Geometric phases in weak measurements

Author: Isabela Dias
Supervisor: Erik Sjöqvist

February 24, 2023

Abstract

We explore a natural extension of the Abelian geometric phase factor in sequences of incomplete measurements to include high dimensional quantum states in the weak measurement scenario. To achieve the goal, we focus on sequential weak measurements of non-commuting projectors and combine the idea of weak measurements and the notion of a non-Abelian geometric phase. In this way, we find that the non-Abelian geometric phase in the weak measurement scenario can be useful to gain information about the space of states and the connection between them, favorable to reproducing the overlap matrix and the Wilson Loop.

Contents

1 Introduction

Quantum information science deals with knowledge from the most fundamental level of nature. Over the past decades, the mathematical methods joined to the improvement of experimental techniques become very efficient to offer coherent control of quantum systems and their interactions [1, 2].

The intensity of interactions in quantum systems appears to be determinant to define the type and amount of information that can be extracted from a system at different scales of distance and temperature. Experimental settings accomplishing small coupling parameters between the system and device lead to the concept of weak measurements. Within this context, the weak measurement of quantum ensembles provides useful partial information or insights regarding the probabilistic nature of quantum theory. This sampling method would be reasonable to retrieve information on the space of states. To access this structure we use the concept of geometric phase (GP).

Anandan and Pines [3] proposed a natural extension of GP to the projective Hilbert space by removing the adiabatic condition. This means no external parameter space is required to describe a cyclic evolution. The application of non-commutativity of these phases is useful to implement a universal set of quantum gates to be used for robust all-geometric quantum computation [4].

This project investigates the connection between weak measurements and GP in the non-Abelian scenario. It becomes a reasonable scheme because the GP could be physically manipulated in some experimental settings. For instance, intriguing amplification phenomena have been studied in interferometer setups. Our scheme is deeply based on the Anandan and Pines approach to reconstructing the non-Abelian holonomies.

This report is organized as follows. Section 2 gives a brief overview of the fundamental concepts in the literature and a mathematical formalism. The next section looks at the main findings. A discussion of the results is given in Section 4. The final considerations are given in Section 5.

2 Conceptual Preliminaries

Most of this section is devoted to outlining the past-present history of the weak measurements and GP in the Abelian and non-Abelian schemes.

2.1 Two State-vector Formalism

Aharonov, Bergmann, and Lebowitz introduced the time-symmetric formulation of quantum theory known as two-state vector formalism (TSVF) [5]. Based on the idea of "reduction of the wave packet", this scheme uses the interplay between the past and future of the strong measurement process with minimum disturbance in between them. A generalized quantum state at a given time t can be described by

$$|\psi(t)\rangle = \langle c_{k''}(t)|\, |a_{k'}(t)\rangle\,, \tag{1}$$

where the state $|a_{k'}\rangle$ is defined by the results of strong von-Neumann measurements performed in the relative past (pre-selection) at the time $t_1 < t$ of a backward evolving quantum state $\langle c_{k''}|$ defined by the results of strong measurements performed on this system in the future at $t_2 > t$ (post-selection). Henceforth, we omitted the time in the notation.
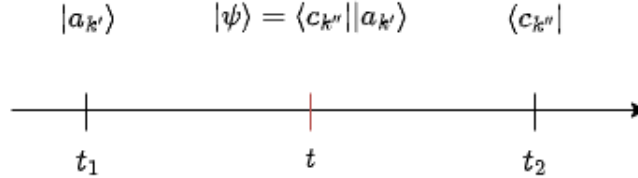
Figure 1: Two-state vector formalism in the weak measurement scheme, $|\psi\rangle, |a_{k'}\rangle$ and $\langle c_{k''}|$ are the generalized quantum state, pre-selected and post-selected states, respectively.

Fig. (1) shows that the key idea is to recover the expressions for prediction from time-symmetric expressions by separating the final (initial) selection procedure from the measurements under consideration by the sequence of strong measurements. For instance, Ref. [6] showed that weakly measuring some systems can minimize decoherence effects.

4

2.2   Weak Measurements

The concept of the weak value was introduced by Aharonov, Albert and Vaidman [7] in the context of the TVSF. Let an ensemble of particles prepared in the initial state $|a_{k'}\rangle$ and the final state $|c_{k''}\rangle$. The system and device interact during a finite time as described by the Hamiltonian

$$H(t) = -g(t)Q\mathcal{P}^B, \tag{2}$$

where $g(t)$ is a time-dependent coupling function, and Q is an operator associated with the canonical variable of the measurement device. We focus on the case where the weakly measured observable $\mathcal{P}^B$ is a rank-one projection operator.

At a time in between we switch on the interaction (2) where the initial state of each measurement apparatus is $[1/\sqrt{\Delta}(2\pi)^{1/4}]\exp(-q^2/4\Delta^2)$. Then, after the post-selection

$$\langle c_{k''}| \, e^{-i\int g(t)q\mathcal{P}^B dt} \, |a_{k'}\rangle \, e^{-q^2/4\Delta^2} \tag{3}$$

with measurement strength

$$\kappa = \frac{1}{\hbar}\int g(t)dt, \tag{4}$$

where $2\pi\hbar$ is Planck's constant.

Now, if the disturbance (Δ) is sufficiently small, we expand the exponential term

$$\langle c_{k''}| \, e^{-i\kappa q\mathcal{P}^B} \, |a_{k'}\rangle \, e^{-q^2/4\Delta^2}$$

$$\cong \langle c_{k''}|a_{k'}\rangle \left[1 + \frac{iq\kappa(\langle c_{k''}|\mathcal{P}^B|a_{k'}\rangle)}{\langle c_{k''}|a_{k'}\rangle}\right] e^{-q^2/4\Delta^2}$$

$$\cong \langle c_{k''}|a_{k'}\rangle \, e^{\left[\frac{iq\kappa(\langle c_{k''}|\mathcal{P}^B|a_{k'}\rangle)}{\langle c_{k''}|a_{k'}\rangle}\right]} e^{-q^2/4\Delta^2}$$

$$\cong \langle c_{k''}|a_{k'}\rangle \, e^{\left[iq\kappa(\mathcal{P}^B_{c,a})_w\right]} e^{-q^2/4\Delta^2} \tag{5}$$

for Δ such that

$$\Delta\kappa \ll \frac{|\langle c_{k''}|a_{k'}\rangle|}{(|\langle c_{k''}|\mathcal{P}^B|a_{k'}\rangle|^{1/n})}, \tag{6}$$

where

$$(\mathcal{P}_{c,a}^{B})_w = \frac{\langle c_{k''}|\mathcal{P}^{B}|a_{k'}\rangle}{\langle c_{k''}|a_{k'}\rangle} = \frac{\langle c_{k''}|b\rangle\,\langle b|a_{k'}\rangle}{\langle c_{k''}|a_{k'}\rangle} \qquad (7)$$

is the weak value of the operator $\mathcal{P}^{B}$ with respect to the pre-selected state $|a_{k'}\rangle$ and the post-selected state $|c_{k''}\rangle$.

The resulting wave packet $\psi(q)$ of the measuring apparatus reads

$$\psi(q) = e^{iq\kappa \mathrm{Re}(\mathcal{P}_{c,a}^{B})_w} e^{\left[-\frac{1}{4\Delta^2}(q+2\kappa\Delta^2\mathrm{Im}(\mathcal{P}_{c,a}^{B})_w)^2\right]}. \qquad (8)$$

Notice that the weak value has both real and imaginary parts.

The post-selection procedure causes a shift in the position of the pointer by a factor

$$\delta q = -\kappa\Delta^2 \mathrm{Im}(\mathcal{P}_{c,a}^{B})_w \qquad (9)$$

and its momentum by

$$\delta p = \hbar\kappa \mathrm{Re}(\mathcal{P}_{c,a}^{B})_w, \qquad (10)$$

where the shift in the canonical variables of the measurement apparatus is related to the real and imaginary parts of the weak values [8]. From (6), the uncertainty for p for each measurement apparatus is $1/2\Delta\kappa$, much bigger than the measured value. However, for an ensemble of N devices sufficiently large, then $(1/\sqrt{N})\Delta p \ll (\mathcal{P}_{c,a}^{B})_w$ can be ascertained with arbitrary accuracy.

