UPPSALA
UNIVERSITET

Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology 2254

An Exploration of Q-Systems

From Spin Chains to Low-Dimensional AdS/CFT

SIMON EKHAMMAR

ACTA
UNIVERSITATIS
UPSALIENSIS
UPPSALA
2023

ISSN 1651-6214
ISBN 978-91-513-1769-4
URN urn:nbn:se:uu:diva-499273

Dissertation presented at Uppsala University to be publicly examined in 10132, Häggsalen, Ångströmslaboratoriet, Lägerhyddsvägen 1, Uppsala, Friday, 2 June 2023 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English. Faculty examiner: Professor Konstantin Zarembo (Niels Bohr Institute, Copenhagen University and Nordita, KTH Royal Institute of Technology and Stockholm University).

Abstract

Ekhammar, S. 2023. An Exploration of Q-Systems. From Spin Chains to Low-Dimensional AdS/CFT. *Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology* 2254. 121 pp. Uppsala: Acta Universitatis Upsaliensis. ISBN 978-91-513-1769-4.

The discovery of integrability in the planar limit of the AdS5/CFT4 correspondence has led to impressive progress in the study of string theory and four-dimensional N=4 Super-Yang-Mills. In particular, with the formulation of the Quantum Spectral Curve (QSC) the spectral problem is now solved. The QSC has demonstrated its versatility and usefulness in various other applications, including the study of Wilson lines, conformal bootstrap, and the calculation of structure constants.

It would be highly desirable to extend the QSC from AdS5/CFT4 to other instances of the AdS/CFT correspondence, a program so far only fully completed for AdS4/CFT3. Achieving this objective requires a solid understanding of the foundation of the QSC, a so-called analytic Q-system. In this thesis and the included papers, we investigate Q-systems and their algebraic and analytic properties. Inspired by the ODE/IM correspondence we propose Q-systems that encode the conserved charges of integrable models with a simply-laced symmetry algebra. In particular, for the case of D(r) a powerful parameterization of the Q-system using pure spinors is employed to efficiently solve compact rational spin chains and find T-functions solving Hirota equations. The extension from D(r) to the non-simply laced algebra B(2)/C(2) is explained and detailed. While many features are similar the relation between the symmetry of the Q-system and that of the integrable model becomes more intricate. By introducing a new method dubbed Monodromy Bootstrap we construct new Quantum Spectral Curves based on gl(2|2). In particular, one curve is conjectured to describe planar string theory on AdS3 with pure RR-flux. We solve the curve in a weak coupling limit both analytically and numerically.

We also discuss the problem of finding the eigenvalue spectrum of operators on the squashed seven-sphere coming from the compactification of eleven-dimensional supergravity. These eigenvalues are of importance for the mass spectrum of fields in AdS4.

Keywords: Quantum Spectral Curve, Integrability, AdS/CFT

Simon Ekhammar, Department of Physics and Astronomy, Theoretical Physics, Box 516, Uppsala University, SE-751 20 Uppsala, Sweden.

ISSN 1651-6214
ISBN 978-91-513-1769-4
URN urn:nbn:se:uu:diva-499273 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-499273)

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Simon Ekhammar, Hongfei Shu and Dmytro Volin, *Extended systems of Baxter Q-functions and fused flags I: simply-laced case,* arXiv:2008.10597.

II Simon Ekhammar and Dmytro Volin, *Bethe Algebra using Pure Spinors*, arXiv:2104.04539.

III Simon Ekhammar and Bengt E.W. Nilsson, *On the squashed seven-sphere operator spectrum*, JHEP 12 (2021) 057, arXiv 2105.05229.

IV Simon Ekhammar and Dmytro Volin, *Monodromy bootstrap for SU(2|2) quantum spectral curves: from Hubbard model to* AdS_3/CFT_2, JHEP 03 (2022) 192, arXiv 2109.06164.

V Andrea Cavaglià, Simon Ekhammar, Nikolay Gromov and Paul Ryan, *Exploring the* AdS_3/CFT_2 *QSC,* arXiv 2211.07810.

Reprints were made with permission from the publishers.

Contents

1. Introduction

This thesis is about the use of symmetries, mainly in the context of exactly solvable models. Such models are rare but of great significance. Among their ranks, we find the harmonic oscillator, the Kepler problem and the hydrogen atom. The more pragmatic person would argue that these models are of interest since they serve as basic stepping stones to more difficult theories. The people who study them daily would surely add that they are fascinating objects on their own.

Of special significance to this text is the appearance of integrability in the duality between superstrings on $\mathrm{AdS}_5\times\mathrm{S}^5$ and conformal four-dimensional $\mathcal{N}=4$ Super-Yang-Mills in the planar limit [1]. While these highly symmetric theories do not describe the everyday world around us, they are instrumental to our understanding of string theory and quantum field theory. The presence of integrability allows for the use of powerful techniques usually not applicable. The full solution of $\mathcal{N}=4$ is yet beyond our reach, but impressive progress has been made in computing the energy of strings, or the so-called conformal dimension in $\mathcal{N}=4$. The most efficient way to do so is through a novel integrability-based formalism: the Quantum Spectral Curve (QSC) [2].

Using the QSC the full non-perturbative spectrum of $\mathcal{N}=4$ is now obtainable by anyone with an internet connection and a Mathematica license [3, 4]. The QSC has also been used to study Wilson lines [5], in conformal bootstrap [6], in the computation of structure constants [7] and to investigate the thermodynamics of string theory [8], to mention a few applications. The QSC was subsequently generalised to the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence and used therein with great success [9].

However, this thesis is not about the use of the $\mathrm{AdS}_5\times\mathrm{S}^5$ QSC, nor is it about AdS_4. This thesis is about extending the QSC beyond these theories and investigating the structures that underlie it.

To better understand the QSC and how we might promote it beyond the realm of AdS_5 and AdS_4 we need to examine the foundation on which it is built. This foundation is known as a Q-system; a collection of functions of one complex parameter called Q-functions. Q-systems are objects of great generality, and they appear in a variety of different applications. It is only when we severely restrict their analytic properties we can use them to describe specific physical models. The defining feature of the QSC is that the Q-functions describing it are functions with branch cuts. It is by this feature we will define the notion of QSC in this text.

The importance of Q-functions, or rather their operatorial counterpart, were recognised by Baxter [10] more than 10 years before $\mathcal{N}=4$ was found from compactifying 10D SYM. Q-operators have since their inception been intensively studied in the maths and physics literature in a variety of contexts; they appear in Yangians and quantum affine Lie algebras [11], they feature in the study of conformal field theory [12–14] and they are intimately connected with ordinary differential equations through the ODE/IM correspondence [15, 16].

Yet, in none of the applications listed above the Q-functions were granted the analytic properties characterising the QSC. Thus as of writing, it remains unclear exactly which Q-systems can be promoted to the world of QSC and AdS/CFT. We currently only have a few isolated points where we understand how to proceed, no landscape of theories has yet been established.

So a line of research emerges: Understand the algebraic relations that define Q-systems for arbitrary symmetry algebras and find a clear recipe for how to dress the Q-functions with analytic properties appropriate for AdS/CFT, or if possible, beyond. We believe that pursuing these tasks simultaneously is advantageous. Indeed, one should naturally expect there to be a non-trivial interplay and lessons should carry over between both exercises.

Let us then detail the results that will be covered in this thesis. On the algebraic side of Q-systems, the task of constructing Q-systems for simple-laced algebras is tackled in Paper I based on intuition coming from ODE/IM. The case of Q-systems for D_r is treated in detail in Paper II. Together with the operatorial results presented in [17] Q-systems for algebras of type D_r are now well functioning for compact rational spin chains, and hopefully ready to be utilised in more challenging theories [18–20].

Generalisations of the QSC framework are considered in Paper IV. Here a general axiomatic approach, called Monodromy Bootstrap, is detailed and used to construct four different QSCs with underlying $\mathfrak{gl}_{2|2}$ symmetry. Among these models, one was identified as a putative candidate to describe the spectrum of planar string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. In Paper V this model is solved in a weak coupling limit both numerically and analytically.

This thesis also includes another excursion into the land of spectrum calculations. In Paper III eigenvalues of certain differential operators on the squashed S^7 are calculated. These eigenvalues describe masses of particles when compactifying eleven-dimensional supergravity to AdS_4. While the motivation for this project is different compared to the other parts of the thesis the tools used have many similarities.

1.1 Thesis Outline

Part I is a review of relevant background. In Chapter 2 we recall the basics of Lie algebras, including a short discussion of Lie superalgebras. We turn to the infinite-dimensional Yangian in Chapter 3 using the RTT formalism. Finally, we formulate the task of computing the spectrum of T-functions in Chapter 4.

Here we consider the functional relations that T-functions satisfy and their connections to Q-functions.

Part II is about the use of algebraic techniques to find the spectrum of integrable models and special differential operators. Chapter 5 discusses how to obtain relations among Q-functions by studying differential equations. Chapter 6 reviews Q-systems with $\mathfrak{gl}_n$ symmetry. In Chapter 7 we review the results of Paper I and Paper II regarding Q-systems for systems with D_r-symmetry. Chapter 8 contains a description of Q-systems for B_2/C_2, the results in this chapter are new and not yet published. In Chapter 9 we review the results of Paper III regarding the spectrum of operators on the squashed seven-sphere. The final leg of Part II is Chapter 10 where we review Q-systems for supersymmetric spin chains.

Part III is dedicated to the study of the Quantum Spectral Curve. In Chapter 11 we very briefly recall the spectral problem before reviewing the $\mathrm{AdS}_5\times\mathrm{S}^5$ QSC in Chapter 12. We discuss its definition and the main techniques on the market. As a bonus, we also quickly discuss how to twist the QSC and produce some results for twisted theories. Finally, Chapter 13 is dedicated to introducing a QSC conjectured to describe planar string theory on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$. This curve was constructed in Paper IV and [21]. We review the construction and how the QSC produces the crossing equations for the so-called dressing phases. In Section 13.4 we review the results of Paper V in which the proposed QSC was solved in a weak coupling regime.

Part I:
Background

2. Lie Algebras

The most well-known symmetries to physicists are described using Lie algebras. This chapter aims to set notation and introduce objects that will later play a key role in our discussion of Q-systems. For a reader seeking to deepen their knowledge, we refer to the excellent books [22, 23].

A Lie algebra $\mathfrak{g}$ is a vector space equipped with an antisymmetric bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ known as the Lie bracket. Let T^a be generators of $\mathfrak{g}$, we will use notation

$$[T^a, T^b] = f^{ab}{}_c T^c\,, \tag{2.1}$$

where $f^{ab}{}_c$ are the so-called structure constants. Here and in the following we will use Einstein summation convention: repeated indices are to be summed over unless explicitly stated. In addition to being antisymmetric, the Lie bracket must also satisfy the Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0\,. \tag{2.2}$$

The simplest choice of structure constants is $f^{ab}{}_c = 0$. Such an algebra is called abelian.

We will be interested in scenarios where the symmetry algebra acts on a vector space V. This requires representations of $\mathfrak{g}$. Formally a representation is a map $\mathfrak{g}\to\mathrm{End}(V)$ respecting the Lie bracket. When V is of finite dimension n the representation of a generator is a $n\times n$ matrix. An important example of a vector space is $\mathfrak{g}$ itself. On this space we have the adjoint representation $\mathrm{ad}_a(T^b)=[T^a,T^b]$.

2.1 Simple Lie Algebras

There are a wide variety of different Lie algebras. We will be interested in the simplest and most well-studied examples. An important class of Lie algebras are simple Lie algebras. These are non-abelian Lie algebras with no nonzero proper ideals. Recall that I is an ideal if $[\mathfrak{g},I]\subseteq I$. An ideal is proper if it is not equal to $\mathfrak{g}$ or 0.

2.1.1 The *ABCD* of Lie algebras

Let us now introduce some important Lie algebras which will play a role in the following.

The Lie algebra $\mathfrak{gl}_n$

A prototypical example of a Lie-algebra is $\mathfrak{gl}_n$. It is customary to denote the generators as $E_i{}^j$ with $i, j = 1, \dots, n$. These generators satisfy the following algebra

$$[E_i{}^j, E_k{}^l] = E_i{}^l \delta_k^j - E_k{}^j \delta_i^l \,, \tag{2.3}$$

where δ_j^i is the Kronecker delta. It is easy to verify that $C = E_i{}^i$ commutes with all other operators. The remaining operators, spanned by $E_i{}^j, i \neq j$ and $E_i{}^i - E_{i+1}{}^{i+1}$ (no sum over i), form the simple Lie algebra $\mathfrak{sl}_n$, also called A_{n-1}.

The defining representation of $\mathfrak{gl}_n$ is given by $E_i{}^j \mapsto e_i{}^j$ where $e_i{}^j$ are the standard matrices $(e_i{}^j)^k{}_l = \delta_i^k \delta_l^j$. In the case of $n = 2$ we find explicitly

$$e_1{}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1{}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2{}^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_2{}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Lie algebras $\mathfrak{so}_n$ and $\mathfrak{sp}_{2r}$

Within $\mathfrak{gl}_n$ we can find interesting subalgebras. The goal of this paragraph is to construct the algebras $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ in this way. Start by noticing that $\tau(E_a{}^b) = -g^{bc} E_c{}^d g_{da}$ with $g^{ac} g_{cb} = \delta_b^a$ also satisfy (2.3). More precisely, τ is a Lie algebra homomorphism:

$$\tau([x,y]) = [\tau(x), \tau(y)] \,. \tag{2.4}$$

We think loosely about τ as "the negative transpose" of $E_i{}^j$. Requiring $g_{ij} = \pm g_{ji}$ we find $\tau^2 = 1$. Since τ squares to 1 it must have eigenvalues ± 1, then $\mathfrak{gl}_n = \mathfrak{gl}_n^{[0]} \oplus \mathfrak{gl}_n^{[1]}$ with $\tau(\mathfrak{gl}_n^{[i]}) = (-1)^i \mathfrak{gl}_n^{[i]}$. The algebra $\mathfrak{gl}_n^{[0]}$ forms a Lie algebra by itself, it is spanned by elements

$$F_i{}^j = E_i{}^j - g^{jl} E_l{}^k g_{ki} \,. \tag{2.5}$$

If $g_{ij} = g_{ji}$ the algebra $\mathfrak{gl}_n^{[0]}$ is a complexification of $\mathfrak{so}(n)$. It is called D_r if $n = 2r$ and B_r if $n = 2r + 1$. We can set $g_{ij} = -g_{ji}$ only if $n = 2r$ since otherwise g_{ij} is not invertible. In this case $\mathfrak{gl}_n^{[0]}$ is called C_r; it is a complex version of the symplectic algebra $\mathfrak{sp}_{2r}$. The commutation relations of $F_i{}^j$ reads

$$\begin{aligned} [F_i{}^j, F_k{}^l] &= F_i{}^l \delta_k^j - F_k{}^j \delta_i^l + g_{ik} g^{lm} F_m{}^j - g^{jl} g_{km} F_i{}^m \\ &= F_i{}^l \delta_k^j - F_k{}^j \delta_i^l - g_{ki} g^{jm} F_m{}^l + g^{lj} g_{im} F_k{}^m \,. \end{aligned} \tag{2.6}$$

The defining representations $f_i{}^j$ of D_r, B_r and C_r are inherited from those of A_r. Using g_{ij} to lower and raise indices according to

$$V^i = g^{ij} V_j \,, \qquad V_i = g_{ij} V^j \,. \tag{2.7}$$

we find

$$f_{ij} = e_{ij} \mp e_{ji} \,, \qquad g_{ij} = \pm g_{ji} \,. \tag{2.8}$$

The Killing form
There exists a non-degenerate inner form on simple Lie algebras known as the Killing form:

$$\kappa_{\mathrm{ad}}(T^a, T^b) = \mathrm{tr}\, \mathrm{ad}_a \circ \mathrm{ad}_b\,. \tag{2.9}$$

This inner form satisfies an important invariance property

$$\kappa_{\mathrm{ad}}([x,y],z) = \kappa_{\mathrm{ad}}(x,[y,z])\,, \tag{2.10}$$

which is a direct consequence of the cyclicity of the trace.

2.1.2 Basic structure of Lie algebras

Having introduced some basic examples of simple Lie algebras we continue to investigate their general structure.

Let $\mathfrak{h}$ be a maximal commuting ad-diagonalisable subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ is known as a Cartan subalgebra. We will denote a set of generators for this algebra as $\{H^i\}_{i=1}^r$ where r is the rank of the algebra. For the rest of the generators we pick a basis in which the adjoint action of H^i acts diagonally. We label the generators not in the Cartan algebra as E^α with α an r-dimensional vector such that

$$[H^i, E^\alpha] = \alpha^i E^\alpha\,. \tag{2.11}$$

α are called *roots*, it turns out that there is only one generator per root.

Since a finite-dimensional algebra has a finite number of roots it is possible to find a hyperplane that divides the roots into two sets, the roots in these two sets are called positive respectively negative roots. From the non-degeneracy of the Killing form it follows that for every positive root α there also exists a negative root $-\alpha$.

Using the Jacobi-identity one sees that $E^{\alpha+\beta} \propto [E^\alpha, E^\beta]$, there is thus a natural notion of adding roots. With this in mind, we define a simple root as a positive root that cannot be obtained as a linear combination of the remaining positive roots using only positive coefficients. There are exactly r positive roots, we will use notation $\alpha_a, a = 1,\dots,r$. As an example, for $\mathfrak{sl}_n$ a choice of generators corresponding to simple roots is given by

$$\{{E_1}^2, {E_2}^3, \dots {E_{n-1}}^n\}\,. \tag{2.12}$$

2.1.3 Finite-dimensional representations of $\mathfrak{sl}_2$

The structure of a simple Lie algebra is heavily constrained by $\mathfrak{sl}_2$ representation theory. This motivates a short reminder, let us use notation $h = {E_1}^1 - {E_2}^2$, $e = {E_1}^2$, $f = {E_2}^1$. The commutation relations are

$$[h,e] = 2e\,, \qquad [h,f] = -2f\,, \qquad [e,f] = h\,. \tag{2.13}$$

Let V be a vector space on which $\mathfrak{g}$ acts. The vector space V, together with the representation of $\mathfrak{g}$, is called a $\mathfrak{g}$-module. However, with a slight abuse of terminology, V is usually called a representation. We will require that $\mathfrak{h}$ acts diagonally.

If our representation is to be finite there must exist a state $|\Lambda\rangle \in V$ such that $e\,|\Lambda\rangle = 0$, $h\,|\Lambda\rangle = \Lambda\,|\Lambda\rangle$. Such a state is called a highest weight state. All other states in V are of the form $f^k\,|\Lambda\rangle$. This implies that all eigenvalues of h_i are of the form $\Lambda - 2k$.

For the representation to be finite we must truncate the tower of states $f^k\,|\Lambda\rangle$ setting $f^{n+1}\,|\Lambda\rangle = 0$ for some n. Killing $f^{n+1}\,|\Lambda\rangle$ is only allowed if we make sure there is no way back from $f^{n+1}\,|\Lambda\rangle$ to the land of the living. This means that we need to impose not only $f^{n+1}\,|\Lambda\rangle = 0$ but also $ef^{n+1}\,|\Lambda\rangle = (n+1)(\Lambda-n)f^n\,|\Lambda\rangle = 0$, so Λ must be a positive integer. This classifies all finite-dimensional representations of $\mathfrak{sl}_2$.

2.1.4 The Chevalley-Serre basis

Using the invariance of the Killing form it is possible to show that after proper normalisation of E^α

$$[E^\alpha, E^{-\alpha}] = \frac{2}{|\alpha|^2}\alpha^i H^i\,, \tag{2.14}$$

with $|\alpha|^2 = (\alpha,\alpha) = \alpha^i\alpha^i$. Inspired by $\mathfrak{sl}_2$ representation theory it is common to introduce notation $h^a = (\alpha_a^\vee)^i H^i$, $e^a = E^{\alpha_a}, f^a = E^{-\alpha_a}$ with $\alpha_a^\vee = \frac{2}{|\alpha_a|^2}\alpha_a$, $\alpha_a^\vee$ is called a coroot. This is convenient since now $\{h^a, e^a, f^a\}$ will form an A_1 algebra due to (2.14). The generators h^a, e^a and f^a form a so-called Chevalley-Serre basis. In this basis the commutation relations are

$$[h^a, e^b] = C_{ab}e^b\,, \quad [h^a, f^b] = -C_{ab}f^b\,, \quad [e^a, f^b] = \delta_{ab}h^a\,, \quad \text{(no sum)}. \tag{2.15}$$

C_{ab} is the Cartan Matrix. The Cartan matrix uniquely defines a simple Lie algebra. The Cartan matrix can be computed as $C_{ab} = (\alpha_a^\vee, \alpha_b)$.

The classification of all Cartan matrices is equivalent to the classification of all simple Lie algebras. The result is 4 infinite series which we have already met; A_r, D_r, B_r and C_r as well as 5 exceptional algebras; E_6, E_7, E_8, G_2 and F_4. It is common to illustrate the Cartan matrix using a Dynkin diagram. A Dynkin diagram for a Lie algebra of rank r has r nodes. To each node one associates a root α_a and draws $C_{ab}C_{ba}$, no sum, lines between node a and b. If one root is longer this is indicated with an additional $>$, if both roots are of the same length no decoration is needed. We collect Dynkin diagrams for all simple Lie algebras we will encounter in Figure 2.1.

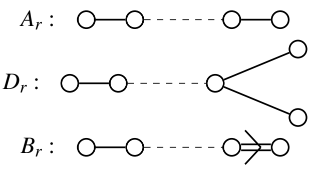


Figure 2.1. Dynkin diagrams for all algebras we will consider.

2.2 Finite-Dimensional Representations

Although the algebras we will eventually aim to study in this thesis are infinite-dimensional we will make ample use of finite-dimensional representations. We will use this section to recall the basic structure of finite-dimensional representations.

Let V be an irreducible representation. That is, the representation of $\mathfrak{g}$ cannot be brought to a block-diagonal form. To each and every state $|\nu\rangle$ in V we can assign an element $\nu \in \mathfrak{h}^\star$ such that

$$h^a\,|\nu\rangle = \nu(h^a)\,|\nu\rangle\,. \tag{2.16}$$

ν is known as a weight, it might not label a state uniquely. Notice that $\alpha \in \mathfrak{h}^\star$ since $[h^a, E^\alpha] = \alpha(h^a)E^\alpha$. The inner form allows us to identify $\mathfrak{h}$ and $\mathfrak{h}^\star$. With the explicit normalisation picked when constructing $[E^\alpha, E^{-\alpha}]$ this identification dictates that

$$\alpha(h^a) = (\alpha, \alpha_a^\vee). \tag{2.17}$$

All eigenvalues of h^a on V must be integers, this we learned when studying $\mathfrak{sl}_2$. We therefore write $\nu = \sum_{a=1}^r \Lambda_a \omega_a, \Lambda_a \in \mathbb{Z}$ with ω_a defined by

$$(\omega_a, \alpha_b^\vee) = \delta_{ab}\,. \tag{2.18}$$

ω_a are called fundamental weights and Λ_a Dynkin labels, we will occasionally write $\nu = \sum_a c_a \omega_a = [c_1 c_2 \ldots c_r]$.

Every finite-dimensional representation has a highest weight state $|\mathrm{HWS}\rangle$ defined by $E^\alpha\,|\mathrm{HWS}\rangle = 0, \alpha > 0$. We can use the Dynkin labels of $|\mathrm{HWS}\rangle$ to label the representation. We will denote a representation as $L(\nu)$ or use notation that indicates the dimension of the representation. For example, $L([1]) = \mathbf{2}$. The representations $L(\omega_a)$ are called fundamental representations.

When considering $\mathfrak{gl}_n$ instead of $\mathfrak{sl}_n$ we introduce λ_i as eigenvalues of ${E_i}^i$. Using $h^i = {E_i}^i - {E_{i+1}}^{i+1}$ as generators of $\mathfrak{sl}_n$ the Dynkin labels are obtained as $\Lambda_a = \lambda_a - \lambda_{a+1}$.

As an example consider $\mathfrak{g} = A_r$. The fundamental representation $L(\omega_1)$ is the defining representation, ${e_i}^j$, the highest weight vector is e_1 with e_i standard vectors, i.e ${e_i}^j e_k = e_i \delta_k^j$. The remaining fundamental representations are found as exterior products of $L(\omega_1)$, namely $L(\omega_a) = \Lambda^a L(\omega_1)$. The highest weight vector of $L(\omega_a)$ is $e_1 \wedge e_2 \cdots \wedge e_a$.

$\mathfrak{g}$	h	$h^\vee$
A_r	$r+1$	$r+1$
D_r	$2r-2$	$2r-2$
B_r	$2r$	$2r-1$
C_r	$2r$	$r+1$

Table 2.1. *Coxeter and dual Coxeter number for ABCD*

Algebra	Positive simple roots	Positive roots
A_r	$\varepsilon_a - \varepsilon_{a+1}, \quad a = 1,\ldots,r$	$\varepsilon_a - \varepsilon_b \quad a < b \leq r+1$
B_r	$\alpha_a = \varepsilon_a - \varepsilon_{a+1} \quad a = 1,\ldots,r-1$ $\alpha_r = \varepsilon_r$	$\varepsilon_a \pm \varepsilon_b \quad a < b \leq r$ $\varepsilon_a \quad a = 1,\ldots,r$
C_r	$\alpha_a = \varepsilon_a - \varepsilon_{a+1} \quad a = 1,\ldots,r-1$ $\alpha_r = 2\varepsilon_r$	$\varepsilon_a \pm \varepsilon_b \quad a < b \leq r$ $2\varepsilon_a \quad a = 1,\ldots,r$
D_r	$\alpha_a = \varepsilon_a - \varepsilon_{a+1} \quad a = 1,\ldots,r-1$ $\alpha_r = \varepsilon_{r-1} + \varepsilon_r$	$\varepsilon_a \pm \varepsilon_b \quad a < b \leq r$

Table 2.2. *Orthogonal basis for the weight lattices*

The highest weight of the adjoint representation is called the highest root θ. Let

$$\theta = \sum_{a=1}^{r} a_a \alpha_a \,, \qquad \frac{2}{(\theta,\theta)}\theta = \sum_{a=1}^{r} a_a^\vee \alpha_a^\vee \,. \tag{2.19}$$

The numbers a_a are Coxeter labels and $c_a^\vee$ are dual Coxeter labels. The Coxeter number h and the dual Coxeter number $h^\vee$ are defined by

$$h = 1 + \sum_{a=1}^{r} a_a \,, \qquad h^\vee = 1 + \sum_{a=1}^{r} a_a^\vee \,. \tag{2.20}$$

We tabulate the (dual) Coxeter number for A_r, B_r, C_r and D_r in table 2.1.

The lattice $Q = \mathrm{span}_{\mathbb{Z}}\{\omega_a\}_{a=1}^r$ is known as the weight space of $\mathfrak{g}$. The root space is a sublattice of Q. Indeed, we can explicitly write roots in terms of fundamental weights as

$$\alpha_a = \omega_b C_{ba} \,. \tag{2.21}$$

Thus the columns of the Cartan matrix allow us to read the decomposition of simple roots in terms of fundamental weights.

It is often useful to describe Q using an orthogonal basis. The basis vectors will be written as ε_a and they satisfy $(\varepsilon_a, \varepsilon_b) = \delta_{ab}$. Table 2.2 gives the positive roots in terms of the orthogonal basis for all the cases we will need.

Characters

Let ω and ν be weights of a Lie algebra $\mathfrak{g}$, we define the formal exponent e^{ω} by the property $e^{\omega}e^{\nu} = e^{\omega+\nu}$. The character of a representation V is the sum

$$\chi(V) = \sum_{\lambda \in V} \text{mult}\,(\lambda)\, e^{\lambda}\,. \tag{2.22}$$

We will often simply write $\chi(\lambda) \equiv \chi(L(\lambda))$.

2.3 Oscillators

To describe representations more explicitly it turns out to be very useful to introduce bosonic and fermionic oscillators, $\mathtt{b}$ and $\mathtt{f}$ respectively. They are defined by

$$[\mathtt{b}_a, \mathtt{b}^b] = \delta_a^b\,, \qquad [\mathtt{b}_a, \mathtt{b}_b] = [\mathtt{b}^a, \mathtt{b}^b] = 0\,, \tag{2.23}$$

$$\{\mathtt{f}_i, \mathtt{f}^j\} = \delta_i^j\,, \qquad \{\mathtt{f}_i, \mathtt{f}_j\} = \{\mathtt{f}^i, \mathtt{f}^j\} = 0\,, \tag{2.24}$$

and $[\mathtt{f}, \mathtt{b}] = 0$. Here $\{\cdot,\cdot\}$ is the anti-commutator; $\{a,b\} = ab + ba$. Using these oscillators we can realize the generators of $\mathfrak{gl}_n$ in two different ways

$$E_i{}^j = \mathtt{b}^i\mathtt{b}_j\,. \qquad \text{or} \qquad E_i{}^j = \mathtt{f}^i\mathtt{f}_j\,. \tag{2.25}$$

This will be used when discussing Lie superalgebras.

We will in Chapter 7 need to understand spinor representations of D_r. In the notation of this chapter, these representations are $L(\omega_{r-1})$ and $L(\omega_r)$ for D_r. We will for future convenience introduce notation $S^- = L(\omega_{r-1}), S^+ = L(\omega_r)$.

