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Defects, renormalization and conformal field theory

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Abstract

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Defects in quantum field theories appear in many different contexts and are of great physical interest. Among other things, defects are used to describe the Kondō effect in condensed matter, confinement in quantum chromodynamics, Rényi entropy in quantum information and cosmic strings in string theory. This thesis is a study on the theoretical framework of defects in quantum field theories, and new analytical methods are developed.

We study renormalization in the prescence of a defect, and how the conformal symmetry at the fixed points of the corresponding renormalization group flow can be used to bootstrap the bulk two-point correlator. We also study the Coleman-Weinberg mechanism, which allows us to flow along the renormalization group to a first-ordered phase transition.

Included in this thesis are several new results. We explain how two-point correlators of mixed bulk-local operators near a boundary can be analytically bootstrapped by exploiting the analytical structure of the conformal blocks. This yields the operator product expansion coefficients in either bootstrap channel. We also consider a sextic bulk-interaction and quartic boundary-interaction near three dimensions, and find the bulk two-point correlator upto two-loops by solving the equation of motion. We then apply the Coleman-Weinberg mechanism to this theory, leading to a spontaneous symmetry breaking of the original O(N)-symmetry. By studying the monodromy of a replica twist defect we learn how a global O(N)-symmetry is broken. We find the anomalous dimensions of the defect-local fields by applying the equation of motion to the bulk-defect operator product expansion. Lastly we study fusion of two scalar Wilson defects. We propose that fusion holds at a quantum level by showing that bare one-point functions stay invariant.

Keywords: defects, boundaries, replica twist defect, conformal field theory, quantum field theory, Coleman-Weinberg mechanism, renormalization group flows

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Dedicated to my beloved wife and son

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I A. Söderberg, *Anomalous Dimensions in the WF O(N) Model with a Monodromy Line Defect*, JHEP03(2018)058, arXiv:1706.02414 [hep-th].

II A.Bissi, T.Hansen and A. Söderberg, *Analytic Bootstrap for Boundary CFT*, JHEP01(2019)010, arXiv:1808.08155 [hep-th].

III V. Procházka and A. Söderberg, *Composite operators near the boundary*, JHEP03(2020)114, arXiv:1912.07505 [hep-th].

IV V. Procházka and A. Söderberg, *Spontaneous symmetry breaking in free theories with boundary potentials*, SciPostPhys.11.2.035, arXiv:2012.00701 [hep-th].

V P. Dey and A. Söderberg, *On analytic bootstrap for interface and boundary CFT*, JHEP07(2021)013, arXiv:2012.11344 [hep-th].

VI A. Söderberg, *Fusion of conformal defects in four dimensions*, JHEP04(2021)087, arXiv:2102.00718 [hep-th].

VII A.Bissi, P. Dey, J. Sisti and A. Söderberg, *Interacting conformal scalar in a wedge*, JHEP10(2022)060, arXiv:2206.06326 [hep-th].

Contents

Introduction

Quantum field theories (QFT's) combine field theory, special relativity and quantum mechanics to give us a theoretical framework of particle physics. In particular, the QFT that describes the interactions of quarks: *quantum chromodynamics* (QCD), yields impressive results of very high accuracy at particle accelerators such as CERN. QFT methods are also used in statistical physics to describe properties of different materials in condensed matter. The Ising model on a lattice is one such example, wherein its continuum limit can be described by a QFT to tell us about e.g. its *phase transitions* (p.t.'s).

Particles in QFT's are the energy states of the fields. These fields are local operators acting on points in spacetime (zero-dimensional). However, QFT's can in general also admit extended objects which covers an entire submanifold $\Sigma^p \subset \mathscr{M}^d$ (parametrized by the coordinates $x_\parallel$) of the theory (with $p \geq 1$). We call such p-dimensional operator a *defect*. From a technical point of view, defects make sense when added to the path integral as an exponential

$$D_p = \exp\left(i \int_{\Sigma^p} d^p x_\parallel \mathscr{L}_{\text{def}}\right) , \tag{1}$$

for some Lagrangian density, $\mathscr{L}_{\text{def}}$, describing fields and interactions exclusive to the defect.[1] Physically, interactions on the defect describe *boundary conditions* (b.c.'s) on the defect.

Defects will not satisfy the same symmetries as local operators. In addition to this, properties (like scaling dimensions) of local operators change if they are localized to the defect. This naturally gives rise to an effective theory for adsorption where particles are glued onto a submanifold, see [1] and references therein. We thus have to differ between the space outside of the defect, called the *bulk* (or the ambient space), and the space along the defect itself

$$x^\mu = x_\parallel^a \oplus x_\perp^i , \tag{2}$$

where $\mu \in \{0, ..., d-1\}$, $a \in \{0, ..., p-1\}$ and $i \in \{1, ..., d-p\}$ (if the time-axis is parallel to the defect).

It is physically important to study defects for various reasons. In condensed matter they give rise to new critical phenomena near the defects. In particular, they can be used to describe impurities in materials (where the underlying microscopic structure of the material may differ). This makes defects important

[1] Just like in a QFT without a defect, the theory on the defect might not have a Lagrangian description (even when the theory without a defect has one).

when studying the Kondō effect (scatterings of electrons in metals due to an impurity) [2, 3].

In string theory we can treat branes as defects,[2] which give rise to b.c.'s for open strings with endpoints on the branes (see [4] and references therein). Moreover, if we consider a QFT in *anti-de Sitter* (AdS) spacetime (negative cosmological constant) with a *conformal field theory* (CFT) dual, then a $(p+1)$-brane in the AdS might intersect the conformal boundary. This intersection then yields a p-dimensional defect in the CFT [5, 6].

Let us mention a couple of specific examples of defects of physical interest:

Wilson loops

Perhaps the most studied defect in the literature is the *Wilson loop* in a gauge theory. If this defect spans the curve γ, then it is defined as [7, 8]

$$W[\gamma] = \mathrm{tr}\left[\mathscr{P}\exp\left(i\oint_{\gamma} A_{\mu}dx^{\mu}\right)\right] , \tag{3}$$

where $\mathscr{P}$ is the path ordering operator and A_{μ} is the gauge field. Among other things, Wilson loops are used to study confinement in QCD (the phenomena that quarks cannot be isolated from each other).

They are also of great importance in *topological quantum field theories* (TQFT's). These are QFT's invariant under topological deformations of spacetime [9, 10, 11, 12]. TQFT's are used in condensed matter to describe the fractional quantum Hall effect in two dimensions

$$G = \frac{e^2\nu}{h} , \quad \nu \in \mathbb{Q} . \tag{4}$$

Here G is the conductance (resistance inversed), e is the charge of the electron and h is Planck's constant. The values of ν describes the quantum steps of G, which are fractional in the case above.

In this thesis we take a closer look at *scalar Wilson lines/surfaces* (or pinning defects) in condensed matter systems. These defects are given by

$$D = \exp\left(i\int_{\mathbb{R}^p} d^p x_{\|} h_{\phi}\phi\right) , \tag{5}$$

where h_{ϕ} describes a localized magnetic field along the defect [13].[3] From a technical point of view, h_{ϕ} can be treated as a coupling constant of finite size localized on the defect. The dimension of the defect is $p=1$ if $d=4-\varepsilon$, and $p=2$ if $d=6-\varepsilon$. Both of these two models have their $O(N)$-symmetry explicitly broken by the scalar Wilson defect.

[2] In string theory we define defects as operators of dimension $p \geq 2$ since the fundamental strings are already one-dimensional.

[3] See [14] for a similar defect in a fermionic QFT.

Boundaries

Another kind of defects that are of physical interest are *boundaries* in QFT's (see [15] and references therein). These are codimension one defects, where there is a physical region on only one side of it. Experimental setups in condensed matter will have boundaries, making our understanding of them crucial. Moreover, they are important for understanding p.t.'s of materials, which is related to the b.c. on the boundary. E.g. the ordinary p.t., where the scalar satisfy Dirichlet b.c.'s

$$\phi|_{x_\perp=0} = 0 , \tag{6}$$

or the special p.t. with Neumann b.c.'s

$$\partial_\perp \phi|_{x_\perp=0} = 0 . \tag{7}$$

A physical example of a *boundary quantum field theory* (BQFT) is graphene. This material consists of a one-atom thick layer of carbon, and thus spans three spacetime dimensions. There is a simple conceptual way of understanding how this material can be described by a BQFT: imagine electrons living on the graphene, affected by photons hitting the material. The electrons are fermions on a three-dimensional surface, while the photons are gauge fields living in a four-dimensional bulk [16]

$$S = -\int_{\mathbb{R}^d} d^d x \frac{(F_{\mu\nu})^2}{4} + \int_{\mathbb{R}^{d-1}} d^{d-1}x_\parallel \bar{\psi}^a_\alpha (i\nabla\!\!\!/^{\alpha\beta} + m\,\delta^{\alpha\beta})\psi^a_\beta . \tag{8}$$

This is in fact a more general version of graphene, where we consider d spacetime dimensions, allow the gauge fields to be non-Abelian and the fermions to enjoy a $SU(N)$-flavour symmetry: $a \in \{1,...,N\}$. $\nabla\!\!\!/^{\alpha\beta}$ is the covariant derivative times the γ-matrices for the fermionic spin structure

$$\nabla\!\!\!/^{\alpha\beta} = \nabla^\mu \gamma_\mu^{\alpha\beta} . \tag{9}$$

The range of the spinor indices vary w.r.t. d, e.g. in four dimensions $\alpha, \beta \in \{1,...,4\}$.

Systems with two boundaries are also of physical interest. E.g. two parallel boundaries (or *a pair of slabs*) can be used describe the *Casimir effect* between two conducting plates [17, 18]. This effect describes an interacting force, or an energy emission/absorption, between the two plates.

Intersecting boundaries (or a *wedge*) also have physical applications. These appear in experimental condensed matter setups. Moreover, if the intersection angle approaches 2π we make contact with conical singularities [19, 20]. These appear e.g. in string theory when studying cosmic strings (line deformations of the spacetime) [21, 22, 23], and in quantum information regarding black hole entropies [24, 25].

Interfaces

There is another kind of codimension one defects, namely *interfaces*. Such systems are referred to as *interface quantum field theories* (IQFT's). The difference between an interface and a boundary is that in an IQFT there is a bulk theory defined on each side of the defect. The two bulk theories may differ from each other. This means that technically there are three QFT's in play: one on each side of the interface, in addition to a lower-dimensional one on the interface. Interfaces can appear in physical systems in various ways, as explained in e.g. the Introduction of paper V.

A particularly interesting class of interfaces are those with a free bulk theory on one side, and an interacting one on the other side. The interface will then couple these two theories to each other, and it is conjectured that it might tell us something about the *renormalization group* (RG) flow between the two bulk theories. Hence such interface is called a *Renormalization group domain wall*. These kinds of defects were first studied in [26] for two dimensions, and in [27] for *higher dimensional spacetime* ($d > 2$).

Replica twist defects

Codimension two defects are special in the sense that they can carry a monodromy action for the fields [28, 29], which means that a bulk field transforms under the global symmetry group as we transport it around the defect. This leads to a symmetry breaking of the global symmetry in the bulk. This was studied in paper I for an $O(N)$-model, where the monodromy action is

$$\phi^i(x_\parallel, r, \theta + 2\pi) = g^{ij}\phi^j(x_\parallel, r, \theta) \ , \quad g^{ij} \in O(N) \ . \tag{10}$$

Here we use polar coordinates for the normal directions: $r > 0$, $\theta \in [0, 2\pi)$. Due to this constraint, we refer to these defects as *($O(N)$-flavoured) monodromy twist defects* (or symmetry defects). The choice of the group element g^{ij} is referred to as the *twist* and characterizes the defect. In fact, above monodromy relation describes a branch cut in the plane of the normal coordinates. We can choose to treat the monodromy defect as a defect of codimension one which spans a half-plane along this branch cut [28].

Monodromy defects can be generalized by modifying the monodromy constraint to branch cuts with n sheets. Let us consider the special case when the QFT's on each sheet are all the same

$$\phi^i_a(x_\parallel, r, \theta + 2\pi) = g^{ij}\phi^j_{a+1}(x_\parallel, r, \theta) \ . \tag{11}$$

We then say that we have *n replicas*, ϕ^i_a with $a \in \{1, ..., n\}$, of the original theory. These defects are called *replica twist defects*, and they enter in QFT's when applying the replica trick used to find the Rényi entropy [30, 31]

$$S_n = \frac{\log Z(n) - n\log Z(1)}{1-n} \xrightarrow{n\to 1} S_{EE} \ , \tag{12}$$

where $Z(n)$ is the partition function for the theory with n replicas. In the $n \to 1$ limit we find the entanglement entropy, S_{EE}, in a QFT, which loosely speaking is a measure on how much information of the total system is preserved on a subregion of the full space of the theory [30, 31].

Thesis outline

This thesis consists of two Parts, wherein the first Part we give an excessive introduction to the subject of defects in QFT's and cover the relevant background topics for the papers included in this thesis. Firstly in Ch. 1 we discuss a *homogeneous QFT* (without a defect), from which we will make analogies and comparisons with when a defect is added to the theory in Ch. 2. Furthermore we will briefly discuss QFT's with several defects in Ch. 3.

In the next Part we explain the technical methods developed in the attached papers of this thesis. While doing this we generalize all of these methods or apply them to new theories. This will not only illustrate the methods, but also provide us with new results not found prior to this thesis.

In Ch. 4 we explain the analytical bootstrap method for bulk two-point functions in a *boundary conformal field theory* (BCFT) developed in paper II and further extended upon in paper V to hold in an *interface conformal field theory* (ICFT). This method extracts the bulk *operator product expansion* (OPE) coefficients from the bootstrap equation for bulk two-point correlators by exploiting the analytical structure of the conformal blocks. We generalize this method to work for mixed correlators where the two external scalars does not need to be identical. We also show how the *boundary operator product expansion* (BOE) coefficients from the bootstrap equation can be extracted in a similar way.

In Ch. 5 we illustrate the methods used in paper IV and VII, and find the bulk two-point function upto two-loops using the *equations of motion* (e.o.m.) In paper VII this was done for a wedge, but here we apply it to a BCFT in $d = 3 - \varepsilon$ with both a bulk- and a boundary-interaction. Lastly we apply the *Coleman-Weinberg* (CW) mechanism to find an effective description for a first-ordered p.t. In paper IV, this was done for a theory with only boundary-interactions, but here we consider in addition a bulk-interaction. This leads to a non-trivial boundary *vacuum expectation value* (v.e.v.), which due to the BOE also extends into the bulk. It leads to a *spontaneous symmetry breaking* (SSB) of the global $O(N)$-symmetry (both in the bulk and on the boundary).

In Ch. 6 we consider a replica twist defect, which is a generalization of the monodromy twist defect considered in paper I. We apply the techniques from paper I to the replica twist defect, and study how its monodromy action (11) breaks the global $O(N)$-symmetry in the bulk. This breaking will occur in the same way as in paper I, and the difference lies in the spin of the defect-local fields. Furthermore, we show how the anomalous dimension of the defect-local fields can be extracted using the e.o.m. and the *defect operator product expansion* (DOE). This gives us a new result which reduces down to that in paper I when we only consider one replica.

Lastly in Ch. 7 we improve the method from paper VI, which concerns fusion of scalar Wilson defects (5) in $d = 4$. In the free theory we generalize this result to hold in $d = 6$. We suggest that the fusion holds at a quantum level (which is what we expect since the path integral stays invariant under fusion) by showing that the bare one-point function in the presence of the two defects is the same as that near the fused defect. The difference lies in renormalization, where the bare coupling constant on the fused defect also takes into account divergences which appear in the *fusion-limit* of the two Wilson defects (when the two defects approach each other). This means that renormalized quantities, such as correlators and β-functions, are allowed to differ.

Part I:
Background

1. Homogeneous quantum field theories

In this Chapter we will briefly discuss a homogeneous QFT without a defect in higher dimensional spacetime ($d > 2$). In the next Chapter we will add a defect to the QFT and compare with this Chapter.

We start with the Poincaré group that follows from frame invariance and special relativity. We then proceed to study global symmetries and their respective conserved currents. Next we study the free propagators found from the e.o.m., which are the building blocks of Feynman diagrams. Moving on to the interacting theory we briefly review how a QFT is renormalized, and how this gives rise to the RG. At f.p.'s of this semigroup (a group without an identity element) the Poincaré symmetry is enhanced to the conformal group. Due to conformal symmetry we can find the powerful and non-perturbative bootstrap equation for four-point functions. Lastly, we discuss how the CW mechanism allows us to flow along the RG away from a conformal second-order p.t. to a non-conformal first-ordered one.

1.1 Symmetries of a quantum field theory

1.1.1 The Poincaré group and the Poincaré algebra

In a d-dimensional QFT we are free to choose a frame (coordinate system). This leads to a symmetry under rotations, boosts and translations. We will consider a relativistic theory with Lorentz boosts.[1] In special relativity we consider invariance under rotations and boosts, which form the *Lorentz group*, $SO(d-1,1)$. It is given by

$$SO(d-1,1) = \{\Lambda^{\mu\nu} \in \mathbb{R}^{(d-1,1)\times(d-1,1)} \mid \Lambda^{\mu\nu}\Lambda_{\nu\rho} = \delta^{\mu}{}_{\rho}\ ,\ \det\Lambda^{\mu\nu} = 1\}\ ,$$

which is graded due to the imaginary time-coordinate, and $\delta^{\mu}{}_{\rho}$ is the Kronecker δ-function (an identity matrix). Here, and throughout the rest of this thesis, we have used Einstein summation to suppress summation over repeating indices. In this case $\mu,\nu \in \{0,...,d-1\}$ with $\mu = 0$ being the time-coordinate. Indices are raised and lowered using the *Minkowksi metric*

$$(\eta^{\mu\nu}) = \mathrm{diag}(-1,1,...,1)\ . \tag{1.1}$$

Adding to the Lorentz group translations yields the *Poincaré group*

$$ISO(d-1,1) = \mathbb{R}^d \rtimes SO(d-1,1)\ . \tag{1.2}$$

[1] In a non-relativistic theory, we consider Galilean boosts.

Lorentzian quantum field theory	Classical statistical physics
Action: S	Hamiltonian: H
Generating functional: $Z = \int \mathscr{D}\phi e^{\frac{i}{\hbar}S[\phi]}$, $\hbar = \dfrac{h}{2\pi}$	Partition function: $Z = \sum_{\text{states}} e^{-\beta H}$, $\beta \equiv \dfrac{1}{k_B T}$
Fundamental fields, e.g. ϕ	Order parameters
Scaling dimensions: Δ	Critical exponents
Vacuum expectation value: $\langle 0\vert\mathscr{O}\vert 0\rangle$	Ensemble average: $\langle\mathscr{O}\rangle$
Amplitude: $\langle T[\mathscr{O}_1 ... \mathscr{O}_n]\rangle$	Correlation function: $\langle\mathscr{O}_1 ... \mathscr{O}_n\rangle$
Masses: M	Correlation lengths: $\xi = \frac{1}{m} \propto \vert T - T_c\vert^{-\nu}$
Momentum cutoff: $\Lambda \gg 1$	Lattice spacing: $a \ll 1$

Table 1.1. *A table with some of the connections between Lorentzian QFT's and classical statistical physics. This is far from a complete list, and we will not explain the correspondence between the quantities in detail.*

Here I denotes inhomogeneous and means that an element cannot be written as a matrix, e.g. a space-time vector transforms under the Poincaré group as

$$x^\mu \to \Lambda^{\mu\nu} x_\nu + a^\mu \ , \tag{1.3}$$

where $a^\mu \in \mathbb{R}^{d-1,1}$ is a constant vector (giving us a translation), and $\Lambda^{\mu\nu} \in SO(d-1,1)$ is a Lorentz transformation.

Performing a Wick rotation $t \to i\tau$ yields Euclidean signature: $\eta^{\mu\nu} \to \delta^{\mu\nu}$, where the Poincaré group is $ISO(d)$. Euclidean signature is used in condensed matter, while Lorentzian signature is more applicable to particle physics. In Tab. 1.1 we list some of the connections between these two research fields.

The Poincaré group is a Lie group, and just like any other Lie group it can be treated as a manifold. Its tangent at identity describes a Lie algebra, where the exponential map is a map from the Lie algebra to the Lie group. This creates a Lie group - Lie algebra correspondence (see [32] and references therein) which is of great importance for physicists when it comes to classifying operators in QFT's (as we will see later on in this thesis).

The Lie algebra consists of several commutation relations for the corresponding Lie group's generators. In physics, eigenvalues of these generators are interesting as they correspond to physical data. E.g. the eigenvalues for the generators of the Lorentz transformations and the translations of the Poincaré group are the full angular momentum (including the quantum spin), $M^{\mu\nu}$, and the momenta, P^μ, respectively

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \ , \quad P_\mu = i\partial_\mu \ . \tag{1.4}$$

Their corresponding eigenvalues are used to classify the *spectrum* of the QFT, i.e. its operators. The non-trivial commutation relations in the Poincaré algebra are given by

$$
\begin{aligned}
[M_{\mu\nu}, P_\rho] &= -i\left(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\right) , \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right) .
\end{aligned}
\tag{1.5}
$$

1.1.2 Global symmetries and Noether's theorem

In addition to the Poincaré group, a QFT may enjoy *global symmetries*. The corresponding symmetry operations transform the fields (rather than the spacetime). Common examples of global symmetries are $O(N)$ for scalars or $SU(N)$ for fermions, where U stands for unitary (meaning that its inverse is the same as its transpose of its complex conjugate).

For each continuous symmetry, there exists a corresponding conserved current. According to *Noether's theorem* they are found in the following way [33]: assume an infinitesimal transformation in the spacetime coordinates (e.g. Poincaré symmetry) and fields (e.g. $O(N)$-symmetry of scalars) which leaves the Lagrangian invariant

$$
\mathscr{O} \to \mathscr{O} + \delta\mathscr{O} , \quad x^\mu \to x^\mu + \delta x^\mu . \tag{1.6}
$$

Then there exist a spin one *Noether current*, J^μ, (for each global symmetry) and a symmetric, spin two *stress-energy (SE) tensor*, $T^{\mu\nu} = T^{\nu\mu}$, (or the energy-momentum tensor) on the form

$$
J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\mathscr{O})}\delta\mathscr{O} - T^{\mu\nu}\delta x_\nu , \quad T^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\mathscr{O})}\partial^\nu\mathscr{O} - g^{\mu\nu}\mathscr{L} . \tag{1.7}
$$

Here $\mathscr{L}$ is the Lagrangian density and $g^{\mu\nu}$ is the spacetime metric. The time-time component, T^{00}, of the SE tensor is the energy density of the theory, and its time-space components, T^{i0}, describe the momentum density. The rest of the terms describe how the energy and the momentum densities are fluctuating in space.

Both of these currents are conserved

$$
\partial_\mu J^\mu = 0 , \quad \partial_\mu T^{\mu\nu} = 0 . \tag{1.8}
$$

This means that their respective *scaling dimensions* (or mass dimensions) are *protected*, i.e. they will not receive any quantum corrections. These can be found from dimensional analysis and are exactly given by

$$
\Delta_J = d - 1 , \quad \Delta_T = d . \tag{1.9}
$$

1.2 Renormalization

1.2.1 Free propagators

In this Section we will study a free theory without interactions. More explicitly we will consider a massive scalar, ϕ^i with $i \in \{1,...,N\}$, satisfying $O(N)$-symmetry and find the two-point function. This propagator is the building block for the Feynman diagrams, which we traditionally compute in the interacting theory.

The action is given by

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \phi^i)^2}{2} + \frac{m^2}{2} (\phi^i)^2 \right) . \tag{1.10}$$

Its e.o.m. is described by the *Klein-Gordon (KG) equation*

$$\frac{\delta S}{\delta \phi^i} = 0 \quad \Rightarrow \quad (-\partial_\mu^2 + m^2)\phi^i = 0 , \tag{1.11}$$

which is found by varying the action (1.10) w.r.t. ϕ. The operator ∂_μ^2 is the *d'Alembert operator*: the Laplacian in Minkowski spacetime. Operator equations only make sense mathematically when acting on propagators, e.g. the two-point function, $D^{ij}(s)$, at separate points $s \equiv x - y$. This yields to the *Dyson-Schwinger (DS) equation* for a scalar [34, 35]

$$\begin{aligned} (-\partial_\mu^2 + m^2) D^{ij}(s) &= \delta^{(d)}(s) , \\ D^{ij}(s) &= \langle T[\phi^i(x)\phi^j(x)]\rangle . \end{aligned} \tag{1.12}$$

In technical terms, $D^{ij}(s)$ is a Green's function. T is the time-ordering operator and $\delta^{(d)}(s)$ is the d-dimensional Dirac δ-function

$$\delta^{(d)}(s) = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{-iks} = \begin{cases} \infty , & \text{if } x = y, \\ 0 , & \text{else.} \end{cases} \tag{1.13}$$

If we express $D^{ij}(s)$ in terms of its Fourier transform

$$D^{ij}(s) = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{-iks} G^{ij}(k) , \tag{1.14}$$

we find the *momentum propagator* from (1.12)

$$G^{ij}(k) = \frac{\delta^{ij}}{k^2 + m^2} . \tag{1.15}$$

The Fourier transform variable, k^μ, describes the momentum of a particle travelling from point x to y $\big(\text{if } \mathrm{Re}(x^0) < \mathrm{Re}(y^0)\big)$ in the spacetime, with its energy in the zeroeth component. We can proceed to integrate over the momenta in

(1.14) to find the spacetime propagator. In order to avoid poles in denominator one has to deform it a bit with a regulator $\varepsilon \ll 1$.

In Euclidean signature the integration is easier to deal with. Firstly, since there is no time coordinate we do not have to deal with time ordered products. Moreover, the integral (1.14) can be solved using a *Schwinger parametrization*

$$\frac{1}{A^n} = \int_0^\infty du \frac{u^{n-1}}{\Gamma_n} e^{-uA} \ , \tag{1.16}$$

with $\Gamma_n \equiv \Gamma(n)$ being the Gamma function, and the Gausian integral in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} d^n x e^{-ax^2+bx} = \left(\frac{\pi}{a}\right)^{\frac{n}{2}} \exp\left(\frac{b^2}{4a}\right) \ . \tag{1.17}$$

The integral (1.14) we are trying to compute is a special case of the more general integral[2]

$$\begin{aligned} I^n_\Delta(k,w,z^2) &= \int_{\mathbb{R}^n} d^n x \frac{e^{ikx}}{[(x-w)^2+z^2]^\Delta} \\ &= \frac{\pi^{\frac{n}{2}}}{2^{\Delta-\frac{n}{2}-1}\Gamma_\Delta} e^{ikw} \left(\frac{|k|}{|z|}\right)^{\Delta-\frac{n}{2}} K_{\Delta-\frac{n}{2}}(|k|z) \\ &= \begin{cases} \dfrac{\pi^{\frac{n}{2}}\Gamma_{\frac{n}{2}-\Delta}}{2^{2\Delta-n}\Gamma_\Delta} \dfrac{e^{ikw}}{|k|^{2\Delta-n}} \ , & \text{if } z=0 \ , \\ \\ \dfrac{\pi^{\frac{n}{2}}\Gamma_{\Delta-\frac{n}{2}}}{\Gamma_\Delta} \dfrac{1}{|z|^{2\Delta-n}} \ , & \text{if } k=0 \ . \end{cases} \end{aligned} \tag{1.18}$$

Here $k, w \in \mathbb{R}^n$ and $z^2, n \in \mathbb{R}$. $K_m(y)$ is the Bessel function of the second kind. It yields

$$D^{ij}(s) = \frac{\delta^{ij}}{(2\pi)^d} I^d_1(-s,0,m^2) = \delta^{ij} \frac{m^{\Delta_\phi} K_{\Delta_\phi}(m|s|)}{(2\pi)^{d-2}|s|^{\Delta_\phi}} \ , \tag{1.19}$$

where Δ_ϕ is the scaling dimension of the free scalar

$$\Delta_\phi = \frac{d-2}{2} \ . \tag{1.20}$$

The free propagators, be it in momentum space (1.15) or Minkowski spacetime (1.19), are used as building blocks in *Feynman diagrams*, where they make up an integrand that is either integrated over exchanged loop momenta or points of vertices in spacetime [36]. E.g. consider an interaction term on the form

[2]The limit $k \to 0$ must be taken before the integration.

$\lambda\left[(\phi^i)^2\right]^n \in \mathscr{L}$ in the Lagrangian, $\mathscr{L}$, for some integer $n \in \mathbb{Z}_{\geq 1}$. Then the two-point function at order λ^m is given by the following Feynman diagram[3]

$$\langle T[\phi^i(x)\phi^j(x)]\rangle = (i\lambda)^m \prod_{i=1}^{m} \int_{\mathbb{R}^d} d^d z_i \langle T\left[\phi^i(x)\phi^j(y)\prod_{j=1}^{m}\left[(\phi^i)^2\right]^n(z_j)\right]\rangle\ .$$

Here, and in the rest of this thesis, we are using units s.t. $\hbar$, $c = 1$. The propagator in the integrand can be split into products of two-point functions (1.19) using Wick's theorem.

1.2.2 Renormalization group

Consider now a field theory with interactions. We will compactly write its action on the form

$$S = \int_{\mathbb{R}^{d-1,1}} d^d x \left[\mathscr{L}_0\left(\{\mathscr{O}_i^0, m_i^0\}\right) + \mathscr{L}_{\text{int}}\left(\{\mathscr{O}_i^0\}, \{g_a^0\}\right)\right]\,, \tag{1.21}$$

where $\mathscr{L}_0$ is the Lagrangian density in the free theory containing the set of bare fundamental fields, $\mathscr{O}_i^0$, and their respective bare masses, m_i^0. $\mathscr{L}_{\text{int}}$ is the interaction Lagrangian density, which contains the terms with interactions, one for each bare coupling constant, g_a^0. We will assume that g_a^0 has mass-dimension $\dim(g_a^0)$ (which can be found from dimensional analysis). We wish to study how this general action runs under the RG flow. The choice of regulator will not be important for this discussion, but we will illustrate it for the deviation from the spacetime dimension ε.[4]

Normally we find the RG flow of a theory by calculating Feynman diagrams for the interactions we consider. We expect the Feynman diagrams to be divergent, containing both *infrared* (IR) and *ultraviolet* (UV) divergences. IR divergences arise from scales of the theory (long distances or low energy in momentum space), while UV divergences arise from *coincident-limits* of fields (short-distance and high energy).

In this thesis we are interested in how to take care of the UV divergences, and will thus assume that no IR divergences are present. To renormalize the theory we assume that the bare quantities ($\mathscr{O}_i^0$, m_i^0 and g_a^0) are all divergent in such a way that times the UV-divergent Feynman diagrams we get a finite expression through a cancellation of the ε-poles.

The bare fields and masses can be found from their respective two-point function, $\langle T[\mathscr{O}_i^0(x)\mathscr{O}_i^0(y)]\rangle$. At first order in the coupling constants, assume

[3] Most commonly we calculate Feynman diagrams in momentum space, where loop momenta is integrated rather than vertex points. We use this notation since we find it more useful when we add defects to the theory. The factors of i come from the e^{iS}-factor in the path integrand. In Euclidean signature, this is exchanged to factors of -1.

[4] One might as well choose a momentum cutoff Λ, in which case poles in ε are exchanged with logarithmic divergences in Λ: $\varepsilon^{-1} \to \log\Lambda$.

$\langle T[\mathscr{O}_i^0(x)\mathscr{O}_i^0(y)]\rangle$ to have poles in ε. We then write the bare field and mass as

$$\begin{aligned}
\mathscr{O}_i^0 &= \sum_j Z^{\mathscr{O}_i^0}{}_{\mathscr{O}_j}\mathscr{O}_j\ , \qquad & Z^{\mathscr{O}_i^0}{}_{\mathscr{O}_j} &= \delta^{\mathscr{O}_i^0}{}_{\mathscr{O}_j} + \frac{(\zeta^{\mathscr{O}_i^0}{}_{\mathscr{O}_j})^a g_a}{\varepsilon} + \mathscr{O}(g^2)\ , \\
m_i^0 &= Z_{m_i} m_i\ , & Z_{m_i} &= 1 + \frac{(\zeta_{m_i})^a g_a}{\varepsilon} + \mathscr{O}(g^2)\ .
\end{aligned} \tag{1.22}$$

Here $\mathscr{O}_i$ and m_i are the renormalized fields and masses respectively, where we allowed the bare fields to mix with renormalized ones. Summation over the index a is implicit. The constants $(\zeta^{\mathscr{O}_i}{}_{\mathscr{O}_j})^a$ and $(\zeta_{m_i})^a$ are tuned s.t. the poles in the renormalized propagators, $\langle T[\mathscr{O}_i(x)\mathscr{O}_i(y)]\rangle$, cancel. $\langle T[\mathscr{O}_i(x)\mathscr{O}_i(y)]\rangle$ is given by

$$\langle T[\mathscr{O}_i(x)\mathscr{O}_i(y)]\rangle = (Z^{\mathscr{O}_i}{}_{\mathscr{O}_j})^{-1}(Z^{\mathscr{O}_i}{}_{\mathscr{O}_k})^{-1}\langle T[\mathscr{O}_j^0(x)\mathscr{O}_k^0(y)]\rangle\ . \tag{1.23}$$

The bare coupling constants are found from their respective diagrams. E.g. if we have an interaction on the form $g_a^0(\phi^0)^n$, then the corresponding bare coupling constant is found from the following propagator of renormalized fields

$$\langle T[\phi(x_1)\ldots\phi(x_n)]\rangle = (Z^{\mathscr{O}_{i_1}^0}{}_{\phi})^{-1}\ldots(Z^{\mathscr{O}_{i_n}^0}{}_{\phi})^{-1}\langle T[\mathscr{O}_{i_1}^0(x_1)\ldots\mathscr{O}_{i_n}^0(x_n)]\rangle\ ,$$

where summation over the repeating indices are implicit. This propagator will in general contain poles in ε. Assume that the following g_a^0 cancel these divergences (together with the bare fields and masses at (1.22))

$$g_a^0 = \mu^{\dim(g_a^0)} Z_{g_a} g_a = \mu^{\dim(g_a^0)}\left(g_a + \frac{(\zeta_{g_a})^{bc} g_b g_c}{\varepsilon} + \mathscr{O}(g^3)\right)\ , \tag{1.24}$$

for some constants $(\zeta_{g_a})^{bc}$ (and where we do not sum over a). We introduced a *renormalization scale*, μ, s.t. the renormalized coupling constants, g_a, are dimensionless. We assumed both *multiplicative* (corresponding to the terms proportional to g_a) and *additive renormalization* (not proportional to g_a). The renormalized coupling constants and masses can potentially be measured in a lab in particle physics, e.g. in quantum electrodynamics the renormalized coupling constant is the electronic charge.

The *renormalization group* consists of the transformations (1.22, 1.24) for the bare quantities. Strictly speaking it is a semigroup, and it measures the *renormalization group flow*. It tells us how the physics change w.r.t. the energy scale μ.

1.2.3 β-functions

Let us now make sense of this RG flow. We will do this by studying how the renormalized coupling constants, g_a, change w.r.t. the RG scale, μ. Per definition, bare couplings do not depend on μ while renormalized do

$$\frac{\partial g_a^0}{\partial \mu} = 0 \ , \quad \beta_a = \mu \frac{\partial g_a}{\partial \mu} = \frac{\partial g_a}{\partial \log \mu} \ . \tag{1.25}$$

The latter derivative is the definition of the (dimensionless) β*-function*, which measures the RG flow. We wish to find the β-functions given the bare couplings (1.24). To do this we will in addition to the above derivatives also use

$$\frac{\partial \log g_a}{\partial \log \mu} = \frac{\beta_a}{g_a} \ , \quad \text{(no summation over } a\text{).} \tag{1.26}$$

Let us assume the couplings are small, allowing for a perturbative expansion around them. Then if we take the logarithm of (1.24) and expand in the coupling constants we find

$$\log g_a^0 = \dim(g_a^0) \log \mu + \log g_a + \frac{(\zeta_{g_a})^{bc}}{\varepsilon} \frac{g_b g_c}{g_a} + \mathscr{O}(g^2) \ . \tag{1.27}$$

Now consider the derivative w.r.t. $\log \mu$ and use the derivatives at (1.25, 1.26)

$$0 = \dim(g_a^0) + \frac{\beta_a}{g_a} + \frac{(\zeta_{g_a})^{bc}}{\varepsilon} \left(\frac{\beta_b g_c}{g_a} + \frac{g_b \beta_c}{g_a} - \frac{g_b g_c \beta_a}{g_a} \right) + \mathscr{O}(g^3) \ .$$

From this system of equations we can find all of the β_a's. The RG flow goes from maxima of the β-functions, towards their minima (i.e. the RG flow goes in the opposite direction of the β-functions). We will see specific examples of RG flows later in.

Let us turn our attention back to the SE tensor from Sec. 1.1.2. The trace of this operator is proportional to the β-functions of the theory [37]

$$T^{\mu}{}_{\mu} = \beta_a \mathscr{O}_a \ . \tag{1.28}$$

Here $\mathscr{O}_a$ is the composite operator of its corresponding interaction. This relation will be important at *fixed points* of the RG flow, when all of the β-functions are zero.

1.2.4 Callan-Symanzik equation

In Sec. 1.2.3 we studied how the renormalized coupling constants change w.r.t. the RG scale, μ. Similarly we can study how a propagator, $D_{\mathscr{O}_i}$, including a renormalized field, $\mathscr{O}_i(x)$, change w.r.t. μ

$$D_{\mathscr{O}_i} = \langle \mathscr{O}_i(x_1) \ldots \rangle \ . \tag{1.29}$$

Here the dots represent an arbitrary product of operators. Let us consider a shift in the RG scale

$$\mu \to \mu + \delta\mu \ . \tag{1.30}$$

This will in turn also shift the renormalized coupling constants, the masses and the fields

$$\begin{aligned} m_i &\to m_i + \delta m_i \ , \\ g_a &\to g_a + \delta g_a \ , \\ \mathscr{O}_i &\to \mathscr{O}_i + \delta Z^{\mathscr{O}_i}{}_{\mathscr{O}_j}\mathscr{O}_j \ . \end{aligned} \tag{1.31}$$

The renormalized propagator (1.29) is then shifted as

$$D_{\mathscr{O}_i} \to \left(\delta^{\mathscr{O}_i}{}_{\mathscr{O}_j} + \delta Z^{\mathscr{O}_i}{}_{\mathscr{O}_j}\right) D_{\mathscr{O}_j} \ . \tag{1.32}$$

Now if we think of $D_{\mathscr{O}_i}$ as a function of μ, g_a and m_i we find

$$\delta D_{\mathscr{O}_i} = \frac{\partial D_{\mathscr{O}_i}}{\partial \mu}\delta\mu + \frac{\partial D_{\mathscr{O}_i}}{\partial g_a}\delta g_a + \frac{\partial D_{\mathscr{O}_i}}{\partial m_j}\delta m_j = \delta Z^{\mathscr{O}_i}{}_{\mathscr{O}_j} D_{\mathscr{O}_j} \ . \tag{1.33}$$

Define the *β-function*, the *mass anomalous dimensions* and the *anomalous dimension matrix* in the following way[5]

$$\beta_a \equiv \mu\frac{\partial g_a}{\partial \mu} \ , \quad \gamma_{m_i} \equiv \mu\frac{\partial m_i}{\partial \mu} \ , \quad \gamma^{ij}_{\mathscr{O}} \equiv -\mu\frac{\partial Z^{O_i}{}_{O_j}}{\partial \mu} \ . \tag{1.34}$$

Then from (1.33) we find the *Callan-Symanzik (CS) equation* [38, 39]

$$\left(\delta^{ij}\mu\frac{\partial}{\partial\mu} + \delta^{ij}\beta_a\frac{\partial}{\partial g_a} + \delta^{ij}\gamma_{m_k}\frac{\partial}{\partial m_k} + \gamma^{ij}_{O}\right) D_{\mathscr{O}_j} = 0 \ . \tag{1.35}$$

This is a partial differential equation for the propagators which describe how they change w.r.t. the renormalization scale.

1.2.5 Operator product expansion

Remember that UV divergence arise from the coincident-limit between fields? This means that if we would use the distance $s \equiv x - y$ between two operators as an infinitesimal cutoff, then we can see from the *operator product expansion* (OPE) [40, 41] how fields mixes in the renormalization procedure in the corresponding coincident-limit

$$\begin{aligned} \mathscr{O}_1(x)\mathscr{O}_2(0) &= \sum_{\mathscr{O}_{\Delta,l}} \tilde{C}_{\Delta,l}(\mu,x)\mathscr{O}_{\Delta,l}(x) \ , \\ \tilde{C}_{\Delta,l}(\mu,x) &= \frac{\lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}_{\Delta,l}}}{|x|^{\Delta^+_{12}-\Delta+l}} C_{\Delta,l}(\mu,x) \ , \\ \mathscr{O}_{\Delta,l}(x) &= x^{\mu_1}\ldots x^{\mu_l}\mathscr{O}^{\mu_1\ldots\mu_l}_{\Delta}(x) \ . \end{aligned} \tag{1.36}$$

[5] Sometimes the mass anomalous dimension is defined with an extra $(m_i)^{-1}$, and the anomalous dimension matrix γ^{ij}_{O} is defined with an extra $(Z^{O_I}{}_{O_J})^{-1}$ multiplied from the left.

Here $\Delta^+_{ij} \equiv \Delta_i + \Delta_j$, and the sum runs over exchanged operators, $\mathscr{O}_{\Delta,l}$, with scaling dimension Δ and *spin l* (full angular momentum). The OPE has divergent terms when the scaling dimensions of the operators satisfy $\Delta < \Delta^+_{12}$. $\lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}_{\Delta,l}}$ is called the *operator product expansion coefficients*, and $C_{\Delta,l}(\mu,x)$ is a differential operator found from the CS eq. (1.35)

$$\left(\delta^{ij}\mu\frac{\partial}{\partial\mu} + \delta^{ij}\beta_a\frac{\partial}{\partial g_a} + \delta^{ij}\gamma_{m_k}\frac{\partial}{\partial m_k} + \gamma^{ij}_{O}\right)\tilde{C}_{\Delta,l}(\mu,x) = 0\,. \tag{1.37}$$

The OPE is very useful when we wish to find propagators involving composite operators. For example, say we want to find $\langle T[\phi(x_1)\phi(x_2)\phi^2(x_3)]\rangle$ knowing $\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle$ upto some order in the coupling constants. Then, instead of calculating new Feynman diagrams involving ϕ^2 as an external operator, we can instead study the coincident-limit of $\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle$ when $x_4 \to x_3$. In this limit we will find poles and logarithms in $x_{34} \equiv x_3 - x_4$. However, the powers of x_{34} can be matched with the OPE (1.36) to read off several different three-point functions. Among these we can find $\langle T[\phi(x_1)\phi(x_2)\phi^2(x_3)]\rangle$. Poles in x_{34} correspond to operators of lower dimension than $2\Delta_\phi$, and from the logarithms we can read off the anomalous dimensions of the exchanged operators.

