

# Classical Gravitational Spinning-Spinless Scattering at $\mathcal{O}(G^2S^\infty)$

Rafael Aoude<sup>1,\*</sup>, Kays Haddad<sup>2,3,†</sup> and Andreas Helset<sup>4,‡</sup>

<sup>1</sup>*Centre for Cosmology, Particle Physics and Phenomenology (CP3), Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium*

<sup>2</sup>*Department of Physics and Astronomy, Uppsala University, Box 516, 75120 Uppsala, Sweden*

<sup>3</sup>*Nordita, Stockholm University and KTH Royal Institute of Technology, Hannes Alfvéns väg 12, 10691 Stockholm, Sweden*

<sup>4</sup>*Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA*



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Making use of the recently derived, all-spin, opposite-helicity Compton amplitude, we calculate the classical gravitational scattering amplitude for one spinning and one spinless object at  $\mathcal{O}(G^2)$  and all orders in spin. By construction, this amplitude exhibits the spin structure that has been conjectured to describe Kerr black holes. This spin structure alone is not enough to fix all deformations of the Compton amplitude by contact terms, but when combined with considerations of the ultrarelativistic limit we can uniquely assign values to the parameters remaining in the even-in-spin sector. Once these parameters are determined, much of the spin dependence of the amplitude resums into hypergeometric functions. Finally, we derive the eikonal phase for aligned-angular-momentum scattering.

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**Introduction.**—Recent years have seen a large mobilization within the scattering amplitudes community toward describing the gravitational coalescence of compact objects. This stems from the necessity for ever-more precise gravitational wave templates in current and upcoming gravitational wave observatories [1–7], and because scattering amplitudes are eminently suited to calculating classical observables in the post-Minkowskian (PM) expansion [8–16]. While scattering-amplitude methods are most directly related to scattering observables, the PM effective-one-body [8,17] and the boundary-to-bound [13,14,18] maps allow for a translation of these observables to quantities pertinent to the inspiral phase of bound systems. Indeed, studies of the energetics of the scattering process demonstrate that scattering information at higher PM orders leads to increasingly accurate predictions for the binding energy—a quantity directly necessary for the generation of gravitational waveforms—particularly when the PM data has been processed through the effective-one-body mapping [9,16].

A huge recent effort has led to unprecedented precision in the PM description of spinless scattering [19–32], tidal effects [33–39], and radiation [18,25–27,40–42]. In this Letter, we focus on yet another pertinent property affecting the dynamics of a binary: the rotational angular momenta of its constituents. The connection between classical rotational

angular momentum and quantum spin appearing in scattering amplitudes is by now well understood [11,43–45]. Classical scattering at 1PM is known to all orders in the spin vectors for Kerr black holes [46–50] and general spinning bodies [45]. Dynamics at 2PM have been understood up to quartic order in spin [39,44,47,51–58]. Until recently, progress past quartic order at 2PM has been restricted owing partly to the absence of a physical opposite-helicity Compton amplitude above this spin order [59]. Several approaches have been taken to remedy these unphysicalities [52,60–63]. Results including spin at 3PM have also begun to emerge [64,65].

Recently, Refs. [63,66] have pushed the state of the art in the scattering of spinning objects at 2PM past the fourth order in spin. In the former Letter, we applied the heavy on-shell variables of Ref. [50] to focus on the classical limit of the Compton amplitude. Doing so allowed us to determine the classical opposite-helicity Compton amplitude free of unphysical poles. We subsequently evaluated the 2PM amplitude up to eighth order in spin, fixing nearly all contact terms by imposing the so-called “black hole spin structure assumption” on the 2PM amplitude; see Eq. (1). Contrasting with our on-shell approach, the authors of Ref. [66] started from a local Lagrangian including operators at fifth order in spin. This enabled them to bypass the issues with unphysical poles in the Compton amplitude and construct the 2PM Hamiltonian for two general spinning bodies up to the fifth order in spin. The shift symmetry proposed in Ref. [66] to describe Kerr black holes is equivalent to imposing the black hole spin structure assumption, and the results of these two works are in agreement where there is overlap.