To build the spinor representation we use all bilinears in $\mathtt{f}^a$ and $\mathtt{f}_a$. In particular, a Chevalley-Serre basis is

$$e^a = \mathtt{f}^{a+1}\mathtt{f}_a\,, \quad a < r\,, \qquad e^r = -\mathtt{f}_{r-1}\mathtt{f}_r\,, \tag{2.26a}$$

$$f^a = \mathtt{f}^a\mathtt{f}_{a+1}\,, \quad a < r-1\,, \qquad f^r = \mathtt{f}^{r-1}\mathtt{f}^r\,, \tag{2.26b}$$

$$h^a = -\mathtt{f}^a\mathtt{f}_a + \mathtt{f}^{a+1}\mathtt{f}_{a+1}\,, \quad a < r-1\,, \qquad h^r = -\mathtt{f}^{r-1}\mathtt{f}_{r-1} - \mathtt{f}^r\mathtt{f}_r + 1\,. \tag{2.26c}$$

Given a vacuum $|0\rangle$ such that $\mathtt{f}_a|0\rangle = 0$ we construct a Fock space spanned by all states of the form $\mathtt{f}^A|0\rangle$ with A a multi-index. This is a reducible representation of D_r, it contains both $L(\omega_{r-1})$ and $L(\omega_r)$. The highest weight vectors of these representations are

$$|\omega_{r-1}\rangle \equiv \mathtt{f}^r|0\rangle\,, \qquad |\omega_r\rangle \equiv |0\rangle\,. \tag{2.27}$$

From the observation that $\{e^a, f^a, h^a\}_a^{r-1}$ generates $\mathfrak{sl}_r$ it follows that we can decompose the spinor representations into $\mathfrak{sl}_r$ representations. Let $V = L(\omega_1)_{\mathfrak{sl}_r}$

then

$$S^{+} = \bigoplus_{i=0}^{[\frac{r}{2}]} \Lambda^{2i} V \,, \qquad\qquad S^{-} = \bigoplus_{i=0}^{[\frac{r-1}{2}]} \Lambda^{2i+1} V \,. \tag{2.28}$$

Another important property of spinors is that they are basic representations. This means that all other fundamental representations can be recovered from taking tensor products of spinor representations. In particular

$$S^{+} \otimes S^{+} = L(2\omega_r) \oplus \bigoplus_{m=1}^{[\frac{r}{2}]} L(\omega_{r-2m}) \,, \tag{2.29}$$

$$S^{+} \otimes S^{-} = L(\omega_r + \omega_{r-1}) \oplus \bigoplus_{m=1}^{[\frac{r-1}{2}]} L(\omega_{r-2m-1}) \,. \tag{2.30}$$

2.4 Lie Superalgebras

Lie superalgebras are important generalisations of ordinary Lie algebras, they describe symmetries relating bosons and fermions. For more information regarding superalgebras see [24, 25].

Mathematically a Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g}$ with a map $[\cdot,\cdot\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which has to satisfy two properties. It needs to be super-skew

$$[x,y\} = -(-1)^{|x||y|}[y,x\} \,, \tag{2.31}$$

and it needs to satisfy the super-Jacobi identity

$$(-1)^{|x||z|}[x,[y,z\}\} + (-1)^{|x||y|}[y,[z,x\}\} + (-1)^{|y||z|}[z,[x,y\}\} = 0 \,, \tag{2.32}$$

where $|x| = 0,1$ is the grading of the element $x \in \mathfrak{g}$. We will call x even if $|x| = 0$ and odd if $|x| = 1$.

The easiest way to memorise (2.32) is to note that it says that the adjoint action is a derivation with a sign when passing odd elements through each other, that is

$$[x,[y,z\}\} = [[x,y\},z\} + (-1)^{|x||y|}[y,[x,z\}\} \,. \tag{2.33}$$

Equivalence between (2.32) and (2.33) follows from (2.31).

$\mathfrak{gl}_{m|n}$

In this thesis, we will only need the Lie superalgebra $\mathfrak{gl}_{m|n}$. The generators of $\mathfrak{gl}_{m|n}$ are often enumerated as $E_i{}^j, i, j = 1, \dots m+n$ with $\mathbb{Z}_2$ grading

$$|E_i{}^j| = |i| + |j| \mod 2 \,, \qquad\qquad |i| = \begin{cases} 0 & i \leq m \\ 1 & i > m \end{cases} \,. \tag{2.34}$$

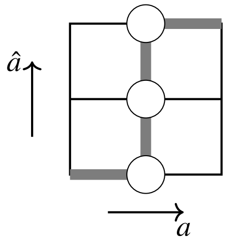


Figure 2.2. A Kac-Dynkin diagram for $\mathfrak{gl}_{2|2}$ depicting the grading $1\hat{1}\hat{2}2$.

The super-commutator is given by

$$[E_i{}^j, E_k{}^l\} = \delta_k^j E_i{}^l - (-1)^{(|i|+|j|)(|k|+|l|)}\delta_i^l E_k{}^j\,. \tag{2.35}$$

An important subalgebra of $\mathfrak{gl}_{m|n}$ is $\mathfrak{sl}_{m|n}$, this algebra is generated by $E_i{}^j, i \neq j, E_i{}^i - (-1)^{|i|+|i+1|}E_{i+1}{}^{i+1}$ (no sum over i). We will also make use of $\mathfrak{psl}(n|n) = \mathfrak{sl}_{n|n}/C$ with $C = E_i{}^i$.

2.4.1 Highest weight representations

Let us split $i = \{a, \hat{a}\}$ with $a = 1, \ldots, m$ and $\hat{a} = \hat{1}, \ldots \hat{n}$, $|a| = 0, |\hat{a}| = 1$. Given an ordering between $\hat{a}$ and a we define a highest weight state $|\mathrm{HWS}\rangle$ of $\mathfrak{gl}_{m|n}$ as

$$E_i{}^j\,|\mathrm{HWS}\rangle = 0\,, \quad i < j\,. \tag{2.36}$$

As an example, let us take $\mathfrak{gl}_{2|2}$. A potential ordering is $1\hat{1}\hat{2}2$, we can graphically illustrate the grading using a Kac-Dynkin diagram, see Figure 2.2.

The eigenvalues of $E_a{}^a$ and $E_{\hat{a}}{}^{\hat{a}}$ on $|\mathrm{HWS}\rangle$ will be named λ_a and ν_a

$$E_a{}^a\,|\mathrm{HWS}\rangle = \nu_a\,|\mathrm{HWS}\rangle\,, \qquad E_{\hat{a}}{}^{\hat{a}}\,|\mathrm{HWS}\rangle = \lambda_a\,|\mathrm{HWS}\rangle\,. \tag{2.37}$$

It is often convenient to realise the superalgebra using oscillators, for $\mathfrak{gl}_{m|n}$ we have

$$E_a{}^b = \mathtt{a}^a\,\mathtt{a}_b\,, \quad E_a{}^{\hat{b}} = \mathtt{a}^a\,\mathtt{f}_b\,, \quad E_{\hat{a}}{}^b = \mathtt{f}^a\,\mathtt{a}_b\,, \quad E_{\hat{a}}{}^{\hat{b}} = \mathtt{f}^a\,\mathtt{f}_b\,. \tag{2.38}$$

Consider as an example again $\mathfrak{gl}_{2|2}$. Introduce the Fock vacuum $|0\rangle$ such that $\mathtt{f}_a\,|0\rangle = 0$, $\mathtt{a}_a\,|0\rangle = 0$. The defining representation is given by $\phi_1 = \mathtt{f}^1\,|0\rangle\,, \psi_1 = \mathtt{a}^1\,|0\rangle\,, \psi_2 = \mathtt{a}^2\,|0\rangle\,, \phi_2 = \mathtt{f}^2\,|0\rangle$. We see that ψ_1 is a highest weight state in, for example, the grading $1\hat{1}\hat{2}2$ while ϕ_1 is a highest weight state in, for example, $\hat{1}12\hat{2}$.

3. Yangians

In the last chapter, we described the simple Lie algebras $\mathfrak{g} = A, B, C, D$. With this background at hand, we now turn our attention to Yangians $\mathscr{Y}(\mathfrak{g})$. Yangians are fascinating objects that make an appearance in many separate instances. The Yangian is the spectrum-generating algebra of integrable spin chains. We will introduce spin chains in this chapter and study them in Part II. Of its more outlandish applications we mention that it appears in the study of the spectrum of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. It also makes an appearance in the study of scattering amplitudes [26].

The Yangian admits three important realisations; Drinfeld's first and second realisation [27, 28] and the RTT realisation. We will only use the RTT formulation, for readers interested in the other realisations see for example [29–31]. For further details regarding Yangians in the RTT framework see [32, 33].

3.1 The RTT Formulation

To describe the RTT formalism we first need to introduce rational R-matrices. While rational R-matrices can be constructed for different representations we will for the sake of this chapter only need those associated with the fundamental representation $L(\omega_1)$. Rational R-matrices associated to $L(\omega_1)$ are rational functions of one parameter u, the spectral parameter, and acts as $R(u) : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$. They satisfy the Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)\,, \tag{3.1}$$

and are invariant under the action of $\mathfrak{g}$

$$[x^a \otimes \mathbb{1} + \mathbb{1} \otimes x^a, R_{12}] = 0\,, \tag{3.2}$$

with x^a generators in the representation $L(\omega_1)$. The notation R_{ab} means that the R-matrix acts in space a and b. We will construct explicit R-matrices later in this section.

To describe the relations that defines $\mathscr{Y}(\mathfrak{g})$ we introduce a generating function T known as the monodromy matrix

$$T = \sum_{n=0}^{\infty} \frac{\hbar^n}{u^n} (t_{(n)})^i{}_j \otimes e_i{}^j\,, \qquad (t_0)^i{}_j = \delta^i_j \mathbb{1}\,. \tag{3.3}$$

where $(t_{(n)})^i{}_j$ eventually will generate the Yangian. Here $\hbar$ is a new book-keeping parameter. It is possible to set $\hbar = 1$, which is common in the maths literature, or $\hbar = \mathtt{i}$ which is more common in the physics literature, we will keep it arbitrary for now. In the context of integrability, the space on which $e_i{}^j$ acts is called the auxiliary space and the space on which $(t_n)^i{}_j$ acts is called the physical space. Using the R-matrix and the monodromy matrix we can now write down the RTT-relation

$$R_{12}(u-v)T_1(u)T_2(v) - T_2(v)T_1(u)R_{12}(u-v) = 0\,. \tag{3.4}$$

The (extended) Yangian $\mathscr{Y}(\mathfrak{g})$ is the algebra generated by polynomials in $(t_{(n)})^i{}_j$ subject to (3.4). We will now proceed to find the R-matrices.

3.1.1 $\mathscr{Y}(\mathfrak{gl}_n)$

Our first observation is that since $L(\omega_1)\otimes L(\omega_1) = L(2\omega_1)\oplus L(\omega_2)$ and R commutes with the action of $\mathfrak{g}$ it follows that R must be written as $R = a(u)\mathbb{P}_S + b(u)\mathbb{P}_A$. $\mathbb{P}_S$ and $\mathbb{P}_A$ are projectors onto the symmetric respectively antisymmetric representation while $a(u)$ and $b(u)$ are rational functions of u. These projectors are readily written using the permutation operator $\mathbb{P}$. Using explicit matrices we have

$$\mathbb{P} = e_i{}^j \otimes e_j{}^i\,, \qquad\qquad \mathbb{P}\, e_i{}^j \otimes e_k{}^l = e_k{}^j \otimes e_i{}^l\,, \tag{3.5}$$

and $\mathbb{P}_S = \frac{1}{2}(\mathbb{1}+\mathbb{P}), \mathbb{P}_A = \frac{1}{2}(\mathbb{1}-\mathbb{P})$. To fix $a(u)$ and $b(u)$ one now requires that R also satisfy Yang-Baxter. The result is explicitly

$$R = \mathbb{1} - \frac{\hbar}{u}\mathbb{P} = \frac{(u-\hbar)}{u}\mathbb{P}_S + \frac{(u+\hbar)}{u}\mathbb{P}_A\,. \tag{3.6}$$

Having found the R-matrix the Yangian algebra follows from (3.4):

$$[T^i{}_j(u), T^k{}_l(v)] = \frac{\hbar}{u-v}\Big(T^k{}_j(u)T^i{}_l(v) - T^k{}_j(v)T^i{}_l(u)\Big)\,. \tag{3.7}$$

This can also be written as a relation for the operators $t_{(n)}$ as

$$\begin{aligned}{}[(t_{(m+1)})^i{}_j, (t_{(n)})^k{}_l] &- [(t_{(m)})^i{}_j, (t_{(n+1)})^k{}_l] \\ &= (t_{(m)})^k{}_j (t_{(n)})^i{}_l - (t_{(n)})^k{}_j (t_{(m)})^i{}_l\,.\end{aligned} \tag{3.8}$$

Note that the choice $(t_0)^i{}_j = \delta^i_j$ made in 3.3 implies that

$$[(t_{(1)})^i{}_j, (t_{(1)})^k{}_l] = \delta^k_j (t_1)^i{}_l - \delta^i_l (t_{(1)})^k{}_j\,, \tag{3.9}$$

and so $(t_{(1)})^i{}_j$ satisfy the $\mathfrak{gl}_n$ algebra.

3.1.2 B, C and D

It is possible to give a unified treatment of $\mathscr{Y}(\mathfrak{g})$ with $\mathfrak{g} = B, C, D$ using the RTT-formalism. The basic observation is that $L(\omega_1) \otimes L(\omega_1)$ contains 3 terms, the R-matrix associated to $L(\omega_1)$ must then be written as a sum of 3 operators.

To explicitly write down the R-matrix we use the fact, as explained in Section 2.1.1, that B_r, C_r and D_r can be obtained from $\mathfrak{gl}_n$ after introducing the metric g_{ij}. This allows us to work with matrices $e_i{}^j, i = 1, \dots n$.

In addition to the permutation operator $\mathbb{P}$ we also need the "trace operator" $\mathbb{K}$

$$\mathbb{K} = g^{ik} g_{lj} e_i{}^j \otimes e_k{}^l \,, \qquad \mathbb{K}\mathbb{K} = n\mathbb{K} \,. \tag{3.10a}$$

The R-matrix is once again obtained by taking an ansatz $R = \sum_\lambda \tau(u) \mathbb{P}^\lambda_{\omega_1 \otimes \omega_1}$ with $\tau(u)$ a rational function of u and $\mathbb{P}$ projectors onto irreps. Solving Yang-Baxter gives

$$R = \mathbb{1} - \frac{\hbar}{u}\mathbb{P} + \frac{\hbar}{u^{[-\beta h^\vee]}}\mathbb{K} \,, \tag{3.11}$$

where $h^\vee$ is the dual Coxeter number, it is tabulated in table 2.1 and $\beta = 1$ for D_r and B_r while $\beta = 2$ for C_r. We have also used standard notation $f^{[n]} = f(u + n\frac{\hbar}{2})$.

The RTT relations become explicitly

$$\begin{aligned} [T^i{}_j(u), T^k{}_l(v)] = {} & \frac{\hbar}{u-v}\Big(T^k{}_j(u) T^i{}_l(v) - T^k{}_j(v) T^i{}_l(u)\Big) \\ & - \frac{\hbar}{u - v - \beta h^\vee \frac{\hbar}{2}} \Big(g^{ki} g_{mn} T^m{}_j(u) T^n{}_l(v) - g_{lj} g^{mn} T^k{}_n(v) T^i{}_m(u)\Big) \,. \end{aligned} \tag{3.12}$$

It is once again possible to verify that $(t_{(1)})^i{}_j$ satisfy the commutation relations of BCD. To see this one focus on the term $\frac{\hbar^2}{uv}$, this reproduces exactly the second line of (2.6) up to some cosmetics due to the index structure of T.

Strictly speaking, the Yangian algebra (3.12) gives the extended Yangian $\mathscr{X}(\mathfrak{g})$ and not the Yangian. The reason is due to the presence of a large centre, see for example [34]. This will not play a part in our discussion.

3.1.3 $\mathscr{Y}(\mathfrak{gl}_{m|n})$

We can also formulate supersymmetric Yangians for $\mathfrak{gl}_{m|n}$ using the RTT formulation [35]. The description turns out to be very similar to $\mathscr{Y}(\mathfrak{gl}_n)$, the R-matrix is given as

$$R = \mathbb{1} - \frac{\hbar}{u}\mathbb{P} \,, \qquad \mathbb{P} = \sum_{i,j}^{n} (-1)^{|j|} e_i{}^j \otimes e_j{}^i \,, \tag{3.13}$$

where $\mathbb{P}$ is now the graded permutation operator. When verifying that (3.13) satisfies Yang-Baxter it is imperative to use the correct conventions of tensor products suitable for super Lie algebras. Explicitly one needs

$$(x \otimes y)(z \otimes w) = (-1)^{|y||z|} xy \otimes yw\,, \tag{3.14}$$

which takes the grading into account. The extension to $\mathfrak{osp}(m|n)$ is a straightforward generalisation of the *BCD* Yangian, see [36].

3.1.4 The transfer matrix

In our study of Lie algebras we encountered the Cartan algebra. While useful the spectrum of $\mathfrak{h}$ is rather dull. Luckily, in the Yangian there exist more interesting commutative algebras. A particularly interesting one is obtained from the so-called transfer matrix defined as

$$\mathbb{T}(u) = \mathrm{tr}_a\, T(u)\,. \tag{3.15}$$

where the subscript a indicates that we are to take the trace over the auxiliary space, not the physical space. $\mathbb{T}$ is important because it commutes with itself for different values of the spectral parameter,

$$[\mathbb{T}(u), \mathbb{T}(v)] = 0\,. \tag{3.16}$$

This means that all elements in the series $\mathbb{T}(u) = \sum_n \frac{\hbar^n}{u^n} \mathbb{t}_{(n)}$ commutes. To obtain (3.16) one traces the RTT relation (3.4) over both auxiliary spaces.

3.2 Representation Theory and Drinfeld Polynomials

With the definition of $\mathscr{Y}(\mathfrak{g})$ taken care of we proceed to study the structure emerging from the Yangian algebra. While certainly not obvious from the RTT-relations used to define it, the Yangian has a surprisingly rich representation theory [37]. Many of its features are similar to those of Lie algebras, but they generalise in non-trivial ways.

The main outcome of this section is the notion of Drinfeld polynomials [28]. They are polynomials describing finite-dimensional representations of Yangians, thus playing the role of Dynkin labels in this new set-up. The Drinfeld polynomials will come to play a crucial part in our study of rational spin chains in Part II.

3.2.1 Representation theory of $\mathscr{Y}(\mathfrak{gl}_n)$

Just as in our discussion of Lie algebras, we start by considering $\mathfrak{g} = \mathfrak{gl}_n$. A representation of $\mathscr{Y}(\mathfrak{gl}_n)$ is a highest weight representation if there exist a state

$|\lambda\rangle$ such that

$$T^i{}_j\,|\lambda\rangle = 0\,, \quad i > j\,, \qquad T^i{}_i\,|\lambda\rangle = \lambda_i(u)\,|\lambda\rangle\,, \quad (\text{no sum over i})\,. \tag{3.17}$$

This clearly mirrors the Lie-algebra definition, recall that $|\lambda\rangle$ is a highest weight state of $\mathfrak{gl}_n$ if it satisfy $E_i{}^j\,|\lambda\rangle = 0, E_i{}^i\,|\lambda\rangle = \lambda_i\,|\lambda\rangle$. For a Lie algebra representation to be finite-dimensional we need $\Lambda_i = \lambda_i - \lambda_{i+1} \in \mathbb{N}_{\geq 0}$ as explained in Chapter 2. The analogue statement for $\mathscr{Y}(\mathfrak{gl}_n)$ is that a representation is finite-dimensional if, and only if, it is a highest weight representation and there exists a polynomial P_a such that

$$\frac{P_a(u+\frac{\hbar}{2})}{P_a(u-\frac{\hbar}{2})} = \frac{\lambda_{a+1}(u)}{\lambda_a(u)}\,. \tag{3.18}$$

The polynomials P_a are known as Drinfeld polynomials.

Let us construct a Drinfeld polynomial explicitly to see how this works in practice. Consider the so-called Lax-matrix

$$L_{1,a}(u,\theta) = \mathbb{1} - \frac{\hbar}{(u-\theta)}\,E_i{}^j \otimes e_j{}^i\,. \tag{3.19}$$

The Lax matrix is nothing but a renormalised and shifted version of the R-matrix, it satisfies RTT as a consequence of the Yang-Baxter equation for R. It follows that we can use the Lax matrix as a monodromy matrix. To be precise we have

$$T^i{}_j = \delta^i_j\,\mathbb{1} - \frac{\hbar}{(u-\theta)} E_j{}^i\,. \tag{3.20}$$

We will now consider a finite-dimensional representation of $\mathfrak{gl}_n$. Introduce the $\mathfrak{gl}_n$ highest weight vector $|\lambda\rangle$. It satisfies $E_i{}^j\,|\lambda\rangle = 0, i < j$ and $E_i{}^i\,|\lambda\rangle = \lambda_i\,|\lambda\rangle$ where we do not sum over i and $\lambda_i - \lambda_{i+1} = \Lambda_i \in \mathbb{Z}_{\geq 0}$. Clearly $|\lambda\rangle$ is a HWS of $\mathscr{Y}(\mathfrak{gl}_n)$, we find

$$\lambda_i(u) = 1 - \frac{\hbar}{(u-\theta)}\lambda_i\,, \qquad P_i = \Big((u-\theta-\hbar\lambda_i)^{[\Lambda_i]_D}\Big)^{[\Lambda_i]}\,, \tag{3.21}$$

where we recall that $f^{[n]} = f(u+n\frac{\hbar}{2})$ and we have introduced the fused product

$$f^{[a]_D} = \begin{cases} f^{[a-1]}\dots f^{[-a+1]} & a \geq 1 \\ \frac{1}{f^{[a-1]}\dots f^{[-a+1]}} & a \leq -1 \end{cases}\,. \tag{3.22}$$

3.2.2 Finite-dimensional representations of $\mathscr{Y}(\mathfrak{g})$

When discussing representation theory of BCD it is very useful to pick an off-diagonal form of g_{ij}, we fix $g_{ij} = \theta_i\,\delta_{i+j,n+1}$. Common choices for θ are $\theta_i = 1$ for $\mathfrak{g} = B_r, D_r$ and $\theta_i = \text{sign}(r-i)$ if $\mathfrak{g} = C_r$.

A highest weight state of $\mathscr{Y}(\mathfrak{g})$ is a state $|\lambda\rangle$ such that

$$T^i{}_j\,|\lambda\rangle = 0 \quad i > j\,, \qquad T^i{}_i\,|\lambda\rangle = \lambda_i(u)\,|\lambda\rangle\;, \qquad \text{(no sum)}\,. \tag{3.23}$$

Furthermore, the representation is finite-dimensional if and only if there exist polynomials P_a such that [38]

$$\frac{P_a^+}{P_a^-} = \frac{\lambda_{a+1}(u)}{\lambda_a(u)}\,, \quad a < r\,, \tag{3.24}$$

$$\frac{P_r^+}{P_r^-} = \frac{\lambda_{r+1}(u)}{\lambda_{r-1}(u)}\,, \mathfrak{g} = D_r\,, \quad \frac{P_r^{[\frac{1}{2}]}}{P_r^{[-\frac{1}{2}]}} = \frac{\lambda_{r+1}(u)}{\lambda_r(u)}\,, \mathfrak{g} = B_r\,, \quad \frac{P_r^{[2]}}{P_r^{[-2]}} = \frac{\lambda_{r+1}(u)}{\lambda_r(u)}\,, \mathfrak{g} = C_r\,. \tag{3.25}$$

Finite dimensional representations of $\mathscr{Y}(\mathfrak{g})$ with $\mathfrak{g} = A_r, B_r, C_r$ or D_r are then also characterised by a set of r Drinfeld polynomials $P_a(u)$.

The simplest modules of the Yangians are those with minimal degree Drinfeld polynomials, that is

$$P_a = (u-\theta)\,, \qquad\qquad P_{b\neq a} = 1\,, \tag{3.26}$$

for some $a = 1,\ldots,r$. The modules described by (3.26) are the fundamental modules of the Yangian algebra, we will denote them as $W_{a,1}(\theta)$ where the index refers to the fact that $W_{a,1}$, when restricted to the underlying Lie algebra $\mathfrak{g}$, always contains $L(\omega_a)$. The modules $W_{a,1}(\theta)$ are special cases of $W_{a,s}(\theta)$, so-called Kirillov-Reshetikhin modules. These are the natural generalisation of $L(s\omega_a)$ for Yangians and they have Drinfeld polynomials

$$W_{a,s}(\theta):\,, \qquad P_a = (u-\theta)^{[\frac{s}{t_a}]_D}\,, \qquad P_{b\neq a} = 1\,, \tag{3.27}$$

with $t_a = \frac{2}{(\alpha_a,\alpha_a)}$. $W_{a,s}$ always contain $L(s\omega_a)$ but is in general a reducible representation of $\mathfrak{g}$. For us, the most important example will be D_r, in this case the fundamental KR modules decompose as

$$\mathfrak{g} = D_r\,, \quad W_{a,1} = L(\omega_a) + L(\omega_{a-2}) + L(\omega_{a-4}) + \ldots\,, \quad a \neq r-1, r\,, \tag{3.28a}$$

$$W_{a,1} = L(\omega_a)\,, a = r, r-1\,. \tag{3.28b}$$

A very observant reader will notice the similarity between (3.28) and (2.29).

4. Analytic Bethe Ansatz

In this chapter, we put focus on the transfer matrix $\mathbb{T}$. We will seek out its spectrum for various choices of auxiliary space, finding that these matrices are related to each other through a system of relations known as a T-system. Essentially the T-system is a system of equations generalising relations among characters of $\mathfrak{g}$. As we will see the analogue with character can be taken even further; the eigenvalues of transfer matrices can naturally be written using a quantum generalisation of eigenvalues [39]. These quantum eigenvalues are expressed as rational combinations of a set of polynomial $\mathbb{q}$-functions satisfying a set of equations known as Bethe equations.

Section 4.1 discusses transfer matrices and T-functions for $\mathfrak{gl}_2$. We derive Hirota bilinear equations, the $\mathfrak{gl}_2$ T-system, which we then proceed to solve using so-called Q-functions. The Q-functions are then used to find the spectrum of $\mathbb{T}$ after using a variation of the analytic Bethe ansatz [40]. Section 4.2 treats the generalisation to *ABCD*. Finally, we present some outlook and comments on the many areas we have not covered in Section 4.3.

An excellent review of T-systems is found in [41].

4.1 Transfer Matrices and T-Functions for $\mathscr{Y}(\mathfrak{gl}_2)$

Our goal in this section is to study transfer matrices of $\mathfrak{gl}_2$ when the representation in the auxiliary space is a symmetric power of the defining representation. We will consider the physical space to be $\mathscr{H} = \bigotimes^L L(\omega_1)$, this is only a technical convenience. We will from now on refer to the physical space as a spin chain.

The reason we set out on this quest is to find functional relations between transfer matrices. That such relations can exist is hinted at by the underlying Lie algebra. We know that $\chi(\omega_1)^2 = \chi(2\omega_1) + \chi(\omega_0)$ for $\mathfrak{sl}_n$, we are now looking for the equivalent statement for the Yangian.

Our first task is to explicitly construct transfer matrices with $L(s\omega_1)$ as auxiliary space [1]. Luckily, it is a simple generalisation of the rational R-matrix

$$L^{(s)}_{1a} = (u + s\frac{\hbar}{2})\mathbb{1} - \hbar\, e_i{}^j \otimes \Pi_s(E_j{}^i)\,, \tag{4.1}$$

[1] We will use $\mathfrak{sl}_2$ notation for convenience

where $\Pi_s(E_i{}^j)$ acts on $L(s\omega_1)$. Practically we can realize (4.1) by embedding the Lax matrix into a $2^{s+1}\times 2^{s+1}$ dimensional matrix as

$$L^{(s)}_{1a} = \mathbb{P}_{s\omega_1}\left((u+s\frac{\hbar}{2})\mathbb{1} - \hbar\, e_i{}^j \otimes \Delta^s(e_j{}^i)\right)\mathbb{P}_{s\omega_1}\,, \tag{4.2}$$

with $\mathbb{P}_{s\omega_1}$ the projector onto $L([s])$ and $\Delta^n(x) = \sum_{i=1}^n \underbrace{\mathbb{1}\otimes\ldots\mathbb{1}}_{i-1}\otimes x\otimes\mathbb{1}$.

To now construct the transfer-matrix acting on $\mathscr{H} = \bigotimes^L L(\omega_1)$ we simply multiply L Lax matrices together to form

$$\mathbb{T}_s(u) = \mathrm{tr}_a\, L^{(s)}_{1,a}(u)L^{(s)}_{2,a}(u)\ldots L^{(s)}_{L,a}(u)\,. \tag{4.3}$$

As advertised at the beginning of this Section we will now generalize the character formula $\chi(\omega_1)^2 = \chi(2\omega_1)+\chi(\omega_0)$ from χ to $\mathbb{T}_s$. The non-trivial part is that we need to take the spectral parameter into account.

The correct object to study is $\mathbb{T}_1^+\mathbb{T}_1^-$. The reason for this is because at $u=\hbar$ the R-matrix degenerates into a projector giving

$$\mathbb{P}_S\,(L_1^{(1)})^-(L_2^{(1)})^+ = (L_2^{(1)})^+(L_1^{(1)})^-\,\mathbb{P}_S\,, \tag{4.4}$$

$$\mathbb{P}_A\,(L_1^{(1)})^+(L_2^{(1)})^- = (L_2^{(1)})^-(L_1^{(1)})^+\,\mathbb{P}_A\,. \tag{4.5}$$

where we recall that we are using notation $f^\pm = f(u\pm\frac{\hbar}{2})$. This allows us to compute

$$\begin{aligned}\mathbb{T}_1^-\mathbb{T}_1^+ = \mathrm{tr}_a\,T_a^-\,\mathrm{tr}_b\,T_b^+ &= \mathrm{tr}_{a,b}\left(\underbrace{(\mathbb{P}_S+\mathbb{P}_A)}_{\mathbb{1}}T_a^-T_b^+\right)\\ &= \mathrm{tr}_{a,b}(\mathbb{P}_S(T_a^-T_b^+)\mathbb{P}_S) + \mathrm{tr}_{a,b}(\mathbb{P}_A(T_a^-T_b^+)\mathbb{P}_A)\,,\end{aligned} \tag{4.6}$$

with $\mathbb{P}$ acting in the auxiliary spaces a,b. This result can be improved upon by using

$$\mathbb{P}_S\,(L_{1a}^{(1)})^-(L_{1b}^{(1)})^+\,\mathbb{P}_S = u\,L^{(2)}_{1,ab}\,, \tag{4.7a}$$

$$\mathbb{P}_A(L_{1a}^{(1)})^+(L_{1a}^{(1)})^-\mathbb{P}_A = (u^2-\hbar^2)\mathbb{P}_A\,, \tag{4.7b}$$

which allows us to state the final result as

$$\mathbb{T}_1^+\mathbb{T}_1^- = u^L\,\mathbb{T}_2 + (u+\hbar)^L(u-\hbar)^L\,. \tag{4.8}$$

With more effort, these relations can be generalised to arbitrary s as

$$\mathbb{T}_s^+\mathbb{T}_s^- = \mathbb{T}_{s+1}\mathbb{T}_{s-1} + (u^{[s+1]})^L(u^{[-s-1]})^L\,, \tag{4.9}$$

with boundary conditions $\mathbb{T}_0 = u^L$. These are the functional equations we sought.