We will come back to study the OPE in the conformal case which contains no physical scales, where $C_{\Delta,l}$ in (1.36) simplifies drastically. In such case the region of convergence of the OPE is known as well [42].

1.3 Conformal field theories

1.3.1 The conformal group and the conformal algebra

At the f.p.'s of the RG the Poincaré symmetry from Sec. 1.1.1 might be extended. I.e. if we consider a massless theory, then when all of the β-functions are zero there is no running of coupling constants. This also leads to a traceless SE tensor (1.28). At this point in the RG there is no dependence on the RG scales. This leads to a scale invariance of the theory, enhancing the Poncaré symmetry group

$$x^\mu \to \lambda x^\mu \quad \lambda \in \mathbb{R} \quad \Rightarrow \quad \mathscr{O}(x) \to \frac{\mathscr{O}(\lambda x)}{\lambda^\Delta}\,. \tag{1.38}$$

Here Δ is the scaling dimension of $\mathscr{O}$, which in the free theory can be found from dimensional analysis.

In condensed matter, the temperature, T, is tuned to the critical temperature, T_c, of a *second-order phase transition* (where the system continuously shifts to a new phase) when all of the masses and β-functions are zero. This means that QFT's at their respective RG f.p.'s are used to describe second-order p.t.'s. See [43, 44] and references therein. In Tab. 1.1 we have the relation between a mass and $|T - T_c|$.

For $d = 2$ it is proven that scale symmetry also implies invariance under *special conformal transformations* (SCT's), while in $d > 2$ it is only conjectured [45, 46]. In two dimensions, SCT's are Möbius transformations, that preserves angles, but not lengths, and in higher dimensions they are its correspondent

$$x^{\mu} \to \frac{x^{\mu} - x_{\nu}^{2} a^{\mu}}{1 - 2a_{\rho}x^{\rho} + a_{\rho}^{2}x_{\sigma}^{2}} . \tag{1.39}$$

Together with the Poincaré transformations (1.3) and the scaling transformation (1.38) they form the conformal $O(d,2)$-symmetry group (in Euclidean space it is $O(d+1,1)$),[6] where only a $SO(d,2) \in O(d,2)$ subgroup can be written as infinitesimal transformations.[7] Note that the conformal group is a Lie matrix group, unlike the inhomogeneous Poincaré group (1.2). A *conformal field theory* (CFT) is a QFT invariant under the scaling transformations (1.38) and the SCT's (1.39) [48, 49].

Let us now look at the conformal algebra. The generators, D and K^{μ}, to the scaling transformations and the SCT's respectively are given by

$$D = ix^{\mu}\partial_{\mu} , \quad K_{\mu} = i(2x_{\mu}x^{\nu}\partial_{\nu} - x_{\nu}^{2}\partial_{\mu}) . \tag{1.40}$$

We refer to D as the *dilation operator*. The Poincaré algebra (1.5) is extended with the non-trivial commutation relations

$$\begin{aligned} [D, P_{\rho}] &= +iP_{\mu} , \\ [D, K_{\rho}] &= -iK_{\mu} , \\ [P_{\mu}, K_{\nu}] &= 2i(\delta_{\mu\nu}D - M_{\mu\nu}) , \end{aligned} \tag{1.41}$$

which yields the *conformal algebra*.

We can understand the conformal algebra in a more compact way by embedding the spacetime in a $(d+2)$-dimensional space

$$(X^{A}) = (X^{+}, X^{-}) \oplus (x^{\mu}) , \quad X^{+} = 1 , \quad X^{-} = x^{\mu}\eta_{\mu\nu}x^{\nu} , \tag{1.42}$$

where $A \in \{+, -, 0, ..., d-1\}$. The inner product is

$$X \cdot Y \equiv X^{A}g_{AB}Y^{B} = x^{\mu}\eta_{\mu\nu}y^{\nu} - \frac{X^{+}Y^{-} + X^{-}Y^{+}}{2} . \tag{1.43}$$

We choose $X^{\pm}$ s.t.

$$X \cdot X \equiv X^{A}g_{AB}X^{B} = 0 . \tag{1.44}$$

Due to this constraint, the vector X^{A} (1.42) is called a *lightcone (LC) coordinate*. The common choice of $X^{\pm}$ is

$$X^{+} = 1 , \quad X^{-} = x^{\mu}\eta_{\mu\nu}x^{\nu} \quad \Rightarrow \quad X \cdot Y = -\frac{(x-y)^{2}}{2} . \tag{1.45}$$

[6] In two dimensions the conformal group is extended to two copies of the infinite-dimensional Virasoro-symmetry: $\mathrm{Vir} \times \overline{\mathrm{Vir}}$ (see [47] and references therein).

[7] Needed for finding the generators.

We define the space' corresponding antisymmetric angular momentum generator in the following way

$$J_{\mu\nu} = M_{\mu\nu} \ , \quad J_{\mu+} = P_\mu \ , \quad J_{\mu-} = K_\mu \ , \quad J_{+-} = D \ . \tag{1.46}$$

With these we can write the full conformal algebra more compactly [50, 51]

$$[J_{AB}, J_{CD}] = -i\left(g_{AC}J_{BD} - g_{AD}J_{BC} - g_{BC}J_{AD} + g_{BD}J_{AC}\right) \ . \tag{1.47}$$

The LC formalism is very useful for finding tensor structures of propagators with spinning fields.

1.3.2 Radial quantization

When we construct a Hilbert space for a QFT we need to consider $(d-1)$-dimensional spacetime foliations, all orthogonal to the time direction. The Hamiltonian operator, which describes time evolution of the theory, then moves states in between these foliations. That said, we are free to choose which foliations to use. In a CFT, a convenient choice is spherical, all centered at the origin. This is called *radial quantization* [52]. As we act with the Dilation operator, D, we are translated from one foliation to another. We can thus treat D as a Hamiltonian: the energy density of the theory.

We will thus label operators by their eigenvalue of D, which are their scaling dimensions, Δ. In addition to this, the operators are also labelled by their quantum spin, l, corresponding to the angular momentum generator, $M_{\mu\nu}$. Since we can translate states in spacetime with P_μ (the generator for translations) it is enough to study states at origo, where we have

$$D|\Delta, l\rangle = i\Delta|\Delta, l\rangle \ , \quad M_{\mu\nu}|\Delta, l\rangle = \Sigma_{\mu\nu}|\Delta, l\rangle \ . \tag{1.48}$$

Here $\Sigma_{\mu\nu}$ is the spin matrix. Due to the conformal algebra (1.41), P_μ and K_μ will act as raising or lowering operators respectively

$$\begin{aligned} DP_\mu|\Delta, l\rangle &= [D, P_\mu]\,|\Delta, l\rangle + iP_\mu D|\Delta, l\rangle = i(\Delta+1)P_\mu|\Delta, l\rangle \ , \\ DK_\mu|\Delta, l\rangle &= [D, K_\mu]\,|\Delta, l\rangle + iK_\mu D|\Delta, l\rangle = i(\Delta-1)K_\mu|\Delta, l\rangle \ . \end{aligned} \tag{1.49}$$

This has a very important consequence: it is enough to know the states with the lowest scaling dimensions, Δ, called *primaries*. These operators are annihilated by the SCT generators, K^μ. The rest, called *descendants*, can be found by applying P_μ. This creates an infinite tower of descendants for each primary. Moreover, due to the nature of P_μ (1.4), the descendants are total derivatives of primaries. All and all, this means that we can label all operators in a CFT with the scaling dimensions and spin of the primaries.

Due to the radial quantization, an operator at origo ($x = 0$) can be described as a state on a sphere (foliation)

$$|\Delta, l\rangle = \mathcal{O}_\Delta^{\mu_1 \dots \mu_l}(0)|0\rangle \ , \tag{1.50}$$

where $|0\rangle$ is the vacuum state. This is the *operator-state correspondence*. In addition, using P_μ we can translate an operator at an arbitrary point in space-time to origo

$$\mathcal{O}_\Delta^{\mu_1 \dots \mu_l}(x)|0\rangle = e^{iP_\mu x^\mu}|\Delta, l\rangle \ , \tag{1.51}$$

which means that we only have to focus on primaries at the origin.

Since the Hamiltonian describes the energy density, its eigenvalues should be real. This means that it should be Hermitian

$$H = H^\dagger \ . \tag{1.52}$$

This yields the *unitary bounds* for the scaling dimensions [53]. For a scalar

$$\Delta \geq \frac{d-2}{2} \ , \tag{1.53}$$

while for an operator with spin l (in the symmetric, traceless representation of the Lorentz group)

$$\Delta \geq d - 2 + l \ . \tag{1.54}$$

Note that the free scalar (1.20) saturates the unitary bound.

Unitarity also implies that the OPE coefficients in (1.36) are real, which in particular means that their squares are not negative

$$\lambda^{\mathcal{O}_1\mathcal{O}_2}{}_{\mathcal{O}_i} \in \mathbb{R} \quad \Rightarrow \quad (\lambda^{\mathcal{O}_1\mathcal{O}_2}{}_{\mathcal{O}_i})^2 \geq 0 \ . \tag{1.55}$$

1.3.3 Two- and three-point functions

Let us now make use of the conformal symmetry and see that it completely fixes all of the two- and three-point functions upto the normalization of the fields, the OPE coefficients and the scaling dimensions and spin of all primaries. Higher-point functions can then be described in terms of the lower-point functions using the OPE. This means that all of the propagators in a CFT is classified by the *CFT data*. That is, the scaling dimensions and spin of the primaries, as well as the OPE coefficients.

We will restrict ourselves to the Euclidean signature, and study the constraints on the *correlators* (propagators in Euclidean signature) that follow from conformal symmetry. Note that for two vectors, X^A and Y^A, only the scalar product $X \cdot Y$ is non-zero (since $X \cdot X = Y \cdot Y = 0$). This means that the two-point function $\langle \mathcal{O}_1(X)\mathcal{O}_2(Y)\rangle$ and three-point function $\langle \mathcal{O}_1(X)\mathcal{O}_2(Y)\mathcal{O}_3(Z)\rangle$ can only depend on $X \cdot Y$ as well as $X \cdot Y$, $Y \cdot Z$ and $Z \cdot X$ respectively. The powers of these products can be found from dimensional analysis and scaling invariance. In particular this fixes the two- and three-point functions by the conformal symmetry.

For scalars, the two-point function is given by (here we are using (1.45))

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{A_d \delta_{\mathcal{O}_1,\mathcal{O}_2}}{(-2X_1 \cdot X_2)^{\frac{\Delta_{12}^+}{2}}} = \frac{A_d \delta_{\mathcal{O}_1,\mathcal{O}_2}}{|x_{12}|^{\Delta_{12}^+}} \ , \tag{1.56}$$

where $x_{ij} \equiv x_i - x_j$ and A_d is a constant which we may normalize to one by a redefinition of the fields. We can see that the two-point function is the massless limit of (1.19) when

$$A_d = \frac{1}{(d-2)S_d} \; , \quad S_d = \frac{2\Gamma_{\frac{d}{2}}}{\pi^{\frac{d}{2}}} \; . \tag{1.57}$$

Here S_d is the *solid angle*: the area of a $(d-1)$-dimensional sphere embedded in a d-dimensional Euclidean space.

In the rest of this Section we will normalize the fields s.t. A_d is one. The three-point function for scalars is then

$$\langle \mathscr{O}_1(x_1)\mathscr{O}_2(x_2)\mathscr{O}_3(x_3)\rangle = \frac{\lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}_3}}{|x_{12}|^{\frac{\Delta_{12}^+ - \Delta_3}{2}}|x_{23}|^{\frac{\Delta_{23}^+ - \Delta_1}{2}}|x_{31}|^{\frac{\Delta_{31}^+ - \Delta_2}{2}}} \; , \tag{1.58}$$

where the constant $\lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}_3}$ is the OPE coefficient entering in the OPE $\mathscr{O}_1 \times \mathscr{O}_2$ at (1.36). This OPE is simplified in a CFT. Firstly, due to the operator-state correspondence, there is an intuitive picture of the it: if we consider two operators at points x and y, then we can view these as states on a sphere enveloping these two points. Using the operator-state correspondence again, these states are described by the exchanged operators in the OPE. Moreover, the β-functions are zero, which simplifies the differential operators, $C_{\Delta,l}(\mu,y)$, in (1.36, 1.37). The sum runs over the primaries, and $C_{\Delta,l}$ generates a tower of descendants for each primary. $C_{\Delta,l}$ can be found by matching the three-point functions with the two-point functions (1.56) using the OPE

$$C_{\Delta,l}(x) \equiv 1 + \frac{x^\mu \partial_\mu}{2} + \mathscr{O}(\partial^2) \; . \tag{1.59}$$

1.3.4 Bootstrap equation

Regarding the four-point function, we are not able to fix it using the conformal symmetry. However, we can use the OPE in different ways, to find the general bootstrap equation. It is a powerful, non-perturbative equation that holds in all CFT's (and is a result of the conformal symmetry).

Let us consider four identical scalars. Using the conformal symmetry we can write the corresponding four-point function in terms of a function, $f(u,v)$, depending on the dimensionless *conformal cross-ratios* u and v.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{A_d f(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} \; , \tag{1.60}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \in (0,1) \; , \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \in (0,1) \; . \tag{1.61}$$

This function satisfy the *bootstrap equation* (or the crossing equation). It is found by applying the OPE in two different ways: either we study the OPE's of $\phi(x_1)\times\phi(x_2)$ and $\phi(x_3)\times\phi(x_4)$ (the s-channel), or $\phi(x_1)\times\phi(x_3)$ and $\phi(x_2)\times\phi(x_4)$ (the t-channel)[8]

$$\begin{aligned} f(u,v) &= v^{\Delta_\phi}\sum_{\mathscr{O}_{\Delta,l}}(\lambda^{\phi\phi}{}_{\mathscr{O}_{\Delta,l}})^2\mathscr{G}_{\Delta,l}(u,v) \\ &= u^{\Delta_\phi}\sum_{\mathscr{O}_{\Delta,l}}(\lambda^{\phi\phi}{}_{\mathscr{O}_{\Delta,l}})^2\mathscr{G}_{\Delta,l}(v,u) = \frac{u^{\Delta_\phi}}{v^{\Delta_\phi}}f(v,u)\ . \end{aligned} \tag{1.62}$$

In particular, this equation is found using only symmetry arguments assuming associativity of the OPE. This means it holds in every CFT. The sum runs over all primaries in the theory, labelled by their scaling dimensions and spin.

This bootstrap equation is solved when we have found the scaling dimensions and spin of all the exchanged primaries, $\mathscr{O}_{\Delta,l}$, together with their respective OPE coefficients, $(\lambda^{\phi\phi}{}_{\mathscr{O}_{\Delta,l}})^2$.

By using the OPE, we can describe higher-point functions in terms of four-point functions. This means that the set of the scaling dimensions and spin of all primaries, together with the set of the OPE coefficients, completely classify a CFT. Together, these two sets form the *CFT data*.

The function $\mathscr{G}_{\Delta,l}(u,v)$ in the bootstrap equation is called a *conformal block*. E.g. in even dimensions it is given in terms of hypergeometric functions [54, 55]. It can be found by comparing above bootstrap equation to the two-point functions (1.56)

$$v^{\Delta_\phi}\mathscr{G}_{\Delta,l}(u,v) = \frac{x_{12}^{\mu_1}\dots x_{12}^{\mu_l}}{|x_{12}|^{l-\Delta}}\frac{x_{34}^{\mu_1}\dots x_{34}^{\mu_l}}{|x_{34}|^{l-\Delta}}C_{\Delta,l}(x_2)C_{\Delta,l}(x_4)\frac{\langle\mathscr{O}_{\Delta,l}(x_2)\mathscr{O}_{\Delta,l}(x_4)\rangle}{A_d}\ .$$

An alternative way of finding $\mathscr{G}_{\Delta,l}(u,v)$ is using the conformal symmetry. More specifically, since it is invariant under conformal transformation it has to satisfy the *conformal Casimir equation* [55]

$$\mathscr{J}^2\frac{\mathscr{G}_{\Delta,l}(v,u)}{|x_{13}|^{2\Delta_\phi}|x_{24}|^{2\Delta_\phi}} = c_{\Delta,l}\frac{\mathscr{G}_{\Delta,l}(v,u)}{|x_{13}|^{2\Delta_\phi}|x_{24}|^{2\Delta_\phi}}\ , \tag{1.63}$$

$$\begin{aligned} \mathscr{J}^2 &= -\frac{(J_1^{AB}+J_2^{AB})^2}{2}\ , \\ c_{\Delta,l} &= \Delta(\Delta-d)+l(l+d-2)\ . \end{aligned} \tag{1.64}$$

Here $\mathscr{J}^2$ is the Casimir of the conformal group, with eigenvalue $c_{\Delta,l}$. J_i^{AB} is the conformal generators (1.46) acting on the point X_i

$$J_i^{AB} = X_i^A\frac{\partial}{\partial X_i^B} - X_i^B\frac{\partial}{\partial X_i^A}\ , \quad \text{(no sum over } i\text{)}. \tag{1.65}$$

[8] In the case of different external operators, the exchanged operators differ in the two channels.

Let us briefly summarize the importance of the bootstrap eq. (1.62). It can be solved exactly in certain two-dimensional CFT's [56], called rational theories. In these theories there is a finite number of primaries.

In higher dimensions it can be studied numerically using semidefinite programming [57] (if the theory is unitary s.t. (1.55) holds), or the method of determinants [58]. Among other things this yields impressive accuracy for the anomalous dimensions in the 3D Ising model [59].

Analytically the bootstrap equation can be studied in the large spin expansion [60, 61], or in Mellin space [62, 63, 64]. It is also possible to project out the OPE coefficients by studying the branch cuts of the bootstrap equation using the *Lorentzian inversion formula* (LIF) [65, 66]. This assumes analyticity of the conformal blocks in the spin. In a similar manner, dispersion relations for the four-point functions have been found by exploiting the analytical structure of the bootstrap equation [67, 68].

Through functional bootstrap we can see how the different approaches to conformal bootstrap are connected to one another [69, 70].

Up until now we have considered a QFT, and studied the UV divergences that appear in the quantized theory. We have seen that these divergences can be taken care of through renormalization, where bare quantities (fields, masses and coupling constants) contain divergences (1.22, 1.24) in such a way s.t. there will be a cancellation of poles. In the process we introduced an RG scale, μ, which gives rise to a RG flow. It describes how the theory change w.r.t. μ through the β-functions (1.25) and the CS eq. (1.35).

At the f.p.'s of this RG, the Poincaré symmetry is enhanced to the conformal group. With this extended symmetry group we are able to find the general and non-perturbative bootstrap eq. (1.62) for the four-point functions (which holds in any CFT). This equation is solved by the CFT data, which completely classify all primaries in the theory, and thus also the CFT itself.

1.4 Coleman-Weinberg mechanism

There is a well-established method, called the *Coleman-Weinberg (CW) mechanism* [71], which allows us to flow along the RG to a *first-order phase transitions* (where the system does not continuously shifts into a new phase) starting from a (conformal) second-order one. In the process we can also find the β-functions upto one-loop [72].

Let us explain this mechanism as we apply it to the *hypercubic anistotropy model* (in Euclidean signature)

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \phi^i)^2}{2} + \lambda^{ijkl} \phi^i \phi^j \phi^k \phi^l \right) , \quad d = 4 - \varepsilon , \tag{1.66}$$

with $i,\ j,\ k,\ l \in \{1,...,N\}$ and the coupling constants

$$\lambda^{ijkl} = \frac{\lambda_1^0}{8} D^{ijkl} + \frac{\lambda_2^0}{4!} \delta^{ijkl} \ . \tag{1.67}$$

The tensor structures are given by

$$\begin{aligned} D^{ijkl} &= \delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} \ , \\ \delta^{ijkl} &= \begin{cases} 1 \ , & \text{if } i = j = k = l, \\ 0 \ , & \text{else.} \end{cases} \end{aligned} \tag{1.68}$$

The model (1.66) satisfy a *hypercubic global symmetry*:
$H(N) = S_N \rtimes \mathbb{Z}_2^N \subset O(N)$. The action of this symmetry permutes the different ϕ^i's (S_N) upto a difference in sign ($\mathbb{Z}_2^N$).[9] Group elements of $H(N)$ (unlike $O(N)$) are not described by a set of continuous parameters, making $H(N)$ a discrete group.

Varying the hypercubic model (1.66) w.r.t. ϕ gives us the e.o.m.

$$\partial_\mu^2 \phi^i = \frac{\lambda_1^0}{2} (\phi^j)^2 \phi^i + \frac{\lambda_2^0}{6} (\phi^i)^3 \ , \quad \text{(no sum over } i\text{)}. \tag{1.69}$$

The β-functions of this theory was found in [73, 74, 75]

$$\begin{aligned} \beta_1 &= -\varepsilon\lambda_1 + \frac{3(N+8)\lambda_1^2 + 2\lambda_1\lambda_2}{\pi^2} + \mathscr{O}(\lambda^3) \ , \\ \beta_2 &= -\varepsilon\lambda_2 + 3\frac{12\lambda_1\lambda_2 + \lambda_2^2}{\pi^2} + \mathscr{O}(\lambda^3) \ . \end{aligned} \tag{1.70}$$

The corresponding RG flow is depicted in Fig.1.1. If we set these β-functions to zero we find the free *Gaussian f.p.* (G)

$$(\lambda_1^*)_G \ , (\lambda_2^*)_G = 0 \ , \tag{1.71}$$

the Ising f.p. (I) with $\mathbb{Z}_2$-symmetry

$$(\lambda_1^*)_I = 0 \ , \quad (\lambda_2^*)_I = \frac{\varepsilon}{3} + \frac{17}{81}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ , \tag{1.72}$$

the *Heisenberg f.p.* (H) with $O(N)$-symmetry

$$(\lambda_1^*)_H = \frac{\varepsilon}{3(N+8)} + \frac{3N+14}{(N+8)^2}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ , \quad (\lambda_2^*)_H = 0 \ , \tag{1.73}$$

and the *cubic f.p.* (C) with $H(N)$-symmetry where both of the couplings are non-trivial

$$\begin{aligned} (\lambda_1^*)_C &= \frac{\varepsilon}{9N} - \frac{19N^2 - 125N + 106}{243N^3}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ , \\ (\lambda_2^*)_C &= \frac{N-4}{3N}\varepsilon + \frac{17N^3 + 92N^2 - 534N + 424}{81N^3}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ . \end{aligned} \tag{1.74}$$

[9] E.g. one element of $H(3)$ is $h = \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ -1 & 0 & 0 \end{pmatrix}$.

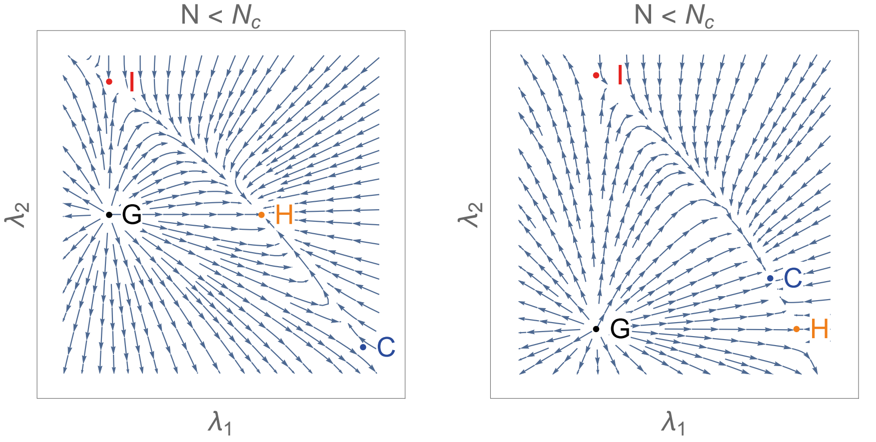

Figure 1.1. The RG flow for the hypercubic anisotropy model (1.66) for N below and above the critical value N_c. The black dot is the Gaussian f.p. (1.71), the red one is I (1.72), the orange one is H (1.73) and the blue one is C (1.74). Note that C is only fully attractive if $N > N_c$.

Due to the factor $(N-4)$ in $(\lambda_2^*)_C$ there is a critical value, N_c, for N when the cubic f.p. is fully attractive.[10]

1.4.1 Path integration

Let us now apply the CW mechanism to the hypercubic anistotropy model (1.66). This was first done in [77]. The way to do this is to split ϕ into a classical and a quantum piece

$$\phi^i = \phi_{cl}\delta^{iN} + \hbar\,\delta\phi^i + \mathscr{O}(\hbar^2)\ . \tag{1.75}$$

In general the classical background, ϕ_{cl}, has N components (called *axial ordering*), but for simplicity we let only one of its components be non-zero (*edge ordering*) [78]. In the rest of this Chapter we will use units s.t. $\hbar$ is one.

If we now expand the action in the quantum fluctuations, $\delta\phi$, then the single-order terms in $\delta\phi$ vanish by virtue of the e.o.m. (1.69). Keeping upto quadratic terms in $\delta\phi$ yields

$$\begin{aligned} S[\phi] &= S[\phi_{cl}] + \delta S[\phi_{cl},\delta\phi] + \mathscr{O}(\delta\phi^3)\ ,\\ \delta S &= \int_{\mathbb{R}^d} \frac{d^dx}{2}\,\delta\phi^i (D^{-1})^{ij}\delta\phi^j\ , \end{aligned} \tag{1.76}$$

where $(D^{-1})^{ij}$ is the inverse of the $\delta\phi$-correlator

$$(D^{-1})^{ij} \equiv \delta^{ij}(-\partial_\mu^2 + m_i^2)\ , \quad \text{(no sum over } i\text{)}, \tag{1.77}$$

[10] Recent analysis on this subject states that $N_c \approx 2.915$ [76].

with the mass

$$m_i^2 \equiv \frac{\lambda_1+\lambda_2}{2}\phi_{cl}^2 + \lambda_1 \phi_{cl}^2 \delta_{i1}\ , \quad \text{(no sum over } i\text{)}. \tag{1.78}$$

Let us now path integrate out $\delta\phi$ keeping ϕ_{cl} fixed

$$Z = \int \mathscr{D}\phi\, e^{-S[\phi]} = \int \mathscr{D}\phi_{cl} e^{-S[\phi_{cl}]} \delta Z[\phi_{cl}]\ . \tag{1.79}$$

Here δZ contains the path integration over $\delta\phi$

$$\begin{aligned}\delta Z &= \int \mathscr{D}\delta\phi\, e^{-\delta S + \mathscr{O}(\delta\phi^3)} \propto \frac{1}{\sqrt{\det(D^{-1})^{ij}}} \\ &= \exp\left(-\frac{\log\det(D^{-1})^{ij}}{2}\right) = \exp\left(-\frac{\operatorname{tr}\log(D^{-1})^{ij}}{2}\right)\ ,\end{aligned} \tag{1.80}$$

where we neglected overall constants and effects from path integrating out higher orders in $\delta\phi$. The trace of an operator is by definition given by the spacetime integral over the operator acting on the Hilbert states

$$\operatorname{tr}\log(D^{-1})^{ij} \equiv \int_{\mathbb{R}^d} d^d x \langle x|\operatorname{tr}_{H(N)} \log(D^{-1})^{ij}|x\rangle\ . \tag{1.81}$$

The trace inside the bracket runs over the group indices i, j.

Let us Fourier transform the Hilbert states

$$|x\rangle = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{-ikx}|k\rangle\ , \tag{1.82}$$

with the normalization

$$\langle k'|k\rangle = (2\pi)^d \delta(k'-k) \quad \Rightarrow \quad \langle x'|x\rangle = \delta(x'-x)\ . \tag{1.83}$$

The operator $(D^{-1})^{ij}$ now acts on the exponential in (1.82), which allows us to factor it out from the bracket in the trace (1.81)

$$\operatorname{tr}\log(D^{-1})^{ij} = \int_{\mathbb{R}^d} d^d x \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \operatorname{tr}_{H(N)} \log(G^{-1})^{ij}(k)\ . \tag{1.84}$$

This is given in terms of the corresponding inverse momentum propagator

$$(G^{-1})^{ij}(k) = \delta^{ij}(k_\mu^2 + m_i^2)\ , \quad \text{(no sum over i)}. \tag{1.85}$$

In order to take the logarithm of this we use its Taylor expansion

$$\begin{aligned}\log\operatorname{diag}(a_1,\ldots,a_N) &= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \operatorname{diag}(a_1-1,\ldots,a_N-1)^n \\ &= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \operatorname{diag}\Big((a_1-1)^n,\ldots,(a_N-1)^n\Big) \\ &= \operatorname{diag}\Big(\log(a_1-1),\ldots,\log(a_N-1)\Big)\ .\end{aligned} \tag{1.86}$$

With this we finally have an expression for the trace

$$\begin{aligned}\operatorname{tr}\log(D^{-1})^{ij} &= \int_{\mathbb{R}^d} d^d x \sum_{i=1}^{N} \frac{P^d(m_i^2,0)}{2} \\ &= \int_{\mathbb{R}^d} d^d x \frac{P_d(m_1^2,0)+(N-1)P_d(m_2^2,0)}{2} \ ,\end{aligned} \tag{1.87}$$

which is expressed in terms of the following master integral

$$P_n(m^2,\hat{m}) \equiv \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} \log(\sqrt{k^2+m^2}+\hat{m}) \ . \tag{1.88}$$

This master integral is divergent, but we regularize it using polar coordinates and introducing a large momentum cutoff $\Lambda \gg 1$ for the radius

$$\begin{aligned}P_n(m^2,\hat{m}) &= \frac{S_n}{(2\pi)^n} \int_0^\Lambda dr\, r^{n-1} \log(\sqrt{r^2+m^2}+\hat{m}) \\ &= \frac{\Lambda^{n+2}}{n(n+2)(m^2-\hat{m}^2)} \left[\frac{\hat{m}}{\sqrt{m^2}} F_1\left(\frac{n}{2}+1; \frac{1}{2}, 1; \frac{n}{2}+2; -\frac{\Lambda^2}{m^2}, \frac{\Lambda^2}{\hat{m}^2-m^2}\right) + \right. \\ &\left. + {}_2F_1\left(1, \frac{n}{2}+1; \frac{n}{2}+2; \frac{\Lambda^2}{\hat{m}^2-m^2}\right)\right] + \frac{\Lambda^n}{n} \log(\sqrt{\Lambda^2+m^2}+\hat{m}) \ .\end{aligned} \tag{1.89}$$

Here S_n is the solid angle (1.57) in n dimensions, $F_1(a;b_1,b_2;c;x,y)$ is an Appell F_1-series and ${}_2F_1$ is a hypergeometric function. Due to the F_1, we cannot directly expand above function around large Λ (or for that matter small m^2, which in our case (1.78) corresponds to an expansion in the coupling constants). This forces us to first expand around small $\hat{m}$. After that we are able to expand around large values of Λ (or alternatively around small values of m^2)

$$\begin{aligned}P_n(m^2,\hat{m}) = \frac{S_n}{(2\pi)^n} &\left(\Lambda^n \left(\frac{\log\Lambda}{n} - \frac{1}{n^2} + \frac{\hat{m}}{(n-1)\Lambda} + \frac{m^2-\hat{m}^2}{2(n-2)\Lambda^2} + \right.\right. \\ &\left. - \frac{m^2\hat{m}}{2(n-3)\Lambda^3} - \frac{m^2(m^2-2\hat{m}^2)}{4(n-4)\Lambda^4} + \ldots \right) \\ &\left. + \frac{\pi}{2}\csc\left(\frac{\pi n}{2}\right)\left(\frac{(m^2)^{\frac{n}{2}}}{n} - \frac{(m^2)^{\frac{n}{2}-1}\hat{m}^2}{2}\right) + \frac{\Gamma_{\frac{1-n}{2}}\Gamma_{\frac{n}{2}}}{2\sqrt{\pi}}(m^2)^{\frac{n-2}{2}}\hat{m}\right) \ .\end{aligned} \tag{1.90}$$

Finally we perform the ε-expansion

$$\begin{aligned}P_d(m^2,0) = \frac{(m^2)^2}{64\pi^2}\left(\log\left(\frac{m^2}{\Lambda^2}\right) - \frac{1}{2}\right) + \frac{m^2\Lambda^2}{32\pi^2} + \\ + \frac{\Lambda^4}{32\pi^2}\left(\log\Lambda - \frac{1}{4}\right) + \ldots \ .\end{aligned} \tag{1.91}$$

Since the second row of this expression is just a general constant, we will neglect it.

1.4.2 Effective potential

Let us now study what effects path integrating out the quantum fluctuation, $\delta\phi$, has on the classical field, ϕ_{cl}. As we just saw, the path integral (1.80) is divergent, which means it needs to be renormalized. To do this, we will first have to study the classical potential of the model (1.66)

$$V(\phi_{cl}) = \left(\lambda_1 + \frac{\lambda_2}{3}\right)\frac{\phi_{cl}^4}{8} \,. \tag{1.92}$$

Note that this potential is defined with a plus sign since the action describes an energy in statistical physics (see Tab. 1.1). From this potential we can define the mass (which in our case is zero)

$$\left.\frac{\partial V}{\partial(\phi_{cl}^2)}\right|_{\phi_{cl}=0} = 0 \,, \tag{1.93}$$

and the coupling constant

$$\left.\frac{\partial^2 V}{\partial(\phi_{cl}^2)^2}\right|_{\phi_{cl}=0} = \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{3}\right) \,. \tag{1.94}$$

These will later be used to introduce a RG scale, μ.

After path integrating out $\delta\phi$ in (1.79) we find an effective potential, V_{eff}, for the classical background

$$Z \propto \int \mathscr{D}\phi_{cl} \exp\left[-\int_{\mathbb{R}^d} d^d x \left(\frac{(\partial\phi_{cl})^2}{2} + V_{\text{eff}}(\phi_{cl})\right)\right] \,, \tag{1.95}$$

where we neglect overall constants and contributions from higher order terms in $\delta\phi$. We thus have from (1.80) that the effective potential is given by

$$\begin{aligned}\int_{\mathbb{R}^d} d^d x V_{\text{eff}}(\phi_{cl}) \equiv \int_{\mathbb{R}^d} d^d x V(\phi_{cl}) &+ \frac{\operatorname{tr}\log(D^{-1})^{ij}}{2} \\ &+ \int_{\mathbb{R}^d} d^d x V_{c.t.}(\phi_{cl}) + \dots \,.\end{aligned} \tag{1.96}$$

Here the trace is given by (1.87). As we will see in a moment, this effective potential captures only one-loop effects. To capture higher loop contributions we need to path integrate out higher orders of $\delta\phi$. $V_{c.t.}(\phi_{cl},\chi)$ contains counterterms which will cancel the divergences in the trace (1.87)

$$V_{c.t.}(\phi_{cl}) = A\phi_{cl}^2 + B\frac{\phi_{cl}^4}{8} \,. \tag{1.97}$$

The constant B will contain the divergent parts of the bare coupling constants. Since we specified to the case when only one component of ϕ_{cl}^i is non-zero, see

(1.75), we cannot distinguish between the λ_1- and λ_2-interaction in $V(\phi_{cl},\chi)$. This means that B will contain the divergent parts of both λ_1^0 and λ_2^0.

To find A we define a mass similar to (1.93)

$$\left.\frac{\partial V_{\text{eff}}}{\partial(\phi_{cl}^2)}\right|_{\phi_{cl}=0} = 0\,, \tag{1.98}$$

which gives us

$$A = -\frac{(N+2)\lambda_1 + N\lambda_2}{128\,\pi^2}\Lambda\,. \tag{1.99}$$

Finding B is more challenging as the second derivative of V_{eff} w.r.t. ϕ_{cl}^2 is divergent in the limit $\phi_{cl}\to 0$ (due to the logarithmic terms in (1.91)). The resolution to this problem is to introduce an RG cutoff μ (of mass dimension one) s.t.[11]

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial(\phi_{cl}^2)^2}\right|_{\phi_{cl}=\mu^{\Delta_\phi}} = \frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{3}\right)\,. \tag{1.100}$$

The constant B found from this equation is a cumbersome expression. However, among other things it contains the following logarithmic divergence

$$B \ni \frac{(N+8)\lambda_1^2 + 2(N+2)\lambda_1\lambda_2 + N\lambda_2^2}{64\,\pi^2}\log\Lambda\,. \tag{1.101}$$

This is the divergent part of the bare coupling constants (with the linear combination (1.92)) giving rise to the β-functions (1.70) (upto one-loop).[12]

When we plug the constants (1.99, 1.101) into the effective potential (1.96) the dependence on the large momentum cutoff, Λ, vanish. The way we introduced μ in (1.100) yields a minimum of V_{eff} at μ

$$\left.\frac{\partial V_{\text{eff}}}{\partial\phi_{cl}}\right|_{\phi_{cl}=\mu^{\Delta_\phi}} = 0\,, \tag{1.102}$$

from which we find that the coupling constants are related as[13]

$$\lambda_2 = -3\,\lambda_1\left(1 + \frac{N-1}{32\,\pi^2}\lambda_1\right) + \mathscr{O}(\lambda_1^3)\,. \tag{1.103}$$

We can see from the RG flow in Fig. 1.1 that this is a point in the RG flow which does not flow towards any of the RG f.p.'s (1.71, 1.72, 1.73, 1.74). If

[11] Note that $\Delta_\phi = 1+\mathscr{O}(\varepsilon)$, meaning we can set $\phi_{cl} = \mu + \mathscr{O}(\varepsilon)$ after taking the derivatives (and neglecting the $\mathscr{O}(\varepsilon)$-terms).

[12] This term in B corresponds to the $(\zeta_{g_a})^{bc}$-terms from the bare coupling constant (1.24).

[13] We also find that the second derivative w.r.t. ϕ_{cl} evaluated at $\phi_{cl} = \mu^{\Delta_\phi}$ is positive using this relation between the coupling constants. It is also worth mentioning that there are other solutions which cannot be treated perturbatively for small coupling constants. However, there might exist some large N-limit under which these solutions are under control.

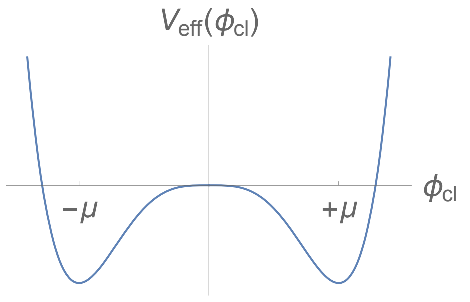

Figure 1.2. A plot of the effective potential (1.104).

we plug in above solution into V_{eff} we find

$$V_{\text{eff}} = \frac{N-1}{256\pi^2}\phi_{cl}^4\left(\log\left(\frac{\phi_{cl}^2}{\mu^2}\right)-\frac{1}{2}\right) . \tag{1.104}$$

A plot of this effective potential is in Fig. 1.2.

Lastly we can perform the *Higgs mechanism* [79, 80, 81, 82]. That is, we expand ϕ_{cl} around its minimum

$$\phi_{cl} = \mu^{\Delta_\phi} + \sigma = \mu + \sigma + \mathscr{O}(\varepsilon) , \tag{1.105}$$

and then expand the potential (1.104) in the *Higgs mode* σ. This gives us a final result for V_{eff}, where we can see that the $H(N)$-symmetry is fully broken

$$\begin{aligned} V_{\text{eff}}(\sigma) = \frac{N-1}{16\pi^2}\lambda_1^2\Bigg(&\frac{\mu^2\sigma^2}{4}+\frac{5\mu\sigma^3}{12}+\frac{11\sigma^4}{48}+ \\ &-3\sum_{m\geq 5}(-1)^m(m+1)_{-5}\frac{\sigma^m}{\mu^{m-4}}\Bigg)+\dots . \end{aligned} \tag{1.106}$$

Here we have removed constant terms and $(x)_n$ is the Pochhammer symbol. From the effective potential we can read off the acquired mass for σ

$$m_\sigma^2 = \frac{N-1}{32\pi^2}\lambda_1^2\mu^2 > 0 . \tag{1.107}$$

Since there can be quantum tunnelling between the vacua (see Fig. 1.2), and since we do not flow towards any f.p.'s in the RG, the theory at the point (1.103) describes a first-order p.t. The theory at this point in the RG flow is non-conformal due to the dimensionfull couplings.

Let us end this Chapter by summary of the CW mechanism [71]. It allows us to flow along the RG from a second-order p.t. towards a first-ordered one.[14]

By path integrating out quantum fluctuations (1.75) we acquire an effective potential, V_{eff}, for the classical field (1.96). This potential contains divergences

[14]Note though that it is not necessary to start with a massless theory.

and thus needs to undergo renormalization, which we did by defining the coupling constants through an RG scale, μ, in (1.100). This also gives us the bare coupling constants (1.101) from which we can find the β-functions [72].

The way we introduce μ in (1.100) tells us that V_{eff} has a minima (1.102) at $\phi_{cl} = \mu^{\Delta_\phi}$. This gives a constraint on the coupling constants (1.103) from which we can keep track of where we are in the RG flow. Note that if we were to only consider one interaction in our model, than this constraint would completely fix the value of the coupling constant. Most likely will this value be outside a perturbative regime where the coupling is small. E.g. if we were to consider the $O(N)$-model (1.66) with $\lambda_2 = 0$, then the non-trivial solution to (1.103) is $\lambda_1 = -\frac{32\pi^2}{N-1}$ which is not infinitesimal for finite values of N. It is thus favourable to consider at least two coupling constants when we apply the CW mechanism.

All and all, the CW mechanism leads to a SSB of the global symmetries. In the example we considered a discrete global symmetry (and moreover only assumed ϕ_{cl} to have one non-trivial component in (1.75)). If we were to consider a continuous global symmetry, e.g. $O(N)$, then the Higgs mechanism (1.105) would yield free massless Goldstone modes in addition to a massive Higgs mode [79, 80, 81, 82]. In such case there is an overall factor of $e^{\eta_k T_k} \in O(N)/O(N-1)$ in (1.105) corresponding to the broken part of the symmetry, where η_k is the Goldstone mode, and the T_k's are the generators for the Lie algebra corresponding to $O(N)/O(N-1)$.[15]

The Higgs mechanism yields that one component of ϕ_{cl}^i (assuming it to consist of N components) have become massive (the Higgs mode, σ). This is a quantum effect of the theory, and we say that σ has generated a mass *radiately*. Moreover, due to this mass scale the conformal symmetry of the original theory will break as well, and we are left with a non-conformal QFT with Poincaré symmetry.