Here, we deal with a special case whereby a projection operator is considered. This type of operator is useful because of the weak coupling between the device and system, and also to combine different types of devices.

2.2.1 Conditional Probability

A natural question that arises is how can we get the probabilities by weakly measuring the quantum ensemble, i.e., in the TSVF.

The weak measurement of a filtering measure P_1 would yield the eigenvalue a_w with probability

$$P(a_w|c,a) = \frac{|\langle c_{k''}|a_w\rangle|^2|\langle a_w|a_{k'}\rangle|^2}{\sum_i |\langle c_{k''}|a_i\rangle|^2|\langle a_i|a_{k'}\rangle|^2} = \frac{|\langle c_{k''}|P_1|a_{k'}\rangle|^2}{\sum_i |\langle c_{k''}|P_1|a_{k'}\rangle|^2}, \qquad (11)$$

where P_1 is the projection operator defined previously. Note that the probability of getting an eigenvalue a_w is a conditional probability that depends on the pre-selected and post-selected states. This equation is very intuitive to show that both pre-selected and post-selected states are equally important to perform weak measurements.

6

2.3 Geometric Phase

In this section, we briefly review the broad perspectives of the non-Abelian geometric phase. When a quantum system evolves under a cyclic evolution it may acquire an additional geometric entity in contrast to the dynamical one called geometric phase (GP). The GP appears in several systems, such as interference experiments with photons, condensed matter systems, and cold atoms [2].

Berry [9] demonstrated that the GP arises from the geometric structure of the space of states after a cyclic evolution in the adiabatic regime. In advance on this subject, Simon [10] connected Berry's GP with the holonomy of a closed path. Wilczek and Zee [11] demonstrated that non-Abelian gauge structures arise in simple quantum systems.

A natural extension of the GP for the non-adiabatic scheme was given by Anandan and Pines [12]. This generalization of the GP allows to explore the relation between the Berry potential and the curvature of projective Hilbert spaces. Here, we highlighted that the holonomy can be interpreted as the indicator of confining behavior in lattice gauge theory [13–17].

2.3.1 The adiabatic GP

This subsection is based on Berry's article [9].

During a cyclic quantum evolution, the slow changing of external parameters gives rise to an adiabatic GP. Let the Hamiltonian H be slowly changed by varying external parameters $\mathbf{R} = (r_1, r_2, ...)$, such that $H(\mathbf{R})$. These parameters can be viewed as points of a manifold M, such that $\mathbf{R} = (r_1, r_2, ...) \in M \to H(\mathbf{R})$. Assuming that the spectrum of H is non-degenerate, the time evolution is governed by the Schrödinger equation

$$H(\mathbf{R}(t)) \left| \psi(t) \right\rangle = i\hbar \frac{d}{dt} \left| \psi(t) \right\rangle. \tag{12}$$

The eigenstates $\left| n(\mathbf{R}) \right\rangle$ of $H(\mathbf{R})$ satisfy

$$H(\mathbf{R}) \left| n(\mathbf{R}) \right\rangle = E_n \left| n(\mathbf{R}) \right\rangle \tag{13}$$

with eigenergies $E_n(\mathbf{R})$. In the adiabatic limit, $\left| \Psi(0) \right\rangle = \left| n(\mathbf{R}(0)) \right\rangle$, i.e., the solution of the Schrödinger equation is an eigenfunction of the lowest energy of H. We can write, at t

$$|\psi(t)\rangle = \mathrm{e}^{\frac{-i}{\hbar}\int_0^t dt' E_n(\mathbf{R}(t))}\mathrm{e}^{i\gamma_n(t)}|n(\mathbf{R}(t))\rangle = \mathrm{e}^{i\phi}\mathrm{e}^{i\gamma_n(t)}|n(\mathbf{R}(t))\rangle, \qquad (14)$$

where is ϕ the dynamical phase. However, the term $\gamma_n(t)$ is non-integrable, i.e., $\gamma_n(t)$ cannot be written as a function of $\mathbf{R}$ and may assume different values for each t.

We can evaluate $\gamma_n(t)$ making the requirement that $|\psi(t)\rangle$ must satisfy the Schrödinger equation. The direct substitution of (14) into (12) leads to

$$\frac{d}{dt}\gamma_n(t) = i\langle n(\mathbf{R}(t))|\nabla_R|n(\mathbf{R}(t))\rangle \cdot \frac{d}{dt}\mathbf{R}(t) \qquad (15)$$

The total phase shift of (14) around the closed path $\mathcal{C}$ is given by

$$|\psi(t)\rangle = e^{i\gamma_n(\mathcal{C})}e^{\frac{-i}{\hbar}\int_0^t dt' E_n(\mathbf{R}(t))}|\psi(0)\rangle, \qquad (16)$$

where we define the GP

$$\gamma_n(\mathcal{C}) = i\oint_{\mathcal{C}}\langle n(\mathbf{R})|\nabla_R|n(\mathbf{R})\rangle \cdot d\mathbf{R} \qquad (17)$$

as the Berry phase. Note that the closed integral in parameter space is dependent on the traversed loop.

We may rewrite (17) as,

$$\gamma_n(\mathcal{C}) = i\oint_{\mathcal{C}}\mathcal{A}^n(\mathbf{R}) \cdot d\mathbf{R}, \qquad (18)$$

where the quantity $\mathcal{A}^n(\mathbf{R})$ is called Berry connection coupled to the slow degrees of freedom. The Berry connection is a vector potential as in classical electrodynamics and its integral around $\mathcal{C}$ is analog to the magnetic flux [11].

2.3.2 GP and Holonomy

Simon [10] was the first to conciliate the concept of holonomy and the GP. The main idea concerns the parallel transport along loops in a fiber bundle. This means that in the geometric view, a loop corresponds to a cyclic path traced by a tangent vector at $\mathbf{R}$ of a manifold M endowed with a linear combination, the idea of parallel transportation along a closed curve [18].

In general, when a vector is parallel transported along a loop, the resulting vector is different from the original one. The difference can be related to the curvature of the connection between them. This scheme was used to describe the effects of the two-dimensional electron gas in a uniform magnetic field, known as the quantum Hall effect [18, 19].

Considering the same loop $\mathcal{C}(t)$ and a choice of basis $|n(\mathbf{R}(0))\rangle$ of the previous section. If the vector $|n(\mathbf{R}(0))\rangle$ is parallel transported along this curve, the single-valued GP (17) is equivalent to the holonomy associated with this connection.

Wilczek and Zee [11] proposed an extension of GP to the non-Abelian structure, which is a matrix-valued holonomy. Let the Hamiltonian $H(\mathbf{R})$ depending on the set of external parameters. For an arbitrary set of basis $|\psi_a(t)\rangle$ we can set

$$H(|n(\mathbf{R}(t))\rangle)|\psi_a(t)\rangle = 0 \tag{19}$$

where this choice can be made locally.

In the adiabatic limit, we consider the solutions of the Schrödinger equation (12) such that $|\eta_a(0)\rangle = |\psi_a(0)\rangle$. Naturally, we can decompose a general state at t

$$|\eta_a(t)\rangle = U_{ab}(t)|\psi_b(t)\rangle, \tag{20}$$

and where the eigenstate $|\eta_a(t)\rangle$ is normalized .