In this Letter, we present the amplitude at  $\mathcal{O}(G^2)$ , all orders in spin, and with an arbitrary mass ratio for

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spinning-spinless scattering based on the Compton amplitude determined in Ref. [63]. Many of the observations made about the 2PM amplitude at finite spin in Ref. [63] are elucidated and shown to hold to all orders in spin.

Our eventual ambition is to find the amplitude describing Kerr black hole scattering. Pinpointing this amplitude lies outside the scope of the effective approach we have taken, but the all-spin amplitude we compute here suggests special values for the contact-term freedoms remaining in the Compton amplitude. In conjunction with the black hole spin structure assumption, consideration of the ultrarelativistic limit of the 2PM amplitude allows for the unique determination of the remaining contact-term coefficients in the even-in-spin sector of the Compton amplitude. The resulting 2PM amplitude is surprisingly simple, with much of the spin dependence resumming into hypergeometric functions. While we cannot conclude that the amplitude presented here describes Kerr black hole scattering, it provides a conjecture that can be tested by future high-spin computations where the identities of the scattering objects may be clearer.

The amplitude presented here expands the analytical knowledge of the two-body problem at 2PM and high-spin orders, and is one part of the pipeline toward precise analytical templates. The next step is the conversion of the amplitude to observables. This can be done using a variety of methods [10,11,14,44]. One approach passes through the eikonal phase [44,47,57,61,65,67–69], which we present for aligned-angular-momentum scattering.

Let us begin by writing the opposite-helicity Compton amplitude in the form most convenient for our present analysis.

*Compton amplitude for heavy spinning particles.*—The all-spin, opposite-helicity gravitational Compton amplitude at leading order in  $\hbar$  was presented by the present authors in Ref. [63]. A vast majority of contact-term contributions were fixed by imposing the black hole spin structure assumption on the 2PM amplitude, which demands that spin structures (parametrized in terms of the ring radii  $\mathbf{a}_i^\mu \equiv S_i^\mu/m_i$  for  $S_i^\mu$  the classical spin vectors and  $m_i$  the masses) appear in the combination

$$(q \cdot \mathbf{a}_i)(q \cdot \mathbf{a}_j) - q^2(\mathbf{a}_i \cdot \mathbf{a}_j), \quad i, j = 1, 2. \quad (1)$$

This assumption is equivalent to the shift symmetry imposed on the 2PM amplitude above  $\mathcal{O}(\mathbf{a}^3)$  in Ref. [66]. In fact, requiring that the 2PM amplitude possesses this symmetry is equivalent to requiring that the opposite-helicity Compton amplitude is invariant under the shift [70]

$$\mathbf{a}^\mu \rightarrow \mathbf{a}^\mu + \xi \frac{q_3^\mu + q_4^\mu}{s_{34}}. \quad (2)$$

At leading order in  $\hbar$ , the most general arbitrary-spin, opposite-helicity Compton amplitude invariant under this shift is (modulo the overall coupling)

$$\mathcal{M}_{\text{cl}}^s = e^{-\mathfrak{s}_1} \sum_{n=0}^{2s} \frac{1}{n!} \bar{K}_n + m^2(w \cdot \mathbf{a})^4 \mathcal{C}, \quad (3)$$

where

$$\bar{K}_n \equiv \begin{cases} K_n, & n \leq 4, \\ K_4 L_{n-4} - K_3 \mathfrak{s}_2 L_{n-5}, & n > 4, \end{cases} \quad (4)$$

$$\mathcal{C} \equiv \sum_{n=0}^{2s-4} \sum_{j=0}^{\lfloor (2s-4-n)/2 \rfloor} d_{n,j} \mathfrak{s}_1^n (\mathfrak{s}_1^2 - \mathfrak{s}_2)^j, \quad (5)$$