[15]The kinetic term of ϕ_{cl} will in general produce derivative-interactions between $(\partial_\mu \eta^k)^2$ and σ. However, in the low-energy limit, $k_\mu \to 0$, of η^k these interactions vanish.

2. A single defect in a quantum field theory

We will now add a defect to the QFT and study the implications caused by it. Our starting point for this discussion is the symmetries preserved by the defect. Spacetime symmetries (Poincaré or conformal) are explicitly broken into a subgroup parallel and orthogonal to it. Global symmetries are not necessarily broken by the defect. However, they can be, and this can either occur spontaneously or explicitly. A global symmetry might also break due to the monodromy of a codimension two defect. We will then study renormalization in the presence of a defect. There will be new UV-divergences as bulk fields approach the defect that needs to be taken into account in bare quantities on the defect. At the f.p.'s of the corresponding RG flow we study how the enhanced conformal symmetry yields a bootstrap equation for two-point functions of bulk-local primaries. We also discuss the CW mechanism for a CFT near a defect. We end this Chapter by a discussion on specific examples of defects: boundaries, interfaces, monodromy and replica twist defects.

2.1 Symmetries

As we have seen in Ch. 1, a homogeneous QFT has $ISO(d-1,1)$-symmetry, and in the conformal case the symmetry is enhanced to $SO(d,2)$. A defect of dimension $p>0$ will be invariant under transformations orthogonal to it. In the case of a flat or a spherical defect (the minimal amount of symmetry breaking caused by a defect), this symmetry group will be $SO(d-p)$ if the time-axis is parallel to the defect

$$ISO(d-1,1) \to SO(d-p)\ , \tag{2.1}$$

and $SO(d-p-1,1)$ if the time-axis is orthogonal to the defect. In Euclidean signature there is no time direction, and thus the defect enjoy $SO(d-p)$-symmetry.

A *defect-local operator* will satisfy a subgroup of the bulk symmetries, with one part describing transformations along the defect, and the other orthogonal to it (the same as the defect itself). The orthogonal symmetry group will act as a global symmetry group for the defect-local fields. For the same reasons as in the Ch. 1, we let the defect-local operators at least be Poincaré invariant. In which case (for a flat or spherical defect), they satisfy a $ISO(p-1,1)\times SO(d-p)$-symmetry if the time-axis is parallel to the defect

$$ISO(d-1,1) \to ISO(p-1,1)\times SO(d-p)\ , \tag{2.2}$$

and $ISO(p) \times SO(d-p-1,1)$-symmetry if the time-axis is orthogonal to the defect. In Euclidean signature local fields on the defect are invariant under $ISO(p) \times SO(d-p)$.

The defect is said to be *conformal* if defect-local operators satisfy conformal symmetry along it, i.e. $O(p,2) \times SO(d-p)$ if the time-axis is parallel to the defect

$$O(d,2) \to O(p,2) \times SO(d-p) \ , \tag{2.3}$$

and $O(p+1,1) \times SO(d-p-1,1)$ if the time-axis is orthogonal to it.[1] In Euclidean signature the corresponding symmetry group is
$O(p+1,1) \times SO(d-p)$. If both the bulk and the defect is conformal, we call the theory a *defect conformal field theory* (DCFT).

Note that for a codimension one defect, i.e. a boundary or an interface, the group of rotations, $SO(d-p)$, around it is trivial. This has important consequences as we shall see in Sec. 2.5 and 2.6, where we study boundaries and interfaces respectively.

Global symmetries in the bulk are in general broken in the presence of the defect. This can occur in several different ways:

- *Explicit symmetry breaking:* a global symmetry is directly broken by an interaction on the defect. E.g. the scalar Wilson defect (5) explicitly breaks the bulk $O(N)$-symmetry down to $O(N-1)$.
- *Spontaneous symmetry breaking:* the potential in the bulk has a non-trivial minima due to the defect, which gives rise to the Higgs mechanism. E.g. for a boundary there is the extraordinary p.t. for the $O(N)$-model near four dimensions where one component of the bulk scalar has a non-trivial v.e.v. [84, 85, 86], or as in paper IV where the CW mechanism is studied in the presence of a boundary.
- *Symmetry breaking by monodromy:* codimension two defects generally carry a monodromy constraint, see e.g. (10), which might break the global symmetries in the bulk. We will study these defects in detail in Sec. 2.7 and Ch. 6.

2.2 Renormalization in the defect-limit

In a homogeneous QFT there are UV divergences from the coincident-limits of fields that can be understood using renormalization and the OPE. In the presence of a defect, there are additional UV-divergences coming from the short-distance limit between the bulk-local operator and the defect itself as a p-dimensional operator (we will refer to this limit as the *defect-limit*). To deal with these new divergences there is an additional step in the renormalization

[1] In two dimensions we can consider one-dimensional defects. Then the group of rotations around the defect is trivial: $SO(d-p) = SO(1) = 1$. In such case only one copy of the extended Virasoro symmetry [83] is preserved by the defect: $\mathrm{Vir} \times \overline{\mathrm{Vir}} \to \mathrm{Vir}$.

procedure, and this was first studied in [87] and expanded upon in paper III for composite operators.[2] Firstly, the bulk UV divergences are taken care of, followed by a similar renormalization procedure near the defects. For certain choices of renormalization schemes, e.g. dimensional regularization, both of these two kinds of UV-divergences can be taken care of at the same time.

Local quantities in the bulk, like masses and interactions, are not affected by the defect due to locality of the bulk theory. This can be seen by translating the bulk fields far away from the defect where they are no longer affected by it. Due to the translation symmetry of bulk-local fields, their properties, such as anomalous dimensions, does not change under this operation.

On the other hand, local quantities (like interactions and masses) on the defect itself are affected by both the coincident-limit of defect-local fields and the defect-limits of bulk-local fields. E.g. bare coupling constants, $\hat{g}_b^0$, on the defect are in general given by (in dimensional regularization and upto quadratic order in the coupling constants)

$$\hat{g}_b^0 = \mu^{\dim(g_b^0)} \left(\hat{g}_b + \frac{(\zeta^1_{\hat{g}_b})^{a_1 a_2} g_{a_1} g_{a_2} + (\zeta^2_{\hat{g}_b})^{a_1 b_1} g_{a_1} \hat{g}_{b_1} + (\zeta^3_{\hat{g}_b})^{b_1 b_2} \hat{g}_{b_1} \hat{g}_{b_2}}{\varepsilon} + \ldots \right) ,$$

which depend on both the renormalized coupling constants (dimensionless) in the bulk, g_a, and on the defect, $\hat{g}_a$. There is still only one renormalization scale, μ. The β-functions corresponding to interactions on the defect are found from the derivative w.r.t. μ

$$\hat{\beta}_b = \mu \frac{\partial \hat{g}_b}{\partial \mu} . \tag{2.4}$$

Similarly, the mass anomalous dimension, $\gamma_{\hat{m}_l}$, and the boundary anomalous dimension matrix, $\gamma^{ij}_{\hat{\mathcal{O}}}$, are defined as

$$\gamma_{\hat{m}_j} \equiv \mu \frac{\partial \hat{m}_j}{\partial \mu} , \quad \gamma^{ij}_{\hat{\mathcal{O}}} \equiv -\mu \frac{\partial Z^{\mathcal{O}_i}{}_{\hat{\mathcal{O}}_j}}{\partial \mu} , \tag{2.5}$$

where $Z^{\mathcal{O}_i}{}_{\hat{\mathcal{O}}_j}$ describes how a bulk-local field mixes with defect-local ones in the defect-limit. This mixing is explained by the *defect operator product expansion* (DOE): the OPE between a bulk-local field and the defect itself

$$\mathcal{O}(x) = \sum_{\hat{\mathcal{O}}_{\hat{\Delta},\hat{l},s}} \mu^{\mathcal{O}}{}_{\hat{\mathcal{O}}_{\hat{\Delta},\hat{l},s}} \frac{x_\perp^{i_1} \ldots x_\perp^{i_s}}{|x_\perp|^{\Delta - \hat{\Delta} + s}} \hat{C}_{\hat{\Delta},\hat{l},s}(\mu, x_\parallel, x_\perp) \hat{\mathcal{O}}_{\hat{\Delta},\hat{l},s}(x_\parallel) . \tag{2.6}$$

The sum runs over defect-local fields, $\hat{\mathcal{O}}_{\hat{\Delta},\hat{l},s}$, of scaling dimension $\hat{\Delta}$, $SO(p-1,1)$-spin $\hat{l}$ and $SO(d-p)$-spin s. There is a new set of OPE coefficients, $\mu^{\mathcal{O}}{}_{\hat{\mathcal{O}}_{\hat{\Delta},\hat{l},s}}$, referred to as *defect operator product expansion coefficients*.

[2]In both of these two papers boundaries are considered, but similar arguments apply for more general defects as well. In Sec. 2.7.1 we will go through the technical details of how this is done for a codimension two defect.

$\hat{C}_{\hat{\Delta},\hat{l},s}$ is a differential operator which depends on the RG scale μ. We identify defect-local fields through the DOE, and they are to be treated as renormalized fields in the defect-limit.

The CS eq. (1.35) for the propagator (1.29) is generalized into (see e.g. [88] in the presence of a boundary)

$$\begin{aligned} 0 = \Bigg(& \delta^{ij}\mu\frac{\partial}{\partial\mu} + \delta^{ij}\beta_a\frac{\partial}{\partial g_a} + \delta^{ij}\gamma_{m_k}\frac{\partial}{\partial m_k} + \gamma^{jj}_{\mathscr{O}} + \delta^{ij}\hat{\beta}_a\frac{\partial}{\partial\hat{g}_b} + \\ & + \delta^{ij}\gamma_{\hat{m}_l}\frac{\partial}{\partial\hat{m}_l} + \gamma^{jj}_{\hat{\mathscr{O}}}\Bigg) D_{O_j} \,, \end{aligned} \tag{2.7}$$

which can be used to find the differential operator in the DOE (2.6) (similar to the case without as defect (1.35)). Although masses and couplings on the defect behave technically similar to the bulk masses and interactions, they physically describe different b.c.'s on the defect.

In a DCFT, the DOE of a scalar simplifies into [89]

$$\begin{aligned} \mathscr{O}(x) &= \sum_{\hat{\mathscr{O}}_{\hat{\Delta},s}} \frac{\mu^{\mathscr{O}}{}_{\hat{\mathscr{O}}_{\hat{\Delta},s}}}{|x_\perp|^{\Delta-\hat{\Delta}+s}} \hat{C}_{d-p}(x_\perp^2\partial_\parallel^2)\hat{\mathscr{O}}_{\hat{\Delta},s}(x_\parallel) \,, \\ \hat{C}_{d-p}(x) &= \sum_{m\geq 0} \frac{x^m}{(-4)^m m! \left(\hat{\Delta} - \frac{p-2}{2}\right)_m} \,, \\ \hat{\mathscr{O}}_{\hat{\Delta},s}(x_\parallel) &= x_\perp^{i_1} \ldots x_\perp^{i_s} \hat{\mathscr{O}}^{i_1\ldots i_s}_{\hat{\Delta}}(x_\parallel) \,. \end{aligned} \tag{2.8}$$

where s is the $SO(d-p)$-spin of the defect-local field. The exchanged defect-local fields are now all primaries (annihilated by the generators of the SCT's along the defect), and the differential operator $\hat{C}_{d-p}(x_\perp^2\partial_\parallel^2)$ generates the towers of descendants.

Note that the exchanged fields does not have any $SO(p-1,1)$-spin. Such spinning operators only appear in the DOE of a bulk-local field with non-trivial $SO(d-1,1)$-spin, say l, wherein such case the DOE contains defect-local operators with $SO(p-1,1)$-spin $\hat{l} \leq l$ [89, 90].

2.2.1 Stress-energy tensor near a defect

If there exist fundamental fields on the defect (which does not exist in the bulk) that are charged under their own global symmetry not present in the bulk theory, then we can apply Noether's theorem in the usual way to find conserved currents localized to the defect. An example of this is the model (8) of Graphene which we mentioned in the introduction.

Naively we could imagine that it is also possible to apply Noether's theorem to the residual parts, $ISO(p-1,1)$ and $SO(d-p)$, of the Poincaré symmetry group (2.2) of a local operator on the defect. Though this will not yield the correct currents as we should instead study the defect-limit of the SE tensor, $T^{\mu\nu}$, in the bulk (which originates from the bulk Poincaré symmetry, $ISO(d-1,1)$). In the DOE of $T^{\mu\nu}$ we find defect-local currents corresponding to the symmetry groups preserved by the defect, see e.g. [89, 91, 92]. This was also studied in paper III for a boundary near three dimensions.

Firstly, let us consider a non-conformal defect (2.2). Then in the DOE of $T^{\mu\nu}$ we find the *pseudo stress-energy tensor*, τ^{ab}, which corresponds to the $ISO(p-1,1)$-symmetry of the defect (see [93] and paper III for a boundary wherein Stokes' theorem have been used)

$$\partial_\mu T^{\mu a}(x) = \delta^{(d-p)}(x_\perp)\partial_b \tau^{ba}(x_\parallel) \ , \quad \Delta_\tau = p + \mathscr{O}(g_a, \hat{g}_b) \ . \tag{2.9}$$

The pseudo SE tensor has $SO(p-1,1)$-spin two, and it measures the energy emitted/absorbed by the defect. Since we expect an energy-flow between the bulk and the defect, τ^{ab} is not conserved. Similar to the bulk SE tensor (1.28), its trace seems to be proportional to the β-functions for interactions, $\hat{\mathscr{O}}_c$, localized to the defect (see [13, 94] and paper III for different kinds of defects)

$$\tau^a{}_a = \hat{\beta}_c \hat{\mathscr{O}}_c \ . \tag{2.10}$$

This operator vanishes in the conformal case (see e.g. paper III), which means that then there is no energy being absorbed/emitted by the defect.

Let us now specify to the conformal case (2.3). By studying the DOE of $T^{\mu\nu}$ we can then find the protected *displacement operator*, D^i, corresponding to the $SO(d-p)$-symmetry. It is found from the following Ward identity of diffeomorphisms (small translations w.r.t. the orthogonal coordinates) of the defect [89]

$$\partial_\mu T^{\mu i}(x) = \delta^{(d-p)}(x_\perp) D^i(x_\parallel) \ , \quad \Delta_D = p + 1 \ . \tag{2.11}$$

The action of this $SO(d-p)$-spin one primary translates the defect in the orthogonal directions.

Likewise, if a continuous global symmetry, G, of the bulk theory is broken down to a subgroup $H \subset G$, then in the DOE of the bulk Noether current, J^{ij}_μ, we also find two interesting operators. For this discussion we specify to $G = O(N)$ and $H = O(N-1)$, where J^{ij}_μ is given by

$$J^{ij}_\mu(x) = \phi^i \partial_\mu \phi^j - \phi^j \partial_\mu \phi^i \ , \quad i,j \in \{1,...,N\} \ . \tag{2.12}$$

We will assume that the N^{th} component of the scalar, ϕ^i, has a non-zero v.e.v. ($\langle \phi^i \rangle \propto \delta^{iN}$) which breaks the $O(N)$-symmetry. The number of broken generators are then

$$\begin{aligned} \dim O(N)/O(N-1) &= \dim O(N) - \dim O(N-1) \\ &= \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N - 1 \ , \end{aligned} \tag{2.13}$$

which means that we can split J_μ^{ij} in two parts: $J_\mu^{\alpha\beta}$ and $J_\mu^{N\alpha}$ with $\alpha,\ \beta \in \{1,...,N-1\}$.[3] Here $J_\mu^{\alpha\beta}$ corresponds to the preserved symmetry H and $J_\mu^{N\alpha}$ to the broken part G/H.

In the non-conformal case (2.2), we find the *pseudo Noether current*, $j_a^{\alpha\beta}$, localized on the defect (similar to τ^{ab})

$$\partial^\mu J_\mu^{\alpha\beta}(x) = \delta^{(d-p)}(x_\perp)\partial^a j_a^{\alpha\beta}(x_\parallel)\ , \quad \Delta_j = p-1+\mathscr{O}(g_a,\hat{g}_b)\ . \tag{2.14}$$

This operator has $ISO(p-1,1)$-spin one and $O(N-1)$-spin two. Similar to τ^{ab}, this operator vanishes in a DCFT (see [92] for a boundary).

Specifying to the conformal case (2.3), we find the protected *tilt operator*, t^α, from the Ward identity of infinitesimal transformations of G [95, 96, 97]

$$\partial^\mu J_\mu^{N\alpha}(x) = \delta^{(d-p)}(x_\perp)t^\alpha(x_\parallel)\ , \quad \Delta_t = p\ , \tag{2.15}$$

which is a primary with $O(N-1)$-spin one. From the two- and four-point function of t^α we can find the metric and Riemannian tensor (respectively) corresponding to the *conformal manifold* G/H [98, 99].

2.3 Conformal bootstrap near a defect

Let us now focus on a DCFT (2.3), and study how the conformal symmetry constrains the correlators. Due to the $SO(p,2)$-symmetry along the defects, defect-local operators behave in a similar manner as in the homogeneous case (see Sec. 1.3). That is, defect one-point functions are trivial, and the defect-defect two-point functions (1.56) are completely fixed (upto field normalization).[4] Due to the DOE (2.8) (or the broken bulk conformal symmetry), the one-point functions in the bulk are no longer trivial, and are allowed to be non-zero. This has important physical consequences, since a non-trivial v.e.v. can break the global symmetries. E.g. the extraordinary p.t. near a boundary.

Bulk-defect two-point functions are fixed upto a DOE coefficient, which can be compared to the OPE coefficients that appear in the three-point functions in a homogeneous CFT. This means that the DOE coefficients, along with the scaling dimension and spin (both transverse and parallel) of defect-local fields are new additions to the CFT data.

The bulk-bulk two-point function is no longer fixed, and using the OPE and DOE in different ways yields a bootstrap equation. Unlike that in the homogeneous case (1.62), it is not a result of crossing symmetry (of one OPE) as completely different operators (either bulk- or defect-local ones) are exchanged in the two bootstrap-channels.

[3] Note that $J_\mu^{NN} = 0$ due to the antisymmetry of $J_\mu^{ij} = -J_\mu^{ji}$.

[4] The normalization of defect-local fields is fixed by that for bulk-local fields (and vice versa).

2.3.1 Bulk one- and bulk-defect two-point functions

Similar to a homogeneous CFT, LC coordinates are effective tools when finding the analytical form of the correlators [89]. We split the LC coordinates into components parallel and orthogonal to the defect. For a flat p-dimensional defect we let

$$X^A = X_{\parallel}^M \oplus X_{\perp}^I \ , \tag{2.16}$$

where $A \in \{+,-,0,...,d-1\}$, $M \in \{+,-,0,...,p-1\}$ and $I \in \{1,...,d-p-1\}$. Splitting the coordinates in this way yields

$$\begin{aligned} X_{\parallel}\cdot Y_{\parallel} &\equiv X_{\parallel}^M g_{MN} Y_{\parallel}^N = x_{\parallel}^a \eta_{ab} y_{\parallel}^b - \eta_{\mu\nu}\frac{x^\mu x^\nu + y^\mu y^\nu}{2} = -\frac{s_{\parallel}^2 + x_{\perp}^2 + y_{\perp}^2}{2} \ , \\ X_{\perp}\cdot Y_{\perp} &\equiv X_{\perp}^I g_{IJ} Y_{\perp}^J = x_{\perp}^i \eta_{ij} y_{\perp}^j = x_{\perp} y_{\perp} \ . \end{aligned}$$

Here $s_{\parallel}^a = x_{\parallel}^a - y_{\parallel}^a$. These are two new scalar products in addition to (1.43). Note that the scalar product for the parallel coordinates also depend on the orthogonal coordinates. Due to the vanishing of the scalar product (1.44) the two scalar products above are related to each other

$$X_{\perp}\cdot X_{\perp} = -X_{\parallel}\cdot X_{\parallel} = -x_{\perp}^2 \ . \tag{2.17}$$

Due to this, the bulk one-point function, $\langle \mathcal{O}(x)\rangle$, can only depend on the combination $X_{\perp}\cdot X_{\perp}$. Using dimensional analysis we find $\langle \mathcal{O}(x)\rangle$ to be on the form (neglecting the field normalization factor)

$$\langle \mathcal{O}(x)\rangle = \frac{\mu^{\mathcal{O}}{}_{\mathbb{1}}}{|X_{\perp}\cdot X_{\perp}|^{\frac{\Delta}{2}}} = \frac{\mu^{\mathcal{O}}{}_{\mathbb{1}}}{|x_{\perp}|^{\Delta}} \ , \tag{2.18}$$

where we used the DOE (2.8) to determine that $\mu^{\mathcal{O}}{}_{\mathbb{1}}$ is the DOE coefficient of the identity exchange on the defect. Similarly, we can determine that the coefficient, $\mu^{\mathcal{O}}{}_{\hat{\mathcal{O}}}$, in $\langle \mathcal{O}(x)\hat{\mathcal{O}}(y)\rangle$ is the DOE coefficient for the exchange of $\hat{\mathcal{O}}$. This correlator can only depend on $X_{\perp}\cdot X_{\perp}$ and $X_{\parallel}\cdot Y_{\parallel}$. Moreover, $\langle \mathcal{O}(x)\hat{\mathcal{O}}(y)\rangle$ should equal $\langle \hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\rangle$ in the defect-limit. This means that the power of $X_{\perp}\cdot X_{\perp}$ should be proportional to $\Delta-\hat{\Delta}$. Using dimensional analysis we can then determine the power of $X_{\parallel}\cdot Y_{\parallel}$ as well

$$\langle \mathcal{O}(x)\hat{\mathcal{O}}(y)\rangle = \frac{\mu^{\mathcal{O}}{}_{\hat{\mathcal{O}}}}{|X_{\perp}\cdot X_{\perp}|^{\frac{\Delta-\hat{\Delta}}{2}}(-2X_{\parallel}\cdot Y_{\parallel})^{\hat{\Delta}}} = \frac{\mu^{\mathcal{O}}{}_{\hat{\mathcal{O}}}}{|x_{\perp}|^{\Delta-\hat{\Delta}}(s_{\parallel}^2 + x_{\perp}^2)^{\hat{\Delta}}} \ . \tag{2.19}$$

The two correlators $\langle \mathcal{O}(x)\rangle$ and $\langle \mathcal{O}(x)\hat{\mathcal{O}}(y)\rangle$ can be compared to the three-point functions (1.58) in a homogeneous CFT as they are both completely fixed by conformal symmetry upto an OPE coefficient.

2.3.2 Bootstrap equation

The two-point function, $\langle \mathscr{O}_1(x)\mathscr{O}_2(y)\rangle$, for two (different) bulk-local scalars is not fixed by the conformal symmetry, and it is often difficult to find using standard Feynman diagram techniques as this requires defect-defect, bulk-defect and bulk-bulk propagators. We will write $\langle \mathscr{O}_1(x)\mathscr{O}_2(y)\rangle$ in terms of a function $f(\xi,\eta)$ which depend on the two cross-ratios ξ and η

$$\begin{aligned}\langle \mathscr{O}_1(x)\mathscr{O}_2(y)\rangle &= A_d \frac{(2X_\perp\cdot X_\perp)^{\frac{\Delta_{21}^-}{4}}(2Y_\perp\cdot Y_\perp)^{\frac{\Delta_{12}^-}{4}}}{(-2X\cdot Y)^{\frac{\Delta_{12}^+}{2}}} f(\xi,\eta) \\ &= A_d \frac{|2x_\perp|^{\frac{\Delta_{21}^-}{2}}|2y_\perp|^{\frac{\Delta_{12}^-}{2}}}{|s|^{\Delta_{12}^+}} f(\xi,\eta)\ ,\end{aligned} \tag{2.20}$$

where $\Delta_{ab}^\pm \equiv \Delta_a \pm \Delta_b$ and A_d is a field normalization constant. The cross-ratios are given by

$$\begin{aligned}\xi &= \frac{-X\cdot Y}{2\sqrt{(X_\perp\cdot X_\perp)(Y_\perp\cdot Y_\perp)}} = \frac{s^2}{4\,|x_\perp|\,|y_\perp|}\ , \\ \eta &= \frac{X_\perp\cdot Y_\perp}{\sqrt{(X_\perp\cdot X_\perp)(Y_\perp\cdot Y_\perp)}} = \frac{(x_\perp y_\perp)}{|x_\perp|\,|y_\perp|} = \cos\varphi\ .\end{aligned} \tag{2.21}$$

The cross-ratio ξ diverges as $x_\perp, y_\perp \to 0$, while η is related to the angle, φ, relative to the defect.

Now we can approach the defect in two different ways using the OPE: either we first use the bulk OPE which allows us to express $\langle \mathscr{O}_1(x)\mathscr{O}_2(y)\rangle$ in terms of bulk one-point functions (which is fixed upto a constant through (2.18)). On the other hand, we can apply the DOE to each bulk operator, which allows us to express $\langle \mathscr{O}_1(x)\mathscr{O}_2(y)\rangle$ in terms of the orthogonal defect-defect two-point functions. This yields a bootstrap equation [89, 100]

$$\begin{aligned} f(\xi,\eta) &= \sum_{\mathscr{O}_{\Delta,l}} \lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}_{\Delta,l}}\,\mu^{\mathscr{O}_{\Delta,l}}{}_{\mathbb{1}}\,\mathscr{G}_{\text{bulk}}(\Delta,l;\xi,\eta) \\ &= \xi^{\frac{\Delta_{12}^+}{2}} \sum_{\hat{\mathscr{O}}_{\hat{\Delta},s}} \mu^{\mathscr{O}_1}{}_{\hat{\mathscr{O}}_{\hat{\Delta},s}}\,\mu^{\mathscr{O}_2}{}_{\hat{\mathscr{O}}_{\hat{\Delta},s}}\,\mathscr{G}_{\text{def}}(\hat{\Delta},s;\xi,\eta)\ .\end{aligned} \tag{2.22}$$

The exchanged bulk-local operators have $SO(d)$-spin l. Since we consider external scalars, the exchanged defect-local operators only have non-trivial $SO(d-p)$-spin, s, and no $SO(p)$-spin. This bootstrap equation looks similar to that for a four-point function in a homogeneous CFT (1.62), with the major difference that here there are two entirely different operators in the two channels: either bulk- or defect-local operators. It yields two entirely different conformal blocks, $\mathscr{G}_{\text{bulk}}(\Delta,l;\xi,\eta)$ and $\mathscr{G}_{\text{def}}(\hat{\Delta},0,s;\xi,\eta)$, in the two channels.

If we compare above bootstrap equation to (2.18, 2.19) we find them to be given by

$$
\begin{aligned}
\mathscr{G}_{\text{bulk}}(\Delta,l;\xi,\eta) &= \frac{1}{|2x_\perp|^{\frac{\Delta_{12}^-}{2}}|2y_\perp|^{\frac{\Delta_{21}^-}{2}}}\frac{s^{\mu_1}\dots s^{\mu_l}}{|s|^{l-\Delta}}C_{\Delta,l}(x)\frac{\langle\mathscr{O}(x)\rangle}{A_d}\ ,\\
\mathscr{G}_{\text{def}}(\hat\Delta,0,s;\xi,\eta) &= 2^{\Delta_{12}^+}\frac{x_\perp^{i_1}\dots x_\perp^{i_s}}{|x_\perp|^{s-\hat\Delta}}\frac{y_\perp^{i_1}\dots y_\perp^{i_s}}{|y_\perp|^{s-\hat\Delta}}\hat C_{d-p}(x_\perp^2\partial_\parallel^2)\hat C_{d-p}(y_\perp^2\partial_\parallel^2)\times\\
&\quad\times\frac{\langle\hat{\mathscr{O}}_{\hat\Delta,0,s}(x)\hat{\mathscr{O}}_{\hat\Delta,0,s}(y)\rangle}{A_d}\ .
\end{aligned}
\tag{2.23}
$$

An alternative approach to find the conformal blocks is to solve their corresponding Casimir equations [89]. The bulk block satisfy the same eq. (1.63) as those in a homogeneous CFT

$$
\mathscr{J}^2\left(\frac{|x_\perp|^{\frac{\Delta_{12}^-}{2}}|y_\perp|^{\frac{\Delta_{21}^-}{2}}}{|s|^{\Delta_{12}^+}}\mathscr{G}_{\text{bulk}}(\Delta,l;\xi,\eta)\right)=c_{\Delta,l}\frac{|x_\perp|^{\frac{\Delta_{12}^-}{2}}|y_\perp|^{\frac{\Delta_{21}^-}{2}}}{|s|^{\Delta_{12}^+}}\mathscr{G}_{\text{bulk}}(\Delta,l;\xi,\eta)\ .
$$

The defect block however satisfy two Casimir equations: one for the parallel LC coordinates, and another for the orthogonal ones

$$
\mathscr{L}^2\frac{\mathscr{G}_{\text{def}}(\Delta,s;\xi,\eta)}{|x_\perp|^{\Delta_1}|y_\perp|^{\Delta_1}}=\hat c_{\hat\Delta,0}\frac{\mathscr{G}_{\text{def}}(\Delta,s;\xi,\eta)}{|x_\perp|^{\Delta_1}|y_\perp|^{\Delta_1}}\ , \tag{2.24}
$$

$$
\mathscr{S}^2\frac{\mathscr{G}_{\text{def}}(\Delta,s;\xi,\eta)}{|x_\perp|^{\Delta_1}|y_\perp|^{\Delta_1}}=\hat c_{0,s}\frac{\mathscr{G}_{\text{def}}(\Delta,s;\xi,\eta)}{|x_\perp|^{\Delta_1}|y_\perp|^{\Delta_1}}\ . \tag{2.25}
$$

The conformal Casimir operators are expressed in terms of the parallel/orthogonal components of the conformal generators, J_1^{AB}, at (1.65)

$$
\mathscr{L}^2=-\frac{(J_1^{MN})^2}{2}\ ,\quad \mathscr{S}^2=-\frac{(J_1^{IJ})^2}{2}\ , \tag{2.26}
$$

with the corresponding eigenvalues

$$
\hat c_{\hat\Delta,s}=\hat\Delta(\hat\Delta-p)+s(s+d-p-2)\ . \tag{2.27}
$$

The Casimir eq.'s (2.24, 2.25) for $\mathscr{G}_{\text{def}}$ can be solved in general [89]

$$
\begin{aligned}
\mathscr{G}_{\text{def}}(\Delta,0,s;\chi,\varphi) &= \frac{\Gamma_{d-p+s-2}\Gamma_{\frac{d-p}{2}-1}}{\Gamma_{\frac{d-p}{2}+s-1}\Gamma_{d-p-2}}\frac{\chi^{-\hat\Delta}}{2^s}\times\\
&\quad\times{}_2F_1\left(\frac{d-p+s}{2}-1,-\frac{s}{2};\frac{d-p-1}{2};\sin\varphi^2\right)\times\\
&\quad\times{}_2F_1\left(\frac{\hat\Delta+1}{2},\frac{\hat\Delta}{2};\hat\Delta+1-\frac{p}{2};(2\chi)^{-2}\right)\ ,
\end{aligned}
\tag{2.28}
$$

which is expressed in the cross-ratio

$$\chi = \frac{-X_{\parallel}\cdot Y_{\parallel}}{2\sqrt{(X_{\perp}\cdot X_{\perp})(Y_{\perp}\cdot Y_{\perp})}} = \frac{s_{\parallel}^2 + x_{\parallel}^2 + y_{\parallel}^2}{4\,|x_{\perp}|\,|y_{\perp}|} = \xi + \frac{\cos\varphi}{2}\ . \tag{2.29}$$

The bootstrap eq. (2.22) have been studied numerically for a codimension one defect assuming only bulk-interactions in [27, 100, 96] and only defect-interactions in [101, 102]. A codimension two defect was numerically bootstrapped in [97]. Note that in the case of semidefinite programming, then the bulk OPE coefficients, $\lambda^{\mathcal{O}_1\mathcal{O}_2}{}_{\mathcal{O}_{\Delta,l}}\mu^{\mathcal{O}_{\Delta,l}}{}_{\mathbb{1}}$, are assumed to be positive. This property is however not guaranteed by assuming the bulk and the defect theory to be unitary.

LIF's have also been developed for conformal defects of codimension strictly greater than one in [103, 104, 105]. The main difference from the homogeneous case is that there is now one LIF for the bulk OPE coefficients [103] (assuming analyticity in the $SO(d,2)$-spin l), and another one for the DOE coefficients [104] (assuming analyticity in the $SO(d-p)$-spin s). Dispersion relations for the bulk two-point function near a defect of codimension strictly greater than one was found in [106]. We will discuss the special case of codimension one defects in Sec. 2.5 where we introduce boundaries.

2.4 Coleman-Weinberg mechanism

We will now study the CW mechanism in the presence of a defect. It has not been worked out in general before this thesis (as far as we are aware). We will go through how path integration in DQFT's works, and how a v.e.v. on the defect will stretch out into the bulk using the DOE (2.6). This will in turn induce a SSB both in the bulk and on the defect. In Sec. 2.5 where we study boundaries, we will come back to this topic and connect this discussion to the results of paper IV.

Let us consider a Euclidean scalar field theory in the presence of a flat defect

$$S = \int_{\mathbb{R}^d} d^dx \left(\frac{(\partial_\mu\phi)^2}{2} + V(\phi)\right) + \int_{\mathbb{R}^p} d^px_{\parallel}\hat{V}(\hat{\phi})\ , \tag{2.30}$$

with a potential in both the bulk, $V(\phi)$, and on the defect, $\hat{V}(\hat{\phi})$. Note that $\hat{V}(\hat{\phi})$ is not always known, wherein in such case we cannot apply the CW mechanism using the technology in this Section. The first thing we want to do is to expand ϕ around a classical background (in both the bulk, ϕ_{cl}, and on the defect, $\hat{\phi}_{cl}$)

$$\phi = \phi_{cl} + \hbar\,\delta\phi + \mathcal{O}(\hbar^2)\ , \quad \hat{\phi} = \hat{\phi}_{cl} + \hbar\,\delta\hat{\phi} + \mathcal{O}(\hbar^2)\ . \tag{2.31}$$

Terms linear in $\delta\phi$ and $\delta\hat{\phi}$ in the action gives us the e.o.m. and the b.c. respectively (something we will see in more detail in Sec. 2.5). Thus it brings the action onto the form

$$\begin{aligned} S[\phi,\hat{\phi}] &= S[\phi_{cl},\hat{\phi}_{cl}] + \hbar^2 \delta S[\phi_{cl},\hat{\phi}_{cl},\delta\phi,\delta\hat{\phi}] + \mathscr{O}(\hbar^3)\,, \\ \delta S[\phi_{cl},\hat{\phi}_{cl},\delta\phi,\delta\hat{\phi}] &= \delta S_{\text{bulk}}[\phi_{cl},\delta\phi] + \delta S_{\text{def}}[\hat{\phi}_{cl},\delta\hat{\phi}]\,, \end{aligned} \tag{2.32}$$

$$\begin{aligned} \delta S_{\text{bulk}}[\phi_{cl},\delta\phi] &= \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \delta\phi)^2}{2} + \frac{m^2(\phi_{cl})}{2}\delta\phi^2 \right)\,, \\ \delta S_{\text{def}}[\hat{\phi}_{cl},\delta\hat{\phi}] &= \int_{\mathbb{R}^p} d^p x_\parallel \frac{\hat{m}(\hat{\phi}_{cl})}{2}\delta\hat{\phi}^2\,. \end{aligned} \tag{2.33}$$

We will now shift to units s.t. $\hbar = 1$. Keeping ϕ_{cl} fixed, the action for $\delta\phi$ is that of a scalar field theory with different masses in the bulk, m^2, and on the defect, $\hat{m}$

$$m^2(\phi_{cl}) = \frac{\partial^2}{\partial\phi_{cl}^2}V(\phi_{cl})\,, \quad \hat{m}(\hat{\phi}_{cl}) = \frac{\partial^2}{\partial\hat{\phi}_{cl}^2}\hat{V}(\hat{\phi}_{cl})\,. \tag{2.34}$$

Note that this is a slight abuse of notation as $\hat{m}$ does not necessarily have the correct units of mass (depending on the dimension, p, of the defect). We note that by varying δS w.r.t. $\delta\hat{\phi}$ we find the e.o.m. and the b.c.[5]

$$\begin{aligned} D^{-1}[\delta\phi] &\equiv (-\partial_\mu^2 + m^2)\delta\phi = 0\,, \\ \text{b.c.}[\delta\hat{\phi}] &\equiv \hat{m}\,\delta\hat{\phi} = 0\,. \end{aligned} \tag{2.35}$$

Path integration of DQFT's has been worked out in [107, 108]. In these works they assume a Gaussian theory with several curved defects inside a curved bulk. In path integrating out $\delta\phi$ in (2.32) we are interested in the case of a single flat defect in a flat spacetime. To do this we write the defect contribution as a dirac δ-function added to the path integral

$$\begin{aligned} Z &= \int \mathscr{D}\phi\, e^{-S[\phi]} = \int \mathscr{D}\phi_{cl}\, e^{-S[\phi_{cl},\hat{\phi}_{cl}]}\delta Z[\phi_{cl},\hat{\phi}_{cl}]\,, \\ \delta Z &= \int \mathscr{D}\delta\phi\, e^{-\delta S} = \int \mathscr{D}\delta\phi\, \delta(\text{b.c.}[\delta\hat{\phi}])e^{-\delta S_{\text{bulk}}} = \\ &= \int \mathscr{D}\delta\phi\, e^{-\delta S_{\text{bulk}}} \int \mathscr{D}\eta\, \exp\left(-\int_{\mathbb{R}^p} d^p x_\parallel\, \eta\, \text{b.c.}[\delta\hat{\phi}] \right)\,. \end{aligned} \tag{2.36}$$

If we complete the square w.r.t. $\delta\phi$ we find[6]

$$\begin{aligned} \delta Z &= Z_{\delta\phi} Z_\eta\,, \\ Z_{\delta\phi} &\equiv \int \mathscr{D}\delta\phi\, e^{-\delta S_{\text{bulk}}}\,, \end{aligned} \tag{2.37}$$

---

[5] In the case of a codimension one defect there might also be an additional contribution to this b.c. coming from the partial integration of the bulk terms (as we shall see in Sec. 2.5).

[6] In completing the square we used

$$\eta\,\text{b.c.}[\hat{\phi}] = (-\partial_\mu^2 + m^2)D_{b.c.}(s_\parallel,0,0)\text{b.c.}[\hat{\phi}]\,\eta = (-\partial_\mu^2 + m^2)\text{b.c.}[\hat{\phi}]\,D_{b.c.}(s_\parallel,0,0)\eta\,,$$

where we commuted the propagator (a function), $D_{b.c.}(s_\parallel,0,0) \equiv (-\partial_\mu^2 + m^2)^{-1}$, with b.c.$[\hat{\phi}]$.

$$Z_\eta \equiv \int \mathscr{D}\eta \, \exp\left(\int_{\mathbb{R}^p} d^p x_\parallel \int_{\mathbb{R}^p} d^p y_\parallel \eta(x_\parallel) D_{b.c.}(s_\parallel,0,0)\eta(y_\parallel)\right) .$$

Here $D_{b.c.}(s_\parallel, x_\perp, y_\perp)$ is the bulk $\delta\phi$-$\delta\phi$ correlator subject to the b.c.'s on the defect. Its defect-limit appears in Z_η. This is a function and is thus not affected by path integrating out $\delta\phi$ in $Z_{\delta\phi}$. Since we introduced the b.c. on the defect as a Dirac δ-function in the path integral (2.36), we can ignore the effects of the defect in $Z_{\delta\phi}$.[7] This means that $Z_{\delta\phi}$ is on the same form as without a defect (see Sec. 1.4), giving us a similar effective potential in the bulk

$$\begin{aligned} Z_{\delta\phi} &\propto \exp\left(-\int_{\mathbb{R}^d} d^d x \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{\log(G^{-1})^{ij}(k)}{2}\right) \\ &= \exp\left(-\int_{\mathbb{R}^d} d^d x \frac{P_d(m^2,0)}{4}\right) , \end{aligned} \tag{2.38}$$

which we expressed in terms of the master integral (1.89). Sometimes we keep ϕ_{cl} fixed in the CW mechanism, giving us an overall factor of $\mathrm{vol}(R^d)$ which we do not care about. However, this is no longer possible in the presence of a defect, and ϕ_{cl} will depend on the normal coordinates, $x^i_\perp$.

Let us now path integrate out η. Following the steps in Sec. 1.4

$$\begin{aligned} Z_\eta &\propto \sqrt{\det D_{b.c.}(s_\parallel,0,0)} = \exp\left(+\frac{\mathrm{tr}\log D_{b.c.}(s_\parallel,0,0)}{2}\right) \\ &= \exp\left(\int_{\mathbb{R}^p} \frac{d^p s_\parallel}{2} \langle s_\parallel | \log D_{b.c.}(s_\parallel,0,0) | s_\parallel \rangle\right) . \end{aligned} \tag{2.39}$$

If we Fourier transform the states

$$|s_\parallel\rangle = \int_{\mathbb{R}^p} \frac{d^p k_\parallel}{(2\pi)^p} e^{-ik_\parallel s_\parallel} |k_\parallel\rangle , \tag{2.40}$$

we find

$$Z_\eta \propto \exp\left(\int_{\mathbb{R}^p} d^p x_\parallel \int_{\mathbb{R}^p} \frac{d^p k_\parallel}{(2\pi)^p} \frac{\log G_{b.c.}(k_\parallel,0,0)}{2}\right) . \tag{2.41}$$

Here $G_{b.c.}(k_\parallel, x_\perp, y_\perp)$ is the momentum propagator (w.r.t. the parallel directions) subject to the b.c. of the defect. This is not known for general defects. However, in the specific case of a boundary, this quantity has been worked out in App. A.2 of paper IV. We will thus revisit this problem after having introduced boundaries in Sec. 2.5.