The goal is to define the holonomy $U(t)$ using the previous assumption (20). The normalization condition requires that

$$\langle\eta_b|\frac{d}{dt}\eta_a\rangle = \langle\eta_b|\frac{d}{dt}U_{ac}|\eta_c\rangle + \langle\eta_b|U_{ac}|\frac{d}{dt}\eta_c\rangle = 0, \tag{21}$$

where the vector potential

$$\langle\psi_b|\frac{d}{dt}\psi_a\rangle = \langle\eta_b|U^{-1}\frac{d}{dt}U|\eta_a\rangle = A_{ab}, \tag{22}$$

where A_{ab} is matrix-valued and depends on the geometry of the space of degenerate levels. We can write the above equation in terms of a path-ordered integral

$$U(t) = \mathcal{P}e^{\int_0^t A(\tau)d\tau}. \tag{23}$$

9

Note that the ordered integral depends only on the path and not on its parametrization. In particular, for a closed path the integral is the Wilson Loop (WL), which is gauge invariant.

If we set a different set of basis

$$|\psi'(t)\rangle = \Omega(t) |\psi(t)\rangle, \tag{24}$$

where $\Omega(t)$ is a function of t.

The A field transform as

$$A'(t) = \frac{d}{dt}\Omega\Omega^{-1} + \Omega A \Omega^{-1}, \tag{25}$$

as a proper gauge potentials.

2.3.3   Non-Adiabatic GP

Anandan and Pines [12] extended the non-adiabatic GP to a cyclic evolution resulting from a sequence of filtering measurements. Consider a cyclic evolution of an n-dimensional subspace of the $(n+m)$-dimensional Hilbert space $\mathcal{H}$. Let $G_{m,n}$ be the Grassmann manifold as a basis manifold containing all the n-dimensional subspaces of $\mathcal{H}$. We call a n-frame a set of n-orthogonal vectors $\{|b_i\rangle\}$ of $V_n \in G_{m,n}$ with associated projection operator

$$P = \sum_{i=1}^{N} |b_i\rangle \langle b_i|, \tag{26}$$

where the operator P is independent of the chosen orthonormal basis and therefore invariant under the unitary group $U(n)$ between the orthonormal basis of V_n.

Now, the manifold $G_{m,n}$ can be connected to the set of rank-n projection operators P uniquely associated with the subspaces V_n. The set of $(n+m)$-frames can be identified with the group $U(n+m)$, and V_n with the equivalence class of $(n+m)$-frames each consisting of n vectors in V_n and m vectors in the orthogonal complement V_m of V_n in $\mathcal{H}$. Naturally, we can also identify the Grassmann manifold as $G_{m,n} = U(n+m)/U(n) \times U(m)$.

Similarly, we define the Stiefel manifold $S_{m,n}$ as the set of n-frames which $S_{m,n} = SU(n+m)/SU(m)$. At this point, we summarize that $U(m+n)$ is a $U(m)$-principal fiber bundle over $S_{m,n}$ with projection map Φ, while $S_{m,n}$

is $U(n)$-principal fiber bundle over $G_{m,n}$ with $U(n) \times U(m)$ as the structure group and projection map $\chi = \Pi\Phi$.

There is a connection in the bundle $S_{m,n}$ over $G_{m,n}$ whose connection one-form with respect to a field of n-frames $\{|b_i\rangle\}$ on $G_{m,n}$ is $B_{ij} = i\langle\tilde{\Psi}_i|d\tilde{\Psi}_j\rangle$. The orthonormality of $\{|b_i\rangle\}$ implies that B_{ij} is a Hermitian matrix, i.e., is in the Lie algebra of $U(n)$.

Anandan [3, 20] produced extensive works showing that this connection gives the non-Abelian GP in the cyclic evolution in a closed curve $\mathcal{C}$ in $G_{m,n}$. The formalism can be extended to n successive incomplete measurements. Two parallel bases in subspaces corresponding to P and $P' \in G_{n,m}$ are related by parallel transport along the geodesic C joining P and P'. For more technical details, see Ref. [12] .

2.3.4 Formalism

Following Ref.[21], consider a sequence $\mathcal{C}$ of discrete points $p_1, p_2, \ldots, p_m$ in Grassmann manifold with arbitrary subspace of dimension K. There is a natural bijection between the Grassmann manifold and the collection of projectors of rank K. Thus we may associate $\mathcal{C}$ to a sequence $\mathcal{C}'$ of projectors $P_1, P_2, \ldots, P_m$. We construct a sequence of filtering measurements

$$\Gamma[\mathcal{C}] = P_1, P_2, \ldots, P_m. \tag{27}$$

Here, we consider an ensemble of particles where the pre- and post-selected states, $\mathcal{F}_a = \text{span}\{|a_{k'}\rangle\}_{k=1}^{N}$ and $\mathcal{F}_c = \text{span}\{|c_{k''}\rangle\}_{k=1}^{N}$, which are non-orthogonal. The set of frames constitutes a Stiefel manifold, which is a fiber bundle with the Grassmanian as the base manifold and the set of K-dimensional unitary matrices as a fiber. We define the overlap matrix $\mathcal{F}_{(c,a)}$ [22]

$$(\mathcal{F}_c|\mathcal{F}_a) = (\mathcal{F}_{c,a}) = \langle c_{k''}|a_{k'}\rangle, \tag{28}$$

where each matrix element corresponds to the inner product between a pair of states in the associated subspaces.

As highlighted by [4, 21], in order to associate the non-adiabatic GP to the product of overlap matrices, we require that the overlap matrices are unitary up to a multiplicative factor. On the contrary, is only possible to get

the holonomy by performing unitary operations on the system. The polar decomposition $|(\mathcal{F}_c|\mathcal{F}_a)|U_{c,a}$ of the overlap, where $|(\mathcal{F}_c|\mathcal{F}_a)| = \sqrt{(\mathcal{F}_c|\mathcal{F}_a)(\mathcal{F}_a|\mathcal{F}_c)}$ leads to definition of the relative phase between frames

$$U_{c,a} = \kappa_{c,a}^{-1}(\mathcal{F}_c|\mathcal{F}_a), \tag{29}$$

where $\kappa_{c,a}$ is a number that represents the transition probability in interferometric setups and $U_{c,a}$ a unitary matrix for a discrete sequence $\mathcal{C}$.

2.4   Spin Coherent States (SCSs)

Arecchi et al.[23] introduced the concept of atomic coherent states for the description of two-level atoms. They are defined by the angular momentum operator in the Hilbert space as irreducible representations of some symmetry Lie group.

Here, we apply the three-spin-$\frac{1}{2}$ setting formalism to the case of general spin j. Consider a quantum device characterized by the unit vector $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in spherical coordinates. In the weak measurements scenario, the filtering measurements select the maximal angular momentum quantum numbers, $m = \pm j$. The selection corresponds to the 2-rank operators $\mathcal{P}^{\mathbf{n}_\alpha} = |j; \mathbf{n}_\alpha\rangle \langle j; \mathbf{n}_\alpha| + |j; -\mathbf{n}_\alpha\rangle |j; -\mathbf{n}_\alpha\rangle = \sum_{k=\pm} |j; k\mathbf{n}_\alpha\rangle \langle j; k\mathbf{n}_\alpha|$. The use of SCSs simplifies the subsequent calculations since $|j; \mathbf{n}\rangle$ can be viewed as a product of $2j$ copies of spin-$\frac{1}{2}$ state $|\frac{1}{2}; \mathbf{n}\rangle$ and $|-j; \mathbf{n}\rangle$ similarly as $2j$ copies of $|-\frac{1}{2}; \mathbf{n}\rangle$ [4, 21].

