tag{4.22}$$

Deriving on both side, we have

$$a\dot{a} + b\dot{b} + c\dot{c} + d\dot{d} = 0. \tag{4.23}$$

So we can simplify (4.19) and (4.20).

$$i(b\dot{a} - a\dot{b} + c\dot{d} - d\dot{c}) = 0 \tag{4.24}$$

Therefore we need

$$b\dot{a} - a\dot{b} + c\dot{d} - d\dot{c} = 0. \tag{4.25}$$

Recalling (3.19) we decide $U(t)$

$$\begin{aligned} U(t) &= V(\vec{\lambda}_t)D(\vec{\omega}t)V^\dagger(\vec{\lambda}_t) \\ &= \begin{bmatrix} (c+di)(c-id)e^{-i\omega_2 t} + (a+bi)(a-ib)e^{-i\omega_1 t} & (a+bi)(-c-id)e^{-i\omega_1 t} + (a+bi)(c+di)e^{-i\omega_2 t} \\ (a-ib)(c-id)e^{-i\omega_2 t} + (id-c)(a-ib)e^{-i\omega_1 t} & (id-c)(-c-id)e^{-i\omega_1 t} + (a+bi)(a-ib)e^{-i\omega_2 t} \end{bmatrix}. \end{aligned} \tag{4.26}$$

We calculate the derivative of $U(t)$

$$\dot{U}(t) = \dot{V}(\vec{\lambda}_t)D(\vec{\omega}t)V^\dagger(\vec{\lambda}_t) + \dot{V}(\vec{\lambda}_t)D(\vec{\omega}t)V^\dagger(\vec{\lambda}_t) + V(\vec{\lambda}_t)D(\vec{\omega}t)\dot{V}^\dagger(\vec{\lambda}_t). \tag{4.27}$$

The first part $\dot{V}(\vec{\lambda}_t)D(\vec{\omega}t)V^\dagger(\vec{\lambda}_t)$ is

$$\begin{bmatrix} (a-ib)e^{-i\omega_1 t}(\dot{a}+i\dot{b}) + (c-id)e^{-i\omega_2 t}(\dot{c}+i\dot{d}) & (-c-id)e^{-i\omega_1 t}(\dot{a}+i\dot{b}) + (a+ib)e^{-i\omega_2 t}(\dot{c}+i\dot{d}) \\ (c-id)e^{-i\omega_2 t}(\dot{a}-i\dot{b}) + (a-ib)e^{-i\omega_1 t}(i\dot{d}-\dot{c}) & (a+ib)e^{-i\omega_2 t}(\dot{a}-i\dot{b}) + (-c-id)e^{-i\omega_1 t}(i\dot{d}-\dot{c}) \end{bmatrix} \tag{4.28}$$

The second part $V(\vec{\lambda}_t)\dot{D}(\vec{\omega}t)V^\dagger(\vec{\lambda}_t)$ is

$$\begin{bmatrix} -i\omega_1(a+ib)(a-ib)e^{-i\omega_1 t} - i\omega_2(c+id)(c-id)e^{-i\omega_2 t} & (c+id)(ib - i\omega_2 a e^{-i\omega_2 t}) - i\omega_1(a+ib)(-c-id)e^{-i_1 t} \\ -i\omega_1(-c+id)(a-ib)e^{-i\omega_1 t} - i\omega_2(a-ib)(c-id)e^{-i\omega_2 t} & (a-ib)(ib - i\omega_2 a e^{-i\omega_2 t} - i\omega_1(-c+id)(-c-id)e^{-i\omega_1 t} \end{bmatrix} \tag{4.29}$$

The third part $V(\vec{\lambda}_t)D(\vec{\omega}t)\dot{V}^\dagger(\vec{\lambda}_t)$ is

$$\begin{bmatrix} (a+ib)e^{-i\omega_1 t}(\dot{a}-i\dot{b}) + (c+id)e^{-i\omega_2 t}(\dot{c}-i\dot{d}) & (c+id)e^{-i\omega_2 t}(\dot{a}+i\dot{b}) + (a+ib)e^{-i\omega_1 t}(-\dot{c}-i\dot{d}) \\ (id-c)e^{-i\omega_1 t}(\dot{a}-i\dot{b}) + (a-ib)e^{-i\omega_2 t}(\dot{c}-i\dot{d}) & (a-ib)e^{-it\omega_2}(\dot{a}+i\dot{b}) + (id-c)e^{-i\omega_1 t}(-\dot{c}-i\dot{d}) \end{bmatrix}. \tag{4.30}$$