with

$$K_n \equiv \frac{y^4}{s_{34} t_{13} t_{14}} \left( \frac{t_{14} - t_{13}}{y} w \cdot \mathbf{a} \right)^n, \quad (6)$$

$$L_m \equiv \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m+1}{2j+1} \mathfrak{s}_1^{m-2j} (\mathfrak{s}_1^2 - \mathfrak{s}_2)^j, \quad (7)$$

and

$$\mathfrak{s}_1 \equiv (q_3 - q_4) \cdot \mathbf{a}, \quad (8)$$

$$\mathfrak{s}_2 \equiv -4(q_3 \cdot \mathbf{a})(q_4 \cdot \mathbf{a}) + s_{34} \mathbf{a}^2. \quad (9)$$

The spin structures  $w \cdot \mathbf{a}$ ,  $\mathfrak{s}_1$ , and  $\mathfrak{s}_2$  are individually invariant under Eq. (2). Therefore, Eq. (3) is itself manifestly invariant under this shift.

We have taken both graviton momenta to be outgoing and the initial massive momentum to be incoming. The graviton labeled by 3 carries negative helicity, and that labeled by 4 carries positive helicity. The momenta are grouped into the Mandelstam variables  $s_{34} = (q_3 + q_4)^2$  and  $t_{1i} = (p_1 - q_i)^2 - m_1^2$ . Finally, we have defined the four-vector  $w^\mu \equiv [4|\bar{\sigma}^\mu|3]/2$  and  $y \equiv 2p_1 w$ .

We have rearranged the contact terms with unfixed coefficients compared to Ref. [63]. As a consequence, the unfixed coefficients here are different from those there. The coefficients here contribute at order  $\mathcal{O}(\mathbf{a}^{n+2j+4})$ . Furthermore, we have written  $\bar{K}_{n \geq 5}$  in terms of  $K_3$  and  $K_4$ , as opposed to  $K_2$  and  $K_3$  as in Ref. [63], by using the recursion relation for  $K_{n \geq 4}$  presented there. The infinite-spin amplitude is trivially found by taking  $s \rightarrow \infty$  in Eqs. (3) and (5).

Having suitably reshuffled the all-spin, opposite-helicity Compton amplitude, we move now to evaluating the  $\mathcal{O}(G^2)$  spinning-spinless amplitude to all orders in spin.

*All-spin scattering.*—The classically relevant part of the one-loop  $2 \rightarrow 2$  amplitude is encoded in the coefficients for triangle topologies, specifically those with one massive and two massless propagators in the loop [19,21,71]. We construct these coefficients out of the Compton amplitude [72] in Eq. (3) and the three-point amplitude describing a

Kerr black hole [47–50,52,59] using generalized unitarity [73–75] (see also Refs. [44,58,63] for an outline of this method applied to the problem at hand). In Ref. [63] the present authors used this method to evaluate the 2PM amplitude for two spinning bodies to eighth order in spin, and to all orders in spin for a spinless probe in a Kerr background. Considering only one object to be spinning, we present here the amplitude to all orders in spin for arbitrary mass ratios.

The scattering amplitude for a spinning particle with mass  $m_1$  and ring radius  $\mathfrak{a}_1$  and a spinless particle with mass  $m_2$  has an even-in-spin and an odd-in-spin part:

$$\mathcal{M}_{2\text{PM}} = \frac{2G^2\pi^2 m_1^2 m_2^2}{\sqrt{-q^2}} (\mathcal{M}_{2\text{PM}}^{\text{even}} + i\omega \mathcal{E}_1 \mathcal{M}_{2\text{PM}}^{\text{odd}}). \quad (10)$$