The overlap matrix takes the form [21]

$$(\mathcal{F}(\theta_c, \phi_c)|\mathcal{F}(\theta_b, \phi_b)) = \begin{pmatrix} R(c,b) & S(c,b) \\ (-1)^{2j}S(c,b)^* & R(c,b)^* \end{pmatrix}, \tag{30}$$

where

$$R(c,b) = \left[\cos\left(\frac{\theta_c - \theta_b}{2}\right)\cos\left(\frac{\phi_c - \phi_b}{2}\right) + i\cos\left(\frac{\theta_c + \theta_b}{2}\right)\sin\left(\frac{\phi_c - \phi_b}{2}\right)\right]^{2j},$$

$$S(c,b) = \left[\sin\left(\frac{\theta_c - \theta_b}{2}\right)\cos\left(\frac{\phi_c - \phi_b}{2}\right) - i\sin\left(\frac{\theta_c + \theta_b}{2}\right)\sin\left(\frac{\phi_c - \phi_b}{2}\right)\right]^{2j}.$$
$$\tag{31}$$

For any $j \in \frac{1}{2}\mathbb{N}$, we can find an irreductible representation of $SU(2)$. If j is half-odd integer, then there is a projective representation of $SO(3)$. If $j \in \frac{1}{2}\mathbb{N}$,

$$(\mathcal{F}(\theta_c, \phi_c)|\mathcal{F}(\theta_b, \phi_b)) = \sqrt{|R(c,b)|^2 + |S(c,b)|^2}U_{c,b}, \tag{32}$$

where $U_{c,b}$ is the relative phase associated to the transformation.

3 Results

Here, we propose two different forms of writing the weak values. The first consists of the components of the pre- and post-selected states, and the projection operator within the TSVF. The formalism can be extended to high-dimensional quantum systems. For instance, it can be applied when a photon carries many degrees of freedom. The second one corresponds, to the vectorial form. In this context, we are interested in the direction of the pointer's device, i.e., the angles of the pre-and post-selected states as well as the projection operator of the weak measurement as degrees of freedom.

3.1 Weak Measurements and Values

From now on, we write the pre- and post-selected SCSs, and the projection operator in terms of vectors in spherical coordinates, $|a_{k'}\rangle = |j; k'\mathbf{n}\rangle$, $\langle c_{k''}| = \langle j; k''\mathbf{m}|$ and $\mathcal{P}^{\mathbf{n}_\alpha} = \sum_{k=\pm} |j; k\mathbf{n}_\alpha\rangle \langle j; k\mathbf{n}_\alpha|$, with indices $k, k', k'' = \pm$ as defined in Section 3. The overlap between two such states is [24]

$$\langle j; k''\mathbf{m}|j; k'\mathbf{n}\rangle = e^{ij\Phi(\mathbf{n},\mathbf{m})} \left(\frac{1 + k''k'\mathbf{n} \cdot \mathbf{m}}{2}\right)^j,\tag{33}$$

where $\Phi(\mathbf{n}, \mathbf{m})$ is a real number. Then, the weak values become

$$(\mathcal{P}^{\mathbf{n}_\alpha}_{k'',k'})_w = \frac{\langle j; k''\mathbf{m}|j; k\mathbf{n}_\alpha\rangle \langle j; k\mathbf{n}_\alpha|j; k'\mathbf{n}\rangle \langle j; k'\mathbf{n}|j; k''\mathbf{m}\rangle}{|\langle j; k''\mathbf{m}|j; k'\mathbf{n}\rangle|^2}$$

$$= \frac{e^{ij\Phi(k\mathbf{n}_\alpha,k''\mathbf{m})}e^{ij\Phi(k'\mathbf{n},k\mathbf{n}_\alpha)}e^{ij\Phi(k''\mathbf{m},k'\mathbf{n})} \left(\frac{1+kk''\mathbf{n}_\alpha\cdot\mathbf{m}}{2}\right)^j \left(\frac{1+kk'\mathbf{n}\cdot\mathbf{n}_\alpha}{2}\right)^j}{\left(\frac{1+k'k''\mathbf{m}\cdot\mathbf{n}}{2}\right)^j},\tag{34}$$

which is defined in terms of the vectors $\mathbf{n}, \mathbf{m}, \mathbf{n}_\alpha$. If the spherical triangles $\{\mathbf{n}, \mathbf{m}, \mathbf{n}_\alpha\}$ is an Euler triangle, then we get the Abelian GP

$$(\mathcal{P}^{\mathbf{n}_\alpha}_{k'',k'})_w = \frac{e^{ij\Delta_P(k''\mathbf{m},k\mathbf{n}_\alpha,k'\mathbf{n})} \left(\frac{1+kk''\mathbf{n}_\alpha\cdot\mathbf{m}}{2}\right)^j \left(\frac{1+kk'\mathbf{n}\cdot\mathbf{n}_\alpha}{2}\right)^j}{\left(\frac{1+k'k''\mathbf{m}\cdot\mathbf{n}}{2}\right)^j},\tag{35}$$

where the quantities $e^{ij\Delta_P(k''\mathbf{m},k\mathbf{n}_\alpha,k'\mathbf{n})}$ are exactly the Pancharatnam GP [25].

3.2 Reconstruction scheme using weak values

The knowledge of a quantum process is an important task in quantum information. In this section, we define our approach and show how it can be used to define the product of overlap matrices. In the following sections, we show how this can be used to get the non-Abelian and Abelian GPs. This means that we can recover partial information about the space of states weakly measuring a quantum system.

Consider the special case in which we perform a weak measurement of the projection operator of rank-2. The resulting product of the overlap matrices (29) takes the form

$$
\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = \begin{pmatrix} \langle j;\mathbf{m}|j;\mathbf{n}_\alpha\rangle & \langle j;\mathbf{m}|j;-\mathbf{n}_\alpha\rangle \\ \langle j;-\mathbf{m}|j;\mathbf{n}_\alpha\rangle & \langle j;-\mathbf{m}|j;-\mathbf{n}_\alpha\rangle \end{pmatrix} \begin{pmatrix} \langle j;\mathbf{n}_\alpha|j;\mathbf{n}\rangle & \langle j;\mathbf{n}_\alpha|j;-\mathbf{n}\rangle \\ \langle j;-\mathbf{n}_\alpha|j;\mathbf{n}\rangle & \langle j;-\mathbf{n}_\alpha|j;-\mathbf{n}\rangle \end{pmatrix}
$$
$$
\times \begin{pmatrix} \langle j;\mathbf{n}|j;\mathbf{m}\rangle & \langle j;\mathbf{n}|j;-\mathbf{m}\rangle \\ \langle j;-\mathbf{n}|j;\mathbf{m}\rangle & \langle j;-\mathbf{n}|j;-\mathbf{m}\rangle \end{pmatrix}. \tag{36}
$$

Writing in the vectorial form, we get the coefficients

$$
A = |\langle j;\mathbf{m}|j;\mathbf{n}\rangle|^2 (\mathcal{P}_{+,+}^{\mathbf{n}_\alpha})_w + |\langle j;\mathbf{m}|j;-\mathbf{n}\rangle|^2 (\mathcal{P}_{+,-}^{\mathbf{n}_\alpha})_w, \tag{37}
$$

$$
B = |\langle j;-\mathbf{m}|j;\mathbf{n}\rangle|^2 (\mathcal{P}_{-,+}^{\mathbf{n}_\alpha})_w + |\langle j;-\mathbf{m}|j;-\mathbf{n}\rangle|^2 (\mathcal{P}_{-,-}^{\mathbf{n}_\alpha})_w, \tag{38}
$$

$$
C = p(\mathcal{P}_{+,+}^{\mathbf{n}_\alpha})_w + q(\mathcal{P}_{+,-}^{\mathbf{n}_\alpha})_w, \tag{39}
$$

$$
D = r(\mathcal{P}_{-,+}^{\mathbf{n}_\alpha})_w + s(\mathcal{P}_{-,-}^{\mathbf{n}_\alpha})_w, \tag{40}
$$

where p, q, r and s are defined in Appendix A and hence, we get $p = r^*$ and $q = s^*$; we find

$$
\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = \begin{pmatrix} A & C \\ D & B \end{pmatrix}. \tag{41}
$$

3.2.1 Holonomy

Our main purpose is to establish the relationship between the weak measurements and the non-Abelian GP, i.e., the holonomy. The notion of holonomy around a loop provides an interesting geometric view of the space of states and the connections between the states. The key idea is to show that we can exploit different combinations of the pre-and post-selected states and the weak measurement device components in order to get the GP [26–30].