Notice the similarity between the bulk contribution (2.38) to the effective potential and that on the defect (2.41). Together they give (upto one-loop)

$$\begin{aligned} V_{\text{eff}}(\phi_{cl}) &= V(\phi_{cl}) + \frac{P_d(m^2,0)}{4} + \text{c.t.'s} + \ldots , \\ \hat{V}_{\text{eff}}(\hat{\phi}_{cl}) &= \hat{V}(\hat{\phi}_{cl}) - \int_{\mathbb{R}^p} \frac{d^p k_\parallel}{(2\pi)^p} \frac{\log G_{b.c.}(k_\parallel,0,0)}{2} + \text{c.t.'s} + \ldots , \end{aligned} \tag{2.42}$$

[7] This is in particular important when calculating the trace corresponding to (1.81).

where 'c.t.'s' are the counter-terms (they differ in the bulk and on the defect). The defect potential can be renormalized by introducing an RG scale by defining the defect couplings in a similar way as in a homogeneous QFT (1.100), which in turn might give us a minima of the defect-local fields

$$\langle \hat{\phi}_{cl} \rangle = \mu^{\Delta_{\hat{\phi}}} \ . \tag{2.43}$$

We find the bulk v.e.v. using the DOE (2.6). Note that we are not at a conformal f.p. in RG. In particular, this changes the operators exchanged in the DOE as well as its differential operator, $\hat{C}_{\hat{\Delta},\hat{l},s}$. However, to lowest order in $x_\perp$ we have (assuming the identity is not exchanged)[8]

$$\langle \phi_{cl} \rangle = \frac{\mu^{\Delta_{\hat{\phi}}}}{x_\perp^{\Delta_\phi - \Delta_{\hat{\phi}}}} + \dots = \mu^{\Delta_{\hat{\phi}}} x_\perp^{\gamma_{\hat{\phi}}} + \dots \ , \tag{2.44}$$

which is given in terms of the defect anomalous dimension, $\gamma_{\hat{\phi}}$, of $\hat{\phi}$. This might induce a SSB of the global symmetries in the bulk, even though its corresponding effective potential has not received any radiative one-loop corrections. An example of this phenomena was studied in paper IV where no bulk potential was considered. Another example of this, with a bulk potential, will be studied in Ch. 5.

2.5 Boundary

Let us now discuss certain special cases of defects in greater detail. We will start with *boundaries*, which are defects of codimension one with a bulk theory defined on only one side of it. See [15, 91, 109] and references therein. We will now refer to the DOE (2.8) as the *boundary operator product expansion* (BOE). In the case when the bulk and the boundary theory are both conformal, we refer to the model as a *boundary conformal field theory* (BCFT).

For codimension one defects the orthogonal symmetry group, $SO(d-p)$, is trivial. In addition to this, it can be proven that in the presence of a codimension one defect, the one-point functions (2.18) are zero for spinning fields [89]. This means that the exchanged operators in the BOE (2.8), and therefore also in the bootstrap equation (2.22), are all scalars [100]

$$\mathscr{O}(x) = \sum_{\hat{\mathscr{O}}} \frac{\mu^{\mathscr{O}}{}_{\hat{\mathscr{O}}}}{|x_\perp|^{\Delta - \hat{\Delta}}} \hat{C}_1(x_\perp^2 \partial_\parallel^2) \hat{\mathscr{O}}(x_\parallel) \ , \tag{2.45}$$

[8] Here we rescaled $\hat{\phi}_{cl}$ s.t. the DOE coefficient for its exchange does not appear.

where the conformal blocks in both channels are known in closed form for any spacetime dimension d [109]

$$
\begin{aligned}
\mathscr{G}_{\text{bulk}}(\Delta;\xi) &= \xi^{\Delta/2}{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2},\Delta-\frac{d-2}{2},-\xi\right) ,\\
\mathscr{G}_{\text{bndy}}(\hat{\Delta};\xi) &= \xi^{-\hat{\Delta}}{}_2F_1\left(\hat{\Delta},\hat{\Delta}-\frac{d-2}{2},2\hat{\Delta}-d+2,-\xi^{-1}\right) .
\end{aligned}
\tag{2.46}
$$

Since there is only one orthogonal coordinate the cross-ratio η in (2.21) is one (either $x_\perp,\, y_\perp > 0$ or $x_\perp,\, y_\perp < 0$), and we are left with only the cross-ratio ξ. Let us mention that the conformal blocks for spinning fields (e.g. the Noether current and the SE tensor) have been worked out in [91], and those for a fermionic two-point function in [110].

The lack of spinning fields in the bootstrap equation have important consequences for analytical bootstrap: since LIF's require analyticity in the spin, we cannot find one in a BCFT. Thus it is important to develop other analytical methods to bootstrap these theories. This has been done, e.g. functional bootstrap [111, 112]. Another method is the *discontinuity method* developed in paper II for a boundary and extended in paper V to work for interfaces. This method is studied in detail in Ch. 4, where it is also generalized to hold for two different external operators.

2.5.1 Propagators

Let us now specify to the free $O(N)$-model near a flat boundary. Its Lagrangian description is given by

$$
S = \int_{\mathbb{R}^d_+} d^dx\left(\frac{(\partial_\mu\phi^i)^2}{2}+\frac{m^2}{2}(\phi^i)^2\right)+\int_{\mathbb{R}^{d-1}} d^{d-1}x_\parallel \frac{\hat{m}}{2}(\hat{\phi}^i)^2 , \tag{2.47}
$$

with $i\in\{1,...,N\}$ and $\mathbb{R}^d_+ = \{x_\parallel\in\mathbb{R}^{d-1}, x_\perp>0\}$. Note that in addition to the standard mass-term in the bulk, m^2, we can now consider a boundary-mass term, $\hat{m}$, as well. As we will see in a moment, $\hat{m}$ physically determines the p.t. (and b.c.) of the theory.

Let us vary the action w.r.t. the bulk and boundary scalars (ϕ and $\hat{\phi}$)

$$
\begin{aligned}
\delta S &= \int_{\mathbb{R}^d_+} d^dx(\partial_a\phi^i\partial_a\delta\phi^i+\partial_\perp\phi^i\partial_\perp\delta\phi^i+m^2\phi^i\delta\phi^i)+\int_{\mathbb{R}^{d-1}} d^{d-1}x_\parallel\hat{m}\hat{\phi}^i\delta\hat{\phi}^i\\
&= \int_{\mathbb{R}^d_+} d^dx(-\partial_\mu^2\phi^i+m^2\phi^i)\delta\phi^i+\int_{\mathbb{R}^{d-1}} d^{d-1}x_\parallel(-\partial_\perp\hat{\phi}^i+\hat{m}\hat{\phi}^i)\delta\hat{\phi}^i .
\end{aligned}
$$

If we set the variations, $\delta\phi$ and $\delta\hat{\phi}$, to zero we find the e.o.m. and the b.c.

$$
\begin{aligned}
(-\partial_\mu^2+m^2)\phi^i &= 0 ,\\
\partial_\perp\hat{\phi}^i &= \hat{m}\hat{\phi}^i .
\end{aligned}
\tag{2.48}
$$

In the bulk we have the KG eq. (1.11), and on the boundary we have Robin b.c.'s. As we can see, there are no scales in these equations if $m^2 = 0$ and either $\hat{m} = 0$ (Neumann b.c.)

$$\partial_\perp \hat{\phi}^i = 0 \,, \tag{2.49}$$

or $\hat{m} \to \pm\infty$ (Dirichlet b.c.)[9]

$$\hat{\phi}^i = 0 \,. \tag{2.50}$$

These are the conformal f.p.'s of the free BQFT (2.47). The theory with Neumann b.c. is called the *special phase transition* in the condensed matter literature, and that with Dirichlet is called the *ordinary phase transition* [15].

In an interacting BQFT we can consider boundary-local interactions, e.g. a quartic interaction, $\left[(\hat{\phi}^i)^2\right]^2$, in $d = 3-\varepsilon$ (see paper III). Such interactions describe a special p.t., but now the Neumann b.c. (2.49) gets modified by the interactions [113]. We will study this theory in more detail in Ch. 5, where we also consider a sextic interaction, $\left[(\phi^i)^2\right]^3$, in the bulk. We call this theory the $\phi^6 - \hat{\phi}^4$ model.

Without any boundary-interactions we can consider an interacting theory with Dirichlet b.c. (2.50). However, this may also have non-trivial consequences. E.g. for a quartic interaction, $\left[(\phi^i)^2\right]^2$, in the bulk in $d = 4-\varepsilon$. Then if the sign of $\hat{m}$ is not the same as the bulk coupling, a SSB of the $O(N)$-symmetry in the bulk will occur. This results in an $O(N-1)$-symmetry of the theory. The conformal f.p. where this is the case is called the *extraordinary phase transtion* [84, 85, 86].

Let us now solve the DS equation in the presence of the boundary (2.47) (without any interactions). At the RG f.p. the DS eq. (1.12) is then accompanied by Dirichlet b.c.

$$\lim_{x_\perp \to 0} D^{ij}_{b.c.}(x,y) = 0 \,, \tag{2.51}$$

or Neumann b.c.

$$\lim_{x_\perp \to 0} \partial_{x_\perp} D^{ij}_{b.c.}(x,y) = 0 \,. \tag{2.52}$$

We solve these problems using the *method of images* originally used in electrostatics [114]. This is done by adding an image field on the other side of the boundary ($-x_\perp$ or $-y_\perp$). Technically, this amounts to adding a Dirac δ-function which is always zero (assuming $x_\perp$ and $y_\perp$ are both strictly greater than zero) on the RHS of the DS eq. (1.12)

$$(-\partial_\mu^2 + m^2) D^{ij}_{b.c.}(x,y) = \delta^{(d)}(s_\parallel, x_\perp - y_\perp) + \omega\, \delta^{(d)}(s_\parallel, x_\perp + y_\perp) \,. \tag{2.53}$$

The constant ω is to be tuned such that the b.c. (2.51) or (2.52) holds. We can now write the propagator $D^{ij}_{b.c}(x,y)$ in terms of two propagators from a homogeneous QFT (1.19)

$$D^{ij}_{b.c.}(x,y) = D^{ij}(s_\parallel, x_\perp - y_\perp) + \omega\, D^{ij}(s_\parallel, x_\perp + y_\perp) \,. \tag{2.54}$$

[9]This can be seen by dividing both sides of the b.c. in (2.48) with $\hat{m}$.

Applying the b.c. yields

$$\omega = \pm 1 \ , \tag{2.55}$$

with a minus sign for Dirichlet b.c., and plus for Neumann.

The BQFT propagator for the more general Robin b.c. (2.48) was found in paper IV

$$\lim_{x_\perp \to 0} (\partial_{x_\perp} - \hat{m}) D^{ij}_{b.c.}(x,y) = 0 \ . \tag{2.56}$$

In such case we had to consider an infinite amount of images to solve the corresponding DS equation. Although not on a closed form, the propagator was found to be given by

$$\begin{aligned} D^{ij}_{b.c.}(s_\parallel, x_\perp, y_\perp) = D^{ij}(s_\parallel, x_\perp - y_\perp) + D^{ij}(s_\parallel, x_\perp + y_\perp) + \\ - 2\hat{m} \int_0^\infty dz e^{-\hat{m}z} D^{ij}(s_\parallel, x_\perp + y_\perp + z) \ . \end{aligned} \tag{2.57}$$

This is on the same form as the wave function in quantum mechanics on a half-line [115]. It reduces down to the correct result for Neumann/Dirichlet b.c. (2.54) in the limits $\hat{m} \to 0$ and $\hat{m} \to \infty$.[10]

The corresponding momentum propagator (Fourier transformed w.r.t. $s_\parallel$) was found in Sec. A.2 of paper IV. Due to its complicated form, we only write its boundary limit here

$$G^{ij}_{b.c.}(k_\parallel, 0, 0) = \frac{\delta^{ij}}{\sqrt{k_\parallel^2 + m^2} + \hat{m}} \ . \tag{2.58}$$

An interesting property that follows from the method of images, is that the propagators (at the conformal f.p.'s (2.51, 2.52)) are symmetric under mirroring the spacetime points through the boundary $x_\perp \to -x_\perp, \ y_\perp \to -y_\perp$ (and picks up a phase if only one of the spacetime points are mirrored $\xi \to -\xi - 1$)

$$D^{ij}_{b.c.}(s_\parallel, -x_\perp, -y_\perp) = D^{ij}_{b.c.}(s_\parallel, x_\perp, y_\perp) \ . \tag{2.59}$$

This is a classical property of the BQFT propagator and it is called *image symmetry*. Nevertheless it was found to hold for the ϕ^4-model near four dimension at a quantum level (at least upto $\mathcal{O}(\varepsilon^2)$) in paper II.

[10] Note that the results of paper IV does not hold for $\hat{m} \to -\infty$ (see its eq. 60). To study the positive large $\hat{m}$ limit we partially integrate s.t. the overall factor of $\hat{m}$ vanishes. We can then see that only the $z = 0$ limit of the integrand survives

$$\begin{aligned} \hat{m} \int_0^\infty dz e^{-\hat{m}z} D^{ij}(s_\parallel, x_\perp + y_\perp + z) = D^{ij}(s_\parallel, x_\perp + y_\perp) + \int_0^\infty dz e^{-\hat{m}z} \partial_z D^{ij}(s_\parallel, x_\perp + y_\perp + z) \\ \xrightarrow{\hat{m} \to +\infty} D^{ij}(s_\parallel, x_\perp + y_\perp) \ . \end{aligned}$$

While on the topic of image symmetry, let us mention the following property of the boundary conformal blocks

$$\mathscr{G}_{\text{bndy}}(\hat{\Delta}; e^{\pm\pi i}(\xi+1)) = e^{\mp\pi i\hat{\Delta}}\mathscr{G}_{\text{bndy}}(\hat{\Delta};\xi) \ . \tag{2.60}$$

This analytical property of the conformal blocks was found and used in paper II and V to constrain the CFT data entering in the bootstrap equation.

2.5.2 Coleman-Weinberg mechanism

Let us now revisit the CW mechanism from Sec. 4.2. In the case of a boundary, the momentum propagator, $G_{b.c.}(k_\parallel,0,0)$, is known (2.58). This brings the contribution to the effective potential (2.42) onto the form

$$\begin{aligned} V_{\text{eff}}(\phi_{cl}) &= V(\phi_{cl}) + \frac{P_d(m^2,0)}{4} + \text{c.t.'s} + \dots \ , \\ \hat{V}_{\text{eff}}(\hat{\phi}_{cl}) &= \hat{V}(\hat{\phi}_{cl}) + \frac{P_{d-1}(m^2,\hat{m})}{2} + \text{c.t.'s} + \dots \ , \end{aligned} \tag{2.61}$$

where we remind the reader that $P_n(m^2,\hat{m})$ is the master integral (1.89). Note that we need to expand $P_n(m^2,\hat{m})$ around small values of $\hat{m}$ (1.90). In particular this makes it difficult to apply the CW mechanism to a theory with the boundary-term, $\hat{m}$, in the action (2.47) not at its RG f.p.'s.

The effective potentials above are on the same form as that derived in paper IV. In that work, the CW mechanism for a $d = 3-\varepsilon$ dimensional BCFT with only a boundary potential ($m^2 = 0$, $\hat{m} \neq 0$) was developed. In particular, this did not yield an effective potential in the bulk, $V_{\text{eff}} = 0$, but only one on the boundary. This gives us a non-trivial v.e.v. (2.43) on the boundary, which extends into the bulk using the BOE (2.44). It leads to a SSB of the global symmetry in both the bulk and on the boundary.

In this case, the relevant master integral (1.89) simplifies

$$\begin{aligned} P_n(0,\hat{m}) = \frac{S_n\Lambda^n}{(2\pi)^n n}&\left(\frac{\Lambda^2}{(n+2)\hat{m}^2}\,{}_2F_1\left(1,\frac{n+2}{2};\frac{n+4}{2};\frac{\Lambda^2}{\hat{m}^2}\right)+\right. \\ &\left. -\frac{\Lambda}{(n+1)\hat{m}}\,{}_2F_1\left(1,\frac{n+1}{2};\frac{n+3}{2};\frac{\Lambda^2}{\hat{m}^2}\right)+\log(\Lambda+\hat{m})\right) \ . \end{aligned} \tag{2.62}$$

Its expansion around large Λ is

$$\begin{aligned} P_n(0,\hat{m}) = \frac{S_n}{(2\pi)^n}&\left(\Lambda^n\left(\frac{\log\Lambda}{n}-\frac{1}{n^2}+\frac{\hat{m}}{(n-1)\Lambda}-\frac{\hat{m}^2}{2(n-2)\Lambda^2}+\right.\right. \\ &\left.\left.+\frac{\hat{m}^3}{3(n-3)\Lambda^3}+\dots\right)+\frac{\pi\,\csc(\pi n)}{n}\hat{m}^n\right) \ . \end{aligned} \tag{2.63}$$

In paper IV this was further expanded in ε for $n=d-1$

$$P_{d-1}(0,\hat{m}) = \frac{\hat{m}^2}{4\pi}\left(\log\left(\frac{\hat{m}}{\Lambda}\right)-\frac{1}{2}\right)+\frac{\Lambda\hat{m}}{8\pi}+\frac{\Lambda^2}{4\pi}\left(\log\Lambda-\frac{1}{2}\right)+\dots\,.$$

In Ch. 5 we will study the CW mechanism applied to the $\phi^6-\hat{\phi}^4$ model. This is an example when there is both a bulk and a boundary effective potential. We will study the one-loop effects, where only the effective potential on the boundary receives a contribution.

2.6 Interface

The other kind of codimension one defects are *interfaces*, where a bulk theory is defined on each side of it (they are allowed to differ from each other). This gives rise to three different QFT: one on each side of the interface, and a $(d-1)$-dimensional one on the interface. When the three theories are all conformal, we refer to the model as a *interface conformal field theory* (ICFT). Having two bulk theories imply some important physical consequences: propagators are never expected to satisfy image symmetry (2.59), and the OPE between two bulk fields on opposite sides of the interface (and thus also the bulk-channel for these correlators) does not converge. So even though the conformal blocks take on the same analytical structure (2.46) as in a BCFT, only the two-point function of bulk fields on the same side of the interface satisfy the bootstrap eq. (2.22). A method to analytically bootstrap such correlators was developed in paper V, and it is a modification of the method developed for a BCFT in paper II.

2.6.1 The folding trick

To make sure that the energy emitted/absorbed by the interface is the same on both sides of it, we let the interface-limit of the normal-parallel components of the SE tensors from the two bulk theories be the same [116]. This creates b.c.'s that relates bulk fields from the two sides with each other

$$\hat{T}_{+}^{\perp a} = \hat{T}_{-}^{\perp a}\,. \tag{2.64}$$

Here $\pm$ denotes fields from the two sides of the interface. Above equation has many different solutions for the boundary-local fields. In paper V we considered the following solution

$$\hat{\phi}_{+}^{a} = \hat{\phi}_{-}^{a}\,,\quad \partial_\perp\hat{\phi}_{+}^{a} = \partial_\perp\hat{\phi}_{-}^{a}\,. \tag{2.65}$$

To deal with this b.c., we apply *the folding trick* [117]. I.e. we collect the two bulk theories on the same side of the interface by making the shift $x_\perp\to -x_\perp$

for only one bulk theory. Effectively, this creates a BQFT for a linear combination of the bulk fields. E.g. for the case above we find Neumann/Dirichlet b.c.'s for the linear combinations $\phi_+^a \pm \phi_-^a$

$$\begin{aligned} \lim_{x_\perp \to 0} [\phi_+^a(x_\parallel, x_\perp) - \phi_-^a(x_\parallel, -x_\perp)] = 0 \, , \\ \lim_{x_\perp \to 0} \partial_\perp [\phi_+^a(x_\parallel, x_\perp) + \phi_-^a(x_\parallel, -x_\perp)] = 0 \, . \end{aligned} \tag{2.66}$$

These b.c.'s can be written in terms of projectors acting on a scalar in the $(N \times 1) \oplus (1 \times N)$-representation of $O(N) \times O(N)$

$$\Phi_\alpha^i = \begin{pmatrix} \phi_-^i \\ \phi_+^i \end{pmatrix}_\alpha \, , \quad (\Pi_\pm)_{\alpha\beta}^{ij} = \frac{\delta^{ij}}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}_{\alpha\beta} \, , \tag{2.67}$$

which yields

$$(\Pi_\pm)_{\alpha\beta}^{ij} \Phi_\beta^j = \begin{pmatrix} \phi_-^i \pm \phi_+^i \\ \pm\phi_-^i + \phi_+^i \end{pmatrix}_\alpha \, . \tag{2.68}$$

Correlators can now be found using the method of images

$$(D_\Phi)_{\alpha\beta}^{ij}(s_\parallel, x_\perp, y_\perp) = \delta_{\alpha\beta} D^{ij}(s_\parallel, x_\perp - y_\perp) + \chi_{\alpha\beta}^{ik} D^{kj}(s_\parallel, x_\perp + y_\perp) \, ,$$

where D^{ij} is the propagator (1.19) in a homogenous QFT. The constant ω in (2.54) is generalized to a matrix difference, $\chi_{\alpha\beta}^{ij}$, between the two projectors $(\Pi_\pm)_{\alpha\beta}^{ij}$ for Dirichlet and Neumann b.c.'s

$$\chi_{\alpha\beta}^{ij} = (\Pi_+)_{\alpha\beta}^{ij} - (\Pi_-)_{\alpha\beta}^{ij} = \delta^{ij} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta} \, . \tag{2.69}$$

This yields the correlator

$$(D_\Phi)_{\alpha\beta}^{ij}(s_\parallel, x_\perp, y_\perp) = \begin{pmatrix} D^{ij}(s_\parallel, x_\perp - y_\perp) & D^{ij}(s_\parallel, x_\perp + y_\perp) \\ D^{ij}(s_\parallel, x_\perp + y_\perp) & D^{ij}(s_\parallel, x_\perp - y_\perp) \end{pmatrix}_{\alpha\beta} \, . \tag{2.70}$$

Note that this propagator, $(D_\Phi)_{\alpha\beta}^{ij}$, is on the same form as the free theory propagator (1.19) in a homogeneous QFT (it is allowed to differ at a quantum level). This is no coincidence, and one way to see this is to add the following Dirac δ-function in the path integral for a free scalar theory (1.10)

$$\begin{aligned} Z &= \int \mathscr{D}\phi \, e^{iS[\phi]} = \int \mathscr{D}\phi_+ \mathscr{D}\phi_- e^{i(S_+[\phi_+] + S_-[\phi_-])} \delta(\hat{\phi}_+ - \hat{\phi}_-) \\ &= \int \mathscr{D}\phi_+ \mathscr{D}\phi_- \mathscr{D}\eta \, e^{i(S_+[\phi_+] + S_-[\phi_-] + \hat{S}[\eta, \hat{\phi}_+, \hat{\phi}_-])} \, , \end{aligned} \tag{2.71}$$

$$\begin{aligned} S_\pm[\phi_\pm] &= \int_{\mathbb{R}_\pm^d} d^d x \left(\frac{(\partial_\mu \phi_\pm^i)^2}{2} + \frac{m^2}{2} (\phi_\pm^i)^2 \right) \, , \\ \hat{S}[\eta, \hat{\phi}_+, \hat{\phi}_-] &= \int_{\mathbb{R}^{d-1}} d^{d-1} x_\parallel \, \eta (\hat{\phi}_+ - \hat{\phi}_-) \, . \end{aligned} \tag{2.72}$$

Here $\mathbb{R}^d_+ = \{x_\parallel \in \mathbb{R}^{d-1}, x_\perp > 0\}$ and $\mathbb{R}^d_- = \{x_\parallel \in \mathbb{R}^{d-1}, x_\perp < 0\}$. Hatted operators denote those living on the interface generated by the δ-function. If we vary the action w.r.t. the fields we find the KG equation (1.11) in the bulk and the same b.c.'s as in (2.65).

2.7 Monodromy twist defect

Let us now study the *($O(N)$-flavoured) monodromy defect*. This object is of codimension two and carry a *monodromy constraint* for the bulk fields [28, 29]. That is, if we transport a bulk field around the defect it transforms under its global symmetry group. E.g. in paper I this was studied for an $O(N)$-model, where the defect carry the monodromy

$$\phi^i(x_\parallel, r, \theta + 2\pi) = g^{ij}\phi^j(x_\parallel, r, \theta) , \quad g^{ij} \in O(N) , \tag{2.73}$$

using polar coordinates in the orthogonal directions: $r > 0$, $\theta \in [0, 2\pi)$. This defect is characterized by its *twist*: the group element g^{ij}.

Since the orthogonal symmetry group $SO(d-p) = SO(2)$ is Abelian, the DOE (2.8) simplifies [29]

$$\phi^i(x) = \sum_{\hat{\mathcal{O}}_s} \mu^{\phi^i}{}_{\hat{\mathcal{O}}_s} \frac{e^{is\theta}}{r^{\Delta - \hat{\Delta}_s}} \hat{C}_2(r^2\partial_\parallel^2)\hat{\mathcal{O}}_s(x_\parallel) . \tag{2.74}$$

Here s is the $SO(2)$-charge of the defect-local field $\hat{\mathcal{O}}_s$. With this at hand, we find that the global symmetry group is broken along the defect by the monodromy action (2.73). In the case of $O(N)$ with $N \geq 2$, this was first studied in paper I, and further studied in [118].

2.7.1 CFT data from the defect-limit

Let us here illustrate the methods of paper III in the presence of a monodromy twist defect. That is, we will study the defect-limit of a bulk-bulk two-point function to read off the bulk-defect two-point functions (or equivalently due to conformal symmetry the scaling dimensions and OPE coefficients (2.18, 2.19)). In paper III this was done for a boundary, but it works just as well for lower dimensional defects as we will show here.

We will consider the same theory as in paper I: the $O(N)$-flavoured monodromy twist defect with monodromy (2.73). Let us first study the bulk two-point function using the DOE (2.74)

$$\begin{aligned}\langle\phi^i(x)\phi^j(y)\rangle = \delta^{ij} \sum_{\hat{\mathcal{O}}_s \hat{\mathcal{O}}_{s'}} \frac{\mu^\phi{}_{\hat{\mathcal{O}}_s}}{r_1^{\Delta_\phi - \hat{\Delta}_s}} \frac{\mu^\phi{}_{\hat{\mathcal{O}}_{s'}}}{r_2^{\Delta_\phi - \hat{\Delta}_{s'}}} e^{-is\theta_1 + is'\theta_2} \times \\ \times C_2(r_1^2\partial_\parallel^2)C_2(r_2^2\partial_\parallel^2)\langle\hat{\mathcal{O}}_s(x_\parallel)\hat{\mathcal{O}}_{s'}(y_\parallel)\rangle .\end{aligned} \tag{2.75}$$

Here we are using the polar coordinates $x_\perp = r_1(\cos\theta_1, \sin\theta_1)$ and $y_\perp = r_2(\cos\theta_2, \sin\theta_2)$ with r_1, $r_2 > 0$ and θ_1, $\theta_2 \in [0, 2\pi)$. Assuming that the defect-defect two-point function is on the form (1.56) (with normalization of ϕ s.t. $A_d = 1$), and only keeping terms at the lowest order in r_1, r_2 yields

$$\langle \phi^i(x)\phi^j(y)\rangle = \delta^{ij}\frac{(\mu^\phi{}_{\hat{\mathscr{O}}_s})^2 e^{is\varphi}}{(r_1r_2)^{\Delta_\phi - \hat{\Delta}_s}|s_\parallel|^{2\hat{\Delta}_s}} + \dots\,, \qquad \varphi \equiv \theta_2 - \theta_1\,, \tag{2.76}$$

where $\hat{\mathscr{O}}_s$ is now the lowest dimensional defect-local operator in the DOE (2.74) of ϕ. Expanding the CFT data in ε

$$\begin{aligned}(\mu^\phi{}_{\hat{\mathscr{O}}_s})^2 &= \mu_s^{(0)} + \varepsilon\,\mu_s^{(1)} + \mathscr{O}(\varepsilon^2)\,,\\ \Delta_\phi &= \Delta_\phi^{(0)} + \varepsilon\,\Delta_\phi^{(1)} + \mathscr{O}(\varepsilon^2)\,,\\ \hat{\Delta}_s &= \hat{\Delta}_s^{(0)} + \varepsilon\,\hat{\Delta}_s^{(1)} + \mathscr{O}(\varepsilon^2)\,,\end{aligned} \tag{2.77}$$

gives us the following ε-expansion

$$\begin{aligned}\langle \phi^i(x)\phi^j(y)\rangle = {}& \frac{\delta^{ij}e^{is\varphi}}{(r_1r_2)^{\Delta_\phi^{(0)} - \hat{\Delta}_s^{(0)}}|s_\parallel|^{2\hat{\Delta}_s^{(0)}}}\Big[\mu_s^{(0)} + \\ &+ \varepsilon\Big(\mu_s^{(1)} + (\hat{\Delta}_s^{(1)} - \Delta_\phi^{(1)})\mu_s^{(0)}\log(r_1r_2) + \\ &- 2\hat{\Delta}_s^{(0)}\mu_s^{(0)}\log|s_\parallel|\Big)\Big] + \dots\,.\end{aligned} \tag{2.78}$$

On the other hand, this correlator was found in [105] upto $\mathscr{O}(\varepsilon)$ using the LIF's developed for a DCFT

$$\langle \phi^i(x)\phi^j(y)\rangle = \delta^{ij}\frac{f(z,\bar{z})}{(r_1r_2)^{\Delta_\phi}}\,, \tag{2.79}$$

with the dynamical function

$$f(z,\bar{z}) = \frac{z^{1-\upsilon}\bar{z}}{1-|z|^2}\left(\frac{1}{1-z} + \varepsilon\frac{\upsilon(\upsilon-1)}{2}\frac{N+1}{N+4}\Phi(z,1,\upsilon)\log|z|\right) + (z \leftrightarrow \bar{z})\,.$$

This is expressed in terms of the Hurwitz-Lerch transcendent

$$\Phi(z,s,a) \equiv \sum_{k\geq 0}\frac{z^k}{(k+a)^s}\,, \tag{2.80}$$

and the complex cross-ratio[11]

$$z = \frac{s_\parallel^2 + r_1^2 + r_2^2 - \sqrt{(s_\parallel^2 + r_1^2 + r_2^2)^2 - 4r_1^2r_2^2}}{2r_1r_2}e^{i\varphi}\,. \tag{2.81}$$

[11] In appendix A of [103] we see that $z = rw$ with $r = \frac{\chi - \sqrt{\chi^2-4}}{2}$, $w = \eta + \sqrt{\eta^2 - 1}$ and $\chi = \frac{s_\parallel^2 + r_1^2 + r_2^2}{r_1r_2}$, $\eta = \cos\phi$.

Expanding (2.79) in ε, r_1 and r_2

$$\langle \phi^i(x)\phi^j(y)\rangle = \delta^{ij}\frac{(r_1 r_2)^{\upsilon}}{|s_{\parallel}|^{2(\upsilon+1)}}(e^{i\upsilon\varphi}+e^{-i\upsilon\varphi})+\dots \, . \tag{2.82}$$

Comparing this with (2.78) tells us that above expression contains two defect-local operators with $SO(2)$-charge

$$s = \pm\upsilon \, , \tag{2.83}$$

both of which has the CFT data

$$\mu^{(0)}_{\pm\upsilon} = 1 \, , \quad \Delta^{(0)}_{\phi} = 1 \, , \quad \hat{\Delta}^{(0)}_{\pm\upsilon} = 1+\upsilon \, . \tag{2.84}$$

Focusing on the defect operator with $s = +\upsilon$, we find at $\mathscr{O}(\varepsilon)$ in (2.79)

$$\begin{aligned}\langle \phi^i(x)\phi^j(y)\rangle \ni \varepsilon\, e^{i\upsilon\varphi}\delta^{ij}\frac{(r_1 r_2)^{\upsilon}}{|s_{\parallel}|^{2(\upsilon+1)}}\bigg(&-\frac{\upsilon H_{\upsilon-1}+1}{2\upsilon}+ \\ &+\frac{(\upsilon-2\gamma_{\phi}-1)N+\upsilon-8\gamma_{\phi}-1}{2(N+4)}\log(r_1 r_2)+ \\ &+\frac{(\upsilon-2)N+\upsilon-5}{N+4}\log|s_{\parallel}|\bigg) \, .\end{aligned} \tag{2.85}$$

Here H_x is the harmonic number. If we now compare this with (2.78) we find

$$\begin{aligned}\mu^{(1)}_{+\upsilon} &= -\frac{\upsilon H_{\upsilon-1}+1}{2\upsilon} \, , \\ \Delta^{(1)}_{\phi} &= -\frac{1}{2}+\gamma_{\phi} \, , \\ \hat{\Delta}^{(1)}_{+\upsilon} &= -\frac{1}{2}+\frac{(\upsilon-1)(N+1)}{2(N+4)} \, .\end{aligned} \tag{2.86}$$

This CFT data, together with (2.84), is consistent with the results from paper I. With this at hand we have found the defect-defect and bulk-defect two-point functions from the bulk-bulk correlator. By studying higher order terms in r_1 and r_2 we can read off the CFT data for other defect-local primaries.

2.8 Replica twist defects

The monodromy constraint (2.73) from the Sec. 2.7 can be generalized to branch cuts with n sheets

$$\phi^i_a(x_{\parallel}, r, \theta+2\pi) = g^{ij}\phi^j_{a+1}(x_{\parallel}, r, \theta) \, , \quad g^{ij}\in O(N) \, , \tag{2.87}$$

where ϕ^i_a, with $a \in \{1,\dots,n\}$ and $i \in \{1,\dots,N\}$, is the fundamental scalar on the a^{th} sheet. The *replica twist defect* is the defect with above monodromy where the QFT on each sheet are all the same. As explained in the Introduction, these defects are important when finding the Rényi entropy in a QFT [30, 31].

If we consider n replicas of the bulk theory, we should identify the $(n+1)^{th}$ bulk field with the first one. This gives us the monodromy constraint for a general ϕ_a^i

$$\phi_{n+1}^i \equiv \phi_1^i \quad \Rightarrow \quad \phi_a^i(x_\parallel, r, \theta + 2\pi n) = g^{ij_1} \prod_{m=2}^{n} g^{j_{m-1} j_m} \phi_a^{j_m}(x_\parallel, r, \theta) \; . \quad (2.88)$$

Since $O(N)$ is closed, $g^{ij_1} \prod_{m=2}^{n} g^{j_{m-1} j_m} \in O(N)$. To avoid clutter we thus write above monodromy constraint as

$$\phi_a^i(x_\parallel, r, \theta + 2\pi n) = g^{ij} \phi_a^j(x_\parallel, r, \theta) \; , \quad g^{ij} \in O(N) \; . \quad (2.89)$$

In Ch. 6 we will study how this monodromy breaks the global $O(N)$-symmetry along the defect. This generalizes the result in paper I, though it makes use of the same method. In this Chapter we will also extract the anomalous dimensions in $d = 4 - \varepsilon$ of the defect-local fields (upto first order in the coupling constant) by applying the e.o.m. to the DOE (2.74).

3. Multiple defects in a quantum field theory

As we saw in Ch. 2, already considering one defect is a difficult task providing us with new non-trivial phenomena. Yet we can always consider more defects, and try to explore the physical consequences in doing this. Even less of the symmetries in the bulk will be preserved by the multiple defects, and thus these models are even more of a challenge to study. Nevertheless, let us briefly discuss such systems, and how they differ from a theory with a single defect.

3.1 Symmetries and OPE's

The Poincaré or conformal symmetry in the bulk is broken in the same way as for one defect, i.e. each defect will be charged under an orthogonal group $SO(d-p)$, and a defect-local field on top of such defect is charged under $SO(p-1,1) \times SO(d-p)$ (assuming flat defects). This is due to localization, where each defect is only affected by the nearby bulk.

A quite interesting scenario is when two defects intersect with each other. Then there will also be a QFT on the intersection. E.g. if we consider two flat defects of dimension $p < d$ and $q \leq p$. Then we can imagine an intersection of dimension $r < q$. The intersection itself will enjoy a $SO(d-p) \times SO(d-q) \times SO(p-r) \times SO(q-r)$-symmetry which arises from orthogonal rotations of the intersection. Local fields on top of this intersection will in addition satisfy a local Poincaré $SO(r-1,1)$- or conformal $SO(r,2)$-symmetry. See e.g. Fig. 1 in paper VII for an illustration on two intersecting boundaries.

As in the case of a single defect, local characteristics, such as anomalous dimensions, β-functions and OPE coefficients, of the bulk theory are not affected by the defects nor possible intersections between them (since the UV divergences these quantities arise from are the coincident-limits of bulk fields).

Likewise, the corresponding characteristics on each defect are not affected by other defects nor any intersections (since these defect quantities arise from their corresponding coincident-limits of defect-local fields and defect-limits of bulk-local fields).

In the case of an intersection between two defects, local quantities on the intersection will be affected by both of the defect theories in addition to the bulk theory. Thus the intersection data will in general depend on the intersection angle (dimensionless) between the two defects.

Regarding propagators: if there is no intersection, then those on the defects behave as in a p- or q-dimensional homogeneous QFT as in Sec. 1.2.2. The ones in the bulk are more complicated as they are now affected by both defects.

If the two defects intersect: then propagators on the intersections behave as in a r-dimensional homogeneous QFT, while those on the defect behave as in a p- or q-dimensional DQFT (see e.g. Sec. 2.3 for the conformal case). Propagators involving bulk fields are now affected by both defects in addition to the intersection theory, giving them a complicated analytic structure.

There will be several new OPE's in play. In addition to the usual bulk-bulk OPE (Sec. 1.3) there is a DOE for each defect (Sec. 2.3) and a defect-defect OPE on each defect (similar to the bulk-bulk OPE (1.36) but for the parallel coordinates). In the case of an intersection, there is also one defect-intersection DOE for each defect and an intersection-intersection OPE. In the conformal case (when the intersection, both of the defects and the bulk are all conformal), these give rise to a conformal bootstrap equation for bulk one- and bulk-intersection two-point correlators [119].

3.2 Correlation functions between defects

Let us consider two non-intersecting conformal defects, D_p and D_q, of dimensions p and q, and discuss the correlation function, $\langle D_p D_q\rangle$, between these two defects. From a Feynman diagrammatic point of view, this correlator is described by correlators of defect-local fields running between two operators on the same defect as well as one operator on each defect (see e.g. Fig. 2 of paper VI). The latter diagrams will describe a Casimir effect which describes how the two defects affect each other.

In order to obtain any Feynman rules we need to know the action for the two defects. Sometimes this is indeed known, e.g. the one-dimensional scalar Wilson line in four dimensions

$$D_1 = \exp\left(-h\int_{\mathbb{R}} dx_{\parallel}\hat{\phi}(x_{\parallel})\right) . \tag{3.1}$$

Here h is a magnetic field along the defect, which can be treated as a coupling constant of finite size [13]. In Ch. 7 we will study these defects in more detail.

However, in general, integral representations of the defects are unknown. It is thus important to have other means of finding $\langle D_p D_q\rangle$. One such idea would be to find a bootstrap equation assuming the two defects are conformal. As seen from symmetry, a conformal defect on its own can be expressed in terms of a series of bulk-local operators [120]. E.g. if we Taylor expand above scalar Wilson line it will be expressed as a series of bulk-local operators.[1] This expansion allows us to decompose $\langle D_p D_q\rangle$ in terms of bulk-local operators

$$\langle D_p D_q\rangle = \sum_{\mathscr{O}_{\Delta,l}} C^{D_p}{}_{\mathscr{O}_{\Delta,l}} C^{D_q}{}_{\mathscr{O}_{\Delta,l}} \mathscr{G}^{p,q}_{\mathscr{O}_{\Delta,l}}(\{\eta_a\}) , \tag{3.2}$$

[1]The Taylor expansion is around $h=0$, and as such the exchanged operators in the summand are not affected by the defect.

where ${C^{D_p}}_{\mathscr{O}_{\Delta,l}}$ and ${C^{D_q}}_{\mathscr{O}_{\Delta,l}}$ are some OPE coefficients that follow from the expansion of the defects in terms of bulk-local operators, $\mathscr{O}_{\Delta,l}$. A general form of the conformal block $\mathscr{G}^{p,q}_{\mathscr{O}_{\Delta,l}}(\{\eta_a\})$ is not known, but methods of finding these blocks for different values of p, q and d have been found in [120, 121]. The number of cross-ratios, η_a, is $\min(d-p, d-q, p+2, q+2)$. At the present day we are not aware of another way of decomposing the defect-defect correlator, and thus whether a bootstrap equation for $\langle D_p D_q \rangle$ exists or not is unknown.

We could imagine that another bootstrap-channel is found by decomposing the two defects in terms of other conformal defects. We call such process a *fusion* of defects. This was explored in paper VI, where two scalar Wilson lines on the form (3.1) were fused, hoping to yield another conformal defect. However, the results in paper VI tells us that this is not the case in general. As it turns out, fusing two scalar Wilson lines yields a non-conformal defect

$$D_f = \exp\left(-2h \sum_{n\geq 0} \frac{R^{2n}}{(2n)!} \int_{\mathbb{R}} dx_{\parallel} \partial_R^{2n} \hat{\phi}(x_{\parallel})\right) . \tag{3.3}$$

One way to understand this statement is that the distance, R, between the two defects is a scale of the theory, and thus has to be preserved after the fusion. This scale then enters in the interactions on the fused defect, making them dimensionfull. In turn, this makes the fused defect action non-conformal.

In Ch. 7 we cover the technical details of how fusion was done in paper VI. We also generalize this fusion to hold for six dimensions (in the free theory) as well as show that it also holds in the interacting theory (at least to first order in the bulk couplings).

3.3 Parallel and intersecting boundaries

Let us now discuss models with two boundaries. These can either be parallel (a *pair of slabs*) or intersect each other (a *wedge*). In both of these systems there is a new parameter. For two parallel boundaries the distance, L, between them will become a new length scale of the theory. On the other hand, for two intersecting boundaries the dimensionless intersection angle, θ, between them plays a central role.

Each boundary is equipped with a b.c., e.g. for a free scalar with a Lagrangian description we can consider

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \phi^i)^2}{2} + \frac{m^2}{2}(\phi^i)^2\right) + \sum_{\pm} \int_{\mathbb{R}^{d-1}} d^{d-1}x_{\parallel} \frac{\hat{m}_{\pm}}{2}(\hat{\phi}^i_{\pm})^2 . \tag{3.4}$$

Here the two boundary masses, $\hat{m}_{\pm}$, are allowed to differ from each other, and $\hat{\phi}^i_{\pm} \equiv \hat{\phi}^i(x_{\parallel})\big|_{x_\perp=\pm L}$. By varying the fields we find the KG eq. (1.11) in the bulk together with the two b.c.'s

$$\partial_\perp \hat{\phi}^i_{\pm} = \hat{m}_{\pm} \hat{\phi}^i_{\pm} . \tag{3.5}$$

Each $\hat{m}_\pm$ has its own RG flow, with the f.p.'s $\hat{m}_\pm = 0$ (Neumann b.c.) and $|\hat{m}_\pm| \to \infty$ (Dirichlet b.c.) as in to Sec. 2.5. Note that this gives different configurations of b.c.'s at the RG f.p.'s. E.g. both boundaries can either be equipped with Neumann or Dirichlet b.c.'s. Alternatively one boundary can have Neumann while the other one has Dirichlet b.c.'s.