Sum these three parts, we find each component of $\dot{U}(t)$:

$$\begin{aligned} U_{11} =& ae^{-i\omega_1 t}(2\dot{a}+\omega_1(b-ia)) - be^{-it\omega_1}(-2\dot{b}+\omega_1(a+ib)) \\ &+ e^{-i\omega_2 t}(c(2\dot{c}-ic\omega_2)+d\omega_2) - d(c\omega_2 - 2\dot{d} + id\omega_2)) \end{aligned} \tag{4.31}$$

$$\begin{aligned} U_{12} =& -(c+id)e^{-i\omega_1 t}(\dot{a}+i\dot{b}) + (c+id)e^{-it\omega_2}(\dot{a}+i\dot{b}) \\ &- (a+ib)e^{-i\omega_1 t}(\dot{c}+i\dot{d}) + (a+ib)e^{-i\omega_2 t}(\dot{c}+i\dot{d}) \\ &+ i\omega_1(a+ib)(c+id)e^{-i\omega_1 t} + i(c+id)(b-\omega_2 ae^{-it\omega_2}) \end{aligned} \tag{4.32}$$

$$\begin{aligned} U_{21} =& -(c-id)e^{-i\omega_1 t}(\dot{a}-i\dot{b}) + (c-id)(-i\omega_2 t)(\dot{a}-i\dot{b}) \\ &- (a-ib)e^{-i\omega_1 t}(\dot{c}-i\dot{d}) + (a-ib)e^{-i\omega_2 t}(\dot{c}-i\dot{d}) \\ &+ \omega_1(c-id)(b+ia)e^{-i\omega_1 t} - i\omega_2(a-ib)(c-id)e^{-i\omega_2 t} \end{aligned} \tag{4.33}$$

$$\begin{aligned} U_{22} =& 2a\dot{a}e^{-i\omega_2 t} + b(ia + 2\dot{b}e^{-i\omega_2 t} + b) - b\omega_2 ae^{-i\omega_2 t} \\ &- i\omega_2 a^2 e^{-i\omega_2 t} + ce^{-i\omega_1 t}(2\dot{c} - ic\omega_1 - d\omega_1) \\ &+ \omega_1 cde^{-i\omega_1 t} + 2d\dot{d}e^{-i\omega_1 t} - i\omega_1 d^2 e^{-i\omega_1 t} \end{aligned} \tag{4.34}$$

Using (3.24), we reversely define the Hamiltonian H

$$\begin{aligned} H_{11} =& i[(a+bi)^2 e^{i\omega_1 t} - (c^2+d^2)e^{-i\omega_2 t}]\{ae^{-i\omega_1 t} \\ &[2\dot{a}+\omega_1(b-ai)] - be^{-i\omega_1}[-2\dot{b}+\omega_1(a+ib) \\ &+ e^{-i\omega_2 t}c(2\dot{c} - ic\omega_2 + d\omega_2) - e^{-i\omega_2 t}d(c\omega_2 - 2\dot{d} + id\omega_2)]\} \end{aligned} \tag{4.35}$$

$$\begin{aligned} H_{12} =& i(c+id)[(a+ib)e^{i\omega_1 t} + (a-ib)e^{-i\omega_2 t}] \\ &[(-c+id)(\dot{a}+i\dot{b})e^{-i\omega_1 t} + (c+id)(\dot{a}+i\dot{b})e^{-i\omega_2 t} \\ &- (a+ib)(\dot{c}+i\dot{d})e^{-i\omega_1 t} + (a+ib)(\dot{c}+i\dot{d})e^{-i\omega_2 t} \\ &+ i\omega_1(a+ib)(c+id)e^{-i\omega_1 t} + i(c+id)(b-\omega_2 ae^{-\omega_2 t})] \end{aligned} \tag{4.36}$$

$$\begin{aligned} H_{21} =& i(-c+id)[(a+ib)e^{-i\omega_1 t} + (a-ib)e^{-i\omega_2 t}] \\ &[-(c-id)e^{-i\omega_1 t}(\dot{a}-i\dot{b})] + (c-id)e^{-i\omega_2 t}(\dot{a}-i\dot{b}) \\ &- (a-ib)e^{-i\omega_1 t}(\dot{c}-i\dot{d}) + (a-ib)e^{-i\omega_2 t}(\dot{c}-i\dot{d}) \\ &+ \omega_1(c-id)(b+ia)e^{-i\omega_1 t} - i\omega_2(a-ib)(c-id)e^{-i\omega_2 t}] \end{aligned} \tag{4.37}$$