Here, we follow the concept of holonomy, and its trace given by Wilczek-Zee [31] and the Wilson Loop (WL) [32], respectively. One of the major limitations to adopting our method for obtaining geometric information is the applicability to direct holonomies [8]. For $j \in \frac{1}{2}\mathbb{N}$, the holonomy $U_{c,b}U_{b,a}U_{a,c}$ is given by (29)

$$\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = [(\kappa_{c,b}U_{c,b})(\kappa_{b,a}U_{b,a})(\kappa_{a,c}U_{a,c})], \tag{42}$$

We can define the WL as a gauge invariant

$$\mathrm{Tr}[\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c}]$$
$$= \kappa_{c,b}\kappa_{b,a}\kappa_{a,c}\mathrm{Tr}[U_{c,b}U_{b,a}U_{a,c}]$$
$$= \kappa WL. \tag{43}$$

Here, κ is defined by

$$\kappa = 2^{-\frac{3}{2}} \left\{ \sum_{k'',k',k} |\langle j;k''\mathbf{m}|j;k\mathbf{n}_\alpha\rangle|^2 |\langle j;k\mathbf{n}_\alpha|j;k'\mathbf{n}\rangle|^2 |\langle j;k'\mathbf{n}|j;k''\mathbf{m}\rangle|^2 \right\}^{\frac{1}{2}}, \tag{44}$$

where the sum extends over all transition probabilities.

Now, we can plug Eqs. (37) and (38) into (43) yielding

$$WL = \kappa^{-1}(A+B) = \kappa^{-1}|\langle j;\mathbf{m}|j;\mathbf{n}\rangle|^2|(\mathcal{P}^{\mathbf{n}_\alpha}_{+,+})_w + \kappa^{-1}\langle j;\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}^{\mathbf{n}_\alpha}_{+,-})_w$$
$$\kappa^{-1}|\langle j;-\mathbf{m}|j;\mathbf{n}\rangle|^2|(\mathcal{P}^{\mathbf{n}_\alpha}_{-,+})_w + \kappa^{-1}\langle j;-\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}^{\mathbf{n}_\alpha}_{-,-})_w, \tag{45}$$

where the WL is s gauge invariant defined on a closed contour. This result shows that we can recover geometric information about the space of states through the non-Abelian GP [11]. However, as highlighted in Ref. [21] the above relation holds only for the case in which the holonomy is a unitary matrix up to a real number, as defined by Eq.(29).

3.3 Sequence of Measurements

In this section, we deal with longer sequences of projection operators. The advantage of performing weak measurements of such sequences concerns the possibility to combine incompatible observables. In Ref. [33], it is shown an implementation of two sequential weak measurements in the interferometric scheme. The operators are observables that could be read on the pointer, as highlighted in Eqs.(9) and (10).

Consider the sequence $\Sigma + i\Gamma = P_1 P_2 ... P_n$ with

$$
\begin{aligned}
\Sigma &= \frac{P_1 P_2 ... P_n + P_n P_{n-1} ... P_1}{2}, \\
\Gamma &= \frac{P_1 P_2 ... P_n - P_n P_{n-1} ... P_1}{2i},
\end{aligned}
\tag{46}
$$

both being Hermitian operators thus corresponding to observables.

Extending the formalism of the previous section, notice that weakly measuring Σ and Γ separately for all combinations of pre- and post-selected $j \in \frac{1}{2}\mathbb{N}$ SCSs, makes it possible to realize the holonomy associated with the path $P_a \to P_n \to P_{n-1} \to ... P_2 \to P_1 \to P_c$. The combination of two incompatible observables has been studied in tests of quantum foundations such as uncertainty bounds, and contextuality [34, 35].

4 Discussion

One of the main goals of this project was to attempt to find the relation between the weak measurements of higher rank operators and the non-Abelian GP.

Wilson [32] proposed an invariant quantity, the WL to quantize gauge field theory on closed paths. Wilczek and Zee realized that the Wilson Loop (WL) can be an indicator of non-Abelian holonomies [31]. Based on these ideas, the WL could be useful to evaluate the non-Abelian GP and also, other related quantities such as the Chern number, and topological invariants.

Our approach relies on the lattice behavior which relates the connection between states and the measurements performed on a quantum system. Thus, after a sequence of non-filtering measurements that correspond to a finite sequence of loops, Eq.(27), we define the WL to give geometric information about the space of states.

In the weak measurement scenario, the post-selection allows us to get information about the space of states. This information is accessible through the non-Abelian GP for the $j \in \frac{1}{2}\mathbb{N}$ SCSs case, where the transition probabilities are simply numbers and equal in magnitude for each step on the Grassmann manifold. This suggests some relation to the geometric structure of the system.

Notice that the filtering operator corresponds to a superposition of the north and south poles of the Bloch sphere. Thus, it is natural that the solid angle composed by the vectors $\{k''\mathbf{m}, k\mathbf{n}_\alpha, k'\mathbf{n}\}$, for different combinations o signs $k'', k', k = \pm$, can be inferred by a weak measurement of spin in the $\mathbf{n}_\alpha$ direction. In this way, the closed paths acquire a specific GP associated with specific trajectories, i.e., different combinations of the angles imply different readouts of the device [36].

Note from the denominator of Eq. (34) that the weak values may assume large values as the post-selected being nearly orthogonal to the pre-selected state. This property is called weak value amplification (WVA) and can provide interesting and advantageous experimental effects. This amplification effect has no classical analog and can be described in terms of quantum interference. For this reason, it is possible to interpret the overlap between the states in terms of transition amplitudes, as follows.

For instance, Ref. [37] highlighted the use of WVA when the device is saturated with a high number of particles or in the case in which the detector cannot differentiate between two signals. For an interesting discussion of this

topic, see [38], where it is shown that by weakly measuring improbable or rare events, it is possible to produce a reciprocal improbable outcome. For instance, the spin of an electron may acquire the value 100 [38]. In Ref. [39], the authors demonstrated the protection of SCSs from decoherence using the weak measurements scheme, which can be improved using rotations around arbitrary axes on the Bloch sphere.

In contrast, the weak values diverge when $\mathbf{m} \to -\mathbf{n}$. This behavior can be interpreted as a limit for which a pre-selected state at the point $\mathbf{n}$ has a vanishing probability for it to pass a post-selection of $-\mathbf{n}$. The antipodal points $\pm\mathbf{n}$ correspond to orthogonal states in which the GP is undefined [8].

The transition probability quantifies the probability of getting a weak value given the pre- and post-selected states. On the other hand, the transition amplitude quantifies the overlap between those states. In Eqs. (39) and (40), the coefficients p, q, r, and s constitute transition amplitudes, in contrast to the transition probabilities in Eqs. (37) and (38).

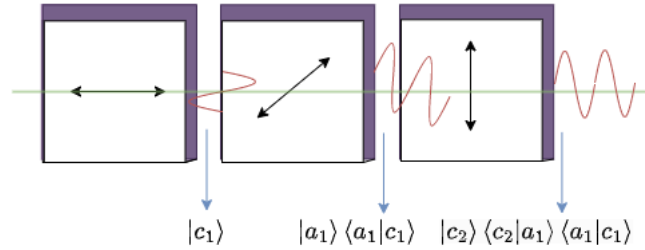Fig.(2) presents a schematic view of our interpretation

Figure 2: Conceptual representation of measurement back-action. The post-selected state $|c_1\rangle$ is weakly measured by sequential filtering measurements. After that, the final state is $|c_2\rangle \langle c_2|a_1\rangle \langle a_1|a_1\rangle$.

where the transition corresponds to the connection between the pre- and post-selected states.

For instance, Aharonov et al. [40] showed that quantum random walks can be described in terms of probability amplitudes. In this context, the amplitude represents the decision on whether a particle takes a given path, depending on the outcome of the measurement. In this sense, it can be interpreted as the partial information remaining after the measurement process. This fact is due to the non-disturbing feature of the weak measurements.

The post-selection plays an important role in the scheme. The presence of transition amplitudes can be interpreted as the remaining or partial information about the past of the post-selected state. This is an important feature because it shows the nature of the weak measurement: the post-selected state carries some information about the interactions at a given time $t < t_2$. Moreover, due to this fact, it is possible to get information about the space of states using the weak measurement scheme.