In the case of a pair of slabs, we may also consider periodic $(+)$ or anti-periodic $(-)$ b.c.'s, which relate the boundary fields on the two boundaries to each other [84, 122, 123]

$$\hat{\phi}^i_- = \pm\hat{\phi}^i_+ \ . \tag{3.6}$$

Similar to models with a single boundary, the method of images can be used to find propagators. Although this is more difficult, as it in general requires an infinite amount of images [124, 125, 126, 127]. It is worth mentioning that for fractional values of θ in the case of a wedge

$$\theta = \frac{a\pi}{b} \ , \quad a, b \in \mathbb{Z}_{\geq 1} \ , \tag{3.7}$$

only a finite number $\left(2\,\text{numerator}\left(\frac{b}{a}\right) - 1\right)$ of images are required, where numerator$\left(\frac{b}{a}\right)$ removes all common factors before taking the numerator. See Fig. 2 in paper VII for an illustrative example.

3.3.1 Wedge

Let us now specify to a wedge. We will refer to these theories as a *wedge quantum field theory*. At the intersection, there is a $(d-2)$-dimensional QFT which we refer to as the *edge*. We will call one of the boundaries the *wall*, and the other (intersecting at an angle θ) the *ramp*.

Through a certain asymptotic, we retrieve a pair of slabs from a wedge. This is achieved by considering an infinitesimally small angle, $\theta \to 0$, and large parallel coordinates along the boundaries, $|x_\parallel| \to \infty$, while at the same time keeping the arc length, $|x_\parallel|\,\theta = 2L$, fixed. This arc length is then the distance between the two parallel boundaries.

In the limit $\theta \to \pi$ where the wedge "unfolds", there is no reason to expect that the edge theory should disappear. Rather, in the general case, the edge theory will split the boundary in two regions (the wall and the ramp theory), becoming an interface on top of the boundary. This limit is thus rather complicated in general. One example of this phenomena is in paper VII (see its eq. (2.31)), where one side of the interface (on the boundary) is equipped with Dirichlet b.c. and the other side with Neumann.

Let us turn our focus to the conformal case, when the edge, the boundaries and the bulk are all conformal. We call such theory a *wedge conformal field theory* (WCFT). As previously stated, edge correlators behave as in a $(d-2)$-dimensional homogeneous CFT, and correlators involving operators on the same boundary behave as in a BCFT (with a $(d-1)$-dimensional bulk). Correlators with operators on different boundaries are less restricted since there is no well-defined OPE between wall and ramp operators. We are left with only the boundary-edge BOE's, which means that only the boundary conformal block decomposition in (2.46) is valid.

Bulk correlators are more difficult. Even the two simplest bulk correlators we can consider: the scalar bulk one-point function, $\langle \mathscr{O}(x) \rangle$, and the scalar bulk-edge two-point function, $\langle \mathscr{O}(x) \hat{\hat{\mathscr{O}}}(y_\parallel) \rangle$, are not fixed by the residual conformal symmetries [119]. Here, and in the rest of this thesis, we denoted edge operators and its corresponding quantities with a double hat. Let us place the wall at $x_{d-1} = 0$ and use polar coordinates for the normal directions

$$x_d = \rho \cos \varphi \,, \quad x_{d-1} = \rho \sin \varphi \,, \quad \varphi = \arctan \left(\frac{x_{d-1}}{x_d} \right) \,, \tag{3.8}$$

with $\rho > 0$ being the radial distance from the edge to the bulk operator, and $\varphi \in (0, \theta)$ being the angle between the bulk operator and the wall. The bulk one-point functions, $\langle \mathscr{O}(x) \rangle$, and the bulk-edge two-point function, $\langle \mathscr{O}(x) \hat{\hat{\mathscr{O}}}(y_\parallel) \rangle$, can be expressed in terms of a function, $f(\varphi)$ and $g(\varphi)$ respectively, which depend on the single cross-ratio φ

$$\langle \mathscr{O}(x) \rangle = \frac{f(\varphi)}{\rho^\Delta} \,, \quad \langle \mathscr{O}(x) \hat{\hat{\mathscr{O}}}(y_\parallel) \rangle = \frac{g(\varphi)}{\rho^{\Delta - \hat{\hat{\Delta}}} (s_\parallel^2 + \rho^2)^{\hat{\hat{\Delta}}}} \,. \tag{3.9}$$

For both of these correlators we can use the wall or ramp bulk-boundary BOE followed by the corresponding boundary-edge BOE to express the bulk operator in terms of edge ones. This gives us two different conformal block decompositions, one where wall operators are exchanged, and another with ramp operators [119]

$$\begin{aligned} f(\varphi) &= \csc^\Delta(\varphi) \left(a_{\mathscr{O}} + \sum_{\hat{\mathscr{O}}} b_{\hat{\mathscr{O}}} \mathscr{G}_{\hat{\Delta},0} \Big(\tan \varphi \Big) \right) \\ &= \csc^\Delta(\theta - \varphi) \left(a_{\mathscr{O}'} + \sum_{\hat{\mathscr{O}}'} b_{\hat{\mathscr{O}}'} \mathscr{G}_{\hat{\Delta}',0} \Big(\tan(\theta - \varphi) \Big) \right) \,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} g(\varphi) &= \sin^{\hat{\hat{\Delta}} - \Delta}(\varphi) \sum_{\hat{\mathscr{O}}} c_{\hat{\mathscr{O}}} \mathscr{G}_{\hat{\Delta}_n, \hat{\hat{\Delta}}} \Big(\tan \varphi \Big) \\ &= \sin^{\hat{\hat{\Delta}} - \Delta}(\theta - \varphi) \sum_{\hat{\mathscr{O}}'} c_{\hat{\mathscr{O}}'} \mathscr{G}_{\hat{\Delta}'_m, \hat{\hat{\Delta}}} \Big(\tan(\theta - \varphi) \Big) \,, \end{aligned} \tag{3.11}$$

where non-primed (hatted) quantities corresponds to wall data, and primed to ramp ones. We call the two different sides of the bootstrap equations for the wall- and ramp-channel. The products of BOE coefficients are

$$
\begin{aligned}
a_{\mathscr{O}}(\theta) &= \mu^{\mathscr{O}}{}_{\mathbb{1}}(\theta)\,, & a_{\mathscr{O}}(\theta') &= \mu^{\mathscr{O}'}{}_{\mathbb{1}}(\theta)\,,\\
b_{\hat{\mathscr{O}}}(\theta) &= \mu^{\mathscr{O}}{}_{\hat{\mathscr{O}}}\hat{\mu}^{\hat{\mathscr{O}}}{}_{\mathbb{1}}(\theta)\,, & b_{\hat{\mathscr{O}}'}(\theta) &= \mu^{\mathscr{O}}{}_{\hat{\mathscr{O}}'}\hat{\mu}^{\hat{\mathscr{O}}'}{}_{\mathbb{1}}(\theta)\,,\\
c_{\hat{\mathscr{O}}}(\theta) &= \mu^{\mathscr{O}}{}_{\hat{\mathscr{O}}}\hat{\mu}^{\hat{\mathscr{O}}}{}_{\hat{\hat{\mathscr{O}}}}(\theta)\,, & c_{\hat{\mathscr{O}}'}(\theta) &= \mu^{\mathscr{O}}{}_{\hat{\mathscr{O}}'}\hat{\mu}^{\hat{\mathscr{O}}'}{}_{\hat{\hat{\mathscr{O}}}}(\theta)\,.
\end{aligned}
\tag{3.12}
$$

We expressed the boundary-edge coefficients, $\hat{\mu}^{\hat{\mathscr{O}}}{}_{\hat{\hat{\mathscr{O}}}}$, with a hat, and explicitly wrote out its dependence on the intersection angle, θ. Finally we have the single conformal block

$$
\mathscr{G}_{\hat{\Delta},\hat{\hat{\Delta}}}(\eta) = \eta^{\hat{\Delta}-\hat{\hat{\Delta}}}\,{}_2F_1\left(\frac{\hat{\Delta}-\hat{\hat{\Delta}}}{2},\frac{\hat{\Delta}-\hat{\hat{\Delta}}+1}{2},\hat{\Delta}-\frac{d-3}{2},-\eta^2\right)\,. \tag{3.13}
$$

Notice that the main difference between the two bootstrap eq.'s (3.10, 3.11) is the exchange of the identity operator in the bulk one-point function. Moreover, the two equations are symmetric under $\varphi \to \theta - \varphi$, which is the same as exchanging the two boundaries. This symmetry tells us that we are free to exchange the b.c.'s on the two boundaries. E.g. a WCFT with Dirichlet b.c. on the wall and Neumann on the ramp is the same as the vice versa.

Free bulk correlators (3.9) were found from the bootstrap eq.'s (3.10, 3.11) in [119]. In particular, if the bulk theory has $O(N)$-symmetry, we expect the bulk one-point function of the fundamental scalar, ϕ^i, to have a vanishing one-point function. This was indeed found for $d < 4$ at the end of Sec. 4 in [119]. Near six dimensions it is possible to have a non-vanishing v.e.v. of ϕ^i. Though this is a quantum effect emerging from a cubic interaction in the bulk, which explicitly breaks the $O(N)$-symmetry.

In paper VII a WCFT was considered, and a way to find the bulk-edge correlator (3.9) in the interacting theory using the e.o.m. was developed. We will illustrate this method for more complicated b.c.'s than those in paper VII in Ch. 5 (in the presence of a single boundary).

Part II:
Defects, conformal symmetry and fusion

4. The discontinuity method in a BCFT

We will now explain the discontinuity method from paper II and V in detail. In these papers, this analytical method was applied to several different BCFTs and ICFTs near four and six dimensions. This method has also been applied to a supersymmetric BCFT in [128], and it projects out the OPE coefficients from the bootstrap equation for two external scalars

$$
\begin{aligned}
f(\xi) &= \sum_{\mathscr{O}} \lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}} \mu^{\mathscr{O}}{}_{\mathbb{1}} \mathscr{G}_{\text{bulk}}(\Delta;\xi) \\
&= \xi^{\frac{\Delta_{12}^+}{2}} \sum_{\hat{\mathscr{O}}} \mu^{\mathscr{O}_1}{}_{\hat{\mathscr{O}}} \mu^{\mathscr{O}_2}{}_{\hat{\mathscr{O}}} \mathscr{G}_{\text{bndy}}(\hat{\Delta};\xi) \,,
\end{aligned}
\tag{4.1}
$$

where the conformal blocks are given by (2.46), which we write out again here

$$
\begin{aligned}
\mathscr{G}_{\text{bulk}}(\Delta;\xi) &= \xi^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2},\Delta-\frac{d-2}{2},-\xi\right) \,, \\
\mathscr{G}_{\text{bndy}}(\hat{\Delta};\xi) &= \xi^{-\hat{\Delta}} {}_2F_1\left(\hat{\Delta},\hat{\Delta}-\frac{d-2}{2},2\hat{\Delta}-d+2,-\xi^{-1}\right) \,.
\end{aligned}
\tag{4.2}
$$

Prior to this thesis, the discontinuity method has only been applied to two-point functions with identical external scalars. Here we will generalize it to mixed correlators where the external scalars might differ. We will also show that it is possible to project out both the bulk OPE and the BOE coefficients from above bootstrap equation, depending on which branch cut we study of the conformal blocks.

4.1 Discontinuities of the conformal blocks

To start with, a hypergeometric function ${}_pF_q(\ldots,z)$ has a branchcut along $z>1$. This means that for the conformal blocks (4.2), the bulk block, $\mathscr{G}_{\text{bulk}}(\Delta;\xi)$, has a branch cut along $\xi<-1$ (this is the same as sending one of the external bulk operators to its *mirror point* $x_\perp \to -x_\perp$) coming from the ${}_2F_1(\ldots,-\xi)$ and the boundary-block, $\mathscr{G}_{\text{bdy}}(\hat{\Delta};\xi)$, has one along $-1<\xi<0$ coming from the ${}_2F_1(\ldots,-\xi^{-1})$. In addition, the factors $\xi^{\Delta/2}, \xi^{-\hat{\Delta}}, \xi^{\frac{\Delta_{12}^+}{2}}$ have branch cuts along the entire negative axis $\xi<0$. See Fig. 4.1. We will show how the bulk OPE coefficients can be found from the branch cut along $\xi<-1$, and the BOE coefficients from the branchcut along $-1<\xi<0$.

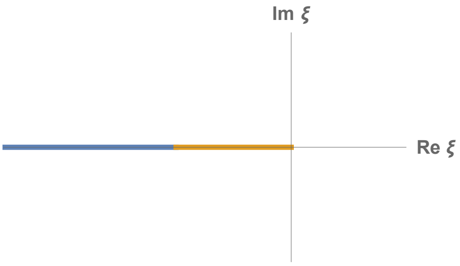

Figure 4.1. The branch cuts of the bootstrap eq. (4.1): the blue line is that of the bulk ${}_2F_1$, while the orange originates from the boundary ${}_2F_1$.

Firstly we define the discontinuity of an arbitrary function $f(\xi)$ along a real interval $\mathscr{I} \subset \mathbb{R}$ as

$$\underset{\xi\in\mathscr{I}}{\operatorname{disc}} f(\xi) = \lim_{\alpha\to 0^+} \left[f(\xi+i\alpha) - f(\xi-i\alpha) \right] , \quad \xi \in \mathscr{I} . \tag{4.3}$$

Note that the boundary-channel lacks a discontinuity along $\xi < -1$ if $\frac{\Delta_{12}^+}{2} - \hat{\Delta} \in \mathbb{Z}$, or equivalently

$$\hat{\Delta}_m = \frac{\Delta_{12}^+}{2} + m , \quad m \in \mathbb{Z} . \tag{4.4}$$

In the case of identical external scalars, $\Delta_1 = \Delta_2$, these operators correspond to the boundary-limit of normal derivatives acting on the scalar. These are then the operators that appear in the conformal block decomposition in a *generalized free field theory* (GFF): the free theory with the bulk correlator given by the massless limit of (2.54) with arbitrary Δ_ϕ. The BOE coefficients are then given by eq. (B.45) in [100].

Likewise, we can move the $\xi^{\frac{\Delta_{12}^+}{2}}$-factor to the bulk-channel, which then does not have a discontinuity along $-1 < \xi < 0$ assuming the exchanged bulk operators have the scaling dimensions

$$\Delta_n = \Delta_{12}^+ + 2n , \quad n \in \mathbb{Z} . \tag{4.5}$$

These operators correspond to scalar double traces: $\phi_1(\partial_\mu)^{2n}\phi_2$. In the case of $\Delta_1 = \Delta_2$, these are the operators that appear in the conformal block decomposition in a GFF, with the bulk OPE coefficients given by eq. (B.44) in [100].

In particular, if we assume exchanged operators of scaling dimensions (4.4, 4.5), we can completely remove one side of the bootstrap eq. (4.1) (in the free theory) by commuting the discontinuity (along either $\xi < -1$ or $-1 < \xi < 0$) with the infinite sums. Due to this wonderful analytical property, we will assume this spectrum of operators when we extract bulk OPE and BOE coefficients from the two bootstrap channels.

It is not fully known when we are allowed to commute the discontinuity with these infinite sums in the bootstrap eq. (4.1). In paper II (see its eq. (2.10)) this was discussed using the radial coordinates [129] for defect conformal blocks. In particular, the radial coordinate for the bulk-channel has a branch cut along $\xi < -1$ while the one for the boundary has one along $-1 < \xi < 0$. This means that it is important to double check the result against some known CFT data. E.g. in paper II and V the anomalous dimension of $\hat{\phi}$ and $\partial_\perp \hat{\phi}$ was found using the image symmetry of the conformal blocks (2.60). This gave results consistent with the older literature.

4.1.1 Discontinuity along $\xi < -1$

Firstly, let us study the discontinuity of the bulk blocks (4.2) along $\xi < -1$ assuming exchanged scalar double traces (4.5)

$$\begin{aligned}\underset{\xi<-1}{\operatorname{disc}}\,\mathscr{G}_{\text{bulk}}(\Delta_n;\xi) = a_n \xi^{\Delta_\phi^{(\mathrm{f})}+1-\frac{\Delta_n}{2}} \times \\ \times {}_2F_1(\Delta_\phi^{(\mathrm{f})}+1-\Delta_1-n, \Delta_\phi^{(\mathrm{f})}+1-\Delta_2-n, \Delta_\phi^{(\mathrm{f})}+2-\Delta_n, -\xi) + \\ + b_n \xi^{\frac{\Delta_n}{2}} {}_2F_1(\Delta_1+n, \Delta_2+n, \Delta_n-\Delta_\phi^{(\mathrm{f})}, -\xi)\,,\end{aligned} \tag{4.6}$$

where

$$\begin{aligned} a_n &= \frac{2\pi^2 i \csc[\pi(\Delta_\phi^{(\mathrm{f})}-\Delta_{12}^+)]\Gamma_{\Delta_n-\Delta_\phi^{(\mathrm{f})}}}{\Gamma_{\Delta_1+n}\Gamma_{\Delta_2+n}\Gamma_{\Delta_1+n-\Delta_\phi^{(\mathrm{f})}}\Gamma_{\Delta_2+n-\Delta_\phi^{(\mathrm{f})}}\Gamma_{\Delta_\phi^{(\mathrm{f})}+2-\Delta_n}}\,, \\ b_n &= e^{\pi i \Delta_{12}^+} - e^{2\pi i \Delta_\phi^{(\mathrm{f})}} - 2i\frac{\sin[\pi(\Delta_1+n-\Delta_\phi^{(\mathrm{f})})]\sin[\pi(\Delta_2+n-\Delta_\phi^{(\mathrm{f})})]}{\sin[\pi(\Delta_{12}^+-\Delta_\phi^{(\mathrm{f})})]}\,, \end{aligned} \tag{4.7}$$

and $\Delta_\phi^{(\mathrm{f})}$ is a constant, which is given by the scaling dimension (1.20) of a fundamental field (without anomalous dimension)

$$\Delta_\phi^{(\mathrm{f})} = \frac{d-2}{2}\,. \tag{4.8}$$

We wish to find an orthogonality relation for $\operatorname{disc}\mathscr{G}_{\text{bulk}}$, but this is difficult due to the two different ${}_2F_1$'s. However, the b_n-term vanishes in the case when we consider two free scalars with scaling dimension $\Delta_1 = \Delta_2 = \Delta_\phi^{(\mathrm{f})}$. In such case it reduces to the discontinuity studied in eq. (4.11) of paper II. Alternatively, this term vanishes when $\Delta_\phi^{(\mathrm{f})}$, Δ_1 and Δ_2 are all integers.

Let us consider one of these two scenarios, where we only need to focus on the a_n-term in $\operatorname{disc}\mathscr{G}_{\text{bulk}}$. In such case we can use the following ${}_2F_1$-identity

$${}_2F_1(a,b,c,z) = (1-z)^{-a}{}_2F_1\left(a, c-b, c, \frac{z}{z-1}\right)\,, \tag{4.9}$$

to rewrite the discontinuity as a Jacobi-polynomial

$$\underset{\xi<-1}{\operatorname{disc}} \mathcal{G}_{\text{bulk}}(\Delta_n;\xi) = c_n \frac{P^{(\Delta_\phi^{(f)},-\Delta_{12}^-)}_{\Delta_1+n-\Delta_\phi^{(f)}-1}(t)}{(1+t)^{\frac{\Delta_{12}^-}{2}}} \ , \tag{4.10}$$

with

$$c_n = -\frac{\pi\, i^{2\Delta_\phi^{(f)}-\Delta_n+1} 2^{\frac{\Delta_{12}^-}{2}+1} \Gamma_{\Delta_n-\Delta_\phi^{(f)}}}{\Gamma_{\Delta_1+n}\Gamma_{\Delta_2+n-\Delta_\phi^{(f)}}} \ , \quad t = -\frac{\xi+2}{\xi} \ . \tag{4.11}$$

Due to orthogonality of the Jacobi polynomial, the discontinuity (4.10) of the bulk block satisfies

$$\int_{-1}^{+1} dt\, \Xi_m(t) \underset{\xi<-1}{\operatorname{disc}} \mathcal{G}_{\text{bulk}}(\Delta_n;\xi) = d_m \delta_{mn} \ , \tag{4.12}$$

where

$$\begin{aligned} d_m &= -\frac{\pi\, i^{2\Delta_\phi^{(f)}-\Delta_m+1} 2^{\Delta_\phi^{(f)}-\frac{\Delta_{12}^-}{2}+2} \Gamma_{\Delta_m-\Delta_\phi^{(f)}+1}}{\Gamma_{\Delta_2+m}\Gamma_{\Delta_1+m-\Delta_\phi^{(f)}}} \ , \\ \Xi_m(t) &= \frac{(1-t)^{\Delta_\phi^{(f)}}}{(1+t)^{\frac{\Delta_{12}^-}{2}}} P^{(\Delta_\phi^{(f)},-\Delta_{12}^-)}_{\Delta_1+n-\Delta_\phi^{(f)}-1}(t) \ . \end{aligned} \tag{4.13}$$

This orthogonality relation is only valid if the coefficient, d_m, on the RHS of (4.12) (which originates from the integration measure) is non-zero and convergent. E.g. in the case of two identical scalars $\Delta_1 = \Delta_2 = \Delta_\phi^{(f)}$, the discontinuity is orthogonal if $n \geq 1$. In which case the ϕ^2-conformal block ($n = 0$) has a no branch cut along $\xi < -1$.

4.1.2 Discontinuity along $-1 < \xi < 0$

Let us now study the analytic structure of the boundary-channel, and consider the discontinuity of the boundary conformal block (4.2) along $-1 < \xi < 0$ assuming boundary operators of dimension (4.4) is exchanged. In such case, we find again two terms with hypergeometric functions. One term vanishes if Δ_{12}^+ and $\Delta_\phi^{(f)}$ are both integers.[1] Assuming this, we can again use the identity (4.9) to write the discontinuity as a Jacobi polynomial

$$\underset{-1<\xi<0}{\operatorname{disc}} \mathcal{G}_{\text{bdy}}(\hat{\Delta}_m;\xi) = \hat{c}_m \frac{P^{(\Delta_\phi^{(f)},-\Delta_\phi^{(f)})}_{\hat{\Delta}_m-\Delta_\phi^{(f)}-1}(t)}{(1+t)^{\Delta_\phi^{(f)}-\frac{\Delta_{12}^+}{2}}} \ , \tag{4.14}$$

[1] E.g. this occurs if $\Delta_1 = \Delta_2 = \Delta_\phi^{(f)}$ in even dimensions.

with

$$\hat{c}_m = -\frac{\pi i e^{\pi i(\Delta_\phi^{(\mathrm{f})}-m)} 2^{\Delta_\phi^{(\mathrm{f})}-\frac{\Delta_{12}^+}{2}+1} \Gamma_{2(\hat{\Delta}_m-\Delta_\phi^{(\mathrm{f})})}}{\Gamma_{\hat{\Delta}_m}\Gamma_{\hat{\Delta}_m-2\Delta_\phi^{(\mathrm{f})}}} . \tag{4.15}$$

The discontinuity (4.14) satisfy the orthogonality relation

$$\int_{-1}^{+1} dt\, \hat{\Xi}_n(t) \underset{-1<\xi<0}{\mathrm{disc}} \mathscr{G}_{\mathrm{bdy}}(\hat{\Delta}_m;\xi) = \hat{d}_n \delta_{mn} , \tag{4.16}$$

where

$$\begin{aligned} \hat{d}_n &= \frac{\sqrt{\pi}\, i\, 2^{\hat{\Delta}_n+n-\Delta_\phi^{(\mathrm{f})}} e^{\pi i(\Delta_\phi^{(\mathrm{f})}-n+1)} \Gamma_{\hat{\Delta}_n-\Delta^{(\mathrm{f})}-\frac{1}{2}}}{\Gamma_{\hat{\Delta}_n-\Delta_\phi^{(\mathrm{f})}}} , \\ \hat{\Xi}_n(t) &= \frac{(1-t)^{\Delta_\phi^{(\mathrm{f})}}}{(1+t)^{\frac{\Delta_{12}^+}{2}}} P^{(\Delta_\phi^{(\mathrm{f})},-\Delta_\phi^{(\mathrm{f})})}_{\hat{\Delta}_m-\Delta_\phi^{(\mathrm{f})}-1}(t) . \end{aligned} \tag{4.17}$$

Similar to the $\underset{\xi<-1}{\mathrm{disc}}\, \mathscr{G}_{\mathrm{bulk}}$, this relation is only valid if the coefficient, $\hat{d}_n$, is non-zero and convergent. In the case of $\Delta_1 = \Delta_2 = \Delta_\phi^{(\mathrm{f})}$, this is the case if $n \geq 2$. Then the conformal block for the $\hat{\phi}$ ($m=0$) and $\partial_\perp \hat{\phi}$ ($m=1$) exchange have no branch cut along $-1 < \xi < 0$.

4.2 OPE coefficients

We will see how the orthogonality relations (4.12, 4.14) can be used to project out the bulk OPE and BOE coefficients from the bootstrap eq. (4.1). The orthogonality relation only holds for free scaling dimensions (without taking into account the anomalous dimensions), and thus we first need to expand the CFT data around the free theory. To illustrate this we consider the ε-expansion, although in principle, the method work for other expansions as well, e.g. that around large N

$$\begin{aligned} \Delta_1 &= \Delta_1^{(\mathrm{f})} + \varepsilon\gamma_1 + \mathscr{O}(\varepsilon^2) , \\ \Delta_2 &= \Delta_2^{(\mathrm{f})} + \varepsilon\gamma_2 + \mathscr{O}(\varepsilon^2) , \end{aligned} \tag{4.18}$$

$$\begin{aligned} \Delta_n &= \Delta_n^{(\mathrm{f})} + \varepsilon\gamma_n + \mathscr{O}(\varepsilon^2) , \\ \hat{\Delta}_m &= \hat{\Delta}_m^{(f)} + \varepsilon\hat{\gamma}_m + \mathscr{O}(\varepsilon^2) , \end{aligned} \tag{4.19}$$

$$\begin{aligned} \lambda^{\mathscr{O}_1\mathscr{O}_2}{}_{\mathscr{O}_n}\mu^{\mathscr{O}_n}{}_{\mathbb{1}} &= \lambda_n^{(f)} + \varepsilon\,\delta\lambda_n + \mathscr{O}(\varepsilon^2) , \\ \mu^{\mathscr{O}_1}{}_{\hat{\mathscr{O}}_m}\mu^{\mathscr{O}_2}{}_{\hat{\mathscr{O}}_m} &= \mu_m^{(f)} + \varepsilon\,\delta\mu_m + \mathscr{O}(\varepsilon^2) . \end{aligned} \tag{4.20}$$

We will then write the bootstrap eq. (4.1) in the following way

$$f(\xi) = G_b + H_b = G_i + H_i + \mathcal{O}(\varepsilon^2) \ , \tag{4.21}$$

where b stands for bulk and i for interface (boundary). We let the anomalous dimensions be contained in G_b and G_i as well as the operators accompanying OPE coefficients of $\mathcal{O}(\varepsilon)$ which does not have a branch cut along $\xi < 0$ (such as $\hat{\phi}$, $\partial_\perp \hat{\phi}$ and ϕ^2)

$$\begin{aligned} G_b &= \sum_n \lambda_n^{(f)} \mathcal{G}_{\text{bulk}}(\Delta_n;\xi) + \varepsilon \sum_{n'} \delta\lambda_{n'} \mathcal{G}_{\text{bulk}}(\Delta_{n'}^{(f)};\xi) \ , \\ G_i &= \sum_m \mu_m^{(f)} \mathcal{G}_{\text{bdy}}(\hat{\Delta}_m;\xi) + \varepsilon \sum_{m'} \delta\mu_{m'} \mathcal{G}_{\text{bulk}}(\hat{\Delta}_{m'}^{(f)};\xi) \ . \end{aligned} \tag{4.22}$$

On the other hand, we let H_b and H_i contain the operators at $\mathcal{O}(\varepsilon)$ which have a non-trivial discontinuity along $\xi < -1$ and $-1 < \xi < 0$ respectively

$$\begin{aligned} H_b &= \varepsilon \sum_{\tilde{n}} \delta\lambda_{\tilde{n}} \mathcal{G}_{\text{bulk}}(\Delta_{\tilde{n}}^{(f)};\xi) \ , \\ H_i &= \varepsilon \sum_{\tilde{m}} \delta\mu_{\tilde{m}} \mathcal{G}_{\text{bulk}}(\hat{\Delta}_{\tilde{m}}^{(f)};\xi) \ . \end{aligned} \tag{4.23}$$

Note that the free scaling dimensions enter in H_b and H_i. If we now take the discontinuity along $\xi < -1$ or $-1 < \xi < 0$, and commute it with the infinite series in H_b and H_i we find

$$\begin{aligned} \varepsilon \sum_{\tilde{n}} \delta\lambda_{\tilde{n}} \operatorname*{disc}_{\xi<-1} \mathcal{G}_{\text{bulk}}(\Delta_n;\xi) &= \operatorname*{disc}_{\xi<-1} (G_i - G_b) \ , \\ \varepsilon \sum_{\tilde{m}} \delta\mu_{\tilde{m}} \operatorname*{disc}_{-1<\xi<0} \mathcal{G}_{\text{bdy}}(\hat{\Delta}_m;\xi) &= \operatorname*{disc}_{-1<\xi<0} (G_b - G_i) \ . \end{aligned} \tag{4.24}$$

By applying the orthogonality relations (4.12, 4.14) we find corrections to the OPE coefficients

$$\begin{aligned} \delta\lambda_{\tilde{n}} &= \frac{d_{\tilde{n}}}{\varepsilon} \int_{-1}^{+1} dt\, \Xi_{\tilde{n}}(t) \operatorname*{disc}_{\xi<-1} (G_i - G_b) \ , \\ \delta\mu_{\tilde{m}} &= \frac{\hat{d}_{\tilde{m}}}{\varepsilon} \int_{-1}^{+1} dt\, \hat{\Xi}_{\tilde{m}}(t) \operatorname*{disc}_{-1<\xi<0} (G_b - G_i) \ . \end{aligned} \tag{4.25}$$

If any of these formulas are used at $\mathcal{O}(\varepsilon^k)$ with $k \geq 2$, then $\varepsilon \to \varepsilon^k$ in above formulas. That said, at some order in the expansion parameters we expect other operators than normal derivatives (4.4) and scalar double traces (4.5) (or to be precise: operators with the same scaling dimensions at $\mathcal{O}(\varepsilon^0)$) to be exchanged. If so, above formulas are not valid anymore.

It is worth mentioning that operators with scaling dimensions (4.4, 4.5) will mix with other kinds of operators,[2] which means that above formulas actually

[2] E.g. $\phi_1(\partial_\mu^2)^n\phi_2$ will mix with $(\partial_\mu^2)^n(\phi_1\phi_2)$ as they both have the same scaling dimension (4.5) in the free theory.

give a sum of OPE coefficients. To go to higher orders, we have to solve the mixing problem by studying several different mixed bulk-bulk correlators. This is in general very difficult.

The first of these formulas (for the bulk OPE coefficients) is the one used in paper II and V (upto $\mathscr{O}(\varepsilon^2)$) as well as in [128], in the case of two identical external scalars $\Delta_1 = \Delta_2 = \Delta_\phi^{(\mathrm{f})}$. However, the second formula for the BOE coefficients has not been used in the literature before.

Though the BOE coefficients were still found in paper II and V by resumming the bulk OPE coefficients,[3] and then using the following orthogonality relation for the boundary blocks themselves (and not their discontinuity)

$$\begin{aligned}\delta_{mn} = \oint\limits_{|w|=\tilde{\varepsilon}} \frac{dw}{2\pi i} w^{n-m-1} {}_2F_1(1-m,-m-\tfrac{d-4}{2},2(1-m),-w)\times \\ \times {}_2F_1(n,n+\tfrac{d-2}{2},2n,-w)\ .\end{aligned} \tag{4.26}$$

Here we integrate over a small circle with infinitesimal radius $\tilde{\varepsilon} \ll 1$ s.t. we can apply the residue theorem to poles of the integrand at $w = \xi^{-1} = 0$. This orthogonality relation was first found in paper II.

All and all, the discontinuity method provides us with the OPE coefficients in terms of the anomalous dimensions. This is a rewarding resolution to the bootstrap eq. (2.22), although it requires calculating difficult infinite sums in G_b and G_i if there are infinitely many operators at the previous orders in the expansion parameters (see Sec. 4.5 in paper II).

4.2.1 BOE coefficients of the $\phi - \phi$ correlator

In the remaining part of this Chapter, we will find the BOE coefficients at $\mathscr{O}(\varepsilon^2)$ that appear in the $\phi - \phi$ correlator (when $\Delta_1 = \Delta_2 = \Delta_\phi^{(\mathrm{f})}$) in $d = 4 - \varepsilon$ using the second formula in (4.25). We will show that this results in an agreement with paper II.

To start with, the bootstrap eq. (4.1) is then solved by a finite number of operators in the free theory, and thus also at $\mathscr{O}(\varepsilon)$ as the BOE coefficients enter squared in the bootstrap equation [100]

$$\begin{aligned}G_b &= 1 \pm (1 \pm \alpha\varepsilon + \varepsilon^2 \delta\lambda_{\phi^2}) \mathscr{G}_{\mathrm{bulk}}(\Delta_{\phi^2}, \xi) + \varepsilon \frac{\alpha}{2} \mathscr{G}_{\mathrm{bulk}}(\Delta_{\phi^4}; \xi)\ , \\ G_i &= (1 \pm 1 + \varepsilon^2 \delta\mu_{\hat{\phi}}) \mathscr{G}_{\mathrm{bdy}}(\Delta_{\hat{\phi}}; \xi) + \\ &\quad + \left(\frac{1 \mp 1}{2} \Delta_\phi^{(\mathrm{f})} + \varepsilon\alpha + \varepsilon^2 \delta\mu_{\partial_\perp \hat{\phi}}\right) \mathscr{G}_{\mathrm{bdy}}(\Delta_{\partial_\perp \hat{\phi}}; \xi)\ .\end{aligned} \tag{4.27}$$

Here $\Delta_0 \equiv \Delta_{\phi^2}$, $\Delta_1 \equiv \Delta_{\phi^4}$, $\hat{\Delta}_0 = \hat{\Delta}_{\hat{\phi}}$, $\hat{\Delta}_1 \equiv \hat{\Delta}_{\partial_\perp \hat{\phi}}$ and α is a free parameter (that is related to the anomalous dimension of ϕ^2). Neumann b.c. correspond to $+1$

[3]Resumming conformal blocks is often difficult, and some methods to do so is in App. C of paper V.

and Dirichlet to -1. The expansion of these blocks are in App. A of paper II. The contributions from new OPE coefficients at $\mathscr{O}(\varepsilon^2)$ are

$$\begin{aligned}
H_b &= \varepsilon^2 \sum_{n\geq 1} \delta\lambda_n \mathscr{G}_{\text{bulk}}(2(n+1);\xi) \ , \\
H_i &= \varepsilon^2 \sum_{m\geq 2} \delta\mu_m \mathscr{G}_{\text{bulk}}(m+1;\xi) \ .
\end{aligned} \tag{4.28}$$

By taking the discontinuity along $-1<\xi<0$ of the difference $G_b - G_i$ we find the integrand of the BOE coefficients (4.25)

$$\begin{aligned}
\delta\mu_{m\geq 2} = \frac{\sqrt{\pi}\,\Gamma_m}{(-4)^m \Gamma_{m-\frac{1}{2}}} \int_{-1}^{+1} dt \left(a_1 \log\left(\frac{1+t}{2}\right) + \right. \\
\left. + a_2 \frac{1-t}{1+t} \log\left(\frac{1-t}{2}\right) + a_3 + a_4 \frac{1-t}{1+t} \right) P_{m-1}^{(+1,-1)}(t) \ ,
\end{aligned} \tag{4.29}$$

with the coefficients

$$\begin{aligned}
a_1 &= \pm\alpha(2\alpha - 1) \ , \\
a_2 &= \alpha(2\alpha - \gamma^{(1)}_{\phi^4} + 1) \ , \\
a_3 &= \alpha(2(3\pm 1)\alpha + 5 \mp 1) + \\
&\quad - 4(\alpha\gamma^{(1)}_{\phi^4} \pm 2\gamma^{(2)}_{\phi} \mp \gamma^{(2)}_{\phi^2} + (1\pm 1)\gamma^{(2)}_{\hat{\phi}} - (1\mp 1)\gamma^{(2)}_{\partial_\perp\hat{\phi}}) \ , \\
a_4 &= 2\gamma^{(2)}_{\phi} \ .
\end{aligned} \tag{4.30}$$

Performing the integration over t yields

$$\begin{aligned}
\delta\mu_{m\geq 2} = \frac{\sqrt{\pi}\,\Gamma_m}{2^{2m-1}(m-1)\Gamma_{m-\frac{1}{2}}} \left(\frac{(-1)^m(m^2-m-1)\alpha(2\alpha - \gamma^{(1)}_{\phi^4} + 1)}{m(m-1)} + \right. \\
\mp \frac{\alpha(2\alpha-1)}{m(m-1)} + 2(1\mp(-1)^m)\gamma^{(2)}_{\phi} + \\
\left. + (-1)^m(\pm\gamma^{(2)}_{\phi^2} + (1\pm 1)\hat{\gamma}^{(2)}_{\hat{\phi}} - (1\mp 1)\hat{\gamma}^{(2)}_{\partial_\perp\hat{\phi}}) \right) \ .
\end{aligned} \tag{4.31}$$

By inserting the anomalous dimensions (see Sec. 4.2 in paper II), this result is in agreement with the corresponding BOE coefficients found in eq. (4.35) of paper II.

5. Correlators from the equation of motion

In this Chapter we illustrate the method used in paper VII. We will consider a BCFT with both a bulk- and a boundary-interaction. Due to the conformal symmetry, the bulk-bulk correlator (2.20) is given by an unknown function $f(\xi)$, which depends on the single cross-ration ξ. By applying the bulk e.o.m. order by order in the couplings we are able to find $f(\xi)$ as an perturbative expansion in these couplings. At each order we will have undetermined constants, which are fixed by the b.c.'s.

This idea is not new in itself, and has been applied to a BCFT prior to this thesis in [130]. In this Chapter we will apply this method to the $\phi^6-\hat{\phi}^4$ model in $d=3-\varepsilon$ [131], governed by the action

$$S=\int_{\mathbb{R}^d_+} d^dx\left(\frac{(\partial_\mu\phi^i)^2}{2}+\frac{g_0}{48}(\phi^i)^6\right)+\int_{\mathbb{R}^{d-1}} d^{d-1}x_\parallel\frac{\lambda_0}{8}(\hat{\phi}^i)^4\,. \tag{5.1}$$

Here $\mathbb{R}^d_+=\{x_\parallel\in\mathbb{R}^{d-1},x_\perp>0\}$ and $i\in\{1,...,N\}$, giving this model $O(N)$-symmetry. This is an interesting BCFT as it has an RG f.p. where both the bulk- and the boundary-interactions are non-zero.[1] We are unaware of any results on the bulk $\phi-\phi$ correlator beyond the free theory. Using the e.o.m. we are able to find this correlator upto first order in g_0 and second order in λ_0 without much effort. Furthermore, we will study the boundary-limit of this correlator using the methods from paper III to read off the anomalous dimension of $\hat{\phi}$.

We will also comment on the operators being exchanged in the bulk and boundary bootstrap channels (4.1), and read off the non-trivial OPE coefficients. From this analysis we will see that we cannot naively use the discontinuity method from Ch. 4.

Lastly, we will apply the CW mechanism and flow along the RG away from the conformal f.p.'s. This gives us an effective potential on the boundary (only taking into account one-loop effects) for a first-order p.t., where a SSB of the $O(N)$-symmetry occurs. Due to the BOE (2.44), the bulk $O(N)$-symmetry is spontaneously broken as well. Applying the Higgs mechanism tells us that there exist massless Goldstone modes invariant under $O(N-1)$-transformations, and a Higgs mode with both a bulk and a boundary mass.

[1]Paper III studies this model at the RG f.p. where only the boundary coupling is non-zero.

5.1 Renormalization group flow

The $\phi^6 - \hat{\phi}^4$ model (5.1) has the e.o.m. and b.c.

$$\begin{aligned} \partial_\mu^2 \phi^i &= \frac{g_0}{8}\phi^4\phi^i \, , \\ \partial_\perp \hat{\phi}^i &= \frac{\lambda_0}{2}\hat{\phi}^2\hat{\phi}^i \, , \end{aligned} \tag{5.2}$$

where we have suppressed the summations over the $O(N)$-indices. The β-function for the bulk coupling is not affected by the boundary and thus it can be directly borrowed from the bulk ϕ^6-theory [132]

$$\beta_g = -2\varepsilon g + \frac{3N+22}{8\pi^2} g + \mathscr{O}(g^2) \, . \tag{5.3}$$

The boundary β-function is (upto two-loops) [131, 88]

$$\begin{aligned} \beta_\lambda = -\varepsilon\lambda - \frac{\pi(N+4)}{8\pi} g + \frac{N+8}{4\pi}\lambda^2 - \frac{(N+4)(N-62)}{64\pi^2}\lambda g + \\ - \frac{(5N+22)\log 2}{2\pi^2}\lambda^3 + \dots \, . \end{aligned} \tag{5.4}$$

Setting both of these β-functions to zero gives us three RG f.p.'s. Aside from the trivial Gaussian f.p.

$$g_G^*, \, \lambda_G^* = 0 \, , \tag{5.5}$$

there is also the *long-range f.p.* (LR) studied in paper III

$$g_G^* = 0 \, , \quad \lambda_{LR}^* = \frac{4\pi\varepsilon}{N+8} + \frac{32\pi(5N+22)\log 2}{(N+8)^3}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \, , \tag{5.6}$$

and two f.p.'s where both the bulk and boundary couplings are non-trivial

$$g^* = \frac{16\pi^2\varepsilon}{3N+22} + \mathscr{O}(\varepsilon^2) \, , \tag{5.7}$$

$$\begin{aligned} \lambda_\pm^* = \pm 2\pi\sqrt{\frac{2(N+4)\varepsilon}{(N+8)(3N+22)}} + \\ + \frac{2\pi}{N+8}\left(1 + \frac{(N+4)(N-62)}{4(3N+22)} + \frac{4(N+4)(5N+22)\log 2}{(N+8)(3N+22)}\right)\varepsilon + \mathscr{O}(\varepsilon^{3/2}) \, . \end{aligned}$$

We will mostly focus on the latter two f.p.'s. Here the bulk-interaction is at the *tricritical point*, i.e. the point in the phase diagram (pressure vs. temperature) where three phases coexist (e.g. solid, liquid and gas). Note that at these tricritical points, $\lambda_\pm^*$ admits an expansion in $\sqrt{\varepsilon}$, while g^* is expanded in ε. Moreover, there exist no f.p. where only the bulk-interaction is non-trivial. The RG flow is depicted in Fig. 5.1, and the tricritical f.p. with λ_+^* is the fully attractive one.