These different sectors are given by

$$\begin{aligned} \mathcal{M}_{2\text{PM}}^{\text{even}} = & m_1 \left[ 3(5\omega^2 - 1)\mathcal{F}_0 + \frac{1}{4}(\omega^2 - 1)\mathcal{F}_2 Q + \frac{8\omega^4 - 8\omega^2 + 1}{\omega^2 - 1}\mathcal{F}_1 Q - \frac{1}{2}\mathcal{F}_2 V + \sum_{k=1}^{\infty} \frac{(8\omega^4 - 8\omega^2 + 1)(-1)^k 2\Gamma[k]}{(\omega^2 - 1)^{k+1} \Gamma[2k]} \mathcal{F}_{k-1} V^k \right] \\ & - m_2 \left[ -3(5\omega^2 - 1)\sqrt{\pi}\mathcal{F}_{-1/2} - \frac{3\sqrt{\pi}}{4}\mathcal{F}_{1/2} Q - \frac{1}{\omega^2 - 1}\mathcal{F}_1 Q + \frac{15\sqrt{\pi}}{4}\mathcal{F}_{1/2} V \right. \\ & + 6\sqrt{\pi} \sum_{k=1}^{\infty} \frac{\omega^{2k}}{(\omega^2 - 1)^{k+1} \Gamma[2k+1] \Gamma[5/2-k]} \left[ {}_2F_1\left(\frac{1}{2} - k, -k; \frac{5}{2} - k; \frac{1}{\omega^2}\right) - \left(k + \frac{3}{2}\right) {}_2F_1\left(\frac{3}{2} - k, -k; \frac{5}{2} - k; \frac{1}{\omega^2}\right) \right] \\ & \left. - \frac{1}{64}(3Q^2 + 30QV + 35V^2) \sum_{k=0}^{\infty} c_k^{(0)} + \frac{1}{16}(Q + 7V)(Q + V) \sum_{k=1}^{\infty} c_k^{(1)} - \frac{1}{64}(Q + V)^2 \sum_{k=2}^{\infty} c_k^{(2)} \right], \end{aligned} \quad (11)$$

for even spin powers and

$$\begin{aligned} \mathcal{M}_{2\text{PM}}^{\text{odd}} = & -m_1 \left[ 4\mathcal{F}_1 + \sum_{k=0}^{\infty} \frac{(2\omega^2 - 1)(-1)^k 8\Gamma[k+1]}{(\omega^2 - 1)^{k+1} \Gamma[2k+1]} \mathcal{F}_k V^k \right] \\ & - m_2 \left\{ \frac{15\sqrt{\pi}}{2}\mathcal{F}_{1/2} + \sum_{k=0}^{\infty} \frac{\omega^{2k}}{(\omega^2 - 1)^{k+1} (1)_k (2k-1)} \right. \\ & \times \left[ {}_2F_1\left(-\frac{1}{2} - k, -k; \frac{3}{2} - k; \frac{1}{\omega^2}\right) - \left(k + \frac{5}{2}\right) {}_2F_1\left(\frac{1}{2} - k, -k; \frac{3}{2} - k; \frac{1}{\omega^2}\right) \right] \left. \right\}, \end{aligned} \quad (12)$$

for odd spin powers. The transfer momentum is given by  $q^\mu$ . We have defined  $\omega \equiv v_1 \cdot v_2$ ,  $\mathcal{E}_1 \equiv \epsilon^{\mu\nu\alpha\beta} v_{1\mu} v_{2\nu} q_\alpha \mathfrak{a}_{1\beta}$ ,  $Q \equiv (q \cdot \mathfrak{a}_1)^2 - q^2 \mathfrak{a}_1^2$ , and  $V \equiv q^2 (v_2 \cdot \mathfrak{a}_1)^2$ . The 2PM amplitude depends on  $Q$  through the hypergeometric function [76]

$$\mathcal{F}_j \equiv \frac{1}{\Gamma[j+1]} {}_0F_1\left(j+1; \frac{Q}{4}\right), \quad (13)$$

while the unfixed contact term coefficients enter in

$$c_k^{(i)} \equiv \frac{1}{4^k} (Q + V)^k \binom{2k}{k-i} \sum_{j=0}^{\infty} \Delta d_{2k,j} Q^j, \quad (14)$$

$$\Delta d_{2k,j} \equiv d_{2k,j} + \frac{16(k-j)(2k+1)}{(2j+2k+4)!}. \quad (15)$$

Finally, the notation  $(j)_m$  indicates the Pochhammer symbol.