$$\begin{aligned} H_{22} =& -2ia\dot{a}[-(a-ib)^2 e^{-\omega_2 t} + c^2 e^{-i\omega_1 t} + d^2 e^{\omega_1 t}]e^{-\omega_2 t} \\ &+ [b(-a+2i\dot{b}e^{-i\omega_2 t} + ib) - iab\omega_2 e^{i\omega_2 t} + a^2\omega_2 e^{-i\omega_2 t} \\ &+ ce^{-i\omega_1 t}(2i\dot{c} + c\omega_1 - id\omega_1) + icd\omega_1 e^{-i\omega_1 t} \\ &+ 2id\dot{d}e^{-i\omega_1 t} + d^2\omega_1 e^{-i\omega_1 t}] \end{aligned} \tag{4.38}$$

We know that from (4.3) and (4.25),we find two constraint of parameters $\{a,b,c,d\}$, that

$$a^2+b^2+c^2+d^2 = 1 \tag{4.39}$$

$$a\dot{b} - \dot{a}b + \dot{c}d - c\dot{d} = 0. \tag{4.40}$$

These constraints implies a set of possible solutions of the parameter functions: $a(t)$, $b(t)$, $c(t)$ and $d(t)$. So the solutions of Hamiltonian will also be a set of selections, whose elements share some common property. In conclusion, we finally develop a series of systems with specific relations of parameters.

4.2 The geometric phase gate system with 3 energy levels

The procedure of designing 3-level non-abadiatic system is similar to 2-level. Instead the representation of $V(\vec{\lambda}_t)$ is an element from $SU(3)$.

To decompose an $SU(3)$ matrix, we use Gell-Mann matrices,which are generators of $SU(3)$ [16]. That for a $U(\vec{\alpha})$ in $SU(3)$, there has

$$U(\vec{\alpha}) = exp\{\sum_{j=1}^{8} \frac{1}{2}\alpha_j T_j\} \tag{4.41}$$

Assuming $\vec{\lambda} = \frac{1}{2}\vec{\alpha}$, we get

$$V(\vec{\lambda}_t) = exp\{i\sum_{j=1}^{8} \lambda_j T_j\} \tag{4.42}$$

where T_j is the set of Gell-Mann matrice.

$$\begin{aligned} T_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ T_4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad T_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad T_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ T_7 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad T_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned} \tag{4.43}$$

The Hermitian adjoint of $V(\vec{\lambda}_t)$ is

$$V^{\dagger}(\vec{\lambda}_t) = V(-\vec{\lambda}_t) = exp\{-i\sum_{j=1}^{8} \lambda_j T_j\}. \tag{4.44}$$

We can further reduce the representation into [17]

$$V(\vec{\lambda}_t) = e^{i\lambda_1 T_3} e^{i\lambda_2 T_2} e^{i\lambda_3 T_3} e^{i\lambda_4 T_5} e^{i\lambda_5 T_3} e^{i\lambda_6 T_2} e^{i\lambda_7 T_3} e^{i\lambda_8 T_8} \tag{4.45}$$

$$V^{\dagger}(\vec{\lambda}_t) = e^{-i\lambda_1 T_3} e^{-i\lambda_2 T_2} e^{-i\lambda_3 T_3} e^{-i\lambda_4 T_5} e^{-i\lambda_5 T_3} e^{-i\lambda_6 T_2} e^{-i\lambda_7 T_3} e^{-i\lambda_8 T_8}. \tag{4.46}$$