However, as highlighted in Ref. [21] the above relation holds only for the case in which the holonomy is a unitary matrix up to a real number, as defined by Eq.(29).

The significance of our findings provides considerable insight into searching for topological and geometric invariants. The deduction of the Wilson Loop in terms of weak values offers a compelling interpretation of invariants and the role of overlap between pre-selected and post-selected states. They can be consider a coming together of topology and geometry. For instance, in the case of SC the amplification effect becomes important in optical settings.

5 Conclusion

We have explored the extension of the Abelian GP in a sequence of incomplete measurements to include more than three states by performing sequential weak measurements. This report also has investigated the notion of non-Abelian GP in the weak measurement scenario.

The results show that we can recover information about the space of states by performing weak measurements. The formalism can be applied to longer sequences of operators. The advantage of performing a sequence of weak measurements concerns the possibility to combine incompatible observables.

The present study has only investigated the case in which the holonomy is a unitary matrix up to a real number. Our approach could be applied to the $j \in \frac{1}{2}\mathbb{N}$ SCSs case, where the transition probabilities are simply numbers and equal in magnitude for each step on the Grassmann manifold. This suggests some relation to the geometric structure of the system.

Our studies can be extended to a topological view in order to search for invariants in quantum materials, such as the Chern number, Bargmann invariant, or the Pancharatnam phase. Future work will investigate the possible implementations of interferometric, polarimetric schemes where κ has a physical interpretation. The motivations are to develop new experimental techniques and improve the amount of information that can be extracted from the quantum systems.

References

[1] V. Neef, J. Pinske, F. Klauck, L. Teuber, M. Kremer, M. Ehrhardt, M. Heinrich, S. Scheel, and A. Szameit, "Three-dimensional non-Abelian quantum holonomy", Nature Physics, 10.1038/s41567-022-01807-5 (2022).

[2] J. Bernstein, *Butterfly in the Quantum World The story of the most fascinating quantum fractal* (IOP Publishing, 2016).

[3] J. Anandan, "Non-adiabatic non-abelian geometric phase", Physics Letters A **133**, 171–175 (1988).

[4] C. J. G. Mommers and E. Sjöqvist, "Universal quantum computation and quantum error correction using discrete holonomies", Physical Review A **105**, 022402 (2022).

[5] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, "Time Symmetry in the Quantum Process of Measurement", Physical Review **134**, B1410–B1416 (1964).

[6] Y. Aharonov and L. Vaidman, "Properties of a quantum system during the time interval between two measurements", Physical Review A **41**, 11–20 (1990).

[7] Y. Aharonov and J. Anandan, "Phase change during a cyclic quantum evolution", Physical Review Letters **58**, 1593–1596 (1987).

[8] E. Sjöqvist, "Geometric phase in weak measurements", Physics Letters, A **359**, 187–189 (2006).

[9] M. V. Berry, "The adiabatic phase and Pancharatnam's phase for polarized light", Journal of Modern Optics **34**, 1401–1407 (1987).

[10] B. Simon, "Holonomy, the Quantum Adiabatic Theorem, and Berry's Phase", Phys. Rev. Lett. **51**, 2167–2170 (1983).

[11] F Wilczek and A Shapere, *Geometric Phases in Physics*, Vol. 5, Advanced Series in Mathematical Physics (WORLD SCIENTIFIC, 1989).

[12] J. Anandan and A. Pines, "Non-abelian geometric phase from incomplete quantum measurements", Physics Letters A **141**, 335–339 (1989).

[13] R Loll, "Loop approaches to gauge field theories", Theoretical and Mathematical Physics **93**, 1415–1432 (1992).

[14] E. Sjöqvist, "Geometric phases in quantum information", International Journal of Quantum Chemistry **115**, 1311–1326 (2015).

[15] Z. Zhou, Y. Margalit, S. Moukouri, Y. Meir, and R. Folman, "An experimental test of the geodesic rule proposition for the noncyclic geometric phase", Science Advances **6**, 10.1126/sciadv.aay8345 (2020).

[16] F. Ghahari, D. Walkup, C. Gutiérrez, J. F. Rodriguez-Nieva, Y. Zhao, J. Wyrick, F. D. Natterer, W. G. Cullen, K. Watanabe, T. Taniguchi, L. S. Levitov, N. B. Zhitenev, and J. A. Stroscio, "An on/off Berry phase switch in circular graphene resonators", Science **356**, 845–849 (2017).

[17] E. Cohen, H. Larocque, F. Bouchard, F. Nejadsattari, Y. Gefen, and E. Karimi, "Geometric phase from Aharonov–Bohm to Pancharatnam–Berry and beyond", Nature Reviews Physics **1**, 437–449 (2019).

[18] C. D.-M. Yvonne Choquet-Bruhat, *Analysis, Manifolds and Physics* (Elsevier, 2000).

[19] J. E. Avron, D. Osadchy, and R. Seiler, "A Topological Look at the Quantum Hall Effect", Physics Today **56**, 38–42 (2003).

[20] J Anandan, "Comment on Geometric Phases for Classical Field Theories", Phys. Rev. Lett. **60**, 2555 (1988).

[21] E. Sjöqvist and J. Åberg, "Weak cloning of an unknown quantum state", Physics Letters, Section A: General, Atomic and Solid State Physics **354**, 396–398 (2006).

[22] C. A. Mead, ""Block diagonalization" as generalization of Pancharatnam-Berry phase relation for multidimensional spaces", Physical Review A **44**, 1473–1476 (1991).

[23] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, "Atomic Coherent States in Quantum Optics", Physical Review A **6**, 2211 (1972).

[24] D. Robert and M. Combescure, "Coherent States and Applications in Mathematical Physics: Second Edition", in *Theoretical and mathematical physics(united states)*, Vol. 2012, 2 (2021), pages 1–575.

[25] S. Pancharatnam, "Generalized theory of interference, and its applications - Part I. Coherent pencils", Proceedings of the Indian Academy of Sciences - Section A **44**, 247–262 (1956).

[26] R. O. Fulp, "Connections on the path bundle of a principal fiber bundle", Differential Geometry and its Applications **4**, 201–237 (1994).

[27] R. Chiao, "Berry's phases in optics: Aharonov-Bohm-like effects and gauge structures in surprising contexts", Nuclear Physics B (Proceedings Supplements) **6**, 298–305 (1989).

[28] H. Lyre, "Berry phase and quantum structure", Studies in History and Philosophy of Science Part B - Studies in History and Philosophy of Modern Physics **48**, 45–51 (2014).

[29] R. Gambini and A. Trias, "Geometrical origin of gauge theories", Physical Review D **23**, 553–555 (1981).

[30] R. Jackiw, "Gauge-covariant conformal transformations", Physical Review Letters **41**, 1635–1638 (1978).

[31] F. Wilczek and A. Zee, "Appearance of Gauge Structure in Simple Dynamical Systems", Physical Review Letters **52**, 2111–2114 (1984).

[32] K. G. Wilson, "Confinement of quarks", Phys. Rev. D **10**, 2445–2459 (1974).

[33] Y. Kim, Y.-S. Kim, S.-Y. Lee, S.-W. Han, S. Moon, Y.-H. Kim, and Y.-W. Cho, "Direct quantum process tomography via measuring sequential weak values of incompatible observables", Nature Communications **9**, 192 (2018).

[34] M. J. W. Hall, A. K. Pati, and J. Wu, "Products of weak values: Uncertainty relations, complementarity, and incompatibility", Physical Review A **93**, 052118 (2016).

[35] F. Piacentini, A. Avella, M. P. Levi, M. Gramegna, G. Brida, I. P. Degiovanni, E. Cohen, R. Lussana, F. Villa, A. Tosi, F. Zappa, and M. Genovese, "Measuring Incompatible Observables by Exploiting Sequential Weak Values", Physical Review Letters **117**, 170402 (2016).