4.1.1 Solving $\mathfrak{gl}_2$ Hirota

We now proceed to solve (4.9). We start by simplifying the equations slightly, let us introduce a function $T_s = \sigma^{[s+1]}\sigma^{[-s-1]}\mathbb{T}_s$. If we require

$$\frac{1}{\sigma^+\sigma^-} = u^L\,, \qquad\qquad \sigma = \left(\frac{1}{\sqrt{2\hbar}}\frac{\Gamma(\frac{u}{2\hbar}+\frac{1}{4})}{\Gamma(\frac{u}{2\hbar}+\frac{3}{4})}\right)^L\,, \tag{4.10}$$

then T satisfy the more compact equation

$$T_s^+T_s^- = T_{s+1}T_{s-1}+1\,. \tag{4.11}$$

We will refer to (4.11) as the Hirota equation. We remark that solving for σ explicitly as done in (4.10) is usually unnecessary, the vital piece of information is the functional relation.

The solution of (4.11) turns out to be remarkably simple, it is

$$T_s = \begin{vmatrix} Q_1^{[s+1]} & Q_1^{[-s-1]} \\ Q_2^{[s+1]} & Q_2^{[-s-1]} \end{vmatrix}\,, \qquad\qquad T_0 = \begin{vmatrix} Q_1^+ & Q_1^- \\ Q_2^+ & Q_2^- \end{vmatrix} = 1\,, \tag{4.12}$$

where $Q_a(u)$ are two functions of the spectral parameter. To verify this claim we write the expression for T_s using index notation as $T_s = \varepsilon^{ab}Q_a^{[s+1]}Q_b^{[-s-1]}$. Here ε^{ab} is the Levi-Civita symbol, it satisfies in particular the following Plücker identity

$$\varepsilon^{ab}\varepsilon^{cd} = \varepsilon^{ac}\varepsilon^{bd} + \varepsilon^{ad}\varepsilon^{cb}\,. \tag{4.13}$$

Using this we find

$$\begin{aligned} T_s^+T_s^- &= \varepsilon^{ab}\varepsilon^{cd}Q_a^{[s+2]}Q_b^{[-s]}Q_c^{[s]}Q_d^{[-s-2]} \\ &= \varepsilon^{ad}Q_a^{[s+2]}Q_d^{[-s-2]}\varepsilon^{cb}Q_c^{[s]}Q_b^{[-s]} + \varepsilon^{ac}Q_a^{[s+2]}Q_c^{[s]}\varepsilon^{bd}Q_b^{[-s]}Q_d^{[-s-2]} \\ &= T_{s+1}T_{s-1}+1\,, \end{aligned} \tag{4.14}$$

which shows (4.9).

4.1.2 QQ-relations and the spectrum of $\mathbb{T}$

Having solved the functional equations we now attack the problem of describing the spectrum of $\mathbb{T}_s$. The Hirota equations (4.9) are very general relations, they will appear in a variety of very different settings. To go back to the case of the $\mathscr{Y}(\mathfrak{gl}_2)$ need to impose appropriate analytic properties on the two functions Q_1, Q_2. This is the spirit of the analytic Bethe ansatz [40]. Guided by the fact that $\mathbb{T}_s$ must be a polynomial we now introduce the analytic ansatz

$$Q_a = \sigma\, q_a\,, \tag{4.15}$$

where q_a are polynomials in u. Using (4.10) they need to satisfy

$$W(q_1,q_2) = u^L\,. \tag{4.16}$$

Here

$$W(Q_1,Q_2) = \begin{vmatrix} Q_1^+ & Q_1^- \\ Q_2^+ & Q_2^- \end{vmatrix}\,, \tag{4.17}$$

is the discrete Wronskian, sometimes called Casarotian.

We must at this stage understand the spectrum of $\mathbb{T}_s$ slightly better. Recall that we are studying a physical space $\mathscr{H} = \bigotimes^L L(\omega_1)$. An important property of $\mathbb{T}$ is that it commutes with $\mathfrak{gl}_2$, thus $\mathbb{T}$ takes the same value on every state in a $\mathfrak{gl}_2$ irrep inside $\mathscr{H}$. Let us write $\mathscr{H} = \sum_{M=0}^{[\frac{L}{2}]} c_a L([L-2M])$. To calculate the eigenvalue of $\mathbb{T}_s$ on $L([L-2M])$ we set $\deg(q) = M$, or in other words, the degree of q_1 counts the number of times we have to use α_1 to go from the vacuum weight $n\omega_1$ to the state under consideration. This should give c_a number of solutions. Putting this on Mathematica one confirms that the number of solutions works out. We will in this thesis always rely on experimental evidence when discussing the number of solutions, for a rigorous discussion for $\mathfrak{gl}_n$ (and $\mathfrak{gl}_{m|n}$) see [42–46].

There exist many other methods to solve for the spectrum of $\mathbb{T}_s$, most will at one point boil down to solving a set of rational equations known as Bethe equations. Explicitly they read

$$\prod_{\substack{j=1\\ j\neq i}}^{M} \frac{u_i - u_j + \hbar}{u_i - u_h - \hbar} = \left(\frac{u_i + \frac{\hbar}{2}}{u_i - \frac{\hbar}{2}}\right)^L\,. \tag{4.18}$$

We end this section with a comment on how to find the Bethe equations from (4.16). First parameterise $q_1 = \prod_{i=1}^{M}(u-u_i)$. One simply shifts the Wronskian relation two different ways, evaluates on zeros of q_1 and divides the resulting equations. Explicitly

$$\begin{cases} q_1^{[2]} q_2 - q_1 q_2^{[2]} = (u^+)^L\,, \\ q_1 q_2^{[-2]} - q_1^{[-2]} q_2 = (u^-)^L\,, \end{cases} \quad \Longrightarrow \quad \left.\frac{q_1^{[2]}}{q_1^{[-2]}}\right|_{q_1=0} = -\left.\left(\frac{u+\frac{\hbar}{2}}{u-\frac{\hbar}{2}}\right)^L\right|_{q_1=0}\,. \tag{4.19}$$

4.1.3 Classical and quantum eigenvalues

In the previous subsection, we discussed how the spectrum of $\mathbb{T}_s$ can be obtained by solving QQ-relations. In Chapter 2 we discussed the spectrum of the Cartan subalgebra which, for example, can be described using characters. The remaining part of this subsection will be dedicated to answering the question:

Can we find a deformed version of classical characters that help us understand T_s? The answer is, happily, confirmatory.

To this end we recall the character of $\mathfrak{gl}_n$ with highest weight $\{\lambda_1, \lambda_2\} = \{s, 0\}$:

$$\chi_s = \sum_{n=0}^{s} \mathrm{x}_1^{s-n} \mathrm{x}_2^n = \mathrm{x}_1^s + \mathrm{x}_1^{s-1}\mathrm{x}_2 + \ldots \mathrm{x}_2^s = \frac{\mathrm{x}_1^{s+1} - \mathrm{x}_2^{s+1}}{\mathrm{x}_1 - \mathrm{x}_2} . \tag{4.20}$$

These characters can be economically packaged into a generating function

$$\frac{1}{1 - t\mathrm{x}_1} \frac{1}{1 - t\mathrm{x}_2} = \sum_{s=0}^{\infty} t^s \chi_s . \tag{4.21}$$

In order to find an analogue of (4.20) and (4.21) our first objective is to eliminate Q_2 from (4.12), this can be accomplished using the QQ-relation (4.17):

$$\begin{aligned} T_s &= Q_1^{[s+1]} Q_1^{[-s-1]} \left(\frac{Q_2^{[-s-1]}}{Q_1^{[-s-1]}} - \frac{Q_2^{[+s+1]}}{Q_1^{[+s+1]}} \right) \\ &= \sum_{n=0}^{s} \left(\frac{Q_1^{[s-n]}}{Q_1^{[-s+n]}} \right)^{[n+1]} \left(\frac{Q_1^{[-n]}}{Q_1^{[n]}} \right)^{[n-s-1]} . \end{aligned} \tag{4.22}$$

This equation already shows clear similarities with (4.20), to get even closer we introduce so-called quantum eigenvalues. They are defined as

$$\Lambda_1 = \frac{Q_1^{[2]}}{Q_1^-} , \qquad \Lambda_2 = \frac{Q_1^{[-2]}}{Q_1} , \tag{4.23}$$

using which (4.22) reads

$$T_s = \sum_{n=0}^{s} (\Lambda_1^{[s-n]_D})^{[n]} (\Lambda_2^{[n]_D})^{[n-s]} . \tag{4.24}$$

A couple of examples are

$$T_1 = \frac{Q_1^{[2]}}{Q_1} + \frac{Q_1^{[-2]}}{Q_1} = \Lambda_1 + \Lambda_2 , \quad T_2 = \Lambda_1^+ \Lambda_1^- + \Lambda_1^+ \Lambda_2^- + \Lambda_2^+ \Lambda_2^- . \tag{4.25}$$

It is of course cumbersome to carry all the additional labels around. To avoid this extracurricular activity one can use the shift operator D with property $Df(u) = f(u + \frac{\hbar}{2})D, f(u)D = Df(u - \frac{\hbar}{2})$ and introduce an operatorial version of the quantum eigenvalues

$$\hat{\Lambda}_1 = D\Lambda_1 D , \qquad \hat{\Lambda}_2 = D\Lambda_2 D . \tag{4.26}$$

We have the following compact formulas

$$\frac{1}{1 - \hat{\Lambda}_2} \frac{1}{1 - \hat{\Lambda}_1} = \sum_{s=0}^{\infty} D^s T_s D^s , \tag{4.27}$$

which provides us with a generalisation of (4.21).

4.2 T-Systems and Bethe Equations

In principle, one could now try to repeat the analysis presented above for the other Lie-algebras we have discussed so far. This task has been completed, for RSOS models, for A_n [47, 48]. Bypassing explicit constructions a u-independent T-system was found in [49] and upgraded into a full-fledged T-system in [50] using a "Baxterisation" process.

The resulting functional equations take a uniform shape for algebras where all roots have the same length, these are known as simply-laced algebras. The T-system becomes

$$T^{+}_{a,s}T^{-}_{a,s} = T_{a+1,s}T_{a,s-1} + \prod_{b:I_{ab}=1} T_{b,s}\,, \tag{4.28}$$

where I_{ab} is the incidence matrix of $\mathfrak{g}$. We now have two labels a,s to specify the auxiliary channel, these labels describe the KR module $W_{a,s}(u)$ of $\mathscr{Y}(\mathfrak{g})$.

In this thesis the only non-simply laced example we will consider is $B_2 \simeq C_2$, the T-system for this algebra is,

$$T^{[-2]}_{1,s}T^{[2]}_{1,s} = T_{1,s-1}T_{1,s+1} + T_{1,2s}\,, \tag{4.29a}$$

$$T^{-}_{2,2s}T^{+}_{2,2s} = T_{2,2s-1}T_{2,2s+1} + T^{-}_{1,s}T^{+}_{1,s}\,, \tag{4.29b}$$

$$T^{-}_{2,2s+1}T^{+}_{2,2s+1} = T_{2,2s}T_{2,2s+2} + T_{1,s}T_{1,s+1}\,. \tag{4.29c}$$

All other T-systems can be found in [41].

In Section 4.1 we solved the T-system using Q-functions. It is also possible to solve it using only the fundamental T-functions, $T_{a,1}$, such formulas were obtained for A_r in [51] and for BCD in [52].

4.2.1 Bethe equations

The diagonalization of T-functions can be reformulated in terms of Bethe equations. These equations take a universal form in terms of the data defining the underlying Lie algebra [53]. To describe the equations we need the symmetrized Cartan matrix $B_{ab} = (\alpha_a, \alpha_b)$. Let $\mathbb{q}_a = \prod_{i=1}^{M_{a|i}}(u-u_{a,i})$, then the Bethe equations for a compact rational spin chain are given as [51, 54, 55]

$$\prod_{b=1}^{r} \frac{\mathbb{q}_b^{[B_{ab}]}}{\mathbb{q}_b^{[-B_{ab}]}}\Bigg|_{\mathbb{q}_a=0} = -\,\frac{P_a^{[\frac{1}{t_a}]}}{P_a^{[-\frac{1}{t_a}]}}\Bigg|_{\mathbb{q}_a=0}\,, \qquad t_a = \frac{2}{(\alpha_a,\alpha_a)}\,, \tag{4.30}$$

with P_a Drinfeld polynomials.

In the case of simply-laced algebras $B_{ab} = C_{ab}$. In particular, we have the well-known A_r equations

$$\frac{\mathbb{q}_a^{[2]}\mathbb{q}_{a-1}^{[-1]}\mathbb{q}_{a+1}^{[-1]}}{\mathbb{q}_a^{[-2]}\mathbb{q}_{a-1}^{[+1]}\mathbb{q}_{a+1}^{[+1]}}\Bigg|_{\mathbb{q}_a=0} = -\,\frac{P_a^{+}}{P_a^{-}}\Bigg|_{\mathbb{q}_a=0}\,, \qquad \mathfrak{g} = A_r\,. \tag{4.31}$$

For clarity, we also write down the non-simply laced case C_2

$$\left.\frac{\mathbb{q}_1^{[2]}}{\mathbb{q}_1^{[-2]}}\frac{\mathbb{q}_2^{[-2]}}{\mathbb{q}_2^{[2]}}\right|_{\mathbb{q}_1=0} = -\left.\frac{P_1^+}{P_1^-}\right|_{\mathbb{q}_1=0}, \qquad \left.\frac{\mathbb{q}_2^{[4]}}{\mathbb{q}_2^{[-4]}}\frac{\mathbb{q}_1^{[-2]}}{\mathbb{q}_1^{[2]}}\right|_{\mathbb{q}_2=0} = -\left.\frac{P_2^{[2]}}{P_2^{[-2]}}\right|_{\mathbb{q}_2=0}. \tag{4.32}$$

To find the spectrum of $\mathbb{T}_{a,s}$ it is possible to generalise the idea of quantum eigenvalues from $\mathfrak{gl}_2$ to $\mathfrak{g}$. To write compact formulas we introduce $\mathbb{Q}_a$ defined to satisfy the same Bethe equations (4.30) as $\mathbb{q}_a$ but with -1 on the RHS.

Let us now list the resulting expressions for $\mathbb{T}_{1,1}$ for A_r and C_2 [11,40,56].

$\mathfrak{g} = A_r$

Introduce $\Lambda_a = \left(\frac{\mathbb{Q}_a^{[+]}}{\mathbb{Q}_a^-}\frac{\mathbb{Q}_{a-1}^{[-2]}}{\mathbb{Q}_{a-1}}\right)^{[\frac{r+3}{2}-a]}$ and $\mathbb{Q}_{r+1} = \mathbb{Q}_0 = 1$, then

$$\mathbb{T}_{1,1} \propto \sum_{a=1}^{r+1} \Lambda_a\,. \tag{4.33}$$

The case of $\mathfrak{g} = C_2$

$$\Lambda_1 = \frac{\mathbb{Q}_1^{[4]}}{\mathbb{Q}_1^{[2]}}, \quad \Lambda_2 = \frac{\mathbb{Q}_2^{[4]}}{\mathbb{Q}_2}\frac{\mathbb{Q}_1}{\mathbb{Q}_1^{[2]}}, \quad \Lambda_{\bar{2}} = \frac{\mathbb{Q}_1}{\mathbb{Q}_1^{[-2]}}\frac{\mathbb{Q}_2^{[-4]}}{\mathbb{Q}_2}, \quad \Lambda_{\bar{1}} = \frac{\mathbb{Q}_1^{[-4]}}{\mathbb{Q}_1^{[-2]}}. \tag{4.34}$$

One finds

$$\mathbb{T}_{1,1} \propto \Lambda_1 + \Lambda_2 + \Lambda_{\bar{2}} + \Lambda_{\bar{1}}, \tag{4.35}$$

$$\mathbb{T}_{2,1} \propto \Lambda_1^-\Lambda_2^+ + \Lambda_1^-\Lambda_{\bar{2}}^+ + \Lambda_1^-\Lambda_{\bar{1}}^+ + \Lambda_2^-\Lambda_{\bar{1}}^+ + \Lambda_{\bar{2}}^-\Lambda_{\bar{1}}^+\,. \tag{4.36}$$

Pursuing similar formulas as the ones presented for arbitrary $T_{a,s}$, and also more complicated representations, leads to tableaux sum formulas. Expressions for $\mathfrak{gl}_n$ were found in [51] and for *BCD* in [57–60].

4.3 Outlook

We have omitted many details and connections in this short introduction. We have throughout our presentation focused on Q-functions and not Q-operators. Q-operators relevant for $\mathscr{Y}(\mathfrak{gl}_n)$ are studied in [61,62], partial generalisation to *BCD* were obtained in [63,64]. Q-operator also makes an appearance in the study of 4D Chern-Simons-Theory [65,66]. For a study of *ABCDE* in this context see [67,68]. T-systems also arise in many other contexts. For example from the Thermodynamic Bethe Ansatz [69–71] and in wall-crossing and computation of BPS spectra [72,73].

Part II: Algebraic Methods

In this part, we start our study of Q-systems and their use in finding the spectrum of integrable models. The ultimate goal of this part is to try and develop an algebraic framework for Q-systems with different types of symmetries. By understanding the underlying algebraic structure we hope to develop a stable ground from which more difficult problems can be attacked. We will mainly compare our constructions against compact rational spin chains. Such chains are of interest in various settings and serve as excellent tests that we are on the right track.

Chapter 5 describes how the methods of ODE/IM can be used to deduce relations among Q-functions. Chapter 6 serves as a review of $\mathfrak{gl}_n$ Q-systems. We will recall their defining QQ-relations and the Wronskian solution. Using Q-functions we discuss how to solve the $\mathfrak{gl}_n$ T-system and rational spin chains. Chapter 7 consider Q-systems for D_r and is based on Paper II. The generalisation of QQ-relations from $\mathfrak{gl}_n$ is described and we comment on how to construct a covariant formalism. We briefly comment on T-functions and the solution of rational spin chains. An extension of the methods to the case of the non-simply laced algebra $B_2 \simeq C_2$ is considered in Chapter 8. The results of Chapter 8 are new. In Chapter 9 we take a break from Q-functions and consider differential equations on the squashed seven-sphere. Finally, we consider supersymmetric Q-systems in Chapter 10.

5. Q-Functions from ODE/IM

The primary objective of this part is to construct Q-systems like the one found for $\mathfrak{gl}_2$ for other algebras. That is, we want to find a system of Q-functions such that after imposing polynomial constraints naturally generalising $W(q_1, q_2) = 1$ we can reproduce the spectrum of integrable models. Q-systems constructed for non-simply laced algebras are usually more involved. We will therefore restrict attention to simply-laced algebras in this chapter.

Important progress towards finding a systematic way to construct Q-systems came from studying relations between ordinary differential equations (ODE) and integrable models (IM). This led to the so-called ODE/IM correspondence [15, 16], first formulate for A_1 and subsequently generalised to all other simple Lie algebras [74].

In Section 5.1 we recall the basics of ODE/IM in the spirit of [75–77]. We explain how distinguished solutions to a specific linear differential operator are labelled by Lie algebra data and how to find Q-functions. In Section 5.2 we use relations among solutions to the differential operator to show how certain Q-functions are composite objects, built from more fundamental Q-functions associated with basic representations of the underlying Lie algebra. Section 5.3 contains a summary and some further comments on Paper I.

For reviews and further references regarding ODE/IM see [78, 79].

5.1 ODE/IM

In ODE/IM one studies the solutions of very special differential operators. Using the form of [80] a primitive example of such a differential operator is

$$\mathscr{L} = \partial_x + \frac{\ell}{x} + \sum_{a=1}^{r} e^a + q(x,z)e^0 , \quad l \in \mathfrak{h}, q(x,z) = x^{Mh} - z, M \in \mathbb{Z} \tag{5.1}$$

with $e^0 = \lambda E^{-\theta}, \lambda \in \mathbb{C}$, θ the highest root and h the Coxeter number [1].

It was proposed in [75] that to find relations among Q-functions one should consider relations between vectors that solve (5.1). To be more precise, for each fundamental representation of the Lie algebra one constructs solutions singled out by their behaviour at infinity and identifies each component in a distinguished basis with a Q-function. This was further developed and expanded upon in [76, 77] where a system of equations called the Ψ-system was derived.

[1] λ can be interpreted as an evaluation parameter of an affine Lie algebra $\hat{\mathfrak{g}}$.

5.1.1 Solutions and Q-functions

Equations of type (5.1) with $\ell = 0$ and their connections to Q-functions for simply-laced algebras are reviewed and studied in Paper I. Before we explain the technical details let us outline what we will be doing. First, we will define solutions ψ to (5.1), among these we will single out those that have the fastest decay in a particular direction in the complex x-plane. Such solutions are important because they are unique, this is so because we cannot add any other solutions to them since that would alter the asymptotic behaviour. We will find that the behaviour at infinity is controlled by eigenvalues of a matrix Λ and can be described using Lie algebra data. We subsequently return to the origin in the x-plane and define, up to a prefactor, Q-functions as $\psi(0)$. This construction is useful because we can now consider tensor products of solutions in various representations. By finding two different tensor products that contain the same irrep and have the same and the fastest possible decay at infinity these solutions must be the same. This implies non-trivial polynomial identities among Q-functions.

Let us now work out the details. First, let $\psi(x,z)$ be a solution to (5.1) in some representation of the $\mathfrak{g}$, that is

$$\mathscr{L}(x,z,\lambda)\,\psi(x,z) = 0\,. \tag{5.2}$$

Introduce $H^\rho \in \mathfrak{h}$ such that $\alpha(H^\rho) = (\alpha,\rho)$ where $\rho = \sum_{a=1}^{r} \omega_a$ is the so-called Weyl vector. H^ρ satisfy

$$[H^\rho, e^a] = e^a\,, \qquad\qquad [H^\rho, e^0] = (1-h)e^0\,, \tag{5.3}$$

where h is the Coxeter number. An important property of the differential equation is

$$\mathsf{q}^{-\frac{n}{hM}H^\rho}\mathscr{L}(\mathsf{q}^{\frac{n}{hM}}x, \mathsf{q}^n z, \lambda)\mathsf{q}^{\frac{n}{hM}H^\rho} = \mathsf{q}^{-\frac{n}{hM}}\mathscr{L}(x,z,e^{2\pi\mathtt{i}n}\lambda)\,, \tag{5.4}$$

where $\mathsf{q} = e^{2\pi\mathtt{i}\frac{M}{M+1}}$. This is sometimes called a Symanzik rotation. Using these rotations we can construct new solutions to (5.1), after rescaling $\lambda \mapsto e^{2\pi\mathtt{i}n}\lambda$, according to

$$\psi^{[2n]}(x,z) = \mathsf{q}^{-\frac{n}{hM}H^\rho}\,\psi(\mathsf{q}^{\frac{n}{hM}}, \mathsf{q}^n z) \tag{5.5}$$

Let us now consider the behaviour of ψ at infinity. Using a gauge transformation one finds

$$p^{H^\rho}\mathscr{L}p^{-H^\rho} = p(x,z)\left(\frac{d}{dS} + \Lambda + \dots\right)\,, \qquad \Lambda = \sum_{a=1}^{r} e^a + e^0\,. \tag{5.6}$$

where $p = q^{\frac{1}{h}}, S = \int_0^x p(x')dx'$. The matrix Λ is one of the main players in the following. It is the spectrum of Λ that decides how fast a solution decreases. For example, for the defining representation of $\mathfrak{sl}_3$ with $\lambda = 1$,

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,. \tag{5.7}$$

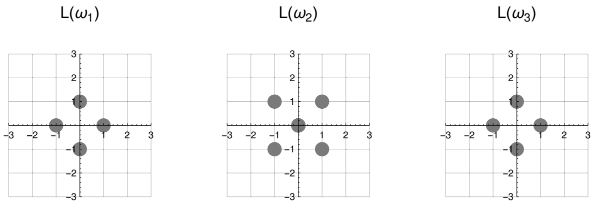


Figure 5.1. Eigenvalues of Λ for the fundamental representations of A_3.

Due to the property

$$\gamma^{\mathrm{ad}_{H^\rho}}\Lambda = \gamma\Lambda\,, \qquad\qquad \gamma = e^{\frac{2\pi\mathrm{i}}{h}}\,, \tag{5.8}$$

all eigenvalues lie on concentric circles. Furthermore, for a fundamental representation $L(\omega_a)$ one find $|\text{max eigenvalue}(\Lambda)| = \mu_a$ where μ_a is the a:th component of the Perron-Frobenius vector of the incidence matrix $I_{ab} = 2\delta_{ab} - C_{ab}$. Recall that the Perron-Frobenius vector is an eigenvector with the largest possible eigenvalue.

Consider $\mathfrak{g} = A_r$, we can now fix λ such that μ_1 is an eigenvalue of Λ for $\psi \in L(\omega_1)$, concretely $\lambda = 1$. Thus there exist a solution $\psi_{(1)}$ such that at $x \to \infty$ the unique fastest decaying solution behaves as

$$\psi_{(1)} \simeq e^{-\mu\int^x p(x')}p^{-H^\rho}\,U_{\mu_1} + \dots\,, \qquad \Lambda U_{\mu_1} = \mu_1 U_{\mu_1}\,. \tag{5.9}$$

With this choice of λ there exists no fully real eigenvalue of magnitude μ_2 for Λ in $L(\omega_2)$. However, $\gamma^{\pm\frac{1}{2}}\mu_2$ are eigenvalues. We denote the solutions that decays fastest along $x = r\gamma^{\pm\frac{1}{2}}, r \to \infty$ as $\psi^{\pm}_{(2)}$. After this the story repeats itself, for $L(\omega_3)$ there is a real eigenvalue and we define $\psi_{(3)}$ as the vector with the fastest decay along $x \to \infty$, etc. We illustrate this behaviour in Figure 5.1. In general, the asymptotic behaviour alternates between representations of adjacent nodes on the Dynkin diagram.

We now finally define a Q-vector as

$$Q_{(a)}(z) = z^{-\frac{H^\rho}{hM}}\,\psi_{(a)}(0,z)\,, \qquad Q^{[n]}_{(a)}(z) \equiv Q_{(a)}(\mathrm{q}^{\frac{n}{2}}z)\,. \tag{5.10}$$

In the rest of this chapter, we will now proceed as follows. We will find identifications between naively different solutions of (5.1). Using this identification we will find polynomial relations between different Q-vectors and their components.

5.2 Fusion of Basic Representations

In the standard approach, one now proceeds to derive functional relations of the schematic form

$$\psi^{+}_{(a)} \wedge \psi^{-}_{(a)} = \bigotimes_{b=1:I_{ab}=1}^{r} \psi_{(b)}\,. \tag{5.11}$$

These equations are known as the Ψ-system [76]. From the Ψ-system one can deduce standard QQ-relations for simply-laced algebras [74] that leads to the Bethe equation (4.30).

We will go about finding Q-systems in a slightly different manner. We will consider solutions $\psi^{[n]}_{\text{basic}}$ with basic labelling a basic representation, $L(\omega_1)$ for A_r and $S^{\pm}$ for D_r. By taking tensor products of these representations we will construct solutions for all other fundamental representations, $L(\omega_a)$. Carefully tuning the superscript we will find a solution with the same asymptotic behaviour as $\psi_{(a)}$. By uniqueness of such a solution these two solutions must thus agree and we will find a polynomial equation between $Q_{(a)}$ and shifted combinations of $Q_{(\text{basic})}$. We will treat $\mathfrak{gl}_n$ as a warm-up example and then consider D_r. For a discussion of the exceptional algebras E_6, E_7, E_8 see Paper I.

5.2.1 The case of $\mathfrak{sl}_n$

The basic representation of $\mathfrak{sl}_n$ is $L(\omega_1)$. We recall that from $L(\omega_1)$ all other fundamental representations are obtained as exterior products $L(\omega_a) = \Lambda^a L(\omega_1)$.

To promote this statement to Q-functions we first need to find the Perron-Frobenius eigenvector of the incidence matrix $I_{ab} = \delta_{a+1,b} + \delta_{a,b+1}$ and its eigenvalues.

To find the eigenvectors and the eigenvalue we will use the identity $[m+2]_q + [m]_q = [2]_q[m+1]_q$ for the q-number $[a]_q = \frac{q^a - q^{-a}}{q - \frac{1}{q}}$. It follows that $v_q = \{[1]_q, [2]_q, \ldots, [r]_q\}$ is an eigenvector of I_{ab} with eigenvalue $[2]_q$ if $[r+2]_q = 0$ which fixes $q = e^{\frac{\pi \mathrm{i}}{r+1}n}, n = 1, 2, \ldots, r$. The Perron-Frobenius eigenvector corresponds to the eigenvector with the largest eigenvalues, thus $\mu_a = [a]_{\gamma^{\frac{1}{2}}}, \gamma = e^{\frac{2\pi \mathrm{i}}{r+1}}$.

Let $\psi_{(1)} \in L(\omega_1)$ be a solution of (5.1). Consider the new solution

$$\bigotimes_{b=1}^{a} \psi^{[a+1-2b]}_{(1)} \tag{5.12}$$

to (5.1). Asymptotically it is an eigenvector of Λ with eigenvalue $\sum_{b=1}^{a} \gamma^{\frac{a+1-2b}{2}} = \mu_a$, see (5.5). Thus projecting it onto $L(\omega_a)$ it must be proportional to $\psi_{(a)}$, we

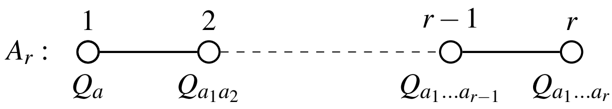


Figure 5.2. Dynkin diagram of A_r and association of Q-functions to fundamental representations.

can adjust the proportionality to be equality. For Q-functions we have

$$\mathbb{P}^{L(\omega_a)}_{\otimes^a L(\omega_1)}\left(\bigotimes_{b=1}^{a} Q_{(1)}^{[a-2b+1]}\right) = Q_{(a)} \tag{5.13}$$

or equivalently, using index notation,

$$Q_{a_1,a_2,\dots,a_n} = W(Q_{a_1}, Q_{a_2}, \dots Q_{a_n}) = |Q_a^{[n+1-2b]}|_{\substack{a=1,\dots,n \\ b=1,\dots,n}} . \tag{5.14}$$

This is the so-called Wronskian solution of the Q-system, we will discuss it more in Chapter 6.