This type of decomposition is also called Euler angle decomposition. We can first expand the matrix exponential of T_{1-7} into a matrix polynomial [18]:

$$exp(i\theta T_j) = I - iT_j sin\lambda_j + T_j^2(cos\lambda_j - 1). \tag{4.47}$$

where θ is a parameter. We obtain

$$exp(i\lambda_j T_2) = \begin{bmatrix} cos\lambda_j & sin\lambda_j & 0 \\ -sin\lambda_j & cos\lambda_j & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.48}$$

$$exp(i\lambda_k T_3) = \begin{bmatrix} e^{i\lambda_k} & 0 & 0 \\ 0 & e^{-i\lambda_k} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.49}$$

$$exp(i\lambda_4 T_5) = \begin{bmatrix} cos\lambda_4 & 0 & sin\lambda_4 \\ 0 & 1 & 0 \\ -sin\lambda_4 & 0 & cos\lambda_4 \end{bmatrix}. \tag{4.50}$$

where $j \in \{2,6\}$ and $k \in \{1,3,5,7\}$ Specifically the matrix exponential of $i\lambda_8 T_8$ is

$$\begin{bmatrix} e^{i\lambda_8/\sqrt{3}} & 0 & 0 \\ 0 & e^{i\lambda_8/\sqrt{3}} & 0 \\ 0 & 0 & e^{-2i\lambda_8/\sqrt{3}} \end{bmatrix}. \tag{4.51}$$

The elements of $V^\dagger$ can be simply obtained through replacing λ_j with $-\lambda_j$.Same as the two-state system,the corresponding time evolution operator $U(t)$ equals to

$$U(t) = VD(\vec{\omega}(t))V^\dagger. \tag{4.52}$$

Now to find the limitation of parameters, use the corresponding parallel transport equation of this system

$$V^\dagger \frac{\partial}{\partial \lambda_j} V \dot{\lambda}_j = 0. \tag{4.53}$$

where $\dot{\lambda}_j$ is the derivative of λ_j respect to time t. The left of the equation is a new matrix. Denote the new matrix as M. Like what we do to $G(\lambda)$, we need the diagonal elements of M equal to zero, that

$$M_{11} = V^\dagger_{11}\dot{V}_{11} + V^\dagger_{12}\dot{V}_{21} + V^\dagger_{13}\dot{V}_{31} = 0 \tag{4.54}$$

$$M_{22} = V^\dagger_{21}\dot{V}_{12} + V^\dagger_{22}\dot{V}_{22} + V^\dagger_{23}\dot{V}_{32} = 0 \tag{4.55}$$

$$M_{33} = V^\dagger_{31}\dot{V}_{13} + V^\dagger_{32}\dot{V}_{23} + V^\dagger_{33}\dot{V}_{33} = 0 \tag{4.56}$$

We give three equations to limit the parameter vector $\vec{\lambda_t}$. A group of possible solutions for each parameter function can be obtained from these equations. With the relation between $V(\vec{\lambda_t})$ and $U(t)$:

$$U(t) = VD(\vec{\omega}(t))V^\dagger, \tag{4.57}$$

we can rebuild the unitary operator $U(t)$. Then recall the expression of Hamiltonian,

$$H = i\dot{U}(t)U^\dagger(t). \tag{4.58}$$

By using the Eq.(4.57), we can also find the corresponding set of Hamiltonians of three state systems, like what we do in the situation of two state system.

The change of the system's geometric structure, or additional conditions will reflect in the time evolution operator. Like, there are some energy states are degenerated [19]. In another word, some of the frequencies in D are identical. We assume $\omega_1 = \omega_2 \neq \omega_3$. Then each element of $U(t)$ is

$$\begin{aligned}
U_{11} =& \sin^2\lambda_4 \cos\lambda_2 \cos\lambda_6 e^{i\lambda_1+i\lambda_3-i\lambda_5-i\lambda_7-\frac{2i\lambda_8}{\sqrt{3}}-i\omega_2 t} + e^{-2i\lambda_7} \\
& (e^{i\lambda_1+i\lambda_3+i\lambda_5-i\omega_1 t}\sin\lambda_6\cos\lambda_2 + e^{i\lambda_1-i\lambda_3-i\lambda_5-i\omega_1 t}\sin\lambda_2\cos\lambda_6\cos\lambda_4) \\
& (\sin\lambda_6\cos\lambda_4\cos\lambda_2 e^{i\lambda_1+i\lambda_3+i\lambda_5} + \sin\lambda_6\sin\lambda_2 e^{i\lambda_1-i\lambda_3-i\lambda_5}) \\
& + (e^{i\lambda_1+i\lambda_3+i\lambda_5}\cos\lambda_2\cos\lambda_4\cos\lambda_6 - e^{i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_2\sin\lambda_6) \\
& (\cos\lambda_2\cos\lambda_4\cos\lambda_6 e^{-i\lambda_1-i\lambda_3-i\lambda_5-i\omega_1 t} - \sin\lambda_2\sin\lambda_6 e^{-i\lambda_1+i\lambda_3+i\lambda_5-i\omega_1 t})
\end{aligned} \tag{4.59}$$

