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# One-loop amplitudes in Einstein-Yang-Mills from forward limits

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Abstract: We present a method to compute the integrands of one-loop Einstein-Yang-Mills amplitudes for any number of external gauge and gravity multiplets. Our construction relies on the double-copy structure of Einstein-Yang-Mills as (super-)Yang-Mills with the so-called YM+$\phi^3$ theory — pure Yang-Mills coupled to bi-adjoint scalars — which we implement via one-loop Cachazo-He-Yuan formulae. The YM+$\phi^3$ building blocks are obtained from forward limits of tree-level input in external gluons and scalars, and we give the composition rules for any number of traces and orders in the couplings $g$ and $\kappa$. On the one hand, we spell out supersymmetry- and dimension-agnostic relations that reduce loop integrands of Einstein-Yang-Mills to those of pure gauge theories. On the other hand, we present four-point results for maximal and half-maximal supersymmetry where all supersymmetry cancellations are exposed. In the half-maximal case, we determine six-dimensional anomalies due to chiral hypermultiplets in the loop.

Keywords: Scattering Amplitudes, Field Theories in Higher Dimensions, Supersymmetric Gauge Theory

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## Contents

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# 1 Introduction

Recent studies of scattering amplitudes revealed a variety of symmetries of and connections between different quantum field theories that are hidden in traditional Lagrangian formulations. The most famous example concerns the double-copy structure of (super-)gravity: the loop integrands of supergravity amplitudes can often be assembled from suitably chosen squares of gauge-theory building blocks. This phenomenon relies on the Bern-Carrasco-Johansson (BCJ) duality between color and kinematics in gauge theories [1–3], see [4] for a comprehensive review. In many cases, the gravitational double copy can be naturally understood from string theory, e.g. from the famous Kawai-Lewellen-Tye (KLT) relations at tree level [5], and from chiral splitting [6, 7] at the level of loop integrands.

The field-theoretic double-copy structure applies to a growing list of theories including Born-Infeld, special Galileons [8, 9] and even open-string theories [10–12]. In particular, amplitudes of Einstein-Yang-Mills (EYM) theories can be obtained from the double copy of (super-)Yang-Mills with the so-called YM+$\phi^3$ theory [13]. EYM refers to Einstein gravity, extended by a dilaton & B-field[1] and minimally coupled to Yang-Mills, including supersymmetric extensions with up to 16 supercharges. The non-supersymmetric ingredient YM+$\phi^3$ of its double copy augments pure Yang-Mills by a minimal coupling to bi-adjoint

[1]The couplings of the dilaton & B-field contribute to the one-loop EYM amplitudes in this work and are natural from the viewpoints of the double copy, supersymmetric extensions and string-theory realizations.

scalars with a cubic self interaction. Similar double-copy descriptions have been found for variants of EYM with spontaneous symmetry breaking [14, 15].

The double-copy structure of EYM implies relations between EYM amplitudes and those of (super-)Yang-Mills. At tree level, such EYM amplitude relations have been analyzed from a multitude of perspectives including open strings interacting with closed strings [16, 17], the Cachazo-He-Yuan (CHY) formalism[2] [24–28], heterotic strings [29], gauge invariance [30, 31] and the color-kinematics duality of YM+$\phi^3$ [32]. Similar double-copy structures and resulting tree-amplitude relations apply to conformal supergravity coupled to gauge theories [12, 33].

There is considerably less literature on *loop-level* amplitudes of EYM: specific four-point one-loop amplitudes with half-maximal supersymmetry and external gluons have been determined in [13] whereas rational one-loop EYM amplitudes can be found in [34] at leading order in the gravitational coupling $\kappa$ and in [35] at general orders. Moreover, all-loop results for one-graviton-$n$-gluon amplitudes have been obtained in [32]. At leading order in $\kappa$, relations between gauge-invariant building blocks of one-loop EYM with higher numbers of gravitons and (super-)Yang Mills amplitudes have been pioneered in [36]. The relations among "partial integrands" in the reference take a universal form for EYM theories with four to sixteen supercharges, but they only capture the contributions from gauge multiplets in the loop.

In this work, we describe a method to extend the one-loop EYM amplitude relations of [36] to arbitrary combinations of gauge and gravity multiplets in the loop and the external legs, i.e. to all orders in $\kappa$. Our results again reduce loop integrands of EYM to dimension-agnostic partial integrands of super-Yang-Mills without any coupling to gravity. In particular, the relations we will derive take a universal form for any non-zero number of supercharges — they are expressible in terms of the gauge-invariant partial integrands of [36] that carry the variable amount of supersymmetry in the double copy. The universality of the relations stems from the non-supersymmetric YM+$\phi^3$ constituent that appears in the double-copy construction of EYM theories with *any* number of supercharges. The dependence of our relations on color factors can be straightforwardly adapted to gauge groups U($N$) and SU($N$).

The amplitude relations in this work are derived from forward limits of tree-level building blocks, where divergences in intermediate expressions are bypassed via ambitwistor-string methods [37–40].[3] This way of implementing forward limits has been applied to explicitly construct loop integrands of gauge theories [43–45] and one-loop matrix elements of higher-mass-dimension operators in the low-energy effective action of superstrings [46]. However, the ambitwistor methods at work lead to an unconventional form of the Feynman propagators in the loop integrand: the inverse propagators of the partial integrands are *linearized* in the loop momentum $\ell$, i.e. given by $2\ell \cdot K + K^2$ instead of $(\ell + K)^2$ for (combinations of) external momenta $K$ [36–39].

[2]Representations of EYM tree amplitudes in the CHY formalism [18–20] were given in [8, 21, 22], also see [23] for an underpinning via ambitwistor strings.

[3]As detailed for instance in [41] and references therein, the significance of forward limits for one-loop amplitudes can be anticipated from the Feynman tree theorem [42].



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We provide the detailed form of the loop integrands in four-point EYM amplitudes with sixteen and eight supercharges in terms of traditional quadratic propagators $(\ell+K)^{-2}$. In particular, we address various configurations of external gauge and gravity multiplets as well as different orders in $\kappa$, i.e. all admissible contributions from gauge and/or gravity multiplets in the loop. The conversion between linearized and quadratic propagators in the loop is performed via elementary partial-fraction manipulations in the examples of this work, see [46–52] for more general recent discussions of this conversion. However, the supersymmetry-agnostic amplitude relations of this work still feature linearized propagators in intermediate steps. We leave it as an open problem to preserve the universal form of the relations for EYM loop integrands while manifesting the quadratic propagators $(\ell+K)^{-2}$ of the super-Yang-Mills building blocks.

**Outline.** This paper is organized as follows: in section 2, we review the EYM double copy including the CHY methods relevant to this work and state the main formulae for our construction of one-loop EYM amplitudes. Their central building blocks are so-called half integrands of YM+$\phi^3$ theory; we spell out the detailed form of their color decomposition in terms of tree-level data for any number of external states in section 3. Based on the four-point examples of YM+$\phi^3$ half integrands in section 4, we proceed to constructing four-point one-loop EYM amplitudes at all orders in the coupling: maximally supersymmetric loop integrands in $D\leq 10$ spacetime dimensions in section 5 and half-maximally supersymmetric ones in $D\leq 6$ in section 6. In both cases, we expose all supersymmetry cancellations and convert the output of the CHY double copy to quadratic Feynman propagators. Moreover, the chiral fermions in six-dimensional EYM theories with 8 supercharges give rise to gauge- and diffeomorphism anomalies whose integrated results can be found in section 6.5.

This work is supplemented by three appendices, starting with a review of the general form of CHY integrands for EYM tree amplitudes in appendix A. Moreover, we have gathered background information on kinematic factors with half-maximal supersymmetry and rational Feynman integrals in the six-dimensional anomalies in appendices B and C, respectively. Finally, some of our results are available in machine-readable form in the supplementary material attached to this paper.

## 2 Review and basics

In this section, we review the basics of the EYM double copy, the CHY formulation of its tree-level amplitudes and the construction of one-loop amplitudes from forward limits of trees in the ambitwistor framework.

### 2.1 Einstein-Yang-Mills as a double copy

The oldest incarnation of the double-copy structure of perturbative gravity is the KLT formula for its $n$-point tree-level amplitudes [5]

$$M^{\rm tree}_{n,{\rm GR}} = \sum_{\rho,\tau\in S_{n-3}} A^{\rm tree}_{\rm YM}(1,\rho,n-1,n)S(\rho|\tau)_1\bar{A}^{\rm tree}_{\rm YM}(1,\tau,n,n-1)\,. \tag{2.1}$$

On the right hand side, $\rho = \rho(2,3,\ldots,n{-}2)$ and $\tau$ are permutations of $n{-}3$ legs in the color-ordered gauge-theory amplitudes $A^{\rm tree}_{\rm YM}(1,2,\ldots,n)$ referring to the coefficient of $\text{Tr}(t^{a_1}t^{a_2}\ldots t^{a_n})$ in the color decomposition. The entries of the $(n{-}3)!\times(n{-}3)!$ KLT matrix $S(\rho|\tau)_1$ [53, 54] are degree-$(n{-}3)$ polynomials in Mandelstam invariants

$$s_{ij} = k_i\cdot k_j = \frac{1}{2}(k_i{+}k_j)^2\,, \qquad s_{ij\ldots p} = \frac{1}{2}(k_i{+}k_j{+}\ldots{+}k_p)^2\,, \tag{2.2}$$

(such as $S(2|2)_1 = -s_{12}$ at four points), where all the external momenta $k_i$ are lightlike throughout this work. The $(n{-}3)!$ permutations of $A^{\rm tree}_{\rm YM}, \bar A^{\rm tree}_{\rm YM}$ form a basis of color-ordered amplitudes via BCJ relations [1] and are therefore sufficient to generate a permutation invariant gravity amplitude via (2.1).

The KLT formula calculates the tree-level amplitudes in a variety of further theories with double-copy structure including EYM [13], Born-Infeld and special Galileons [8, 9] as well as even open-string theories [10–12]. In general, (2.1) yields the tree amplitudes $M^{\rm tree}_{n,B\otimes C}$ in the double-copy theory $B\otimes C$ such as general relativity (GR) from the color-ordered amplitudes $A^{\rm tree}_B, \bar A^{\rm tree}_C$ in theories $B$ and $C$, say Yang-Mills (YM). Since (2.1) features the outer products of the polarizations of theories $B$ and $C$, the spins of the external legs add up under double copy. The color degrees of freedom common to theories $B$ and $C$ in turn are stripped off.

In the case of $B\otimes C = \text{EYM}$, the constituent theories are $B = \text{YM}$ and an extended gauge theory $C = \text{YM}+\phi^3$ [13] whose Lagrangian ($\mu,\nu = 0,1,\ldots,D{-}1$ in $D$ spacetime dimensions)

$$\begin{aligned}\mathcal{L}_{\text{YM}+\phi^3} = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_\mu\phi^A)^a(D^\mu\phi^A)^a - \frac{g^2}{4}f^{abe}f^{ecd}\phi^{aA}\phi^{bB}\phi^{cA}\phi^{dB}\\ &+\frac{\lambda g}{3!}f^{abc}\hat f^{ABC}\phi^{aA}\phi^{bB}\phi^{cC}\end{aligned} \tag{2.3}$$

with coupling constants $g,\lambda$ involves gluons $A_\mu = A^a_\mu t^a$ coupled to bi-adjoint scalars $\Phi = \phi^{aA}t^a\otimes T^A$ with two species of gauge-group generators $t^a, T^A$ and respective structure constants $f^{abc}, \hat f^{ABC}$. Our conventions for the non-linear gluon field-strengths $F^a_{\mu\nu}$ and gauge-covariant derivatives $D_\mu$ are

$$\begin{aligned}F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}A^b_\mu A^c_\nu\\ (D_\mu\phi^A)^a &= \partial_\mu\phi^{aA} + g f^{abc}A^b_\mu\phi^{cA}\,.\end{aligned} \tag{2.4}$$

In adapting the KLT formula (2.1) to EYM, the color-ordering of the $\text{YM}+\phi^3$ amplitudes is performed w.r.t. the generators $t^a$ common to the scalars and the gauge bosons (rather than the $T^A$ exclusive to scalars). In other words, the color-dressed $n$-point amplitudes[4]

$$M^{\rm tree}_{n,\text{YM}+\phi^3} = \sum_{\rho\in S_{n-1}} \text{Tr}(t^1t^{\rho(2)}t^{\rho(3)}\ldots t^{\rho(n)})A^{\rm tree}_{\text{YM}+\phi^3}(1,\rho(2,3,\ldots,n)) \tag{2.5}$$

[4]For gauge-group generators $t^a$ and $T^A$, we will often abbreviate the adjoint indices $a_i\to i$ and $A_i\to i$ referring to the color degrees of freedom of the $i^{\rm th}$ external leg and write $t^i$ and $T^i$ in the place of $t^{a_i}$ and $T^{A_i}$, respectively.

decompose into color-ordered amplitudes $A_{\mathrm{YM}+\phi^3}$ entering (2.1) that still depend on the $T^A$ of the external scalars. In case of scalars in the first legs $1,2,\ldots,r$, further color decomposition w.r.t. $T^A$ gives rise to doubly-partial amplitudes $m^{\mathrm{tree}}_{\mathrm{YM}+\phi^3}$,

$$A^{\mathrm{tree}}_{\mathrm{YM}+\phi^3}(1,\rho) = \sum_{\gamma\in S_{r-1}} \mathrm{Tr}(T^1 T^{\gamma(2)} T^{\gamma(3)} \ldots T^{\gamma(r)}) m^{\mathrm{tree}}_{\mathrm{YM}+\phi^3}(1,\rho|1,\gamma) + \text{multitrace}\,. \tag{2.6}$$

The multitrace terms receive contributions from the contractions $\phi^{aA}\phi^{bB}\phi^{cC}\phi^{dD}\delta_{AC}\delta_{BD}$ in (2.3) and gluon propagators. They realize all cyclically inequivalent partitions of the $r$ scalars into up to $\lfloor \frac{r}{2} \rfloor$ traces, e.g. three permutations of $\mathrm{Tr}(T^1T^2)\mathrm{Tr}(T^3T^4)$ at four points. In the double copy to EYM, the $T^A$ are re-interpreted as the gauge-group generators of the YM states. For $r$ external gluons in the first legs $1,2,\ldots,r$ and $n{-}r$ external gravitons, the color-decomposition (2.6) as well as the KLT formula (2.1) carry over to

$$M^{\mathrm{tree}}_{n,r,\mathrm{EYM}} = \sum_{\gamma\in S_{r-1}} \mathrm{Tr}(T^1 T^{\gamma(2)} \ldots T^{\gamma(r)}) A^{\mathrm{tree}}_{\mathrm{EYM}}(1,\gamma(2,\ldots,r)) + \text{multitrace} \tag{2.7}$$

$$A^{\mathrm{tree}}_{\mathrm{EYM}}(1,\gamma(2,\ldots,r)) = \sum_{\rho,\tau\in S_{n-3}} A^{\mathrm{tree}}_{\mathrm{YM}}(1,\rho,n{-}1,n) S(\rho|\tau)_1 m^{\mathrm{tree}}_{\mathrm{YM}+\phi^3}(1,\tau,n,n{-}1|1,\gamma)\,. \tag{2.8}$$

The results of this work concern the explicit form of the analogous one-loop amplitudes. More specifically, we present a general method to determine $n$-point loop integrands of EYM from one-loop building blocks of (super-)Yang-Mills and spell out the detailed form of four-point examples. Our relations between loop integrands apply to supersymmetric EYM theories in any number of spacetime dimensions $D$ compatible with the variable amount of supersymmetry, and the four-point examples in sections 5 and 6 preserve 8 or 16 supercharges.

The double copy for supersymmetric EYM has all the supersymmetries on the YM side while taking $\mathrm{YM}+\phi^3$ as a purely bosonic theory with color structure $t^a$ for each gluon and $t^a \otimes T^A$ for each scalar. Furthermore, in the double copy to EYM, the coupling constants $g$ and $\lambda$ from the constituent theories are mapped onto the gauge coupling $g$ and gravitational coupling $\kappa$ of EYM theory according to

$$(g^2,\lambda) \to \left(\frac{\kappa}{4},\ 4\,\frac{g}{\kappa}\right)\,. \tag{2.9}$$

In four-point one-loop amplitudes of EYM, for instance, (2.9) leads to the powers of $\kappa$ and $g$ listed in table 1. In the examples of sections 5 and 6, we will determine the contributions at the orders of $g^m\kappa^{4-m}$ $(0\leq m\leq 4)$ for different numbers of external gluons and gravitons.

### 2.2 The Einstein-Yang-Mills double copy in the CHY formalism

There are several equivalent formulations of the double-copy structure of tree-level amplitudes including the KLT formula (2.1), the BCJ double copy based on cubic-vertex diagrams [1–3] and the CHY formalism [18–20]. The backbone of the CHY formulae for $n$-point tree-level amplitudes are moduli-space integrals over punctures $\sigma_1,\sigma_2,\ldots,\sigma_n$ on the Riemann sphere

| YM+$\phi^3$ | YM | EYM |
|---|---|---|
| $g^4\lambda^4$ | $g^4$ | $g^4$ |
| $g^4\lambda^3$ | $g^4$ | $\frac{1}{4}\kappa g^3$ |
| $g^4\lambda^2$ | $g^4$ | $\frac{1}{16}\kappa^2 g^2$ |
| $g^4\lambda$ | $g^4$ | $\frac{1}{64}\kappa^3 g$ |
| $g^4$ | $g^4$ | $\frac{1}{256}\kappa^4$ |

**Table 1**. In four-point one-loop amplitudes of $\mathrm{YM}\otimes(\mathrm{YM}+\phi^3)=\mathrm{EYM}$, the couplings $g$ and $\lambda$ on the left-hand side of the double copy are mapped to the following powers of the couplings $\kappa$ and $g$ in EYM via (2.9).



$\mathbb{C}\cup\{\infty\}$ which are completely localized by the scattering equations. The latter are imposed by the delta functions in the measure

$$\mathrm{d}\mu_n^{\mathrm{tree}} = \frac{\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\ldots\,\mathrm{d}\sigma_n}{\mathrm{vol}\,\mathrm{SL}_2(\mathbb{C})}\prod_{i=1}^{n}{}'\delta\Bigg(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{s_{ij}}{\sigma_{ij}}\Bigg)\,, \qquad \sigma_{ij}=\sigma_i-\sigma_j\,. \tag{2.10}$$

The inverse $\mathrm{vol}\,\mathrm{SL}_2(\mathbb{C})$ and the prime along with the product instruct to drop any three $\mathrm{d}\sigma_i\,\mathrm{d}\sigma_j\,\mathrm{d}\sigma_k$ and the associated delta functions, and the respective punctures can be fixed to $(\sigma_i,\sigma_j,\sigma_k)\to(0,1,\infty)$ after inserting the Jacobian $|\sigma_{ij}\sigma_{ik}\sigma_{jk}|^2$. The CHY reformulation of the KLT double copy then reads

$$M_{n,B\otimes C}^{\mathrm{tree}} = \int \mathrm{d}\mu_n^{\mathrm{tree}}\, I_B^{\mathrm{tree}}(\{1,2,\ldots,n\})\, I_C^{\mathrm{tree}}(\{1,2,\ldots,n\})\,, \tag{2.11}$$

where the integral is over the moduli space $\mathcal{M}_{0,n}$ of $n$ marked points on the Riemann sphere. The so-called half integrands $I_B^{\mathrm{tree}}, I_C^{\mathrm{tree}}$ are functions of the $\sigma_j$ that both transform with weight two under Möbius transformations $\sigma_j\to\frac{a\sigma_j+b}{c\sigma_j+d}$ with $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{C})$. Moreover, the half integrands $I_B^{\mathrm{tree}}, I_C^{\mathrm{tree}}$ depend on momenta and polarization or color degrees of freedom of the particles enclosed in $\{\ldots\}$.

#### 2.2.1 Basic half integrands for color and kinematics

The CHY formulae for bi-adjoint scalars, YM and gravity are based on two types of half integrands:

- Color degrees of freedom are encoded in

  $$I_{\phi^3}^{\mathrm{tree}}(\{1,2,\ldots,n\}) = \sum_{\rho\in S_{n-1}}\mathrm{Tr}(t^1 t^{\rho(2)} t^{\rho(3)}\ldots t^{\rho(n)})\mathrm{PT}(1,\rho(2,3,\ldots,n)) \tag{2.12}$$

  with Parke-Taylor factor

  $$\mathrm{PT}(1,2,\ldots,n) = \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{n-1,n}\sigma_{n1}}\,. \tag{2.13}$$

Upon color ordering w.r.t. two species of gauge-group generators, Parke-Taylor integrals

$$m_{\phi^3}(1,2,\ldots,n|\rho(1,2,\ldots,n)) = \int \mathrm{d}\mu_n^{\text{tree}}\,\mathrm{PT}(1,2,\ldots,n)\mathrm{PT}(\rho(1,2,\ldots,n)) \tag{2.14}$$

yield the doubly-partial amplitudes of biadjoint scalars. We will frequently apply this formula to determine Parke-Taylor integrals from the straightforward Feynman-diagram computation of $m_{\phi^3}$, for instance using the Berends-Giele recursion of [55].

- The dependence on polarization vectors $\epsilon_j$ (subject to transversality $\epsilon_j \cdot k_j = 0$) in YM and gravity is carried by the reduced Pfaffian

$$I^{\text{tree}}_{\text{YM}}(\{1,2,\ldots,n\}) = \mathrm{Pf}'\Psi_n(\{1,2,\ldots,n\}) = \frac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}\big[\Psi_n(\{1,2,\ldots,n\})\big]^{ij}_{ij}\,, \tag{2.15}$$

where $^{ij}_{ij}$ instruct to remove the $i^{\text{th}}$ and $j^{\text{th}}$ rows and columns from the $2n \times 2n$ matrix $\Psi_n$,

$$\begin{aligned} \Psi_n(\{1,2,\ldots,n\}) &= \begin{pmatrix} A & -C^t \\ C & B \end{pmatrix} \qquad & B_{ij} &= \begin{cases} \frac{\epsilon_i\cdot\epsilon_j}{\sigma_{ij}} &: \ i \neq j \\ 0 &: \ i = j \end{cases} \\ A_{ij} &= \begin{cases} \frac{s_{ij}}{\sigma_{ij}} &: \ i \neq j \\ 0 &: \ i = j \end{cases} \qquad & C_{ij} &= \begin{cases} \frac{\epsilon_i\cdot k_j}{\sigma_{ij}} &: \ i \neq j \\ -\sum_{m\neq i}^n \frac{\epsilon_i\cdot k_m}{\sigma_{im}} &: \ i = j \end{cases} . \end{aligned} \tag{2.16}$$

The reduced Pfaffian is linear in all of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and, on the support of the scattering equations in (2.10), independent on the choice of $i, j \in \{1,2,\ldots,n\}$ in (2.15) and invariant under linearized gauge transformations $\epsilon_m \to k_m$.