5.2.2 The case of D_r

We will now repeat the same argument for D_r. We first need the Perron-Frobenius vector of the incidence matrix of D_r. The computation is essentially equivalent to the one of $\mathfrak{gl}_n$, the result is

$$\mu_a = [a]_{\gamma^{\frac{1}{2}}} , \qquad\qquad \mu_{r-1} = \mu_r = \frac{1}{2}[r-1]_{\gamma^{\frac{1}{2}}} . \tag{5.15}$$

with $\gamma = e^{\frac{2\pi\mathtt{i}}{h}}$ and we recall $h = 2r-2$, see 2.1. We learned in Section 2.3 that we can construct all fundamental representations by taking tensor products between two spinors. Consider therefore

$$\psi_{(a)}^{[m]} \otimes \psi_{(b)}^{[-m]} , \qquad\qquad a,b = r-1, r . \tag{5.16}$$

From the identity

$$\mu_{r-1-m} = (\gamma^{\frac{m}{2}} + \gamma^{-\frac{m}{2}})\mu_r , \tag{5.17}$$

we learn that (5.16) have the same eigenvalue as $\psi_{(r-1-m)}$ by tuning $m \geq 1$ properly. Thus by picking a, b properly and projecting upon $L(\omega_a)$ we can write Q-functions associated with vectors as polynomials in Q-functions associated with spinors, we note that this also includes the trivial representation since $\gamma^{\frac{r-1}{2}} + \gamma^{-\frac{r-1}{2}} = 0$.

To write this in explicit formulas we introduce notation Ψ for the Q-functions of the spinor representations and projectors $\Gamma^{\pm}$ such that $\Gamma^{\pm}\Psi \in S^{\pm}$ [2]. To send

[2] The similarity in notation between ψ and Ψ is unfortunate.

spinors into anti-symmetric tensors we need Γ-matrices Γ^I with I a multi-index appropriate for describing antisymmetric tensors and the matrix C giving us an inner product for spinors. In particular, let us write

$$\overline{\Psi} = \Psi^T C\,. \tag{5.18}$$

We will give convenient descriptions of all these objects in Section 7.2.1. The idea of "fusing" spinors to construct vectors is then finally formulated as

$$\overline{\Psi}^{[-r+1+|I|]}\Gamma^I\Gamma^{\pm}\Psi^{[r-1-|I|]} = V^I\,, \quad |I| \leq r-2 \tag{5.19}$$

$$\overline{\Psi}^{[-r+1]}\Gamma^{\pm}\Psi^{[r-1]} = 1\,. \tag{5.20}$$

Furthermore, we notice that if we pick m smaller than the values prescribed in (5.19) we find a solution in $L(\omega_a)$ with an eigenvalue bigger than μ_a, but such a solution does not exist. That means that projection onto this irrep must vanish. This gives the following projection relations:

$$\overline{\Psi}^{[-m]}\Gamma^I\Gamma^{\pm}\Psi^{[m]} = 0\,, \quad |I| \leq r-2\,, \quad m \in \{-r+2+|I|, \ldots, r-2-|I|\}\,. \tag{5.21}$$

5.3 Summary

In this section, we have utilised the ODE/IM correspondence to find relations between coefficients of vectors solving special differential equations. Following [75] we interpreted these coefficients as Q-functions of integrable models. Using this identification we derived a set of algebraic equations relating Q-vectors in fundamental representations to those of Q-vectors in basic representations. Our main assumption in the following is that the relations derived in Section 5.2 are universal. That is, they do not depend on the specifics of the linear problem we started from [3]. We will therefore after this chapter once again use the notation $f^{[n]} = f(u+n\frac{\hbar}{2})$.

The relations we found are but a subset of all relations one can derive from ODE/IM. In Paper I many more relations are studied and discussed. In that paper, the idea of a Fused Flag was introduced to bring order and structure to all relations. We will not explore this construction more in the rest of the text since the equations we have obtained are sufficient. We refer to Paper I for further details.

[3] An incomplete proof of this can be found in Paper I.

6. Review of $\mathfrak{gl}_n$ Q-Systems

In this chapter, we review $\mathfrak{gl}_n$ Q-systems [81–83], see also Paper I for additional references and historical remarks. These are the most well-studied type of Q-systems and will serve as inspiration for all other cases treated in this thesis. Furthermore, $\mathfrak{gl}_n$ Q-systems are very close to $\mathfrak{gl}_{m|n}$ Q-systems which we will need to formulate the QSC in Part III.

In Section 6.1 we present the primary results of this chapter: the $\mathfrak{gl}_n$ Q-system and its Wronskian solution which we already met in Chapter 5. In Section 6.2 we put this formalism to use by solving the $\mathfrak{gl}_n$ T-system and compact rational spin chains.

6.1 $\mathfrak{gl}_n$ QQ-Relations

A $\mathfrak{gl}_n$ Q-system consists of functions Q_A satisfying

$$W(Q_{Aa}, Q_{Ab}) = Q_{Aab}\, Q_A\,. \tag{6.1}$$

Here A is a multi-index of $\{1,2,\ldots,n\}$, including also the empty set. We will refer to (6.1) as QQ-relations or $\mathfrak{gl}_n$ QQ-relations when we want to be more specific. This formulation is a direct generalisation of the $\mathfrak{gl}_2$ Q-system we discussed in Part I. For $\mathfrak{gl}_2$ we had two Q-functions, Q_1, Q_2. We now have introduced a set of Q-functions for each of the fundamental representations of $\mathfrak{gl}_n$, these are components of Q-vectors.

We will rescale the Q-functions to enforce

$$Q_{\emptyset} = Q_{\overline{\emptyset}} = 1\,. \tag{6.2}$$

See Section 6.1.2 for further discussion of symmetries.

6.1.1 The Wronskian solution

We now turn to the task of solving (6.1) in terms of a basic set of Q-functions. The solution is of course exactly the one already encountered in Chapter 5,

$$Q_{a_1 a_2 \ldots a_m} = W(Q_{a_1}, Q_{a_2}, \ldots, Q_{a_m})\,. \tag{6.3}$$

To show (6.3) from (6.1) is very useful to introduce differential forms. Take an abstract differential basis $\theta^a, \theta^{a_1 a_2 \ldots a_n} \equiv \theta^{a_1} \wedge \theta^{a_2} \cdots \wedge \theta^{a_n}$ and introduce notation

$$Q_{(n)} \equiv Q_{(1)}^{[n-1]} \wedge Q_{(1)}^{[n-3]} \wedge \cdots \wedge Q_{(1)}^{[-n+1]}\,. \tag{6.4}$$

Furthermore, define the operation $\star$ by

$$\star\,(\theta^{a_1\dots a_n}) = \varepsilon^{a_1 a_2 \dots a_n}\,. \tag{6.5}$$

We now generalise the Plücker identity we wrote down for ε^{ab} in Chapter 4 to the case of arbitrary many indices. Using the differential form notation it reads

$$\star(\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_n)\,\mathbf{b} = \sum_i \star(\mathbf{a}_1 \dots \mathbf{a}_{i-1} \wedge \mathbf{b} \wedge \mathbf{a}_{i+1} \dots \mathbf{a}_n)\,\mathbf{a}_i\,, \tag{6.6}$$

with $\mathbf{a}_i, \mathbf{b}$ arbitrary one-forms.

With this technical tool in hand, the proof of (6.1) is but a calculation away. We consider only Q_{abc}, generalisation to arbitrary indices is straightforward. Since only three indices enter we are free to restrict a to $1,2,3$ and so we have $Q_{(1)} = Q_a \zeta^a = Q_1 \theta^1 + Q_2 \theta^2 + Q_3 \theta^3$. Using the Plücker identities (6.6) gives

$$\begin{aligned}
&\star\,(Q_{(1)}^{[2]} \wedge Q_{(1)} \wedge \theta^a)\,\star\,(Q_{(1)} \wedge Q_{(1)}^{[-2]} \wedge \theta^b) \\
&\qquad = \star(Q_{(1)}^{[2]} \wedge Q_{(1)} \wedge Q_{(1)}^{[-2]})\,\star\,(Q_{(1)} \wedge \theta^a \wedge \theta^b) \\
&\qquad + \star(Q_{(1)}^{[2]} \wedge Q_{(1)} \wedge \theta^b)\,\star\,(Q_{(1)} \wedge Q_{(1)}^{[-2]} \wedge \theta^a)\,.
\end{aligned} \tag{6.7}$$

Translating (6.7) into the language of components reveals

$$Q_{abc} = \mathrm{Wr}(Q_a, Q_b, Q_c)\,. \tag{6.8}$$

6.1.2 Symmetries of $\mathfrak{gl}_n$ Q-system

The Q-system exhibits 3 distinct types of symmetries: H-rotations, gauge transformations and Hodge duality. Understanding these symmetries will be crucial in Part III.

H-rotations

H-rotations are linear transformations

$$Q_a \to H_a{}^b Q_b\,, \qquad Q_A \to (H_A{}^B)^{[A-1]} Q_B \qquad (H_a{}^b)^+ = (H_a{}^b)^-\,, \tag{6.9}$$

where $H_A{}^B = |H_a{}^b|_{\substack{a\in A\\ b\in B}}$. We emphasise that H_a is $\hbar$ periodic, a necessary condition for the QQ-relations to be satisfied after the transformation.

Gauge transformations

Gauge transformations are u-dependent rescalings of Q-functions. There are two independent such transformations. A convenient parameterisation of these gauge transformations is given by

$$Q_A \to g_+^{[A]} g_-^{[-A]}\, Q_A\,. \tag{6.10}$$

There is a slight overlap between gauge transformations and H-rotations when H is proportional to the identity matrix.

Hodge duality
Finally we have *Hodge duality*, it acts as

$$Q_A \to Q^A \equiv \varepsilon^{AB} Q_B\,. \tag{6.11}$$

where the sum over multi-index is defined to be only over ordered sets. Hodge duality is, as opposed to the other symmetries, a discrete symmetry.

6.2 The T-System and Spin Chains

Having obtained the Q-functions we now return to the $\mathfrak{gl}_N$ T-system. We recall that this is a set of functional identities of the form

$$T_{a,s}^- T_{a,s}^+ = T_{a,s+1} T_{a,s-1} + T_{a+1,s} T_{a-1,s}\,, \tag{6.12}$$

with boundary conditions $T_{a,-1} = 0, T_{a,0} = 1$.

The main observation is that we can solve these relations using the following parameterisation

$$T_{a,s} = \star\big(Q_{(a)}^{[s+\frac{n}{2}]} \wedge Q_{(n-a)}^{[-s-\frac{n}{2}]}\big)\,, \tag{6.13}$$

In particular $T_{a,0} = Q_{\bar{\emptyset}}^{[a-\frac{n}{2}]}$. In components, this equation reads

$$T_{a,s} = Q_A^{[s+\frac{n}{2}]} (Q^A)^{[-s-\frac{n}{2}]} \qquad\qquad |A| = a\,. \tag{6.14}$$

To verify this identity one uses the Plücker identity (6.6) yet again.

6.2.1 Compact spin chains

Let us now consider the task of solving compact rational spin chains. To find the spectrum of a spin chain we need to impose proper analytic properties on the Q-functions. There exists usually no systematic way of finding the correct analytic properties to solve a specific problem. This part of the process involves guesswork and comparison against known results. For spin chains one can in principle deduce the correct analytic properties by studying Q-operators [61]. We will not discuss Q-operators and will instead argue for analytic properties by considering Bethe equations.

We recall that the Bethe equations of a rational $\mathfrak{gl}_n$ spin chain

$$\left.\frac{\mathbb{q}_a^{[2]}}{\mathbb{q}_a^{[-2]}} \frac{\mathbb{q}_{a+1}^-}{\mathbb{q}_{a+1}^+} \frac{\mathbb{q}_{a-1}^-}{\mathbb{q}_{a-1}^+}\right|_{\mathbb{q}_a=0} = -\left.\frac{P_a^+}{P_a^-}\right|_{\mathbb{q}_a=0}\,, \tag{6.15}$$

where P_a are Drinfeld polynomials characterising the representation we are considering. In order to reproduce (6.15) from the Q-functions appearing in

(6.1) one uses the analytic ansatz

$$Q_A = \sigma_{|A|} q_A\,, \qquad \prod_{b=1}^{n-1} \sigma_{(b)}^{[-C_{ab}]_D} = P_a\,, \tag{6.16}$$

where q_A needs to be polynomial. Identify $\mathbb{q}_{(a)} = q_{\leftarrow a}$, for example $\mathbb{q}_{(3)} = q_{123}$. Among the many QQ-relations, we find

$$W(\mathbb{q}_a, q_{\leftarrow a-1,a+1}) = P_a\, \mathbb{q}_{a+1} \mathbb{q}_{a-1}\,. \tag{6.17}$$

Using the same trick of shifting and evaluating at zeros of $\mathbb{q}_a$ we recover (6.15).

The most efficient method to solve rational spin chains utilises supersymmetric extensions [84]. We will here present a less powerful method but one that easily generalises to the case of Q-systems for D_r and B_2/C_2 to be discussed in Chapter 7 and Chapter 8.

The trick is simply to solve $W(Q_1,\ldots,Q_n) = 1$. This gives the so-called Wronskian Bethe Equation

$$W(q_1,q_2,\ldots,q_n) = \prod_{a=1}^{r} P_a^{[r+1-a]_D}\,. \tag{6.18}$$

After fixing H-rotations (6.18) gives a discrete spectrum. However, in general there will be too many solutions. The reason for this is that (6.18) alone cannot guarantee polynomiality of q_A. To ensure polynomiality we need also to verify that the remaining QQ-relations are compatible with the prescribed structure. Using $W(Q_{a_1},\ldots,Q_{a_m}) = Q_{a_1,\ldots,a_m}$ and then (6.16) one finds that this requires

$$\text{Remainder}\left(\frac{W(q_{a_1},q_{a_2},\ldots,q_{a_m})}{\prod_{a=1}^{m-1} P_a^{[m-a]_D}}\right) = 0\,. \tag{6.19}$$

We call these conditions kinematic constraints.

7. D_r and Pure Spinors

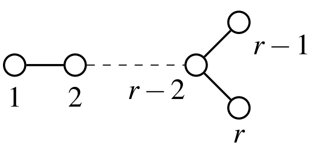
We now turn to our main example of Q-systems outside of A_r, namely D_r. The goal of this section is to review the construction of the D_r Q-system first proposed in Paper I and subsequently detailed in Paper II. Our goal will not be to reproduce these articles but to highlight some results and explain some calculations in a simplified setting. Our approach is purely functional. We remark that D_r Q-systems have also been studied using different methods. An article by Ferrando, Frassek and Kazakov [17] detailing the structure of Q-systems with D_r-symmetry appeared shortly before Paper I, see also [63]. It is based on an operatorial construction of Q-functions and contains results, among others, that overlap with those presented in this chapter.

We will take the same route as in Chapter 6 and start by giving QQ-relations in Section 7.1. In Section 7.2 we review how to find the QQ-relations from the construction of Chapter 5. In Setion 7.3 we find the character solution of the D_4 T-system. Finally, in Section 7.4 we describe how to find the spectrum of compact rational spin chains with D_r symmetry.

We illustrate the D_r Cartan matrix, the Dynkin diagram as well as our labelling conventions in Figure 7.1.

7.1 The Pure Spinor Q-System

Based on the ODE/IM perspective presented in Chapter 5 and experience from the $\mathfrak{gl}_n$ Q-system from Chapter 6 we want to build a Q-system out of basic representations. The two basic representations of D_r are the two spinor representations S^+, S^- and in Chapter 5 we found that we can find all other Q-functions from these representations. Thus we are in the end left with 2^r Q-functions to study. This is still a large number, we expect there to only be r independent functions, just as there are r $\mathbb{q}_a$-functions appearing in the Bethe equations. We

$D_r :$


$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & & & \\ -1 & 2 & -1 & \dots & & & \\ 0 & -1 & 2 & \dots & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ & & & \dots & 2 & -1 & -1 \\ & & & \dots & -1 & 2 & 0 \\ & & & \dots & -1 & 0 & 2 \end{pmatrix}$$

Figure 7.1. Dynkin diagram and Cartan matrix for D_r

must therefore find a way to cut the number of independent spinor Q-functions down to only r functions. This task will be accomplished in this section.

In Section 2.3 we wrote down an oscillator realisation of the spinor representation and found that S^+ and S^- decompose into antisymmetric tensors of $\mathfrak{sl}_r$. We will therefore use the form notation introduced in Chapter 6. Combining both S^+ and S^- into a Dirac spinor we write

$$\Psi = \sum_{n=0}^{r} \Psi_{(n)}\,, \qquad\qquad \Psi_{(n)} = \Psi_A\,\theta^A\,, \quad |A| = n\,, \tag{7.1}$$

where $a = 1,\ldots r$ and A is a multi-index containing a. Furthermore, when writing $\Psi_A\theta^A$ we only sum over ordered indices. For example, if $r = 3$ we have $\Psi_{(2)} = \Psi_{12}\theta^{12} + \Psi_{13}\theta^{13} + \Psi_{23}\theta^{23}$. With this notation we are now ready to write down a D_r version of the $\mathfrak{gl}_n$ Q-system.

7.1.1 Pure spinor QQ-relations

To build a Q-system out of the spinor components Ψ_A we constrain them to satisfy

$$W(\Psi_{Aa},\Psi_{Ab}) = W(\Psi_{Aab},\Psi_A)\,, \tag{7.2}$$

We will call these relations pure spinor QQ-relations. They are to be supplemented with the quantisation condition

$$W(\Psi_1,\Psi_2,\ldots,\Psi_r) = \Psi_\emptyset^{[r-2]_D}\,. \tag{7.3}$$

These are the relations proposed in Paper II to generalise the $\mathfrak{gl}_n$ Q-system. We will shortly describe them in more detail, but we first turn to the analogue of the Wronskian solution in $\mathfrak{gl}_n$.

As already mentioned we would like to express all our spinorial Q-functions in terms of only r independent functions. In our current set-up those r-functions are Ψ_a. Given Ψ_a it is possible to deduce $\Psi_\emptyset$ from the quantisation condition (7.3). With $\Psi_\emptyset$ fixed Ψ_{ab} is computed through the Wronskian relation

$$W(\Psi_{ab},\Psi_\emptyset) = W(\Psi_a,\Psi_b)\,, \tag{7.4}$$

while all remaining Q-functions follow from

$$\Psi_{(2n)} = \frac{1}{n!}\frac{\Psi_{(2)}\wedge\cdots\wedge\Psi_{(2)}}{\Psi_\emptyset^{n-1}}\,, \tag{7.5a}$$

$$\Psi_{(2n-1)} = \frac{1}{(n-1)!}\frac{\Psi_{(1)}\wedge\Psi_{(2)}\wedge\cdots\wedge\Psi_{(2)}}{\Psi_\emptyset^{n-1}}\,. \tag{7.5b}$$

The proof that (7.5) solves (7.2) can be found in Paper II, it uses the Plücker identities (6.6) but is slightly lengthier than the $\mathfrak{gl}_n$ version. The identities (7.5) was employed already by Cartan [85] in a different but surprisingly related setting. He called spinors satisfying these properties pure spinors, this is the reason for our choice of name.

7.2 D_r Covariant Formalism

In Chapter 5 we found how to "fuse" spinors into vectors. We will now outline how this formulation leads to the Q-system presented in Section 7.1 and also deduce the expected identity

$$V^{i_1,i_2\dots,i_k} = W(V^{i_1}, V^{i_2}, \dots V^{i_k})\,. \tag{7.6}$$

This section is technical, and we will at many times try to take small shortcuts to quickly reach the more important expressions. We refer to Paper II for a complete treatment.

7.2.1 Spinors and their properties

To explicitly realize the spinor representations we can use the oscillator construction of Section 2.3. We introduce the following notation

$$\mathtt{f}^a = \Gamma^a\,, \qquad\qquad \mathtt{f}_a = \Gamma_a\,. \tag{7.7}$$

and use indices $i \in \{1,2,\dots r,-r,-r+1,\dots -1\}$ and $a \in \{1,2,\dots,r\}$ as in Chapter 3. Let us introduce an off-diagonal metric $g^{ij} = \delta^{i+j,0}$ and packages Γ^a and Γ_a into Γ^i, in particular $\Gamma^{-a} = \Gamma_a$. We notice that these Γ-matrices satisfy the Clifford algebra $\{\Gamma^i, \Gamma^j\} = g^{ij}$. Let us also introduce the chirality matrices $\Gamma^{\pm}$. When acting on the basis θ introduced in (7.1) we have

$$\Gamma^a\theta^A = \theta^{aA}\,, \qquad \Gamma_a\theta^A = \partial_a\theta^A\,, \qquad \Gamma^{\pm}\theta^A = \frac{1\pm(-1)^{|A|}}{2}\theta^A\,, \tag{7.8}$$

where ∂ is a fermionic derivative.

To construct Γ^I with I a multi-index we simply anti-symmetrize single-index Γ-functions. For example, $\Gamma^{ij} = \frac{1}{2}\left(\Gamma^i\Gamma^j - \Gamma^j\Gamma^i\right)$. Finally, C is constructed as

$$C = \left(\Gamma^r + \Gamma^{-r}\right)\dots\left(\Gamma^1 + \Gamma^{-1}\right)\,, \quad C\,\theta^A = (-1)^{\frac{(r-|A|)(r-|A|-1)}{2}} \star\theta^A\,. \tag{7.9}$$

where $\star\theta^A = \varepsilon^{A\bar{A}}\theta^{\bar{A}}$ and $\bar{A}$ is the complement of A.

7.2.2 Finding the pure spinor Q-system

Let us recall the results from Chapter 5. We found that

$$\overline{\Psi}^{[-r+1+|I|]}\Gamma^I\Gamma^{\pm}\Psi^{[r-1-|I|]} = V^I\,, \quad |I| \le r-1\,, \tag{7.10a}$$

$$\overline{\Psi}^{[-r+1]}\Gamma^{\pm}\Psi^{[r-1]} = 1\,. \tag{7.10b}$$

and

$$\overline{\Psi}^{[-m]}\Gamma^I\Gamma^{\pm}\Psi^{[m]} = 0\,, \quad |I| \le r-2\,, \quad m \in \{-r+2+|I|,\dots,r-2-|I|\}\,. \tag{7.11}$$

We want to verify that (7.10) and (7.11) imply the pure spinor QQ-system of Section 7.1. To further simplify the presentation we will here focus on D_4.

Using the relations $\overline{\Psi}\Gamma^+\Psi = 0, \overline{\Psi}\Gamma^a\Gamma^-\Psi = 0$ it is immediate to verify the parameterisation (7.5). Next, the spinor relations (7.4) follows from

$$\overline{\Psi}^-\Gamma^{ab}\Gamma^+\Psi^+ = \overline{\Psi}^-\Gamma^{ab}\Gamma^-\Psi^+ .$$

This leaves only the quantisation condition (7.3). To find this equation we need the following computation

$$\begin{aligned}
&\overline{\Psi}^{[-m]}\Gamma^+\Psi^{[m]} \\
&= \Psi^{[m]}_{1234}\Psi^{[-m]}_{\emptyset} - \Psi^{[m]}_{12}\Psi^{[-m]}_{34} + \Psi^{[m]}_{13}\Psi^{[-m]}_{24} - \Psi^{[m]}_{14}\Psi^{[-m]}_{23} + (m \leftrightarrow -m) \\
&= \Psi^{[m]}_{\emptyset}\Psi^{[-m]}_{\emptyset} \star \left(\frac{\Psi^{[m]}_{(4)}}{\Psi^{[m]}_{\emptyset}} - \frac{\Psi^{[m]}_{(2)} \wedge \Psi^{[-m]}_{(2)}}{\Psi^{[m]}_{\emptyset}\Psi^{[-m]}_{\emptyset}} + \frac{\Psi^{[-m]}_{(4)}}{\Psi^{[-m]}_{\emptyset}} \right) \\
&= \frac{\Psi^{[m]}_{\emptyset}\Psi^{[-m]}_{\emptyset}}{2} \star \left(\frac{\Psi^{[m]}_{(2)}}{\Psi^{[m]}_{\emptyset}} - \frac{\Psi^{[-m]}_{(2)}}{\Psi^{[-m]}_{\emptyset}} \right)^2 \\
&= \begin{cases} 0 & m = 0,1,2\,, \\ \frac{\Psi^{[3]}_{(1)} \wedge \Psi^{[1]}_{(1)} \wedge \Psi^{[-1]}_{(1)} \wedge \Psi^{[-3]}_{(1)}}{\Psi^{[1]}_{\emptyset}\Psi^{[-1]}_{\emptyset}} & m = 3\,, \end{cases} \quad ,
\end{aligned} \tag{7.12}$$

from which we see that $\overline{\Psi}^{[-3]}\Gamma^+\Psi^{[3]} = 1$ indeed implies the quantisation condition (7.3). This completes our derivation of the pure spinor Q-system from the covariant formalism.

7.2.3 Vectors and spinors

Let us now discuss how to find the vectors V^i which we defined in (7.10). Using a very similar computation as (7.12) it is possible to find that

$$V^a = \varepsilon^{abcd}\frac{\Psi^{[2]}_b\Psi_c\Psi^{[-2]}_d}{\Psi_{\emptyset}}\,, \qquad V^{ab} = \varepsilon^{abcd}\Psi^+_c\Psi^-_d\,. \tag{7.13}$$

We find that this implies $W(V^a,V^b) = V^{ab}$ after using (7.3). How do we find V_a? There is a neat trick, we can see from (7.5) that

$$(\partial_a - \mu_{ab}\theta^b)\Gamma^+\Psi = 0\,, \qquad \mu_{ab} \equiv \frac{\Psi_{ab}}{\Psi_{\emptyset}}\,. \tag{7.14}$$

This allows us to derive

$$V_a = \overline{\Psi}^{[-2]}\Gamma_a\Gamma^+\Psi^{[2]} = \mu^{[2]}_{ab}\overline{\Psi}^{[-2]}\Gamma^b\Gamma^+\Psi^{[2]} = \mu^{[2]}_{ab}V^b\,, \tag{7.15}$$

and gives a very useful relation

$$V_a = \mu_{ab}^{[m]} V^b \,, \qquad\qquad m = -2, 0, 2\,. \tag{7.16}$$

Using this relation we can compute

$$W(V_a, V^b) = \mu_{ac}^{+} V^{cb} = V_a{}^b \,, \tag{7.17}$$

and in a similar fashion one finds, as was already announced,

$$V^{i_1, i_2, \ldots, i_k} = W(V^{i_1}, V^{i_2}, \ldots, V^{i_k})\,. \tag{7.18}$$

7.3 D_4 Character Solution of the T-System

We recall the T-system for an algebra of type D_4,

$$T_{1,s}^{-} T_{1,s}^{+} = T_{1,s+1} T_{1,s-1} + T_{2,s} \,, \tag{7.19a}$$
$$T_{2,s}^{-} T_{2,s}^{+} = T_{2,s+1} T_{2,s-1} + T_{1,s} T_{3,s} T_{4,s} \,, \tag{7.19b}$$
$$T_{3,s}^{-} T_{3,s}^{+} = T_{3,s+1} T_{3,s-1} + T_{2,s} \,, \tag{7.19c}$$
$$T_{4,s}^{-} T_{4,s}^{+} = T_{4,s+1} T_{4,s-1} + T_{2,s} \,. \tag{7.19d}$$

In [17] and Paper I a solution to this system was found using bilinear combinations of Q-functions

$$T_{1,s} = (V^i)^{[-s-3]} V_i^{[s+3]} \,, \qquad T_{2,s} = \frac{1}{2} (V^{ij})^{[-s-3]} V_{ij}^{[s+3]} \,, \tag{7.20}$$

$$T_{3,s} = \overline{\Psi}^{[-s-3]} \Gamma^{-} \Psi^{[s+3]} \,, \qquad T_{4,s} = \overline{\Psi}^{[-s-3]} \Gamma^{+} \Psi^{[s+3]} \,. \tag{7.21}$$

These relations can be rewritten in many different ways by the use of various QQ-relations. For a variety of results see [17] and Paper II. Instead of simply repeating the formulas from those papers, let us consider how one can go about solving the T-system, this will also be important practice before Chapter 8.

We will discuss the so-called character solution of the system. The character solution is given by constant $T_{a,s}$-functions encoding the decomposition of the KR-module $W_{a,s}$ into $\mathfrak{g}$-irreps. On the level of Q-functions, the character solution amounts to setting

$$V^i = B_i \, \mathtt{x}_i^{\frac{u}{\hbar}} \,, \quad \text{no sum}\,, \tag{7.22}$$

with $\mathtt{x}_{-a} = \frac{1}{\mathtt{x}_a}$. To most easily find the character solution we use that

$$T_{1,m-3} = (V^i)^{[-m]} V_i^{[m]} = 0\,, \quad m = 2, 1, 0\,, \quad T_{1,0} = (V^i)^{[-3]} V_i^{[3]} = 1\,. \tag{7.23}$$

These equations are consequences of (7.16). For example

$$\begin{aligned}(V^i)^{[-2]}V_i^{[2]} &= (V^a)^{[-2]}(V^b)^{[2]}\left(\mu_{ab}^{[2]} - \mu_{ab}^{[-2]}\right) \\ &= (V^{[-2]})^a(V^{[2]})^b\left(W(\Psi_a,\Psi_b)^+ + W(\Psi_a,\Psi_b)^-\right) \\ &= 0\,,\end{aligned} \tag{7.24}$$

where the last line follows from $(V^a)^{[m]}\Psi_a = 0$ for $m = -2, 0, 2$, see (7.13).