[36] Y. Wang, K. Snizhko, A. Romito, Y. Gefen, and K. Murch, "Observing a topological transition in weak-measurement-induced geometric phases", Physical Review Research **4**, 23179 (2022).

[37] J. P. Torres and L. J. Salazar-Serrano, "Weak value amplification: a view from quantum estimation theory that highlights what it is and what isn't", Scientific Reports **6**, 19702 (2016).

[38] Y. Aharonov, D. Z. Albert, and L. Vaidman, "How the result of a measurement of a component of the spin of a spin- 1/2 particle can turn out to be 100", Physical Review Letters **60**, 1351–1354 (1988).

[39] T. Vanderbruggen, R. Kohlhaas, A. Bertoldi, S. Bernon, A. Aspect, A. Landragin, and P. Bouyer, "Feedback Control of Trapped Coherent Atomic Ensembles", Physical Review Letters **110**, 210503 (2013).

[40] Y. Aharonov, L. Davidovich, and N. Zagury, "Quantum random walks", Physical Review A **48**, 1687–1690 (1993).

[41] J Klauder and B Skagerstam, *Coherent States* (WORLD SCIENTIFIC, 1985).

[42] A. Peres, editor, *Quantum Theory: Concepts and Methods* (Springer Netherlands, Dordrecht, 2002).

A Appendix

A.1 Product of the overlap matrices

In this section, we evaluate the product of the overlap matrices Here, we consider a special case of a given rank-2 projector operator.

$$
\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = \begin{pmatrix} \langle c_1|b_1 \rangle & \langle c_1|b_2 \rangle \\ \langle c_2|b_1 \rangle & \langle c_2|b_2 \rangle \end{pmatrix} \begin{pmatrix} \langle b_1|a_1 \rangle & \langle b_1|a_2 \rangle \\ \langle b_2|a_1 \rangle & \langle b_2|a_2 \rangle \end{pmatrix} \begin{pmatrix} \langle a_1|c_1 \rangle & \langle a_1|c_2 \rangle \\ \langle a_2|c_1 \rangle & \langle a_2|c_2 \rangle \end{pmatrix}
$$
$$
= \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \quad (47)
$$

where each matrix element represents the resulting overlap element. The formalism can be extended to any number of components of the pre and post-selected states, and the projection operator within the TSVF. The formalism can be extended to high-dimensional quantum systems and also to other degrees of freedom.

If we consider the weak value of a projector operator 2-rank

$$
(\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w = \frac{\langle c_1|\mathcal{P}^{\mathbf{n}}|a_1 \rangle}{\langle c_1|a_1 \rangle} = \frac{\langle c_1|b_1 \rangle \langle b_1|a_1 \rangle}{\langle c_1|a_1 \rangle} + \frac{\langle c_1|b_2 \rangle \langle b_2|a_1 \rangle}{\langle c_1|a_1 \rangle}, \quad (48)
$$

then we can write the matrix elements A, B, C and D as

$$
A = |\langle c_1|a_1 \rangle|^2 (\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w + |\langle c_1|a_2 \rangle|^2 (\mathcal{P}^{\mathbf{n}}_{c_1,a_2})_w, \quad (49)
$$

$$
B = |\langle c_2|a_1 \rangle|^2 (\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + |\langle c_2|a_2 \rangle|^2 (\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w, \quad (50)
$$

$$
C = \langle c_1|a_1 \rangle \langle a_1|c_2 \rangle (\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w + \langle c_1|a_2 \rangle \langle a_2|c_2 \rangle (\mathcal{P}^{\mathbf{n}}_{c_1,a_2})_w, \quad (51)
$$

$$
D = \langle c_2|a_1 \rangle \langle a_1|c_1 \rangle (\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + \langle c_2|a_2 \rangle \langle a_2|c_1 \rangle (\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w. \quad (52)
$$

Now, in order to get the relation between the matrix elements, we write

$$
p = \langle c_1|a_1 \rangle \langle a_1|c_2 \rangle,
$$
$$
q = \langle c_1|a_2 \rangle \langle a_2|c_2 \rangle. \quad (53)
$$

26

Then, the matrix elements C and D become

$$C = p(\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w + q(\mathcal{P}^{\mathbf{n}}_{c_1,a_2})_w, \tag{54}$$

$$D = p^*(\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + q^*(\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w, \tag{55}$$

where the coefficients C and D are related by the factors p and q .

A.2 Vectorial form of SCSs

Here, we deduce the expression for the overlap between two SCSs states.

Let $\mathbf{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ be a unit vector pointing in the spatial direction corresponding to a polar angle θ and azimuth ϕ, in spherical coordinates. We choose a normalized vector $|\mathbf{n}\rangle$ and for convenience, we choose $|\mathbf{n}\rangle$ to be an eigenvector of J_z. Thus $J_z|\mathbf{n}\rangle = m|\mathbf{n}\rangle$ and so we may denote $|\mathbf{n}\rangle$ by $|m; \mathbf{n}\rangle$ where $m = -j, -j+1, ..., j$. Hence, we also introduce the related expression

$$U(\theta, \phi) = e^{-i\phi J_z} e^{-i\theta J_y}, \tag{56}$$

and correspondingly we define,

$$|\theta, \phi\rangle = R(\theta, \phi)|m; \mathbf{n}\rangle. \tag{57}$$

However, we are interested only in the extreme values of m, i.e., when $m = \pm j$. The overlap becomes

$$\langle \theta', \phi' | \theta'', \phi'' \rangle = \langle m; \mathbf{n} | e^{i\theta'' J_y} e^{i(\phi'-\phi'') J_z} e^{-i\theta' J_y} | m; \mathbf{n} \rangle$$

$$= \sum_{n=-j}^{j} \langle m; \mathbf{n} | e^{i\theta'' J_y} | n; \mathbf{n} \rangle \langle n; \mathbf{n} | e^{-i\theta' J_y} | m; \mathbf{n} \rangle\, e^{i(\phi'-\phi'')n}, \tag{58}$$

which can be expressed in terms of the reduced Wigner coefficients of the spin-j representation [41],

$$\mathcal{D}^j_{m,n} = \langle n | e^{-i\theta J_y} | m \rangle. \tag{59}$$

For the special case in which $m = j$, this expression leads to the result[42],

$$\langle \theta', \phi' | \theta'', \phi'' \rangle = \left[\cos\left(\frac{\theta'}{2}\right) \cos\left(\frac{\theta''}{2}\right) e^{i(\phi'-\phi'')/2} + \sin\left(\frac{\theta'}{2}\right) \sin\left(\frac{\theta''}{2}\right) e^{-i(\phi'-\phi'')/2} \right]^{2j}. \tag{60}$$

Now, consider two SCSs states $\langle \mathbf{m}| = \langle (\theta_b, \phi_b)|$ and $|\mathbf{n}\rangle = |(\theta_a, \phi_a)\rangle$. The squared modulus of the overlap between them is defined by

$$|\langle \mathbf{m}|\mathbf{n}\rangle|^2 = \left(\cos\left(\frac{\theta_b}{2}\right)\cos\left(\frac{\theta_a}{2}\right) + \cos(\phi_b - \phi_a)\sin\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_a}{2}\right) \right)^2$$

$$+ \left(\sin(\phi_b - \phi_a)\sin\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_a}{2}\right) \right)^2 =$$

$$= \cos^2\left(\frac{\theta_a}{2}\right)\cos^2\left(\frac{\theta_b}{2}\right) + \sin^2\left(\frac{\theta_a}{2}\right)\sin^2\left(\frac{\theta_a}{2}\right)$$

$$+ 2\cos(\phi_a - \phi_b)\cos\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_b}{2}\right). \quad (61)$$

Using the following trigonometric relation

$$\cos\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_b}{2}\right) = \frac{1}{2}\sin(\theta_a)\frac{1}{2}\sin(\theta_b), \quad (62)$$

we get,

$$|\langle \mathbf{m}|\mathbf{n}\rangle|^2 = \frac{1}{2} + \frac{1}{2}\cos(\theta_a)\frac{1}{2}\cos(\theta_b) + \frac{1}{2}\cos(\phi_a - \phi_b)\sin(\theta_a)\sin(\theta_b)$$

$$= e^{ij\phi(\mathbf{n},\mathbf{m})}\frac{1}{2}(1 + \mathbf{n}\cdot\mathbf{m}). \quad (63)$$

Expressed in terms of the vectors

$$|\langle \mathbf{m}|\mathbf{n}\rangle|^2 = e^{ij\phi(\mathbf{n},\mathbf{m})}\left(\frac{1 + \mathbf{n}\cdot\mathbf{m}}{2}\right)^{j/2}, \quad (64)$$

where $\phi(\mathbf{n}, \mathbf{m})$ is a real phase.