### 2.2.2 Integrands of EYM from half integrands of YM+$\phi^3$

The CHY formula (2.11) for EYM tree-level amplitudes with $r$ gauge bosons and $n{-}r$ gravitons reads [8, 21]

$$M^{\text{tree}}_{n,\text{EYM}} = \int \mathrm{d}\mu_n^{\text{tree}}\,\mathrm{Pf}'\Psi_n(\{1,2,\ldots,n\})\,I^{\text{tree}}_{\text{YM}+\phi^3}(\{1,2,\ldots,r\};\{r{+}1,\ldots,n\})\,, \tag{2.17}$$

and supersymmetric EYM amplitudes can be obtained by replacing the Pfaffian (2.15) by its fermionic completion.[5] The non-supersymmetric half integrand $I^{\text{tree}}_{\text{YM}+\phi^3}$ refers to YM+$\phi^3$ theory with Lagrangian (2.3). It depends on the gauge-group generators $T^j$ of the external scalars $j = 1,2,\ldots,r$ in the first set of labels and the polarizations $\epsilon_j$ of the external gauge bosons $j = r{+}1,\ldots,n$ in the second set. By analogy with the color decomposition (2.6),

[5]For ten-dimensional SYM, the supersymmetrization of the Pfaffian may be imported from the open pure-spinor superstring [56], based on the correlation functions of massless vertex operators in [10, 57, 58]. Alternatively, the simplified spin-field correlation functions of [45] yield an analogue of the Pfaffian (2.16) with two and four fermions and arbitrary numbers of bosons among the external states, also see [59] for two fermions.

it will be convenient to separately analyze color-ordered half integrands $J^{\rm tree}_{{\rm YM}+\phi^3}$ for single and double traces,

$$\begin{aligned} J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\}) &= I^{\rm tree}_{{\rm YM}+\phi^3}\big|_{{\rm Tr}(T^1T^2\ldots T^r)} \\ J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,\ldots,p|p{+}1,\ldots,r;\{r{+}1,\ldots,n\}) &= I^{\rm tree}_{{\rm YM}+\phi^3}\big|_{{\rm Tr}(T^1\ldots T^p){\rm Tr}(T^{p+1}\ldots T^r)}\,, \end{aligned} \tag{2.18}$$

where vertical bars are used to separate multiple traces (e.g. $J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots|\ldots|\ldots;\{r{+}1,\ldots,n\})$ for triple traces). The curly-bracket notation $\{\ldots\}$ refers to permutation-invariant functions of the data of the enclosed particles, e.g. $J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots;\{i,j,\ldots\}) = J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots;\{j,i,\ldots\})$. By contrast, the $J^{\rm tree}_{{\rm YM}+\phi^3}$ in (2.18) are only cyclically invariant in each of the slots associated with trace structures, e.g.

$$J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots|i_1,i_2,\ldots,i_p|\ldots;\{\ldots\}) = J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots|i_2,i_3,\ldots,i_p,i_1|\ldots;\{\ldots\})\,. \tag{2.19}$$

The single-trace instances of the half integrands are given by [21]

$$J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\}) = {\rm PT}(1,2,\ldots,r){\rm Pf}\,\Psi_n(\{r{+}1,\ldots,n\})\,, \qquad r\geq 2 \tag{2.20}$$

and combine the Parke-Taylor factor (2.13) from $I^{\rm tree}_{\phi^3}$ with the Pfaffian from $I^{\rm tree}_{\rm YM}$. The $2(n{-}r)\times 2(n{-}r)$ matrix $\Psi_n(\{r{+}1,\ldots,n\})$ slightly generalizes the definition (2.16) of the matrix $\Psi_n(\{1,\ldots,n\})$: the sum over $m$ in any $C_{ii} = -\sum_{m=1}^n \epsilon_i\cdot k_m/\sigma_{im}$ entering (2.20) runs over all of $\{1,2,\ldots,n\}$ instead of the shorter list $\{r{+}1,\ldots,n\}$ of gluon labels. Note that (2.20) only applies to $r\geq 2$ scalars. For $n$ gluons, the results of appendix A lead to the half integrand

$$J^{\rm tree}_{{\rm YM}+\phi^3}(\emptyset;\{1,2,\ldots,n\}) = {\rm Pf}'\,\Psi_n(\{1,2,\ldots,n\}) = I^{\rm tree}_{\rm YM}(\{1,2,\ldots,n\})\,, \tag{2.21}$$

and tree amplitudes with a single external scalar ($r=1$) vanish. For single-trace amplitudes of $r=n$ scalars in turn, we recover the result of the pure $\phi^3$ theory in (2.12),

$$J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,\ldots,n;\emptyset) = J^{\rm tree}_{\phi^3}(1,2,\ldots,n) = {\rm PT}(1,2,\ldots,n)\,. \tag{2.22}$$

In the double-trace situation, the simplest cases with zero and one gluon are [8, 21]

$$J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,\ldots,p|p{+}1,\ldots n;\emptyset) = s_{12\ldots p}{\rm PT}(1,2,\ldots,p){\rm PT}(p{+}1,\ldots,n) \tag{2.23}$$

$$\begin{aligned} J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,\ldots,p|p{+}1,\ldots n{-}1;\{n\}) &= {\rm PT}(1,2,\ldots,p){\rm PT}(p{+}1,\ldots,n{-}1) \\ &\times\Bigg[\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\frac{(s_{j,n}\epsilon_n\cdot k_i - s_{i,n}\epsilon_n\cdot k_j)\sigma_{ij}}{\sigma_{i,n}\sigma_{n,j}} + s_{12\ldots p}\sum_{j=1}^{n-1}\frac{\epsilon_n\cdot k_j}{\sigma_{j,n}}\Bigg]\,, \end{aligned} \tag{2.24}$$

and more general half integrands of ${\rm YM}+\phi^3$ can be found in appendix A.

### 2.2.3 Kleiss-Kuijf relations

We note for future reference that partial-fraction relations between Parke-Taylor factors imply so-called Kleiss-Kuijf relations [60] for $J^{\rm tree}_{{\rm YM}+\phi^3}$: within each trace, different cyclic orderings are related by

$$J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots|i,P,j,Q|\ldots;\{r{+}1,\ldots,n\}) = (-1)^{|Q|}\sum_{R\in P\sqcup\tilde{Q}} J^{\rm tree}_{{\rm YM}+\phi^3}(\ldots|i,R,j|\ldots;\{r{+}1,\ldots,n\})\,. \tag{2.25}$$

The shuffle product $\shuffle$ is defined recursively by

$$Pa\shuffle Qb = (P\shuffle Qb)a + (Pa\shuffle Q)b \tag{2.26}$$

for any two words $P = (p_1, p_2, \ldots, p_{|P|})$ and $Q = (q_1, q_2, \ldots, q_{|Q|})$ concatenated with words $a$ and $b$ of length one and $P\shuffle\emptyset = \emptyset\shuffle P = P$ in case of the empty word $\emptyset$. The length of the word $Q$ is denoted by $|Q|$, and we use a tilde-notation for the reversal $\tilde{Q} = (q_{|Q|}, \ldots, q_2, q_1)$. Intuitively, the shuffle product $P\shuffle Q$ collects all possibilities to interleave the letters in $P$ and $Q$ while preserving the order among the $p_i$ and $q_j$.

### 2.3 One-loop CHY formulae and forward limits

The CHY formulae for tree-level amplitudes were underpinned by ambitwistor-string theories in the Ramond-Neveu-Schwarz [61, 62] and pure-spinor formulations [63, 64]. The ambitwistor-string prescription for one-loop amplitudes is centered on moduli-space integrals over punctured genus-one surfaces or tori, and all integrations are again localized by scattering equations involving a $D$-dimensional loop momentum $\ell$. In particular, the manipulations of [37, 39] localize the integrand at the cusp $\tau \to i\infty$, where the torus with modular parameter $\tau$ degenerates to a nodal Riemann sphere. This limit reduces the genus-one scattering equations for the punctures to $(i = 1, 2, \ldots, n)$

$$\frac{\ell \cdot k_i}{\sigma_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0 \tag{2.27}$$

which are the forward limits $k_\pm \to \pm\ell$ of the genus-zero scattering equations in (2.10) with two additional legs $+, -$. Accordingly, $D$-dimensional $n$-point one-loop amplitudes in theories $B \otimes C$ with double-copy structure are given by [37, 39]

$$M^{\text{1-loop}}_{n,B\otimes C} = \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu^{\text{tree}}_{n+2}\, I^{\text{1-loop}}_B(\{1,2,\ldots,n\};\ell)\, I^{\text{1-loop}}_C(\{1,2,\ldots,n\};\ell) \tag{2.28}$$

in terms of forward limits of $(n+2)$-point CHY integrals at tree level. For theories with an ambitwistor-string description, the half integrands $I^{\text{1-loop}}_B, I^{\text{1-loop}}_C$ can be obtained from correlation functions on a torus in its degeneration limit to a nodal Riemann sphere. For instance, the maximally supersymmetric four-point correlators known from type-I and type-II superstrings [65] give rise to

$$I^{\text{1-loop}}_{\text{YM,max}} = I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,4\};\ell) = t_8(1,2,3,4) \sum_{\rho\in S_4} \mathrm{PT}(+,\rho(1,2,3,4),-)\,, \tag{2.29}$$

where the external polarizations conspire to the permutation-symmetric $t_8$-tensor contracting the linearized field strengths $f_j^{\mu\nu}$ (not to be confused with structure constants)

$$\begin{aligned} t_8(1,2,3,4) &= \mathrm{tr}(f_1f_2f_3f_4) - \frac{1}{4}\mathrm{tr}(f_1f_2)\mathrm{tr}(f_3f_4) + \mathrm{cyc}(2,3,4) \\ &= s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4) \\ f_i^{\mu\nu} &= k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu\,. \end{aligned} \tag{2.30}$$

Similar to (2.29), the half integrands $I^{\text{1-loop}}_{\text{YM}}(\{1,2,\ldots,n\};\ell)$ of gauge theories with 0 to 16 supercharges can be expressed in terms of $(n{+}2)$-point Parke-Taylor factors (2.13) involving the double points $(\sigma_+,\sigma_-)\to(0,\infty)$ of the nodal Riemann sphere. This can for instance be seen from the conformal-field-theory origin of these correlators[6] [62] and the manipulations of Ramond-Neveu-Schwarz spin sums in [43] based on [66].

Alternatively, (2.29) as well as generalizations to higher multiplicity and reduced supersymmetry [39, 43, 59] can be obtained from forward limits of genus-zero half integrands. For one-loop amplitudes in pure YM, the forward limit of the Pfaffian of (2.16) was shown to match the appropriate correlation functions on the torus [39] and to yield combinations of Parke-Taylor factors that reproduce known amplitude representations [44]. Apart from aligning the momenta $k_\pm\to\pm\ell$, the forward limit in pairs of external gluons $+,-$ amounts to the replacement (see section 3.1.1 for the analogous color replacement rules for scalars)

$$\begin{aligned}&\sum_{+,-}\epsilon_+^\mu\epsilon_-^\nu=\Delta^{\mu\nu}\,,\qquad \eta_{\mu\nu}\Delta^{\mu\nu}=D{-}2\\&V_\mu W_\nu\Delta^{\mu\nu}=V\cdot W\,,\qquad V,W\in\{k_i,\epsilon_i:\ i=1,2,\ldots,n\}\,.\end{aligned}\tag{2.31}$$

Similarly, by combining forward limits (2.31) in gluon polarizations with those in fermions and scalars yields $n$-point correlators with supersymmetric multiplets in the loop such as (2.29) [45].

The key idea in this work is to extend the construction of one-loop half integrands from forward limits of tree-level ones to the YM+$\phi^3$ theory. As will be detailed in section 3.1, forward limits of the tree-level half integrands of YM+$\phi^3$ in the gluons and scalars yield the non-supersymmetric half integrands

$$\begin{aligned}I^{\text{1-loop}}_{\text{YM}+\phi^3}(\{1,\ldots,r\};\{r{+}1,\ldots,n\};\ell)=\lim_{k_\pm\to\pm\ell}\sum_{+,-}\big[&I^{\text{tree}}_{\text{YM}+\phi^3}(\{1,2,\ldots,r,+,-\};\{r{+}1,\ldots,n\})\\&+I^{\text{tree}}_{\text{YM}+\phi^3}(\{1,2,\ldots,r\};\{r{+}1,\ldots,n,+,-\})\big]\end{aligned}\tag{2.32}$$

in one-loop amplitudes of EYM with $r$ external gluons and $n{-}r$ external gravitons:

$$M^{\text{1-loop}}_{n,\text{EYM},\alpha}=\int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int\mathrm{d}\mu^{\text{tree}}_{n+2}\,I^{\text{1-loop}}_{\text{YM},\alpha}(\{1,\ldots,n\};\ell)\,I^{\text{1-loop}}_{\text{YM}+\phi^3}(\{1,\ldots,r\};\{r{+}1,\ldots,n\};\ell)\tag{2.33}$$

The amount of supersymmetry will be indicated by the parameter $\alpha$ on the left-hand side and in the subscript of the YM half integrand on the right-hand side (e.g. $\alpha=$max or $\frac{1}{2}$-max). In the same way as $I^{\text{tree}}_{\text{YM}+\phi^3}$ are available in Parke-Taylor form [24, 27, 28, 68],[7] the forward limits in (2.32) can be brought into the $n!$-term form

$$I^{\text{1-loop}}_{\text{YM}+\phi^3}(\ell)=\sum_{\rho\in S_n}N_{+|\rho(12\ldots n)|-}(\ell)\mathrm{PT}(+,\rho(1,2,\ldots,n),-)\,,\tag{2.34}$$

[6] One can again import simplified correlators from the chiral-splitting formulation [6, 7] of superstrings to the ambitwistor setup such as the four-point expression (2.29) from [65] and the multiparticle correlators of [66] for external bosons and of [67] for the entire gauge multiplet.

[7] It follows from the work of Aomoto in the mathematics literature [69] that an arbitrary half integrand in the tree-level formula (2.11) with $\mathrm{SL}_2(\mathbb{C})$-weight two in all of $\sigma_1,\sigma_2,\ldots,\sigma_n$ can be decomposed in terms of Parke-Taylor factors.



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where $N_{+|\rho(12\ldots n)|-}(\ell)$ are local combinations of the color degrees of freedom, momenta and polarizations of the YM+$\phi^3$ states. Together with the Parke-Taylor representations of $I^{\text{1-loop}}_{\text{YM},\alpha}$ [43–45], all the $\mathrm{d}\mu^{\text{tree}}_{n+2}$-integrals in (2.33) can be straightforwardly performed in terms of doubly-partial amplitudes of biadjoint $\phi^3$ via (2.14). In this procedure, the forward limit $k_\pm \to \pm\ell$ of the integrals over $\sigma_j$ should be performed *after* summing the permutations to avoid divergences. More specifically, forward-limit divergences occur in non-supersymmetric theories and can be addressed in the ambitwistor framework using the methods of [44].[8] The supersymmetric examples in sections 5 and 6 will be unaffected by subtleties related to forward-limit divergences, and the discussion of half integrands of YM+$\phi^3$ in sections 3 and 4 are completely supersymmetry-agnostic.

### 2.4 Linearized versus quadratic propagators

Given that doubly-partial amplitudes (2.14) at tree level are functions of $k_i \cdot k_j$ with all the $k_j^2$ set to zero, the forward limit in (2.28) and (2.33) can only involve the loop momentum via $k_i \cdot \ell$ and not via $\ell^2$. Hence, the square of $\ell$ only enters the loop integrand of $M^{\text{1-loop}}_{n,B\otimes C}$ as a global prefactor $\ell^{-2}$ outside the $\mathrm{d}\mu^{\text{tree}}_{n+2}$-integral, see [37, 39] for its origin. This can be reconciled with the Feynman propagators $(\ell+K)^{-2}$ (with combinations $K$ of external momenta) expected for loop diagrams in massless field theories by the following rewriting of an $n$-gon integral,

$$\begin{aligned}\int \frac{2^{n-1}\,\mathrm{d}^D\ell}{\ell^2\ell_1^2\ell_{12}^2\ldots\ell_{12\ldots n-1}^2} &= \sum_{i=0}^{n-1}\int \frac{2^{n-1}\,\mathrm{d}^D\ell}{\ell_{12\ldots i}^2}\prod_{j\neq i}^{n}\frac{1}{\ell_{12\ldots j}^2-\ell_{12\ldots i}^2} \\ &= \sum_{i=0}^{n-1}\int\frac{\mathrm{d}^D\ell}{\ell^2}\prod_{j=0}^{i-1}\frac{1}{s_{j+1,j+2,\ldots,i,-\ell}}\prod_{j=i+1}^{n-1}\frac{1}{s_{i+1,i+2,\ldots,j,\ell}}\,,\end{aligned} \tag{2.35}$$

where our notation for composite momenta and $\ell$-dependent Mandelstam invariants is

$$k_{12\ldots p} = \sum_{j=1}^{p} k_j\,, \qquad \ell_{12\ldots p} = \ell + k_{12\ldots p}\,, \qquad s_{12\ldots p,\pm\ell} = s_{12\ldots p} \pm \ell\cdot k_{12\ldots p}\,. \tag{2.36}$$

The first step of (2.35) is based on partial-fraction manipulations, and we have shifted the loop momentum by external momenta in passing to the second line such that the only quadratic propagator is $\ell^{-2}$ rather than $(\ell+K)^{-2}$. The transition to linearized propagators in (2.35) can be straightforwardly extended to massive corners. As visualized in figure 1, each of the $n$ terms in the partial-fraction decomposition (2.35) of the $n$-gon can be thought of as the $(n{+}2)$-point tree-level diagram obtained from cutting one of the $n$-gon propagators.

With the $n$ possibilities to cut a given $n$-gon and its $(n{-}1)!$ cyclically inequivalent orderings, the linearized $n$-gon propagators can be associated with $n!$ tree-level diagrams of half-ladder topology. These are the master diagrams in the sense of the BCJ color-kinematics duality [1, 2], i.e. their kinematic numerators w.r.t. linearized propagators generate all the lower-gon numerators by kinematic Jacobi identities. The $n!$ Parke-Taylor coefficients

[8] In general non-supersymmetric one-loop CHY formulae, the regularization of forward-limit divergences can be traced back to the dropout of singular solutions of the scattering equations [38, 40].

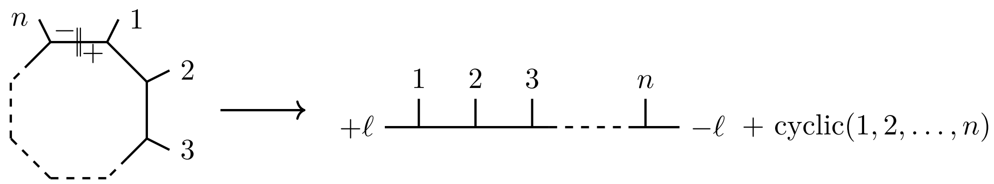


**Figure 1**. Terms in the partial-fraction representation of $n$-point loop integrals in (2.35) can be interpreted as $(n{+}2)$-point tree-level diagrams.

$N_{+|\rho(12\ldots n)|-}(\ell)$ in (2.34) with $\rho \in S_n$ are the master numerators associated with the ladder diagram in figure 1 and its permutations in $1,2,\ldots,n$. The kinematic Jacobi identities that determine the remaining cubic-diagram numerators (of $(n{-}1)$-gons, $(n{-}2)$-gons etc.) can be traced back to properties of the $\mathrm{d}\mu^{\rm tree}_{n+2}$-integrals over Parke-Taylor factors in (2.33).

However, the forward limit of individual doubly-partial amplitudes (2.14) will usually not recombine to quadratic propagators. It requires the sum over $n$ terms in (2.35) to recover the $n$-gon integral with Feynman propagators $(\ell+k_{12\ldots p})^{-2}$ by reversing the shifts of loop momentum and the partial-fraction manipulations. It will be an important cross-check for the $I^{\text{1-loop}}_{\mathrm{YM}+\phi^3}$ to be derived from forward limits (2.32) that their $\mathrm{d}\mu^{\rm tree}_{n+2}$-integrals against $I^{\text{1-loop}}_{\mathrm{YM},\alpha}$ admit a recombination to quadratic propagators. We will do so at the level of color-ordered EYM amplitudes where, in analogy with the tree-level notation (2.18),

$$\begin{aligned} &J^{\text{1-loop}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,j|j{+}1,\ldots,p|\ldots;\{r{+}1,\ldots,n\};\ell) \\ &\quad= I^{\text{1-loop}}_{\mathrm{YM}+\phi^3}(\{1,2,\ldots,r\};\{r{+}1,\ldots,n\};\ell)\big|_{\mathrm{Tr}(T^1T^2\ldots T^j)\mathrm{Tr}(T^{j+1}\ldots T^p)\ldots}\,. \end{aligned} \tag{2.37}$$

We can express the loop integrand of color-ordered EYM amplitudes in terms of so-called partial integrands in YM theory

$$a^{\text{1-loop}}_{\mathrm{YM},\alpha}(+,\rho(1,\ldots,n),-) = \lim_{k_\pm\to\pm\ell}\int \mathrm{d}\mu^{\rm tree}_{n+2}\mathrm{PT}(+,\rho(1,\ldots,n),-)I^{\text{1-loop}}_{\mathrm{YM},\alpha}(\{1,\ldots,n\};\ell) \tag{2.38}$$

which were introduced in [36] as gauge-invariant building blocks of loop integrands in linearized-propagator representations. The idea is to calculate the $\mathrm{d}\mu^{\rm tree}_{n+2}$-integral separately for each Parke-Taylor factor in the decomposition (2.34) of the YM+$\phi^3$ half integrand (2.37) and to thereby obtain the decomposition

$$\begin{aligned} &A^{\text{1-loop}}_{\mathrm{EYM},\alpha}(1,2,\ldots,j|j{+}1,\ldots,p|\ldots;\{r{+}1,\ldots,n\}) \\ &= \int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int \mathrm{d}\mu^{\rm tree}_{n+2}\,J^{\text{1-loop}}_{\mathrm{YM}+\phi^3}(1,\ldots,j|j{+}1,\ldots,p|\ldots;\{r{+}1,\ldots,n\};\ell)I^{\text{1-loop}}_{\mathrm{YM},\alpha}(\{1,\ldots,n\};\ell) \\ &= \int\frac{\mathrm{d}^D\ell}{\ell^2}\sum_{\rho\in S_n} N_{+|\rho(12\ldots n)|-}(\ell)\big|_{\mathrm{Tr}(T^1T^2\ldots T^j)\mathrm{Tr}(T^{j+1}\ldots T^p)\ldots}\,a^{\text{1-loop}}_{\mathrm{YM},\alpha}(+,\rho(1,2,\ldots,n),-)\,. \end{aligned} \tag{2.39}$$

The partial integrands $a^{\text{1-loop}}_{\mathrm{YM},\alpha}$ with maximal and half-maximal supersymmetry are available from forward limits of doubly-partial amplitudes [36, 43] and will be reviewed in sections 5.1 and 6.1, respectively. While the $a^{\text{1-loop}}_{\mathrm{YM},\alpha}$ are still given in terms of linearized propagators, the single- and multi-trace amplitudes $A^{\text{1-loop}}_{\text{EYM},\alpha}$ admit quadratic-propagator representations. By combining the contributions of all the $n!$ permutations $\rho$ in (2.39), we will find the expected quadratic-propagator expressions for EYM loop-integrands at $n=4$ for different amounts $\alpha$ of supersymmetry, and separately at each order in the couplings $g$ and $\kappa$ (see table 1).

## 3 YM+$\phi^3$ half integrands at one loop: all-multiplicity results

The first step in the construction of one-loop amplitudes in EYM theories is to obtain the non-supersymmetric half integrands $I^{\text{1-loop}}_{\text{YM}+\phi^3}$ in their CHY representation (2.33). In this section we investigate the color decomposition of these one-loop half integrands in YM+$\phi^3$ theory, with single- and multi-trace coefficients $J^{\text{1-loop}}_{\text{YM}+\phi^3}$ entering the color-ordered EYM amplitudes in (2.39). In particular, we explain how to choose the forward limits of tree-level half integrands in (2.32) that contribute to a given color-ordered one-loop half integrand $J^{\text{1-loop}}_{\text{YM}+\phi^3}$. Four-point examples can be found in the next section 4.

### 3.1 First look at one-loop YM+$\phi^3$ half integrands from forward limits

In YM+$\phi^3$ theory, various forward limits of tree-level half integrands in both scalars and gluons contribute to the same color-ordered one-loop half integrand. The basic premise for the choice of these contributions is to include all forward limits of $(n{+}2)$-point tree-level half integrands that have the external states, color structure and powers of the couplings $\lambda, g$ compatible with the desired $n$-point one-loop half integrand.

#### 3.1.1 Color management in forward limits

In EYM theories with gauge groups SU($N$) and U($N$), forward limits in color degrees of freedom may change the number of traces. At the level of the double-copy constituents YM+$\phi^3$, this concerns the trace structure of the generators $T^A$ specific to the bi-adjoint scalars which changes upon forward limits in the scalars. The sum $\sum_{+,-}$ over adjoint degrees of freedom $T^+ = T^{A_+}$ and $T^- = T^{A_-}$ of scalar legs $+$ and $-$ in (2.32) is implemented through the completeness relations

$$\begin{aligned} \sum_{+,-} (T^+)_i{}^j (T^-)_k{}^l &= \delta_i^l \delta_k^j && : \ \text{U}(N) \\ \sum_{+,-} (T^+)_i{}^j (T^-)_k{}^l &= \delta_i^l \delta_k^j - \frac{1}{N}\delta_i^j \delta_k^l && : \ \text{SU}(N)\,, \end{aligned} \tag{3.1}$$

with fundamental indices $i,j,k,l = 1,2,\ldots,N$. Depending on the relative positions of legs $+$ and $-$, the forward limits of traces follow from one of

$$\begin{aligned} \sum_{+,-} \text{Tr}(P,+,Q,-) &= \text{Tr}(P)\text{Tr}(Q) + c_1 \text{Tr}(PQ) \\ \sum_{+,-} \text{Tr}(P,+,-) &= c_2 \text{Tr}(P) \\ \sum_{+,-} \text{Tr}(+,-) &= N c_2 \\ \sum_{+,-} \text{Tr}(P,+)\text{Tr}(Q,-) &= \text{Tr}(PQ) + c_1 \text{Tr}(P)\text{Tr}(Q)\,. \end{aligned} \tag{3.2}$$

| factor | $\mathrm{U}(N)$ | $\mathrm{SU}(N)$ |
|---|---|---|
| $c_1$ | $0$ | $-\frac{1}{N}$ |
| $c_2$ | $N$ | $\frac{N^2-1}{N}$ |

**Table 2**. The color factors $c_1$ and $c_2$ subject to $c_2 - c_1 = N$ arise in the forward limits of traces over generators of the gauge groups $\mathrm{U}(N)$ and $\mathrm{SU}(N)$, see (3.2).