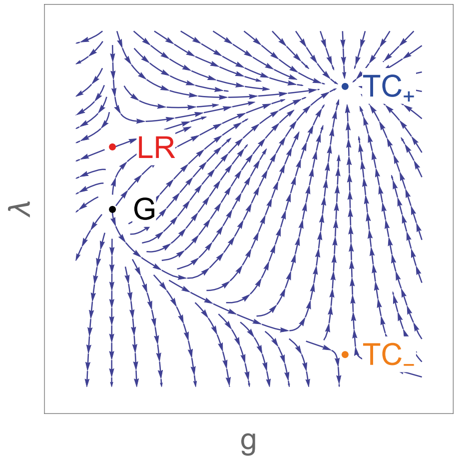

Figure 5.1. The RG flow of the $\phi^6 - \hat{\phi}^4$ model (5.1). The black dot (G) is the trivial Gaussian f.p. (5.5), the red one is the LR f.p. (5.6), the orange (TC_-) is the tricritical point (5.7) with λ_-^*, and the blue (TC_+) that with λ_+^*. The blue f.p. is fully attractive.

5.2 The correlator upto $\mathscr{O}(\sqrt{\varepsilon})$

Let us consider the tricritical f.p. (5.7) and find the bulk-bulk correlator using the e.o.m. and the b.c. (5.2). From conformal symmetry we know that it has to be on the form[2]

$$\langle \phi^i(x)\phi^j(y)\rangle = A_d \delta^{ij} \frac{F(\xi)}{|4x_\perp y_\perp|^{\Delta_\phi}} , \tag{5.8}$$

with the cross-ratio, ξ, given by (2.21)

$$\xi = \frac{s^2}{4x_\perp y_\perp} . \tag{5.9}$$

We wish to find $F(\xi)$ from (5.2). Let us first solve it upto $\mathscr{O}(\sqrt{\varepsilon})$

$$\partial_\mu^2 \langle \phi^i(x)\phi^j(y)\rangle = \mathscr{O}(\varepsilon) , \tag{5.10}$$

$$\langle \partial_\perp \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle = \frac{(N+2)\lambda_\pm^*}{2} \langle \hat{\phi}^2(x_\parallel)\rangle_{(f)} \langle \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle_{(f)} + \mathscr{O}(\varepsilon) = \mathscr{O}(\varepsilon) ,$$

where the subscript (f) means the correlator from the free theory. Note that $\langle \hat{\phi}^2(x_\parallel)\rangle$ is trivial due to the conformal symmetry along the boundary. At this order we are asked to solve the classical KG eq. (1.11) with Neumann b.c.

[2] If we compare with (2.20) we have that $F(\xi) = A_d \delta^{ij} \xi^{\Delta_\phi} f(\xi)$. We find it more convenient to work with $F(\xi)$ when we are solving the e.o.m. due to the lack of $s_\parallel$ in the prefactor of $\langle \phi^i(x)\phi^j(y)\rangle$. Of course, it does not matter which notation we use. The end result for $\langle \phi^i(x)\phi^j(y)\rangle$ is the same.

(2.49). The bulk e.o.m. gives us

$$\frac{A_d}{x_\perp^{\Delta_\phi+2}(4y_\perp)^{\Delta_\phi}}\Bigg(\xi(\xi+1)F''(\xi)+\Delta_\phi(\Delta_\phi+1)F(\xi)+ \\ +\left((\Delta_\phi+1)(2\xi+1)-\left(\Delta_\phi-\frac{d-2}{2}\right)\frac{x_\perp}{y_\perp}\right)F'(\xi)\Bigg)=0\,. \tag{5.11}$$

Since $F(\xi)$ should only depend on the cross-ratio, the $\frac{x_\perp}{y_\perp}$-term has to vanish. This puts a constraint on the scaling dimension of the bulk field

$$\Delta_\phi=\frac{d-2}{2}+\mathscr{O}(\varepsilon)\,. \tag{5.12}$$

This is indeed the correct result for a free bulk scalar (1.20), which should not get affected by the boundary coupling. The differential eq. (5.11) then yields

$$4\xi(\xi+1)F''(\xi)+2d(2\xi+1)F'(\xi)+d(d-2)F(\xi)=0\,, \tag{5.13}$$

which can be solved upto two constants A and B

$$F(\xi)=A\,\xi^{-\Delta_\phi}+B(\xi+1)^{\Delta_\phi}\,. \tag{5.14}$$

By applying the Neumann b.c. to (5.8) we can fix one of the constants

$$\frac{A_d\Delta_\phi y_\perp}{2(s_\parallel^2+y_\perp^2)^{\frac{d}{2}}}(A-B)=0 \quad\Rightarrow\quad B=A\,. \tag{5.15}$$

At this point we have

$$F(\xi)=A(\xi^{-\Delta_\phi}+(\xi+1)^{\Delta_\phi})\,. \tag{5.16}$$

Since this has the same form of a free scalar, its conformal block decomposition is also the same. That is, in the bulk-channel only the identity operator and ϕ^2 is exchanged, and in the boundary-channel only $\hat{\phi}$ is exchanged [100]

$$\begin{aligned}\lambda^{\phi\phi}{}_{\mathbb{1}}&=A+\mathscr{O}(\varepsilon)\,,\\ \lambda^{\phi\phi}{}_{\phi^2}\mu^{\phi^2}{}_{\mathbb{1}}&=A+\mathscr{O}(\varepsilon)\,,\\ (\mu^{\phi}{}_{\hat{\phi}})^2&=2A+\mathscr{O}(\varepsilon)\,.\end{aligned} \tag{5.17}$$

Moreover, it tells us that neither ϕ^2 nor $\hat{\phi}$ receives an anomalous dimension

$$\gamma_{\phi^2}=\mathscr{O}(\varepsilon)\,,\quad \gamma_{\hat{\phi}}=\mathscr{O}(\varepsilon)\,. \tag{5.18}$$

The constant A can be fixed by normalization, which we will chose to be

$$\lambda^{\phi\phi}{}_{\mathbb{1}}=1\,,\quad \text{(exactly)}\quad\Rightarrow\quad A=1\,, \tag{5.19}$$

which gives us

$$
\begin{aligned}
F(\xi) &= \xi^{-\Delta_\phi} + (\xi+1)^{\Delta_\phi} \ , \\
\lambda^{\phi\phi}{}_{\phi^2}\mu^{\phi^2}{}_{\mathbb{1}} &= 1 + \mathscr{O}(\varepsilon) \ , \\
(\mu^{\phi}{}_{\hat{\phi}})^2 &= 2 + \mathscr{O}(\varepsilon) \ .
\end{aligned}
\tag{5.20}
$$

This is the same correlator as that found in Sec. 2.5. Note that there are no non-trivial Feynman diagrams at $\mathscr{O}(\sqrt{\varepsilon})$ for the bulk-bulk correlator. Thus the corrections to the CFT data is expected to be trivial at $\mathscr{O}(\sqrt{\varepsilon})$.

5.3 The correlator at $\mathscr{O}(\varepsilon)$

At $\mathscr{O}(\varepsilon)$, the e.o.m. (5.2) becomes

$$
\begin{aligned}
\partial_\mu^2 \langle \phi^i(x)\phi^j(y)\rangle &= \frac{(N+4)(N+2)g^*}{8}\langle \phi^2(x)\rangle^2_{(f)}\langle \phi^i(x)\phi^j(y)\rangle_{(f)} + \mathscr{O}(\varepsilon^{3/2}) \ , \\
\langle \partial_\perp \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle &= \frac{\lambda^*_\pm}{2}\langle \hat{\phi}^2\hat{\phi}^i(x_\parallel)\phi^j(y)\rangle_\lambda + \mathscr{O}(\varepsilon^{3/2}) \ .
\end{aligned}
\tag{5.21}
$$

Here the subscript λ denotes the correlator at $\mathscr{O}(\lambda^*_\pm)$. In order to solve this differential equation we need $\langle \phi^2(x)\rangle_{(f)}$ and $\langle \hat{\phi}^2\hat{\phi}^i(x_\parallel)\phi^j(y)\rangle_\lambda$.

Firstly let us find $\langle \phi^2(x)\rangle_{(f)}$. It can be found from the $\phi\times\phi$ bulk OPE[3]

$$
\langle \phi^i(x)\phi^j(y)\rangle_{(f)} = A_d\delta^{ij}\sum_{\mathscr{O}}\frac{1}{|s|^{2\Delta_\phi}}C_{\Delta,l}(x)\langle \mathscr{O}(x)\rangle_{(f)} \ , \tag{5.22}
$$

where we rescaled the exchanged bulk operator as $\mathscr{O} \to \lambda^{\phi\phi}{}_{\mathscr{O}}\mathscr{O}$. We will compare this with the coincident-limit of the correlator (5.8, 5.20)

$$
\langle \phi^i(x)\phi^j(y)\rangle_{(f)} = A_d\delta^{ij}\left(\lim_{y\to x}\frac{1}{s^{\Delta_\phi}} + \frac{1}{|2x_\perp|^{2\Delta_\phi}}\right) \ , \tag{5.23}
$$

from which we can see that the first term in (5.22) corresponds to the identity exchange, and the second to the v.e.v. of ϕ^2

$$
\langle \phi^2(x)\rangle_{(f)} = A_d\frac{\mu^{\phi^2}{}_{\mathbb{1}}}{|2x_\perp|^{2\Delta_\phi}} \ , \quad \mu^{\phi^2}{}_{\mathbb{1}} = 1 \ . \tag{5.24}
$$

This is on the form (2.18) we expect from conformal symmetry, and is consistent with the CFT data in (5.20).[4]

[3]Remember that spinning operators have trivial one-point functions in the presence of a codimension one defect [89].
[4]We rescaled the one-point function with a factor $2^{-2\Delta_\phi}$ for simplicity.

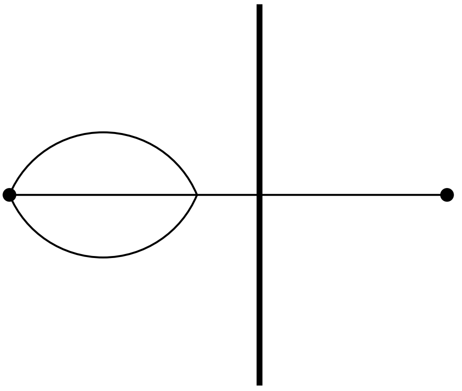

Figure 5.2. The diagram at $\mathscr{O}(\lambda)$ for the $\hat{\phi}^3$-ϕ correlator, where the LHS of the thick line represents the boundary, and the RHS the bulk.

Next up we need to find $\langle \hat{\phi}^2 \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle_\lambda$. It is given by the Feynman diagram in Fig. 5.2

$$\begin{aligned}\langle \hat{\phi}^2 \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle_\lambda &= -16(N+2)\lambda^*_\pm A_d \delta^{ij}\times \\ &\quad \times \int_{\mathbb{R}^{d-1}} d^{d-1}z_\parallel \frac{1}{|x_\parallel - z_\parallel|^{3\Delta_\phi}[(z_\parallel - y_\parallel)^2 + y_\perp^2]^{\Delta_\phi}} \\ &= -16(N+2)\lambda^*_\pm A_d \delta^{ij} J^{d-1}_{3\Delta_\phi,\Delta_\phi}(-s_\parallel, y_\perp^2)\ .\end{aligned} \tag{5.25}$$

In the second line we performed the shift $z_\parallel \to z_\parallel + x_\parallel$, and wrote it in terms of the following master integral

$$\begin{aligned}J^n_{a,b}(z,w^2) &\equiv \int_{\mathbb{R}^n} \frac{d^n x}{x^{2a}[(x-z)^2+w^2]^b} = \\ &= \frac{\Gamma_{a+b}}{\Gamma_a\Gamma_b}\int_0^1 du(1-u)^{a-1}u^{b-1} I^n_{a+b}(0,0,u(1-u)z^2+uw^2) \\ &= \frac{\pi^{\frac{n}{2}}\Gamma_{a+b-\frac{n}{2}}\Gamma_{\frac{n}{2}-a}}{\Gamma_b\Gamma_{\frac{n}{2}}(z^2+w^2)^{a+b-\frac{n}{2}}}\, {}_2F_1\left(a+b-\frac{n}{2}, n-\frac{a}{2}, \frac{n}{2}, \frac{z^2}{z^2+w^2}\right) \\ &= \frac{\pi^{\frac{n}{2}}\Gamma_{a+b-\frac{n}{2}}\Gamma_{\frac{n}{2}-a}\Gamma_{\frac{n}{2}-b}}{\Gamma_a\Gamma_b\Gamma_{n-a-b}} \frac{1}{|z|^{2(a+b)-n}}\ , \quad \text{if } w=0.\end{aligned} \tag{5.26}$$

It gives us

$$\begin{aligned}\langle \hat{\phi}^2 \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle_{\lambda^*_\pm} &= A_d \delta^{ij} \mu^\phi{}_{\hat{\phi}^3} \frac{y_\perp}{(s_\parallel^2 + y_\perp^2)^{\frac{3}{2}}} + \mathscr{O}(\varepsilon)\ , \\ \mu^\phi{}_{\hat{\phi}^3} &= \frac{(N+2)\lambda^*_\pm}{2\pi^2} + \mathscr{O}(\varepsilon)\ .\end{aligned} \tag{5.27}$$

This has the expected form (2.19) from conformal symmetry.

We have now everything we need to solve the e.o.m. at $\mathscr{O}(\varepsilon)$. We will consider the ε-expansion

$$\begin{aligned} \Delta_\phi &= \frac{d-2}{2} + \varepsilon\,\gamma_\phi + \mathscr{O}(\varepsilon^2)\;, \\ F(\xi) &= F_{(0)}(\xi) + \varepsilon\, F_{(1)}(\xi) + \mathscr{O}(\varepsilon^{\frac{3}{2}})\;, \end{aligned} \tag{5.28}$$

where $F_{(0)}(\xi)$ is the solution of $F(\xi)$ at (5.20) upto $\mathscr{O}(\sqrt{\varepsilon})$. By expanding the bulk e.o.m. (5.21) in ε we find a differential equation for $F_{(1)}(\xi)$

$$\begin{aligned} 0 = \frac{\delta^{ij}\varepsilon}{8\pi x_\perp^{\frac{5}{2}}\sqrt{y_\perp}} \Bigg\{ &\xi(\xi+1)F''_{(1)}(\xi) + \frac{3(2\xi+1)}{2}F'_{(1)}(\xi) + \frac{3}{4}F_{(1)}(\xi) + \\ &- \frac{1}{2}\left(\frac{(N+2)(N+4)g^*}{256\pi^2\varepsilon} - \gamma_\phi\right)\left(\frac{1}{\sqrt{\xi}} + \frac{1}{\sqrt{\xi+1}}\right) + \\ &- \gamma_\phi \left[\left(\frac{1}{\xi^{\frac{3}{2}}} - \frac{1}{(\xi+1)^{\frac{3}{2}}}\right) + \frac{x_\perp}{y_\perp}\left(\frac{1}{\xi^{\frac{3}{2}}} + \frac{1}{(\xi+1)^{\frac{3}{2}}}\right)\right]\Bigg\}\;. \end{aligned} \tag{5.29}$$

If we demand that $F_{(1)}(\xi)$ should only depend on the cross-ratio ξ, then we need to set the $\frac{x_\perp}{y_\perp}$-term to zero. It yields

$$\gamma_\phi = 0\;. \tag{5.30}$$

This agrees with the older literature [132]. It can also be seen by assuming that Δ_ϕ is not affected by the boundary coupling λ. Then in the case without a boundary there are no non-trivial Feynman diagrams for the $\phi - \phi$ correlator at $\mathscr{O}(g)$. Thus γ_ϕ has to be zero upto $\mathscr{O}(\varepsilon)$.

Above constraint brings (5.29) into

$$\begin{aligned} 0 = \xi(\xi+1)F''_{(1)}(\xi) &+ \frac{3(2\xi+1)}{2}F'_{(1)}(\xi) + \frac{3}{4}F_{(1)}(\xi) + \\ &- \frac{(N+2)(N+4)g^*}{512\pi^2\varepsilon}\left(\frac{1}{\sqrt{\xi}} + \frac{1}{\sqrt{\xi+1}}\right)\;, \end{aligned} \tag{5.31}$$

which has the solution

$$\begin{aligned} F_{(1)}(\xi) = &\frac{(N+2)(N+4)g^*}{256\pi^2\varepsilon}\left(\frac{1}{\sqrt{\xi}} + \frac{1}{\sqrt{\xi+1}}\right)\log(\sqrt{\xi}+\sqrt{\xi+1}) \\ &+ \frac{A}{\sqrt{\xi}} + \frac{B}{\sqrt{\xi+1}}\;. \end{aligned} \tag{5.32}$$

There will be a $x_\perp$-pole in the boundary-limit of $\partial_\perp\phi^i(x)$, and thus we cannot naively apply the b.c. (5.21). To understand the origin of this pole we study the BOE (2.8) of $\partial_\perp \phi^i$ (which is a descendant of ϕ^i in the bulk)

$$\begin{aligned}\langle \partial_\perp \phi^i(x)\phi^j(y)\rangle &= A_d \delta^{ij} \sum_{\hat{\mathscr{O}}} \partial_{x_\perp} \frac{\mu^\phi{}_{\hat{\mathscr{O}}}}{|x_\perp|^{\Delta_\phi - \hat{\Delta}}} B_{\hat{\Delta}}(x_\perp^2 \partial_{x_\parallel}^2)\langle \hat{\mathscr{O}}^i(x_\parallel)\phi^j(y)\rangle \\ &= A_d \delta^{ij} \sum_{\hat{\mathscr{O}}} \left(\frac{(\hat{\Delta}-\Delta_\phi)(\mu^\phi{}_{\hat{\mathscr{O}}})^2}{|x_\perp y_\perp|^{\Delta_\phi - \hat{\Delta}+1}(s_\parallel^2 + y_\perp^2)^{\hat{\Delta}}} + \mathscr{O}(x_\perp^{1-\Delta_\phi+\hat{\Delta}}) \right) .\end{aligned} \tag{5.33}$$

Here we plugged in the form (2.19) of the bulk-boundary correlator and expanded in $x_\perp$. To the lowest orders in $x_\perp$ the boundary fields $\hat{\phi}$ and $\partial_\perp \hat{\phi}$ are exchanged. In the ε-expansion of their CFT data we have

$$\begin{aligned} (\mu^\phi{}_{\hat{\phi}})^2 &= 2 + \mathscr{O}(\varepsilon) , & \Delta_{\hat{\phi}} &= \Delta_\phi^{(f)} + \varepsilon \gamma_{\hat{\phi}} + \mathscr{O}(\varepsilon^{\frac{3}{2}}) , \\ (\mu^\phi{}_{\partial_\perp \hat{\phi}})^2 &= \varepsilon (\mu^\phi{}_{\partial_\perp \hat{\phi}})^2_\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}}) , & \Delta_{\partial_\perp \hat{\phi}} &= \Delta_\phi^{(f)} + 1 + \mathscr{O}(\varepsilon) , \end{aligned} \tag{5.34}$$

where we remind the reader that $\Delta_\phi^{(f)}$ is given by (4.8). Plugging this into (5.33) yields

$$\begin{aligned}\langle \partial_\perp \phi^i(x)\phi^j(y)\rangle &= \frac{\varepsilon A_d \delta^{ij} \gamma_{\hat{\phi}}}{x_\perp \sqrt{s_\parallel^2 + y_\perp}} + \langle \partial_\perp \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle + \mathscr{O}(\varepsilon^{\frac{3}{2}}, x_\perp) , \\ \langle \partial_\perp \hat{\phi}^i(x_\parallel)\phi^j(y)\rangle &= \varepsilon A_d \delta^{ij} \frac{y_\perp}{(s_\parallel^2 + y_\perp)^{\frac{3}{2}}} (\mu^\phi{}_{\partial_\perp \hat{\phi}})^2_\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}}) .\end{aligned}$$

So from the pole in $x_\perp$ we can read off the anomalous dimension of $\hat{\phi}$, while the $x_\perp^0$-term should be matched with the b.c. in (5.21).

For the solution (5.32) this yields

$$\begin{aligned}\gamma_{\hat{\phi}} &= -\frac{(N+2)(N+4)g^*}{512\pi^2 \varepsilon} , \\ (\mu^\phi{}_{\partial_\perp \hat{\phi}})^2_\varepsilon &= A - B .\end{aligned} \tag{5.35}$$

The b.c. (5.21) with (5.27) now gives us

$$B = A - \frac{(N+2)(\lambda_\pm^*)^2}{4\pi^2 \varepsilon} . \tag{5.36}$$

Finally we normalize the theory according to (5.19)

$$A = 0 . \tag{5.37}$$

This gives us

$$\begin{aligned}F_{(1)}(\xi) = {}& \frac{(N+2)(N+4)g^*}{256\pi^2 \varepsilon} \left(\frac{1}{\sqrt{\xi}} + \frac{1}{\sqrt{\xi+1}} \right) \log(\sqrt{\xi} + \sqrt{\xi+1}) \\ & - \frac{(N+2)(\lambda_\pm^*)^2}{4\pi^2 \varepsilon \sqrt{\xi+1}} .\end{aligned} \tag{5.38}$$

Note that the $\lambda^*_\pm$-term breaks the image symmetry ($x_\perp \to -x_\perp$ or $\xi \to \xi+1$) from Sec. 2.5. We retrieve the same $\phi - \phi$ correlator as in paper III when $g^* \to 0$, $\lambda^*_\pm \to \lambda^*_{LR}$ and $\varepsilon \to \varepsilon^2$. Note that by using the e.o.m. we just had to calculate one tree-level Feynman diagram for the $\hat{\phi}^3 - \phi$ correlator at $\mathscr{O}(\sqrt{\varepsilon})$ rather than two two-loop $\phi - \phi$ diagrams at $\mathscr{O}(\varepsilon)$.

5.4 Conformal block decomposition

Let us now decompose (5.38) in the conformal blocks at (2.46), and read off the OPE coefficients. For this purpose we use the same notation as in Ch. 4

$$f(\xi) = \xi^{\Delta_\phi} F(\xi) , \tag{5.39}$$

which admits the conformal block decompositions at (4.1).

We consider the following ε-expansions

$$\begin{aligned} \lambda^{\phi\phi}{}_{\phi^{2n}} \mu^{\phi^{2n}}{}_{\mathbb{1}} &\equiv \delta_{n,1} + \varepsilon\, \delta\lambda_n + \mathscr{O}(\varepsilon^{\frac{3}{2}}) , \\ (\mu^{\phi}{}_{\partial_\perp^m \hat{\phi}})^2 &\equiv 2\,\delta_{m,0} + \varepsilon\, \delta\mu_m + \mathscr{O}(\varepsilon^{\frac{3}{2}}) . \end{aligned} \tag{5.40}$$

Starting with the bulk-channel, we note that the conformal blocks for ϕ^{2n}, $n \in \{1,2,3\}$, are given by

$$\begin{aligned} \mathscr{G}_{\text{bulk}}(\Delta_{\phi^2};\xi) &= \sqrt{\frac{\xi}{\xi+1}} - \frac{\varepsilon}{2}\left(\log\left(\frac{\xi}{\xi+1}\right) - \frac{\gamma_{\phi^2} g^*}{\varepsilon}\log\xi\right) + \mathscr{O}(\varepsilon^{\frac{3}{2}}) , \\ \mathscr{G}_{\text{bulk}}(\Delta_{\phi^4};\xi) &= \sqrt{\frac{\xi}{\xi+1}}\log(\sqrt{\xi}+\sqrt{\xi+1}) + \mathscr{O}(\sqrt{\varepsilon}) , \\ \mathscr{G}_{\text{bulk}}(\Delta_{\phi^6};\xi) &= 3\left(\log(\sqrt{\xi}+\sqrt{\xi+1}) - \sqrt{\frac{\xi}{\xi+1}}\right) + \mathscr{O}(\sqrt{\varepsilon}) . \end{aligned}$$

Since $F_{(1)}(\xi)$ in (5.38) has no $\log(\xi)$-term we immediately find

$$\gamma_{\phi^2} = \mathscr{O}(\varepsilon^{\frac{3}{2}}) , \tag{5.41}$$

which has to be the case as there are no non-trivial Feynman diagrams for the bulk $\phi^2 - \phi^2$ correlator at $\mathscr{O}(g^*)$ in the homogeneous theory without a boundary. This means that (5.38) is a linear combination of the three blocks above at $\mathscr{O}(\varepsilon^0)$

$$\sqrt{\xi}\, F_{(1)}(\xi) = \sum_{n=1}^{3} \delta\lambda_n \mathscr{G}_{\text{bulk}}(n;\xi) , \tag{5.42}$$

with the bulk OPE coefficients

$$\begin{aligned}
\delta\lambda_1 &= \frac{(N+4)(N+2)g^*}{256\pi^2\varepsilon} - \frac{(N+2)(\lambda_\pm^*)^2}{4\pi^2\varepsilon} \,, \\
\delta\lambda_2 &= \frac{(N+4)(N+2)g^*}{256\pi^2\varepsilon} \,, \\
\delta\lambda_3 &= \frac{(N+4)(N+2)g^*}{768\pi^2\varepsilon} \,.
\end{aligned} \tag{5.43}$$

Note that there is no mixing in the bulk-channel due to the e.o.m. (5.2).

To decompose in boundary blocks we need to look at the full $F(\xi)$ at (5.28). We can expand the conformal block from the free theory (for $\hat{\phi}$) in ε using the Mathematica package *HypExp* [133, 134]. By expanding around large ξ we can then read off the same anomalous dimension at (5.35) as well as the following BOE coefficents

$$\begin{aligned}
\delta\mu_0 &= \frac{(N+2)(N+4)g^*}{256\pi^2\varepsilon}\log 4 - \frac{(N+2)(\lambda_\pm^*)^2}{4\pi^2\varepsilon} \,, \\
\delta\mu_1 &= -\frac{(N+2)(\lambda_\pm^*)^2}{16\pi^2\varepsilon} \,.
\end{aligned} \tag{5.44}$$

Due to the b.c. (5.2) there is no mixing between boundary operators. Note that the $\partial_\perp\hat{\phi}$-exchange ($\delta\mu_1$) arises due to the boundary-interaction.

At the next order there are five Feynman diagrams for the $\phi-\phi$ correlator at $\mathscr{O}(\varepsilon^{\frac{3}{2}})$ (three at $\mathscr{O}(g\lambda)$ and two at $\mathscr{O}(\lambda^3)$). To find this correlator using the e.o.m. we need to calculate two $\phi^5-\phi$ diagrams at $\mathscr{O}(\sqrt{\varepsilon})$ and four $\hat{\phi}^3-\phi$ diagrams at $\mathscr{O}(\varepsilon)$ (two at $\mathscr{O}(g)$ and two at $\mathscr{O}(\lambda^2)$). To avoid the calculation of all of these (three-loop) Feynman diagrams we could instead try to bootstrap the theory using the discontinuity method from Ch. 4.

Since we want to preserve $O(N)$-symmetry, we expect the exchanged bulk primaries to contain an even amount of ϕ's (as we have already seen at $\mathscr{O}(\varepsilon)$). Schematically they would be on the form $\mathscr{O}_n \equiv \partial_\mu^{2n_1}\phi^{2n_2}$ with $n_1 \in \mathbb{Z}_{\geq 0}$, $n_2 \in \mathbb{Z}_{\geq 1}$ and $n \equiv 2n_1 + n_2$ (the exact location of the derivatives are not specified). At $\mathscr{O}(\varepsilon^0)$, the corresponding scaling dimensions are integers

$$\Delta_n^{(0)} = 2(n_2\Delta_\phi^{(0)} + n_1) = n \in \mathbb{Z}_{\geq 1} \,. \tag{5.45}$$

Following Ch. 4, we study the discontinuity along $\xi < -1$ of the bulk blocks hoping to find an orthogonality relation

$$\underset{\xi<-1}{\operatorname{disc}}\, \mathscr{G}_{\text{bulk}}(n;\xi) \propto P_{\frac{n-3}{2}}^{(\frac{1}{2},0)}(t) \,. \tag{5.46}$$

However, this Jacobi polynomial is not orthogonal since the argument $\frac{n-3}{2} \in \frac{\mathbb{Z}_{\geq -1}}{2}$ is not strictly an integer or a half-integer for $n \in \mathbb{Z}_{\geq 1}$.[5]

[5]We also to expressed this discontinuity in terms of other polynomials, e.g. Chebyshev or Gegenbauer, but we always found them to be non-orthogonal for the same reason.

On the other hand, we could try to project out BOE coefficients by studying the discontinuity along $\xi \in (-1,0)$. As we have already learned from Sec. 4.1.2, the discontinuity of the boundary block will contain two hypergeometric functions if normal derivatives in $d=3$ are exchanged (or operators with half-integer scaling dimensions: $\hat{\Delta}_m^{(0)} \in \mathbb{Z}_{\geq}0+\frac{1}{2}$). This makes it difficult to find an orthogonality relation for this discontinuity.[6]

5.5 CFT data upto $\mathscr{O}(\varepsilon)$

In this Section we present a summary of the CFT data found upto $\mathscr{O}(\varepsilon)$. Firstly, we found that the bulk fields ϕ and ϕ^2 does not recieve any anomalous dimensions at this order. Furthermore, we normalized the bulk OPE coefficient for the identity exchange to be

$$\lambda^{\phi\phi}{}_{\mathbb{1}} = 1\,, \quad \text{(exactly)}. \tag{5.47}$$

At the bulk tricritical point (5.7), $\hat{\phi}$ receives an anomalous dimension

$$\Delta_{\hat{\phi}} = \frac{1-\varepsilon}{2} - \frac{(N+2)(N+4)}{32(3N+22)}\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}})\,, \tag{5.48}$$

which is in agreement with [131]. In addition to this, we have the non-trivial OPE coefficients

$$\begin{aligned}
\lambda^{\phi\phi}{}_{\phi^2}\mu^{\phi^2}{}_{\mathbb{1}} &= 1 + \frac{(N-24)(N+2)(N+4)}{16(N+8)(3N+22)}\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}})\,,\\
\lambda^{\phi\phi}{}_{\phi^4}\mu^{\phi^4}{}_{\mathbb{1}} &= \frac{(N+2)(N+4)}{16(3N+22)}\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}})\,,\\
\lambda^{\phi\phi}{}_{\phi^6}\mu^{\phi^6}{}_{\mathbb{1}} &= \frac{(N+2)(N+4)}{48(3N+22)}\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}})\,,
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
(\mu^{\phi}{}_{\hat{\phi}})^2 &= 2 + \frac{(N+2)(N+4)}{3N+22}\left(\frac{\log 2}{8} - \frac{2}{N+8}\right)\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}})\,,\\
(\mu^{\phi}{}_{\partial_\perp\hat{\phi}})^2 &= \frac{(N+2)(N+4)}{2(N+8)(3N+22)}\varepsilon + \mathscr{O}(\varepsilon^{\frac{3}{2}})\,.
\end{aligned} \tag{5.50}$$

At the LR f.p. (5.6), when $g^* = 0$, $\hat{\phi}$ does not receive any anomalous dimension upto $\mathscr{O}(\varepsilon^2)$. This is in agreement with paper III. At this f.p. only ϕ^2 is

[6] We encounter the same problem with multiple ${}_2F_1$'s if we assume exchanged boundary operators with scaling dimensions $\hat{\Delta}_m^{(0)} \in \frac{\mathbb{Z}_{\geq}1}{2}$ or $\hat{\Delta}_m^{(0)} \in \mathbb{Z}_{\geq}1$.

exchanged in the bulk-channel, and the non-trivial OPE coefficients are

$$\begin{aligned}
\lambda^{\phi\phi}{}_{\phi^2}\mu^{\phi^2}{}_{\mathbb{1}} &= 1 - \frac{4(N+2)}{(N+8)^2}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ , \\
(\mu^{\phi}{}_{\hat{\phi}})^2 &= 2 - \frac{4(N+2)}{(N+8)^2}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ , \\
(\mu^{\phi}{}_{\partial_\perp\hat{\phi}})^2 &= \frac{(N+2)}{(N+8)^2}\varepsilon^2 + \mathscr{O}(\varepsilon^3) \ .
\end{aligned} \tag{5.51}$$

5.6 Coleman-Weinberg Mechanism

Finally let us study the CW mechanism applied to the $\phi^6 - \hat{\phi}^4$ model (5.1) and flow along the RG away from the conformal f.p.'s. If we vary the field

$$\phi^i = \phi^i_{cl} + \hbar\,\delta\phi^i + \mathscr{O}(\hbar^2) \ , \tag{5.52}$$

we find (in addition to the e.o.m. and b.c. at (5.2))

$$\begin{aligned}
S[\phi_{cl},\delta\phi] &= S[\phi_{cl}] + \hbar^2\delta S[\phi_{cl},\delta\phi] + \mathscr{O}(\hbar^3) \ , \\
\delta S[\phi_{cl},\delta\phi] &= \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu\delta\phi^i)^2}{2} + \delta V(\phi_{cl},\delta\phi)\right) + \int_{\mathbb{R}^{d-1}} d^{d-1}x\,\delta\hat{V}(\hat{\phi}_{cl},\delta\hat{\phi}) \\
&= \int_{\mathbb{R}^d_+} \frac{d^d x}{2}\delta\phi^i(-\partial^2|_{b.c.})^{ij}\delta\phi^j \ .
\end{aligned}$$

From this point on, we set $\hbar = 1$. The potential terms are given by

$$\delta V(\phi^2_{cl},\delta\phi^2) = \frac{\delta\phi^i(m^2)^{ij}\delta\phi^j}{2} \ , \quad \delta\hat{V}(\hat{\phi}_{cl},\delta\hat{\phi}) = \frac{\delta\hat{\phi}^i\hat{m}^{ij}\delta\hat{\phi}^j}{2} \ , \tag{5.53}$$

with the bulk and boundary masses for $\delta\phi$ (keeping ϕ_{cl} constant)

$$\begin{aligned}
(m^2)^{ij} &= m_1^2\delta^{ij} + a\,\phi^i_{cl}\phi^j_{cl} \ , \qquad m_1^2 = \frac{g\,\phi^4_{cl}}{8} \ , \qquad a = \frac{g\,\phi^2_{cl}}{2} \ , \\
\hat{m}^{ij} &= \hat{m}_1\delta^{ij} + \hat{a}\,\hat{\phi}^i_{cl}\hat{\phi}^j_{cl} \ , \qquad \hat{m}_1 = \frac{\lambda\,\hat{\phi}^2_{cl}}{2} \ , \qquad \hat{a} = \lambda \ .
\end{aligned} \tag{5.54}$$

This brings the differential operator $(-\partial^2|_{b.c.})^{ij}$ to the form

$$\begin{aligned}
(-\partial^2|_{b.c.})^{ij} &= -\delta^{ij}\partial^2 + (m^2)^{ij} + \delta(x_\perp)\text{b.c.}^{ij} \ , \\
\text{b.c.}^{ij} &= -\delta^{ij}\partial_\perp + \hat{m}^{ij} \ .
\end{aligned} \tag{5.55}$$

Following Sec. 2.4 we write the b.c. as a dirac δ-function when we path integrate out $\delta\phi$. In the effective potentials for ϕ_{cl} we find

$$\begin{aligned}
V_{\text{eff}}(\phi_{cl}) &\ni -\int_{\mathbb{R}^d}\frac{d^d k}{(2\pi)^d}\frac{\text{tr}_{O(N)}\log G^{ij}(k)}{2} \ , \\
\hat{V}_{\text{eff}}(\hat{\phi}_{cl}) &\ni -\int_{\mathbb{R}^{d-1}}\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\frac{\text{tr}_{O(N)}\log G^{ij}_{b.c.}(k_\parallel,0,0)}{2} \ ,
\end{aligned} \tag{5.56}$$

where the trace runs over the $O(N)$-indices. The momentum propagator in the homogeneous theory is given by

$$G^{ij}(k) = \frac{\delta^{jk}}{k^2+m_1^2} - \frac{a\,\phi_{cl}^i\phi_{cl}^j}{(k^2+m_1^2)(k^2+m_2^2)}\ , \tag{5.57}$$

and that satisfying the b.c. is (see App. A in paper IV for details on this)[7]

$$\begin{aligned}
G_{b.c.}^{ij}(k_\parallel,0,0) &= H_{m_1}^{ij}(k_\parallel) + a\,\phi_{cl}^i\phi_{cl}^k\,\frac{H_{m_1}^{kj}(k_\parallel)-H_{m_2}^{kj}(k_\parallel)}{m_1^2-m_2^2}\ ,\\
H_m^{ij}(k_\parallel) &= \left(\hat{m}^{ij}+\sqrt{k_\parallel^2+m^2}\,\delta^{ij}\right)^{-1}\\
&= \frac{1}{\sqrt{k_\parallel^2+m^2}+\hat{m}_1}\left(\delta^{ij}-\frac{\hat{a}\,\hat{\phi}_{cl}^i\hat{\phi}_{cl}^j}{\sqrt{k_\parallel^2+m^2}+\hat{m}_2}\right)\ .
\end{aligned} \tag{5.58}$$

Here we defined

$$m_2^2 \equiv m_1^2 + a\,\phi_{cl}^2 = \frac{5\,g\,\phi_{cl}^4}{8}\ , \quad \hat{m}_2 \equiv \hat{m}_1 + \hat{a}\,\hat{\phi}_{cl}^2 = \frac{3\,\lambda\,\hat{\phi}_{cl}^2}{2}\ . \tag{5.59}$$

To find the logarithms in (5.56) we use

$$\begin{aligned}
\log(a\,\mathbb{1}) &= \mathbb{1}\log(a)\ ,\\
\log(\mathbb{1}+b|\phi_{cl}\rangle\langle\phi_{cl}|) &= \sum_{n\geq 1}\frac{(-1)^{n+1}b^n}{n}\phi^{2(n-1)}|\phi_{cl}\rangle\langle\phi_{cl}|\\
&= \frac{|\phi_{cl}\rangle\langle\phi_{cl}|}{\phi_{cl}^2}\log(1+b\,\phi_{cl}^2)\ .
\end{aligned} \tag{5.60}$$

The logarithm of the bulk propagator (5.57) is thus

$$\begin{aligned}
\log G^{ij}(k) &= \log\left(\frac{\delta^{jk}}{k^2+m^2}\right) + \log\left(\delta^{jk} - \frac{a\phi^j\phi^k}{k^2+m_2^2}\right)\\
&= -\delta^{jk}\log(k^2+m_1^2) + \frac{\phi_{cl}^j\phi_{cl}^k}{\phi^2}\log\left(1-\frac{a\phi_{cl}^2}{k^2+m_2^2}\right)\\
&= -\left(\delta^{jk}-\frac{\phi_{cl}^j\phi_{cl}^k}{\phi^2}\right)\log(k^2+m_1^2) - \frac{\phi^j\phi^k}{\phi^2}\log(k^2+m_2^2)\ ,
\end{aligned} \tag{5.61}$$

giving us the trace in (5.56)

$$\mathrm{tr}_{O(N)}\log G^{ij}(k) = -(N-1)\log(k^2+m_1^2) - \log(k^2+m_2^2)\ . \tag{5.62}$$

---

[7] H^{ij} is found by making the ansatz $H^{ij} = \hat{b}\,\delta^{ij} + \hat{c}\,\hat{\phi}_{cl}^i\hat{\phi}_{cl}^j$, and then finding the coefficients $\hat{b}$, $\hat{c}$ from $(H^{-1})^{ij}H^{jk} = \delta^{ik}$.

Likewise, the trace of the boundary propagator (5.58) is

$$\mathrm{tr}_{O(N)} \log G^{ij}_{b.c.}(k_\parallel, 0, 0) = -(N-1)\log(\hat{m}_1 + \sqrt{k_\parallel^2 + m_1^2}) - \log(\hat{m}_2 + \sqrt{k_\parallel^2 + m_2^2}) \, .$$

All and all, it allows us to express the effective potentials in (5.56) in terms of the master integral (1.88, 1.90) (upto one-loop)

$$V_{\mathrm{eff}}(\phi_{cl}) = V(\phi_{cl}) + \frac{(N-1)P_d(m_1^2, 0) + P_d(m_2^2, 0)}{4} + V_{c.t.}(\phi_{cl}) + \dots \, , \tag{5.63}$$
$$\hat{V}_{\mathrm{eff}}(\hat{\phi}_{cl}) = \hat{V}(\hat{\phi}_{cl}) + \frac{(N-1)P_{d-1}(m_1^2, \hat{m}_1) + P_{d-1}(m_2^2, \hat{m}_2)}{2} + \hat{V}_{c.t.}(\hat{\phi}_{cl}) + \dots \, .$$

The ε-expansion of the master integrals in $d = 3 - \varepsilon$ is

$$P_d(m^2, 0) = -\frac{|m|^3}{12\pi} + \frac{m^2\Lambda}{4\pi^2} + \frac{\Lambda^3}{6\pi^2}\left(\log\Lambda - \frac{1}{3}\right) + \dots \, , \tag{5.64}$$
$$P_{d-1}(m^2, \hat{m}) = \frac{\hat{m}^2 - m^2}{8\pi}\log\left(\frac{m^2}{\Lambda^2}\right) + \frac{m^2}{8\pi} - \frac{\hat{m}|m|}{2\pi} + \frac{\Lambda^2}{2\pi}\left(\log\Lambda - \frac{1}{2}\right) + \dots \, .$$

The lack of a $\log(m^2)$-term in the bulk is due to the the bulk coupling, g, having no non-trivial RG f.p. at one-loop. In particular this means that the bulk potential will stay the same.