In Ref. [63] it was observed that the odd-in-spin parts of the spinning-spinless 2PM amplitude were uniquely fixed

by imposing Eq. (1), up to  $\mathcal{O}(\mathfrak{a}^7)$ . The results in this section demonstrate that the unfixed contact-term coefficients in Eq. (5) do not enter the odd-in-spin sector of the 2PM amplitude for spinning-spinless scattering at any order in spin. It is actually easy to understand why this happens. Parity-even contributions to the 2PM amplitude with odd powers of spin are reexpressible such that they contain exactly one Levi-Civita symbol. However, the contact terms depend on three four-vectors  $q_3^\mu$ ,  $q_4^\mu$ , and  $\mathfrak{a}_1^\mu$ , which, after inserting the contact terms into the cut, become  $q^\mu$ ,  $p_2^\mu$ , and  $\mathfrak{a}_1^\mu$ . There are thus only three vectors that can be contracted into the Levi-Civita symbol, so any Levi-Civita symbol coming from contact terms in Eq. (5) that has quenched Lorentz indices vanishes when one of the particles is not spinning.

Armed with this all-spin amplitude, we are in a position to make some statements about the unfixed even-in-spin coefficients in Eq. (5).

*Contact terms and high-spin resummation.*—Ultimately, we would like to understand which set of contact terms in

the Compton amplitude describes black hole physics. Answering this question definitively requires performing a matching computation to a quantity that unambiguously describes black hole dynamics. In the absence of such an object to which we can compare, we content ourselves with identifying sets of contact terms that impart special qualities to the 2PM amplitude. Equation (1) [equivalently the 2PM analog of Eq. (2) [66]] is one such special property, already almost entirely eliminating contact-term freedoms. The ultrarelativistic ( $\omega \rightarrow \infty$ ) limit gives us another handle with which we can uniquely fix the parameters appearing in Eq. (11).

Expanded in powers of the spin, the full spinning-spinless 2PM amplitude can be written as [77]

$$\mathcal{M}_{2\text{PM}} = G^2 m_1^2 m_2^2 \frac{\pi^2}{\sqrt{-q^2}} \times \sum_{n=0}^{\infty} \sum_{k=0}^n [M_k^{(2n)} + i\omega \mathcal{E}_1 M_k^{(2n+1)}] Q^{n-k} V^k, \quad (16)$$

where the  $M_k^{(i)}$  are form factors in the  $\mathcal{O}(\mathfrak{a}_i^j)$  spin sector depending only on the masses and  $\omega$ . We noted in Ref. [63] that certain values of the remaining contact-term coefficients were suggested by the ultrarelativistic limit of the 2PM amplitude [78]. It was observed there that up to  $\mathcal{O}(\mathfrak{a}^6)$  it was possible to improve the ultrarelativistic limits—that is, cause to scale with lower powers of  $\omega$  as  $\omega \rightarrow \infty$ —of some even-in-spin form factors by choosing certain values for the remaining coefficients [79]. Inspecting Eq. (11) we can see that this is always possible.

The contributions to Eq. (11) from the contact terms always enter at  $\mathcal{O}(\omega^0)$ . As such, no values for the coefficients in Eq. (5) can affect the ultrarelativistic scalings of the form factors with  $k = 0, 1$ ; these scale as  $\mathcal{O}(\omega^2)$  and  $\mathcal{O}(\omega^0)$ , respectively, for any values of the coefficients. However, in the absence of the  $c_k^{(i)}$  in the last line of Eq. (11), the form factors with  $k \geq 2$  would scale as  $\mathcal{O}(\omega^{-2})$  when  $\omega \rightarrow \infty$ , a behavior that is worsened to  $\mathcal{O}(\omega^0)$  by the  $c_k^{(i)}$ . Requiring the quickest possible decay of all even-in-spin form factors as  $\omega \rightarrow \infty$  is thus equivalent to setting

$$d_{2k,j} = -\frac{16(k-j)(2k+1)}{(2j+2k+4)!}, \quad (17)$$

which imposes  $c_k^{(i)} = 0$  [see Eq. (14)]. Equation (17) produces agreement between Eq. (3) and the classical limit of the opposite-helicity Compton amplitude of Ref. [59] up to fourth order in spin; past this order, the latter possesses unphysical poles.