$$\begin{aligned}
U_{12} =& -\sin\lambda_6\sin^2\lambda_4\cos\lambda_2 e^{i\lambda_1+i\lambda_3-i\lambda_5+i\lambda_7-\frac{2i\lambda_8}{\sqrt{3}}-it\omega_2} \\
& + e^{2i\lambda_7}(-e^{-i\lambda_1-i\lambda_3-i\lambda_5-i_1 t}\cos\lambda_2\cos\lambda_4\sin\lambda_6 - e^{-i\lambda_1+i\lambda_3+i\lambda_5-i\omega_1 t}\sin\lambda_2\cos\lambda_6) \\
& (-\sin\lambda_2\sin\lambda_6 e^{i\lambda_1-i\lambda_3-i\lambda_5} + \cos\lambda_6\cos\lambda_2\cos\lambda_4 e^{i\lambda_1+i\lambda_3+i\lambda_5}) \\
& + (e^{i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_2\cos\lambda_6 + e^{i\lambda_1+i\lambda_3+i\lambda_5}\sin\lambda_6\cos\lambda_2\cos\lambda_4) \\
& (\cos\lambda_2\cos\lambda_6 e^{i\lambda_1+i\lambda_3+i\lambda_5-it\omega_2} - \sin\lambda_2\sin\lambda_6\cos\lambda_4 e^{i\lambda_1-i\lambda_3-i\lambda_5-it\omega_1})
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
U_{13} =& \sin\lambda_4\cos\lambda_2\cos\lambda_4 e^{i\lambda_1+i\lambda_3-it\omega_2} - \sin\lambda_2\sin\lambda_4 e^{i\lambda_1-i\lambda_3-i\lambda_7+\frac{i\lambda_8}{\sqrt{3}}-it\omega_1} \\
& (e^{i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_2\cos\lambda_6 + e^{i\lambda_1+i\lambda_3+i\lambda_5}\sin\lambda_6\cos\lambda_2\cos\lambda_4) - \sin\lambda_4\cos\lambda_2 \\
& e^{-i\lambda_1-i\lambda_3+i\lambda_7+\frac{i\lambda_8}{\sqrt{3}}-it\omega_1}(e^{i\lambda_1+i\lambda_3+i\lambda_5}\cos\lambda_2\cos\lambda_4\cos\lambda_6 - e^{i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_2\sin\lambda_6)
\end{aligned} \tag{4.61}$$

$$\begin{aligned}
U_{21} =& -\sin\lambda_2\sin^2\lambda_4\cos\lambda_2 e^{-i\lambda_1+i\lambda_3-i\lambda_5-i\lambda_7-\frac{2i\lambda_8}{\sqrt{3}}-it\omega_2} \\
& + (-e^{-i\lambda_1+i\lambda_3+i\lambda_5}\sin\lambda_2\cos\lambda_4\cos\lambda_6 - e^{-i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_6\cos\lambda_2) \\
& (\cos\lambda_2\cos\lambda_4\cos\lambda_6 e^{-i\lambda_1-i\lambda_3-i\lambda_5-it\omega_1} - \sin\lambda_2\sin\lambda_6 e^{-i\lambda_1+i\lambda_3+i\lambda_5-it\omega_1}) \\
& + e^{-2i\lambda_7}(e^{i\lambda_1+i\lambda_3+i\lambda_5-i\omega_1 t}\cos\lambda_2\cos\lambda_6 + e^{i\lambda_1-i\lambda_3-i\lambda_5-i\omega_1 t}\sin\lambda_2\sin\lambda_6\cos\lambda_4) \\
& (-\sin\lambda_2\cos\lambda_4\cos\lambda_6 e^{-i\lambda_1+i\lambda_3+i\lambda_5} + \sin\lambda_6\cos\lambda_2 e^{-i\lambda_1-i\lambda_3-i\lambda_5})
\end{aligned} \tag{4.62}$$