We employ the shorthand notation

$$\mathrm{Tr}(i_1, i_2, \ldots, i_n) = \mathrm{Tr}(T^{A_{i_1}} T^{A_{i_2}} \ldots T^{A_{i_n}}) \tag{3.3}$$

and use capital letters from the second half of the alphabet for words $P = (i_1, i_2, \ldots, i_{|P|})$ of length $|P|$ as in section 2.2.3. Moreover, the factors $c_1$ and $c_2$ subject to $c_2 - c_1 = N$ differ for gauge groups $\mathrm{U}(N), \mathrm{SU}(N)$ and are listed in table 2. The first and last line of (3.2) illustrate that the number of traces may be raised or lowered under forward limits in color factors.

### 3.1.2 Coupling dependence in forward limits

Even though we defined the half integrands $J^{\text{tree}}_{\text{YM}+\phi^3}$ to exclude the couplings in the $\text{YM}+\phi^3$ Lagrangian (2.3), one can straightforwardly associate powers of $g$ and $\lambda$ to each trace- and external-state configuration of $J^{\text{tree}}_{\text{YM}+\phi^3}$: single-trace integrands of $n$ external scalars generate diagrams with $n{-}2$ cubic vertices $\sim \phi^3$, so they are associated with the powers $(g\lambda)^{n-2}$. Cubic vertices involving one or three gluons in turn involve a factor of $g$ rather than $g\lambda$. As exemplified in figure 2, this reduces the power-counting of $\lambda$ by one whenever an external scalar is replace by an external gluon and by two for each additional trace. Hence, we associate

$$m\text{-trace } J^{\text{tree}}_{\text{YM}+\phi^3} \ @ \ r \geq 2 \text{ scalars } \& \ n{-}r \text{ gluons } \ \leftrightarrow \ g^{n-2}\lambda^{r-2m} \,, \tag{3.4}$$

for instance $g^{n-2}\lambda^{r-2}$ to single-trace examples and $g^{n-2}\lambda^{r-4}$ to double-trace examples with $r \geq 2$ scalars and $n{-}r$ gluons.

By the color-factor identities (3.2), forward limits in a scalar can introduce or eliminate one trace. As a result, the tree-level power counting of (3.4) yields a bandwidth of three different powers of $\lambda$ in each $J^{\text{1-loop}}_{\text{YM}+\phi^3}$ with $m \geq 2$ traces, where the lowest power $\lambda^{r-2m}$ also arises from forward limits in a pair of gluons,

$$\begin{aligned} &m\text{-trace } J^{\text{1-loop}}_{\text{YM}+\phi^3} \ @ \ r \geq 2 \text{ scalars } \& \ n{-}r \text{ gluons with } m \geq 2 \\ &\leftrightarrow \begin{cases} g^n\lambda^{r-2m+4} \ \& \ g^n\lambda^{r-2m+2} \ \& \ g^n\lambda^{r-2m} : \text{scalar forward limit} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad g^n\lambda^{r-2m} : \text{gluon forward limit} \,. \end{cases} \end{aligned} \tag{3.5}$$

For single-trace half integrands at one loop, the option with the highest power of $\lambda$ in (3.5) is absent since there is no underlying forward limit with fewer traces. Hence, one arrives at



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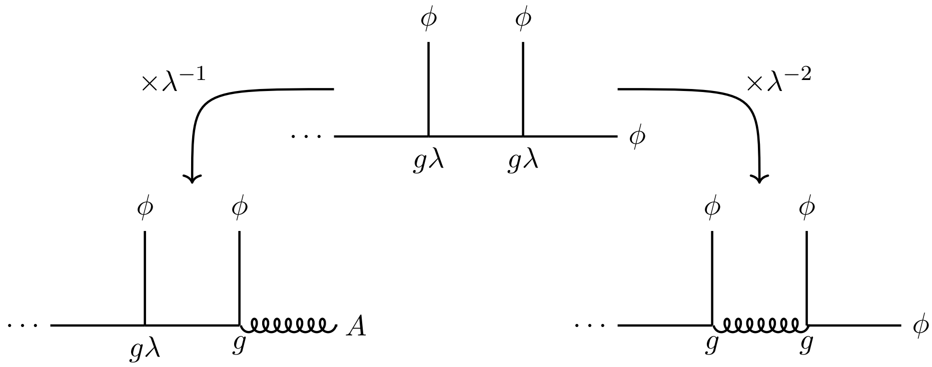


**Figure 2**. Trading an external scalar for a gluon effectively removes a power of $\lambda$ (left panel). Similarly, increasing the number of traces effectively removes two powers of $\lambda$ (right panel).

only two dependences on the couplings for single traces, where the lower power $\lambda^{r-2}$ also arises from forward limits in gluons.

$$\begin{aligned} &\text{single-trace } J^{\text{1-loop}}_{\text{YM}+\phi^3} \text{ @ } r \geq 2 \text{ scalars \& } n{-}r \text{ gluons} \\ &\leftrightarrow \begin{cases} g^n\lambda^r \ \& \ g^n\lambda^{r-2} : \text{scalar forward limit} \\ \qquad\qquad\ g^n\lambda^{r-2} : \text{gluon forward limit}\,. \end{cases} \end{aligned} \tag{3.6}$$

Upon double copy through the CHY-representation (2.39) of EYM amplitudes, the dictionary (2.9) for the couplings leads to the following power-counting on $g$ and $\kappa$,

$$\begin{aligned} &\text{single-trace } A^{\text{1-loop}}_{\text{EYM},\alpha} \text{ @ } r \geq 2 \text{ gluons \& } n{-}r \text{ gravitons} \\ &\leftrightarrow \begin{cases} g^r\kappa^{n-r} \ \& \ g^{r-2}\kappa^{n+2-r} : \text{gluon forward limit} \\ \qquad\qquad\quad g^{r-2}\kappa^{n+2-r} : \text{graviton forward limit} \end{cases} \\ &m\text{-trace } A^{\text{1-loop}}_{\text{EYM},\alpha} \text{ @ } r \geq 2 \text{ gluons \& } n{-}r \text{ gravitons with } m \geq 2 \\ &\leftrightarrow \begin{cases} g^{r-2m+4}\kappa^{n+2m-4-r} \ \& \ g^{r-2m+2}\kappa^{n+2m-2-r} \ \& \ g^{r-2m}\kappa^{n+2m-r} : \text{gluon forward limit} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad g^{r-2m}\kappa^{n+2m-r} : \text{graviton forward limit}\,. \end{cases} \end{aligned} \tag{3.7}$$

### 3.2 Explicit single-trace YM+$\phi^3$ half integrands at one loop

We shall now spell out the detailed decomposition of single-trace YM+$\phi^3$ half integrands at one loop into forward limits of their color-ordered tree-level counterparts $J^{\text{tree}}_{\text{YM}+\phi^3}$. This will be done separately for the two orders in the coupling noted in (3.6), i.e. for both half integrands on the right-hand side of

$$\begin{aligned} J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\};\ell) &= J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\};\ell)\big|_{g^n\lambda^r} \\ &\quad + J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\};\ell)\big|_{g^n\lambda^{r-2}}\,. \end{aligned} \tag{3.8}$$

#### 3.2.1 No external gluons

In order to simplify the bookkeeping, we shall focus on the case with $r = n$ scalars first. The contribution $\sim \lambda^n$ to (3.8) can then be obtained by adding up all the forward limits that yield the single-trace $\mathrm{Tr}(1,2,\ldots,n)$ on the right-hand side of (3.2). As we shall see, both $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ gauge groups give rise to the same $\lambda^n$ contributions

$$J^{\text{1-loop}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n;\emptyset;\ell)\big|_{g^n\lambda^n} = N\big[J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n,+,-;\emptyset)+J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n,-,+;\emptyset)\\ +\mathrm{cyc}(1,2,\ldots,n)\big]\,. \tag{3.9}$$

With the Parke-Taylor form (2.22) of the single-trace half integrands of the pure $\phi^3$-theory, this reproduces the color factors in the ambitwistor-string formulae for planar one-loop super-Yang-Mills amplitudes [37, 39]. Still, it is instructive to see how it arises from the forward-limit computations and a careful tracking of all single traces at the $\lambda^n$ order in (3.2): intermediate steps towards (3.9) give

$$\begin{aligned} &J^{\text{1-loop}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n;\emptyset;\ell)\big|_{g^n\lambda^n}\\ &= c_2\big[J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n,+,-;\emptyset)+J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n,-,+;\emptyset)\big]\\ &\quad + c_1\sum_{j=1}^{n-1} J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,j,+,j{+}1,\ldots,n,-;\emptyset)+\mathrm{cyc}(1,2,\ldots,n)\,, \end{aligned} \tag{3.10}$$

where the special case $J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n,-,+;\emptyset)+\mathrm{cyc}(1,2,\ldots,n,-)=0$ of the Kleiss-Kuijf relations (2.25) identifies the coefficient of $c_1$ to be minus the coefficient of $c_2$. By virtue of the relation $c_2-c_1=N$ universal to $\mathrm{SU}(N)$ and $\mathrm{U}(N)$, one arrives at the simplified expression (3.9).

At the subleading order of $\lambda^{n-2}$ in turn, the $N$-dependence varies between $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ through the color factor $c_2$ in table 2 (with $Nc_2=N^2$ for $\mathrm{U}(N)$ and $Nc_2=N^2-1$ for $\mathrm{SU}(N)$),

$$\begin{aligned} &J^{\text{1-loop}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,n;\emptyset;\ell)\big|_{g^n\lambda^{n-2}}\\ &= Nc_2 J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,\ldots,n|+,-;\emptyset)+J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,\ldots,n;\{+,-\})\\ &\quad +\sum_{j=1}^{n-1}\big[J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,2,\ldots,j,+|-,j{+}1,\ldots,n;\emptyset)+\mathrm{cyc}(1,2,\ldots,n)\big]\,. \end{aligned} \tag{3.11}$$

The forward limit in the gluon legs $\{+,-\}$ is understood to incorporate the sum over $D-2$ physical polarization vectors $\epsilon_+,\epsilon_-$ as in (2.31).

In the expression (3.11) for the $\lambda^{n-2}$-order, the gluonic forward limits $J^{\text{tree}}_{\mathrm{YM}+\phi^3}(\ldots;\{+,-\})$ and the scalar forward limits $J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,\ldots,j,+|-,j{+}1,\ldots,n;\emptyset)$ capture different terms in the partial-fraction decomposition of various Feynman integrals. For instance, the $n$-gon diagram in figure 3 with one gluon propagator and otherwise scalar propagators yields one tree diagram corresponding to a gluonic forward limit and $n{-}1$ tree diagrams corresponding to scalar forward limits under partial-fraction decomposition. As a result, the recombination of the loop integrand of (2.39) to quadratic propagators relies on having the correct relative normalization of the terms $J^{\text{tree}}_{\mathrm{YM}+\phi^3}(\ldots;\{+,-\})$ and $J^{\text{tree}}_{\mathrm{YM}+\phi^3}(1,\ldots,j,+|-,j{+}1,\ldots,n;\emptyset)$ at the same order of $g,\lambda$ in (3.11). Since this recombination has to occur for any value of $N$, its first term with coefficient $Nc_2$ will separately yield quadratic propagators after performing the $d\mu^{\text{tree}}_{n+2}$ integral in (2.39).

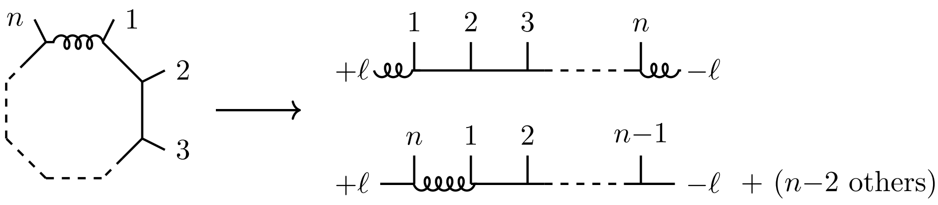

**Figure 3**. In the partial-fraction decomposition of an $n$-gon diagram with one gluon propagator and otherwise scalar propagators, the associated tree-level diagrams describe one gluonic forward limit with internal scalars and $n{-}1$ scalar forward limits with one internal gluon line.

One can also confirm from one-loop Feynman-diagram computations that the only admissible dependence of color-ordered EYM amplitudes on the group-theory data can occur via Kronecker-deltas in the fundamental indices $\delta^i_i = N$ and in the adjoint ones $\delta^a_a = Nc_2$. This can be manifested in the forward-limit approach of this work by applying Kleiss-Kuijf relations, both at the single-trace and at the multitrace level.

#### 3.2.2 Adjoining external gluons

The single-trace expressions (3.9) and (3.11) can be straightforwardly generalized to external gluons. The two contributions in (3.8) at different orders in the couplings then read (for $r \geq 2$ scalars)

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\};\ell)\big|_{g^n\lambda^r} \\ &= N\big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r,+,-;\{r{+}1,\ldots,n\}) \\ &\qquad + J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r,-,+;\{r{+}1,\ldots,n\}) + \text{cyc}(1,2,\ldots,r)\big] \end{aligned} \tag{3.12}$$

as well as

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n\};\ell)\big|_{g^n\lambda^{r-2}} \\ &= Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r|+,-;\{r{+}1,\ldots,n\}) \\ &\quad + J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{r{+}1,\ldots,n,+,-\}) \\ &\quad + \sum_{j=1}^{r-1}\big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,j,+|-,j{+}1,\ldots,r;\{r{+}1,\ldots,n\}) + \text{cyc}(1,2,\ldots,r)\big]\,. \end{aligned} \tag{3.13}$$

In particular, the simplification of the leading order in $\lambda$ literally follows the discussion around (3.10) since the tree-level half integrands (2.20) with external gluons obey the same Kleiss-Kuijf relations as in the case without gluons.

While $\text{YM}+\phi^3$ amplitudes and half integrands with a single scalar vanish, the purely gluonic cases can be obtained by truncating (3.13) to its first two lines,

$$\begin{aligned} J^{\text{1-loop}}_{\text{YM}+\phi^3}(\emptyset;\{1,2,\ldots,n\};\ell) &= J^{\text{1-loop}}_{\text{YM}+\phi^3}(\emptyset;\{1,2,\ldots,n\};\ell)\big|_{g^n} \\ &= Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}(+,-;\{1,2,\ldots,n\}) + J^{\text{tree}}_{\text{YM}+\phi^3}(\emptyset;\{1,2,\ldots,n,+,-\})\,, \end{aligned} \tag{3.14}$$

see (2.20) and (2.21) for the respective tree-level building blocks. The first and second term on the right-hand side of (3.14) describe diagrams with scalars and gluons in the loop, respectively.

### 3.3 Explicit multi-trace YM+$\phi^3$ half integrands at one loop

The single-trace results of the previous section will now be generalized to multiple traces. By (3.5), there are three possible dependences on the couplings

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\dots|\text{Tr}_m;P;\ell) \\ &= J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\dots|\text{Tr}_m;P;\ell)\,\big|_{g^n\lambda^{r-2m+4}} + J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\dots|\text{Tr}_m;P;\ell)\,\big|_{g^n\lambda^{r-2m+2}} \\ &\quad + J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\dots|\text{Tr}_m;P;\ell)\,\big|_{g^n\lambda^{r-2m}} \end{aligned} \tag{3.15}$$

for a total of $n-r$ gluons in $P=\{r+1,\dots,n\}$ and $r$ scalars in the union of the cyclically ordered sets $\text{Tr}_1,\text{Tr}_2,\dots,\text{Tr}_m$ with $m\geq 2$. In the same way as the combinatorial structure of the single-trace formulae (3.9) and (3.11) does not change upon addition of gluons in (3.12) and (3.13), also the multitrace results can be presented in a unified way for any choice of $P$.

#### 3.3.1 Double trace

In the double-trace example, i.e. (3.15) at $m=2$, the three different orders in couplings contribute with

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\dots,s|s{+}1,\dots,r;P;\ell)\,\big|_{g^n\lambda^{r}} \\ &= \sum_{\substack{Q\in\text{cyc}(1,2,\dots,s)\\ R\in\text{cyc}(s+1,\dots,r)}} \big[J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+,R,-;P)+(+\leftrightarrow -)\big] \end{aligned} \tag{3.16}$$

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\dots,s|s{+}1,\dots,r;P;\ell)\,\big|_{g^n\lambda^{r-2}} \\ &= N\Bigg\{\sum_{Q\in\text{cyc}(1,2,\dots,s)} J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+,-|s{+}1,\dots,r;P) \\ &\qquad + \sum_{R\in\text{cyc}(s+1,\dots,r)} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\dots,s|R,+,-;P)+(+\leftrightarrow -)\Bigg\} \end{aligned} \tag{3.17}$$

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\dots,s|s{+}1,\dots,r;P;\ell)\,\big|_{g^n\lambda^{r-4}} \\ &= Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\dots,s|s{+}1,\dots,r|+,-;P) \\ &\quad + J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\dots,s|s{+}1,\dots,r;P\cup\{+,-\}) \\ &\quad + \sum_{j=1}^{s-1}\big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\dots,j,+|-,j{+}1,\dots,s|s{+}1,\dots,r;P)+\text{cyc}(1,2,\dots,s)\big] \\ &\quad + \sum_{j=1}^{r-s-1}\big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,\dots,s|s{+}1,\dots,s{+}j,+|-,s{+}j{+}1,\dots,r;P)+\text{cyc}(s{+}1,\dots,r)\big]\,. \end{aligned} \tag{3.18}$$

While the orders of $g^n\lambda^r$ and $g^n\lambda^{r-4}$ follow from straightforward application of (3.2), the result (3.17) for the intermediate order of $g^n\lambda^{r-2}$ is based on additional simplifications: the color identities (3.2) in the first place lead to

$$\begin{aligned}
&J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,s|s{+}1,\ldots,r;P;\ell)\,\big|_{g^n\lambda^{r-2}} \\
&= c_2\Bigg\{\sum_{Q\in\text{cyc}(1,2,\ldots,s)} J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+,-|s{+}1,\ldots,n;P) \\
&\qquad + \sum_{R\in\text{cyc}(s+1,\ldots,r)} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,s|R,+,-;P) + (+\leftrightarrow -)\Bigg\} \\
&\quad + c_1\sum_{j=1}^{s-1} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,j,+,j{+}1,\ldots,s,-|s{+}1,\ldots,r;P) \\
&\quad + c_1\sum_{j=1}^{r-s-1} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,s|s{+}1,\ldots,s{+}j,+,s{+}j{+}1,\ldots,r,-;P) \\
&\quad + c_1\sum_{\substack{Q\in\text{cyc}(1,2,\ldots,s)\\ R\in\text{cyc}(s+1,\ldots,r)}}\left[J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+|R,-;P) + J^{\text{tree}}_{\text{YM}+\phi^3}(Q,-|R,+;P)\right],
\end{aligned} \tag{3.19}$$

but the last line cancels by $J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,m,+|\ldots;P) + \text{cyc}(1,2,\ldots,m) = 0$, i.e. by Kleiss-Kuijf relations. Moreover, the same type of Kleiss-Kuijf relations implies that the coefficients of $c_1$ in the third and fourth line of (3.19) conspire to minus the coefficient of $c_2$ in the first two lines. Based on $c_2 - c_1 = N$, one can then confirm the global prefactor in (3.17); the same recombination was already noted in the single-trace context of (3.10).

As mentioned before, one can anticipate from one-loop Feynman diagrams that the only $N$-dependence in (3.16) to (3.18) has to occur via one of $\delta^i_i = N$ and $\delta^a_a = Nc_2$.

### 3.3.2 Any number of traces

The multitrace generalizations of the double-trace results (3.16) to (3.18) are given as follows for $r$ scalars in $\text{Tr}_1,\ldots,\text{Tr}_m$ with $m\geq 2$ and $n{-}r$ gluons in $P=\{r{+}1,\ldots,n\}$

$$\begin{aligned}
&J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\ldots|\text{Tr}_m;P;\ell)\,\big|_{g^n\lambda^{r-2m+4}} \\
&= \sum_{1\leq i<j}^{m}\ \sum_{\substack{Q\in\text{cyc}(\text{Tr}_i)\\ R\in\text{cyc}(\text{Tr}_j)}} J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+,R,-|\text{Tr}_1|\ldots|\widehat{\text{Tr}_i}|\ldots|\widehat{\text{Tr}_j}|\ldots|\text{Tr}_m;P) + (+\leftrightarrow -)
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\ldots|\text{Tr}_m;P;\ell)\,\big|_{g^n\lambda^{r-2m+2}} \\
&= N\sum_{j=1}^{m}\ \sum_{Q\in\text{cyc}(\text{Tr}_j)} J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+,-|\text{Tr}_1|\ldots|\widehat{\text{Tr}_j}|\ldots|\text{Tr}_m;P) + (+\leftrightarrow -)
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
&J^{\text{1-loop}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\ldots|\text{Tr}_m;P;\ell)\,\big|_{g^n\lambda^{r-2m}} \\
&= Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\ldots|\text{Tr}_m|+,-;P) + J^{\text{tree}}_{\text{YM}+\phi^3}(\text{Tr}_1|\text{Tr}_2|\ldots|\text{Tr}_m;P\cup\{+,-\}) \\
&\quad + \sum_{j=1}^{m}\Bigg\{\sum_{\substack{\text{Tr}_j=QR\\ |Q|,|R|\neq 0}} J^{\text{tree}}_{\text{YM}+\phi^3}(Q,+|R,-|\text{Tr}_1|\ldots|\widehat{\text{Tr}_j}|\ldots|\text{Tr}_m;P) + \text{cyc}(\text{Tr}_j)\Bigg\}.
\end{aligned} \tag{3.22}$$

The notation $\widehat{\mathrm{Tr}}_j$ instructs to omit the respective trace, and the sum in the last line is over all possibilities to split $\mathrm{Tr}_j = (c_1, c_2, \ldots, c_r)$ into non-empty words $Q = (c_1, \ldots, c_{|Q|})$ and $R = (c_{|Q|+1}, \ldots, c_r)$ with $|Q| = 1, 2, \ldots, r{-}1$. The prefactor $N$ of (3.21) again follows from Kleiss-Kuijf relations;[9] together with the first term in (3.22), the complete color dependence lines up with the factors $\delta^i_i = N$ or $\delta^a_a = N c_2$ expected from Feynman diagrams.

### 3.4 Parke-Taylor form of the tree-level building blocks

The forward-limit representations of $J^{\text{1-loop}}_{\text{YM}+\phi^3}$ in sections 3.2 and 3.3 are particularly convenient if the $\mathrm{d}\mu^{\text{tree}}_{n+2}$ integration over the punctures in one-loop CHY formula (2.33) can be performed via (2.14) in terms of doubly-partial amplitudes. This is the case when all the contributing $J^{\text{tree}}_{\text{YM}+\phi^3}$ are organized in terms of $(n{+}2)$-point Parke-Taylor factors as we assumed in (2.34) and in passing to the last line of (2.39).

In order to make use of the Parke-Taylor decompositions of tree-level half integrands in the literature [24, 27, 28] or the `Mathematica` package [68], we relegate the forward limit $k_\pm \to \pm\,\ell$ to the last step of the computation, i.e. *after* performing the $\mathrm{d}\mu^{\text{tree}}_{n+2}$-integral in terms of doubly-partial amplitudes. This has been implicitly assumed in introducing the partial integrands $a^{\text{1-loop}}_{\text{YM},\alpha}$ in (2.39).

#### 3.4.1 Examples at leading order in $\lambda$ with external gluons

The simplest non-trivial examples of Parke-Taylor decompositions (2.34) arise for the $\lambda^r$-order of the single-trace half integrands (3.12) with external gluons. Their tree-level constituents are given in (2.20) and in the one-gluon case for instance reduce to [16, 24]

$$J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r,+,-;\{p\}) = \sum_{j=0}^{r} \epsilon_p\cdot(k_-{+}k_1{+}\ldots{+}k_j)\mathrm{PT}(-,1,2,\ldots,j,p,j{+}1,\ldots,r,+)\,. \tag{3.23}$$

The resulting one-loop half integrand in (3.12) then becomes

$$\begin{aligned} &J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{p\};\ell)\big|_{g^n\lambda^r} \\ &= N\Big\{\epsilon_p{\cdot}\ell\big[\mathrm{PT}(-,1,\ldots,r,+,p) - \mathrm{PT}(+,1,\ldots,r,-,p)\big] \\ &\qquad + \sum_{j=1}^{r-1}\epsilon_p{\cdot}k_{12\ldots j}\big[\mathrm{PT}(-,1,2,\ldots,j,p,j{+}1,\ldots,r,+) + \mathrm{PT}(+,1,2,\ldots,j,p,j{+}1,\ldots,r,-)\big]\Big\} \end{aligned} \tag{3.24}$$

and thereby reproduces the expression for $a_{\text{EYM}}(+,1,2,\ldots,r,-;p)$ in [36], averaged over $(+\leftrightarrow-)$, also see the appendix of the reference for the two-gluon case. The generalizations of (3.23) to higher numbers of gluons can be found in [24, 27, 28, 68] and give the $g^n\lambda^r$-order of $J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,\ldots,r;\{r{+}1,\ldots,n\};\ell)$ for arbitrary $n, r$ via (3.12).