Let us then write out the expression for $T_{1,s}$, it becomes

$$T_{1,s} = \sum_{a=1}^{4} B_a B_{-a}(\mathrm{x}_a^{s+3} + \frac{1}{\mathrm{x}_a^{s+3}})\,. \tag{7.25}$$

It is immediate to solve the equations $T_{1,-3} = T_{1,-2} = T_{1,-1} = 0$ and $T_{1,0} = 1$,

$$B_a B_{-a} = \frac{\varepsilon^{abcd}(\mathrm{x}_b^2 + \frac{1}{\mathrm{x}_b^2})(\mathrm{x}_c + \frac{1}{\mathrm{x}_c})(\mathrm{x}_d^0 + \frac{1}{\mathrm{x}_d^0})}{|\Delta|}\,, \quad \Delta = |\mathrm{x}_a^{4-b} + \frac{1}{\mathrm{x}_a^{4-b}}|\,, \tag{7.26}$$

with no sum over a. Here $|M_a^b|$ denotes the determinant of the matrix M_a^b. Let us recall the character for $L(s\omega_1)$, it is given by

$$\chi(s\omega_1) = \frac{\left|\mathrm{x}^{s\,\delta_{b,1}+4-b} + \frac{1}{\mathrm{x}_a^{s\,\delta_{b,1}+4-b}}\right|}{\Delta}\,, \tag{7.27}$$

and so we find $T_{1,s} = \chi(s\omega_1)$. The same calculations for spinors reveals that $T_{3,s} = \chi(s\omega_3), T_{4,s} = \chi(s\omega_4)$. To find $T_{2,s}$ is now simply a bit of algebra. It turns out to be more efficient to compute the difference $T_{2,s} - T_{2,s-1}$. Using the solution (7.26) one obtains

$$T_{2,s} - T_{2,s-1} = \chi(s\omega_2)\,, \tag{7.28}$$

and the initial value $T_{2,0} = 1$. In conclusion, the character ansatz leads to

$$T_{a,s} = \chi(s\,\omega_a)\,, \quad a = 1,3,4\,, \qquad T_{2,s} = \sum_{m=0}^{s}\chi(m\omega_2)\,. \tag{7.29}$$

This is indeed the decomposition of $W_{a,s}(\theta)$ over D_r.

7.4 Compact Rational Spin Chains

To solve a compact rational spin chain we employ the analytic ansatz

$$V^I = \sigma_{|I|}v^I\,, \quad |I| \le r-2\,, \tag{7.30a}$$

$$\Psi_A = \sigma_{r-1}\,\psi_A\,, \quad |A|\ \text{odd}\,, \qquad \Psi_A = \sigma_r\,\psi_A\,, \quad |A|\ \text{even}\,, \tag{7.30b}$$

with ψ_A and v^I polynomial and no sums. Here the "dressing phases" σ_a satisfy

$$\prod_{b=1}^{r} \sigma_b^{[-C_{ab}]_D} = P_a \,. \tag{7.31}$$

We then reformulate the quantization (7.3) and the basic QQ-relations (7.4) as

$$W(\psi_1, \ldots, \psi_r) = \psi_\emptyset^{[r-2]_D} \prod_{a=1}^{r-1} P_a^{[a]_D} \,, \tag{7.32}$$

$$P_r\, W(\psi_a, \psi_b) = P_{r-1}\, W(\psi_{ab}, \psi_\emptyset) \,. \tag{7.33}$$

The first equation (7.32) was proposed in [17] to play the role of the Wronskian Bethe equations of $\mathfrak{gl}_n$ for D_r. But, as also noticed in [17], this equation is not enough, it will generally produce too many solutions. The fix is to take (7.4) into account. We furthermore note that in general one also needs to include kinematic constraints. They arise from the fusion of spinors into vectors and are given by

$$\text{Remainder}\left(\frac{W(\psi_{a_1}, \ldots, \psi_{a_k})}{\psi_\emptyset^{[k-2]_D} \prod_{b=r-k+1}^{r-1} P_b^{[b+k-r]}} \right) = 0 \,. \tag{7.34}$$

Let us consider a homogeneous spin chain. We recall that a finite-dimensional representation of $\mathscr{Y}(\mathfrak{g})$ is specified by a set of Drinfeld polynomials. Given these polynomials, we define

$$\lambda_{\max} = \sum_{a=1}^{r} \omega_a \deg P_a \,. \tag{7.35}$$

This is the weight of the ground state of the spin chain. Let λ be the weight of the state we want to consider. To fix the degree of ψ_a and $\psi_\emptyset$ we introduce

$$\gamma_a = \omega_r - \varepsilon_a \,, \qquad \gamma_\emptyset = \omega_r \,, \tag{7.36}$$

where ε_a is the orthogonal basis of Chapter 2. Then we have

$$\deg(\psi_a) = (\omega_{r-1}, \lambda_{\max} + \rho) - (\gamma_a, \lambda + \rho) \,, \tag{7.37}$$

$$\deg(\psi_\emptyset) = (\omega_r, \lambda_{\max} + \rho) - (\gamma_\emptyset, \lambda + \rho) \,, \tag{7.38}$$

where $\rho = \sum_{a=1}^{r} \omega_a$ is the Weyl vector. It was checked in Paper II that this reproduces the spectrum of a rational spin chain to a moderately large length, for example, all 1456 expected solutions for $P_1 = u^7, P_{b\neq 1} = 1$ was recovered.

8. Q-System for B_2/C_2

In this chapter, we discuss the most basic example of a non-simply laced algebra; $B_2 \simeq C_2$. Q-systems for $B_2 \simeq C_2$ have been discussed in the context of ODE/IM [77] and a solution of the $B_2 \simeq C_2$ T-systems using Wronskian like expressions were found in [86]. Still, in comparison to simply-laced algebras, B_2 remains largely unexplored. The methods used in this section naturally explain the connection between [77] and [86].

The goal of this Chapter is to show how a Q-system for $B_2 \simeq C_2$ can be constructed as a natural generalisation of the D_r Q-system detailed in Paper II and reviewed in Chapter 7. The results in this chapter are new and not included in any of the attached papers.

In Section 8.1 we will define the $B_2 \simeq C_2$ Q-system. Using the Q-functions we will solve the T-system in Section 8.2. In particular, we give regarding the character solution of the T-system. Thereafter in Section 8.3 we explain how to build a Baxter equation. In Section 8.4 we solve some simple rational spin chains and we conclude in Section 8.5.

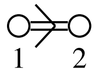
8.1 Proposal for the B_2 Q-System

Let us recall that $B_2 \simeq C_2$ have two fundamental representations, the vector **5** and the spinor **4**. We give the Dynkin diagram and the character of these two representations in Figure 8.1.

We will use labelling similar to the one introduced for D_r: V^i will be a vector with $i = 1, 2, 0, \bar{2}, \bar{1}$ with $\bar{a} = -a$ and Ψ a 4-dimensional spinor. There is now only one type of spinor, we write it using the superspace notation as

$$\Psi = \Psi_\emptyset + \Psi_a \theta^a + \frac{1}{2}\Psi_{ab}\theta^{ab}\,. \tag{8.1}$$

B_2 :



$$\chi([10]_{B_2}) = \mathrm{x}_1 + \mathrm{x}_2 + 1 + \frac{1}{\mathrm{x}_2} + \frac{1}{\mathrm{x}_1}\,,$$
$$\chi([01]_{B_2}) = \sqrt{\mathrm{x}_1}\sqrt{\mathrm{x}_2} + \frac{\sqrt{\mathrm{x}_1}}{\sqrt{\mathrm{x}_2}} + \frac{\sqrt{\mathrm{x}_1}}{\sqrt{\mathrm{x}_2}} + \frac{1}{\sqrt{\mathrm{x}_1\mathrm{x}_2}}$$

Figure 8.1. Dynkin Diagram for B_2 and the character of **5** and **4** using the orthogonal basis 2.2 .

We will also need matrices C, g and γ^i. We will define these as follows

$$\gamma^{\pm 1} = \sigma^{\mp} \otimes \mathbb{1}\,, \qquad \gamma^{\pm 2} = \sigma^{z} \otimes \sigma^{\mp}\,, \qquad \gamma^{0} = \frac{1}{2}\sigma^{z} \otimes \sigma^{z}\,,$$

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad g_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and σ^z the standard Pauli matrix. With the definitions taken care of we now proceed to construct the B_2/C_2 Q-system. Inspired by the ODE/IM correspondence [77] and the covariant formulation of D_r presented in Chapter 7 we propose the following covariant set of equations

$$\overline{\Psi}^{[-2]}\gamma^i\Psi^{[2]} = V^i\,, \tag{8.2}$$

$$V^{ij} \equiv \mathrm{Wr}(V^i, V^j) = \overline{\Psi}^{[-1]}\gamma^{ij}\Psi^{[1]}\,. \tag{8.3}$$

We immediately note that (8.2) and (8.3) are not independent. For them to be consistent the following constraints must be satisfied

$$\overline{\Psi}^{[-1]}\Psi^{[1]} = 0\,, \qquad\qquad \overline{\Psi}^{[-3]}\Psi^{[3]} = 1\,. \tag{8.4}$$

This looks very familiar to the quantization conditions we have encountered previously, it is a good indication that we are on the right track. Let us also note that if we write out $\overline{\Psi}^{[-1]}\overline{\Psi} = 0$ in components we obtain

$$W(\Psi_{12}, \Psi_{\emptyset}) = W(\Psi_1, \Psi_2)\,, \tag{8.5}$$

which is nothing but the D_r QQ-relations we familiarised ourselves with in Section 7.1. Indeed, it is not hard to see that Ψ_1 and Ψ_2 are the two only independent functions since we can fix Ψ_{12} and $\Psi_{\emptyset}$ from (8.4).

8.2 The T-System and The Character Solution

We already wrote down the T-system for $B_2 \simeq C_2$ in Section 4.2. Let us write it here with slightly different notation as

$$T^{[-2]}_{\mathbf{5},s} T^{[2]}_{\mathbf{5},s} = T_{\mathbf{5},s-1} T_{\mathbf{5},s+1} + T_{\mathbf{4},2s}\,, \tag{8.6a}$$

$$T^{-}_{\mathbf{4},2s} T^{+}_{\mathbf{4},2s} = T_{\mathbf{4},2s-1} T_{\mathbf{4},2s+1} + T^{-}_{\mathbf{5},s} T^{+}_{\mathbf{5},s}\,, \tag{8.6b}$$

$$T^{-}_{\mathbf{4},2s+1} T^{+}_{\mathbf{4},2s+1} = T_{\mathbf{4},2s} T_{\mathbf{4},2s+2} + T_{\mathbf{5},s} T_{\mathbf{5},s+1}\,. \tag{8.6c}$$

The subscript **5** and **4** is to remind us which representations are being traced over in the auxiliary space.

From (8.4) it is reasonable to expect that $\overline{\Psi}^{[-3-2s]}\Psi^{[3+2s]}$ describes the spectrum of a transfer-matrix, see (7.20). This expectation turns out to be correct, but with an amusing twist; it is equal to $T_{\mathbf{5},s}$, not $T_{\mathbf{4},s}$. The full T-system (8.6) is solved by the following ansatz:

$$T_{\mathbf{5},s} = \overline{\Psi}^{[-3-2s]}\Psi^{[3+2s]}\,, \tag{8.7}$$

$$T_{\mathbf{4},s} = V_i^{[-3-s]}V^i_{[3+s]} + \frac{(-1)^s}{2}W^{[-3-s]}W^{[3+s]}\,, \tag{8.8}$$

where $W = \overline{\Psi}^{[-2]}\Psi^{[2]}$ is simply short-hand notation. To verify this statement is an algebraic exercise using the definition of Ψ.

8.2.1 The character solution

Having found a bilinear ansatz for the T-system we proceed to consider the character solution. This section is slightly technical, anyone not interested in the details can find the final result given in (8.14). As for D_4 we take an ansatz for the Q-functions in terms of x_a. In our current case it is favourable to start from the spinor Q-function

$$\Psi_A = A_A \frac{\prod_{a=1}^{r} \mathrm{x}_a^{\frac{u}{2\hbar}}}{\prod_{a\in A} \mathrm{x}_a^{\frac{u}{\hbar}}}\,. \tag{8.9}$$

Just as for D_4 it is much easier to solve the constraints (8.4) than the full Q-system to quickly obtain the character solution. We write

$$\overline{\Psi}^{[-3-2s]}\Psi^{[3+2s]} =$$
$$A_0A_{12}\Big(\mathrm{x}_1^{\frac{3}{2}+s}\mathrm{x}_2^{\frac{3}{2}+s} - \frac{1}{\mathrm{x}_1^{\frac{3}{2}+s}\mathrm{x}_2^{\frac{3}{2}+s}}\Big) + A_1A_2\Big(\frac{\mathrm{x}_2^{\frac{3}{2}+s}}{\mathrm{x}_1^{\frac{3}{2}+s}} - \frac{\mathrm{x}_1^{\frac{3}{2}+s}}{\mathrm{x}_2^{\frac{3}{2}+s}}\Big)\,, \tag{8.10}$$

Enforcing the constraints $T_{\mathbf{5},-1} = 0, T_{\mathbf{5},0} = 1$ fixes

$$A_0A_{12} = \frac{1}{\tilde{\Delta}}\frac{1}{\sqrt{\mathrm{x}_1\mathrm{x}_2}}(\mathrm{x}_1-\mathrm{x}_2)\,, \qquad A_1A_2 = \frac{1}{\tilde{\Delta}}\frac{\mathrm{x}_1\mathrm{x}_2-1}{\sqrt{\mathrm{x}_1\mathrm{x}_2}}\,. \tag{8.11}$$

where $\tilde{\Delta} = |\mathrm{x}_a^{3-b} - \frac{1}{\mathrm{x}_a^{3-b}}|_{\substack{a=1,2\\ b=1,2}}$ is Vandermonde-like factor. To find $T_{\mathbf{4},s}$ we start by finding V^i and W from their respective definitions in terms of Ψ, see (8.3) and then use (8.7). We obtain the following expression

$$T_{\mathbf{5},s} = \frac{1}{\sqrt{\mathrm{x}_1\mathrm{x}_2}\tilde{\Delta}}\left((\mathrm{x}_1-\mathrm{x}_2)\left(\mathrm{x}_1^{s+\frac{3}{2}}\mathrm{x}_2^{s+\frac{3}{2}} - \frac{1}{\mathrm{x}_1^{s+\frac{3}{2}}\mathrm{x}_2^{s+\frac{3}{2}}}\right) + (\mathrm{x}_1\mathrm{x}_2-1)\Big(\frac{\mathrm{x}_2^{s+\frac{3}{2}}}{\mathrm{x}_1^{s+\frac{3}{2}}} - \frac{\mathrm{x}_1^{s+\frac{3}{2}}}{\mathrm{x}_2^{s+\frac{3}{2}}}\Big)\right) \tag{8.12}$$

and a significantly bulkier expression for $T_{\mathbf{4},s}$. We can now verify that these solutions indeed reproduce the correct Lie-algebra decomposition of KR-modules. The trick to making this comparison easier is to write down the character in the orthogonal basis of C_2. Consider therefor a representation $L([ab]_{C_2})$ of C_2 and associated with it $\lambda = \{a+b, b\}$. The character of a finite-dimensional representation of C_2 is given by

$$\chi(\lambda_{C_2}) = \frac{1}{\tilde{\Delta}} \begin{vmatrix} \mathrm{x}_1^{\lambda_1+2} - \frac{1}{\mathrm{x}_1^{\lambda_1+2}} & \mathrm{x}_1^{\lambda_2+1} - \frac{1}{\mathrm{x}_1^{\lambda_2+1}} \\ \mathrm{x}_2^{\lambda_1+2} - \frac{1}{\mathrm{x}_2^{\lambda_1+2}} & \mathrm{x}_2^{\lambda_2+1} - \frac{1}{\mathrm{x}_2^{\lambda_2+1}} \end{vmatrix} . \tag{8.13}$$

By considering $[0s]$ and expanding the determinant we reproduce (8.12). For $T_{\mathbf{4},s}$ it is useful to compute $T_{\mathbf{4},s} - T_{\mathbf{4},s-2}$. After some, rather long, algebra this yields the following identification

$$T_{\mathbf{5},s} = \chi([0s]_{C_2}), \qquad T_{\mathbf{4},s} = \sum_{m=0}^{[\frac{s}{2}]} \chi([s-2m, 0]_{C_2}). \tag{8.14}$$

This agrees perfectly with the results in the literature [41].

8.3 A Baxter Equation

A happy surprise is that for B_2/C_2 one can find a powerful equation relating Q-functions, quantum eigenvalues and fundamental T-functions. It was found in [86] that one should consider the following generating functional

$$\begin{aligned} \mathscr{B} = & \\ & (1-\Lambda_{\bar{1}}^{[-4]} D^{-2})(1-\Lambda_{\bar{2}}^{[-4]} D^{-2})(1-\Lambda_{\bar{2}}^{[-4]}\Lambda_{2}^{[-6]} D^{[-4]})(1-\Lambda_{2}^{[-4]} D^{-2})(1-\Lambda_{1}^{[-4]} D^{-2}) \\ & = 1 - T_{\mathbf{4},1}^{[-4]} D^{-2} + T_{\mathbf{5},1}^{[-5]} D^{-4} - T_{\mathbf{5},1}^{[-7]} D^{-8} + T_{\mathbf{4},1}^{[-8]} D^{-10} - D^{-12} \end{aligned} \tag{8.15}$$

Where Λ are the quantum eigenvalues introduced already in Chapter 4, we repeat them here and also highlight how they are connected to V^i;

$$\begin{aligned} \Lambda_1 &= \frac{\mathbb{Q}_4^{[4]}}{\mathbb{Q}_{\mathbf{4}}^{[2]}} = \frac{(V^1)^{[4]}}{(V^1)^{[2]}}, & \Lambda_2 &= \frac{\mathbb{Q}_{\mathbf{5}}^{[4]}}{\mathbb{Q}_{\mathbf{5}}} \frac{\mathbb{Q}_{\mathbf{4}}}{\mathbb{Q}_{\mathbf{4}}^{[2]}} = \frac{(V^{12})^{[3]}}{(V^{12})^{[1]}} \frac{V^1}{(V^1)^{[2]}} \\ \Lambda_{\bar{1}} &= \frac{\mathbb{Q}_4^{[-4]}}{\mathbb{Q}_{\mathbf{4}}^{[-2]}} = \frac{(V^1)^{[-4]}}{(V^1)^{[-2]}}, & \Lambda_{\bar{2}} &= \frac{\mathbb{Q}_{\mathbf{4}}}{\mathbb{Q}_{\mathbf{4}}^{[-2]}} \frac{\mathbb{Q}_{\mathbf{5}}^{[-4]}}{\mathbb{Q}_{\mathbf{5}}} = \frac{V^1}{(V^1)^{[-2]}} \frac{(V^{12})^{[-3]}}{(V^{12})^{[-1]}}. \end{aligned} \tag{8.16}$$

With the expressions for Λ in hand we can now explicitly see what combinations of our Q-functions are killed by the Baxter operator (8.15). Direct computations yields

$$\mathscr{B} V^i = 0, \qquad \mathscr{B} (-1)^{\frac{u}{\hbar}} W = 0. \tag{8.17}$$

This allows us to find explicit determinant representations of the T-functions by expanding a trivial determinant. With notation $U^\lambda = \{V^i, (-1)^{\frac{u}{\hbar}} W\}$ and writing

$$|abcdef| = |\hat{U}^\lambda_{[g]}|_{\substack{g=a,b,\dots,f \\ \lambda=1,2,\dots,6}} . \tag{8.18}$$

the T-functions are obtained as

$$T_{\mathbf{4},1} = \frac{|620\bar{2}\bar{4}\bar{6}|^{[-2]}}{|420\bar{2}\bar{4}\bar{6}|^{[-2]}} , \qquad T_{\mathbf{5},1} = \frac{|640\bar{2}\bar{4}\bar{6}|^{[-1]}}{|420\bar{2}\bar{4}\bar{6}|^{[-1]}} , \tag{8.19}$$

with $\bar{a} = -a$. These are the Weyl type formulas of [86]. One can calculate $[0\bar{2}\bar{4}\bar{6}\bar{8}\overline{10}] = (-1)^{\frac{u}{\hbar}}$.

8.4 Wronskian Bethe Equations

In this section, we study the spectrum of rational spin chains with $B_2 \simeq C_2$ symmetry. We will work with a generalisation of the Wronskian Bethe approach discussed for $\mathfrak{gl}_n$ in Section 6.2 and D_r in Section 7.4.

We will consider the following Yangian representation, or spin chain,

$$\mathscr{H} = \bigotimes^{L_4} W_{\mathbf{4},1} \otimes \bigotimes^{L_5} W_{\mathbf{5},1} . \tag{8.20}$$

Where $W_{\mathbf{4},1} = \mathbf{4}, W_{\mathbf{5},1} = \mathbf{5}$ as $\mathfrak{g}$ representations. This is a consequence of the R-matrices we wrote down in Chapter 3. The analytic Bethe ansatz in this case is

$$\Psi_A = \sigma_{\mathbf{5}}\, \psi_A , \qquad V^i = \sigma_{\mathbf{4}}\, v^i , \tag{8.21}$$

where ψ_A and v^i are polynomials of the spectral parameter u.

The Bethe equations for a rational spin chain are

$$\left. \frac{\mathbb{q}_{\mathbf{4}}^{[2]}}{\mathbb{q}_{\mathbf{4}}^{[-2]}} \frac{\mathbb{q}_{\mathbf{5}}^{[-2]}}{\mathbb{q}_{\mathbf{5}}^{[-2]}} \right|_{\mathbb{q}_{\mathbf{4}}=0} = -\frac{P_{\mathbf{5}}^{+}}{P_{\mathbf{5}}^{-}} , \qquad \left. \frac{\mathbb{q}_{\mathbf{5}}^{[4]}}{\mathbb{q}_{\mathbf{5}}^{[-4]}} \frac{\mathbb{q}_{\mathbf{4}}^{[-2]}}{\mathbb{q}_{\mathbf{4}}^{[-2]}} \right|_{\mathbb{q}_{\mathbf{5}}=0} = -\frac{P_{\mathbf{5}}^{[2]}}{P_{\mathbf{5}}^{[-2]}} . \tag{8.22}$$

Write $\Psi_\emptyset = \mathbb{Q}_{\mathbf{5}} = \sigma_{\mathbf{5}} \mathbb{q}_{\mathbf{5}}$ and $V^1 = \mathbb{Q}_{\mathbf{4}} = \sigma_{\mathbf{4}} \mathbb{q}_{\mathbf{4}}$, we reproduce (8.22) from (8.2) and (8.3) by imposing

$$\frac{\sigma_{\mathbf{5}}^{+} \sigma_{\mathbf{5}}^{-}}{\sigma_{4}^{+} \sigma_{4}^{-}} = P_{\mathbf{4}} , \qquad \frac{\sigma_{\mathbf{4}}}{\sigma_{\mathbf{5}}^{[2]} \sigma_{\mathbf{5}}^{[-2]}} = P_{\mathbf{5}} . \tag{8.23}$$

To find the Wronskian Bethe equations we use the quantization condition $T_{\mathbf{5},1} = 1$ and simplify it slightly using (8.5) to

$$\begin{vmatrix} \psi_\emptyset^{[3]} & 0 & \psi_\emptyset^{[-1]} & 0 \\ 0 & \psi_\emptyset^{[1]} & 0 & \psi_\emptyset^{[-3]} \\ \psi_2^{[3]} & \psi_2^{[1]} & \psi_2^{[-1]} & \psi_2^{[-3]} \\ \psi_1^{[3]} & \psi_1^{[1]} & \psi_1^{[-1]} & \psi_1^{[-3]} \end{vmatrix} = \psi_\emptyset^{+} \psi_\emptyset^{-} P_{\mathbf{4}} P_{\mathbf{5}}^{+} P_{\mathbf{5}}^{-} . \tag{8.24}$$

L	2	3	4	5	6	7
$\otimes^L W_{\mathbf{4},1}$	3(0.04 s)	6(0.34 s)	20(1.08 s)	50(1.5 s)	175(10 s)	490(166.39)
$\otimes^L W_{\mathbf{5},1}$	3(0.27)	7(1.13)	25(8.47)	81(86.15)	-	-

Table 8.1. *Table of the number of solutions found from the Wronskian equation and the time spent to find them.*

The equation immediately teaches us one important fact: fusion of two spinors gives a vector, this follows from the fact that $P_{\mathbf{4}} = (u+\frac{\hbar}{2})^L(u-\frac{\hbar}{2})^L$ is indistinguishable from $P_{\mathbf{5}} = u^L$.

Finally, we need a prescription to compute the degree of the spinor given $L_{\mathbf{4}}, L_{\mathbf{5}}$ in (8.20) and the quantum numbers of the state we are considering. We will use B_2 notation and encode this information in a magnon vector M and a highest weight $\lambda_{\max}$

$$\lambda_{\max} = L_{\mathbf{4}}\, \omega_1^\vee + L_{\mathbf{5}}\, \omega_2^\vee\,, \qquad\qquad M = M_{\mathbf{4}}\, \alpha_1^\vee + M_{\mathbf{5}}\, \alpha_2^\vee\,, \tag{8.25}$$

The formula for the degree of the Q-functions can be written in a group theoretical way, let $\lambda_a = \{\omega_2, \omega_2-\alpha_2, \omega_2-\alpha_1-\alpha_2, \omega_2-\alpha_1-2\alpha_2\}$ encode the weights of the spinor representation. We find

$$\deg(\psi_a) = (\lambda_a, M) + (\omega_a - \lambda_a, \lambda_{\max} + \rho^\vee)\,, \qquad \rho^\vee = \sum_a^2 \omega_a^\vee\,. \tag{8.26}$$

Recall that when we solved the A_r and D_r spin chains we needed to also impose kinematic constraints. These constraints are of course also needed in our current setup. They are found from (8.2) and (8.3) after plugging in the analytical ansatz, the result is

$$\operatorname{Res}\left(\frac{\overline{\psi}^{[-2]}\gamma^i\psi^{[2]}}{P_{\mathbf{5}}}\right) = 0\,. \tag{8.27}$$

We put (8.24) on a standard laptop and asked Mathematica to solve these equations, the result is shown in table 8.1. When solving for a spin chain with nodes in the vector representation we also used (8.27).

8.5 Conclusions

A better understanding of $B_2 \simeq C_2$ Q-systems should be of help in the Gaudin-based approach to CFTs in 3 dimensions [87], another possible application is in the study of fishnets and fishchains for ABJM [88–92]. Furthermore, the Baxter equation looks amendable to the functional SoV approach [93–95]. So far the functional SoV methods have not been extended beyond $\mathfrak{gl}_n$. It seems that $B_2 \simeq C_2$ could be an ideal testing ground to understand more general algebras.

9. Operators on the Squashed Seven-Sphere

In this chapter, we take a break from the discussion of Q-systems. We will instead discuss algebraic methods to obtain the spectrum for several different operators on the squashed seven-sphere. This chapter is a review of Paper III We will omit most technical details and only sketch the methods used.

Section 9.1 discusses eleven-dimensional supergravity and Freund-Rubin [96] compactification. In Section 9.2 the squashed seven-sphere is introduced and we outline how to use its coset space structure to our advantage. Finally, in Section 9.3 we summarize the results of Paper III and comment on recent related progress in the literature.

Many of the details regarding eleven-dimensional supergravity and compactification are expertly reviewed in [97].

9.1 Eleven-Dimensional Supergravity

In eleven dimensions there exists a unique two-derivative $\mathcal{N} = 1$ supergravity [98]. Eleven-dimensional supergravity of course naively fails to describe the four-dimensional world around us. To build a realistic model we need to explain how to deal with the seven extra dimensions present in the theory. One way of resolving the issue is through compactification, wherein the additional dimensions are compact and small, thus unnoticeable to the human eyes. In eleven-dimensional supergravity it was realised by Freund and Rubin [96] that it is possible to turn on background fields in such a way that compactification happens spontaneously, i.e it is a consequence of Einstein's equations. The resulting space-time is of type $\mathrm{AdS}_4 \times \mathrm{M}^7$ with M^7 a compact Einstein space. Unfortunately, AdS_4 does not seem to describe our world.

The question we would like to address in this chapter is that of finding the mass spectra on AdS_4. Starting from eleven-dimensional fields Φ one considers a perturbation around their vacuum value, $\Phi = \langle \Phi \rangle + \hat{\Phi}$. Let x be coordinates on AdS_4 and y coordinates on M^7, the field $\hat{\Phi}$ is schematically split as $\hat{\Phi}(x,y) = \hat{\Phi}(x)Y(y)$ where Y are eigenvalues of the mass operators on M^7. For explicit expressions for the operators see [97]. For our purposes we need only know that the differential operators appearing in M are the Laplace-de Rham operator $\Delta_p = \mathrm{d}\,\delta + \delta\,\mathrm{d}$ which acts on antisymmetric tensor, the Dirac operator $\not{D} = -\mathrm{i}\not{\nabla}$ which acts on spinors and the Lichnerowicz operator Δ_L which acts on symmetric traceless tensors. Here d is the exterior derivative, $\delta = (-1)^p \star \mathrm{d} \star$ its adjoint and the Lichnerowicz operator is obtained from linearizing Einstein's equations, see [97] for an explicit expression.

9.2 The Squashed Seven-Sphere

When M^7 is a coset-space G/H there exists a powerful method due to Salam and Strathdee [99] to study differential operators, see also [97] for a review. One splits the generators T_A of G into $T_{\bar{a}}$ describing H and their orthogonal complement T_a. Using T_a the action of a covariant derivative is given as

$$-T_a Y = \nabla_a Y + \frac{1}{2} f_a{}^{bc} \Sigma_{bc} Y \,. \tag{9.1}$$

where Σ are generators of $Spin(7)$ for the relevant representation. The strength of (9.1) is that it allows us to trade covariant derivatives for the generators T_a and algebraic expressions.