$$\begin{aligned}
U_{22} =& \sin\lambda_2\sin\lambda_6\sin^2\lambda_4 e^{-i\lambda_1+i\lambda_3-i\lambda_5+i\lambda_7-\frac{2i\lambda_8}{\sqrt{3}}-it\omega_2} \\
& + e^{2i\lambda_7}(-e^{-i\lambda_1+i\lambda_3+i\lambda_5}\sin\lambda_2\cos\lambda_4\cos\lambda_6 - e^{-i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_6\cos\lambda_2) \\
& (-\sin\lambda_2\cos\lambda_6 e^{-i\lambda_1+i\lambda_3+i\lambda_5-it\omega_1} - \sin\lambda_6\cos\lambda_2\cos\lambda_4 e^{-i\lambda_1-i\lambda_3-i\lambda_5-it\omega_1}) \\
& + (e^{i\lambda_1+i\lambda_3+i\lambda_5-i_1 t}\cos\lambda_2\cos\lambda_6 - e^{i\lambda_1-i\lambda_3-i\lambda_5-i\omega_2 t}\sin\lambda_2\sin\lambda_6\cos\lambda_4) \\
& (\cos\lambda_2\cos\lambda_6 e^{-i\lambda_1-i\lambda_3-i\lambda_5} - \sin\lambda_2\sin\lambda_6\cos\lambda_4 e^{-i\lambda_1+i\lambda_3+i\lambda_5})
\end{aligned} \tag{4.63}$$

$$\begin{aligned}
U_{23} =& -\sin\lambda_2\sin\lambda_4\cos\lambda_4 e^{-i\lambda_1+i\lambda_3-it\omega_2} - \sin\lambda_4\cos\lambda_2 e^{-i\lambda_1-i\lambda_3+i\lambda_7+\frac{i\lambda_8}{\sqrt{3}}-it\omega_1} \\
& (-e^{-i\lambda_1+i\lambda_3+i\lambda_5}\sin\lambda_2\cos\lambda_4\cos\lambda_6 - e^{-i\lambda_1-i\lambda_3-i\lambda_5}\sin\lambda_6\cos\lambda_2) \\
& - \sin\lambda_2\sin\lambda_4 e^{i\lambda_1-i\lambda_3-i\lambda_7+\frac{i\lambda_8}{\sqrt{3}}-it\omega_1} \\
& (e^{-i\lambda_1-i\lambda_3-i\lambda_5}\cos\lambda_2\cos\lambda_6 - e^{-i\lambda_1+i\lambda_3+i\lambda_5}\sin\lambda_2\sin\lambda_6\cos\lambda_4)
\end{aligned} \tag{4.64}$$

$$\begin{aligned}
U_{31} =& \sin\lambda_4\cos\lambda_4\cos\lambda_6 e^{-i\lambda_5-i\lambda_7-i\sqrt{3}\lambda_8-it\omega_2} - e^{i\lambda_5-2i\lambda_7}\sin\lambda_6\sin\lambda_4 \\
& (\sin\lambda_2\cos\lambda_4\cos\lambda_6 e^{i\lambda_1-i\lambda_3-i\lambda_5-it\omega_2} + \sin\lambda_6\cos\lambda_2 e^{i\lambda_1+i\lambda_3+i\lambda_5-it\omega_1}) \\
& - e^{i\lambda_5}\sin\lambda_4\cos\lambda_6(\cos\lambda_2\cos\lambda_4\cos\lambda_6 e^{-i\lambda_1-i\lambda_3-i\lambda_5-it\omega_1} \\
& - \sin\lambda_2\sin\lambda_6 e^{-i\lambda_1+i\lambda_3+i\lambda_5-it\omega_1})
\end{aligned} \tag{4.65}$$

$$
\begin{aligned}
U_{32} = & -\sin\lambda_4 \sin\lambda_6 \cos\lambda_4 e^{-i\lambda_5+i\lambda_7-i\sqrt{3}\lambda_8-it\omega_2} - e^{i\lambda_5+2i\lambda_7} \sin\lambda_4 \cos\lambda_6 \\
& (-\sin\lambda_2 \cos\lambda_6 e^{-i\lambda_1+i\lambda_3+i\lambda_5-it\omega_1} - \sin\lambda_6 \cos\lambda_2 \cos\lambda_4 e^{-i\lambda_1-i\lambda_3-i\lambda_5-it\omega_1}) \\
& - e^{i\lambda_5} \sin\lambda_4 \sin\lambda_6 (\cos\lambda_2 \cos\lambda_6 e^{i\lambda_1+i\lambda_3+i\lambda_5-it\omega_1} \\
& - \sin\lambda_2 \sin\lambda_6 \cos\lambda_4 e^{i\lambda_1-i\lambda_3-i\lambda_5-it\omega_1})
\end{aligned} \tag{4.66}
$$

$$
U_{33} = \sin^2\lambda_4 \cos\lambda_2 \cos\lambda_6 e^{-i\lambda_1-i\lambda_3+i\lambda_5+i\lambda_7+i\sqrt{3}\lambda_8-it\omega_1} + e^{-it\omega_1} \cos^2\lambda_4. \tag{4.67}
$$