The classical potentials are

$$V(\phi_{cl}) = \frac{g}{48}\phi_{cl}^6 \, , \quad \hat{V}(\hat{\phi}_{cl}) = \frac{\lambda}{8}\hat{\phi}_{cl}^4 \, . \tag{5.65}$$

These satisfy

$$\begin{aligned} \left.\frac{\partial^2 V}{\partial(\phi_{cl}^2)^2}(\phi_{cl})\right|_{\phi_{cl}^i=0} &= 0 \, , & \left.\frac{\partial^3 V}{\partial(\phi_{cl}^2)^3}(\phi_{cl})\right|_{\phi_{cl}^i=0} &= \frac{g}{8} \, , \\ \left.\frac{\partial \hat{V}}{\partial(\hat{\phi}_{cl}^2)}(\phi_{cl})\right|_{\hat{\phi}_{cl}^i=0} &= 0 \, , & \left.\frac{\partial^2 \hat{V}}{\partial(\phi_{cl}^2)^2}(\phi_{cl})\right|_{\hat{\phi}_{cl}^i=0} &= \frac{\lambda}{4} \, . \end{aligned} \tag{5.66}$$

Based on these (and using the BOE (2.44)), we introduce the RG scale through

$$\begin{aligned} \left.\frac{\partial^2 V_{\mathrm{eff}}}{\partial(\phi_{cl}^2)^2}\right|_{\phi_{cl}^i=0} &= 0 \, , & \left.\frac{\partial^3 V_{\mathrm{eff}}}{\partial(\phi_{cl}^2)^3}\right|_{\phi_{cl}^i=\mu^{\Delta_{\hat{\phi}}} x_\perp^{\gamma_{\hat{\phi}}} \delta^{iN}+\dots} &= \frac{g}{8} \, , \\ \left.\frac{\partial \hat{V}_{\mathrm{eff}}}{\partial(\hat{\phi}_{cl}^2)}\right|_{\hat{\phi}_{cl}^i=0} &= 0 \, , & \left.\frac{\partial^2 \hat{V}_{\mathrm{eff}}}{\partial(\phi_{cl}^2)^2}\right|_{\hat{\phi}_{cl}^i=\mu^{\Delta_{\hat{\phi}}}\delta^{iN}} &= \frac{\lambda}{4} \, , \end{aligned} \tag{5.67}$$

where $\gamma_{\hat{\phi}}$ is the boundary anomalous dimension (5.35) (with the coupling constant not tuned to the RG f.p.).

We define the counter-terms in (5.63) as

$$V_{c.t.} = A\,\phi_{cl}^4 + \frac{B\,\phi_{cl}^6}{48}\,, \quad \hat{V}_{c.t.} \quad = \hat{A}\,\phi_{cl}^2 + \frac{\hat{B}\,\hat{\phi}_{cl}^4}{8}\,. \tag{5.68}$$

By implementing (5.67) on the effective potentials (5.63) (together with (5.64, 5.68)) we are able to tune the constants A, B, $\hat{A}$ and $\hat{B}$ s.t. the divergences in Λ vanish. In particular, we find that $\hat{B}$ contains

$$\hat{B} \ni \left((N+8)\lambda^2 - \frac{(N+4)g}{2}\right)\frac{\log\Lambda}{2}\,. \tag{5.69}$$

This is in fact the divergent part of the bare boundary coupling. Using the technology from Sec. 1.2.2 (under the exchange $\log\Lambda \to \varepsilon^{-1}$) we find exactly the same β-functions (5.4) (upto one-loop).

It brings the effective potential to

$$V_{\text{eff}}(\phi_{cl}) = V(\phi_{cl}) + \dots\,, \tag{5.70}$$

$$\hat{V}_{\text{eff}}(\hat{\phi}_{cl}) = \frac{\lambda\,\hat{\phi}_{cl}^4}{8} + \left((N+8)\lambda^2 - \frac{(N+4)g}{2}\right)\frac{\hat{\phi}_{cl}^4}{2}\left(\log\left(\frac{\hat{\phi}_{cl}^2}{\mu}\right) - \frac{3}{2}\right) + \dots\,.$$

The way we introduced the RG scale, μ, in (5.67) tells us that $\hat{V}_{\text{eff}}$ has a minimum at this point

$$\hat{\phi}_{cl}^i\frac{\partial\hat{V}_{\text{eff}}}{\partial\hat{\phi}_{cl}^i}\Bigg|_{\hat{\phi}_{cl}^2=\mu+\mathscr{O}(\varepsilon)} = 0 \quad \Rightarrow \quad g = -\frac{8\pi\lambda}{N+4} + \frac{2(N+8)\lambda^2}{N+4}\,. \tag{5.71}$$

This relation between the coupling constants does not flow to any of the f.p.'s in the RG flow in Fig. 5.1. Inserting this into $\hat{V}_{\text{eff}}$ yields

$$\hat{V}_{\text{eff}} = \frac{\lambda\,\hat{\phi}_{cl}^4}{8}\left(\log\left(\frac{\hat{\phi}_{cl}^2}{\mu}\right) - \frac{1}{2}\right) + \dots\,. \tag{5.72}$$

A plot of the boundary effective potential for $N=2$ is in Fig. 5.3, where we can see that it has an $O(N)$-invariant minima at $\sqrt{\mu}$ as in (2.43). This means that $\hat{\phi}_{cl}$ has received a non-trivial v.e.v.

$$\langle\hat{\phi}_{cl}^i\rangle = \mu^{\Delta_{\hat{\phi}}}\delta^{iN} = \sqrt{\mu}\,\delta^{iN} + \mathscr{O}(\varepsilon,\lambda)\,. \tag{5.73}$$

As was explained in Sec. 2.4 the BOE will in turn induce a v.e.v. in the bulk (2.44) (to lowest order in $x_\perp$)

$$\langle\phi_{cl}^i\rangle = \mu^{\Delta_{\hat{\phi}}}x_\perp^{\gamma_{\hat{\phi}}} + \dots = \sqrt{\mu}\,\delta^{iN} + \mathscr{O}(\varepsilon,\lambda,x_\perp)\,. \tag{5.74}$$

This means that a SSB of the $O(N)$-symmetry occurs, leaving us with $O(N-1)$-symmetry. We can thus apply the Higgs mechanism and expand around this minimum

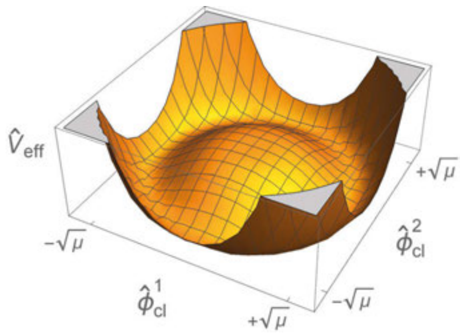

Figure 5.3. The effective potential, $\hat{V}_{\text{eff}}(\hat{\phi}_{cl})$, on the boundary for $N = 2$.

$$\begin{aligned} \phi^i_{cl} &= e^{\eta^k T^k}(\sqrt{\mu}+\sigma)\delta^{iN} + \mathscr{O}(\varepsilon,\lambda,x_\perp)\ , \\ \hat{\phi}^i_{cl} &= e^{\hat{\eta}^k T^k}(\sqrt{\mu}+\hat{\sigma})\delta^{iN} + \mathscr{O}(\varepsilon,\lambda)\ . \end{aligned} \tag{5.75}$$

Here σ is the Higgs mode, η^k is the massless Goldstone mode and T^k is a generator of the Lie algebra corresponding to $O(N)/O(N-1)$, with $k \in \{1,\dots,N-1\}$ due to (2.13). Expanding around the v.e.v. (5.75) yields kinetic terms to η^k and σ in the bulk

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \sigma)^2}{2} + \frac{(\sqrt{\mu}+\sigma)^2(\partial_\mu \eta^k)^2}{2} + V_{\text{eff}} \right) + \int_{\mathbb{R}^{d-1}} d^{d-1}x_\parallel \hat{V}_{\text{eff}}\ .$$

In the low-energy limit ($k_\mu \to 0$) of η^k, its mixed interactions with σ vanish and thus becomes free. If we expand the effective potentials in σ we find interactions which break the $O(N)$-symmetry down to $O(N-1)$ both in the bulk and on the boundary

$$V_{\text{eff}} = \frac{g}{48}\sum_{n=1}^{6}\binom{6}{n}\mu^{3-\frac{n}{2}}\sigma^n\ , \tag{5.76}$$

$$\hat{V}_{\text{eff}} = \frac{\lambda\,\mu}{2}\sigma^2 + \frac{5\lambda\sqrt{\mu}}{6}\sigma^3 + \frac{11\lambda}{24}\sigma^4 - 6\lambda\sum_{n\geq 5}\frac{(-1)^n(n+1)_{-5}}{\mu^{\frac{n}{2}-2}}\sigma^n\ ,$$

where we neglected constant terms. This is an effective field theory for a first-order p.t. The Higgs mode has received the bulk and boundary masses (using the relation (5.71) between the coupling constants)

$$m_\sigma^2 = \frac{5\,g\,\mu^2}{8} = -\frac{5\,\pi\,\lambda\,\mu^2}{N+4}\left(1 - \frac{N+8}{4\pi}\lambda\right)\ , \quad \hat{m}_\sigma = \lambda\,\mu\ . \tag{5.77}$$

For the fields to be physical (with positive energy) we require $m_\sigma^2 > 0$. This leads to $\lambda < 0$ (and $g > 0$ due to (5.71)) if we consider infinitesimal couplings, $\lambda \ll 1$, and finite values of $N \in \mathbb{Z}_{\geq 1}$.

6. The $O(N)$-flavoured replica twist defect

Here we will apply the methods in paper I (which concerns a monodromy twist defect) to a replica twist defect in an $O(N)$-model. By studying the monodromy of the defect we will show how the $O(N)$-symmetry is broken in the same way as in paper I. The difference lies in the $SO(2)$-charges of the defect-local fields, which now depend on the number of replicas, n.

In addition to this we apply the bulk e.o.m. to the DOE (2.74) to extract the anomalous dimensions of defect-local fields. This idea was first developed in [135] where it was applied to the OPE in a homogeneous CFT. It was generalized to a codimension one defect in a free scalar theory in [27], and later to a boundary in an interacting theory in [130] (see also Sec. 3.2 in paper V). For a monodromy defect this method has been applied in [136] for a $\mathbb{Z}_2$-twist, and in paper I for an $O(N)$-twist. This method was generalized in [137, 138] for codimension one and two defects (respectively) to extract the anomalous dimension of defect-local tensor operators. In this Chapter we apply it to the $O(N)$-flavoured replica twist defect in a CFT near four dimensions.

6.1 Monodromy of replica twist defects

As we saw in Sec. 2.7, codimension two defects are special in the sense that they can carry a monodromy action for the bulk-local fields. We will consider a replica twist defect in an $O(N)$-model, where there are n replicas of the bulk theory. Its monodromy action is given by (2.89)

$$\phi_a^i(x_\parallel, r, \theta + 2\pi n) = g^{ij}\phi_a^j(x_\parallel, r, \theta)\ , \quad g^{ij} \in O(N)\ . \tag{6.1}$$

By conjugation, a general $O(N)$-element is given by

$$(g^{ij}) = \mathrm{diag}(R_1^\pm, ..., R_p^\pm, \pm 1)\ , \tag{6.2}$$

where the last $O(1) = \mathbb{Z}_2$-element is not present if N is even, and $R_\alpha^\pm$, $\alpha \in \{1, ..., p\}$, is a general $O(2)$-matrix characterized by an angle ϑ_a

$$R_\alpha^\pm = \begin{pmatrix} \pm\cos\vartheta_\alpha & \mp\sin\vartheta_\alpha \\ \sin\vartheta_\alpha & \cos\vartheta_\alpha \end{pmatrix}\ , \quad \vartheta_\alpha \in [0, 2\pi)\ . \tag{6.3}$$

Here $R_\alpha^+ \in SO(2) \subset O(2)$ is a proper rotation (with determinant one), and $R_\alpha^- \in O(2)$ an improper one (with determinant minus one). We allow each element in the twist (6.2) to differ in this aspect ($\pm$).

To understand how the global $O(N)$-symmetry is broken along the defect we make use of the DOE (2.74)

$$\phi^j(x) = \sum_s \mu^{\phi^j}{}_{\hat{\mathcal{O}}_s} \frac{e^{is\theta}}{r^{\Delta_\phi - \hat{\Delta}_s}} \hat{C}_2(r^2\partial_\parallel^2)\hat{\mathcal{O}}_s(x_\parallel) \, , \tag{6.4}$$

where we have characterized the defect-local fields by their $SO(2)$-charge s. The idea is to study which values of s is valid for the monodromy (6.1) to hold. From this we can find the subgroups of $O(N)$ under which defect-local fields might be charged under.

Due to the form of the twist (6.2) we need to study the cases $R_1^{\pm}$ and ± 1 separately. We will start with the latter

$$\phi_a^N(x_\parallel, r, \theta + 2\pi n) = \pm \phi_a^N(x_\parallel, r, \theta) \, . \tag{6.5}$$

If we now apply the DOE (6.4) to both sides of this equation we find

$$e^{2\pi i n s} = \pm 1 \quad \Rightarrow \quad s = \frac{m_\pm}{2n} \in \mathbb{Q} \, . \tag{6.6}$$

Here $m_\pm$ is even/odd for ± 1 respectively.

Let us now proceed with $R_1^{\pm}$, where we find the system of equations

$$\begin{cases} \phi_a^1(x_\parallel, r, \theta + 2\pi n) = \pm \cos\vartheta_1 \phi_a^1(x_\parallel, r, \theta) \mp \sin\vartheta_1 \phi_a^2(x_\parallel, r, \theta) \, , \\ \phi_a^2(x_\parallel, r, \theta + 2\pi n) = \sin\vartheta_1 \phi_a^1(x_\parallel, r, \theta) + \cos\vartheta_1 \phi_a^2(x_\parallel, r, \theta) \, . \end{cases} \tag{6.7}$$

We now use the DOE (6.4) and compare powers of r. This is the same as comparing the terms including the same defect-local operators $\hat{\mathcal{O}}_s(x_\parallel)$ on the two sides.[1] We find

$$\begin{cases} e^{2\pi i n s} \mu^{\phi^1}{}_{\hat{\mathcal{O}}_s} = \pm \cos\vartheta_1 \mu^{\phi^1}{}_{\hat{\mathcal{O}}_s} \mp \sin\vartheta_1 \mu^{\phi^2}{}_{\hat{\mathcal{O}}_s} \, , \\ e^{2\pi i n s} \mu^{\phi^1}{}_{\hat{\mathcal{O}}_s} = \sin\vartheta_1 \mu^{\phi^1}{}_{\hat{\mathcal{O}}_s} + \cos\vartheta_1 \mu^{\phi^2}{}_{\hat{\mathcal{O}}_s} \, . \end{cases} \tag{6.8}$$

When one of the trigonometric functions in $R_1^{\pm}$ vanish (i.e. $\vartheta_1 \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$), we find charges in the same class as (6.6). This is expected as in such case we have $R_1^{\pm} = \mathrm{diag}(\pm 1, \pm 1)$ ($\pm$ signs not related) after conjugation.

Otherwise, the first equation is generally solved by

$$\mu^{\phi^1}{}_{\hat{\mathcal{O}}_s} = \mp \frac{\sin\vartheta_1}{e^{2\pi i n s} \mp \cos\vartheta_1} \mu^{\phi^2}{}_{\hat{\mathcal{O}}_s} \, . \tag{6.9}$$

Then from the second equation of (6.8) we find

$$(e^{2\pi i n s} - \cos\vartheta_1)(e^{2\pi i n s} \mp \cos\vartheta_1) = \mp \sin\vartheta_1^2 \, . \tag{6.10}$$

This equation has different solutions for s depending on whether $R_1^{\pm}$ is proper or improper. In the proper case we find

$$(e^{2\pi i n s} - \cos\vartheta_1)^2 = -\sin\vartheta_1^2 \quad \Rightarrow \quad e^{2\pi i n s} = \cos\vartheta_1 \pm i \sin\vartheta_1 \, , \tag{6.11}$$

[1] Since the anomalous dimensions are included in $\hat{\Delta}_s$ it is safe to assume no mixing.

which has the solution

$$s = \frac{m}{n} + \frac{\vartheta_1}{2\pi n}\,, \quad m \in \mathbb{Z}\,. \tag{6.12}$$

If we insert this charge back into (6.9) we find

$$\mu^{\phi^1}{}_{\hat{\mathscr{O}}_s} = \pm i \mu^{\phi^2}{}_{\hat{\mathscr{O}}_s}\,. \tag{6.13}$$

The corresponding two defect-local operators both have $SO(2)$-charge given by (6.12). Let us now move on to the improper solution of (6.10). In such case

$$(e^{2\pi i n s} - \cos\vartheta_1)(e^{2\pi i n s} + \cos\vartheta_1) = \sin\vartheta_1^2 \quad \Rightarrow \quad e^{4\pi i n s} = \pm 1\,.$$

This is solved by (6.6). It corresponds to one defect-local operators with even $SO(2)$-charge (w.r.t. $m_\pm$), and another with odd charge.

To summarize:

- The $\mathbb{Z}_2$-element in (6.2) gives us one defect operator with either even or odd (w.r.t. $m_\pm$) $SO(2)$-charge (6.6).
- Each proper R_a^+ gives us a pair of defect operators with the fractional charge (6.12). Their DOE coefficients are related by (6.13).
- Lastly, each improper R_a^- gives us one defect operator with even (w.r.t. $m_\pm$) charge (6.6), and another defect operator with odd charge.

With this information at hand we can see how the bulk $O(N)$-symmetry is broken along the defect by counting the number of defect fields with the same $SO(2)$-charge. That is, the DOE splits into several different sums, where each sum runs over different classes of $SO(2)$-charges. If we assume that in total there exist:

- n_+ defect fields with charge $s \in \frac{\mathbb{Z}}{n}$,
- n_- with $s \in \frac{\mathbb{Z}}{n} + \frac{1}{2n}$,
- $2n_1$ with $s \in \frac{\mathbb{Z}}{n} + \frac{\vartheta_1}{2\pi n}$,
- $2n_2$ with $s \in \frac{\mathbb{Z}}{n} + \frac{\vartheta_2}{2\pi n}$,

  $\vdots$
- $2n_q$ with $s \in \frac{\mathbb{Z}}{n} + \frac{\vartheta_q}{2\pi n}$,

where $0 \leq n_\pm \leq N$, $0 \leq n_r \leq \frac{N}{2}$, $r \in \{1,...,q\}$ and $0 \leq q \leq p$ with p from the twist (6.2), then the $O(N)$-symmetry is broken down to

$$O(N) \to O(n_+) \times O(n_-) \times O(2n_1) \times ...O(2n_q)\,. \tag{6.14}$$

This means that the defect-local fields are in irreducible representations of these subgroups of $O(N)$.

6.2 Anomalous dimensions from the equation of motion

Let us now extract the anomalous dimensions of the defect-local operators from the DOE (6.4) using the e.o.m. (in the process we will also find the DOE coefficients in the free theory).

In $d = 4 - \varepsilon$ we can consider a quartic bulk-interaction

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \phi^i)^2}{2} + \frac{\lambda}{8} \phi^4 \right) , \tag{6.15}$$

where $\phi^4 \equiv [(\phi^i)^2]^2$. The bulk coupling has the non-trivial *Wilson-Fisher* (WF) f.p. [139]

$$\lambda^* = \frac{(4\pi)^2 \varepsilon}{N+8} + \mathscr{O}(\varepsilon^2) , \tag{6.16}$$

which gives us the following e.o.m. at the conformal f.p.

$$\partial_\mu^2 \phi^i = \lambda^* (\phi^j)^2 \phi^i , \quad \lambda^* = \frac{8\pi^2 \varepsilon}{N+8} + \mathscr{O}(\varepsilon^2) . \tag{6.17}$$

This yields the following DS equation

$$\partial_{y^\mu}^2 \langle \hat{\mathscr{O}}_s(x_\parallel) \phi^i(y) \rangle = \lambda^* \langle \hat{\mathscr{O}}_s(x_\parallel) (\phi^j)^2 \phi^i(y) \rangle . \tag{6.18}$$

We will start by studying the free theory where RHS of this equation is zero. For simplicity we will consider the bulk-defect correlator, which using the DOE (6.4) can be written as

$$\langle \hat{\mathscr{O}}_s(x_\parallel) \phi^i(y) \rangle = \mu^{\phi^i}{}_{\hat{\mathscr{O}}_s} e^{is\theta} \sum_{m \geq 0} \frac{a_{\hat{\Delta}_s, m}}{r^{\Delta_\phi - \hat{\Delta}_s - 2m}} \partial_\parallel^{2m} \frac{A_d}{|s_\parallel|^{2m}} . \tag{6.19}$$

The LHS of the classical DS eq. (6.18) is then

$$\begin{aligned} \partial_\mu^2 \langle \hat{\mathscr{O}}_s(x_\parallel) \phi^i(y) \rangle &= (\partial_\parallel^2 + \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2) \langle \phi^i(x) \hat{\mathscr{O}}_s(y_\parallel) \rangle \\ &= \mu^{\phi^i}{}_{\hat{\mathscr{O}}_s} e^{is\theta} \sum_{m \geq 0} \frac{a_{\hat{\Delta}_s, m-1} + (\hat{\Delta}_s - \Delta_\phi + 2m + s)(\hat{\Delta}_s - \Delta_\phi + 2m - s) a_{\hat{\Delta}_s, m}}{r^{\Delta_\phi - \hat{\Delta}_s - 2(m-1)}} \times \\ &\quad \times \partial_\parallel^{2m} \frac{A_d}{|s_\parallel|^{2m}} . \end{aligned} \tag{6.20}$$

In the free theory this is to be zero on its own. By comparing powers of r and avoiding trivial solutions, we find

$$\frac{(\hat{\Delta}_s - \Delta_\phi)^2 - 2m(2(\Delta_\phi + 1) - d) - s^2}{(-4)^m m! (\hat{\Delta}_s - \frac{d-2}{2})_m} = 0 , \quad \Delta_\phi = \frac{d-2}{2} . \tag{6.21}$$

This has several different solutions. The Pochhammer symbol in the denominator is zero if

$$\hat{\Delta}_s = \Delta_\phi - k - m , \quad k \in \mathbb{Z}_{\geq 1} , \tag{6.22}$$

which is only valid for non-unitary theories on the defect (as it violates the unitary bound (1.53) in $p=d-2$ dimensions). Another solution is when the numerator of (6.21) is zero

$$\hat{\Delta}_s=\Delta_\phi \pm s\ . \tag{6.23}$$

We will now move on to the interacting theory, and study the RHS of the DS eq. (6.18)

$$\lambda^*\langle \hat{\mathscr{O}}_s(x_\parallel)(\phi^j)^2\phi^i(y)\rangle=(N+2)\lambda^*\langle\phi^2(y)\rangle\langle\hat{\mathscr{O}}_s(x_\parallel)\phi^i(y)\rangle+\mathscr{O}(\varepsilon^2)\ . \tag{6.24}$$

In order to extract CFT data from this equation, we need the one-point function of ϕ^2 (at $\mathscr{O}(\varepsilon^0)$), which is found from the coincident-limit of $\langle\phi(x)\phi(y)\rangle$.

6.2.1 Dyson-Schwinger equation

The $\phi-\phi$ correlator can be found from the DS eq. (1.12) in radial coordinates

$$(\partial_\parallel^2+\partial_{r_1}^2+r_1^{-1}\partial_{r_1}+r_1^{-2}\partial_{\theta_1}^2)D^{ij}(s_\parallel,r_1,r_2,\varphi)=r_1^{-1}\delta^{ij}\delta^{(d-2)}(s_\parallel)\delta(r_1-r_2)\delta(\varphi)\ ,$$

where $D^{ij}(s_\parallel,r_1,r_2,\varphi)\equiv\langle\phi^i(x_\parallel,r_1,\theta_1)\phi^j(y_\parallel,r_2,\theta_2)\rangle$ and $\varphi\equiv\theta_2-\theta_1$. Details on how this differential equation is solved are in App. B of [140]

$$\begin{aligned}D^{ij}(s_\parallel,r_1,r_2,\varphi)=A_d\delta^{ij}\sum_s\frac{\Gamma_{\hat{\Delta}_s}}{\Gamma_{\Delta_\phi}\Gamma_{\hat{\Delta}_s-\Delta_\phi+1}}\frac{e^{is\varphi}}{(r_1r_2)^{\Delta_\phi}}(4\xi)^{-\hat{\Delta}_s}\times\\ \times{}_2F_1\left(\hat{\Delta}_s,\hat{\Delta}_s-\frac{p-1}{2},2\hat{\Delta}_s-p+1,-\xi^{-1}\right)\ .\end{aligned} \tag{6.25}$$

Here $\hat{\Delta}_s$ is given by

$$\hat{\Delta}_s=\Delta_\phi+|s|\ , \tag{6.26}$$

which is a subclass of the free theory solution (6.23), and the cross-ratio is

$$\xi=\frac{s_\parallel^2+(r_1-r_2)^2}{4r_1r_2}\ . \tag{6.27}$$

Before we find $\langle\phi^2\rangle$, let us extract the DOE coefficients. We do this by comparing (6.25) to the expression found from the DOE (6.4)

$$D^{ij}(s_\parallel,r,r',\delta\theta)=\sum_s(\mu^{\phi^i}{}_{\hat{\mathscr{O}}_s})^\dagger\mu^{\phi^j}{}_{\hat{\mathscr{O}}_s}\frac{e^{is\varphi}}{(r_1r_2)^{\Delta_\phi}}\frac{A_d}{|s_\parallel|^{2\hat{\Delta}_s}}\ .$$

If we know expand (6.25) in r_1,r_2 we find

$$(\mu^{\phi^i}{}_{\hat{\mathscr{O}}_s})^\dagger\mu^{\phi^j}{}_{\hat{\mathscr{O}}_s}=\delta^{ij}\frac{\Gamma_{\hat{\Delta}_s}}{\Gamma_{\Delta_\phi}\Gamma_{\hat{\Delta}_s-\Delta_\phi+1}}\ . \tag{6.28}$$

These are on the same form as in paper I.

Now we will proceed with finding $\langle\phi^2\rangle$. In $d=4$ the summand simplify using the following ${}_2F_1$-identity

$$
{}_2F_1\left(|s|+\frac{1}{2},|s|+1,2|s|+1,-\xi^{-1}\right)=4^{|s|}\sqrt{\frac{\xi}{\xi+1}}\frac{\xi^{|s|}}{(\sqrt{\xi}+\sqrt{\xi+1})^{2|s|}}\,.
$$

The coincident-limit, $r_2\to r_1\equiv r$, $\varphi\to 0$, of (6.25) is then

$$
D^{ij}(s_\parallel,r,r,0)=\frac{A_d\delta^{ij}}{|s_\parallel|\sqrt{s_\parallel^2+4r^2}}\sum_s\left(\frac{2r}{|s_\parallel|+\sqrt{s_\parallel^2+4r^2}}\right)^{|s|}\,. \tag{6.29}
$$

After the change in variables $s=\frac{t}{n}$ (s.t. $t\in\mathbb{Z}+\upsilon$ with $\upsilon\equiv\frac{\vartheta}{2\pi}$), this is re-summed in the same way as in App. B.2 of paper I. We find

$$
\begin{aligned}
D^{ij}(s_\parallel,r,r,0)&=\frac{2A_d\delta^{ij}}{|s_\parallel|\sqrt{s_\parallel^2+4r^2}}\frac{\left(\frac{2r}{|s_\parallel|+\sqrt{s_\parallel^2+4r^2}}\right)^{\frac{2\upsilon}{n}}}{1-\left(\frac{2r}{|s_\parallel|+\sqrt{s_\parallel^2+4r^2}}\right)^{\frac{2}{n}}}\\
&=A_d\delta^{ij}\left(\frac{n}{s_\parallel^2}-\frac{2\upsilon-1}{2r|s_\parallel|}+\frac{6\upsilon(\upsilon-1)-n^2+1}{12nr^2}+\mathscr{O}(s_\parallel)\right)\,.
\end{aligned}
$$

We can identify one-point functions of a bulk-local operator $\mathscr{O}$ (times $|s_\parallel|^{\Delta-2\Delta_\phi}\lambda^{\phi\phi}{}_{\mathscr{O}}\mu^{\mathscr{O}}{}_{\mathbb{1}}$) by comparing above expansion with the bulk OPE (1.36). The first term corresponds to the identity exchange, the second to the bulk scalar ϕ itself (since the bulk $O(N)$-symmetry is broken by the defect)

$$
\langle\phi^i(x_\parallel,r,\theta)\rangle=\frac{A_d\mu^{\phi}{}_{\mathbb{1}}\delta^{iN}}{r^2}+\mathscr{O}(\varepsilon)\,,\quad \mu^{\phi}{}_{\mathbb{1}}=\frac{1}{2}-\upsilon\,, \tag{6.30}
$$

and the last one to ϕ^2 which we are interested in

$$
\langle\phi^2(x_\parallel,r,\theta)\rangle=\frac{A_d\mu^{\phi^2}{}_{\mathbb{1}}}{r^2}+\mathscr{O}(\varepsilon)\,,\quad \mu^{\phi^2}{}_{\mathbb{1}}=\frac{6\upsilon(\upsilon-1)-n^2+1}{12n}\,. \tag{6.31}
$$

As a sanity check we see that this reduces to the result in paper I when there is only one replica: $n=1$.

6.2.2 Anomalous dimensions

We are now ready to find the anomalous dimensions of the defect-local operators, $\hat{\mathscr{O}}_s$, that appear in the DOE (6.4) using the DS eq. (6.18). The RHS of the DS eq. (6.24) is given by

$$\langle \hat{\mathscr{O}}_s(x_\parallel)\lambda(\phi^j)^2\phi^i(y)\rangle = \mu^{\phi^i}{}_{\hat{\mathscr{O}}_s} e^{is\theta}\sum_{m\geq 0}\frac{(N+2)A_d\mu^{\phi^2}{}_{\mathbb{1}}a_{\hat{\Delta}_s,m}\lambda}{r^{\Delta_\phi-\hat{\Delta}_s-2(m-1)}}\partial_\parallel^{2m}\frac{A_d}{|s_\parallel|^{2m}}\ .$$

Powers in r can now be compared to the LHS of (6.18), i.e. (6.20), which gives us

$$a_{\hat{\Delta}_s,m-1}+(\hat{\Delta}_s-\Delta_\phi+2m+s)(\hat{\Delta}_s-\Delta_\phi+2m-s)a_{\hat{\Delta}_s,m} = (N+2)A_d\mu^{\phi^2}{}_{\mathbb{1}}a_{\hat{\Delta}_s,m}\lambda\ .$$

Let us now expand both sides in ε (where we use as input that the bulk anomalous dimension, γ_ϕ, start first at $\mathscr{O}(\varepsilon^2)$ [139])

$$\Delta_\phi = \frac{d-2}{2}+\mathscr{O}(\varepsilon^2)\ ,\quad \hat{\Delta}_s = \frac{d-2}{2}+|s|+\varepsilon\,\hat{\gamma}_s+\mathscr{O}(\varepsilon^2)\ , \tag{6.32}$$

which yields the anomalous dimension of the defect-local fields

$$\hat{\gamma}_s = \frac{N+2}{N+8}\frac{\mu^{\phi^2}{}_{\mathbb{1}}}{|s|} = \frac{N+2}{N+8}\frac{6\upsilon(\upsilon-1)-n^2+1}{12n|s|}\ . \tag{6.33}$$

This is a new result, which reduces down to the result in paper I when $n=1$.

7. Fusion of two defects

In this Chapter we study fusion of two scalar Wilson defects (5). In the free theory we generalize the results of paper VI to work in six dimensions. We will then move on to study an interacting theory with cubic interactions in $d = 6-\varepsilon$.

By studying the one-point function of bulk fields in the presence of the two defects we find the RG flow for the defect couplings. This is a slight generalization of the corresponding results in [94], and we use the more traditional way of calculating Feynman diagrams [13] assuming the bulk interactions are small w.r.t. those on the defects. In particular, we find that the couplings on the two defects are not affected by each other, which is what we expect since their corresponding β-functions measure divergences in their respective defect-limit of bulk-local fields as well as in the coincident-limit of defect-local fields on the corresponding defect.

Specifying to the real-valued f.p.'s of the defects, we compare the bare one-point function in the presence of the two defects with that near the fused defect (at first order in the bulk couplings). We find that they are exactly the same, and there are no modifications needed to the fused defect. The underlying reason for this is that the path integral for the two defects is the same as that for the fused defect. We check that this is indeed the case for line defects in $d = 4-\varepsilon$ with a quartic bulk-interaction as well.

The difference between the model with two defects and that with the fused defect lies in renormalization of the theory. Diagrams with bulk vertices connecting the two defects have logarithmic divergences in the *fusion-limit* (as the distance between the defects goes to zero). Such divergences are absorbed in the bare coupling constant on the fused defect, giving us different β-functions and renormalized correlators. So in addition to UV divergences in the coincident-limit of defect-local fields and in the defect-limit of bulk-local fields, the β-functions on the fused defect also take into account UV divergences in the fusion-limit of the two defects.

7.1 Model

Let us first introduce the model we consider. In the bulk we have

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \phi^i)^2}{2} + \frac{(\partial_\mu \sigma)^2}{2} + \frac{g_1}{2}\sigma(\phi^i)^2 + \frac{g_2}{3!}\sigma^3 \right) , \tag{7.1}$$

where $d = 6-\varepsilon$ and $i \in \{1,...,N\}$. The scalars ϕ^i are invariant under $O(N)$. We consider two parallel surface defects, $D_\pm$, of dimension $p=2$, spanned along $\hat{x}^a_\parallel$, $a \in \{1,2\}$. They are separated by a distance $2R$, $R \equiv |R_i|$, in the orthogonal directions $\hat{x}^i_\perp$, $i \in \{1,...,d-p\}$

$$D_\pm = \exp\left(-\int_{\mathbb{R}^p} d^p x \left[h^\phi_\pm \hat{\phi}^{i_\pm}(x_\pm) + h^\sigma_\pm \hat{\sigma}(x_\pm)\right]\right) . \tag{7.2}$$

Here $x_\pm \equiv x_a \hat{x}^a_\parallel \pm R_i \hat{x}^i_\perp$ and $h^\phi_\pm$, $h^\sigma_\pm$ are couplings (or magnetic fields) localized on the respective defects. Due to their $\phi^{i_\pm}$-interaction, the $O(N)$-symmetry of the model is broken down to $O(N-2)$ by the defects (in the case when $i_+ = i_-$ the symmetry is broken down to $O(N-1)$). This is an explicit symmetry breaking caused by the defect interactions, and thus differs from e.g. the extraordinary p.t. near a boundary (which is a SSB).

The effective action is given by

$$S_{\text{eff}} = S + \sum_\pm \log D_\pm . \tag{7.3}$$

Since the β-functions for the bulk couplings arise from divergences in the coincident-limit of the bulk fields, they are not affected by the defect couplings. This means that we can borrow these results from the bulk theory (7.1) without the defects [141, 142]

$$\begin{aligned} \beta_1 &= -\frac{\varepsilon}{2} g_1 + \frac{(N-8)g_1^3 - 12 g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3} + \mathcal{O}(g^4) , \\ \beta_2 &= -\frac{\varepsilon}{2} g_2 - \frac{4N g_1^3 - N g_1^2 g_2 + 3 g_2^3}{4(4\pi)^3} + \mathcal{O}(g^4) . \end{aligned} \tag{7.4}$$

In the case when $N=0$ and the ϕ^i-fields are not present, there is a negative sign in front of the g_2^3-term in β_2. Due to this we find no real-valued RG f.p.

By including the $O(N)$-scalars we can expand in large $N \gg 1$

$$\begin{aligned} \beta_1 &= -\frac{\varepsilon}{2} g_1 + \frac{N g_1^3}{12(4\pi)^3} + \mathcal{O}(g^4) , \\ \beta_2 &= -\frac{\varepsilon}{2} g_2 - \frac{4N g_1^3 - N g_1^2 g_2}{4(4\pi)^3} + \mathcal{O}(g^4) . \end{aligned} \tag{7.5}$$

Setting these β-functions to zero yields a Gaussian f.p. and two non-trivial, real-valued f.p.'s[1]

$$g_2^* = 6 g_1^* + \mathcal{O}\left(\varepsilon, \frac{1}{N}\right) , \quad g_1^* = \pm\sqrt{\frac{6(4\pi)^3\varepsilon}{N}} + \mathcal{O}\left(\varepsilon, \frac{1}{N}\right) . \tag{7.6}$$

Note that these f.p.'s go as $\sqrt{\varepsilon}$, which differ from the WF f.p. (6.16) in $d = 4-\varepsilon$. The RG flow is depicted in Fig. 7.1.

[1] In particular, this result is valid for $N \geq 1039$ [142]

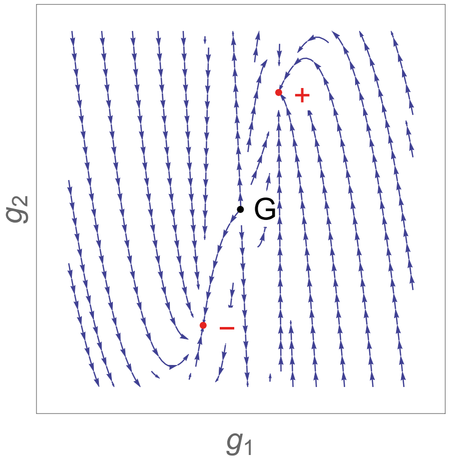

Figure 7.1. The RG flow of the cubic $O(N)$-model (5.1) at large N near six dimensions. The black dot (G) is the trivial Gaussian f.p. (5.5) and the two red dots ($\pm$) are the attractive f.p.'s at (7.6).

We will proceed with finding the f.p.'s of the defect-interactions in (7.2). The corresponding β-functions measure divergence in the respective near distance limits. This means that e.g. the β-functions on D_+ does not depend on the interactions on D_-. In turn this tells us that the defect β-functions are the same on the two defects, and it can be found from the theory with only one defect. We will calculate Feynman diagrams in the theory with both defects (considering small bulk-interactions) to show that this is indeed the case. We allow the defect couplings, $h_\pm^\phi$ and $h_\pm^\sigma$, to be of finite size.

7.2 Free theory

Correlators in the presence of the two defects (7.2) are found by expanding $D_\pm$ in its interactions and then applying Wick's theorem. This was done for a single insertion of a bulk field in paper VI. In general, it gives us

$$\langle D_+ D_- \ldots \rangle = \langle D_+ \rangle \langle D_- \rangle \langle D_+ D_- \rangle \left(\sum_\pm \langle D_\pm \ldots \rangle_N + \delta \langle D_+ D_- \ldots \rangle_N \right) . \tag{7.7}$$

Here the dots represent any combination of operators. $\langle D_\pm \rangle$ describes self-interactions on $D_\pm$, and $\langle D_+ D_- \rangle$ is a non-perturbative (w.r.t. R) Casimir effect between the defects. See Fig. 2 in paper VI for a diagrammatic representation of these correlators.

For the purposes of this Chapter we are not interested in $\langle D_\pm\rangle$ and $\langle D_+D_-\rangle$. Thus we normalize the correlators by dividing with these factors

$$\langle D_+D_-\ldots\rangle_N \equiv \frac{\langle D_+D_-\ldots\rangle}{\langle D_+\rangle\langle D_-\rangle\langle D_+D_-\rangle} = \sum_\pm \langle D_\pm\ldots\rangle_N + \delta\langle D_+D_-\ldots\rangle_N \ . \tag{7.8}$$

The remaining three correlators, $\langle D_\pm\ldots\rangle_N$ and $\delta\langle D_+D_-\ldots\rangle_N$, can be found using standard Feynman diagrams techniques, where $\langle D_\pm\ldots\rangle$ is the one-point function in the presence of one of the defects, $D_\pm$, and $\delta\langle D_+D_-\ldots\rangle$ is the sum of Feynman diagrams connecting the two defects. We will see an example of a $\delta\langle D_+D_-\ldots\rangle$-diagram later in when we take into account the bulk-interactions.

The one-point function of ϕ^i and σ in the presence of the two defects $D_\pm$ is given by the third diagram in Fig. 2 of paper VI

$$\begin{aligned} \langle D_+D_-\phi^i(x)\rangle_N &= \sum_\pm \langle D_\pm\phi^i(x)\rangle_N \ , \\ \langle D_\pm\phi^i(x)\rangle_N &= -h^\phi_\pm \delta^{ii_\pm} K_\pm(x) \ , \end{aligned} \tag{7.9}$$

$$\begin{aligned} \langle D_+D_-\sigma(x)\rangle_N &= \sum_\pm \langle D_\pm\sigma(x)\rangle_N \ , \\ \langle D_\pm\sigma(x)\rangle_N &= -h^\sigma_\pm K_\pm(x) \ , \end{aligned} \tag{7.10}$$

where the integral $K_\pm$, which is the corresponding integral from paper VI, is given by

$$K_\pm(x) = \int_{\mathbb{R}^p} d^pz\langle\sigma(x)\sigma(z_\pm)\rangle\big|_{h^\phi_\pm,h^\sigma_\pm=0} \ . \tag{7.11}$$

The integrand is the same as the connected part of $\langle D_+D_-\sigma(x)\sigma(y)\rangle_N$. It is not affected by the defects interactions, and is thus the massless limit of the correlator (1.19)

$$\langle\sigma(x)\sigma(y)\rangle\big|_{h^\phi_\pm,h^\sigma_\pm=0} = \langle D_+D_-\sigma(x)\sigma(y)\rangle_N^{\text{conn}} = \frac{A_d}{|x-y|^{2\Delta_\phi}} \ . \tag{7.12}$$

The integrals $K_\pm$ are special cases of the master integral (1.18)

$$K_\pm(x) = A_d I^p_{\Delta_\phi}(0,x_\parallel,x_\perp\mp R) = \frac{A_d\pi^{\frac{p}{2}}\Gamma_{\Delta_\phi-\frac{p}{2}}}{\Gamma_{\Delta_\phi}}\frac{1}{|x_\perp\mp R|^{2\Delta_\phi-p}} \ . \tag{7.13}$$

In the interacting theory we will find it useful to Fourier transform w.r.t. the normal distances, $s^\pm_\perp \equiv x_\perp \mp R$, to the defects

$$\begin{aligned} \prod_{c=\pm}\int_{\mathbb{R}^{d-p}} ds^c_\perp e^{ik^c_\perp s^c_\perp} K_\pm(x) &= \frac{A_d\pi^{\frac{p}{2}}\Gamma_{\Delta_\phi-\frac{p}{2}}}{\Gamma_{\Delta_\phi}}\delta(k_\pm) I^{d-p}_{\Delta_\phi-\frac{p}{2}}(k^\mp_\perp,0,0) \\ &= \frac{\delta(k_\pm)}{k^2_\mp} \ , \quad \text{(exactly)}. \end{aligned} \tag{7.14}$$

The momenta $k_\pm$ is that flowing between the bulk field and the defect $D_\pm$. It describes how momenta is being absorbed/emitted by the two defects. The Dirac δ-function tell us that the momenta is only affected by one of the defects.

Note that the one-point functions (7.9, 7.10) are the forms we expect a one-point function to have from conformal symmetry (2.18),[2] from which we can read off the BOE coefficients

$$\mu^{\phi^i}{}_{\mathbb{1}_\pm} = -h^\phi_\pm \delta^{ii_\pm} \frac{\Gamma_{\Delta_\phi - 1}}{4\pi^{\Delta_\phi}} \ , \quad \mu^\sigma{}_{\mathbb{1}_\pm} = -h^\sigma_\pm \frac{\Gamma_{\Delta_\phi - 1}}{4\pi^{\Delta_\phi}} \ . \tag{7.15}$$

Here the $\mathbb{1}_\pm$ subscript denotes the identity exchange on the respective defect.