[9] More specifically, the cancellation of the last line of (3.19) straightforwardly generalizes to insertions of $+$ and $-$ into any pair of $\mathrm{Tr}_i, \mathrm{Tr}_j$ with $1 \le i < j \le m$. Similarly, the conspiration of $c_1, c_2$ in the first four lines of (3.19) to $c_2 - c_1 = N$ occurs for the terms where $+,-$ are inserted into the same $\mathrm{Tr}_j$ with $j = 1, 2, \ldots, m$. All of these manipulations are again based on the Kleiss-Kuijf relations (2.25) that hold for any number of traces or gluon insertions.



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### 3.4.2 Examples at subleading order in $\lambda$ with external scalars

The one-loop EYM amplitude relations in [36] are limited to gauge multiplets in the loop and lowest orders in the gravitational coupling. Already the amplitude relations for the subleading orders (3.13) of single-trace $J^{\text{1-loop}}_{\text{YM}+\phi^3}$ in the double copy are a new result of this work. In absence of external gluons, the relevant Parke-Taylor decompositions include [24]

$$\begin{aligned}J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,j,+|-,j{+}1,\ldots,n;\emptyset) &= \sum_{i=1}^{j}\sum_{k=j+1}^{n}(-1)^{i+k}s_{ik}\\ &\quad\times\sum_{\substack{Q\in(1,2,\ldots,i-1)\\ \sqcup(j,j-1,\ldots,i+1)}}\ \sum_{\substack{R\in(k+1,k+2,\ldots n)\\ \sqcup(k-1,\ldots,j+1)}} J^{\text{tree}}_{\text{YM}+\phi^3}(Q,i,k,R,-,+;\emptyset)\\ J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,n|+,-;\emptyset) &= \sum_{j=1}^{n-1}(-1)^{j-1}s_{j,\ell}\sum_{\substack{Q\in(j-1,\ldots,2,1)\\ \sqcup(j+1,j+2,\ldots,n-1)}} J^{\text{tree}}_{\text{YM}+\phi^3}(j,Q,n,+,-;\emptyset)\,,\end{aligned} \tag{3.25}$$

where the double traces signal that the EYM amplitudes obtained from double copy via (2.39) feature gravity multiplets in the loop. Note that the right-hand sides of (3.25) admit a variety of alternative Parke-Taylor representations[10] related by scattering equations. As usual in worldsheet approaches to field-theory amplitudes, modifying the moduli-space integrand via scattering equations or total derivatives might reorganize the cubic-diagram expansion of the loop integrand and amount to *generalized gauge transformations* in the lingo of the literature on the color-kinematics duality.

The forward limit in a pair of gluons simplifies the Parke-Taylor decomposition of [24] to

$$\begin{aligned}&J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r;\{+,-\})\\ &= -\sum_{j=2}^{r-2}\left[s_{12\ldots j}\text{PT}(1,2,\ldots,j,+,j{+}1,\ldots,r,-)+\text{cyc}(1,2,\ldots,r)\right]\\ &\quad+\frac{4-D}{2}\sum_{j=1}^{r-1}(-1)^{j-1}s_{j,\ell}\sum_{\substack{Q\in(j-1,\ldots,2,1)\\ \sqcup(j+1,j+2,\ldots,r-1)}}\left[\text{PT}(j,Q,r,-,+)-\text{PT}(j,Q,r,+,-)\right].\end{aligned} \tag{3.26}$$

This expression is obtained after bringing the Mandelstam invariants $s_{ij}$ with $1\leq i<j\leq r$ into an $\frac{r}{2}(r{-}3)$-element basis and exposes that the $\ell$-dependent terms in the second line cancel in $D=4$ spacetime dimensions. We have verified (3.26) up to and including $r=6$ external scalars, and its validity for higher $r$ is conjectural.

In fact, the second line of (3.26) can be identified with a multiple of the scalar forward limit $J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,r|+,-;\emptyset)$ in (3.25): the latter is symmetric under $(+\leftrightarrow-)$ including $\ell\to-\ell$,[11] so the two Parke-Taylor factors in the square bracket of (3.26) yield the same

[10] More specifically, the right-hand sides of (3.25) are attained by rewriting the products $\text{PT}(i_1,i_2,\ldots,i_p)\text{PT}(j_1,j_2,\ldots,j_q)$ entering the expression (2.23) for $J^{\text{tree}}_{\text{YM}+\phi^3}(i_1,i_2,\ldots,i_p|j_1,j_2,\ldots,j_q;\emptyset)$ in terms of $(p{+}q)$-point Parke-Taylor factors. In doing so through the identities in section 7 of [24], the cyclic symmetry in $i_1,i_2,\ldots,i_p$ and $j_1,j_2,\ldots,j_q$ is no longer manifest. Our expression for $J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,\ldots,j,+|-,j{+}1,\ldots,n;\emptyset)$ in (3.25) is tailored to avoid $s_{j,\ell}$ involving loop momenta.

[11] Following our earlier comment, this symmetry is manifest from the left-hand side of (3.25), but it requires a substantial amount of scattering equations to verify the symmetry on the right-hand side of (3.25).

$J^{\rm tree}_{\rm YM+\phi^3}$ upon summation over $j$ and $Q$, i.e.

$$J^{\rm tree}_{\rm YM+\phi^3}(1,2,\ldots,r;\{+,-\}) = -\sum_{j=2}^{r-2}\big[s_{12\ldots j}{\rm PT}(1,2,\ldots,j,+,j{+}1,\ldots,r,-)+{\rm cyc}(1,2,\ldots,r)\big] \\ +(D-4)J^{\rm tree}_{\rm YM+\phi^3}(1,2,\ldots,r|+,-;\emptyset)\,. \tag{3.27}$$

On these grounds, the $\lambda^{r-2}$-order of one-loop half integrands (3.13) simplifies to

$$\begin{aligned} &J^{\rm 1\text{-}loop}_{\rm YM+\phi^3}(1,2,\ldots,n;\emptyset;\ell)\big|_{g^n\lambda^{n-2}} \\ &= (Nc_2+D-4)J^{\rm tree}_{\rm YM+\phi^3}(1,2,\ldots,n|+,-;\emptyset) \\ &\quad -\sum_{j=2}^{n-2}\big[s_{12\ldots j}{\rm PT}(1,2,\ldots,j,+,j{+}1,\ldots,n,-)+{\rm cyc}(1,2,\ldots,n)\big] \\ &\quad +\sum_{j=1}^{n-1}\big[J^{\rm tree}_{\rm YM+\phi^3}(1,2,\ldots,j,+|-,j{+}1,\ldots,n;\emptyset)+{\rm cyc}(1,2,\ldots,n)\big] \end{aligned} \tag{3.28}$$

with the $J^{\rm tree}_{\rm YM+\phi^3}$ on the right-hand side given by (3.25).

Based on (3.25) to (3.28), the $\lambda^{n-2}$-order of one-loop half integrands (3.13) without gluon insertions is available in Parke-Taylor form. Generalizations to additional gluons, traces or to different power counting in $g, \lambda$ can be obtained by inserting the tree-level results of [24, 27, 28, 68] into (3.16) to (3.18) and (3.20) to (3.22).

# 4 YM+$\phi^3$ half integrands at one loop: four-point examples

We shall now specialize the general approach of the previous section to four external legs and spell out all color-ordered one-loop half integrands in YM+$\phi^3$ theory, separately at each order in the couplings $g$ and $\lambda$. Each subsection is dedicated to a different combination of external scalars and gluons.

## 4.1 No external gluons

We begin with the one-loop four-scalar amplitude in YM+$\phi^3$ theory. In this case, we have a single- and a double trace sector.

### 4.1.1 Single-trace sector

There are two different orders in the couplings contributing to the single-trace sector by (3.8)

$$J^{\rm 1\text{-}loop}_{\rm YM+\phi^3}(1,2,3,4;\emptyset;\ell) = J^{\rm 1\text{-}loop}_{\rm YM+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^4} + J^{\rm 1\text{-}loop}_{\rm YM+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^2}\,. \tag{4.1}$$

At order $g^4\lambda^4$ the color-ordered half integrand comprises cyclic combinations of ${\rm PT}(+,\ldots,-)$ in (3.9) known as one-loop Parke-Taylor factors [37],

$$J^{\rm 1\text{-}loop}_{\rm YM+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^4} = N\big[{\rm PT}(+,1,2,3,4,-)+{\rm PT}(-,1,2,3,4,+)+{\rm cyc}(1,2,3,4)\big]\,. \tag{4.2}$$

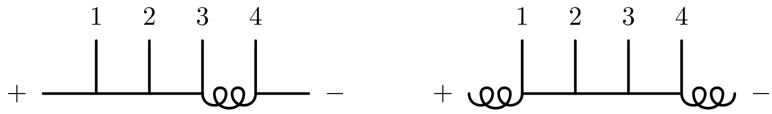


**Figure 4**. The half integrand in (4.3) receives contributions from forward limits in both scalars (drawn in the left panel) and gluons (drawn in the right panel).

According to (3.11), the half integrand at the subleading order $g^4\lambda^2$ receives contributions from forward limits in both scalars and gluons,

$$\begin{aligned} J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^2} = Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,4|+,-;\emptyset) + J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,4;\{+,-\}) \\ + \Big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,+|-,4;\emptyset) + J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,+|-,3,4;\emptyset) \\ + J^{\text{tree}}_{\text{YM}+\phi^3}(1,+|-,2,3,4;\emptyset) + \text{cyc}(1,2,3,4)\Big] \,. \end{aligned} \tag{4.3}$$

By the discussion below (3.10), the relative normalization between the forward limits in scalars and gluons is fixed by requiring a quadratic-propagator representation of $A^{\text{1-loop}}_{\text{EYM},\alpha}(1,2,3,4)$ resulting from (2.39). More specifically, changing the prefactor of $J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,4;\{+,-\})$ on the right-hand side of (4.3) would spoil the recombination of linearized propagators to quadratic ones in four-gluon one-loop EYM amplitudes (for any amount of supersymmetry $\alpha$). The two classes of tree-level diagrams associated with the forward limit in a gluon or scalar are drawn in figure 4 (cf. figure 3).

The tree-level building blocks on the right-hand side of (4.3) are given by the following specializations of (3.25)[12]

$$\begin{aligned} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,4|+,-;\emptyset) &= s_{1,\ell}\text{PT}(4,3,2,1,-,+) + s_{3,\ell}\text{PT}(4,1,2,3,-,+) \\ &\quad - s_{2,\ell}\big[\text{PT}(4,1,3,2,-,+) + \text{PT}(4,3,1,2,-,+)\big] \end{aligned} \tag{4.4}$$

$$\begin{aligned} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,+|-,4;\emptyset) &= -s_{14}\text{PT}(3,2,1,4,-,+) - s_{34}\text{PT}(1,2,3,4,-,+) \\ &\quad + s_{24}\big[\text{PT}(1,3,2,4,-,+) + \text{PT}(3,1,2,4,-,+)\big] \end{aligned} \tag{4.5}$$

$$\begin{aligned} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,+|-,3,4;\emptyset) &= s_{13}\text{PT}(2,1,3,4,-,+) - s_{14}\text{PT}(2,1,4,3,-,+) \\ &\quad - s_{23}\text{PT}(1,2,3,4,-,+) + s_{24}\text{PT}(1,2,4,3,-,+) \end{aligned} \tag{4.6}$$

[12]Note that (4.4) to (4.6) are forward limits of the six-point tree-level identities (74) and (75) of [24] whose right-hand sides obscure the cyclic symmetries of the double-trace structures. Cyclic permutations in the underlying tree-level expressions lead to alternative representations of (4.4) to (4.6) which may also alter the power-counting of loop momenta, see for instance the factor of $s_{4,\ell}$ in the following equivalent of (4.5):

$$\begin{aligned} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3,+|-,4;\emptyset) = -s_{24}\text{PT}(+,3,2,4,-,1) - s_{4,\ell}\text{PT}(2,3,+,4,-,1) \\ + s_{34}\big[\text{PT}(2,+,3,4,-,1) + \text{PT}(+,2,3,4,-,1)\big] \,. \end{aligned}$$



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and of (3.26),

$$\begin{aligned}J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,3,4;\{+,-\}) = &-s_{12}\big[{\rm PT}(1,2,+,3,4,-)+{\rm PT}(1,2,-,3,4,+)\big]\\ &-s_{23}\big[{\rm PT}(2,3,+,4,1,-)+{\rm PT}(2,3,-,4,1,+)\big]\\ &+\frac{4-D}{2}\Big\{s_{1,\ell}{\rm PT}(1,2,3,4,-,+)+s_{3,\ell}{\rm PT}(3,2,1,4,-,+)\\ &-s_{2,\ell}\big[{\rm PT}(2,1,3,4,-,+)+{\rm PT}(2,3,1,4,-,+)\big]-(+\leftrightarrow -)\Big\}\,.\end{aligned} \tag{4.7}$$

Following the general discussion around (3.27), the last two lines of (4.7) are proportional to (4.4) on the support of scattering equations. Hence, the half integrand (4.3) can be simplified to

$$\begin{aligned}J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^2} = &(Nc_2+D-4)J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,3,4|+,-;\emptyset)\\ &+\Big[J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,3,+|-,4;\emptyset)+J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,+|-,3,4;\emptyset)\\ &+J^{\rm tree}_{{\rm YM}+\phi^3}(1,+|-,2,3,4;\emptyset)-s_{12}{\rm PT}(1,2,+,3,4,-)+{\rm cyc}(1,2,3,4)\Big]\,,\end{aligned} \tag{4.8}$$

in lines with (3.28). Finally, with the expressions in (4.5) and (4.6), the last two lines of (4.8) conspire to a permutation sum of Parke-Taylor factors,

$$\begin{aligned}J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^2} = &(Nc_2+D-4)J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,3,4|+,-;\emptyset)\\ &+2s_{13}\sum_{\rho\in S_4}{\rm PT}(+,\rho(1,2,3,4),-)\,.\end{aligned} \tag{4.9}$$

#### 4.1.2 Double-trace sector

The double-trace sector of the one-loop half integrand with four external scalars is compatible with three different powers $\lambda^4, \lambda^2, \lambda^0$, see (3.15),

$$\begin{aligned}J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2|3,4;\emptyset;\ell) = &J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4\lambda^4}\\ &+J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4\lambda^2}+J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4}\,.\end{aligned} \tag{4.10}$$

At leading order $g^4\lambda^4$ in the couplings, (3.16) specializes to

$$\begin{aligned}J^{\text{1-loop}}_{\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4\lambda^4} = 2\big[&{\rm PT}(1,2,+,3,4,-)+{\rm PT}(2,1,+,3,4,-)\\ &+{\rm PT}(1,2,+,4,3,-)+{\rm PT}(2,1,+,4,3,-)\big]\,.\end{aligned} \tag{4.11}$$

Also the subleading order $g^4\lambda^2$ entirely stems from tree-level half integrands with six scalars (this time distributed over two traces). More specifically, (3.17) with $P=\emptyset$ as well as $s=2$ and $r=4$ yields

$$\begin{aligned}J^{\text{1-loop}}_{{\rm YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4\lambda^2} = 2N\big[&J^{\rm tree}_{{\rm YM}+\phi^3}(1,2,+,-|3,4;\emptyset)+J^{\rm tree}_{{\rm YM}+\phi^3}(2,1,+,-|3,4;\emptyset)\\ &+J^{\rm tree}_{{\rm YM}+\phi^3}(1,2|3,4,+,-;\emptyset)+J^{\rm tree}_{{\rm YM}+\phi^3}(1,2|4,3,+,-;\emptyset)\big]\,,\end{aligned} \tag{4.12}$$

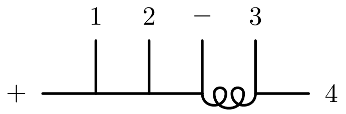


**Figure 5**. The one-loop half integrand in (4.12) has contributions from forward limits of tree-level half integrands associated to diagrams with six scalars and a double trace.

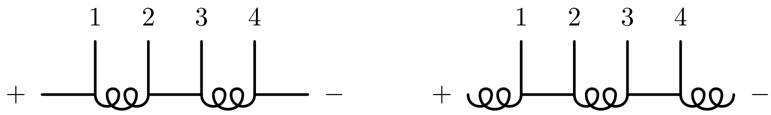


**Figure 6**. The half integrand in (4.14) has contributions from forward limits in both scalars and gluons. Typical diagrams from $J^{\text{tree}}_{\text{YM}+\phi^3}(+,1|3,4|2,-;\emptyset)$ and $J^{\text{tree}}_{\text{YM}+\phi^3}(1,2|3,4;\{+,-\})$ are depicted in the left and right panel, respectively.

where all terms on the right-hand side are permutations of

$$\begin{aligned} J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,+,-|3,4;\emptyset) = s_{24}\big[&\text{PT}(1,+,2,4,3,-)+\text{PT}(+,1,2,4,3,-)\big] \\ &- s_{14}\text{PT}(+,2,1,4,3,-) - s_{4,\ell}\text{PT}(1,2,+,4,3,-)\,. \end{aligned} \tag{4.13}$$

A typical diagram contributing to the forward limit in (4.12) is depicted in figure 5.

At the lowest order in $\lambda$, we also have contributions with gluons in the forward limit from the second line of (3.18),

$$\begin{aligned} J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4} = Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}&(1,2|3,4|+,-;\emptyset) + J^{\text{tree}}_{\text{YM}+\phi^3}(1,2|3,4;\{+,-\}) \\ &+ J^{\text{tree}}_{\text{YM}+\phi^3}(+,1|3,4|2,-;\emptyset) + J^{\text{tree}}_{\text{YM}+\phi^3}(+,2|3,4|1,-;\emptyset) \\ &+ J^{\text{tree}}_{\text{YM}+\phi^3}(+,3|1,2|4,-;\emptyset) + J^{\text{tree}}_{\text{YM}+\phi^3}(+,4|1,2|3,-;\emptyset)\,, \end{aligned} \tag{4.14}$$

see figure 6 for typical diagrams that contribute. It would be interesting to investigate multitrace generalizations of (4.8), e.g. whether the expression for $J^{\text{tree}}_{\text{YM}+\phi^3}(1,2|3,4;\{+,-\})$ simplifies after peeling off $(D-4)J^{\text{tree}}_{\text{YM}+\phi^3}(1,2|3,4|+,-;\emptyset)$.

### 4.2 One external gluon

For one-loop half integrands of YM + $\phi^3$ with three external scalars and one external gluon $p$, (3.8) admits two different powers of the coupling $\lambda$ from the forward limit of trees,

$$J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell) = J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda^3} + J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda}\,. \tag{4.15}$$

Following the lines of [36], at order $g^4\lambda^3$ we use (3.12) to obtain

$$\begin{aligned} J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda^3} = N\,\big[&J^{\text{tree}}_{\text{YM}+\phi^3}(+,1,2,3,-;\{p\}) + J^{\text{tree}}_{\text{YM}+\phi^3}(-,1,2,3,+;\{p\})\big] \\ &+ \text{cyc}(1,2,3)\,, \end{aligned} \tag{4.16}$$

where the Parke-Taylor decomposition (3.24) for the $J^{\text{tree}}_{\text{YM}+\phi^3}$ on the right-hand side yields

$$\begin{aligned} J^{\text{tree}}_{\text{YM}+\phi^3}(+,1,2,3,-;\{p\}) = -(\epsilon_p\cdot\ell)\text{PT}(+,1,2,3,-,p) &+ (\epsilon_p\cdot k_1)\text{PT}(+,1,p,2,3,-) \\ &+ (\epsilon_p\cdot k_{12})\text{PT}(+,1,2,p,3,-)\,. \end{aligned} \tag{4.17}$$

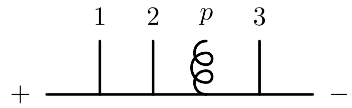


**Figure 7**. The forward limits contributing to the integrand in (4.16) are performed in scalars as illustrated in this figure.

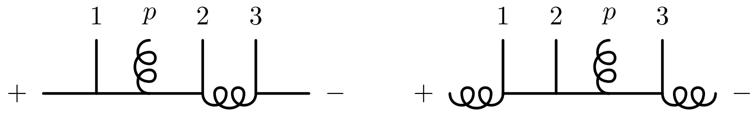


**Figure 8**. The half integrand in (4.18) receives contributions from forward limits in both scalars (left panel) and gluons (right panel).

A typical diagram at the order of $g^4\lambda^3$ is depicted in figure 7.

The half integrand at the order of $g^4\lambda$ receives contributions from forward limits in both scalars and gluons (see figure 8 for typical diagrams in both cases), and (3.13) specializes to

$$\begin{aligned}&J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda}\\&=J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3;\{p,+,-\})+Nc_2\,J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,3|+,-;\{p\})\\&\quad+\big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,+|-,3;\{p\})+J^{\text{tree}}_{\text{YM}+\phi^3}(1,+|-,2,3;\{p\})+\text{cyc}(1,2,3)\big]\,.\end{aligned}\tag{4.18}$$

The tree-level half integrands on the right-hand side have both been brought into Parke-Taylor form in sections 5 and 8 of [24].

### 4.3 Two external gluons

Also for two external scalars and two external gluons $p,q$, one-loop half integrands of $\text{YM}+\phi^3$ exhibit two different powers of the coupling $\lambda$ from the forward limit of trees,

$$J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)=J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4\lambda^2}+J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4}\,.\tag{4.19}$$

As in the previous case, at the leading order in $\lambda$ we use (3.12) to obtain

$$J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4\lambda^2}=2N\big[J^{\text{tree}}_{\text{YM}+\phi^3}(1,2,+,-;\{p,q\})+J^{\text{tree}}_{\text{YM}+\phi^3}(2,1,+,-;\{p,q\})\big]\,,\tag{4.20}$$

see figure 9 for typical diagrams that contribute. The $J^{\text{tree}}_{\text{YM}+\phi^3}$ on the right-hand side are of the following form,

$$\begin{aligned}J^{\text{tree}}_{\text{YM}+\phi^3}(2,1,+,-;\{p,q\})=\;&\text{PT}(1,2,p,q,-,+)\big((\epsilon_p\cdot\ell_{12})(\epsilon_q\cdot\ell)-\tfrac{1}{2}(\epsilon_p\cdot\epsilon_q)(s_{p,\ell}-s_{pq})\big)\\&+\text{PT}(1,p,2,q,-,+)\big((\epsilon_p\cdot\ell_1)(\epsilon_q\cdot\ell)-\tfrac{1}{2}(\epsilon_p\cdot\epsilon_q)(s_{1p}+s_{p,\ell})\big)\\&+\text{PT}(1,p,q,2,-,+)\big((\epsilon_p\cdot\ell_1)(\epsilon_q\cdot\ell_{1p})-\tfrac{1}{2}(\epsilon_p\cdot\epsilon_q)(s_{1p}+s_{p,\ell})\big)\\&+\text{PT}(p,1,2,q,-,+)\big((\epsilon_p\cdot\ell)(\epsilon_q\cdot\ell)-\tfrac{1}{2}(\epsilon_p\cdot\epsilon_q)s_{p,\ell}\big)\\&+\text{PT}(p,1,q,2,-,+)\big((\epsilon_p\cdot\ell)(\epsilon_q\cdot\ell_{1p})-\tfrac{1}{2}(\epsilon_p\cdot\epsilon_q)s_{p,\ell}\big)\\&+\text{PT}(p,q,1,2,-,+)\big((\epsilon_p\cdot\ell)(\epsilon_q\cdot\ell_p)-\tfrac{1}{2}(\epsilon_p\cdot\epsilon_q)s_{p,\ell}\big)\\&+(p\leftrightarrow q)\,,\end{aligned}\tag{4.21}$$

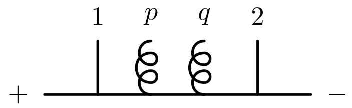


**Figure 9**. The forward limits contributing to the integrand in (4.20) are performed in scalars of tree-level half integrands with two gluons and four scalars.

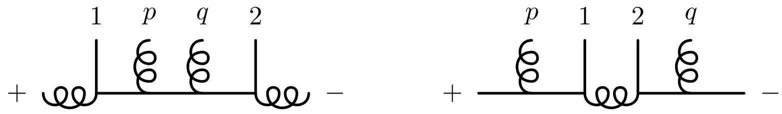


**Figure 10**. Forward limits in gluons and scalars that contribute to the half integrand (4.22) are illustrated in the left and right panel, respectively.