The reverse engineering method also works for a partly degenerated system. We cannot use this method for a fully degenerated system, like $\omega_1 = \omega_2 = \omega_3$. Because then $U(t)$ will become trivial. When the time evolution operator is simplified, the arbitrary form of Hamiltonian is still very complicated. The geometric phase will have a more complex representing form depending on the Hamiltonian.

4.3 Discussion and analysis

From the process shown above, we explain each step of how to use reverse engineering to of set up the Hamiltonian of non-adiabatic system with two or three energy states. These specific modified systems will only have the geometric phases.

Furthermore, because therein principle exists a method to decompose $SU(N)$ element with several parameters [17] in theoretical it is possible to define the Hamiltonian of a quantum system with any number of energy state. However, when there are more than two energy states, the calculation will be somewhat messy and difficult due to introducing too many parameters. It is also hard to do some analysis with the results of three energy state systems. However, we can still do some discussion on two energy state systems.

The limitations we established for two energy state system is a group of ordinary differential equations:

$$
a^2 + b^2 + c^2 + d^2 = 1 \tag{4.68}
$$

$$
a\dot{b} - \dot{a}b + \dot{c}d - c\dot{d} = 0. \tag{4.69}
$$

It might be difficult to find analytic expressions of c, d with a, b. However, in practice, we can work out the relations among these four parameters by adding more conditions and explicitly expressing H. For exampleif we chose $a\dot{b} - \dot{a}b = 0$ and $\dot{c}d - c\dot{d} = 0$, that means $\frac{a}{b}$ and $\frac{c}{d}$ are all constants independence of time. The number of parameters reduces to two. Changing these constants will also produce an array of Hamiltonians.

5 Conclusion

In this thesis, we introduced a theoretical method to realize quantum computation, which relies on the geometric properties of the given quantum system. After explaining the procedure of building a quantum system, we examine it by deciding the limitations of two non-adiabatic systems. First, we studied the evolution of a quantum system from a geometric perspective. We build a relation between the space of physics states and the space of state vectors with fiber bundles. We give the evolution process of a physics state a differential geometric interpretation. Through studying the geometric structures of these two space and their relations, we find the parallel transport in base space, the space of physics state will cause a phase transition for the condition in total space, the space of state vectors. The phase is divided into two parts: the dynamic and the geometric phases. Meanwhile, the geometric phase is the connection of the manifold corresponding to vector state space, which means it only depends on the system itself at any time.

Then we developed a general method to realize a specific quantum gate with a quantum system. The problem is to find a way to design the quantum system with a special Hamiltonian that only has a geometric phase. This requirement can be satisfied by deliberately defining the time evolution matrix $U(t)$ to vanish the dynamic phase. Hence we use the idea of reverse engineering to define $U(t)$ and H. This method is that at the beginning, we parameterize the matrix $U(t)$ and assuming the $U(t)$ satisfies the parallel transport equation, we study the characteristics of the parameters. We will obtain some limitations of these parameters. Next, we use the idea of reverse engineering again to build a Hamiltonian with $U(t)$. Combining with the limitations, we obtained before, finally, we defined a set of Hamiltonian, which gives us a series of quantum systems.

Finally, we selected two examples to calculate to examine our method and analyze these geometric phase quantum systems. Mainly we focused on the non-adiabatic systems. We separately calculated the limitations of two energy states and three energy systems and decided on their corresponding Hamiltonian. We also analyzed these systems while some of their frequencies are identical.

Through our calculation and analysis, we found that reverse engineering is a practical and helpful method for the more straightforward system. However, the reverse engineering method will produce detailed results for the system with many energy states, especially for non-adiabatic systems. The limitation of the three-energy state system is that a set of complex ordinary differential equations might be hard to find a definitive solution. Therefore future work will improve this method for non-adiabatic systems with more than three energy states. It is possible to utilize popular technology like machine learning in future research. Some generated machine learning models can be trained to generate Hamiltonian for a given geometric phase.

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