We will be interested in the case when M^7 is the squashed seven-sphere. Luckily, the squashed seven-sphere is a coset manifold of type

$$G/H = Sp_2 \times Sp_1^C / Sp_1^A \times Sp_1^{A+B} \,, \tag{9.2}$$

where Sp_2 is split as $Sp_1^A \times Sp_1^B$ and Sp_1^{A+B} is the diagonal subgroup of Sp_1^B and Sp_1^C. Let us take an orthonormal basis, in [100] it was found that for the squashed-seven sphere the structure constants f_{abc} are given as

$$f_{abc} = -\frac{2}{3} m\, a_{abc} \,, \tag{9.3}$$

with a_{abc} the octonion structure constants. a_{abc} is a fully antisymmetric tensor such that

$$a_{abc} = 1 \,, \qquad abc = 456, 041, 052, 063, 162, 135, 243 \,. \tag{9.4}$$

The octonion structure constants can be nicely encoded in 7 dimensional Γ-matrices. Introduce $A = \{a, 8\}$ with $a = 1, \ldots, 8$ then

$$(\Gamma_a)_b{}^c = a_{abc} \,, \qquad (\Gamma_a)_b{}^8 = \mathtt{i}\delta_{ab} \,, \qquad (\Gamma_a)_8{}^b = -\mathtt{i}\delta_{ab} \,. \tag{9.5}$$

In particular, we can introduce the Killing spinor η such that

$$\nabla_a \eta = -\frac{\mathtt{i}m}{2} \Gamma_a \eta \,, \qquad \bar{\eta}\eta = 1 \,. \tag{9.6}$$

In particular,

$$a_{abc} = \mathtt{i}\bar{\eta}\Gamma_{abc}\eta \,. \tag{9.7}$$

This allows us to derive expressions such as $\nabla a_{bcd} = m c_{abcd}$ with $c_{abcd} = \frac{1}{6}\varepsilon_{abcdefg} a_{efg}$.

9.2.1 The spectrum of scalars and vectors

Let us review the typical steps in the computation treating the scalar and vector. This computation can be found both in [97] and in Paper III. We repeat the step here once again because they clearly illustrate the main technical tricks needed for other more complicated operators. For a complementary method see [101].

9.2.2 The scalar

Let Y be a scalar of $Spin(7)$, we find $\Delta_0 = -\nabla_a \nabla^a$. Using (9.1) we thus have

$$\Delta_0 Y = -T_a T_a Y \,. \tag{9.8}$$

We can write $T_a T^a = (\mathscr{C}_G - \mathscr{C}_H)$ with $\mathscr{C}$ the quadratic Casimir for either G or H. This fixes the spectrum of scalars.

9.2.3 The vector

We now turn to vectors, let us introduce Y_a such that $\Delta_1 Y_a = \lambda_1^2 Y_a$ and $\nabla^a Y_a = 0$. For vectors the Laplace-de Rham operator acts as $\Delta_1 Y_a = -\Box Y_a + R_a{}^b Y_b$ with $\Box \equiv \nabla_a \nabla^a$. The squashed seven-sphere is an Einstein manifold with $R_a{}^b = 6m^2 Y_a$. By squaring (9.1) one derives that

$$\Box Y_a + \frac{2m}{3} a_{abc} \nabla_b Y_c - 6m^2 a_{ab}{}^c Y_c = -\mathscr{C}_2 Y_a \,, \tag{9.9}$$

where the explicit value $\mathscr{G}_H = \frac{12}{5}$ have been used. With this equation at hand one find that

$$(\lambda_1^2 - \mathscr{C}_G) Y_a = \frac{2m}{3} a_{abc} \nabla_b Y_c \,. \tag{9.10}$$

At this stage, one uses the following trick: square the operator $(DY)_a = a_{abc} \nabla_b Y_c$. After some algebra it is revealed that

$$(D^2 Y)_a - 4m(DY)_a - \lambda_1^2 Y_a = 0 \,. \tag{9.11}$$

Using (9.10) an algebraic equation on λ_1^2 is derived which is easily solved.

9.3 Results and Outlook

When considering more complicated operators compared to those of Section 9.2 it becomes very convenient to decompose them into G_2 irreps. After this is performed the methods of Section 9.2 can be applied, although the computations are significantly longer.

The accomplishments of Paper III was to find the eigenvalues of Δ_0,Δ_1,Δ_2,Δ_3,$\not{D}_{1/2}$ and Δ_L. Surprisingly, it was found that the number of eigenvalues was not sufficient to distinguish all operators on the squashed seven-sphere found in [102]. However, the full spectrum was recently obtained in [103], using an orthogonal method. It was found that the list of eigenvalues obtained in Paper III is complete for the operators considered and that there are non-trivial multiplicities present [1]. This recent progress opens up the possibilities of further explorations of these models, see for example [105].

[1] See also [104].

10. Review of Supersymmetric Q-Systems

We now return to Q-systems and integrable models. We will continue with the theme of supersymmetry from Chapter 9 but now in the context of spin chains [106, 107].

Supersymmetric Q-system were first introduced in [108], see also [109, 110]. We recall their definition in Section 10.1. As a warm-up for the Quantum Spectral Curve, to be introduced in the next Part, we explain how to solve a rational $\mathfrak{psu}_{1,1|2}$ Q-systems in Section 10.2 using the methods of [111, 112].

10.1 Supersymmetric Spin Chains

To obtain a Q-system for $\mathfrak{gl}_{m|n}$ from $\mathfrak{gl}_{m+n}$ one can use a trick called fermionisation (or "bosonisation") [113]. The prescription goes as follows: Let Q_M be a $\mathfrak{gl}_{m+n}$ Q-system, split the index $M = \{A, I\}, |A| = m, |I| = n$ and define

$$Q_{A|I} = \varepsilon^{\bar{J}I} Q_{A\bar{J}} \,. \tag{10.1}$$

The functions $Q_{A|I}$ form what we will call a $\mathfrak{gl}_{m|n}$ Q-system. For future reference, we write the QQ-relations in this new notation:

$$W(Q_{Aa|Ii}, Q_{A|I}) = Q_{Aa|I}\, Q_{A|Ii} \,, \tag{10.2a}$$

$$W(Q_{Aa|I}, Q_{Ab|I}) = Q_{Aab|I}\, Q_{A|I} \,, \tag{10.2b}$$

$$W(Q_{A|Ii}, Q_{A|Ij}) = Q_{A|I}\, Q_{A|Iij} \,. \tag{10.2c}$$

It is standard to refer to the indices A as bosonic, so that for example $Q_{a|\emptyset}$ is a bosonic Q-function, and the indices I as fermionic. This is only a naming convention, there is nothing fermionic about Q-functions.

10.1.1 Symmetries

Symmetries of the supersymmetric Q-system follow naturally from the $\mathfrak{gl}_n$ system described in Chapter 6.

Rotations

We can rotate bosonic Q-functions and fermionic Q-functions independently. The transformations are

$$Q_{a|\emptyset} \mapsto (H_{\mathrm{b}})_a{}^b Q_{b|\emptyset} \,, \qquad Q_{\emptyset|i} \mapsto (H_{\mathrm{f}})_i{}^j Q_{\emptyset|j} \,, \tag{10.3}$$

and extend to other Q-functions in a natural way.

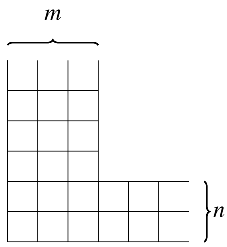


Figure 10.1. The supersymmetric L-hook for $\mathfrak{su}_{m|n}$, we associate $T_{a,s}$ to each vertex.

Gauge-transformations
The supersymmetric Q-system stays invariant under the same type of gauge transformations as its parent $\mathfrak{gl}_{m+n}$ Q-system. Gauge transformations can be parameterised as

$$Q_{A|I} \mapsto g_1^{[|A|-|I||]} g_2^{[-|A|+|I|]} Q_{A|I} \,. \tag{10.4}$$

Hodge duality
Finally, we have Hodge duality. This transformation acts as

$$Q_{A|I} \mapsto Q^{A|I} = (-1)^{|A||\bar{I}|} \epsilon^{\bar{A}A} \epsilon^{\bar{I}I} Q_{\bar{A}|\bar{I}} \,. \tag{10.5}$$

Compact rational spin chains and their T-system
The T-functions of a compact supersymmetric spin chain are labelled as $T_{a,s}$. The functional relations between $T_{a,s}$ are the same as those of a $\mathfrak{gl}_{m+n}$ system,

$$T_{a,s}^{+} T_{a,s}^{-} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1} \,, \tag{10.6}$$

but the shape of the T-system must be modified to that of an L-hook, see Figure 10.1. Intuitively we find two different infinite directions since in a supersymmetric system we can consider an arbitrary number of symmetrised bosons or an arbitrary number of antisymmetrised fermions.

10.2 The $\mathfrak{psu}_{1,1|2}$ Spin Chain

Throughout this thesis, we have so far only considered compact rational spin chains. In this section, we will take a look at a non-compact supersymmetric spin chain.

We will consider the example of $\mathfrak{psu}_{1,1|2}$, the reason for this choice is mainly due to its relevance for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ to be described in Chapter 13. Luckily, the generalisation from $\mathfrak{psu}_{1,1|2}$ to $\mathfrak{psu}_{2,2|4}$ relevant for $\mathrm{AdS}_5 \times \mathrm{S}^5$ is relatively straightforward.

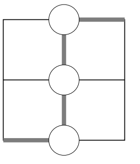
From this point on we set $\hbar = \mathtt{i}$ as is appropriate for AdS/CFT integrability.

To construct a representation of $\mathfrak{psu}_{1,1|2}$ we use oscillators. Let us introduce the notation

$$\mathtt{a}_1 = \mathtt{b}^\dagger\,, \qquad \mathtt{a}^1 = -\mathtt{b}\,, \qquad \mathtt{a}_2 = \mathtt{a}\,, \qquad \mathtt{a}^2 = \mathtt{a}^\dagger\,, \tag{10.7}$$

which gives the following generators

$$E = \begin{pmatrix} -\mathtt{b}\mathtt{b}^\dagger & -\mathtt{b}\mathtt{f}_a & -\mathtt{b}\mathtt{a} \\ \mathtt{f}_a^\dagger\mathtt{b}^\dagger & \mathtt{f}_a^\dagger\mathtt{f}_b & \mathtt{f}_a^\dagger\mathtt{a} \\ \mathtt{a}^\dagger\mathtt{b}^\dagger & \mathtt{a}^\dagger\mathtt{f}_a & \mathtt{a}^\dagger\mathtt{a} \end{pmatrix}$$



.

We will introduce the representation V_f which we define by its highest weight state $\phi_1 = |\mathrm{HWS}\rangle = \mathtt{f}_1^\dagger\,|0\rangle$ where $|0\rangle$ is the Fock vacuum $\mathtt{a}\,|0\rangle = \mathtt{b}\,|0\rangle = \mathtt{f}_a\,|0\rangle = 0$. We note that since $\mathtt{a}^\dagger\mathtt{b}^\dagger$ can act arbitrarily many times on $|0\rangle$ the representation is infinite-dimensional.

10.2.1 Bethe equations

The Bethe equations of a non-compact supersymmetric spin chain depend on the grading. To describe a spin chain with Hilbert space $\mathscr{H} = \bigotimes_{i=1}^L V_\mathrm{f}$ [114, 115] we introduce distinguished Q-functions $\mathbb{q}_{A|I} = \prod_i^{M_{A|I}}(u - u^i_{A|I})$ with $M_{A|I}$ an integer. The so-called auxiliary equations are as follows

$$\left.\frac{\mathbb{q}_{1|1}^+}{\mathbb{q}_{1|1}^-}\right|_{\mathbb{q}_{1|\emptyset}=0} = \left.\frac{\mathbb{q}_{1|1}^+}{\mathbb{q}_{1|1}^-}\right|_{\mathbb{q}_{\emptyset|1}=0} = \left.\frac{\mathbb{q}_{1|1}^+}{\mathbb{q}_{1|1}^-}\right|_{\mathbb{q}_{1|12}=0} = \left.\frac{\mathbb{q}_{1|1}^+}{\mathbb{q}_{1|1}^-}\right|_{\mathbb{q}_{12|1}=0} = 1\,. \tag{10.8}$$

While the remaining two equations, which we will call momentum-carrying, are given as

$$\left(\frac{u-\frac{\mathtt{i}}{2}}{u+\frac{\mathtt{i}}{2}}\right)^L \left.\frac{\mathbb{q}_{1|1}^{[2]}}{\mathbb{q}_{1|1}^{[-2]}}\frac{\mathbb{q}_{\emptyset|1}^-}{\mathbb{q}_{\emptyset|1}^+}\frac{\mathbb{q}_{12|1}^-}{\mathbb{q}_{12|1}^+}\right|_{\mathbb{q}_{1|1}=0} = -1\,, \tag{10.9a}$$

$$\left(\frac{u+\frac{\mathtt{i}}{2}}{u-\frac{\mathtt{i}}{2}}\right)^L \left.\frac{\mathbb{q}_{1|1}^{[2]}}{\mathbb{q}_{1|1}^{[-2]}}\frac{\mathbb{q}_{1|\emptyset}^-}{\mathbb{q}_{1|\emptyset}^+}\frac{\mathbb{q}_{1|12}^-}{\mathbb{q}_{1|12}^+}\right|_{\mathbb{q}_{1|1}=0} = -1\,. \tag{10.9b}$$

We will say that (10.9a) is in the $\mathfrak{su}_2$ sector and (10.9b) is in the $\mathfrak{sl}_2$ sector.

10.2.2 The Q-system

We now introduce the Q-system, it is composed of $Q_{a|\emptyset}, Q_{\emptyset|i}, Q_{a|i}, Q_{12|i}, Q_{a|12}$ and $Q_{12|12}$. The structure of the Q-functions mirrors that of representation theory to some extent. The indices a can be thought about as describing compact

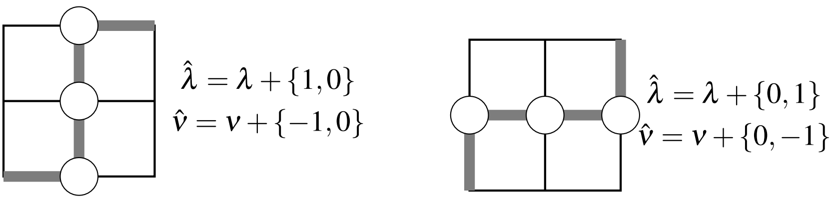


Figure 10.2. Instructions for how to find the shifted weights for states either in the grading $1\hat{1}\hat{2}2$ or $\hat{1}12\hat{2}$. For definition of weights see Section 2.4

directions while i describes non-compact directions. However, only finite-dimensional transformations act on the Q-functions. We will henceforth set $Q_{\emptyset|\emptyset} = Q_{12|12} = 1$.

To describe the spin chain we need to employ an analytic ansatz and specify the generalisations of Drinfeld polynomials for super spin chains. We will be very non-ambitious and avoid all definitions. By simply comparing against (10.9a) and (10.9b) we find the following prescription

$$Q_{a|\emptyset} = \frac{1}{u^L} q_{a|\emptyset}\,, \qquad\qquad Q_{\emptyset|i} = u^L q_{\emptyset|i}\,. \tag{10.10a}$$

To prepare ourselves for the AdS$_5$/CFT$_4$ QSC we introduce the notation

$$\mathbf{P}_a = Q_{a|\emptyset}\,, \qquad\qquad \mathbf{Q}_i = Q_{\emptyset|i}\,, \tag{10.11}$$

$$\mathbf{P}^a = Q^{a|\emptyset} = -\varepsilon^{ab} Q_{b|12}\,, \qquad \mathbf{Q}^i = Q^{\emptyset|i} = -\varepsilon^{ij} Q_{12|j}\,. \tag{10.12}$$

These functions have the following asymptotics

$$\mathbf{P}_a \simeq A_a u^{-\hat{\lambda}_a}\,, \qquad \mathbf{Q}_i \simeq B_i u^{-\hat{\nu}_i - 1}\,, \qquad Q_{a|i} \simeq \mathtt{i}\,\frac{A_a B_i}{\hat{\lambda}_i + \hat{\nu}_i}\, u^{-\hat{\lambda}_a - \hat{\nu}_i}\,, \tag{10.13}$$

where $\hat{\lambda}_a$ and $\hat{\nu}_i$ are so called shifted weights. They are equal to the standard weights of the state under consideration up to integer shifts. We summarise the off-sets in Figure 10.2 for the two gradings considered so far.

10.2.3 Solving the $\mathfrak{sl}_2$ sector

Let us consider a state of the schematic form $\left|\phi_1 \mathscr{D}^S \phi_1\right\rangle$ with $\mathscr{D} = \mathtt{a}^\dagger \mathtt{b}^\dagger$. In this sector we have

$$\hat{\lambda} = \{2,1\}\,, \qquad\qquad \hat{\nu} = \{-S-2, S-1\}\,, \tag{10.14a}$$

from which one can calculate

$$A_1 A^1 = -\mathtt{i}\, S(S+1)\,, \qquad\qquad A_2 A^2 = \mathtt{i}\, S(S+1)\,. \tag{10.15}$$

The first step to finding the full Q-system is to solve for a subset of functions $\mathbb{Q}_{a,s}$ called *distinguished Q-functions*. The distinguished Q-functions are defined as

$$\mathbb{Q}_{a,s} = Q_{1\ldots a|1\ldots s}\,. \tag{10.16}$$

An important property of $\mathbb{Q}_{a,s}$ is that they are rational functions of u. There exists an efficient method due to Marboe and Volin to find $\mathbb{Q}_{a,s}$ [111]. The idea is to embed the Q-system of $\mathfrak{gl}_{2|2}$ into a larger supersymmetric Q-system, we will not need it to solve the $\mathfrak{sl}_2$ sector.

Due to the high amount of symmetry it is possible to demand that $\mathbf{P}_1, \mathbf{P}^2$ are even and $\mathbf{P}_2, \mathbf{P}^1$ odd, this forces

$$\mathbf{P}_1 = \frac{A_1}{u^2}\,, \qquad \mathbf{P}_2 = \frac{A_2}{u}\,, \qquad \mathbf{P}^1 = A^1 u\,, \qquad \mathbf{P}^2 = A^2\,. \tag{10.17}$$

It is not hard to find all distinguished Q-functions explicitly in this case, they are given by

$$\left[\begin{array}{c|c|c} \mathbb{Q}_{2,0} & \mathbb{Q}_{2,1} & \mathbb{Q}_{2,2} \\ \hline \mathbb{Q}_{1,0} & \mathbb{Q}_{1,1} & \mathbb{Q}_{1,2} \\ \hline \mathbb{Q}_{0,0} & \mathbb{Q}_{0,1} & \mathbb{Q}_{0,2} \end{array}\right] = \left[\begin{array}{c|c|c} \frac{c^2}{(u^{[2]_D})^2} & \nabla^{S+1}(u^{[S]_D})^2 & 1 \\ \hline \frac{c}{u^2} & -\nabla^{S}(u^{[S]_D})^2 & 1 \\ \hline 1 & \nabla^{S-1}(u^{[S]_D})^2 & 1 \end{array}\right],$$

where $\nabla f = f^+ - f^-$. All other Q-functions follow now from QQ-relations. Let us note that using the QQ-relation $Q^+_{a|i} = (\delta^b_a - \mathbf{P}_a\mathbf{P}^b)Q^-_{b|i}$ it is possible to derive a 2-order finite difference equation for $Q_{1|i}$. It takes the form

$$\frac{1}{(\mathbf{P}_1\mathbf{P}^2)^+}\,Q^{[2]}_{1|i} - \left(\frac{1-(\mathbf{P}_1\mathbf{P}^1)^+}{(\mathbf{P}_1\mathbf{P}^2)^+} + \frac{1+(\mathbf{P}_1\mathbf{P}^1)^-}{(\mathbf{P}_1\mathbf{P}^2)^-}\right)Q_{1|i} + \frac{1}{(\mathbf{P}_1\mathbf{P}^2)^-}\,Q^{[-2]}_{1|i} = 0\,. \tag{10.18}$$

Using the form of $\mathbf{P}$ we find

$$(u^+)^2\,Q^{[2]}_{1|1} - (2u^2 - S(S+1) - \frac{1}{2})\,Q_{1|1} + (u^-)^2\,Q^{[-2]}_{1|1} = 0\,, \tag{10.19}$$

which is the famous $\mathfrak{sl}_2$ Baxter equation and the functions $\nabla^S(u^{[S]_D})^2$ are known as Hahn polynomials [32, 116–118].

Part III: Quantum Spectral Curve

The AdS/CFT duality [119–121] is one of the most influential developments in theoretical high-energy physics during the last three decades. Roughly it states that string theory or M-theory on anti-de Sitter space is dual to a conformal field theory. We will not review the correspondence, suffice it to say, the duality is very challenging to confirm. However, in specific settings progress has been made due to an integrable structure. In these cases, powerful tools have been developed that allow us to study the correspondence in detail. It is the purpose of this chapter to explain and explore how the formalism of Q-systems studied and developed in the previous part plays a key role in the integrability approach to AdS/CFT. This will lead us to the most powerful way to tackle the spectral problem of AdS/CFT to date, the Quantum Spectral Curve (QSC).

Integrability was first observed in the duality between type IIB superstrings on $\mathrm{AdS}_5\times\mathrm{S}^5$ and four-dimensional $\mathscr{N}=4$ Super Yang-Mills theory. We will briefly review some aspects of integrability in this context in Chapter 11. This review will be severely incomplete, luckily many of the key developments until 2010 are expertly reviewed in [122] which we refer to for more details and references. In Chapter 12 we will formulate the $\mathrm{AdS}_5\times\mathrm{S}^5$ Quantum Spectral Curve. Following the literature, we will investigate the curve in various limits. For clarity we will reproduce a variety of well-known results. As a bonus, we will also present some novel calculations for deformations of the curve. In Chapter 13 we turn our attention to planar string theory on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$. This theory is expected to be integrable, just like its higher dimensional cousin $\mathrm{AdS}_5\times\mathrm{S}^5$. We will describe a conjectured QSC from Paper IV and its predictions.

11. Integrability in AdS/CFT

In this chapter, we introduce integrability in the context of AdS/CFT. A brief recollection of the spectral problem in $\mathscr{N}=4$ and its relation to supersymmetric spin chains is presented in Section 11.1. The road to the QSC is outlined in Section 11.2. Section 11.3 mentions some of the advancements of integrability in the AdS$_4$/CFT$_3$ correspondence.

11.1 $\mathscr{N}=4$ and AdS$_5$/CFT$_4$

Four-dimensional $\mathscr{N}=4$ SYM is a special theory in many ways. It has a maximal amount of supersymmetry and it is conformal [123, 124], possessing the superconformal symmetry $\mathfrak{psu}(2,2|4)$. A pleasant fact is that we can write down the $\mathscr{N}=4$ Langrangian explicitly. It is most easily obtained by compactification of 10 dimensional $\mathscr{N}=1$ SYM [125]. The bosonic piece reads

$$S_{\mathscr{N}=4,\mathrm{bos}}=\frac{1}{2g_{\mathrm{YM}}^2}\int d^4x\left(-\frac{1}{2}\operatorname{tr}\mathscr{F}^2+\operatorname{tr}\mathscr{D}_\mu\Phi_I\mathscr{D}^\mu\Phi^I-\frac{1}{2}\operatorname{tr}[\Phi_I,\Phi_J][\Phi^I,\Phi^J]\right). \tag{11.1}$$

Here $\Phi^I, I=1,\dots,6$ are Lorentz scalars but vectors under the bosonic $\mathfrak{so}_6\simeq\mathfrak{su}_4$ subalgebra, $\mathscr{D}_\mu$ are covariant derivatives and $\mathscr{F}_{\mu\nu}$ is a field strength and a singlet of $\mathfrak{so}_6$. Both Φ^I and $\mathscr{F}_{\mu\nu}$ are $N\times N$ matrices of an $\mathfrak{su}(N)$ gauge group, this is the reason we took a trace in (11.1). We will often use the complex scalars Z,X,Y instead of Φ^I. The fields Φ^I and F combine with additional fermions into a supermultiplet $\mathscr{V}_{\mathrm{F}}$, sometimes called the singleton representation. The components of $\mathscr{V}_{\mathrm{F}}$ are schematically

$$\mathscr{V}_{\mathrm{F}}=\{\mathscr{D}^n\mathscr{F}^+,\mathscr{D}^n\mathscr{F}^-,\mathscr{D}^n\psi^i,\mathscr{D}^n\overline{\psi}_i,\mathscr{D}^n\Phi^I\}. \tag{11.2}$$

Here $\mathscr{F}^\pm$ is the (anti)-self-dual piece of the field strength and $\psi^i,\overline{\psi}_i$ are (co)-spinors transforming in the (dual-)defining representation of $\mathfrak{su}_4$.

We will in the following focus on single trace operators. These are gauge-invariant operators of schematic form $\mathscr{O}=\operatorname{tr}W_1W_2\dots$ with $W_i\in\mathscr{V}_{\mathrm{F}}$. Conformal invariance dictates that the two-point function is

$$\left\langle\mathscr{O}(x)\overline{\mathscr{O}}(y)\right\rangle\propto\frac{1}{|x-y|^{2\Delta}}, \tag{11.3}$$

where Δ is the conformal dimension of $\mathscr{O}$. On a representation theory level Δ is the eigenvalue of the dilation operator $D\in\mathfrak{psu}(2,2|4)$. At zero coupling the conformal dimension is simply the bare dimension of the operators, but has a non-trivial dependence on the coupling when it is non-zero.

A remarkable simplification takes place when restricting to the so-called planar limit; $N \to \infty, g_{\rm YM} \to 0$ while keeping the 't Hooft parameter $\lambda = g_{\rm YM}^2 N$ fixed. At one-loop level Minahan and Zarembo [1] showed that for the $\mathfrak{so}_6$-sector, which contains operators of type $\mathscr{O} = \chi_{I_1,I_2,\ldots I_n} \operatorname{tr}\left(\Phi^{I_1}\Phi^{I_2}\ldots\Phi^{I_n}\right)$, finding Δ is the same problem as diagonalising an $\mathfrak{so}_6$-spin chain. That is, the one-loop contribution to Δ is an eigenvalue of the Hamiltonian

$$H = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{L} \left(\mathbb{K}_{\ell,\ell+1} + 2 - 2\mathbb{P}_{\ell,\ell+1}\right) . \tag{11.4}$$

We note that this statement is non-trivial because integrability requires precisely the correct ratio between the coefficients of $\mathbb{K}$ and $\mathbb{P}$ and this number cannot be fixed by appealing to $\mathfrak{so}_6$-symmetry.

It was soon realized [126–129] that integrability extends to the full $\mathcal{N} = 4$ theory at 1-loop. The spectrum of Δ can thus be computed from supersymmetric Bethe equations. For example, for operators of schematic type $\operatorname{tr} Z^{L-M} X^M$, $\operatorname{tr} Z^{L-M} \psi^M$ or $\operatorname{tr} \mathscr{D}_+^M Z^L$ there is only one Bethe equation

$$\left(\frac{u_i + \frac{\mathtt{i}}{2}}{u_i - \frac{\mathtt{i}}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq i}}^{M} \left(\frac{u_i - u_j + \mathtt{i}}{u_i - u_j - \mathtt{i}}\right)^\eta . \tag{11.5}$$

with $\eta = 1, 0$ respectively -1. The energy γ is computed from

$$\gamma = 2g^2 \sum_{i=1}^{M} \frac{1}{u_i^2 + \frac{1}{4}} , \tag{11.6}$$

where we have introduced yet another coupling constant commonly denoted as g, it is defined by

$$g^2 = \frac{\lambda}{16\pi^2} = \frac{g_{\rm YM}^2 N}{16\pi^2} . \tag{11.7}$$

The results from gauge theory were complemented by progress on the string theory side. Using a coset-space formulation of the superstring [130] it was realised that the equation of motion admits a Lax representation [131]. This shows classical integrability of the superstring. The classical regime was then brought to the quantum realm by focusing on large charge operators such as the BMN operators [132]. For those operators it was possible to match string and gauge theory results [133, 134].

11.2 ABA and TBA

Soon there is a growing amount of evidence that integrability could extend to all loops. This led to the expectation that there should exist a set of Bethe equations generalising those of a rational super spin chain and smoothly interpolating between gauge theory and string theory. To this end, an Asymptotic Bethe Ansatz (ABA) was developed and Bethe equations for the full theory were proposed by Beisert and Staudacher [135]. The ABA contains two new ingredients we have not yet seen in this thesis: the first is the Zhukovsky variable x defined as

$$x+\frac{1}{x}=\frac{u}{g}\,. \tag{11.8}$$

The second is the dressing phase σ_{BES}, constructed by Beisert, Eden and Staudacher [117]. With these two new objects the Bethe equations for the one-loop anomalous dimension (11.5) can be extended to all loops, and the resulting equation is

$$\left(\frac{x_i^+}{x_i^-}\right)^L=\prod_{\substack{j=1\\j\neq i}}^{M}\left(\frac{x_i^+-x_j^-}{x_i^--x_j^+}\right)^{\eta}\frac{1-\frac{1}{x_i^+x_j^-}}{1-\frac{1}{x_i^-x_j^+}}\sigma_{\text{BES}}^2(x_i,x_j)\,. \tag{11.9}$$

Here as usual $x_i^{\pm}=x(u_i\pm\frac{\mathtt{i}}{2})$ and the anomalous part of the conformal dimension is computed as

$$\gamma=2\mathtt{i}\,g\left(\sum_{i=1}^{M}\frac{1}{x_i^+}-\frac{1}{x_i^-}\right)\,. \tag{11.10}$$

Beisert subsequently showed that these Bethe equations arise from a centrally extended $\mathfrak{su}_{2|2}$ S-matrix [136, 137]. This structure was also found from string theory [138, 139].

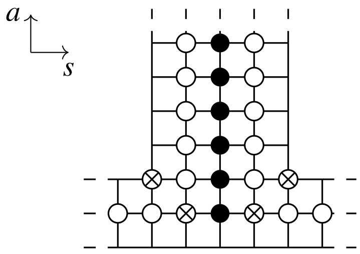
11.2.1 Wrapping it up

As the name indicates ABA is only valid in the asymptotic regime when the length of the single trace operator is large, $L\gg 1$. Thus it misses finite size corrections [140, 141]. However, corrections are also suppressed at weak coupling and the ABA gives reliable results to a relatively high loop order. For the simple operator $\mathscr{O}=\operatorname{tr}Z\mathscr{D}^2Z$, usually referred to as the ($\mathfrak{sl}_2$) Konishi-operator,

$$\gamma_{\text{ABA}}=12g^2-48g^4+336g^6-(2820+288\zeta_3)g^8+\dots \tag{11.11a}$$

$$\gamma_{\text{Exact}}=12g^2-48g^4+336g^6+(-2496+576\zeta_3-1440\zeta_5)g^8+\dots \tag{11.11b}$$

where the exact result was first computed in [142]. It was shown in [143] that the discrepancy between the ABA and the exact computation exactly corresponds to finite size corrections known as Lüscher corrections.



$$T^+_{a,s}T^-_{a,s} = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s}$$

$$Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$$

Figure 11.1. T-system and Y-system for the Mirror TBA of $AdS_5 \times S^5$.