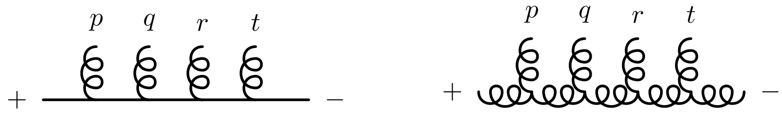


**Figure 11**. The four-gluon one-loop half integrand has a contribution with a scalar loop (left panel) and a gluon loop (right panel).

which is equivalent to the $n = 2$ instance of (28) in [36].

At the subleading order in $\lambda$, (3.13) specializes to

$$\begin{aligned} J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4} &= J^{\text{tree}}_{\text{YM}+\phi^3}(1,2;\{p,q,+,-\}) + Nc_2 J^{\text{tree}}_{\text{YM}+\phi^3}(1,2|+,-;\{p,q\}) \\ &\quad + J^{\text{tree}}_{\text{YM}+\phi^3}(1,+|2,-;\{p,q\}) + J^{\text{tree}}_{\text{YM}+\phi^3}(2,+|1,-;\{p,q\})\,, \end{aligned} \tag{4.22}$$

and typical diagrams associated with the gluonic and scalar forward limit are depicted in figure 10.

### 4.4 Four external gluons

In case of four external gluons $p, q, r, t$, half integrands of YM + $\phi^3$ only receive contributions at the order of $g^4$ according to (3.14). This time, scalars and gluons in the loop occur at the same powers in the couplings,

$$J^{\text{1-loop}}_{\text{YM}+\phi^3}(\{p,q,r,t\};\ell) = Nc_2\, J^{\text{tree}}_{\text{YM}+\phi^3}(+,-;\{p,q,r,t\}) + J^{\text{tree}}_{\text{YM}}(\emptyset;\{p,q,r,t,+,-\})\,, \tag{4.23}$$

but with an extra factor of $Nc_2$ in case of the scalars in the loop. The master numerators in the Parke-Taylor decomposition of $Nc_2\, J^{\text{tree}}_{\text{YM}+\phi^3}(+,-;\{p,q,r,t\})$ can be found in the supplementary material attached to this paper. The master numerators associated with $J^{\text{tree}}_{\text{YM}}(\emptyset;\{p,q,r,t,+,-\})$ in turn are described in section 3 of [44]. The associated master diagrams are depicted in figure 11.

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# 5 Four-point EYM amplitudes at one loop with maximal supersymmetry

In this and the following section, we apply the method of this work to obtain expressions for one-loop integrands of four-point amplitudes in EYM theories that expose the simplifications due to supersymmetry. The calculations are driven by the results of section 4 for the four-point one-loop half integrands of YM+$\phi^3$ which apply to EYM with any number of supercharges. We perform separate calculations for the color-ordered EYM amplitudes (2.39) at different orders in the couplings $g$ and $\kappa$ which are related to the orders of the couplings $g$ and $\lambda$ of YM+$\phi^3$ via (2.9), see table 1.

Following the last line of (2.39), we first obtain the loop integrands of EYM in terms of one-loop half integrands of YM+$\phi^3$ on the nodal Riemann sphere and partial integrands $a(\ldots)$ in YM theory. While the partial integrands are defined in terms of linearized propagators, we combine different terms in the permutation sum of (2.39) to attain the conventional form of Feynman integrals with quadratic propagators. The maximal supersymmetry of the partial integrands in this section leads to extra simplifications in comparison to the half-maximally supersymmetric case in section 6.

The recombination of the loop integrands in (2.39) to quadratic propagators is a first consistency check of our method and the expressions for the half integrands $J^{\text{1-loop}}_{\text{YM}+\phi^3}$ in the previous section. Moreover, we have verified all EYM amplitudes with external gravitons to respect linearized diffeomorphism invariance. This is non-trivial for the polarization vectors from $J^{\text{1-loop}}_{\text{YM}+\phi^3}$ that we denote by $\bar{\epsilon}_i$ (in contradistinction to the polarizations $\epsilon_i$ of the YM half integrands). All the explicit results for four-point EYM amplitudes in this section are checked to vanish under the linearized gauge transformation $\bar{\epsilon}_p \to p$ that double copies to linearized diffeomorphisms.

The results of this section exemplify that the no-triangle property [70] of maximally supersymmetric YM and supergravity does not apply to EYM with 16 supercharges. In spite of their maximal supersymmetry, the one-loop integrands of EYM amplitudes in (5.7), (5.9) and later equations feature triangle and bubble diagrams.[13] The loop integrands presented in this section can also be found in the supplementary material attached to this paper.

## 5.1 Partial integrands with maximal supersymmetry

Maximally supersymmetric EYM theory has 16 supercharges and can be defined in spacetime dimensions $D \leq 10$. Its four-dimensional incarnation is said to have $\mathcal{N} = 4$ supersymmetry. In the CHY construction of EYM one-loop amplitudes, all the supersymmetries arise from the YM half integrand which takes the simple form (2.29) at four points. The simplicity of the underlying correlation function on a torus was first revealed in [65], and the associated

[13] We refrain from re-interpreting diagrams by introducing spurious propagators $1 = \frac{(\ell+K)^2}{(\ell+K)^2}$ (as one may need to manifest the color-kinematics duality in the results of this section). For instance, the triangle with propagators $\frac{1}{\ell^2\ell_1^2\ell_{12}^2} = \frac{\ell_{123}^2}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2}$ is not counted as a box with the inverse propagator $\ell_{123}^2$ in the numerator.



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partial integrands in terms of linearized propagators are given by [36][14]

$$
\begin{aligned}
a^{\text{1-loop}}_{\text{YM,max}}(1,2,3,4,-,+) &= \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} \\
a^{\text{1-loop}}_{\text{YM,max}}(1,2,3,-,4,+) &= \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{12,\ell}s_{4,\ell}} + \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{12,\ell}s_{3,\ell}} + \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{14,\ell}s_{3,\ell}} + \frac{t_8(1,2,3,4)}{s_{4,\ell}s_{14,\ell}s_{3,\ell}} \\
a^{\text{1-loop}}_{\text{YM,max}}(1,2,-,3,4,+) &= \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{12,\ell}s_{124,\ell}} + \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{14,\ell}s_{124,\ell}} + \frac{t_8(1,2,3,4)}{s_{4,\ell}s_{14,\ell}s_{124,\ell}} \\
&\quad + \frac{t_8(1,2,3,4)}{s_{4,\ell}s_{34,\ell}s_{134,\ell}} + \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{14,\ell}s_{134,\ell}} + \frac{t_8(1,2,3,4)}{s_{4,\ell}s_{14,\ell}s_{134,\ell}} \,. 
\end{aligned} \tag{5.1}
$$

The $t_8$-tensor defined in (2.30) prescribes a dimension-agnostic contraction of $D$-dimensional polarization vectors and momenta. Given that also the YM+$\phi^3$ ingredients of the previous sections are dimension agnostic, the results of this section apply to any dimensional reduction of ten-dimensional EYM with maximal supersymmetry to $D \leq 10$.

### 5.2 No external gravitons

The amplitude with four external gluons has a single- and a double-trace sector.

#### 5.2.1 Single-trace sector

By the two contributions (4.1) to the YM+$\phi^3$ half integrand, there are two different combinations of couplings in the single-trace sector,

$$
A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3,4;\emptyset) = A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3,4;\emptyset)\big|_{g^4} + A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3,4;\emptyset)\big|_{\kappa^2 g^2} \,. \tag{5.2}
$$

The first term describing a four-gluon amplitude with the maximally supersymmetric gauge multiplet in the loop coincides with the one-loop amplitude in SYM [65] ($g^4$ does not leave any room for gravitational exchange). The YM+$\phi^3$ half integrand is the one-loop Parke-Taylor factor given in (4.2). In combination with the partial integrands (5.1), we find

$$
\begin{aligned}
A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3,4;\emptyset)\big|_{g^4} &= A^{\text{1-loop}}_{\text{YM,max}}(1,2,3,4) \\
&= N\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int \mathrm{d}\mu_6^{\text{tree}}\, I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,4\};\ell) \\
&\quad \times \big[\mathrm{PT}(+,1,2,3,4,-)+\mathrm{PT}(-,1,2,3,4,+)+\mathrm{cyc}(1,2,3,4)\big] \\
&= N\, t_8(1,2,3,4)\int \frac{\mathrm{d}^D\ell}{\ell^2}\left(\left(\frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}}+\frac{1}{s_{1,\ell}s_{14,\ell}s_{143,\ell}}\right)+\mathrm{cyc}(1,2,3,4)\right) \\
&= 8\,N\, t_8(1,2,3,4)\int \frac{\mathrm{d}^D\ell}{\ell^2}\left(\frac{1}{\ell_1^2\ell_{12}^2\ell_{123}^2}+\frac{1}{\ell_1^2\ell_{14}^2\ell_{143}^2}\right) \\
&= 16\,N\, t_8(1,2,3,4)\int \frac{\mathrm{d}^D\ell}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} \,,
\end{aligned} \tag{5.3}
$$

where we used (2.35) to obtain the quadratic propagators $\ell_{12\ldots p}^{-2} = (\ell + k_{12\ldots p})^{-2}$. The last step is based on the reflection property $k_2 \leftrightarrow k_4$ of the scalar box with propagators $\ell^{-2}\ell_1^{-2}\ell_{12}^{-2}\ell_{123}^{-2}$.

[14] Actually, only the first line of (5.1) is independent under the Kleiss-Kuijf relations while the second and the last two lines can be obtained from $-a^{\text{1-loop}}_{\text{YM,max}}((4⧢1,2,3),-,+)$ and $a^{\text{1-loop}}_{\text{YM,max}}((1,2⧢4,3),-,+)$, respectively.

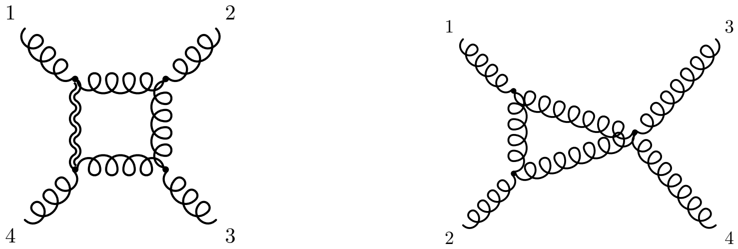


**Figure 12**. At order $\kappa^2 g^2$ the four-gluon EYM amplitude at one loop has a single- and a double-trace sector. The single-trace sector in (5.4) is exclusively furnished by the box diagrams in the left panel with one propagator of the gravity multiplet (double wavy line). The tree diagrams in its partial-fraction decomposition correspond to forward limits in different particles, see figure 4 for their YM+$\phi^3$ analogue. The double-trace sector in (5.7) of the maximally supersymmetric four-gluon amplitude is built from the triangle diagrams in the right panel. All graphs in this figure and later ones do not represent Feynman diagrams but instead illustrate the propagator structure of the maximally supersymmetric loop integrands.

At order $\kappa^2 g^2$, the relevant half integrand is given by (4.9) and leads to the following single-trace contribution to the four-gluon amplitude in EYM theory:

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3,4;\emptyset)\big|_{\kappa^2 g^2} \qquad (5.4)\\
&= \frac{1}{16}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^2}\\
&= s_{13}\, t_8(1,2,3,4) \int \frac{\mathrm{d}^D\ell}{\ell^2}\left[\frac{1}{\ell_1^2\ell_{12}^2\ell_{123}^2} + \text{perm}(2,3,4)\right]\\
&= 2s_{13}\, t_8(1,2,3,4) \int \frac{\mathrm{d}^D\ell}{\ell^2}\left[\frac{1}{\ell_1^2\ell_{12}^2\ell_{123}^2} + \text{cyc}(2,3,4)\right]
\end{aligned}
$$

As illustrated in figure 12, the result stems from box graphs with a graviton propagator, and the Mandelstam invariant $s_{13}$ can be attributed to its two gravitational vertices $\sim \kappa^2$. Note that the $\kappa^2 g^2$ order does not share the prefactor of $N$ in the $g^4$ order in (5.3), and the first line $\sim (Nc_2{+}D{-}4)$ of the half integrand (4.9) does not contribute in the maximally supersymmetric case.

By the permutation-symmetric combination of boxes in (5.4) and $s_{13} + \text{cyc}(1,2,3) = 0$, our result obeys Kleiss-Kuijf relations $A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3,4;\emptyset)\big|_{\kappa^2 g^2} + \text{cyc}(1,2,3) = 0$. This ensures that the accompanying color factors combine to permutations of contracted structure constants $\hat f^{A_1 A_2 B} \hat f^{B A_3 A_4}$ as expected from the left panel of figure 12.

### 5.2.2 Double-trace sector

Based on the half integrand (4.10) of YM+$\phi^3$, the double-trace sector of the four-gluon amplitude can come with three different powers of the couplings

$$
\begin{aligned}
A^{\text{1-loop}}_{\text{EYM,max}}(1,2|3,4;\emptyset) &= A^{\text{1-loop}}_{\text{YM,max}}(1,2|3,4;\emptyset)\big|_{g^4} \qquad (5.5)\\
&\quad + A^{\text{1-loop}}_{\text{EYM,max}}(1,2|3,4;\emptyset)\big|_{\kappa^2 g^2} + A^{\text{1-loop}}_{\text{EYM,max}}(1,2|3,4;\emptyset)\big|_{\kappa^4}\,.
\end{aligned}
$$



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At order $g^4$, we recover the double-trace amplitude of SYM which is determined by the half integrand in (4.11)

$$A^{\text{1-loop}}_{\text{YM,max}}(1,2|3,4;\emptyset)\big|_{g^4} = 32\, t_8(1,2,3,4) \int \frac{\mathrm{d}^D\ell}{\ell^2} \left( \frac{1}{\ell_1^2 \ell_{12}^2 \ell_{123}^2} + \mathrm{cyc}(2,3,4) \right) . \tag{5.6}$$

Since this is proportional to a permutation sum of the single-trace amplitude (5.3), we can verify the relation of [70] between planar and non-planar one-loop gauge-theory amplitudes which only holds for the $\kappa \to 0$ limit of EYM amplitudes.

At the order of $\kappa^2 g^2$ in the double-trace sector, the net effect of the gravitational vertices is to cancel one of the propagators of the box diagrams: the YM+$\phi^3$ half integrand constructed in (4.12) leads to triangle diagrams

$$\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(1,2|3,4;\emptyset)\big|_{\kappa^2 g^2} \\
&= \frac{1}{16} \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm \ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4\lambda^2} \\
&= -\frac{N}{2}\, t_8(1,2,3,4) \int \frac{\mathrm{d}^D\ell}{\ell^2} \left[ \left( \frac{1}{\ell_1^2 \ell_{12}^2} + (1 \leftrightarrow 2) \right) + (1,2 \leftrightarrow 3,4) \right] \\
&= -2\, N\, t_8(1,2,3,4) \int \frac{\mathrm{d}^D\ell}{\ell^2 \ell_1^2 \ell_{12}^2} .
\end{aligned} \tag{5.7}$$

These triangular contributions are illustrated diagrammatically in the right panel of figure 12. The recombination in terms of quadratic propagators is based on the identity

$$4 \int \frac{\mathrm{d}^D\ell}{\ell^2} \frac{f(\ell)}{\ell_P^2 \ell_{PQ}^2} = \int \frac{\mathrm{d}^D\ell}{\ell^2} \left[ \frac{f(\ell)}{s_{P,\ell} s_{PQ,\ell}} + \frac{f(\ell - k_P)}{s_{Q,\ell} s_{QR,\ell}} + \frac{f(\ell - k_P - k_Q)}{s_{R,\ell} s_{RP,\ell}} \right] \tag{5.8}$$

for multiparticle momenta subject to $k_P + k_Q + k_R = 0$ which can be derived from partial-fraction identities and shifts of the loop momenta similar to those employed to obtain (2.35). In passing to the last line of (5.7), we have used elementary properties of the scalar triangle integrals which allow to identify the four terms in the square bracket of the third line.

The triangles in (5.7) are the first example where one-loop EYM amplitudes with 16 supercharges violate the no-triangle property of maximally supersymmetric YM and supergravity [70]. As we will see in (5.9) and below, generic one-loop integrands of maximally supersymmetric EYM additionally involve bubble diagrams which go even further beyond the no-triangle property.

At order $\kappa^4$ we use the YM+$\phi^3$ half integrand constructed in (4.14) to obtain:

$$\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(1,2|3,4;\emptyset)\big|_{\kappa^4} \\
&= \frac{1}{256} \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm \ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4} \\
&= \frac{t_8(1,2,3,4)}{256} \int \frac{\mathrm{d}^D\ell}{\ell^2} \left\{ \left[ \frac{8 s_{14}^2}{\ell_1^2 \ell_{12}^2 \ell_{123}^2} + \frac{4 s_{12}}{\ell_1^2 \ell_{12}^2} + \frac{4 s_{12}}{\ell_3^2 \ell_{34}^2} + \frac{(D-3+N c_2)}{2 \ell_{12}^2} + (3 \leftrightarrow 4) \right] + (1 \leftrightarrow 2) \right\} \\
&= \frac{t_8(1,2,3,4)}{16} \int \frac{\mathrm{d}^D\ell}{\ell^2} \left\{ \frac{s_{14}^2}{\ell_1^2 \ell_{12}^2 \ell_{123}^2} + \frac{s_{13}^2}{\ell_1^2 \ell_{12}^2 \ell_{124}^2} + \frac{2 s_{12}}{\ell_1^2 \ell_{12}^2} + \frac{(D-3+N c_2)}{8 \ell_{12}^2} \right\} .
\end{aligned} \tag{5.9}$$

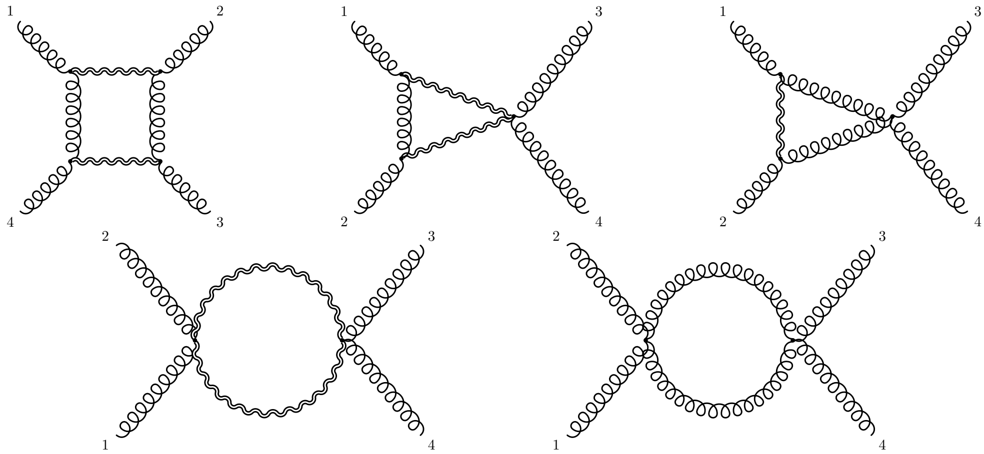


**Figure 13**. The expression (5.9) for the $\kappa^4$ order of the four-gluon EYM amplitude at one loop mixes box integrals with triangles and bubbles.

In addition to the partial-fraction manipulations (2.35) and (5.8) we use the identity

$$2\int \frac{\mathrm{d}^D\ell}{\ell^2}\frac{1}{\ell_P^2} = \int \frac{\mathrm{d}^D\ell}{\ell^2}\left[\frac{1}{s_{P,\ell}}+\frac{1}{s_{Q,\ell}}\right] \tag{5.10}$$

for multiparticle momenta $k_P + k_Q = 0$ to obtain the quadratic propagators of the bubble integral in (5.9). Again, we have used relabelling symmetries of the boxes, triangles and bubbles in passing to the last line of (5.9). The contributions to the one-loop amplitude with four external gluons at the $\kappa^4$ order are depicted diagrammatically in figure 13.

### 5.3 One external graviton

The structure of the YM+$\phi^3$ half integrand in (4.15) admits two contributions to the EYM four-point amplitude with one external graviton and three gluons,

$$A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3;\{p\}) = A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3;\{p\})\big|_{\kappa g^3} + A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3;\{p\})\big|_{\kappa^3 g}\,. \tag{5.11}$$

At order $\kappa g^3$, the YM+$\phi^3$ half integrand in (4.16) leads to

$$\begin{aligned} &A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3;\{p\})\big|_{\kappa g^3} \\ &= \frac{1}{4}\int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,p\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda^3} \\ &= 4\,N\,t_8(1,2,3,p)\int\frac{\mathrm{d}^D\ell}{\ell^2}\left[\frac{(\bar\epsilon_p\cdot\ell)}{\ell_1^2\ell_{12}^2\ell_{123}^2}+\text{cyc}(1,2,3)\right]. \end{aligned} \tag{5.12}$$

As already derived in [36], maximal supersymmetry leads to a cyclic orbit of box integrals as depicted in figure 14. The recombination to quadratic propagators is done by supplementing (2.35) with a shift $\ell\to\ell-k_{12\ldots i}$ of the loop momentum in the numerator in the $i^{\text{th}}$ term of the second line. Linearized gauge invariance of (5.12) can be shown via shifts of loop momenta in the cyclic orbit of

$$\delta_{\bar\epsilon_p\to p}\left(\frac{2(\bar\epsilon_p\cdot\ell)}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2}\right) = \frac{\ell^2-\ell_{123}^2}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} = \frac{1}{\ell_1^2\ell_{12}^2\ell_{123}^2}-\frac{1}{\ell^2\ell_1^2\ell_{12}^2}\,. \tag{5.13}$$

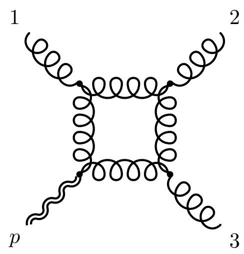


**Figure 14**. The box graphs in the three-gluon amplitude at order $\kappa g^3$ given by (5.12).

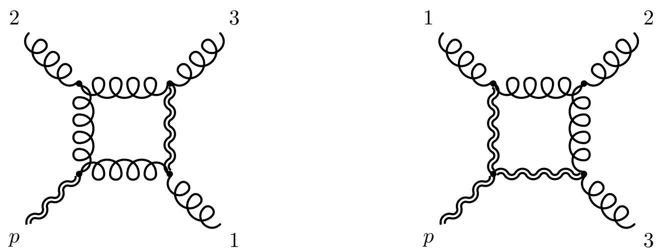


**Figure 15**. The box graphs in the three-gluon amplitude at order $\kappa^3 g$ given by (5.14).

For the order $\kappa^3 g$ we use the half integrand (4.18) to get,

$$\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3;\{p\})\big|_{\kappa^3 g} \qquad (5.14)\\
&= \frac{1}{64}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm \ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,3,p\};\ell) J^{\text{1-loop}}_{\text{YM+}\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda}\\
&= \frac{1}{4} t_8(1,2,3,p)\left(k_1\cdot \bar{f}_p \cdot k_2\right)\int \frac{\mathrm{d}^D\ell}{\ell^2}\left[\frac{1}{\ell_1^2 \ell_{12}^2 \ell_{123}^2} + \text{cyc}(1,2,3)\right],
\end{aligned}$$

where we introduce the linearized field strength in (2.30) to manifest gauge invariance of

$$k_1 \cdot \bar{f}_p \cdot k_2 = s_{1p}(\bar{\epsilon}_p \cdot k_2) - s_{2p}(\bar{\epsilon}_p \cdot k_1)\,. \qquad (5.15)$$

The configurations of vertices $\sim \kappa^3 g$ in the box diagrams of (5.14) are depicted in figure 15. Moreover, the kinematic factor (5.15) manifests the permutation antisymmetry of $A^{\text{1-loop}}_{\text{EYM,max}}(1,2,3;\{p\})\big|_{\kappa^3 g}$ in 1, 2, 3, consistent with the color structure $\hat{f}^{A_1 A_2 A_3}$ expected by the figure.

### 5.4 Two external gravitons

With two external gravitons, the YM+$\phi^3$ half integrand (4.19) introduces the following coupling dependence:

$$A^{\text{1-loop}}_{\text{EYM,max}}(1,2;\{p,q\}) = A^{\text{1-loop}}_{\text{EYM,max}}(1,2;\{p,q\})\big|_{\kappa^2 g^2} + A^{\text{1-loop}}_{\text{EYM,max}}(1,2;\{p,q\})\big|_{\kappa^4} \qquad (5.16)$$

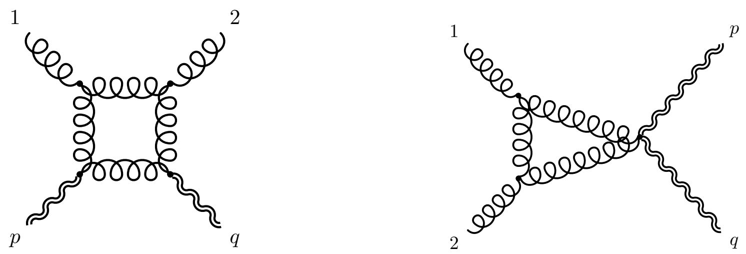


**Figure 16**. The $\kappa^2 g^2$ order of the two-gluon-two-graviton amplitude in (5.17) is given by the depicted boxes and triangles.