Equations (1) and (17) jointly endow the all-spin 2PM amplitude with a remarkably compact form. The former condition allows for the spin dependence to be written only in terms of  $Q$ ,  $V$ , and  $\mathcal{E}_1$ , relegating spin effects to the hyperplane orthogonal to  $q^\mu$ , up to subleading-in- $\hbar$  effects.

Moreover, apart from the  $\mathcal{O}(V^0)$  portion of the even-in-spin sector, the latter condition causes the resummation of all  $Q$  dependence into hypergeometric functions.

Further still, at  $\mathcal{O}(V^k)$  for fixed  $k \geq 2$  (1) in the even (odd)-in-spin sector, all  $Q$  dependence is encapsulated in precisely one  $\mathcal{F}_j$ . This explains an observation made in Ref. [63] in the odd-in-spin sector that certain form factors at different spin orders are proportional to each other. It also shows that this proportionality exists in the even-in-spin sector as well, specifically for the form factors whose ultrarelativistic behavior is improved by Eq. (17). The constants of proportionality can be obtained by expanding the amplitude in  $Q$  at a fixed order in  $V$ :

$$M_k^{(2n+1)} = \frac{4^{k-n} M_k^{(2k+1)}}{(1)_{n-k} (k+1)_{n-k}}, \quad 1 \leq k \leq n, \quad (18a)$$

$$M_k^{(2n)} = \frac{4^{k-n} M_k^{(2k)}}{(1)_{n-k} (k)_{n-k}}, \quad 2 \leq k \leq n. \quad (18b)$$

Because of the lower bounds on  $k$ , these proportionalities can only be observed when spin orders higher than four are considered. We correspondingly dub the resulting resummation the “high-spin resummation.”

Fixing the coefficients in Eq. (5) in the odd-in-spin sector requires consideration of  $\mathcal{O}(\mathfrak{a}_1^{2n+1} \mathfrak{a}_2^{i>0})$  sectors of the 2PM amplitude. The possibility of nonvanishing odd-in-spin coefficients is itself interesting: it implies that certain properties of the object described by those values can only be probed at  $\mathcal{O}(G^2)$  in the classical limit by scattering with another spinning body. Such a phenomenon is actually not novel, as we observed in Ref. [63] that some coefficients imposing Eq. (1) are left unfixed by the spinning-spinless sector of the scattering. Similar behavior can also be seen in the spinning tidal results of Ref. [45].

*Aligned-angular-momentum eikonal phase.*—The eikonal phase allows for the relation of the amplitude to observables such as the linear impulse, the spin kick, and, when the motion is planar, the scattering angle [44,57,67]. When one of the scattering objects is spinning, the condition of planar motion is satisfied when the spin vector is orthogonal to the plane formed by the impact parameter  $\mathbf{b}$  and the asymptotic center-of-mass three-momentum  $\mathbf{p}$ ; that is, when the rotational and orbital angular momenta are aligned. For brevity, we will present the eikonal phase in this setup.

In terms of the amplitude, the eikonal phase at  $\mathcal{O}(G^2)$  is

$$\chi = \frac{1}{4m_1 m_2 \sqrt{\omega^2 - 1}} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{i\mathbf{b} \cdot \mathbf{q}} \mathcal{M}_{2\text{PM}}. \quad (19)$$