To take finite size corrections into account a new method is needed, namely the mirror Thermodynamic Bethe Ansatz (TBA) [144–146][1]. For a pedagogical introduction to TBA see for example [148]. The TBA equations that replace the ABA are a set of integral equations on a set of functions $T_{a,s}$. The naming is no coincidence, $T_{a,s}$ are natural generalisations of the T-functions we encountered when considering spin chains. It is possible to show that the TBA equations imply that T-functions satisfy the canonical T-system equations $T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$ and forces $T_{a,s}$ to live on the T-hook diagram seen in Figure 11.1. While the T-functions of the TBA satisfy the same functional equations as those associated with a rational spin chain they are no longer polynomial functions of u, but have complicated analytic properties. This makes it difficult to solve the T-system. It was therefore a great success when the $\mathfrak{psu}(2,2|4)$ T-system was reformulated in terms of Q-functions. This task was accomplished in a series of papers [113, 149–151] culminating with the formulation of the Quantum Spectral Curve [2].

11.2.2 The dressing phase

The dressing phase is the most complicated object that appears in the ABA, unlike the other constituents it cannot be written as a rational function of the Zhukovsky variable. Viewed as a function of u $\sigma_{\text{BES}}(u,v)$ is a function defined on a sheet with two branch cuts outside of which it is analytic. Analytic continuation through the cuts is controlled by Janik's crossing equation [152]

$$\sigma^{\gamma_c}_{\text{BES}}\sigma_{\text{BES}} = \frac{y^-}{y^+}\frac{x^- - y^+}{x^+ - y^+}\frac{1-\frac{1}{x^-y^-}}{1-\frac{1}{x^+y^-}}\,. \tag{11.12}$$

We illustrate the analytic structure of σ_{BES} and display the curve γ_c in Figure 11.2. This is all the information we will need regarding the dressing phase, for a review of further properties and explicit expressions see [153].

[1] The name mirror theory is from [147]

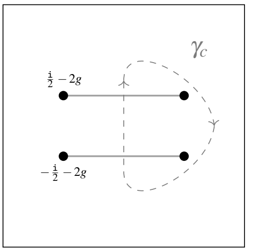


Figure 11.2. An illustration of the analytic structure of σ_{BES} and the curve γ_c. σ_{BES} have 4-branch points at $\pm\frac{i}{2}\pm 2g$, we connect them with short branch cuts.

11.3 Beyond AdS_5

Given the success of integrability in $\text{AdS}_5/\text{CFT}_4$ there exist many attempts to extend the same methods to other string backgrounds. While not every background is expected to allow for integrability [154] there are examples beyond AdS_5.

The most prominent example is the case of $\text{AdS}_4/\text{CFT}_3$, here integrability makes an appearance in the duality between type IIA strings on $\text{AdS}_4\times\mathbb{CP}$ and ABJM theory [155]. In this case, the underlying algebraic structure is $\mathfrak{osp}_{6|4}$. Integrability in the weak coupling limit was established in [156]. A TBA was formulated in [157] and then reformulated as a QSC in [9, 158].

12. Review of the $\mathrm{AdS}_5\times\mathrm{S}^5$ QSC

In the last section, we sketched how the spectral problem in $\mathrm{AdS}_5\times\ \mathrm{S}^5$ exhibits an integrable structure and can eventually be expressed in the language of Q-functions as the Quantum Spectral Curve. We will now describe this formalism in detail.

The main aim of this section is to introduce important aspects of the QSC in the well-understood setting of $\mathscr{N}=4$ large N gauge theory. This will provide us with the necessary background needed for Chapter 13.

The formalism of QSC is by now almost a decade old and there are a collection of pedagogical introductions available to which we refer for further details [159–161].

12.1 Formulation of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ Quantum Spectral Curve

The QSC is an analytic $\mathfrak{psu}_{2,2|4}$ Q-system, it consists of $Q_{A|I}$ where A and I are subsets of $\{1,2,3,4\}$. Not all of the original $2^8=256$ Q-functions are non-trivial, we will from now on set $Q_{\emptyset|\emptyset}=1$ and implement the $\mathfrak{p}$ of $\mathfrak{psu}$ with $Q_{\bar{\emptyset}|\bar{\emptyset}}=1$. We will utilise the following standard notation

$$\mathbf{P}_a\equiv Q_{a|\emptyset}\,,\qquad \mathbf{P}^a\equiv Q^{a|\emptyset}\,,\qquad \mathbf{Q}_i\equiv Q_{\emptyset|i}\,,\qquad \mathbf{Q}^i\equiv Q^{\emptyset|i}\,,\qquad (12.1)$$

which was already introduced for the $\mathfrak{psu}_{1,1|2}$ spin chain in Section 10.2.

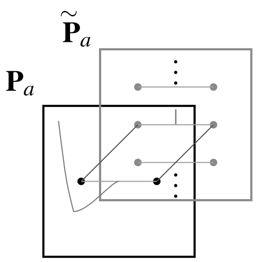
In the following, there will be a frequent need to use some basic QQ-relations. For convenience, we recall the most important relations

$$Q^+_{a|i}-Q^-_{a|i}=\mathbf{P}_a\,\mathbf{Q}_i\,,\qquad Q_{a|i}Q^{a|j}=-\delta_i^j\,,\qquad Q_{a|i}Q^{b|i}=-\delta_a^b\,,\qquad (12.2a)$$

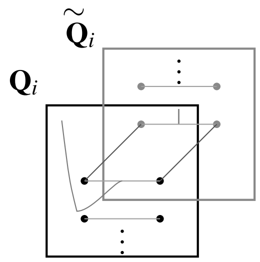
$$\mathbf{Q}_i=-\mathbf{P}^aQ^\pm_{a|i}\,,\qquad \mathbf{P}_a=-\mathbf{Q}^iQ^\pm_{a|i}\,,\qquad \mathbf{P}_a\mathbf{P}^a=\mathbf{Q}_i\mathbf{Q}^i=0\,.\qquad (12.2b)$$

The novel feature of the QSC Q-functions is their analytic properties. The Q-functions of the QSC are functions with branch cuts; they live on an infinite genus Riemann surface. Importantly there exists a gauge, which we will henceforth assume, in which the analytic properties of the Q-functions are remarkably simple. We will spend the rest of this section outlining the structure of the Q-functions in this gauge.

The QSC formalism states that there exists a sheet on which $\mathbf{P}_a$ and $\mathbf{P}^a$ are analytic outside of a short cut between $\pm 2g$ in the u plane. We will write



(a) Analytic Structure of $\mathbf{P}_a$



(b) Analytic Structure of $\mathbf{Q}_i$

$\widetilde{\mathbf{P}}^a$ for analytic continuation of $\mathbf{P}_a$ through this cut. In general $\widetilde{f}$ will mean analytic continuation around the point $2g$ on the real axis. It will turn out that all cuts are quadratic, details are given in Subsection 12.1.1, hence we need not distinguish if we crossed the cut from below or above and the notation $\tilde{f}$ is not ambiguous. The result of analytic continuation of $\mathbf{P}_a$ is encapsulated in the $\mathbf{P}\mu$ system:

$$\widetilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b\,, \qquad \widetilde{\mathbf{P}}^a = \mu^{ab}\mathbf{P}_b\,, \qquad \widetilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\widetilde{\mathbf{P}}_b - \widetilde{\mathbf{P}}_a\mathbf{P}_b\,, \tag{12.3}$$

where $\mu_{ab}\mu^{bc} = \delta_a^c$ and μ_{ab} is an anti-symmetric matrix with a tower of branch-points at $\pm 2g \pm \mathtt{i}n, n \in \mathbb{N}$. The analytic structure of $\mathbf{P}_a$ is illustrated in Figure 12.1a. Finally, we require that μ_{ab} is a *mirror-periodic function*:

$$\widetilde{\mu}_{ab} = \mu_{ab}^{[2]}\,. \tag{12.4}$$

A common gauge choice in the literature is to require that $\mathrm{Pf}(\mu) = 1$.

The analytic structure of $\mathbf{Q}_i$ nicely complements that of $\mathbf{P}_a$, they are functions analytic outside of a long cut: $(-\infty, -2g] \cup [2g, \infty)$. If one prefers to work with the defining sheet of $\mathbf{P}_a$ this implies that $\mathbf{Q}_i$ is analytic in the upper half-plane and $\widetilde{\mathbf{Q}}_i$ is analytic in the lower half-plane. We illustrate this structure in Figure 12.1b. The analogue of the $\mathbf{P}\mu$-system is called the $\mathbf{Q}\omega$-system, it is given as

$$\widetilde{\mathbf{Q}}_i = \omega_{ij}\mathbf{Q}^j\,, \qquad \widetilde{\mathbf{Q}}^i = \omega^{ij}\mathbf{Q}_j\,, \qquad \widetilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i\widetilde{\mathbf{Q}}_j - \mathbf{Q}_j\widetilde{\mathbf{Q}}_i\,. \tag{12.5}$$

where ω is a periodic matrix defined as

$$\omega_{ij} = Q^-_{a|i}\mu^{ab}Q^-_{b|j}\,. \tag{12.6}$$

Using the definition of ω and the relation $\mathbf{P}_a = -\mathbf{Q}^i Q^+_{a|i}$ it is possible to verify that the $\mathbf{P}\mu$ system implies the $\mathbf{Q}\omega$ system and vice versa.

Both $\mathbf{P}_a$ and $\mathbf{Q}_i$ are analytic in the upper half-plane, we require that the same is true for all other Q-functions. The remaining analytic properties can be deduced from those of $\mathbf{P}_a$ and $\mathbf{Q}_i$ using QQ-relations. As an example let us find the cut structure of $Q_{a|i}$ on the defining short cut sheet. Formally inverting the finite difference equation for $Q_{a|i}$ yields $Q_{a|i} = -\sum_{m=0}^{\infty} D^{2m+1}\,\mathbf{P}_a \mathbf{Q}_i$ which shows that $Q_{a|i}$ has a ladder of branch points at $\pm 2g - \frac{\mathtt{i}}{2} - \mathtt{i}n$, $n \in \mathbb{Z}_{\geq 0}$.

12.1.1 Further details and technical comments

In this section, we make a few technical remarks regarding the QSC construction. They are meant to clarify some of the properties already introduced or be of use at various stages when discussing large volume and weak coupling in the following sections.

Quadratic cuts

All cuts of the AdS_5 QSC are quadratic, this means that we travel to the same sheet if encircle the branch point at $2g$ either clockwise or anti-clockwise. To verify this claim we compute what happens if we cross the cut twice:

$$\tilde{\tilde{\mathbf{P}}}_a = \widetilde{\mu}_{ab}\widetilde{\mathbf{P}}^c = \mu_{ab}\widetilde{\mathbf{P}}^c = \mu_{ab}\mu^{bc}\mathbf{P}_c = \mathbf{P}_a\,. \tag{12.7}$$

Here we used $\mathbf{P}_a\mathbf{P}^a = 0, \mathbf{P}_a\widetilde{\mathbf{P}}^a = 0$ in the second step. We also immediately have that μ_{ab} have a quadratic cut from the $\mathbf{P}\mu$ system. The calculations for $\mathbf{Q}_i$ and ω_i are identical.

A Baxter equation for $\mathbf{Q}_i$

It is possible to write down a finite difference equation for $\mathbf{Q}_i$ with coefficients built only from $\mathbf{P}_a$ and $\mathbf{P}^a$, this was first noticed in [162]. To find the finite difference equation we repeat the trick of expanding a trivial determinant

$$\begin{vmatrix} (\mathbf{P}^a)^{[4]} & (\mathbf{P}^a)^{[2]} & \mathbf{P}^a & (\mathbf{P}^a)^{[-2]} & (\mathbf{P}^a)^{[-4]} \\ (\mathbf{P}^1)^{[4]} & (\mathbf{P}^1)^{[2]} & \mathbf{P}^1 & (\mathbf{P}^1)^{[-2]} & (\mathbf{P}^1)^{[-4]} \\ (\mathbf{P}^2)^{[4]} & (\mathbf{P}^2)^{[2]} & \mathbf{P}^2 & (\mathbf{P}^2)^{[-2]} & (\mathbf{P}^2)^{[-4]} \\ (\mathbf{P}^3)^{[4]} & (\mathbf{P}^3)^{[2]} & \mathbf{P}^3 & (\mathbf{P}^3)^{[-2]} & (\mathbf{P}^3)^{[-4]} \\ (\mathbf{P}^4)^{[4]} & (\mathbf{P}^4)^{[2]} & \mathbf{P}^4 & (\mathbf{P}^4)^{[-2]} & (\mathbf{P}^4)^{[-4]} \end{vmatrix} = 0\,. \tag{12.8}$$

Using $\mathbf{Q}_i = -\mathbf{P}^a Q_{a|i}^{\pm}$ and $Q_{a|i}^{+} - Q_{a|i}^{-} = \mathbf{P}_a\mathbf{Q}_i$ we can recast this as an equation on $\mathbf{Q}_i$.

$$\begin{aligned} &T_2\,\mathbf{Q}_i^{[4]} - \left(T_1 - T_2\,\mathbf{P}_a^{[2]}(\mathbf{P}^a)^{[4]}\right)\mathbf{Q}_i^{[2]} \\ &\qquad + \left(T_0 + T_{-1}\,\mathbf{P}_a(\mathbf{P}^a)^{[-2]} - T_{-2}\,\mathbf{P}_a(\mathbf{P}^a)^{[-4]}\right)\mathbf{Q}_i \\ &\qquad\qquad - \left(T_{-1} + T_{-2}\mathbf{P}_a^{[-2]}(\mathbf{P}^a)^{[-4]}\right)\mathbf{Q}_i^{[-2]} + T_{-2}\,\mathbf{Q}_i^{[-4]} = 0\,. \end{aligned} \tag{12.9}$$

Where $T_a = \left|(P^b)^{[6-2c-\theta(3-c+a)]}\right|_{\substack{b=1,\dots,4 \\ c=1,\dots,4}}$ and θ is the Heaviside function

The Symmetric Sector
The subalgebra preserving the complex scalar $Z = \Phi^1 + \mathtt{i}\Phi^2$ is $\mathfrak{psu}(2|2)^2$. Operators invariant under the exchange of the two $\mathfrak{psu}(2|2)$ are said to be in the symmetric sector. Q-functions describing these operators can, with the use of symmetry transformations, be brought to

$$\mathbf{P}_a = \chi_{ab}\mathbf{P}^b\,, \qquad \mathbf{Q}_i = \chi_{ij}\mathbf{Q}^j\,, \qquad \chi = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,. \tag{12.10}$$

Notice that χ is morally the matrix C in Chapter 7.

12.1.2 Monodromy Bootstrap

In [163] it was suggested that the structure of the QSC potentially can be bootstrapped with a few axioms. This method was subsequently put to use in Paper IV and there dubbed Monodromy Bootstrap.

To explain the idea of Monodromy Bootstrap we start by noticing that we have not treated the upper half-plane and lower half-plane democratically. Indeed, all our functions are upper half-plane analytic (UHPA) but not lower half-plane analytic (LHPA).

To restore the broken symmetry we introduce two new LHPA Q-system: ${Q_{A|}}^{I}, {Q^{A}}_{|I}$. We require that these systems are obtainable from $Q_{A|I}$ by a symmetry transformation. We need two systems because due to the branch cut there are two distinct away to travel to the lower-half plane: through the short cut or around it. We refer to the choice of avoiding short cuts as physical kinematics and the choice of avoiding long cuts as mirror kinematics. Explicitly we find

$$Q_{a|\emptyset} = {Q_{a|}}^{\emptyset}\,, \qquad Q_{\emptyset|i} = \omega_{ij}\,{Q_{\emptyset|}}^{j}\,, \qquad \text{(physical kinematics)}\,, \tag{12.11a}$$

$$Q_{a|\emptyset} = \mu_{ab}\,{Q^{b}}_{|\emptyset}\,, \qquad Q_{\emptyset|i} = {Q^{\emptyset}}_{|i}\,, \qquad \text{(mirror kinematics)}\,. \tag{12.11b}$$

To complete our journey around the branch point we also need to prescribe what happens when we consider ${Q_{A|}}^{I}$ in mirror kinematics and ${Q^{A}}_{I}$ in physical kinematics. We once again require that we can construct an UHPA system using a symmetry transformation. The equations close by requiring that this system is the Hodge dual Q-system $Q^{A|I}$.

$${Q^{a}}_{|\emptyset} = Q^{a|\emptyset}\,, \qquad {Q^{\emptyset}}_{|i} = \omega_{ij}Q^{\emptyset|j}\,, \qquad \text{(physical kinematics)}\,, \tag{12.12a}$$

$${Q_{a|}}^{\emptyset} = \mu_{ab}\,Q^{b|\emptyset}\,, \qquad {Q_{\emptyset|}}^{i} = Q^{\emptyset|i}\,, \qquad \text{(mirror kinematics)}\,. \tag{12.12b}$$

We have only written down symmetry transformations for a handful of functions, the transformations of the remaining functions follow from the general structure of symmetry transformations presented in Section 10.1.

Let us now explain how to recover all expressions from Section 12.1. By tracing the path around the branch point and making sure to always use Q-functions in an appropriate gauge we obtain the expressions for $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ from (12.3) and (12.5). From $(Q^{a}{}_{|i})^{-} = \mu^{ab} Q^{-}_{b|i} = \omega_{ij}(Q^{a|j})^{-}$ we obtain (12.6). Finally, we also need to find the discontinuity equations for μ and ω. This equation is obtained after noting that $(Q^{i}{}_{|a})^{-}$ does not have a cut on the real axis. Let $\Delta(f) = \widetilde{f} - f$, then

$$0 = \Delta\left((Q^{a}{}_{|i})^{-}\right) = \Delta(\omega_{ij}(Q^{-})^{a|j}) = (Q^{a|j})^{+}\Delta\left(\omega_{ij} - \widetilde{\mathbf{Q}}_i \mathbf{Q}_j\right). \tag{12.13}$$

Since $Q_{a|i}$ is invertible this gives the discontinuity of ω_{ij} in (12.5). The discontinuity of μ is obtained in the same way.

12.1.3 The spectral problem for single-trace operators

So far we have described the universal structure of the QSC. This information is not enough to constrain the QSC to give distinct solutions, we need to impose further analytic constraints. The exact form of these constraints depends on what problem we would like to study, finding them can be challenging. In some cases they can be derived from TBA or other independent methods, in other scenarios, one is forced to resolve to guesswork.

In this section, we describe how to fix analytic properties to solve the spectral problem for single trace operators in undeformed $\mathcal{N} = 4$. We will discuss twisted generalisations of this in Section 12.4. Historically the appropriate analytic properties were in this case suggested by TBA.

For the spectral problem of single trace operators all Q-functions are required to have polynomial asymptotics. Explicitly

$$\mathbf{P}_a \simeq A_a u^{-\hat{\lambda}_a}, \quad \mathbf{P}^a \simeq A^a u^{\hat{\lambda}_a - 1}, \quad \mathbf{Q}_i \simeq B_i u^{-\hat{\nu}_i - 1}, \quad \mathbf{Q}^i \simeq B^i u^{\hat{\nu}_i}, \tag{12.14}$$

where $\hat{\lambda}_a, \hat{\nu}_i$ are shifted weights, see Section 10.2. A standard choice in the literature is to work in the so-called non-compact ABA grading, we display the shifted weights for this choice in Figure 12.2. Since the QSC describes the spectrum at any coupling, $\hat{\nu}_i$ will in general not be an integer. It is possible to find the asymptotic behaviour of all other Q-functions from the Q-system.

Due to the asymptotics (12.14) the QQ-relations at large u are polynomial, this gives

$$A_a A^a = \mathtt{i}\frac{\prod_{i=1}^{4}(\hat{\lambda}_a + \hat{\nu}_i)}{\prod_{b\neq a}(\hat{\lambda}_a - \hat{\lambda}_b)}, \quad B_i B^i = \mathtt{i}\frac{\prod_{a=1}^{4}(\hat{\nu}_i + \hat{\lambda}_a)}{\prod_{j\neq i}(\hat{\nu}_i - \hat{\nu}_j)}, \quad \text{(no sum)}. \tag{12.15}$$

By definition, $\mathbf{P}_a$ is a single cut function with polynomial asymptotics. A very useful property for such a function is that we can resolve the first branch

$$\hat{\lambda} = \frac{1}{2}\{J_1+J_2-J_3+2, J_1-J_2+J_3,$$
$$-J_1+J_2+J_3+2, -J_1-J_2-J_3\}$$
$$\hat{\nu} = \frac{1}{2}\{-\Delta+S_1+S_2-2, -\Delta-S_1-S_2,$$
$$\Delta+S_1-S_2-2, \Delta-S_1+S_2\}$$

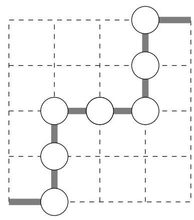


Figure 12.2. Map between quantum numbers and shifted weights in the non-compact ABA grading.

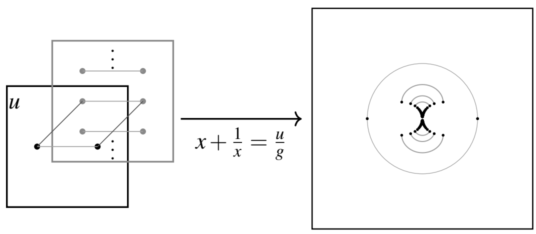


Figure 12.3. We can resolve the first branch-cut using the Zhukovsky parameter x.

cut using the Zhukovsky parameter. We illustrate this parameterisation in Figure 12.3. A useful parameterisation of $\mathbf{P}_a$ valid for most states is [3]

$$\mathbf{P}_a = \frac{1}{(gx)^L}\left(\sum_{n=0}^{-\hat{\lambda}_a+L} d_{a,n}(xg)^n + \sum_{n=1}^{\infty} c_{a,s}\left(\frac{g}{x}\right)^n\right), \tag{12.16a}$$

$$\mathbf{P}^a = \left(\sum_{n=0}^{\hat{\lambda}_a-1} d^H_{a,n}(xg)^n + \sum_{n=1}^{\infty} c^H_{a,n}\left(\frac{g}{x}\right)^n\right), \tag{12.16b}$$

which is a convergent series until the first branch points in the x-plane. The factors of g have been inserted so as to have $d_{a,n} \simeq c_{a,s} \simeq \mathcal{O}(g^0)$ at weak coupling.

12.2 The Large Volume Limit

Fully explicit analytic solutions of the QSC are not known. To solve the equations we need to take some form of limit. Here we consider a large volume limit to reproduce the ABA. We discuss the large volume solution of the $\mathrm{AdS}_5\times\mathrm{S}^5$ QSC to prepare ourselves for the case of $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ discussed in Paper IV and Paper V and reviewed in Section 13.3.5. The remainder of this section essentially follows [163] up to some cosmetic rearrangements.

The large volume limit is $\Delta, J_1 \to \infty$. The main assumption is that in this limit Q-functions will scale in the same way as their asymptotics. Introducing

$\varepsilon = u^{-\frac{\Delta}{2}} = u^{-\frac{J_1}{2}}$ we have the following scaling

$$Q_{a|i} \simeq \begin{pmatrix} 1 & 1 & \varepsilon^2 & \varepsilon^2 \\ 1 & 1 & \varepsilon^2 & \varepsilon^2 \\ \frac{1}{\varepsilon^2} & \frac{1}{\varepsilon^2} & 1 & 1 \\ \frac{1}{\varepsilon^2} & \frac{1}{\varepsilon^2} & 1 & 1 \end{pmatrix}, \quad \mathbf{P}_a \simeq \{\varepsilon, \varepsilon, \frac{1}{\varepsilon}, \frac{1}{\varepsilon}\}, \quad \mathbf{Q}_i \simeq \{\frac{1}{\varepsilon}, \frac{1}{\varepsilon}, \varepsilon, \varepsilon\}, \tag{12.17}$$

as well as $\omega^{ij} \sim 1$. In particular, this means that in large volume we find

$$\mu_{12} \simeq Q^-_{12|12}\omega^{12}. \tag{12.18}$$

12.2.1 Finding μ and ω

Let $\mathbb{Q} = \prod_{i=1}^N (u - u_i)$ encode the zeros of μ_{12}^+ and define $F^2 = \frac{\mathbb{Q}^-}{\mathbb{Q}^+}\frac{\mu_{12}^{[2]}}{\mu_{12}}$. From the fact that $\frac{\tilde{\mu}_{12}}{\mu_{12}} = \frac{Q^+_{12|12}}{Q^-_{12|12}} = \frac{Q^{+\,12}_{12|}}{Q^{-\,12}_{12|}}$ it follows that F is analytic outside of a short cut on both the first sheet and the second sheet. In other words: F is a single-valued function of the Zhukovsky variable without zeros or poles for $|x| > 1$ and with $F(\infty) = 1$. It satisfies the Riemann-Hilbert problem $F\widetilde{F} = \frac{\mathbb{Q}^+}{\mathbb{Q}^-}$ from which we can find F as

$$F = \pm \frac{B_{(+)}}{B_{(-)}}. \tag{12.19}$$

Here $B_{(\pm)}$ and $R_{(\pm)} = \widetilde{B}_{(\pm)}$, are given as

$$B_{(\pm)} = \prod_{n=1}^N \sqrt{\frac{g}{x_k^\mp}}(\frac{1}{x} - x_k^\mp), \qquad R_{(\pm)} = \prod_{n=1}^N \sqrt{\frac{g}{x_k^\mp}}(x - x_k^\mp). \tag{12.20}$$

This then allows us to fix ω^{12} and μ_{12}. We also obtain an expression for $Q_{12|12}$ from (12.17). Explicitly

$$\omega^{12} \propto \frac{\bar{f}}{f^{[2]}}, \qquad \mu_{12} \propto \mathbb{Q}^- f\bar{f}^{[-2]}, \qquad Q_{12|12} \propto \mathbb{Q}(f^+)^2. \tag{12.21}$$

The function f satisfies

$$\frac{f}{f^{[2]}} = \frac{B_{(+)}}{B_{(-)}}, \qquad f = \prod_{n=0}^\infty \frac{B_{(+)}^{[2n]}}{B_{(-)}^{[2n]}}, \qquad \bar{f} = \prod_{n=0}^\infty \frac{B_{(-)}^{[-2n]}}{B_{(+)}^{[-2n]}}. \tag{12.22}$$

At this stage, we can also verify that the dispersion relation of $\mathcal{N} = 4$ is correctly encoded in the large volume solution. To do so we compute the large u limit of $Q_{12|12}$, using the shifted weights $Q_{12|12} \simeq u^{-\hat{\lambda}_1 - \hat{\lambda}_2 - \hat{\nu}_1 - \hat{\nu}_2} \simeq u^{\Delta - J_1}$

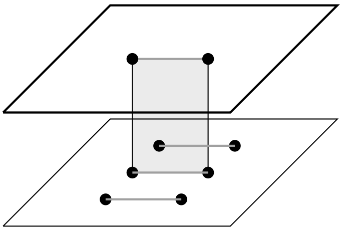


Figure 12.4. Analytic structure of σ, see (12.26).

and $\mathbb{Q} \simeq u^N$. The asymptotics of f is most easily obtained from (12.22). It follows that

$$f \simeq u^{\mathtt{i} g \sum_{i=1}^{N}\left(\frac{1}{x_i^+}-\frac{1}{x_i^-}\right)}\,, \qquad \Delta - J_1 - N = 2\,\mathtt{i}\,g \sum_{i=1}^{N}\left(\frac{1}{x_i^+}-\frac{1}{x_i^-}\right), \tag{12.23}$$

which reproduces the dispersion relation in (11.10).

12.2.2 Identifying the dressing phase

To obtain $\mathbf{P}_a$ we start by computing its analytic continuation using both UHPA and LHPA Q-functions

$$\widetilde{\mathbf{P}}_\alpha \simeq Q_{\alpha|12}\,\omega^{12}\,, \qquad\qquad \widetilde{\mathbf{P}}_\alpha \simeq \frac{1}{\omega^{12}} Q_{\alpha|}{}^{12}\,. \tag{12.24}$$

This implies that the combination $\widetilde{\mathbf{P}}/f^{[2]}\overline{f}^{[-2]}$ have only one cut on the real axis. Furthermore, the cut is quadratic and $\mathbf{P}_a$ is regular with polynomial asymptotics so we can parameterise

$$\mathbf{P}_\alpha \propto x^{-\frac{L}{2}} R_{\alpha|\emptyset} B_{\alpha|12}\,\sigma\,, \qquad Q_{\alpha|12} \propto x^{\frac{L}{2}} B_{\alpha|\emptyset} R_{\alpha|12} \frac{f f^{[2]}}{\sigma}\,, \tag{12.25}$$

where $R_{\alpha|\emptyset} = \widetilde{B}_{\alpha|\emptyset}$ contains the zeros of $\mathbf{P}_\alpha$ on the first sheet. $B_{\alpha|12} = \widetilde{R}_{\alpha|12}$ contains the zeros of $\mathbf{P}_\alpha$ on the second sheet and σ is a function satisfying

$$\sigma\widetilde{\sigma} = f^{[2]}\overline{f}^{[-2]}\,, \qquad\qquad \sigma(\infty) = 1\,. \tag{12.26}$$

We illustrate the analytic structure of σ in Figure 12.4.

All other functions can now be found with the use of QQ-relations, we will refrain from giving them all explicitly, they can be found in [135]. However, we do want to reproduce the ABA equation quoted in (11.9). From a Q-system point of view this equation comes from

$$\begin{aligned} W(Q_{12|12}, Q_{13|12}) =& Q_{1|12} Q_{123|12} \\ &\Longrightarrow \left.\frac{Q_{12|12}^{[2]}}{Q_{12|12}^{[-2]}} \frac{Q_{1|12}^-}{Q_{1|12}^+} \frac{Q_{123|12}^-}{Q_{123|12}^+}\right|_{Q_{12|12}=0} = -1\,, \end{aligned} \tag{12.27}$$

in the case when there are no auxiliary excitation, $R_{1|\emptyset} = B_{1|12} = 1$, and we pick $\eta = 1$ in (11.9). To write down this equation we also need to find $Q_{123|12}$. Luckily, we are in the symmetric sector $Q_{123|12} \propto Q_{1|12}$. Then using the expressions obtained so far we have

$$\frac{Q_{12|12}^{[2]}}{Q_{12|12}^{[-2]}} \frac{Q_{1|12}^{-}}{Q_{1|12}^{+}} \frac{Q_{123|12}^{-}}{Q_{123|12}^{+}} = \left(\frac{x^-}{x^+}\right)^L \frac{\mathbb{Q}^{[2]}}{\mathbb{Q}^{[-2]}} \left(\frac{\sigma^+}{\sigma^-}\right)^2 , \tag{12.28}$$

which matches perfectly with (11.9) if we identify $\frac{\sigma^+}{\sigma^-} = \prod_{i=1}^{N} \sigma_{\text{BES}}(u, u_i)$.