Using the YM+$\phi^3$ half integrand (4.20) we obtain the following combination of boxes and triangles at order $\kappa^2 g^2$, see figure 16 for an illustration of both contributions.

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(1,2;\{p,q\})\big|_{\kappa^2 g^2} \qquad (5.17)\\
&= \frac{1}{8}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,p,q\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4\lambda^2}\\
&= 2N\, t_8(1,2,p,q) \int \frac{\mathrm{d}^D\ell}{\ell^2} \Bigg\{\Bigg[\Bigg(\frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell)}{\ell_p^2\ell_{p1}^2\ell_{p12}^2} + \frac{(\bar\epsilon_p\cdot(\ell+k_1))(\bar\epsilon_q\cdot\ell)}{2\,\ell_1^2\ell_{1p}^2\ell_{1p2}^2} - \frac{(\bar\epsilon_p\cdot\bar\epsilon_q)}{4\,\ell_1^2\ell_{12}^2}\Bigg)\\
&\qquad\qquad + (p\leftrightarrow q)\Bigg] + (1\leftrightarrow 2)\Bigg\}.
\end{aligned}
$$

For the order $\kappa^4$ contribution we use (4.22) to get

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(1,2;\{p,q\})\big|_{\kappa^4} \qquad (5.18)\\
&= \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM,max}}(\{1,2,p,q\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4}\\
&= \frac{1}{256} t_8(1,2,p,q) \int \frac{\mathrm{d}^D\ell}{\ell^2} \Bigg\{\Bigg[\Bigg[(Nc_2{+}D{-}3)\left(\frac{1}{2}\frac{(\bar\epsilon_p\cdot\bar\epsilon_q)}{\ell_{pq}^2} - \frac{2(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell_p)}{\ell_p^2\ell_{pq}^2}\right)\\
&\quad + \frac{4s_{12}(\bar\epsilon_p\cdot\bar\epsilon_q)}{\ell_1^2\ell_{12}^2} + \frac{4s_{12}(\bar\epsilon_p\cdot\bar\epsilon_q) + 4(\bar\epsilon_p\cdot q)(\bar\epsilon_q\cdot p)}{\ell_p^2\ell_{pq}^2}\\
&\quad - 8s_{12}\frac{2(\bar\epsilon_p\cdot\ell_1)(\bar\epsilon_q\cdot\ell) + (\bar\epsilon_p\cdot\ell_1)(\bar\epsilon_q\cdot k_1) + (\bar\epsilon_p\cdot k_2)(\bar\epsilon_q\cdot\ell)}{\ell_1^2\ell_{12}^2\ell_{12p}^2}\\
&\quad + 8s_{1q}\frac{(\bar\epsilon_p\cdot\bar\epsilon_q)s_{1q} + (\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot p) + (\bar\epsilon_p\cdot k_2)(\bar\epsilon_q\cdot k_1) - (\bar\epsilon_p\cdot q)(\bar\epsilon_q\cdot\ell) - (\bar\epsilon_p\cdot k_1)(\bar\epsilon_q\cdot k_2)}{\ell_1^2\ell_{12}^2\ell_{12p}^2}\\
&\quad - 4s_{12}\frac{2(\bar\epsilon_p\cdot\ell_1)(\bar\epsilon_q\cdot\ell) + (\bar\epsilon_p\cdot\ell_1)(\bar\epsilon_q\cdot k_1) + (\bar\epsilon_p\cdot k_2)(\bar\epsilon_q\cdot\ell)}{\ell_1^2\ell_{1p}^2\ell_{1p2}^2}\\
&\quad + 4s_{1q}\frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot p) + (\bar\epsilon_p\cdot q)(\bar\epsilon_q\cdot\ell) + (\bar\epsilon_p\cdot k_1)(\bar\epsilon_q\cdot p)}{\ell_1^2\ell_{1p}^2\ell_{1p2}^2}\Bigg] + (1\leftrightarrow 2)\Bigg] + (p\leftrightarrow q)\Bigg\}.
\end{aligned}
$$

The contributing diagrams are illustrated in figure 17. For both of (5.17) and (5.18), we have verified gauge invariance in both $\bar\epsilon_p$ and $\bar\epsilon_q$.

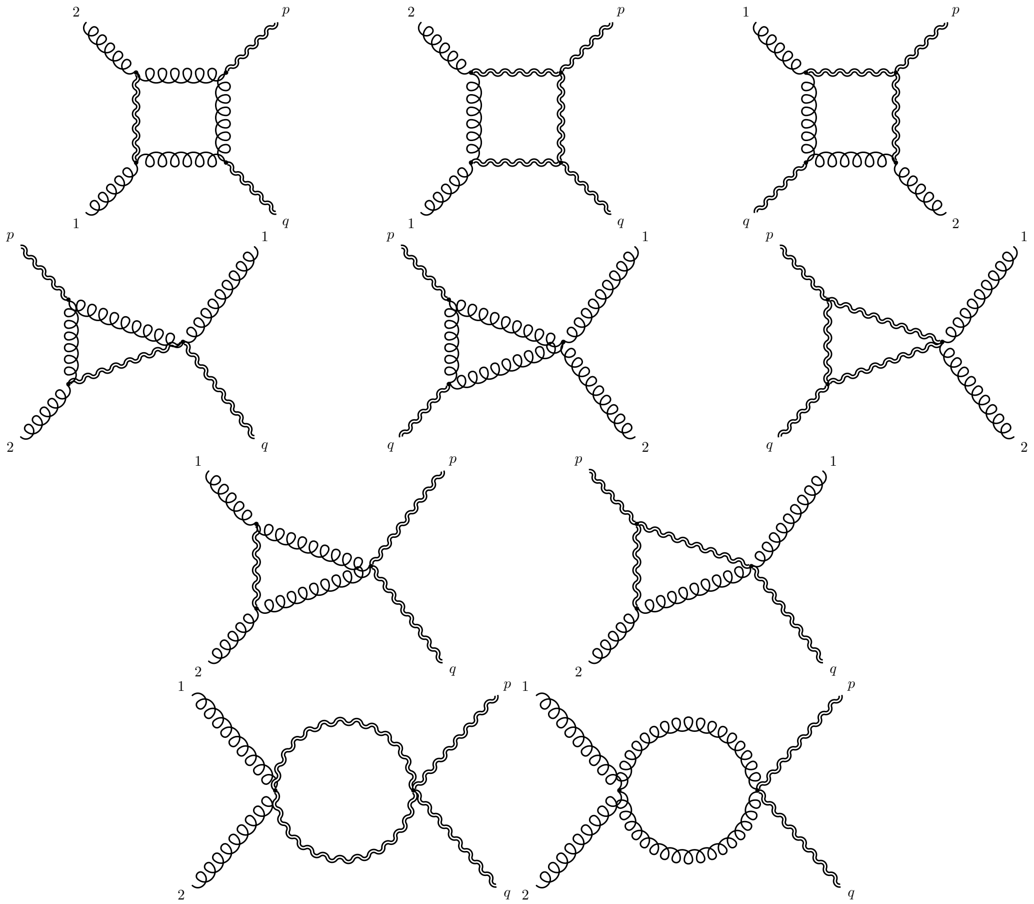

**Figure 17**. At the $\kappa^4$ order of the two-gluon and two-graviton amplitude of EYM, various boxes, triangles and bubbles contribute to the integrand (5.18).

### 5.5 Four external gravitons

Finally, the entire one-loop four-graviton amplitude is proportional to $\kappa^4$, but it can be organized into contributions from gauge and gravity multiplets in the loop,

$$\begin{aligned} A^{\text{1-loop}}_{\text{EYM,max}}(\{p,q,r,t\}) &= A^{\text{1-loop}}_{\text{EYM,max}}(\{p,q,r,t\})\big|_{\kappa^4} \\ &= A^{\text{1-loop}}_{\text{EYM,max}}(\{p,q,r,t\})\big|_{\text{graviton loop}} + A^{\text{1-loop}}_{\text{EYM,max}}(\{p,q,r,t\})\big|_{\text{gluon loop}}\,, \end{aligned} \tag{5.19}$$

see (4.23) for the analogous organization of the YM+$\phi^3$ half integrand. The contribution to (5.19) from a gluon loop follows from inserting the Parke-Taylor form of $J^{\text{tree}}_{\text{YM}+\phi^3}(+,-;\{p,q,r,t\})$ in the supplementary material attached to this paper into (2.39):

$$\begin{aligned} &A^{\text{1-loop}}_{\text{EYM,max}}(\{p,q,r,t\})\big|_{\text{gluon loop}} \\ &= \frac{1}{256}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}}\, I^{\text{1-loop}}_{\text{YM,max}}(\{p,q,r,t\};\ell) N c_2\, J^{\text{tree}}_{\text{YM}+\phi^3}(+,-;\{p,q,r,t\}) = \end{aligned} \tag{5.20}$$

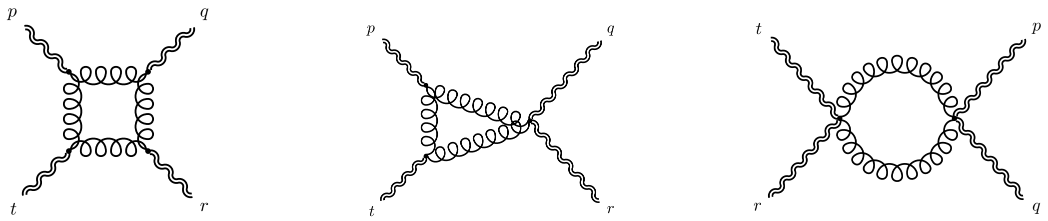


**Figure 18**. The contribution (5.19) of a gauge-multiplet loop to the four-graviton amplitude contains box, triangle and bubble graphs.

$$
\begin{aligned}
=\frac{Nc_2}{64}t_8(p,q,r,t)\int\frac{\mathrm{d}^D\ell}{\ell^2}\bigg\{\frac{1}{2}\bigg(&\frac{(\bar\epsilon_p\cdot\bar\epsilon_q)(\bar\epsilon_r\cdot\bar\epsilon_t)}{\ell_{pq}^2}+\frac{(\bar\epsilon_p\cdot\bar\epsilon_r)(\bar\epsilon_q\cdot\bar\epsilon_t)}{\ell_{pr}^2}+\frac{(\bar\epsilon_p\cdot\bar\epsilon_t)(\bar\epsilon_q\cdot\bar\epsilon_r)}{\ell_{pt}^2}\bigg)\\
&+\bigg(\bigg[2\frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell_p)(\bar\epsilon_r\cdot\ell_{pq})(\bar\epsilon_t\cdot\ell)}{\ell_p^2\ell_{pq}^2\ell_{pqr}^2}-(\bar\epsilon_p\cdot\bar\epsilon_q)\bigg(\frac{(\bar\epsilon_r\cdot\ell)(\bar\epsilon_t\cdot\ell_r)}{\ell_r^2\ell_{tr}^2}+\frac{(\bar\epsilon_r\cdot\ell_t)(\bar\epsilon_t\cdot\ell)}{\ell_t^2\ell_{tr}^2}\bigg)\bigg]\\
&\quad+\mathrm{perm}(p,q,r,t)\bigg)\bigg\}.
\end{aligned}
$$

The different diagrams with a gluon loop are depicted in figure 18, and their interplay gives rise to a gauge-invariant amplitude under $\bar\epsilon_p\to p$.

One can similarly obtain the contributions from a graviton in the loop via

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM,max}}(\{p,q,r,t\})\big|_{\text{graviton loop}} \qquad (5.21)\\
&=\frac{1}{256}\int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int\mathrm{d}\mu_6^{\text{tree}}\,I^{\text{1-loop}}_{\text{YM,max}}(\{p,q,r,t\};\ell)\,J^{\text{tree}}_{\text{YM}+\phi^3}(\emptyset;\{p,q,r,t,+,-\})\,,
\end{aligned}
$$

where the master-numerator decomposition of $J^{\text{tree}}_{\text{YM}+\phi^3}(\emptyset;\{p,q,r,t,+,-\})$ — the forward limit of a six-point Pfaffian (2.15) — can be found in section 3 of [44].

## 6 Four-point one-loop EYM amplitudes with half-maximal supersymmetry

In this section, we investigate four-point one-loop amplitudes in EYM theory with half-maximal supersymmetry, i.e. 8 supercharges instead of 16. This kind of half-maximal supersymmetry can be realized in spacetime dimensions $D\le 6$ and is referred to as $\mathcal{N}=2$ supersymmetry in four dimensions. Our procedure to construct the loop integrands of EYM is very similar to the one presented for the maximally supersymmetric case in section 5: we start by introducing half integrands $I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}$ tailored to a chiral hypermultiplet in the loop[15] and spell out the associated half-maximally supersymmetric partial integrands of YM. Based on the CHY formula (2.39), one-loop amplitudes of EYM are constructed from the double copy of these YM partial integrands with the YM+$\phi^3$ building blocks of

[15] Maximally supersymmetric gauge multiplets can be decomposed into a vector multiplet of half-maximal supersymmetry (one vector and two Weyl fermions in six dimensions) and two hypermultiplets (two scalars and a single Weyl fermion in six dimensions). Accordingly, one can extract the contribution of a vector multiplet in the loop from the linear combinations (6.4).

sections 3 and 4. In all cases, we convert the linearized propagators in the loop integrand of (2.39) to conventional quadratic ones.

All supercharges in our EYM results are from the half-maximally supersymmetric YM theory which is double-copied with the universal, non-supersymmetric YM+$\phi^3$ theory. By double copy of the hypermultiplets of YM with the scalars and gluons of YM+$\phi^3$, the internal states for the EYM results in this section are hypermultiplets in the adjoint representation of U($N$) and gravitino supermultiplets. In six dimensions, the gravitino multiplet has 8 bosonic and 8 fermionic on-shell degrees of freedom from the double copy of a hypermultiplet with the 4 physical polarizations of a $D = 6$ gluon. The contributions to the EYM loop integrand from adjoint vector multiplets and graviton multiplets with 8 supercharges each can be reconstructed from combining the results of this section with the maximally supersymmetric loop integrands of section 5.

Massless gravitino multiplets generically conflict with local supersymmetry unless they are embedded into a larger gravity multiplet or rendered massive via compactification, see for instance [71] or section 2.6 of [4]. Hence, one can view the expressions of this section for gravitino multiplets in the loop as formal building blocks that compactly encode the difference between supergravity multiplets with 16 or 8 supercharges in the loop.

The half-maximally supersymmetric EYM amplitudes in this section are once more expressed in terms of dimension-agnostic gluon polarization vectors. Hence, the loop integrands in this section apply to both the maximal dimension $D = 6$ for 8 supercharges and dimensional reductions thereof. We also track parity-odd contributions from the chiral fermions in the loop in terms of the six-dimensional Levi-Civita tensor. The running of chiral fermions in fact leads to gauge and diffeomorphism anomalies in some of the four-point amplitudes. We perform the loop integrals for these six-dimensional anomalies and obtain rational functions of the momenta as expected.

The reduction from 16 to 8 supercharges leads to longer expressions for the EYM loop integrands in this section as compared to those in section 5. Hence, we only present a subset of the possible four-point amplitudes with external gluons or gravitons in the main text and relegate some cases (or certain orders in $g, \kappa$) to the supplementary material attached to this paper.

### 6.1 Partial integrands with half-maximal supersymmetry

The four-point one-loop half integrand of YM with half-maximal supersymmetry will again be used in Parke-Taylor form [43]

$$
\begin{aligned}
&I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}(\{\hat{1},2,3,4\};\ell)\\
&=\frac{1}{2}\sum_{\rho\in S_4}\text{PT}(+,\rho(1,2,3,4),-) \qquad (6.1)\\
&\quad\times\Big\{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4}+\ell_\mu\Big[\text{sgn}^\rho_{23}s_{23}C^\mu_{1|23,4}+\text{sgn}^\rho_{24}s_{24}C^\mu_{1|24,3}+\text{sgn}^\rho_{34}s_{34}C^\mu_{1|34,2}\Big]\Big\},
\end{aligned}
$$

where the coefficients depend on the permutation $\rho$ in the Parke-Taylor ordering via

$$\mathrm{sgn}^{\rho}_{ij} = \begin{cases} +1: & i \text{ is on the right of } j \text{ in } \rho(1,2,3,4) \\ -1: & i \text{ is on the left of } j \text{ in } \rho(1,2,3,4)\,. \end{cases} \tag{6.2}$$

The vectorial and tensorial kinematic factors $C^{\mu}_{1|23,4}$ and $C^{\mu\nu}_{1|2,3,4}$ introduced in [72, 73] are multilinear in $D$-dimensional gluon polarization vectors, see appendix B for a brief review. In contrast to the scalar $t_8(1,2,3,4)$ that we encountered in the maximally supersymmetric case, they are not permutation invariant and obey

$$C^{\mu\nu}_{1|2,3,4} = C^{\mu\nu}_{1|3,2,4} = C^{\mu\nu}_{1|2,4,3}\,, \qquad C^{\mu}_{1|23,4} = -C^{\mu}_{1|32,4} \tag{6.3}$$

as well as more complicated identities under exchange of leg 1 in front of the vertical bar [73] and contractions with external momenta, see for instance (6.13). Moreover, they exhibit simple poles in $s_{ij}$ from their expansion in terms of polarization vectors and momenta in the supplementary material attached to this paper.

The analogous half integrand associated with a half-maximally supersymmetric vector multiplet in the loop (instead of a hypermultiplet) is a linear combination of (6.1) with the maximally supersymmetric one (2.29) [39, 43]

$$I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{1},2,3,4\};\ell)\big|_{\text{vector}} = I^{\text{1-loop}}_{\text{YM},\max}(\{1,2,3,4\};\ell) - 2I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{1},2,3,4\};\ell)\,. \tag{6.4}$$

Their relative coefficients follow from the fact that the maximally supersymmetric gauge multiplet with $8+8$ on-shell degrees of freedom decomposes into one vector multiplet and two hypermultiplets associated with 8 supercharges.

### 6.1.1 Singling out an anomaly leg

While the vector $C^{\mu}_{1|23,4}$ is gauge invariant with respect to $\epsilon^{\mu}_j \to k^{\mu}_j \ \forall\ j = 1,2,3,4$, the parity-odd part of the tensor $C^{\mu\nu}_{1|2,3,4}$ has an anomalous gauge variation in the first leg,

$$\delta_{\epsilon_1\to k_1} C^{\mu\nu}_{1|2,3,4} = 2i\eta^{\mu\nu}\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\,, \qquad \delta_{\epsilon_j\to k_j} C^{\mu\nu}_{1|2,3,4} = 0 \text{ for } j=2,3,4\,, \tag{6.5}$$

where $\varepsilon_6$ is the six-dimensional Levi-Civita tensor contracting the six vectors in the brackets.

By the asymmetric gauge variation (6.5) of the tensor, the half integrand (6.1) is gauge invariant in external legs $2,3,4$ but anomalous in the first leg [43, 45]

$$\delta_{\epsilon_1\to k_1} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{1},2,3,4\};\ell) = i\ell^2\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\sum_{\rho\in S_4}\mathrm{PT}(+,\rho(1,2,3,4),-)\,. \tag{6.6}$$

The hat above leg 1 in the notation on the left-hand side of (6.1) keeps track of the external leg that carries the anomaly. While generic kinematic half integrand $I^{\text{1-loop}}_{\text{YM},\alpha}$ in (2.39) are supposed to be permutation symmetric, the anomaly introduces a mild asymmetry [43, 45][16]

$$
\begin{aligned}
&I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{1},2,3,4\};\ell) - I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{2},1,3,4\};\ell) \\
&= -i\ell^2 \varepsilon_6(\epsilon_1,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4) \sum_{\rho\in S_4} \text{PT}(+,\rho(1,2,3,4),-)
\end{aligned} \tag{6.7}
$$

while maintaining permutation invariance in the unhatted legs $2,3,4$ in (6.1). As we will see in section 6.5, the factor of $\ell^2$ in the asymmetry (6.7) and the anomalous gauge variation $\delta_{\epsilon_1\to k_1}$ of (6.1) implies that all the Feynman integrals obtained from integration over the $\sigma_j$ evaluate to rational functions of the momenta, i.e. no logarithms in the Mandelstam invariants.

### 6.1.2 Partial integrands

A Kleiss-Kuijf basis of partial integrands (2.38) for an internal hypermultiplet resulting from (6.1) can be assembled from permutations in $\{2,3,4\}$ of

$$
\begin{aligned}
a^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\hat{1},2,3,4,-,+) &= \frac{\ell_\mu C^\mu_{1|23,4}}{s_{1,\ell}s_{4,\ell}} - \frac{\ell_\mu C^\mu_{1|34,2}}{s_{1,\ell}s_{12,\ell}} \\
&\quad - \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - (\ell_\mu C^\mu_{1|24,3}s_{24} + \ell_\mu C^\mu_{1|23,4}s_{23} + \ell_\mu C^\mu_{1|34,2}s_{34})}{2s_{1,\ell}s_{4,\ell}s_{12,\ell}}
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
a^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(4,\hat{1},2,3,-,+) &= -\frac{\ell_\mu C^\mu_{1|23,4}}{s_{4,\ell}s_{14,\ell}} \\
&\quad - \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu C^\mu_{1|24,3}s_{24} - \ell_\mu C^\mu_{1|23,4}s_{23} + \ell_\mu C^\mu_{1|34,2}s_{34}}{2s_{4,\ell}s_{3,\ell}s_{14,\ell}}
\end{aligned} \tag{6.9}
$$

$$
\begin{aligned}
a^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(3,4,\hat{1},2,-,+) &= -\frac{\ell_\mu C^\mu_{1|34,2}}{s_{2,\ell}s_{34,\ell}} \\
&\quad - \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu C^\mu_{1|24,3}s_{24} + \ell_\mu C^\mu_{1|23,4}s_{23} - \ell_\mu C^\mu_{1|34,2}s_{34}}{2s_{2,\ell}s_{3,\ell}s_{34,\ell}}
\end{aligned} \tag{6.10}
$$

$$
\begin{aligned}
a^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(2,3,4,\hat{1},-,+) &= \frac{\ell_\mu C^\mu_{1|34,2}}{s_{1,\ell}s_{2,\ell}} - \frac{\ell_\mu C^\mu_{1|23,4}}{s_{1,\ell}s_{23,\ell}} \\
&\quad - \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - (\ell_\mu C^\mu_{1|24,3}s_{24} + \ell_\mu C^\mu_{1|23,4}s_{23} + \ell_\mu C^\mu_{1|34,2}s_{34})}{2s_{1,\ell}s_{2,\ell}s_{23,\ell}}\,.
\end{aligned} \tag{6.11}
$$

Vector multiplets in the loop can be accommodated by taking linear combinations with the maximally supersymmetric partial integrands in (5.1) according to (6.4). In (6.8) to (6.11) and similar equations below, the hat indicates the leg that carries the anomaly i.e. singles out the variant of the underlying half integrand (6.1).

[16]The asymmetry of the half integrand can be traced back to the assignment of superghost pictures to the vertex operators of the RNS ambitwistor string [62] that determine $I^{\text{1-loop}}_{\text{YM},\alpha}$ through their genus-one correlators. More precisely, the prescription for the parity-odd part of genus-one amplitudes in the reference requires one of the vertex operators to appear in the superghost picture $-1$. In contrast to the remaining vertex operators in the zero picture, the gauge variation in the $-1$ picture yields a derivative in moduli space and ultimately leads to the factor of $\ell^2$ in (6.7).