For aligned-angular-momentum scattering, the amplitude in Eq. (10) greatly simplifies because  $V = 0$ . Further imposing Eq. (17), the aligned-angular-momentum eikonal phase is a sum of two terms:



$$\chi_{\text{aam}} = -\frac{G^2 \pi m_1 m_2}{4bx(1-x^2)^{3/2}(\omega^2-1)} \left( \frac{\chi_{\text{aam}}^{\text{even}}}{x\sqrt{(1-x^2)(\omega^2-1)}} + \omega \chi_{\text{aam}}^{\text{odd}} \right), \quad (20)$$

where

$$\begin{aligned} \chi_{\text{aam}}^{\text{even}} = & -2m_1 \sqrt{1-x^2} [x^2 \omega^2 [x^2 \omega^2 + 6(\omega^2-1)] + [1 - (1-x^2)^{3/2}] (\omega^2-1)^2] \\ & + m_2 x^2 [3(\omega^2-1)(5(x^2-1)\omega^2 - 2x^2 + 1) - 2x^2 \sqrt{1-x^2}], \end{aligned} \quad (21)$$

$$\chi_{\text{aam}}^{\text{odd}} = 8m_1 [\omega^2 x^2 + (\omega^2-1)(1 - (1-x^2)^{3/2})] + 3x^2 m_2 [2 + 5(\omega^2-1)\sqrt{1-x^2}]. \quad (22)$$

We have defined  $x \equiv a_1/b$ . The unstylized  $a_1$  and  $b$  represent the magnitudes of the spatial spin and impact-parameter vectors, respectively. The sign on the eikonal phase in the odd-in-spin sector depends on the direction of the spin vector. The conversion from hypergeometric functions in the amplitude to elementary functions in the eikonal phase occurs by expanding the former in  $Q$ , Fourier transforming each term in the series individually, then resumming the result in impact-parameter space.

The spinless-probe limit of this eikonal phase agrees with the first line of Eq. (11) in Ref. [80] for all non-negative powers of the spin.

*Summary.*—Exploiting the cured opposite-helicity Compton amplitude derived in Ref. [63], we have presented in Eqs. (10), (11), and (12) the 2PM amplitude describing the scattering of a spinning and a spinless body. By construction, the amplitude exhibits Eq. (1)—a structure observed in Kerr black hole scattering at low spin orders [44,51,53,55,58,81]—to all spin orders. Our result includes the most general set of contact terms that adheres to Eq. (1), and demonstrates explicitly that these contact terms do not contribute at any odd spin order in spinning-spinless scattering at 2PM.

Analyzing Eq. (11), we noticed that it is always possible to improve the ultrarelativistic behavior of even-in-spin form factors by selecting appropriate values for the coefficients in Eq. (5). These values are given in Eq. (17). Additionally, these values for the coefficients lead to a compact resummation of nearly all  $Q$  dependence of the amplitude into hypergeometric functions. At finite spin, this resummation is signalled by the proportionality of form factors at different spin orders—see Eq. (18).

Finally, we presented the eikonal phase for aligned-angular-momentum scattering to all orders in spin and for a general mass ratio. This quantity can be easily converted to pertinent observables [44,57,67].

We have found a compact form for the  $\mathcal{O}(G^2 S^\infty)$  spinning-spinless amplitude by imposing only two constraints upon it. Despite the elegance of the result, its relevance to Kerr black hole scattering remains to be elucidated. The ubiquity of hypergeometric functions in Eqs. (11) and (12) is intriguing; Ref. [82] connected the

hypergeometric functions solving the radial part of the Teukolsky equation to a hidden conformal symmetry in the near region of a Kerr black hole. Investigation of this connection may provide hints as to the Compton or 2PM amplitude that truly describes a Kerr black hole. Apart from scrutinizing solutions to the Teukolsky equation (see also Ref. [61]), classical self-force calculations with a spinning test mass would be valuable to testing the assumptions we have imposed on the amplitude.

The remaining coefficients in Eq. (5) that are not determined by Eq. (17) require an analysis of the spinning-spinning amplitude. We leave this for future Letter.

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\*Corresponding author.

rafael.aoude@uclouvain.be

†Corresponding author.

kays.haddad@physics.uu.se

‡Corresponding author.

ahelset@caltech.edu

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