7.3 Interacting theory

We will now proceed to the interacting theory, and find the β-functions of the defect couplings as well as the corresponding RG f.p.'s.

The one-point functions at first order in the bulk couplings are given by the two Feynman diagrams in Fig. 7.2. If a diagram contains n defect points of the same field, we have to divide the symmetry factor with a factor $n!$ to avoid overcounting (which is seen from the integration of the defect points). We find

$$\begin{aligned} \langle D_\pm \phi^i(x)\rangle^{(1)}_N &= 2\left(-\frac{g_1}{2}\right)(-h^\phi_\pm)(-h^\sigma_\pm)\delta^{ii_\pm} L^\pm_\pm(x) \ , \\ \delta\langle D_+ D_- \phi^i(x)\rangle^{(1)}_N &= 2\left(-\frac{g_1}{2}\right)\sum_{a=\pm}(-h^\phi_a)(-h^\sigma_{-a})\delta^{ii_a} L^+_-(x) \ , \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} \langle D_\pm \sigma(x)\rangle^{(1)}_N &= \left(\frac{2}{2}\left(-\frac{g_1}{2}\right)(-h^\phi_\pm)(-h^\phi_\pm)\delta^{i_\pm i_\pm} + \right. \\ &\quad \left. + \frac{3!}{2}\left(-\frac{g_2}{3!}\right)(-h^\sigma_\pm)(-h^\sigma_\pm)\right) L^\pm_\pm(x) \ , \\ \delta\langle D_+ D_- \sigma(x)\rangle^{(1)}_N &= \left(2\left(-\frac{g_1}{2}\right)(-h^\phi_+)(-h^\phi_-)\delta^{i_+ i_-} + \right. \\ &\quad \left. + 3!\left(-\frac{g_2}{3!}\right)(-h^\sigma_+)(-h^\sigma_-)\right) L^+_-(x) \ . \end{aligned} \tag{7.17}$$

Here L^a_b, with $a,b = \pm$, is the following integral

$$\begin{aligned} L^a_b(x) &\equiv \int_{\mathbb{R}^d} d^d z \ \langle \phi(x)\phi(z)\rangle|_{h=0} K_a(z) K_b(z) \\ &= A_d \int_{\mathbb{R}^{d-p}} d^{d-p} z_\perp K_a(z) K_b(z) I^p_{\Delta_\phi}(0, x_\parallel, (z_\perp - x_\perp)^2) \\ &= \frac{A_d^3 \pi^{\frac{3p}{2}} \Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \int_{\mathbb{R}^{d-p}} \frac{d^{d-p} z_\perp}{(|z_\perp|\,|z_\perp + x_\perp - aR|\,|z_\perp + x_\perp - bR|)^{2\Delta_\phi - p}} \ , \end{aligned} \tag{7.18}$$

[2] The defects are placed at the orthogonal coordinates $\pm R_i$, hence a shift in the denominator.

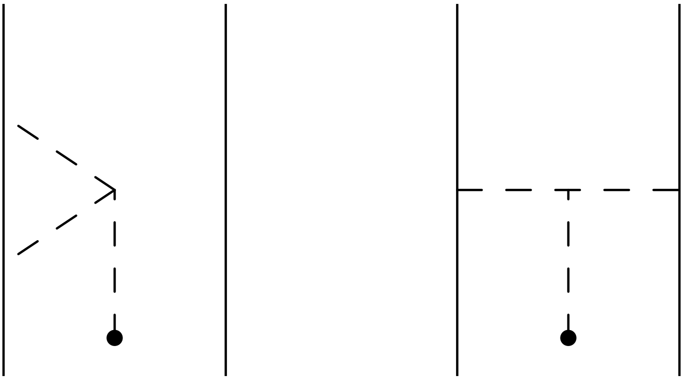

Figure 7.2. The two diagrams that contribute to the one-point functions of ϕ^i and σ. The dot is the external bulk point, the dotted lines are either $\phi-\phi$ or $\sigma-\sigma$ correlators and the solid lines are the two surface defects.

where in the last step we shifted $z_\perp \to z_\perp + x_\perp$.

We find it easier to study the UV divergences of these integrals in momentum space, where we Fourier transform w.r.t. $s_\perp^\pm$

$$M^a_b(k_\perp^\pm) \equiv \prod_{c=\pm} \int_{\mathbb{R}^{d-p}} d^{d-p} s_\perp^c e^{i k_\perp^c s_\perp^c} L^a_b(x) \,. \tag{7.19}$$

This integral can then be performed using only the master integral (1.18).

$$\begin{aligned} M^\pm_\pm(k_\perp^\pm) &= \frac{A_d^3 \pi^{\frac{3p}{2}} \Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \delta(k_\perp^\pm) \int_{\mathbb{R}^{d-p}} \frac{d^{d-p} z_\perp}{|z_\perp|^{2\Delta_\phi - p}} I^p_{2\Delta_\phi - p}(k_\perp^\mp, -z_\perp, 0) \\ &= \frac{A_d^3 2^{d+p-4\Delta_\phi} \pi^{\frac{d}{2}+p} \Gamma_{\frac{d+p}{2} - 2\Delta_\phi} \Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi} \Gamma_{2\Delta_\phi - p} |k_\perp^\mp|^{\frac{d+p}{2} - 2\Delta_\phi}} \delta(k_\perp^\pm) I^{d-p}_{\Delta_\phi - \frac{p}{2}}(-k_\perp^\mp, 0, 0) \\ &= \frac{\delta(k_\perp^\pm)}{8\pi^2 (k_\perp^\mp)^2} \left(\frac{1}{\varepsilon} - \log|k_\perp^\mp| + \mathscr{A} \right) , \end{aligned} \tag{7.20}$$

$$\begin{aligned} M^+_-(k_\perp^\pm) &= \frac{A_d^3 \pi^{\frac{3p}{2}} \Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \int_{\mathbb{R}^{d-p}} \frac{d^{d-p} z_\perp}{|z_\perp|^{2\Delta_\phi - p}} \prod_{a=\pm} I^p_{\Delta_\phi - \frac{p}{2}}(k_\perp^a, -z_\perp, 0) \\ &= \frac{A_d^3 4^{d-2\Delta_\phi} \pi^{d+\frac{p}{2}} \Gamma^2_{\frac{d}{2} - \Delta_\phi} \Gamma_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi} |k_\perp^+|^{d-2\Delta_\phi} |k_\perp^-|^{d-2\Delta_\phi}} I^{d-p}_{\Delta_\phi - \frac{p}{2}}(-k_\perp^+ - k_\perp^-, 0, 0) \\ &= \frac{1}{(k_\perp^+)^2 (k_\perp^-)^2 (k_\perp^+ + k_\perp^-)^2} \,, \quad \text{(exactly).} \end{aligned} \tag{7.21}$$

Here $\mathscr{A}$ is the following constant

$$\mathscr{A} = \log\left(\frac{2\sqrt{\pi}}{e^{\frac{\gamma_E}{2}-1}}\right) , \tag{7.22}$$

which can be absorbed in the coupling constants (by defining minimal subtraction scheme couplings) without affecting the RG flow. Thus we will not care about it.

Note that the diagram M_-^+ (in $\delta\langle D_+D_-\phi^i(k_\pm)\rangle_N^{(1)}$) connecting the two defects is convergent. So only the diagrams $M_\pm^\pm$ (in $\langle D_\pm\phi^i(k_\pm)\rangle_N^{(1)}$), which are affected by one of the defects, are divergent. This means that in the renormalization procedure, the couplings on D_+ are not affected by those on D_- (and vice versa). This is the expected result since the bare couplings on D_+ should capture UV divergences in the coincident-limit of defect-local fields in addition to divergences in the limit as bulk-local fields approach D_+.

To find the bare defect couplings we add the free theory correlators (7.9, 7.10) to those at first order in the coupling constants (7.16). We then make the following ansatz for the bare coupling constants

$$\begin{aligned} h_\pm^\phi &= \mu^{\frac{\varepsilon}{2}}\tilde{h}_\pm^\phi\left(1 + a_\pm^\phi\frac{\tilde{h}_\pm^\sigma\tilde{g}_1}{\varepsilon}\right) , \\ h_\pm^\sigma &= \mu^{\frac{\varepsilon}{2}}\left(\tilde{h}_\pm^\sigma + b_\pm^\phi\frac{(\tilde{h}_\pm^\phi)^2\tilde{g}_1}{\varepsilon} + b_\pm^\sigma\frac{(\tilde{h}_\pm^\sigma)^2\tilde{g}_2}{\varepsilon}\right) , \end{aligned} \tag{7.23}$$

where the constants $a_\pm^\phi, b_\pm^\phi$ and $b_\pm^\sigma$ are tuned s.t. that the ε-poles in the correlators vanish. Coupling constants with a tilde are renormalized ones (dimensionless), and μ is the RG scale. By expanding the correlators in the bulk couplings and in ε we find (by matching powers of $k_\pm$)

$$b_\pm^\phi = b_\pm^\sigma = \frac{a_\pm^\phi}{2} , \quad a_\pm^\phi = -\frac{1}{8\pi^2} . \tag{7.24}$$

From which we find the β-functions (see Sec. 1.2)[3]

$$\beta_\pm^\phi = -\frac{\varepsilon}{2}\tilde{h}_\pm^\phi - \frac{\tilde{h}_\pm^\phi\tilde{h}_\pm^\sigma\tilde{g}_1}{8\pi^2\varepsilon} , \quad \beta_\pm^\sigma = -\frac{\varepsilon}{2}\tilde{h}_\pm^\sigma - \frac{(\tilde{h}_\pm^\phi)^2\tilde{g}_1}{16\pi^2\varepsilon} - \frac{(\tilde{h}_\pm^\sigma)^2\tilde{g}_2}{16\pi^2\varepsilon} . \tag{7.25}$$

Setting these to zero gives us a Gaussian f.p. where both defect couplings are zero,[4] in addition to the following non-trivial ones

$$\left((h_\pm^\phi)^*, (h_\pm^\sigma)^*\right) \in \left\{\left(0, -\frac{8\pi^2\varepsilon}{g_2^*}\right), \left(\pm 4\pi^2\varepsilon\frac{\sqrt{2g_1^* - g_2^*}}{(g_1^*)^{\frac{3}{2}}}, -\frac{4\pi^2\varepsilon}{g_1^*}\right)\right\} .$$

[3]Here we used that the bulk β-functions are given by (7.5).

[4]The defect couplings can be zero while those in the bulk (7.6) are not.

The first one is the same as that found in [94]. The bulk couplings are tuned to their respective f.p.'s (7.6), where we find four complex f.p.'s

$$\left((h_\pm^\phi)^*, (h_\pm^\sigma)^*\right) = \left(\pm i\sqrt{\frac{\pi N \varepsilon}{6}}, \pm\frac{1}{2}\sqrt{\frac{\pi N \varepsilon}{6}}\right) , \tag{7.26}$$

and two real-valued f.p.'s where only $h_\pm^\sigma$ is non-trivial

$$\left((h_\pm^\phi)^*, (h_\pm^\sigma)^*\right) = (0, h^*) , \quad h^* = \mp\frac{1}{6}\sqrt{\frac{\pi N \varepsilon}{6}} . \tag{7.27}$$

The sign of h^* is opposite to the bulk-couplings at their f.p. (7.6). If we restrict ourselves to real-valued f.p.'s then the $\phi^{i_\pm}$-term on the defects (7.2) vanish

$$D_\pm = \exp\left(-h^* \int_{\mathbb{R}^p} d^p x\, \hat{\sigma}(x_\pm)\right) . \tag{7.28}$$

Note that since $N \gg 1$, none of the f.p.'s (7.26, 7.27) have to be small.

By studying the derivative of $\beta_\pm^\sigma$ we can check whether the real-valued f.p. is attractive or not

$$\partial_{\tilde{h}_\pm^\sigma} \beta_\pm^\sigma \Big|_{\tilde{h}_\pm^\phi = 0,\ \tilde{h}_\pm^\sigma = h^*} = \frac{\varepsilon}{2} , \tag{7.29}$$

which does not depend on the sign of h^* at (7.27). Since this is positive, the f.p.'s at (7.27) are minima of the β-function and they are thus attractive.

The one-point functions of ϕ^i are trivial at this f.p. (giving us $O(N)$-symmetry), while those for σ can be resummed in ε

$$\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N = \sum_{a=\pm} \langle D_a \sigma(k_\perp^\pm)\rangle_N + \delta\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N + \mathscr{O}(g^2) ,$$

$$\begin{aligned} \langle D_\pm \sigma(k_\perp^\pm)\rangle_N &= -h^* \frac{\delta(k_\perp^\pm)}{(k_\perp^\mp)^{2-\frac{\varepsilon}{2}}} , \\ \delta\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N &= -\frac{(h^*)^2 g_2^*}{(k_\perp^+)^2 (k_\perp^-)^2 (k_\perp^+ + k_\perp^-)^2} . \end{aligned} \tag{7.30}$$

Note that the RG scale has completely vanished at the f.p. We have

$$(h^*)^2 g_2^* = \pm \frac{\pi^{\frac{5}{2}}}{\sqrt{N}} \left(\frac{2\varepsilon}{3}\right)^{\frac{3}{2}} , \tag{7.31}$$

which is at a subleading order in N. This means that $\delta\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N$ is small compared to $\langle D_\pm \sigma(k_\perp^\pm)\rangle_N$.

In Euclidean space we find

$$\begin{aligned}\langle D_\pm \sigma(x)\rangle_N &= \prod_{c=\pm}\int_{\mathbb{R}^{d-p}} \frac{d^{d-p}k_\perp^c}{(2\pi)^{d-p}} e^{-ik_\perp^c s_\perp^c} \langle D_\pm \sigma(k_\perp^\pm)\rangle_N \\ &= -\frac{h^*}{(2\pi)^{d-p}} I^{d-p}_{1-\frac{\varepsilon}{4}}(-s_\perp^\pm,0,0) + \mathscr{O}(\varepsilon^{\frac{3}{2}}) \\ &= -\frac{h^*}{(2\pi)^{2-\frac{\varepsilon}{2}} |s_\perp^\pm|^{2-\frac{\varepsilon}{2}}} ,\end{aligned} \tag{7.32}$$

which agrees with the free theory result at $\mathscr{O}(\sqrt{\varepsilon})$, and has the correct scaling dimension of σ. From this we can also read off the BOE coefficients

$$\mu^\sigma{}_{\mathbb{1}_\pm} = -\frac{h^*}{(2\pi)^{2-\frac{\varepsilon}{2}}} + \mathscr{O}(g^2) \, . \tag{7.33}$$

7.4 Fusion

Let us now fuse the two defects (7.28). This can be done by Taylor expanding the two defects w.r.t. each component of R_i (remember that the two defects are placed at $\pm R_i$ along the orthogonal coordinates)

$$D_\pm = \exp\left(-h \prod_{i=1}^{d-p} \sum_{n_i \geq 0} \frac{(\pm 1)^{n_i} R_i^{n_i}}{n_i!} \lim_{R_i \to 0} \partial_i^{n_i} \int_{\mathbb{R}^p} d^p x \, \sigma(x_+) \right) . \tag{7.34}$$

Adding the exponents gives us the fused defect

$$D_f = D_+ D_- = \exp\left(-2h \prod_{i=1}^{d-p} \sum_{n_i \geq 0} \frac{R_i^{2n_i}}{(2n_i)!} \int_{\mathbb{R}^p} d^p x \, \partial_i^{2n_i} \hat{\sigma}(x_+) \right) . \tag{7.35}$$

This is the multivariate version of the result in paper VI. Since this is just a Taylor expansion, we find the path integral, which generates all of the correlators, to be the same for D_+D_- as for D_f (see Sec. 3 of paper VI for a proof on this). Note that the entire tower of terms w.r.t. R_i has to be kept to find the same path integral. R_i should be treated as a distance scale of the theory, and thus we keep it even after fusion of the two defects.

Let us also mention that two straight parallel lines are conformally equivalent to two concentric circles. This means that above fusion is also true for two concentric circular Wilson lines. Although not commented upon, this was seen in paper VI. I.e. its eq.'s (2.21) and (5.4) are the same.

Since the path integral is the same for D_+D_- and D_f, we expect the fusion (7.35) to hold even in an interacting theory. We will check that this indeed the case by showing that the expansion of $\langle D_+D_-\sigma\rangle_N$ in R is the same as $\langle D_f\sigma\rangle_N$ upto $\mathscr{O}(g_2)$ (before renormalization of the couplings).

To do this we need the full $\langle D_+ D_- \sigma \rangle_N^{(1)}$ at $\mathscr{O}(g_2)$ in Euclidean space, see (7.17). We are left to find

$$\delta \langle D_+ D_- \sigma(x) \rangle_N^{(1)} = -(h^*)^2 g_2^* L_-^+ \ . \tag{7.36}$$

We know from its Fourier transform (7.21) that L_-^+ is free of UV divergences. So we are free to set $\varepsilon = 0$ before integration over $z_\perp$ in (7.18)

$$L_-^+ = \int_{\mathbb{R}^4} \frac{d^4 z_\perp}{64\pi^6} \frac{1}{z_\perp^2 (z_\perp + s_\perp^+)^2 (z_\perp + s_\perp^-)^2} \ . \tag{7.37}$$

This integral has been done in the amplitude literature [143]. Its a rather lengthy expression for general R, but by specifying to one dimensional $x_\perp$ and R

$$R^i = R\,\delta^{i1} \ , \quad R > 0 \ , \tag{7.38}$$

it simplifies to[5]

$$L_-^+ = \frac{1}{64\pi^4 R} \left(\frac{1}{s_+} \log\left(\frac{|s_-|}{2R} \right) + \frac{1}{s_-} \log\left(\frac{|s_+|}{2R} \right) \right) \ . \tag{7.39}$$

The full one-point function of σ upto $\mathscr{O}(g_2)$ is thus

$$\langle D_+ D_- \sigma(k_\perp^\pm) \rangle_N = -\frac{h^*}{4\pi^2} \sum_{a=\pm} \left[\frac{1}{(s_\perp^a)^2} + \frac{h^* g_2^*}{8\pi^2} \left(\frac{\log|s_\perp^a|}{(s_\perp^a)^2} + \frac{1}{2R|s_\perp^{-a}|} \log \left| \frac{s_\perp^a}{2R} \right| \right) \right] \ .$$

In the expansion of L_-^+ in R we find a logarithmic divergence

$$\delta \langle D_+ D_- \sigma(x) \rangle_N^{(1)} \ni -\frac{(h^*)^2 g_2^*}{32\pi^4} \log(R) \sum_{n \geq 0} \frac{R^{2n}}{x_\perp^{2(n+1)}} \ . \tag{7.40}$$

Note that this is not an IR divergence since R is a distance scale. Still it should not be absorbed in the bare couplings on $D_\pm$. To avoid this logarithmic divergence we instead expand the integrands (7.18) of L_b^a in R before we integrate over $z_\perp$. In this way we capture the logarithmic divergence in R as a pole in ε

$$\begin{aligned} \langle D_+ D_- \sigma(x) \rangle_N^{(1)} = -h^2 g_2 A_d^3 \pi^{\frac{3p}{2}} \frac{\Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \sum_{n \geq 0} a_n R^{2n} \times \\ \times J^{d-p}_{\Delta_\phi - \frac{p}{2}, 2\Delta_\phi - p + n}(-x_\perp, 0) \ , \end{aligned} \tag{7.41}$$

where $J^n_{a,b}(z, w^2)$ is the master integral (5.26), and a_n is the constant

$$\begin{aligned} a_n &= \frac{(2\Delta_\phi - p)_n}{n!} + \frac{(4\Delta_\phi - 2p)_{2n}}{(2n)!} \\ &= \frac{2(n+1)(2n^2 + 4n + 3)}{3} + \mathscr{O}(\varepsilon) \ . \end{aligned} \tag{7.42}$$

[5] This integral can also be done using Feynman parametrization. Then the integrals over the Feynman parameters simplify greatly in the case of (7.38).

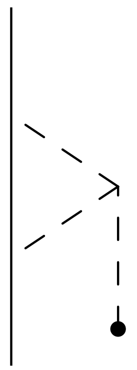

Figure 7.3. The single diagram at $\mathscr{O}(g_2)$ in $\langle D_f \sigma(x)\rangle$.

It gives us the ε-expansion

$$\begin{aligned}\langle D_+ D_- \sigma(x)\rangle_N^{(1)} = &-\frac{h^2 g_2}{16\pi^4 x_\perp^2}\left(\frac{1}{\varepsilon}+\log(x_\perp^2)+\mathscr{A}'\right)+\\ &+\frac{h^2 g_2}{96\pi^4}\sum_{n\geq 1}\frac{2n^2+4n+3}{n}\frac{R^{2n}}{x_\perp^{2(n+1)}}\,,\end{aligned} \tag{7.43}$$

which is not renormalized. We specified to (7.38) to simplify the Taylor expansion of $|z_\perp + s_\perp^\pm|^{-2\Delta_\phi+p}$ in R, and the constant $\mathscr{A}'$ is given by

$$\mathscr{A}' = \log(e^{\gamma_E+1}\pi)\,. \tag{7.44}$$

We will now compute $\langle D_f \sigma(x)\rangle_N^{(1)}$ and see that it exactly equals (7.41). If we specify to (7.38), then

$$D_f = \exp\left(-2h\sum_{n\geq 0}\frac{R^{2n}}{(2n)!}\int_{\mathbb{R}^p} d^p x \partial_R^{2n}\hat{\sigma}(x_+)\right)\,. \tag{7.45}$$

$\langle D_f \sigma(x)\rangle_N^{(1)}$ is found from the single Feynman diagram in Fig. 7.3

$$\begin{aligned}\langle D_f \sigma(x)\rangle_N^{(1)} = &-\frac{4h^2 g_2}{2}\sum_{m,n\geq 0}\frac{R^{2(m+n)}}{(2m)!(2n)!}\int_{\mathbb{R}^d} d^d z\prod_{i=1}^{2}\int_{\mathbb{R}^p} d^p w_i \times\\ &\times \langle\sigma(x)\sigma(z)\rangle_{h=0}\lim_{R'\to 0}\partial_{R'}^{2m}\left.\langle\sigma(z)\sigma(w_\parallel\hat{x}_\parallel + R'\hat{x}_\perp^1)\rangle\right|_{h=0}\times\\ &\times\lim_{R''\to 0}\partial_{R''}^{2n}\left.\langle\sigma(z)\sigma(w_\parallel\hat{x}_\parallel + R''\hat{x}_\perp^1)\rangle\right|_{h=0}\,.\end{aligned}$$

Performing the integration over the parallel coordinates, and differentiating

$$\begin{aligned}\lim_{R'\to 0}\partial_{R'}^{2m}[(z_\perp + x_\perp - R')^2]^{-\Delta_\phi+\frac{p}{2}} &= (2m)!\,b_m|z_\perp + x_\perp|^{-2\Delta_\phi+p-2m}\,,\\ b_m &\equiv \frac{(2\Delta_\phi+p)_{2m}}{(2m)!}\,,\end{aligned} \tag{7.46}$$

gives us

$$\langle D_f \sigma(x)\rangle_N^{(1)} = -2h^2 g_2 A_d^3 \pi^{\frac{3p}{2}} \frac{\Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \sum_{n\geq 0} c_n R^{2n} J^{d-p}_{\Delta_\phi - \frac{p}{2}, 2\Delta_\phi - p + n}(-x_\perp, 0) \,,$$
$$c_n = \sum_{m=0}^{n} b_m b_{n-m} = \frac{a_n}{2} \,. \tag{7.47}$$

This is exactly the same as (7.41), i.e.

$$\langle D_f \sigma(x)\rangle_N^{(1)} = \langle D_+ D_- \sigma(x)\rangle_N^{(1)} \,. \tag{7.48}$$

Thus the fusion (7.45) seems to hold even in the interacting theory. Note that using D_f, instead of $D_\pm$, simplified the Feynman diagram calculation as we did not need to calculate L_-^+ in (7.37) .

Let us now renormalize $\langle D_f \sigma(x)\rangle_N^{(1)}$ by making the following ansatz for the coupling constant on the fused defect

$$h = \mu^{\frac{\varepsilon}{2}} \tilde{h}_f \left(1 + b\frac{\tilde{h}_f \tilde{g}_2}{\varepsilon}\right) \,, \tag{7.49}$$

where b is found by cancelling the ε-pole in (7.41). For this we need the free theory result

$$\begin{aligned}\langle D_f \sigma(x)\rangle_N^{(0)} &= -2A_d \pi^{\frac{p}{2}} h^2 g_2 \frac{\Gamma_{\Delta_\phi - \frac{p}{2}}}{\Gamma_{\Delta_\phi}} \sum_{n\geq 0} \frac{(2\Delta_\phi - p)_{2n}}{(2n)!} \frac{R^{2n}}{x_\perp^{2\Delta_\phi - p + 2n}} \\ &= -\frac{h^2 g_2}{2\pi^2} \sum_{n\geq 1} (2n+3) \frac{R^{2n}}{x_\perp^{2(n+1)}} + \mathscr{O}(\varepsilon) \,.\end{aligned} \tag{7.50}$$

By studying the full one-point function near the fused defect

$$\langle D_f \sigma(x)\rangle_N = \langle D_f \sigma(x)\rangle_N^{(0)} + \langle D_f \sigma(x)\rangle_N^{(1)} \,, \tag{7.51}$$

with the bare coupling (7.49) gives us

$$b = -\frac{1}{8\pi^2} \,. \tag{7.52}$$

This differ from the corresponding coefficient, $b_\pm^\sigma$, in the bare couplings (7.23) on the two defects $D_\pm$ since the bare coupling constant on D_f also take into account divergences in the fusion-limit. This gives rise to a different β-function on D_f

$$\beta_f = -\frac{\varepsilon}{2}\tilde{h}_f - \frac{\tilde{h}_f^2 \tilde{g}_2}{8\pi^2 \varepsilon} \,, \tag{7.53}$$

which has a Guassian f.p. in addition to the following non-trivial f.p.

$$h_f^* = -\frac{4\pi^2\varepsilon}{\tilde{g}_2} = \mp\frac{1}{12}\sqrt{\frac{\pi N\varepsilon}{6}}\ . \tag{7.54}$$

The sign of this coupling is opposite to the bulk couplings at the non-trivial f.p. (7.6). For large values of N this coupling is of finite size.

Finally let us performing the sum over n in (7.43) to get the renormalized correlator at $\mathscr{O}(\varepsilon)$

$$\begin{aligned}\langle D_f\sigma(x)\rangle_N^{(1)} = &-\frac{(h_f^*)^2 g_2^*}{8\pi^4}\left(\frac{\log(x_\perp^2)}{2x_\perp^2} + \frac{1}{4x_\perp^2}\log\left(\frac{x_\perp^2 - R^2}{x_\perp^2}\right)\right. \\ &\left.+\frac{R^2}{3x_\perp^2(x_\perp+R)^2} - \frac{R^2}{6(x_\perp-R)^2(x_\perp+R)^2}\right)\ .\end{aligned} \tag{7.55}$$

It would be interesting to check whether the fusion (7.45) holds at higher orders in the bulk couplings. At $\mathscr{O}(g_2^2)$ there is a diagram in $\delta\langle D_+D_-\sigma(x)\rangle_N^{(2)}$ with a bulk vertex which connects the two defects. We wish to expand its integrand in R before doing any of the integrations over the normal coordinates of the bulk vertices (since otherwise we end up with a L_-^+ integral (7.37) giving us $\log(R)$-terms (7.40)). However, when we Taylor expand the integrand we end up with a rational function in the normal coordinates that we should integrate over (at each order in R). This makes the calculation of this diagram rather difficult as the numerator will contain mixing of e.g. $z_\perp^2 w_\perp^2$ and $(z_\perp w_\perp)^2$ terms (both are to be integrated over $z_\perp, w_\perp \in \mathbb{R}^{d-p}$).

7.5 Line defects near four dimensios

We will end this Chapter by showing that fusion also seems to hold at a quantum level in $d = 4-\varepsilon$. This calculation is similar to that in Sec. 7.4. Near four dimensions we can consider a quartic interaction in the bulk (6.15), with two $p = 1$ dimensional line defects on the form (7.2) (without the σ-term). The bulk coupling has the non-trivial RG f.p. (6.16), and on the defects we have the following attractive f.p.'s [13]

$$h_\pm^* = \pm\sqrt{N+8} \pm \frac{4N^2+45N+170}{4(N+8)^{\frac{3}{2}}}\varepsilon + \mathscr{O}(\varepsilon^2)\ , \tag{7.56}$$

which is of finite size.

Fusing the defects (7.2) with a multivariate Taylor expansion yields

$$D_f = \exp\left(-h\prod_{i=1}^{d-p}\sum_{n_i\geq 0}\frac{\delta^{ji_+} + (-1)^n\delta^{ji_-}}{n_i!}R_i^{n_i}\int_{\mathbb{R}^p} d^p x\partial_i^{n_i}\hat{\phi}^j(x_+)\right)\ . \tag{7.57}$$

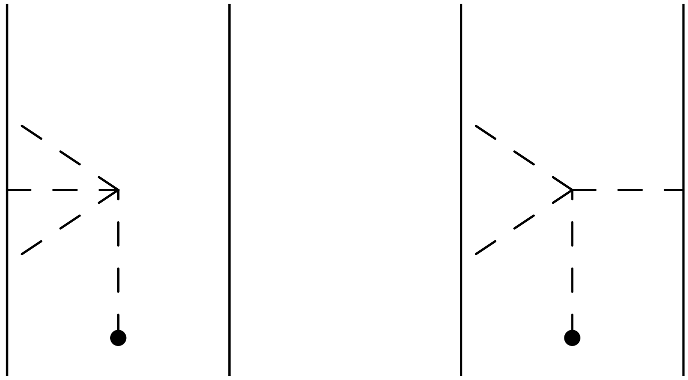

Figure 7.4. The diagrams that contribute to the one-point function of ϕ^i in $d = 4 - \varepsilon$.

This reduces to the form (7.35) when $i_\pm = N$ (under the exchange $\sigma \to \phi^N$).

At $\mathscr{O}(\lambda)$ we find $\langle D_+ D_- \phi^i(x)\rangle_N^{(1)}$ from the diagrams in Fig. 7.4

$$\begin{aligned}\langle D_+ D_- \phi^i(x)\rangle_N^{(1)} = (-h)^3 \left(-\frac{\lambda}{8}\right) \sum_{a=\pm} \bigg(\frac{4!}{3!}\delta^{i i_a}\tilde{L}_a^a + \\ + \frac{8!}{2}(2\,\delta^{i_+ i_-}\delta^{i i_{+a}} + \delta^{i i_{-a}})\tilde{L}_{-a}^{+a}\bigg)\ .\end{aligned} \tag{7.58}$$

This is expressed in terms of the following integral

$$\tilde{L}_b^a = \int_{\mathbb{R}^d} d^d z\, \langle \phi(x)\phi(z)\rangle|_{h=0} K_a(z)^2 K_b(z)\ . \tag{7.59}$$

We will now perform the following steps (in order):

1. Specify to $i_\pm \equiv N$ and one-dimensional R (7.38) (for simplicity).
2. Integrate over the parallel coordinate, $z_\parallel \in \mathbb{R}$, in $\tilde{L}_b^a$.
3. Expand in R.
4. Integrate over $z_\perp \in \mathbb{R}^{d-1}$.
5. Expand in ε.

Doing this yields

$$\begin{aligned}\langle D_+ D_- \phi^i(x)\rangle_N^{(1)} = \frac{h^3 \lambda\, \delta^{iN}}{32\pi^3 |x_\perp|}\bigg(\frac{1}{\varepsilon} + 3(\log|x_\perp| + \mathscr{A}) + \\ + \frac{1}{2}\sum_{n\geq 0} \frac{(n+1)(n+2)}{n(2n+1)} \frac{R^{2n}}{x_\perp^{2n}}\bigg)\ , \\ \mathscr{A} = \log(\sqrt{2}\pi e^{\frac{\gamma_E}{2}+1})\ .\end{aligned} \tag{7.60}$$

On the other hand, for the fused defect we have

$$
\begin{aligned}
\langle D_f \phi^i(x) \rangle_N^{(1)} &= (-h)^3 \left(-\frac{\lambda}{8}\right) \frac{4!}{3!} \delta^{iN} \int_{\mathbb{R}^d} d^d z \, \langle \phi(x)\phi(z) \rangle|_{h=0} \times \\
&\quad \times \prod_{i=1}^{3} \int_{\mathbb{R}^p} d^p w_i \sum_{n_i \geq 0} \frac{R^{2n_i}}{(2n_i)!} \partial_i^{2n_i} \, \langle \phi(z)\phi(w_i) \rangle|_{h=0} \qquad (7.61) \\
&= \frac{4\pi^{2p} A_d^4 h^3 \lambda \, \Gamma^4_{\Delta_\phi - \frac{p}{2}} \delta^{iN}}{\Gamma^4_{\Delta_\phi}} \sum_{n \geq 0} d_n R^{2n} J^{d-p}_{\Delta_\phi - \frac{p}{2}, 3\Delta_\phi - \frac{3p}{2} + n}(-x_\perp, 0) \, ,
\end{aligned}
$$

with the constants[6]

$$
d_n \equiv \sum_{m=0}^{n} b_{n-m} c_m = \frac{(n+1)(n+2)}{2} + \mathscr{O}(\varepsilon) \, . \qquad (7.62)
$$

Here b_m, c_m are given by (7.46, 7.47). If we expand $\langle D_f \phi^i(x) \rangle_N^{(1)}$ at (7.61) in ε we find a perfect agreement with $\langle D_+ D_- \phi^i(x) \rangle_N^{(1)}$ at (7.60). This suggests that the fusion (7.45) is valid in $d = 4 - \varepsilon$ as well.

$\langle D_f \phi^i(x) \rangle_N^{(1)}$ is renormalized by the bare coupling

$$
h = \mu^{\frac{\varepsilon}{2}} \tilde{h}_f \left(1 + \frac{\tilde{\lambda} \, \tilde{h}_f^2}{8\pi^2 \varepsilon} + \mathscr{O}(\tilde{\lambda}^2) \right) \, , \qquad (7.63)
$$

which gives us the following β-function

$$
\beta_f = -\frac{\varepsilon}{2} \tilde{h}_f + \frac{\tilde{h}_f^3 \tilde{\lambda}}{4\pi^2} + \mathscr{O}(\tilde{\lambda}^2) \, . \qquad (7.64)
$$

This β-function has the non-trivial f.p.

$$
h_f^* = \pm \sqrt{\frac{N+8}{8}} + \mathscr{O}(\varepsilon) \, . \qquad (7.65)
$$

Note that this differs from (7.56) by a factor of $\sqrt{8}$ at $\mathscr{O}(\varepsilon^0)$. Finally we can resum $\langle D_f \phi^i(x) \rangle_N^{(1)}$ to find

$$
\begin{aligned}
\langle D_f \phi^i(x) \rangle_N^{(1)} &= \frac{(h_f^*)^3 \lambda^* \delta^{iN}}{32\pi^3} \left(\frac{3}{|x_\perp|} \log|x_\perp| + \frac{1}{|x_\perp|} \log\left(\frac{x_\perp^2 - R^2}{x_\perp^2}\right) + \right. \\
&\quad \left. + \frac{3}{4R} \operatorname{arctanh}\left(\frac{R}{|x_\perp|}\right) - \frac{1}{2|x_\perp|} - \frac{x_\perp^2}{4(x_\perp^2 - R^2)} \right) + \mathscr{O}(\varepsilon^2) \, ,
\end{aligned}
$$

where

$$
(h_f^*)^3 \lambda^* = \pm \sqrt{\frac{N+8}{2}} \pi \varepsilon + \mathscr{O}(\varepsilon^2) \, . \qquad (7.66)
$$

[6] In general this constant is given by two ${}_2F_1$ hypergeometric functions. For $n = 0$ the corrections in ε are zero.

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Svensk sammanfattning

En kvantfältteori (QFT) är en fysikalisk teori som används för att teoretiskt beskriva hur partiklar interagerar med varandra på atomnivå, det vill säga partikelfysik. En välkänd sådan teori är kvantkromodynamik (QCD) som beskriver hur kvarkar[1] interagerar med varandra. Denna teori testas aktivt i partikelacceleratorn LHC i CERN, där den ger mycket hög förutsägbarhet. En QFT används även för att beskriva statistiska system inom kondenserad materia, och ger på så sätt effektiva modeller för olika material.

I en QFT kombineras tre olika fysikområden: fältteori, speciell relativitetsteori och kvantmekanik. Fältteorier beskriver ett fälts, eller en vågs, rörelse istället för enskilda partiklars rörelse. De definieras utifrån rörelseekvationer, även kallade fältekvationer. Den mest välkända fältteorin är elektromagnetismen, som uppfyller de berömda Maxwell-ekvationerna, efter James Clerk Maxwell 1861 och 1862.

Den speciella relativitetsteorin appliceras vid höga hastigheter relativt ljusets och utvecklades av Albert Einstein 1905. Den innefattar den berömda formeln "$E = mc^2$". Denna formel beskriver all massa som energi, vilket innebär att partiklar kan skapas och annihileras så länge energin är konstant.

Slutligen har vi kvantmekaniken, vilken beskriver fysiken på atomnivå. Kvantmekaniken ger statistiska modeller. Till skillnad från den klassiska mekaniken ges alltså inga exakta resultat, utan sannolikheter för olika utfall. Kvantmekaniken är även betydelsefull inom kemin.

I denna avhandling studeras defekter i QFT. En defekt är ett objekt av högre dimension, till exempel en linje eller en yta, istället för en punktpartikel. En defekt kan betraktas som en QFT av lägre dimension inuti den ursprungliga teorin. Detta leder till en utmaning att studera defekter.

Defekter har flera fysikaliska tillämpningar. Exempelvis används de för att beskriva Kondō effekten i metaller, efter Jun Kondō 1964. Kondō effekten beskriver hur elektroner interagerar mot orenheter i metallen.[2] En vanligt förekommande defekt är Wilsonloopen, efter Kenneth Wilson 1974. Den används för att beskriva kvarkinneslutning i QCD, vilket innebär att kvarkar inte kan isoleras utan istället alltid förekommer i grupper om två och två (mesoner) eller tre och tre (baryoner). Wilsonloopar används även i en topologiska QFT[3] för att beskriva det fysikaliska fenomenet fraktionerad kvantmekanisk Halleffekt. Detta fenomen förutspår konduktansen för en elektron i två dimen-

[1] Beståndsdelarna i en proton eller en neutron.

[2] Områden där metallens mikroskopiska atomstruktur avviker.

[3] En QFT som är invariant under topologiska transformationer av rumtiden

sioner och dess upptäckt ledde till nobelpriset i fysik 1998 som gick till Robert Laughlin, Horst Störmer och Daniel Tsui.

En annan sorts defekt är en rand i en QFT. Randen halverar rumtiden dessa teorier lever i. Den kan användas för att modellera ämnet grafen. Detta ämne består av ett endimensionellt tunt lager kol som beskrivs av randen. Andre Geim och Konstantin Novoselov tilldelades nobelpriset i fysik 2010 för deras studier på grafen. En rand kan också användas för att beskriva Casimireffekten, efter Hendrik Casimir 1948, vilken beskriver kraften som uppstår mellan två metallplattor i kvantelektrodynamik.

Denna avhandling är en genomgående och teoretisk studie av defekter i deras allmänhet i QFT. Avhandlingen fokuserar inte enbart på system med en defekt, utan även system med flertalet defekter. I avhandlingen presenteras de sju vetenskapliga artiklarna I - VII jag varit med och skrivit under min doktorand. Dessutom redogörs generaliseringar av flera av dessa artiklar, vilka har givit flertalet nya intressanta resultat.

Vi studerar renormering, en metod för att hantera divergenser som förekommer i en QFT, när en defekt är närvarande. Detta gjordes i detalj i artikel III för en rand och leder till ett renormeringsflöde, vilket beskriver hur en QFT beter sig vid olika energiskalor.

Det finns speciella fixpunkter i detta flöde där en QFT har en utökad konform symmetri vilket resulterar i en konform fältteori (CFT). Den konforma symmetrin innebär bland annat att vi kan skala om rumtiden. Genom att studera dessa symmetritransformationer kan vi lära oss mycket om en CFT. Bland annat leder det till bootstrapekvationen, vilken används för att beskriva egenskaper hos de fält som befinner sig i teorin (CFT-data). En metod för hur denna ekvation kan lösas i närheten av en rand utvecklades i artikel II och V. Metoden beskrivs i kapitel 4 där den generaliseras så att den håller för två olika fält, istället för två identiska fält. Vi visar även hur liknande metoder kan användas för att utvinna ännu mer CFT-data ur bootstrapekvationen.

Utöver detta så appliceras även Coleman-Weinberg-mekaniken (CW-mekaniken), efter Sidney Coleman och Erick Weinberg 1973, vilken låter oss följa flödeslinjer i renormeringsflödet från den konforma fixpunkten. Vilket ger en effektiv teori för att beskriva första ordningens fasövergång. Ett exempel på en sådan fasövergång är när is blir till flytande vatten vid noll grader Celsius. I artikel IV studeras CW-mekaniken för en rand, och i denna avhandling diskuteras även hur den skulle se ut för en mer allmän defekt. I kapitel 5 illustreras CW-mekaniken för en mer komplicerad rand än den i artikel IV.

I kapitel 5 appliceras även metoden som används i artikel VII. Denna artikel behandlar korrelatorer[4] för ett system nära två korsande ränder och hur de kan hittas genom rörelseekvationen. I kapitel 5 appliceras denna metod till ett system med en rand, vilket blir aningen mer komplicerat på grund av särskilda tekniska skäl.

[4]Beskriver hur ett fält fortplantar sig i rummet.

Artikel I är en studie på hur en särskild sorts defekt, av kodimension två, bryter symmetrier av den ursprungliga teorin. Detta resultat generaliseras i kapitel 6 för att gälla en mer allmän defekt, en replika vridningsdefekt. En sådan defekt genererar kopior, eller replikor, av den ursprungliga teorin och används för att hitta en fysikalisk sammanflätningsentropi. Denna entropi är ett mått på hur mycket information det finns i en QFT.

Slutligen diskuteras artikel VI som behandlar fusion av två defekter. Även denna artikel generaliseras i avhandlingen. I kapitel 7 tas först fram ett mer allmänt resultat av denna fusion, som sedan visas hålla även när teorin kvantiseras. Detta görs för skalära Wilson-defekter, vilka beskriver ett magnetiskt fält längs en linje eller en yta.

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