### 6.1.3 The anomalous kinematic factor $P_{1|2|3,4}$

As we will see below in (6.16) and (6.17), we will also encounter scalar kinematic factors $P_{1|a|b,c} = P_{1|a|c,b}$ introduced in [72, 73],

$$s_{12}P_{1|2|3,4} = \frac{1}{4}\big[\mathrm{tr}(f_1f_3)\mathrm{tr}(f_2f_4) + \mathrm{tr}(f_1f_4)\mathrm{tr}(f_2f_3) - \mathrm{tr}(f_1f_2)\mathrm{tr}(f_3f_4)\big] - \mathrm{tr}(f_1f_3f_2f_4) \\ + i\big[(\epsilon_1\cdot k_2)\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4) + (1\leftrightarrow 2)\big] - is_{12}\varepsilon_6(\epsilon_1,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\,, \tag{6.12}$$

where the traces refer to the Lorentz indices of the linearized field strengths, e.g. $\mathrm{tr}(f_1f_2) = (f_1)_\mu{}^\nu(f_2)_\nu{}^\mu$. They are symmetric in the last two legs, $P_{1|2|3,4} = P_{1|2|4,3}$, and related to the vectors and tensors in (6.1) via identities like

$$\begin{aligned}(k_1)_\mu C^\mu_{1|23,4} &= P_{1|3|2,4} - P_{1|2|3,4}\\ (k_1)_\mu C^{\mu\nu}_{1|2,3,4} &= -k_2^\nu P_{1|2|3,4} - k_3^\nu P_{1|3|2,4} - k_4^\nu P_{1|4|2,3}\\ \eta_{\mu\nu}C^{\mu\nu}_{1|2,3,4} &= 2(P_{1|2|3,4} + P_{1|3|2,4} + P_{1|4|2,3})\,.\end{aligned} \tag{6.13}$$

Similar to (6.5), the parity-odd terms in (6.12) exhibit an anomalous gauge variation in the first leg,

$$\delta_{\epsilon_1\to k_1}P_{1|2|3,4} = 2i\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4), \quad \delta_{\epsilon_j\to k_j}P_{1|2|3,4} = 0 \text{ for } j=2,3,4\,. \tag{6.14}$$

The anomalies of the six-dimensional loop integrands due to (6.5) and (6.14) are discussed in more detail in section 6.5.1.

Note that spinor-helicity components of $C^\mu_{1|23,4}, C^{\mu\nu}_{1|2,3,4}$ and $P_{1|2|3,4}$ upon dimensional reduction to $D=4$ vanish outside the MHV sector as required by supersymmetry, see section 5.1 of [73] for the non-zero MHV expressions.

## 6.2 No external gravitons

We shall now determine the loop integrand of the four-gluon amplitude from the half-maximally supersymmetric ingredients (6.1) and (6.8) to (6.11). As before, separate calculations are performed for the single- and double-trace sector.

### 6.2.1 Single-trace sector

As in the maximally supersymmetric case, the two contributions (4.1) to the YM+$\phi^3$ half integrand give two different dependences on the couplings in the single-trace sector,

$$A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset) = A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset)\big|_{g^4} + A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset)\big|_{\kappa^2g^2}\,, \tag{6.15}$$

where the hat above the first leg tracks the anomalous leg in the underlying half integrand (6.1). Using the half integrand in (4.2) and the half-maximally supersymmetric partial integrands in (6.8) to (6.11), we obtain the single-trace sector of a half-maximally supersymmetric four-gluon amplitude in EYM theory at order $g^4$:

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(\hat{1},2,3,4;\emptyset)\big|_{g^4} \qquad (6.16)\\
&= \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{1},2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^4}\\
&= 4N \int \frac{\mathrm{d}^D\ell}{\ell^2}\Bigg\{ \frac{\ell_\mu C^\mu_{1|23,4}}{\ell_1^2\ell_{14}^2} - \frac{\ell_\mu C^\mu_{1|23,4}}{\ell_4^2\ell_{41}^2} + \frac{\ell_\mu C^\mu_{1|34,2}}{\ell_2^2\ell_{21}^2} - \frac{\ell_\mu C^\mu_{1|34,2}}{\ell_1^2\ell_{12}^2} - \frac{P_{1|2|3,4}}{\ell_4^2\ell_{43}^2} - \frac{P_{1|4|2,3}}{\ell_2^2\ell_{23}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(C^\mu_{1|34,2}s_{34} + C^\mu_{1|24,3}s_{24} + C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{12}^2\ell_{123}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu(C^\mu_{1|34,2}s_{34} + C^\mu_{1|24,3}s_{24} + C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{14}^2\ell_{143}^2}\Bigg\}.
\end{aligned}
$$

This expression matches the representation of the four-gluon amplitude in half-maximally supersymmetric YM obtained in [73] after symmetrizing the loop integrand in the reference w.r.t. $2 \leftrightarrow 4$. The scalar kinematic factors $P_{1|a|b,c}$ have been introduced in section 6.1.3.

Similarly, we use the YM+$\phi^3$ half integrand (4.9) at subleading order in $\lambda$ to get:[17]

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(\hat{1},2,3,4;\emptyset)\big|_{\kappa^2 g^2} \qquad (6.17)\\
&= \frac{1}{16}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\max}(\{\hat{1},2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3,4;\emptyset;\ell)\big|_{g^4\lambda^2}\\
&= \frac{1}{2}s_{13}\int \frac{\mathrm{d}^D\ell}{\ell^2}\Bigg\{ -t_8(1,2,3,4)\left(\frac{1}{\ell_2^2\ell_{23}^2} + \frac{1}{\ell_3^2\ell_{32}^2}\right)\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(s_{23}C^\mu_{1|23,4} + s_{24}C^\mu_{1|24,3} + s_{34}C^\mu_{1|34,2})}{\ell_1^2\ell_{12}^2\ell_{123}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(s_{23}C^\mu_{1|23,4} + s_{24}C^\mu_{1|24,3} - s_{34}C^\mu_{1|34,2})}{\ell_1^2\ell_{12}^2\ell_{124}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(-s_{23}C^\mu_{1|23,4} + s_{24}C^\mu_{1|24,3} + s_{34}C^\mu_{1|34,2})}{\ell_1^2\ell_{13}^2\ell_{132}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(-s_{23}C^\mu_{1|23,4} - s_{24}C^\mu_{1|24,3} + s_{34}C^\mu_{1|34,2})}{\ell_1^2\ell_{13}^2\ell_{134}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(s_{23}C^\mu_{1|23,4} - s_{24}C^\mu_{1|24,3} - s_{34}C^\mu_{1|34,2})}{\ell_1^2\ell_{14}^2\ell_{142}^2}\\
&\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu(s_{23}C^\mu_{1|23,4} + s_{24}C^\mu_{1|24,3} + s_{34}C^\mu_{1|34,2})}{\ell_1^2\ell_{14}^2\ell_{143}^2}\\
&\quad - P_{1|4|2,3}\left(\frac{1}{\ell_3^2\ell_{32}^2} + \frac{1}{\ell_2^2\ell_{23}^2}\right) - P_{1|2|3,4}\left(\frac{1}{\ell_3^2\ell_{34}^2} + \frac{1}{\ell_4^2\ell_{43}^2}\right) - P_{1|3|2,4}\left(\frac{1}{\ell_4^2\ell_{42}^2} + \frac{1}{\ell_2^2\ell_{24}^2}\right)\Bigg\}.
\end{aligned}
$$

[17] We have discarded a tadpole diagram proportional to $A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int\frac{\mathrm{d}^D\ell}{\ell^2}$ in deriving the result (6.17) which integrates to zero in dimensional regularization.

Since the loop integral over the expression in the curly brackets is permutation invariant w.r.t. $2,3,4$, the color-ordered amplitude (6.17) obeys Kleiss-Kuijf relations just like its maximally supersymmetric counterpart in (5.4).

### 6.2.2 Double-trace sector

The double-trace sector of the half-maximally supersymmetric four-gluon amplitude, based on the half integrand (4.10) of YM+$\phi^3$, introduces three different powers of the couplings:

$$\begin{aligned}A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset) &= A^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{g^4} \\ &\quad + A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{\kappa^2 g^2} + A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{\kappa^4}\,.\end{aligned} \tag{6.18}$$

For the order $g^4$ of the double-trace sector we get

$$\begin{aligned}&A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{g^4} \\ &= \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}(\{\hat{1},2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2|3,4;\{\emptyset\};\ell)\big|_{g^4\lambda^4} \\ &= 8\int \frac{\mathrm{d}^D\ell}{\ell^2}\bigg\{ \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(C^\mu_{1|24,3}s_{24} + C^\mu_{1|34,2}s_{34} + C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{12}^2\ell_{123}^2} \\ &\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(C^\mu_{1|24,3}s_{24} - C^\mu_{1|34,2}s_{34} + C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{12}^2\ell_{124}^2} \\ &\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} - \ell_\mu(C^\mu_{1|24,3}s_{24} + C^\mu_{1|34,2}s_{34} - C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{13}^2\ell_{132}^2} \\ &\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu(C^\mu_{1|24,3}s_{24} - C^\mu_{1|34,2}s_{34} + C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{13}^2\ell_{134}^2} \\ &\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu(C^\mu_{1|24,3}s_{24} + C^\mu_{1|34,2}s_{34} - C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{14}^2\ell_{142}^2} \\ &\quad + \frac{\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4} + \ell_\mu(C^\mu_{1|24,3}s_{24} + C^\mu_{1|34,2}s_{34} + C^\mu_{1|23,4}s_{23})}{\ell_1^2\ell_{14}^2\ell_{143}^2} \\ &\quad - P_{1|4|2,3}\Big(\frac{1}{\ell_2^2\ell_{23}^2} + \frac{1}{\ell_3^2\ell_{32}^2}\Big) - P_{1|2|3,4}\Big(\frac{1}{\ell_3^2\ell_{34}^2} + \frac{1}{\ell_4^2\ell_{43}^2}\Big) - P_{1|3|2,4}\Big(\frac{1}{\ell_2^2\ell_{24}^2} + \frac{1}{\ell_4^2\ell_{42}^2}\Big)\bigg\}\,.\end{aligned} \tag{6.19}$$

This is again proportional to a permutation sum of the single-trace amplitude (6.16), so the results of this section respect the supersymmetry-agnostic relations of [70] between planar and non-planar one-loop amplitudes at zeroth order in $\kappa$.

Similarly, for the order of $\kappa^2 g^2$ in the double-trace sector, we obtain

$$\begin{aligned}&A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{\kappa^2 g^2} \\ &= \frac{1}{16}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}(\{\hat{1},2,3,4\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2|3,4;\emptyset;\ell)\big|_{g^4\lambda^2} =\end{aligned} \tag{6.20}$$

$$
\begin{aligned}
= \frac{N}{4}\int \frac{\mathrm{d}^D\ell}{\ell^2}\Big\{ & 2s_{3,\ell}\ell_\mu C^\mu_{1|34,2}\left[\frac{1}{\ell_1^2\ell_{12}^2}+\frac{1}{\ell_2^2\ell_{21}^2}\right] \\
&+\frac{\ell_\mu(s_{34}C^\mu_{1|34,2}-s_{23}C^\mu_{1|23,4}-s_{24}C^\mu_{1|24,3})-\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4}}{\ell_3^2\ell_{34}^2} \\
&+\frac{\ell_\mu(-s_{34}C^\mu_{1|34,2}-s_{23}C^\mu_{1|23,4}-s_{24}C^\mu_{1|24,3})-\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4}}{\ell_4^2\ell_{43}^2} \\
&+\frac{\ell_\mu(s_{23}C^\mu_{1|23,4}+s_{24}C^\mu_{1|24,3}-s_{34}C^\mu_{1|34,2})-\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4}}{\ell_1^2\ell_{12}^2} \\
&+\frac{\ell_\mu(s_{23}C^\mu_{1|23,4}+s_{24}C^\mu_{1|24,3}-s_{34}C^\mu_{1|34,2})-\ell_\mu\ell_\nu C^{\mu\nu}_{1|2,3,4}}{\ell_2^2\ell_{21}^2}\Big\}.
\end{aligned}
$$

The $\kappa^4$ order of the double-trace sector (6.18) is more lengthy and therefore relegated to the supplementary material attached to this paper.

### 6.3 One external graviton

For all results with external gravitons and half-maximal supersymmetry, we chose to have a graviton carry the anomaly of $I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}$ and therefore write $\hat{p}$ in the place of $p$. The analogous expressions with a gluon in the anomaly leg can be found in the supplementary material attached to this paper.

The amplitude with one external graviton is based on the half integrand in (4.15) and therefore contains two different combinations of couplings,

$$
A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2,3;\{\hat{p}\}) = A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2,3;\{\hat{p}\})\big|_{\kappa g^3} + A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2,3;\{\hat{p}\})\big|_{\kappa^3 g}\,. \tag{6.21}
$$

At order $\kappa g^3$ we find

$$
\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2,3;\{\hat{p}\})\big|_{\kappa g^3} \qquad (6.22)\\
&= \frac{1}{4}\int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}(\{1,2,3,\hat{p}\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2,3;\{p\};\ell)\big|_{g^4\lambda^3} \\
&= 2N\int\frac{\mathrm{d}^D\ell}{\ell^2}\Big\{\Big[-\frac{(\bar{\epsilon}_p\cdot\ell)\ell_\mu C^\mu_{p|12,3}}{\ell_p^2\ell_{p3}^2}-\frac{(\bar{\epsilon}_p\cdot\ell_3)\ell_\mu C^\mu_{p|12,3}}{\ell_3^2\ell_{3p}^2}-\frac{(\bar{\epsilon}_p\cdot\ell)P_{p|3|1,2}}{\ell_1^2\ell_{12}^2} \\
&\quad+\frac{(\bar{\epsilon}_p\cdot\ell)\ell_\mu\ell_\nu C^{\mu\nu}_{p|1,2,3}-(\bar{\epsilon}_p\cdot\ell)(s_{23}\ell_\mu C^\mu_{p|23,1}+s_{13}\ell_\mu C^\mu_{p|13,2}+s_{12}\ell_\mu C^\mu_{p|12,3})}{\ell_p^2\ell_{p1}^2\ell_{p12}^2}\Big]+\text{cyc}(1,2,3)\Big\},
\end{aligned}
$$

which we confirmed to be invariant under linearized gauge transformations $\bar{\epsilon}_p\to p$ in the YM+$\phi^3$ half integrand. The analogous integrand at order $\kappa^3 g$ can be found in the supplementary material attached to this paper. At both orders $\kappa g^3$ and $\kappa^3 g$, the anomalous variations $\epsilon_j\to k_j$ on the supersymmetric side can be found in section 6.5.2.

### 6.4 Two external gravitons

The amplitude with two external gravitons, based on the half integrand (4.19) of YM+$\phi^3$, can come with two different powers of the couplings:

$$
A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2;\{\hat{p},q\}) = A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2;\{\hat{p},q\})\big|_{\kappa^2 g^2} + A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2;\{\hat{p},q\})\big|_{\kappa^4} \tag{6.23}
$$

The half-maximally supersymmetric amplitude at order $\kappa^2 g^2$ is found to be

$$\begin{aligned}
&A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(1,2;\{\hat{p},q\})\big|_{\kappa^2 g^2} \qquad (6.24)\\
&= \frac{1}{8}\int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu_6^{\text{tree}} I^{\text{1-loop}}_{\text{YM},\frac{1}{2}-\text{max}}(\{1,2,\hat{p},q\};\ell) J^{\text{1-loop}}_{\text{YM}+\phi^3}(1,2;\{p,q\};\ell)\big|_{g^4\lambda^2}\\
&= \frac{N}{2}\int \frac{\mathrm{d}^D\ell}{\ell^2}\bigg\{ \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot k_2)\ell_\mu C^\mu_{p|q2,1}}{\ell_p^2\ell_{p1}^2} + \frac{(\bar\epsilon_p\cdot\ell_1)(\bar\epsilon_q\cdot k_2)\ell_\mu C^\mu_{p|q2,1}}{\ell_1^2\ell_{p1}^2}\\
&\quad + \frac{(\bar\epsilon_p\cdot\bar\epsilon_q)\Big[s_{pq}\ell_\mu(C^\mu_{p|q2,1}+C^\mu_{p|12,q}) + s_{p2}\ell_\mu(C^\mu_{p|q2,1}-C^\mu_{p|q1,2}) - \ell_\mu\ell_\nu C^{\mu\nu}_{p|q,1,2}\Big]}{2\ell_1^2\ell_{12}^2}\\
&\quad + \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell_p)\Big[\ell_\mu\ell_\nu C^{\mu\nu}_{p|q,1,2} - (s_{p2}\ell_\mu C^\mu_{p|q1,2} + s_{pq}\ell_\mu C^\mu_{p|12,q} + s_{p1}\ell_\mu C^\mu_{p|q2,1})\Big]}{\ell_p^2\ell_{pq}^2\ell_{pq1}^2}\\
&\quad + \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot(\ell-k_2))\Big[\ell_\mu\ell_\nu C^{\mu\nu}_{p|q,1,2} - (s_{p1}\ell_\mu C^\mu_{p|q2,1} - s_{p2}\ell_\mu C^\mu_{p|q1,2} + s_{pq}\ell_\mu C^\mu_{p|12,q})\Big]}{\ell_p^2\ell_{p1}^2\ell_{p1q}^2}\\
&\quad + \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell)\Big[\ell_\mu\ell_\nu C^{\mu\nu}_{p|q,1,2} + (s_{p1}\ell_\mu C^\mu_{p|q2,1} + s_{p2}\ell_\mu C^\mu_{p|q1,2} - s_{pq}\ell_\mu C^\mu_{p|12,q})\Big]}{\ell_p^2\ell_{p1}^2\ell_{p12}^2}\\
&\quad - \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot(\ell-p))P_{p|q|1,2}}{\ell_1^2\ell_{12}^2} - \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell)P_{p|2|q,1}}{\ell_q^2\ell_{q1}^2} - \frac{(\bar\epsilon_p\cdot\ell)(\bar\epsilon_q\cdot\ell_1)P_{p|1|q,2}}{\ell_1^2\ell_{1q}^2} + (1\leftrightarrow 2)\bigg\}.
\end{aligned}$$

Our result for the integrand at order $\kappa^4$ as well as the one for the four-graviton amplitude can be found in the supplementary material attached to this paper.

### 6.5 Gauge anomalies in six dimensions

In this section, we integrate the anomalous gauge variations of the six-dimensional versions of the amplitudes in this section. These anomalies result from the variations (6.5) and (6.14) of the tensor $C^{\mu\nu}_{1|2,3,4}$ and scalar $P_{1|2|3,4}$ which yield rational functions of the momenta upon loop integration.

#### 6.5.1 Amplitudes with no external gravitons

First, we consider the variation $\epsilon_1\to k_1$ of the four-gluon amplitude at order $g^4$. In the single-trace sector, the anomaly due to the expression (6.16) is given by

$$\begin{aligned}
&\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset)\big|_{g^4} \qquad (6.25)\\
&= 8iN\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\int \mathrm{d}^D\ell\left\{\frac{\eta_{\mu\nu}\ell^\mu\ell^\nu}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} + \frac{\eta_{\mu\nu}\ell^\mu\ell^\nu}{\ell^2\ell_1^2\ell_{14}^2\ell_{143}^2} - \frac{1}{\ell^2\ell_4^2\ell_{43}^2} - \frac{1}{\ell^2\ell_2^2\ell_{23}^2}\right\}\\
&= 8iN\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\int \mathrm{d}^D\ell\left\{\frac{\eta_{\mu\nu}\ell^\mu\ell^\nu}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} - \frac{1}{\ell_1^2\ell_{12}^2\ell_{123}^2} + (2\leftrightarrow 4)\right\},
\end{aligned}$$

where a naive contraction $\eta_{\mu\nu}\ell^\mu\ell^\nu\to\ell^2$ would give rise to a vanishing loop integrand. However, all the integrals with measure $\mathrm{d}^D\ell$ in this work are understood in dimensional regularization where $D=2m-2\varepsilon$ is displaced from integer values $2m\in\mathbb{N}$ by some infinitesimal parameter $\varepsilon$. In an expansion around six dimensions (with $m=3$), the inverse propagators $\ell^2_{12\ldots}$ are $(6-2\varepsilon)$-dimensional contractions while the anomalous contributions in the numerator are lacking the formal $(-2\varepsilon)$-dimensional components of $\ell$,

$$\eta_{\mu\nu}\ell^\mu\ell^\nu = \ell^2 - \ell^2_{(-2\varepsilon)}\,. \tag{6.26}$$

In this way, the integrand of (6.25) is found to be non-vanishing and proportional to $\ell^2_{(-2\varepsilon)}$,

$$\begin{aligned}&\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset)\big|_{g^4} \\ &= -8iN\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\int \mathrm{d}^{6-2\varepsilon}\ell\left\{\frac{\ell^2_{(-2\varepsilon)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} + (2\leftrightarrow 4)\right\},\end{aligned} \tag{6.27}$$

The key formulae for the evaluation of $\ell^2_{(-2\varepsilon)}$ integrals in $D=6-2\varepsilon$ dimensions are reviewed in appendix C. Specifically, the identity (C.5) for the integral in (6.27) yields

$$\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset)\big|_{g^4} = \frac{8}{3}\pi^3\,N\,\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\,, \tag{6.28}$$

consistent with the anomaly of the four-gluon amplitude in [73]. Here and below, we discard the $\mathcal{O}(\varepsilon)$ contributions that vanish in $D=6$.

Based on the permutation sum of the computations in (6.25) to (6.28), the double-trace sector (6.19) of the one-loop four-gluon amplitude is given by

$$\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{g^4} = 16\,\pi^3\,\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\,. \tag{6.29}$$

At the order $\kappa^2 g^2$ of the single-trace amplitude in (6.17), the same mechanism leads to

$$\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2,3,4;\emptyset)\big|_{\kappa^2 g^2} = \pi^3\,s_{13}\,\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\,. \tag{6.30}$$

For the anomaly at the $\kappa^2 g^2$ order of the double-trace sector in turn, the variation of (6.20) introduces

$$\begin{aligned}&\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{\kappa^2 g^2} \\ &= -\frac{iN}{2}\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\int\frac{\mathrm{d}^{6-2\varepsilon}\ell}{\ell^2}\,\eta_{\mu\nu}\ell^\mu\ell^\nu\left\{\frac{1}{\ell_3^2\ell_{34}^2}+\frac{1}{\ell_4^2\ell_{43}^2}+\frac{1}{\ell_1^2\ell_{12}^2}+\frac{1}{\ell_2^2\ell_{21}^2}\right\}.\end{aligned} \tag{6.31}$$

With the rewriting (6.26) of the six-dimensional $\eta_{\mu\nu}\ell^\mu\ell^\nu$, we obtain one-mass bubble integrals that vanish in dimensional regularization such as

$$\int\frac{\mathrm{d}^{6-2\varepsilon}\ell}{\ell_3^2\ell_{34}^2} = 0\,. \tag{6.32}$$

The $(-2\varepsilon)$-dimensional parts of $\eta_{\mu\nu}\ell^\mu\ell^\nu$ in (6.31) introduce rational versions (C.6) of scalar triangles which yield the anomaly

$$\begin{aligned}&\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\text{max}}(\hat{1},2|3,4;\emptyset)\big|_{\kappa^2 g^2} \\ &= \frac{iN}{2}\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\int\mathrm{d}^{6-2\varepsilon}\left\{\left(\frac{\ell^2_{(-2\varepsilon)}}{\ell^2\ell_3^2\ell_{34}^2}+(3\leftrightarrow 4)\right)+(1,2\leftrightarrow 3,4)\right\} \\ &= \frac{\pi^3}{6}\,N\,s_{12}\,\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4)\,.\end{aligned} \tag{6.33}$$

Finally, the expression for the order $\kappa^4$ of the double-trace sector in the supplementary material attached to this paper and the integrals in appendix C give rise to the following anomaly

$$\begin{aligned}\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(\hat{1},2|3,4;\emptyset)\big|_{\kappa^4} &= \frac{\pi^3}{256}\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,k_4,\epsilon_4) \\ &\quad\times\left(\frac{2(Nc_2{+}D{-}3)}{15}s_{12}^2-\frac{16}{3}s_{14}s_{13}\right).\end{aligned} \tag{6.34}$$

### 6.5.2 Amplitudes with one external graviton

For the amplitude with one external graviton at order $\kappa g^3$ that was given in (6.22), the sum of the anomalous variations from the tensor and scalar building blocks conspire to

$$\begin{aligned}&\delta_{\epsilon_p\to p} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(1,2,3;\{\hat{p}\})\big|_{\kappa g^3} \\ &= -2iN\,\varepsilon_6(k_1,\epsilon_1,k_2,\epsilon_2,k_3,\epsilon_3)\int \mathrm{d}^{6-2\varepsilon}\ell\,(\bar{\epsilon}_p\cdot\ell)\left\{\frac{\ell^2_{(-2\varepsilon)}}{\ell^2\ell_p^2\ell_{p1}^2\ell_{p12}^2}+\text{cyc}(1,2,3)\right\}.\end{aligned} \tag{6.35}$$

By the expression (C.8) for the vector integral, the anomaly is proportional to the cyclic sum over $\bar{\epsilon}_p\cdot(k_1{+}2k_2{+}3k_3)$ which vanishes by $\bar{\epsilon}_p\cdot p=0$,

$$\delta_{\epsilon_p\to p} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(1,2,3;\{\hat{p}\})\big|_{\kappa g^3}=0\,. \tag{6.36}$$

The same conclusion can be reached for a gluon in the anomalous leg,

$$\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(\hat{1},2,3;\{p\})\big|_{\kappa g^3}=0\,. \tag{6.37}$$