This leads to the result,

$$\langle \mathbf{m}|\mathbf{n}\rangle = e^{ij\Phi(\mathbf{n},\mathbf{m})}\left(\frac{1 + \mathbf{n}\cdot\mathbf{m}}{2}\right)^{j}. \quad (65)$$

A.3   Transition Amplitudes and Probabilities

Using the concept of relative phase defined in (29), we can find the transition amplitude by writing

$$\kappa_{c,b} = \left[\frac{Tr(\mathcal{F}_{c,b}\mathcal{F}_{b,c})}{2} \right]^{\frac{1}{2}}$$

$$= \left[|\langle c_1|b_1\rangle|^2 + |\langle c_1|b_2\rangle|^2 + |\langle c_2|b_1\rangle|^2 + |\langle c_2|b_2\rangle|^2 \right]^{\frac{1}{2}}, \tag{66}$$

$$\kappa_{b,a} = \left[|\langle b_1|a_1\rangle|^2 + |\langle b_1|a_2\rangle|^2 + |\langle b_2|a_1\rangle|^2 + |\langle b_2|a_2\rangle|^2 \right]^{\frac{1}{2}}, \tag{67}$$

$$\kappa_{a,c} = \left[|\langle a_1|c_1\rangle|^2 + |\langle a_1|c_2\rangle|^2 + |\langle a_2|c_1\rangle|^2 + |\langle a_2|c_2\rangle|^2 \right]^{\frac{1}{2}}. \tag{68}$$

In the vectorial form, we write,

$$\kappa_{c,b} = \left[\sum_{k'',k} \frac{|\langle j; k'\mathbf{m}|j; k\mathbf{n}_\alpha\rangle|^2}{2} \right]^{\frac{1}{2}}, \tag{69}$$

$$\kappa_{b,a} = \left[\sum_{k,k'} \frac{|\langle j; k\mathbf{n}_\alpha|j; k'\mathbf{n}\rangle|^2}{2} \right]^{\frac{1}{2}}, \tag{70}$$

$$\kappa_{a,c} = \left[\sum_{k',k''} \frac{|\langle j; k'\mathbf{n}|j; k''\mathbf{m}\rangle|^2}{2} \right]^{\frac{1}{2}}. \tag{71}$$

In this way, κ can be interpreted in terms of transition probabilities.

We can define the transition amplitudes as

$$\langle c_1|a_1\rangle = \langle j; \mathbf{m}|j; \mathbf{n}\rangle, \tag{72}$$

$$\langle c_1|a_2\rangle = \langle j; \mathbf{m}|j; -\mathbf{n}\rangle, \tag{73}$$

$$\langle c_2|a_1\rangle = \langle j; -\mathbf{m}|j; \mathbf{n}\rangle, \tag{74}$$

$$\langle c_2 | a_2 \rangle = \langle j; -\mathbf{m} | j; -\mathbf{n} \rangle. \tag{75}$$

And then we define the product of transition amplitudes

$$p = \langle c_1 | a_1 \rangle \langle a_1 | c_2 \rangle = \langle j; \mathbf{m} | j; \mathbf{n} \rangle \langle j; \mathbf{n} | j; -\mathbf{m} \rangle, \tag{76}$$

$$q = \langle c_1 | a_2 \rangle \langle a_2 | c_2 \rangle = \langle j; \mathbf{m} | j; -\mathbf{n} \rangle \langle j; -\mathbf{n} | j; -\mathbf{m} \rangle, \tag{77}$$

$$r = \langle c_2 | a_1 \rangle \langle a_1 | c_1 \rangle = \langle j; -\mathbf{m} | j; \mathbf{n} \rangle \langle j; \mathbf{n} | j; \mathbf{m} \rangle, \tag{78}$$

$$s = \langle c_2 | a_2 \rangle \langle a_2 | c_1 \rangle = \langle j; -\mathbf{m} | j; -\mathbf{n} \rangle \langle j; -\mathbf{n} | j; \mathbf{m} \rangle. \tag{79}$$

And similarly, for the general case, we can identify that $p = r^*$ and $q = s^*$. Rewriting the matrix elements A, B, C, and D,

$$A = |\langle j; \mathbf{m} | j; \mathbf{n} \rangle|^2 (\mathcal{P}_{+,+}^{\mathbf{n}_\alpha})_w + |\langle j; \mathbf{m} | j; -\mathbf{n} \rangle|^2 (\mathcal{P}_{+,-}^{\mathbf{n}_\alpha})_w, \tag{80}$$

$$B = |\langle j; -\mathbf{m} | j; \mathbf{n} \rangle|^2 (\mathcal{P}_{-,+}^{\mathbf{n}_\alpha})_w + |\langle j; -\mathbf{m} | j; -\mathbf{n} \rangle|^2 (\mathcal{P}_{-,-}^{\mathbf{n}_\alpha})_w, \tag{81}$$

$$C = p(\mathcal{P}_{+,+}^{\mathbf{n}_\alpha})_w + q(\mathcal{P}_{+,-}^{\mathbf{n}_\alpha})_w, \tag{82}$$

$$D = r(\mathcal{P}_{-,+}^{\mathbf{n}_\alpha})_w + s(\mathcal{P}_{-,-}^{\mathbf{n}_\alpha})_w, \tag{83}$$

where A more detailed explanation can be found in Appendix A and $p = r^*$ and $q = s^*$.

A.4 Reconstruction scheme for SCSs

For the SCSs, the coefficients $A, B, C,$ and D are given by the elements of the overlap matrices or rotation matrices

$$A = R(c,a)R(a,c)\left(\frac{R(c,b)R(b,a) - S(c,b)S(b,a)^*}{R(c,a)}\right)$$
$$+ S(c,a)S(a,c)\left(\frac{R(c,b)S(b,a) + S(c,b)R(b,a)^*}{S(c,a)}\right), \tag{84}$$

$$B = S(c,a)^* S(a,c)^*\left(\frac{-S(c,b)^* R(b,a) - R(c,b)^* S(b,a)^*}{-S(c,a)^*}\right)$$
$$+ S(c,a)^* S(a,c)^*\left(\frac{-S(c,b)^* R(b,a) - R(c,b)^* S(b,a)^*}{R(c,a)^*}\right), \tag{85}$$

$$C = R(c,a)S(a,c)\left(\frac{R(c,b)R(b,a) - S(c,b)S(b,a)^*}{R(c,a)}\right)$$
$$+ S(c,a)R(a,c)^*\left(\frac{R(c,b)S(b,a) + S(c,b)R(b,a)^*}{S(c,a)}\right), \tag{86}$$

$$D = -S(c,a)^* R(a,c)\left(\frac{-S(c,b)^* R(b,a) - R(c,b)^* S(b,a)^*}{-S(c,a)^*}\right)$$
$$- S(c,a)S(a,c)^*\left(\frac{-S(c,b)^* S(b,a) + R(c,b)^* R(b,a)^*}{R(c,a)^*}\right), \tag{87}$$

which we can get that $A = B^*$ and $C = -D^*$. Using (53), we can define the

$$D = (-1)^{2j}\{p(\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + q(\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w\}^*. \tag{88}$$

31