The integrand for the amplitude with one external graviton at order $\kappa^3 g$ can be found in the supplementary material attached to this paper. In six dimensions a gauge transformation in the leg $p$ in the half-maximally supersymmetric half integrand together with (C.5) results in the anomaly

$$\begin{aligned}&\delta_{\epsilon_p\to p} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(1,2,3;\{\hat{p}\})\big|_{\kappa^3 g} \\ &=\frac{i}{8}\,\varepsilon_6(k_1,\epsilon_1,k_2,\epsilon_2,k_3,\epsilon_3)\int\frac{\mathrm{d}^{6-2\varepsilon}\ell}{\ell^2} \\ &\quad\times\left[(k_1\cdot\bar{f}_p\cdot k_3)\frac{\ell^2_{(-2\varepsilon)}}{\ell_p^2\ell_{p1}^2\ell_{p13}^2}-(k_1\cdot\bar{f}_p\cdot k_2)\frac{\ell^2_{(-2\varepsilon)}}{\ell_p^2\ell_{p1}^2\ell_{p12}^2}+\text{cyc}(1,2,3)\right] \\ &=\frac{\pi^3}{8}\,(k_1\cdot\bar{f}_p\cdot k_2)\varepsilon_6(k_1,\epsilon_1,k_2,\epsilon_2,k_3,\epsilon_3)\,,\end{aligned} \tag{6.38}$$

(see (2.30) for the linearized field strength $\bar{f}_p$) and similarly

$$\delta_{\epsilon_1\to k_1} A^{\text{1-loop}}_{\text{EYM},\frac{1}{2}-\max}(\hat{1},2,3;\{p\})\big|_{\kappa^3 g}=\frac{\pi^3}{8}\,(k_1\cdot\bar{f}_p\cdot k_2)\varepsilon_6(k_2,\epsilon_2,k_3,\epsilon_3,p,\epsilon_p)\,. \tag{6.39}$$

### 6.5.3 Amplitudes with two external gravitons

For the amplitude with two external gravitons at order $\kappa^2 g^2$ in half-maximal supersymmetry the anomaly in six dimensions is:

$$\begin{aligned}
&\delta_{\epsilon_p\to p} A^{\text{1-loop}}_{4,\text{EYM},\frac{1}{2}-\text{max}}(1,2;\{\hat{p},q\})\big|_{\kappa^2 g^2} \qquad (6.40)\\
&= iN\varepsilon_6(k_q,\epsilon_q,k_1,\epsilon_1,k_2,\epsilon_2)\int \frac{\mathrm{d}^{6-2\varepsilon}\ell}{\ell^2}\,\ell^2_{(-2\varepsilon)}\left\{\frac{(\bar{\epsilon}_p\cdot\bar{\epsilon}_q)}{2\ell_1^2\ell_{12}^2}+\frac{(\bar{\epsilon}_p\cdot\bar{\epsilon}_q)}{2\ell_2^2\ell_{21}^2}\right.\\
&\qquad -\frac{(\bar{\epsilon}_p\cdot\ell)(\bar{\epsilon}_q\cdot(\ell+p+k_1))}{\ell_p^2\ell_{p1}^2\ell_{p1q}^2}-\frac{(\bar{\epsilon}_p\cdot\ell)(\bar{\epsilon}_q\cdot(\ell+p))}{\ell_p^2\ell_{pq}^2\ell_{pq1}^2}-\frac{(\bar{\epsilon}_p\cdot\ell)(\bar{\epsilon}_q\cdot(\ell+p))}{\ell_p^2\ell_{pq}^2\ell_{pq2}^2}\\
&\qquad \left.-\frac{(\bar{\epsilon}_p\cdot\ell)(\bar{\epsilon}_q\cdot\ell)}{\ell_p^2\ell_{p1}^2\ell_{p12}^2}-\frac{(\bar{\epsilon}_p\cdot\ell)(\bar{\epsilon}_q\cdot(\ell-k_1))}{\ell_p^2\ell_{p2}^2\ell_{p2q}^2}-\frac{(\bar{\epsilon}_p\cdot\ell)(\bar{\epsilon}_q\cdot\ell)}{\ell_p^2\ell_{p2}^2\ell_{p21}^2}\right\}\\
&= -\frac{N}{24}\pi^3\varepsilon_6(k_q,\epsilon_q,k_1,\epsilon_1,k_2,\epsilon_2)(\bar{f}_p)_{\mu\nu}(\bar{f}_q)^{\mu\nu}\,.
\end{aligned}$$

The integrals were calculated as indicated in appendix C, and we have rewritten the kinematic factor in terms of linearized field strengths via $(\bar{f}_p)_{\mu\nu}(\bar{f}_q)^{\mu\nu} = 2(\bar{\epsilon}_p\cdot\bar{\epsilon}_q)s_{pq} - 2(\bar{\epsilon}_p\cdot q)(\bar{\epsilon}_q\cdot p)$.

## 7 Conclusion

We have introduced a method to determine one-loop integrands of EYM theories with any number of external gauge and gravity multiplets to all orders in the couplings $\kappa$ and $g$. Our construction is based on the double copy of (possibly supersymmetric) gauge theories with YM+$\phi^3$ theory, implemented via one-loop CHY formulae involving forward limits of tree-level integrands on the Riemann sphere. More specifically, the forward limits of YM+$\phi^3$ building blocks yield new relations between loop integrands of EYM and those of YM theories, see (2.32) and (2.39) for the main formulae. These relations take a universal form for any number of supersymmetries and spacetime dimensions.

We have worked out the composition rules for tree-level building blocks in color-ordered EYM loop integrands with any number of traces. At four points, we have applied our method to determine one-loop EYM amplitudes with 8 and 16 supercharges and exposed their supersymmetry cancellations. In particular, the linearized Feynman propagators resulting from the CHY integrals are recombined to conventional quadratic ones. Moreover, we have evaluated the rational expressions for six-dimensional gauge and diffeomorphism anomalies in the half-maximally supersymmetric case due to chiral hypermultiplets in the loop.

This methods and results of this work suggest a variety of follow-up research directions:

- *higher multiplicity*: with the availability of supersymmetric YM loop integrands at $n \geq 5$ points [36, 43, 45, 74, 75], there is no obstruction to constructing higher-point one-loop EYM amplitudes from our method. It is conceivable that the half integrands of YM+$\phi^3$ in section 3 admit further all-multiplicity simplifications as exemplified in (3.28) for $n$ external scalars at subleading order in the coupling $\lambda$.
- *higher loops*: based on ambitwistor-string methods, the integrands of two-loop amplitudes in gauge theories and (super-)gravity can be derived from double-forward

limits of tree-level building blocks [76–78]. It would be interesting to perform the same double-forward limits in the half-integrands of YM+$\phi^3$ and deduce two-loop EYM amplitudes from the setup in the references.

- *more general supergravities:* the double-copy structure of EYM theories generalizes to magical, general homogeneous or gauged $\mathcal{N}=2$ supergravities [79, 80]. The non-supersymmetric double-copy constituents in the references augment YM+$\phi^3$ by fundamental fermions and mass terms that preserve the color-kinematics duality. It would be a rewarding line of follow-up research to investigate worldsheet descriptions of the double copies and one-loop integrands of these $\mathcal{N}=2$ supergravities.

- *comparison with conventional string theories*: it would give a valuable crosscheck of our results to match the one-loop EYM amplitudes in this work with the point-particle limit $\alpha' \to 0$ of genus-one amplitudes of heterotic and type-I superstrings. In particular, the EYM amplitude relations in this work call for comparison with the $\alpha' \to 0$ limit of the relations for mixed open- and closed-string type-I amplitudes at genus one in [81].

- *uplift to higher-mass-dimension operators*: it would be rewarding to incorporate $\alpha'$-corrections into our forward-limit approach to EYM one-loop amplitudes as done for loop integrands of pure SYM and supergravity in [46]. Adapting the methods of the reference to genus-one correlators mixing gauge and gravity multiplets should yield one-loop matrix elements of higher-mass-dimension operators[18] $D^{2k}F^mR^n$ and the non-analytic contributions to the $\alpha'$-expansion of the respective genus-one string amplitudes.

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## A YM+$\phi^3$ half integrands at tree-level

In (2.18), (2.20), (2.23) and (2.24) we gave some specific examples for the half integrands of YM+$\phi^3$ in the CHY representation (2.17) of EYM tree-level amplitudes. These can be deduced from a general expression for multiple traces given in [8] that we review here. We denote the cyclically ordered set of scalars in the $i^{\text{th}}$ trace as $\text{Tr}_i$. The color-decomposed

[18] The shorthand $D^{2k}F^mR^n$ refers to effective operators involving $m$ powers of the non-abelian gluon field strength $F$, $n$ powers of the Riemann tensor $R$ and $2k$ gauge- and diffeomorphism-covariant derivatives.

YM+$\phi^3$ half integrand with $r$ gluons $\{p_1, p_2, \ldots, p_r\}$ and scalars in $m$ traces $\mathrm{Tr}_1, \mathrm{Tr}_2, \cdots \mathrm{Tr}_m$ is given by

$$J^{\text{tree}}_{\text{YM}+\phi^3}(\mathrm{Tr}_1|\ldots|\mathrm{Tr}_m;\{p_1,\ldots,p_r\}) = \mathrm{PT}(\mathrm{Tr}_1)\ldots\mathrm{PT}(\mathrm{Tr}_m)\mathrm{Pf}'\Pi(\mathrm{Tr}_1,\ldots,\mathrm{Tr}_m,\{p_1,\ldots,p_r\})\,. \tag{A.1}$$

The antisymmetric $2(r+m) \times 2(r+m)$-matrix $\Pi(\mathrm{Tr}_1, \ldots, \mathrm{Tr}_m, \{p_1, \ldots, p_r\})$ is constructed from the $(2r \times 2r)$-matrix $\Psi(\{p_1, \ldots, p_r\})$ in the YM half integrand (2.16) by adding rows and columns for each of the traces:

$$\Pi := \begin{array}{cccc|l} b\in\{p_1,\ldots,p_r\} & j\in\{1,\ldots,m\} & b\in\{p_1,\ldots,p_r\} & j'\in\{1,\ldots,m\} & \\ \hline A_{ab} & \Pi_{a,j} & (-C)^T_{ab} & \Pi_{a,j'} & a \in \{p_1,\ldots,p_r\} \\ \Pi_{i,b} & \Pi_{i,j} & \tilde{\Pi}_{i,b} & \Pi_{i,j'} & i \in \{1,\ldots,m\} \\ C_{ab} & \tilde{\Pi}_{a,j} & B_{ab} & \tilde{\Pi}_{a,j'} & a \in \{p_1,\ldots,p_r\} \\ \Pi_{i',b} & \Pi_{i',j} & \tilde{\Pi}_{i',b} & \Pi_{i',j'} & i' \in \{1,\ldots,m\} \end{array} \tag{A.2}$$

The submatrices are defined as

$$\Pi_{i,b} = \sum_{c\in\mathrm{Tr}_i} \frac{k_c\cdot k_b}{\sigma_{c,b}}, \qquad \tilde{\Pi}_{i,b} = \sum_{c\in\mathrm{Tr}_i} \frac{k_c\cdot \epsilon_b}{\sigma_{c,b}}, \qquad \Pi_{i',b} = \sum_{c\in\mathrm{Tr}_i} \frac{\sigma_c k_c\cdot k_b}{\sigma_{c,b}}, \qquad \tilde{\Pi}_{i',b} = \sum_{c\in\mathrm{Tr}_i} \frac{\sigma_c k_c\cdot \epsilon_b}{\sigma_{c,b}}$$
$$\Pi_{i,j} = \sum_{c\in\mathrm{Tr}_i, d\in\mathrm{Tr}_j} \frac{k_c\cdot k_d}{\sigma_{c,d}}, \qquad \Pi_{i',j} = \sum_{c\in\mathrm{Tr}_i, d\in\mathrm{Tr}_j} \frac{\sigma_c k_c\cdot k_d}{\sigma_{c,d}}, \qquad \Pi_{i',j'} = \sum_{c\in\mathrm{Tr}_i, d\in\mathrm{Tr}_j} \frac{\sigma_c k_c\cdot k_d \sigma_d}{\sigma_{c,d}}\,. \tag{A.3}$$

The modified Pfaffian Pf$'$ of $\Pi$ can be evaluated in several equivalent ways

$$\mathrm{Pf}'\,\Pi := \mathrm{Pf}\,[\Pi]^{ij'}_{ij'} = \frac{(-1)^a}{\sigma_a}\,\mathrm{Pf}\,[\Pi]^{j'a}_{j'a} = \frac{(-1)^a}{\sigma_a}\,\mathrm{Pf}\,[\Pi]^{ia}_{ia} = \frac{(-1)^{a+b}}{\sigma_{a,b}}\,\mathrm{Pf}\,[\Pi]^{ab}_{ab}\,, \tag{A.4}$$

where $[\ldots]^{ab}_{ab}$ once more instructs to remove the $a^{\text{th}}$ and $b^{\text{th}}$ rows and columns. Note that in the absence of scalars, the matrix $\Pi$ reduces to the matrix $\Psi$ in (2.16) and thus the YM+$\phi^3$ half integrand reduces to a YM half integrand. Similarly, (A.2) smoothly reduces to the $2m \times 2m$ matrix $\left(\begin{smallmatrix} \Pi_{i,j} & \Pi_{i,j'} \\ \Pi_{i',j} & \Pi_{i',j'} \end{smallmatrix}\right)$ with $i, j, i', j' \in \{1, 2, \ldots, m\}$ in absence of gluons.

## B Building blocks of the half-maximally supersymmetric integrand

The tensor $C^{\mu\nu}_{1|2,3,4}$ and the vectors $C^{\mu}_{1|ab,c}$ are the central building blocks of the half-maximally supersymmetric integrands in section 6. Together with a closely related scalar $C_{1|abc}$, they are defined in terms of more elementary tensors $t$ [72, 73]

$$\begin{aligned} C_{1|234} &= t_{1,234} + t_{12,34} + t_{123,4} - t_{124,3} - t_{14,23} - t_{142,3} + t_{143,2} \\ C^{\mu}_{1|23,4} &= t^{\mu}_{1,23,4} + t^{\mu}_{12,3,4} - t^{\mu}_{13,2,4} + k^{\mu}_3 t_{123,4} - k^{\mu}_2 t_{132,4} + k^{\mu}_4 \left[t_{14,23} - t_{214,3} + t_{314,2}\right] \\ C^{\mu\nu}_{1|2,3,4} &= t^{\mu\nu}_{1,2,3,4} + 2\left[k^{(\mu}_2 t^{\nu)}_{12,3,4} + (2 \leftrightarrow 3,4)\right] - 2\left[k^{(\mu}_2 k^{\nu)}_3 t_{213,4} + (2,3|2,3,4)\right]\,, \end{aligned} \tag{B.1}$$

which in turn are defined in terms of Berends-Giele currents $\mathfrak{e}$ and $\mathfrak{f}$ as

$$
\begin{aligned}
t_{A,B} &= -\frac{1}{2}(\mathfrak{f}_A)_{\mu\nu}\mathfrak{f}_B^{\mu\nu} \\
t^\mu_{A,B,C} &= [\mathfrak{e}^\mu_A t_{B,C} + (A \leftrightarrow B,C)] + \frac{i}{4}\varepsilon^\mu_6(\mathfrak{e}_A,\mathfrak{f}_B,\mathfrak{f}_C) \qquad \text{(B.2)}\\
t^{\mu\nu}_{A,B,C,D} &= 2\left[\mathfrak{e}_A^{(\mu}\mathfrak{e}_B^{\nu)}t_{C,D} + (A,B|A,B,C,D)\right] + \frac{i}{2}\left[\mathfrak{e}_B^{(\mu}\varepsilon_6^{\nu)}(\mathfrak{e}_A,\mathfrak{f}_C,\mathfrak{f}_D) + (B\leftrightarrow C,D)\right].
\end{aligned}
$$

These currents are labelled by words $P=(p_1,p_2,\ldots,p_{|P|})$ and recursively defined by

$$
\begin{aligned}
\mathfrak{e}^\mu_P &= \frac{1}{2s_P}\sum_{XY=P}\left[\mathfrak{e}^\mu_Y(k_Y\cdot\mathfrak{e}_Y) + (\mathfrak{e}_Y)_\nu\mathfrak{f}_X^{\mu\nu} - (X\leftrightarrow Y)\right] \\
\mathfrak{f}^{\mu\nu}_P &= k^\mu_P\mathfrak{e}^\nu_P - k^\nu_P\mathfrak{e}^\mu_P - \sum_{XY=P}\left(\mathfrak{e}^\mu_X\mathfrak{e}^\nu_Y - \mathfrak{e}^\mu_Y\mathfrak{e}^\nu_X\right), \qquad \text{(B.3)}
\end{aligned}
$$

starting with the single-particle cases $\mathfrak{e}^\mu_j = \epsilon^\mu_j$ and $\mathfrak{f}^{\mu\nu}_j = k^\mu_j\epsilon^\nu_j - k^\nu_j\epsilon^\mu_j$. Vector indices are symmetrized according to the normalization convention $2k_2^{(\mu}k_3^{\nu)} = k_2^\mu k_3^\nu + k_2^\nu k_3^\mu$, and the sums over deconcatenations $P=XY$ exclude the empty words $X=\emptyset$ and $Y=\emptyset$.

The propagators $s_P^{-1}$ in the recursion (B.3) for $\mathfrak{e}^\mu_P$ expose simple poles $t^\mu_{12,3,4}\sim s_{12}^{-1}$, and one might naively expect a pole structure of $(s_{12}s_{123})^{-1}$ and $(s_{23}s_{123})^{-1}$ for $t_{123,4}$. The propagators $s_{123}^{-1}$ diverge in the momentum phase-space of four massless particles and are in fact absent from $t_{123,4}$ based on the Minahaning procedure [72, 73, 82].[19] Similarly, the poles $s_{12}^{-1}$ and $s_{34}^{-1}$ of $(\mathfrak{f}_{12})_{\mu\nu}$ and $\mathfrak{f}^{\mu\nu}_{34}$ do not occur simultaneously in $t_{12,34}$. On these grounds, all of $C_{1|234}, C^\mu_{1|23,4}$ and $C^{\mu\nu}_{1|2,3,4}$ only have simple poles in $s_{ij}$.

More details on the symmetries and relations of these building blocks can be found in [43, 73].

## C Feynman integrals in six-dimensional anomalies

In this appendix, we review the expressions for the Feynman integrals in the anomalous gauge variations of half-maximally supersymmetric EYM amplitudes presented in section 6.5. As a common feature of the Feynman integrals in such anomalies, their loop integrand is proportional to $\ell^2_{(-2\varepsilon)}$, the formal $(-2\varepsilon)$-dimensional part of the $(6-2\varepsilon)$-dimensional square $\ell^2$. These factors of $\ell^2_{(-2\varepsilon)}$ arise along with the propagators of box and triangle integrals via

$$\int d^{6-2\varepsilon}\ell\left\{\frac{f(\ell)\ell^\mu\ell^\nu\eta_{\mu\nu}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} - \frac{f(\ell-k_1)}{\ell^2\ell_2^2\ell_{23}^2}\right\} = -\int d^{6-2\varepsilon}\ell\,\frac{f(\ell)\ell^2_{(-2\varepsilon)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} \qquad \text{(C.1)}$$

$$\int d^{6-2\varepsilon}\ell\left\{\frac{f(\ell)\ell^\mu\ell^\nu\eta_{\mu\nu}}{\ell^2\ell_1^2\ell_{12}^2} - \frac{f(\ell-k_1)}{\ell^2\ell_2^2}\right\} = -\int d^{6-2\varepsilon}\ell\,\frac{f(\ell)\ell^2_{(-2\varepsilon)}}{\ell^2\ell_1^2\ell_{12}^2}, \qquad \text{(C.2)}$$

where $f(\ell)$ is a polynomial in the loop momentum. Such integrals can be related to the poles in dimensionally regulated integrals in higher dimensions with the relation [83–85]

$$\int\frac{d^{2m-2\varepsilon}\ell}{i\pi^{m-\varepsilon}}(-\ell^2_{(-2\varepsilon)})^r F(\ell_{(2m)},\ell^2_{(-2\varepsilon)}) = \frac{\Gamma(r-\varepsilon)}{\Gamma(-\varepsilon)}\int\frac{d^{2m+2r-2\varepsilon}\ell}{i\pi^{m+r-\varepsilon}}F(\ell_{(2m)},\ell^2_{(-2\varepsilon)}), \qquad \text{(C.3)}$$

[19] Following J. Minahan's prescription to relax momentum conservation in intermediate steps [82], factors of $s_{123}$ are temporarily taken to be non-zero and cancel between numerators and denominators of $t_{123,4}$ [72, 73].



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valid for $m, r \in \mathbb{N}$ and arbitrary functions $F(\ell_{(2m)}, \ell^2_{(-2\varepsilon)})$ of the $2m$- and the $(-2\varepsilon)$-dimensional components of the loop momentum. Specifically, we use this relation with $r=1$ and $m=3$,

$$\int \mathrm{d}^{6-2\varepsilon}\ell\, \ell^2_{(-2\varepsilon)} F(\ell_{(6)}, \ell^2_{(-2\varepsilon)}) = \frac{\varepsilon}{\pi} \int \mathrm{d}^{8-2\varepsilon}\ell\, F(\ell_{(6)}, \ell^2_{(-2\varepsilon)}) \,, \tag{C.4}$$

where the resulting scalar boxes, triangles and bubbles in $8-2\varepsilon$ dimensions can be obtained with standard methods [85], for instance

$$\int \mathrm{d}^{6-2\varepsilon}\ell\, \frac{\ell^2_{(-2\varepsilon)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} = \frac{\varepsilon}{\pi} \int \mathrm{d}^{8-2\varepsilon}\ell\, \frac{1}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} = \frac{i\pi^3}{6} + \mathcal{O}(\varepsilon) \tag{C.5}$$

$$\int \mathrm{d}^{6-2\varepsilon}\ell\, \frac{\ell^2_{(-2\varepsilon)}}{\ell^2\ell_1^2\ell_{12}^2} = \frac{\varepsilon}{\pi} \int \mathrm{d}^{8-2\varepsilon}\ell\, \frac{1}{\ell^2\ell_1^2\ell_{12}^2} = -\frac{i\pi^3 s_{12}}{12} + \mathcal{O}(\varepsilon) \tag{C.6}$$

$$\int \mathrm{d}^{6-2\varepsilon}\ell\, \frac{\ell^2_{(-2\varepsilon)}}{\ell^2\ell_{12}^2} = \frac{\varepsilon}{\pi} \int \mathrm{d}^{8-2\varepsilon}\ell\, \frac{1}{\ell^2\ell_{12}^2} = \frac{i\pi^3 s_{12}^2}{15} + \mathcal{O}(\varepsilon) \,, \tag{C.7}$$

where the $\mathcal{O}(\varepsilon)$ terms on the right-hand sides are not tracked in the six-dimensional anomalies of section 6.5. Additionally, the anomalies in half-maximally supersymmetric EYM amplitudes with external gravitons feature vector and tensor integrals with $\ell^2_{(-2\varepsilon)}$ insertions. We employ Passarino-Veltman reduction [86], in particular its implementation in the `Mathematica` package `FeynCalc` [87], to obtain these vector and tensor integrals from scalar ones. Upon inserting the expressions (C.5) to (C.7) for scalar integrals, we arrived at

$$\int \mathrm{d}^{6-2\varepsilon}\ell\, \frac{\ell^2_{(-2\varepsilon)}\ell^\mu_{(6)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} = \frac{\varepsilon}{\pi} \int \mathrm{d}^{8-2\varepsilon}\ell\, \frac{\ell^\mu_{(6)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} = \frac{i\pi^3}{24}\left[k_2^\mu + 2k_3^\mu + 3k_4^\mu\right] + \mathcal{O}(\varepsilon) \tag{C.8}$$

$$\begin{aligned} \int \mathrm{d}^{6-2\varepsilon}\ell\, \frac{\ell^2_{(-2\varepsilon)}\ell^\mu_{(6)}\ell^\nu_{(6)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} &= \frac{\varepsilon}{\pi} \int \mathrm{d}^{8-2\varepsilon}\ell\, \frac{\ell^\mu_{(6)}\ell^\nu_{(6)}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} \\ &= \frac{i\pi^3}{120}\big[2k_2^\mu k_2^\nu + 3(k_2^\mu k_3^\nu + k_2^\nu k_3^\mu) + 4(k_2^\mu k_4^\nu + k_2^\nu k_4^\mu) \\ &\qquad + 6k_3^\mu k_3^\nu + 8(k_3^\mu k_4^\nu + k_3^\nu k_4^\mu) + 12 k_4^\mu k_4^\nu + \eta^{\mu\nu} s_{13}\big] + \mathcal{O}(\varepsilon) \,. \end{aligned} \tag{C.9}$$



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