Let us take the time to verify that σ solves Janick's crossing equation. This is a straightforward application of (12.26):

$$\left(\frac{\sigma^+}{\sigma^-}\right)^{\gamma_c} = \frac{\sigma^-}{\sigma^+} \frac{R_{(-)}^-}{R_{(+)}^-} \frac{B_{(-)}^+}{B_{(+)}^+} = \frac{\sigma^-}{\sigma^+} \frac{R_{(-)}^-}{R_{(-)}^+} \frac{B_{(+)}^-}{B_{(+)}^+} . \tag{12.29}$$

So that after using the explicit expressions we find

$$\left(\frac{\sigma^+}{\sigma^-}\right)^{\gamma_c} \left(\frac{\sigma^+}{\sigma^-}\right) = \prod_{k=1}^{N} \frac{x^- - x_k^+}{x^+ - x_k^+} \frac{\frac{1}{x^-} - x_k^-}{\frac{1}{x^+} - x_k^-} = \frac{R_{(-)}^-}{R_{(-)}^{(+)}} \frac{B_{(+)}^-}{B_{(+)}^+} . \tag{12.30}$$

For a full derivation of the BES phase starting from σ see [164].

12.3 Weak Coupling Solution

In this section we describe how to solve the QSC at weak coupling. We will focus on so-called twist-two operators.

There currently exist two different methods on the market to extract the anomalous dimension at weak coupling. The first method is based on solving the $\mathbf{P}\mu$-system. It was originally developed for states in the $\mathfrak{sl}_2$ sector [165] and later streamlined and generalised to arbitrary states [118]. We describe it in Section 12.3.3. The other method, first described in [166], is based on constructing $\mathbf{Q}_i$ and imposing regularity conditions. We will describe it in Section 12.3.4.

When taking the weak coupling limit the cuts of Q-functions will collapse. This leads to poles in various functions, as a very simple example consider

$$g(x - \frac{1}{x}) = \sqrt{u - 2g}\,\sqrt{u + 2g} \simeq u - \frac{2g^2}{u} - \frac{2g^4}{u^3} + \mathscr{O}(g^6) . \tag{12.31}$$

Thus, although the Q-functions of the QSC are regular functions at finite coupling in weak coupling calculations we deal with poles.

12.3.1 The $\mathfrak{sl}_2$ sector

For clarity of exposition we will consider the following operators

$$\mathscr{O}_S = \operatorname{tr} Z \mathscr{D}^S Z + \text{perm}\,. \tag{12.32}$$

The shifted weights controlling the asymptotics of the Q-functions are given as

$$\hat{\lambda}_a = \{3,2,1,0\}\,, \quad \hat{\nu}_i = \{-3,-2-S,S-1,0\} - \frac{\gamma}{2}\{1,1,-1,-1\}\,. \tag{12.33a}$$

The ansatz (12.16) can be slightly simplified because the states under consideration are parity invariant. This is reflected in the Q-system as

$$\mathbf{P}_a(-u) = \mathbb{g}_a{}^b \mathbf{P}_b(u)\,, \tag{12.34}$$

with $\mathbb{g}$ a diagonal matrix.

At zero coupling the states $\operatorname{tr} Z\mathscr{D}^n Z$ are short, this is encoded in the QSC by $A_1 A^1 \simeq A_4 A^4 \simeq g^2, B_1 B^1 \simeq B_4 B^4 \simeq g^2$. We will gauge-fix so that $\mathbf{P}_1 \simeq g^2, \mathbf{Q}_4 \simeq g^2$.

12.3.2 The Q-system at zero coupling

The Q-system at zero coupling is that of a rational spin chain. To solve it we can use the methods explained in Section 10.2.

In particular, for the operators $\operatorname{tr} Z\mathscr{D}^n Z$ it is possible to find all distinguished Q-functions explicitly [118]. Since $\mathbf{P}_1^{(0)} = 0, \mathbf{Q}_4^{(0)} = 0$, we have $\mathbb{Q}_{a,0} = 0, a > 0, \mathbb{Q}_{a,4} = 0, a < 4$. Let us introduce notation $c = S(S+1)$ then in a convenient gauge the remaining distinguished Q-functions are

$$\left[\begin{array}{c|c|c|c|c}
0 & \frac{c^2(c-2)^2}{(u^{[3]_D})^2} & \nabla^{S+2} u^{[S]_D} & 1 & 1 \\
\hline
0 & \frac{c^2(c-2)}{(u^{[2]_D})^2} & -\nabla^{S+1}(u^{[S]_D})^2 & 1 & 0 \\
\hline
0 & \frac{c(c-2)}{u^2} & \nabla^{S}(u^{[S]_D})^2 & 1 & 0 \\
\hline
0 & (c-2) & -\nabla^{S-1}(u^{[S]_D})^2 & 1 & 0 \\
\hline
1 & u^S & \nabla^{S-2}(u^{[S]_D})^2 & 1 & 0
\end{array}\right],$$

with $\nabla f = f^+ - f^-$. The ordering is such that $\mathbb{Q}_{0,0}$ is in the lower left corner and $\mathbb{Q}_{4,4}$ is in the upper right corner, see Section 10.2.

With the distinguished Q-functions at our disposal one proceeds to find $\mathbf{Q}_i$ and $Q_{a|i}$. This can be done systematically by for example solving for $\mathbf{Q}_i$ using $\mathbb{Q}_{0,i}$ and then obtaining $Q_{a|1}$ using $\mathbb{Q}_{a,1}$. From this one easily constructs $\mathbf{P}_a$ and $Q_{a|i}$.

To construct μ_{ab} at weak coupling we take an ansatz with

$$\mu_{ab}^{(0)} = (Q_{ab|12}^{(0)})^-(\omega^{(0)})^{12}\,, \qquad (\omega^{(0)})^{ij} = 2\,(\omega^{(0)})^{12}\,\delta_{12}^{ij}\,. \tag{12.35}$$

One then proceeds to impose the equations $\widetilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b, \widetilde{\mathbf{P}}^a = \mu^{ab}\mathbf{P}_b$, this fixes Δ_1 and we can proceed to higher orders. Notice that $(\omega^0)^{12}$ scales with g, for the operators under consideration we have $(\omega^0)^{12} \simeq \frac{1}{g^4}$.

12.3.3 $\mathbf{P}\mu$-system approach

Using mirror periodicity we can rewrite the discontinuity of μ_{ab} as

$$\mu_{ab}^{[2]} = (\delta_a^c - \mathbf{P}_a\mathbf{P}^c)\mu_{cd}(\delta_b^d - \mathbf{P}^d\mathbf{P}_b)\,. \tag{12.36}$$

From (12.2) it follows that this equation is automaticaly satisfied for $\mu_{ab} = Q^-_{a|i}Q^-_{b|j}h^{ij}$ for h^{ij} any periodic function. The idea is to use that we already know $(Q^{(0)}_{a|i})^-$ which allows us to parameterise

$$\mu_{ab} = \frac{1}{2}Q^{(0)}_{ab|ij}h^{ij}\,, \qquad\qquad h^{ij} = \sum_{m=0}^{\infty}(h^{(m)})^{ij}g^{2m-4}\,. \tag{12.37}$$

Inserting (12.37) into (12.36) and using $Q_{ab|ij}Q^{ab|kl} = 4\delta_{ij}^{kl}$ a finite difference equation on $(h^{(m)})^{ij}$ is obtained

$$(h^{(m)})^{ij} - ((h^{(m)})^{ij})^{[2]} = \frac{1}{4}(Q^{(0)})^{ab|ij}S_{ab}^{(m)}\,, \tag{12.38}$$

$$S_{ab}^{(m)} = -\left.\left(\sum_{n=0}^{m-1} g^{2n-4}(\delta_a^c - \mathbf{P}_a\mathbf{P}^c)\mu_{cd}^{(n)}(\delta_b^d - \mathbf{P}^d\mathbf{P}_b)\right)\right|_{g^{2m-4}}\,. \tag{12.39}$$

The general solution is written as $h^{ij} = h_{\mathrm{P}}^{ij} + h_{\mathrm{H}}^{ij}$ with

$$(h_{\mathrm{P}}^{(m)})^{ij} = \frac{1}{1-D^2}\left(\frac{1}{4}(Q^{(0)})^{ab|ij}S_{ab}^{(m)}\right)\,, \quad (h_{\mathrm{H}}^{(m)})^{ij} = \sum_{n=0}^{\infty}(\phi_n^{(m)})^{ij}\mathscr{P}_n\,, \tag{12.40}$$

where $\mathscr{P}_{m\geq 1} = \sum_{n=-\infty}^{\infty}\frac{1}{(u+\mathtt{i}n)^m}, \mathscr{P}_0 = 1$ are periodic functions. To solve $\frac{1}{1-D^2}$ one must introduce so-called η-functions, see [167].

Since μ have a square-root cut on the real axis the combinations $\tilde{\mu}+\mu$ and $\frac{(\tilde{\mu}-\mu)}{x-\frac{1}{x}}$ do not have cuts on the real axis. This is very useful because it imposes that these combinations cannot have poles at the origin, recall that all poles of the QSC are due to the collapse of branch cuts. In other words, close to $u = 0$ we have

$$\mu_{ab}^{[2]} + \mu_{ab} = \mathrm{reg}\,, \qquad \mu_{ab}^{[2]} - \mu_{ab} = \sqrt{u-2g}\sqrt{u+2g}\times\mathrm{reg}\,. \tag{12.41}$$

Imposing these conditions fixes almost all of the coefficients ϕ. In particular, the regularity constraint sets an upper limit of the degree of the pole that is allowed to appear in μ [3].

One can now compute $\widetilde{\mathbf{P}}_a$ in two different ways, either from $\widetilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$ or from the ansatz (12.16) using $\tilde{x} = \frac{1}{x}$. By matching these two expansions we can fix the remaining coefficients.

12.3.4 Gluing $\mathbf{Q}$

We can find $Q_{a|i}$ in essentially the same way as we found μ in the previous section. Parameterise $Q^-_{a|i} = (Q^{(0)}_{a|j})^-(b)^j_i$, then from $Q^-_{a|i} - (\delta^b_a + \mathbf{P}_a\mathbf{P}^b)Q^+_{b|i} = 0$ we can solve for b perturbatively [166].

After this $Q^{a|i}$ can be constructed as the negative inverse transpose of $Q_{a|i}$ or by repeating the same exercise for $Q^{a|i}$. Having found $Q_{a|i}, Q^{a|i}$ and with $\mathbf{P}_a, \mathbf{P}^a$ at hand we can construct $\mathbf{Q}_i, \mathbf{Q}^i$. We will construct them so that $\mathbf{Q}_i = \chi_{ij}\mathbf{Q}^j$.

To now close the equations we can use the following trick: We know that $\tilde{\mathbf{Q}}_i$ have to be a LHPA function. Furthermore, it has to solve the same Baxter equation (12.9) as $\mathbf{Q}_i(-u)$ [1]. It follows that $\tilde{\mathbf{Q}}_i$ and $\mathbf{Q}_i(-u)$ must be related by an `i`-periodic matrix ${G_i}^j$,

$$\widetilde{\mathbf{Q}}_i(u) = {G_i}^j\mathbf{Q}_j(-u)\,. \tag{12.42}$$

One can severely constrain the form of ${G_i}^j$ by considering its asymptotics, we refer [168] for details. In the case at hand it turns out that (12.42) implies $\widetilde{\mathbf{Q}}_1 - \alpha\,\mathbf{Q}_3(-u)$ [4] with α constant.

Having found $\mathbf{Q}_i$ we can impose the same consistency equations that we used when discussing $\mathbf{P}\mu$, that is, at $u = 0$ we must have

$$\widetilde{\mathbf{Q}}_i + \mathbf{Q}_i = \text{reg}\,, \qquad \widetilde{\mathbf{Q}}_i - \mathbf{Q}_i = \sqrt{u-2g}\sqrt{u+2g}\times\text{reg}\,. \tag{12.43}$$

Imposing these conditions fixes all coefficients.

12.3.5 Explicit results

Implementing the above algorithm in Mathematica one finds perfect agreement with known results [169]

$$\Delta_1 = 8\,S_1(S)\,, \quad \Delta_2 = -16\left(2S_1\left(S_2 + S_{-2}\right) - 2S_{-2,1} + S_{-3} + S_3\right)\,, \tag{12.44}$$

with $S_{ab\ldots}$ harmonic sums, see for example [170]. For Konishi, $S = 2$, one also finds a perfect match with (11.11).

Of course, results for much higher loop orders are available in the literature. We only wrote down these results to make an explicit comparison with conjectured QSC for AdS_3 to be presented in Chapter 13. Using the approach of the $\mathbf{P}\mu$-system the anomalous dimension for twist-two operators have been found in terms of harmonic numbers up to seven loops [111, 171], see also [172].

In general, for large S limit the anomalous dimension scale as

$$\gamma \sim f(g)\log(S) + \ldots \tag{12.45}$$

[1] To make this statement we have utilised both the fact that we have parity and that we are in the symmetric sector. In more general cases one should consider $\overline{\mathbf{Q}}^i$

with $f(g)$ the so-called scaling function, see [173] for a review. Interestingly, $f(g)$ is expected to be fully reproduced from ABA, that is, wrapping effects does not contribute to $f(g)$. We will come back to this in Chapter 13 where we find something very different.

12.3.6 Numerical solution

To go beyond weak coupling we must generally give up on finding an analytic solution and resort to numerical methods. An efficient method to solve the QSC was proposed in [4]. Impressively, the method has been successfully used to study the conformal dimension even for complex spin.

The ansatz for $\mathbf{P}_a$ takes the same form as in the weak coupling solution,

$$\mathbf{P}_a = \sum_{n=\hat{\lambda}_a}^{\texttt{cutP}} \frac{c_{a,n}}{x^n}, \tag{12.46}$$

with `cutP` a fixed cut-off. Next we find $Q_{a|i}$ at large u as

$$Q_{a|i} \simeq \sum_{n=\hat{\lambda}_a+\hat{\nu}_i}^{\texttt{cutQai}} \frac{B_{a|i,n}}{u^n}. \tag{12.47}$$

The coefficients $B_{a|i,n}$ can be fixed from expanding $Q^+_{a|i} - (\delta^b_a - \mathbf{P}_a\mathbf{P}^b)Q^-_{b|i}$ at large u. Take $u_0 \in (-2g, 2g)$, after using the asymptotic expression to estimate $Q_{a|i}\left(u_0 + \mathrm{i}(\texttt{shQai} + \frac{1}{2})\right)$ for large `shQai` we can descend to the real axis by using $Q^-_{a|i} = (\delta^a_b + \mathbf{P}_a\mathbf{P}^b)Q^+_{b|i}$. Once we have obtained $Q_{a|i}(u_0 + \frac{\mathrm{i}}{2})$ it is possible to construct $\mathbf{Q}_i$ on the cut using $\mathbf{Q}_i = -\mathbf{P}^a Q^+_{a|i}$. On true solutions of the QSC we must have that $\widetilde{\mathbf{Q}}_1(u) - \alpha\mathbf{Q}_3(-u) = 0$. To find this point in parameter space we reformulate the gluing condition (12.42) as a minimization problem for the quantity

$$S = \sum_{m=1}^{\texttt{nbrPoints}} |\tilde{\mathbf{Q}}_1(u_m) - \alpha\mathbf{Q}_3(-u_m)|^2, \tag{12.48}$$

with $u_m \in (-2g, 2g)$ and `nbrPoints` the number of discrete points on the cut we choose to evaluate at.

12.4 Twisting the Curve

The weak coupling expansion presented in the previous section can readily be generalised to the case of the twisted QSC. The twisted QSC was first described in [83] and has been used in for example [174, 175] and to study the fishnet limit of AdS_5 [176].

$$\hat{\lambda} = \{3,2,2,-1\}$$

$$\hat{\nu}\Big|_{g=0} = \{-2,-3,0,-1\}$$

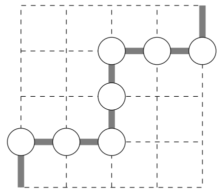


Figure 12.5. Shifted weights and grading used for the $\mathfrak{su}_2$ states in the $\mathbb{Z}_2$ orbifold.

As a non-trivial example of how to twist the QSC we will in this section consider the simple case of adding a twist of -1, still preserving $\mathcal{N}=2$. This is of interest because of recent endeavours to find integrable structures in $\mathcal{N}=2$ theories [177, 178], see also [179].

The idea to twist and deform $\mathcal{N}=4$ to find new integrable modes has a long history. Bethe equations of twisted $\mathcal{N}=4$ SYM was discussed in [180] and the case of an orbifold was treated in detail in [181]. A twisted TBA relevant for β-deformations and orbifolds was studied in [182–184] and Y-systems for β-deformations and orbifolds are constructed in [185, 186].

12.4.1 Weak coupling results

Let us outline the orbifold construction very schematically, we refer to [181] for details. We focus on the simplest possible case with a twist that squares to 1. In the twisted theory we can study operators of the form $\mathcal{O} = \mathrm{tr}(\gamma^s W_1 W_2 \dots W_L)$ with $\mathcal{W}_i \in V_{\mathrm{f}}$ and γ a new novel operator that implements the twist $s = 0,1$.

As an example we consider

$$\mathcal{O} = \mathrm{tr}\,\gamma ZXX + \mathrm{perm}\,, \tag{12.49}$$

The presence of γ gives additional exponential prefactors in the asymptotics of Q-functions, in our case we twist only the R-symmetry so that

$$\mathbf{P}_a \simeq \mathrm{x}_a^{\mathrm{i}u} A_a u^{-\hat{\lambda}_a}\,, \qquad \mathbf{Q}_i \simeq B_i u^{-\hat{\nu}_i}\,, \tag{12.50}$$

We will now fix the twist as

$$\mathrm{x}_a = \{1,1,-1,-1\}\,. \tag{12.51}$$

To avoid branch cut ambiguities we will set $\mathrm{x}_3^{\mathrm{i}u} = e^{-\pi u}, \mathrm{x}_4^{\mathrm{i}u} = e^{\pi u}$. We illustrate the appropriate grading and the shifted weights in Figure 12.5. The shifted weights of a twisted Q-system are computed as [83]

$$\hat{\lambda}_a = \lambda_a - \sum_{b \prec a} \delta_{\mathrm{x}_a,\mathrm{x}_b} + \sum_{i \prec a} \delta_{\mathrm{y}_i,\mathrm{x}_a}\,, \quad \hat{\nu}_i = \nu_i - \sum_{j \prec i} \delta_{\mathrm{y}_i,\mathrm{y}_j} + \sum_{a \prec i} \delta_{\mathrm{y}_i,\mathrm{x}_a}\,. \tag{12.52}$$

Solving for the distinguished Q-functions at $g=0$ one find up to overall prefactors

$$
\left[\begin{array}{c|c|c|c|c}
\frac{1}{(u^{[4]_D})^3}\times\left(u^4-\frac{3}{2}u^2-\frac{15}{16}\right) & \frac{1}{(u^{[3]_D})^3}\times(u^2-1)u^{[3]_D} & 1 & 1 & 1 \\
\hline
\mathtt{x}_3^{\mathtt{i}u}\times\frac{1}{(u^{[3]_D})^3}\times u & \mathtt{x}_3^{\mathtt{i}u}\frac{1}{(u^{[2]_D})^3}\times u^{[2]_D} & \mathtt{x}_3^{\mathtt{i}u} & \mathtt{x}_3^{\mathtt{i}u} & \mathtt{x}_3^{\mathtt{i}u} \\
\hline
\left(\frac{1}{u^{[2]_D}}\right)^3\times 1 & \frac{1}{u^3}\times u & u^2-\frac{1}{4} & u^2-\frac{1}{2} & u^2-\frac{3}{4} \\
\hline
\frac{1}{u^3} & 1 & u^3 & u^2-\frac{3}{4} & 0 \\
\hline
1 & 0 & 0 & 0 & 0
\end{array}\right]
$$

From which all other Q-functions can be worked out. Using then the method of Section 12.3.3 we computed the anomalous dimension to $\mathscr{O}(g^{12})$, the result is

$$
\begin{aligned}
\gamma = {} & 8g^2-32g^4+256g^6+g^8\left(-2560+512\zeta_3-640\zeta_5\right) \\
& +g^{10}\left(28672-4096\zeta_3-1536\zeta_3^2-5120\zeta_5+13440\zeta_7\right) \\
& +g^{12}\Big(-344064-6144\zeta_3^2+75776\zeta_5+24576\zeta_3 \\
& \qquad +46080\zeta_3\zeta_5+41216\zeta_7-217728\zeta_9\Big)+\mathscr{O}(g^{14})\,.
\end{aligned}
\tag{12.53}
$$

We observe wrapping at g^6 as compared to the ABA. Our results coincide perfectly with the computation in [187].

As an additional exercise, we generalised also the twist-two operators discussed earlier. The distinguished Q-functions can be taken as

$$
\left[\begin{array}{c|c|c|c|c}
0 & \frac{p_1(u)}{(u^{[3]_D})^3} & \tilde{\nabla}^2\nabla^S(u^{[S]_D})^S & 1 & 1 \\
\hline
0 & \mathtt{x}_3^{\mathtt{i}u}\frac{p_2(u)}{(u^{[2]_D})^2} & -\mathtt{i}\tilde{\nabla}\nabla^S(u^{[S]_D})^S\mathtt{x}_3^{\mathtt{i}u} & \mathtt{x}_3^{\mathtt{i}u} & 2\mathtt{x}_3^{\mathtt{i}u} \\
\hline
0 & \frac{1}{u^3}\times u & \nabla^S(u^{[S]_D})^2 & 1 & 4 \\
\hline
0 & c-2 & -\nabla^{S-1}(u^{[S]_D})^2 & 1 & 0 \\
\hline
1 & u^2 & \nabla^{S-2}(u^{[S]_D})^2 & 1 & 0
\end{array}\right]
$$

where $\tilde{\nabla}f=f^++f^-$ and p_1,p_2 are polynomials fixed by QQ-relations. Explicitly computing $S=2,4,6,\ldots,58$ for even S we found a perfect match with [184]:

$$
\gamma=g^2\,\Delta_1+g^4\,\Delta_2+g^6\,\Delta_3+g^6\Delta_3^{\text{wrapping}}+\mathscr{O}(g^8)\,, \tag{12.54}
$$

with

$$
\Delta_3^{\text{wrapping}}=64\,\frac{S_1(S)(S_2(S-1)-S_{-2}(S-1)-S_{-2}(S+1)-S_2(S+1))}{S(S+1)}\,, \tag{12.55}
$$

and Δ_1, Δ_2 given in (12.44). The undeformed Δ_3 was calculated in [169]. We notice that twisting the QSC does not alter the large S behaviour, we still have the same scaling function.

12.5 Outlook

In this brief review of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ QSC we have omitted many interesting applications. We have fully failed to discuss the connection between QSC and the classical spectral curve [188], we have not discussed small expansion in S and analytic continuation in quantum numbers [162, 168, 189–192]. The QSC has been successfully used to treat Wilson lines [5], the quark-anti quark potential [193] and to compute the Hagedorn temperature [8, 194]. There has also been widespread interest in trying to compute structure constants using the formalism [7, 195–198].

There are many other areas in which the QSC could be of use. Q-functions plays an important role in the computation of one-points functions [199–203], for reviews see [204, 205]. It would be exciting to extend the QSC to study determinant operators along the line of [206]. More general twists than the ones discussed in Section 12.4 are also of importance, this invites an exploration of the QSC with Drinfeld-Reshetikhin twist following [207], see also [208]

13. $\mathfrak{gl}(2|2)$ QSC and AdS$_3$/CFT$_2$

We have already mentioned that integrability appears in the AdS$_4$/CFT$_3$ correspondence, it is thus natural to look for further low-dimensional examples. Since three is just around the corner from four this is where we will venture in this chapter.

The most promising AdS$_3$ spaces are those of the form AdS$_3\times$S$^3\times$M$_4$ with M$_4$ =T^4 or M$_4$ = S$^3\times$S^1. The superisometry algebra of AdS$_3\times$S$^3\times$S$^3\times$S^1 is $\mathfrak{d}_{2,1;\alpha}\oplus\mathfrak{d}_{2,1;\alpha}$, here $\mathfrak{d}_{2,1;\alpha}$ is a one-parameter exceptional superalgebra. Unfortunately, no Q-system has been formulated for this algebra. Luckily, there exist a limit in which $\mathfrak{d}_{2,1;\alpha}$ becomes $\mathfrak{psu}_{1,1|2}$, in this case we are studying AdS$_3\times$S$^3\times$T^4. Since we understand $\mathfrak{psu}$ Q-systems there is still a fighting chance.

In Paper IV a method to construct QSCs when the underlying symmetry group is $\mathfrak{gl}_{2|2}$ was proposed. Using that method a conjectured QSC for AdS$_3\times$S$^3\times$T^4 was presented. The same curve was also proposed independently in [21] at the same time. Subsequently, a TBA for AdS$_3\times$S$^3\times$T^4 was constructed in [209], it is an outstanding question if the QSC and TBA agree or not.

In Section 13.1 we briefly review AdS$_3$ integrability, for further information see the review [210]. Thereafter in Section 13.2 we describe the procedure of Monodromy Bootstrap. Section 13.3 is dedicated to the conjectured curve and Section 13.4 discusses how to solve it at weak coupling. In Section 13.5 we discuss the dressing phases of AdS$_3$ in some more detail. Finally, Section 13.6 contains some conclusions and outlook.

13.1 ABA for AdS$_3\times$S^3

Green-Schwarz actions for strings on AdS$_3\times$S^3 were constructed in [211–214] and shown to admit a Lax formulation in [215], see also [216]. From the Lax matrix a set of finite gap equations were constructed. In AdS$_5$ the finite gap equations controlling the classical string are known to arise as a limit of spin chain Bethe equations, it was soon realised that the finite gap equations of AdS$_3$ could be obtained in the strong-coupling limit of a novel $\mathfrak{d}^2_{2,1;\alpha}$ spin chain [115]. At zero coupling this spin chain decomposes into two separate spin chains, unaware of each other existence, but when the coupling is turned on they start interacting. The promotion from zero coupling to finite coupling once again involves the introduction of the Zhukovsky variable. From the $\mathfrak{d}^2_{2,1;\alpha}$ spin chain an exact centrally extended $\mathfrak{su}^2_{1|1}$ S-matrix could be constructed [217], sharing many similarities with the centrally extended $\mathfrak{su}(2|2)$ S-matrix of $\mathcal{N}=4$, see also [218]. Bethe equations were finally obtained by diagonalising the S-matrix [219]. The calculations were subsequently redone for the case of a $\mathfrak{psu}(1,1|2)^2$ spin chain with massive excitations expected to describe $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ [220]. We write out the Bethe equations obtained at the end of this section in (13.1) and (13.2).

In AdS_3 one must deal not only with massive modes but also massless. From a spin chain point of view, the massless sector is rather mysterious, although some progress has been made [221,222].

Focusing instead on the string theory side a non-perturbative S-matrix satisfying Yang-Baxter and incorporating both massive and massless modes was found from symmetry considerations [223–226], see also [227,228] for earlier results on massive mixed flux and [229] for an alternative approach. Unfortunately, we will not have much to say about massless modes and mixed flux.

In the case of massive scattering, there are two dressing phases: $\sigma^{\bullet\bullet}$ and $\hat{\sigma}^{\bullet\bullet}$. We divide the Bethe equations of the massive sector into two sets, one for each $\mathfrak{psu}_{1,1|2}$. We emphasise that these two sets of equations are still coupled. We will call the first set the undotted sector, the Bethe equations of this sector are written in an $\mathfrak{su}_2$ grading, see Figure 13.1, and read [220]

$$1=\prod_{j=1}^{K_2}\frac{y_{1,k}-x_j^+}{y_{1,k}-x_j^-}\prod_{j=1}^{K_{\dot{2}}}\frac{1-\frac{1}{y_{1,k}\dot{x}_j^-}}{1-\frac{1}{y_{1,k}\dot{x}_j^+}}, \tag{13.1a}$$

$$\begin{aligned}\left(\frac{x_k^+}{x_k^-}\right)^L=&\prod_{\substack{j=1\\ j\neq k}}^{K_2}\frac{x_k^+-x_j^-}{x_k^--x_j^+}\frac{1-\frac{1}{x_k^+x_j^-}}{1-\frac{1}{x_k^-x_j^+}}\sigma^{\bullet\bullet}(x_k,x_j)^2\prod_{j=1}^{K_1}\frac{x_k^--y_{1,j}}{x_k^+-y_{1,k}}\prod^{K_3}\frac{x_k^--y_{3,j}}{x_k^+-y_{3,j}}\\ &\times\prod_{j=1}^{K_{\dot{2}}}\frac{1-\frac{1}{x_k^+\dot{x}_j^+}}{1-\frac{1}{x_k^-\dot{x}_j^-}}\frac{1-\frac{1}{x_k^+\dot{x}_j^-}}{1-\frac{1}{x_k^-\dot{x}_j^+}}\hat{\sigma}^{\bullet\bullet}(x_k,\dot{x}_j)^2\prod_{j=1}^{K_{\dot{1}}}\frac{1-\frac{1}{x_k^-y_{\dot{1},j}}}{1-\frac{1}{x_k^+y_{\dot{1},j}}}\prod_{j=1}^{K_{\dot{3}}}\frac{1-\frac{1}{x_k^-y_{\dot{3},j}}}{1-\frac{1}{x_k^+y_{\dot{3},j}}},\end{aligned} \tag{13.1b}$$

$$1=\prod_{j=1}^{K_2}\frac{y_{3,k}-x_j^+}{y_{3,k}-x_j^-}\prod_{j=1}^{K_{\dot{2}}}\frac{1-\frac{1}{y_{3,k}\dot{x}_j^-}}{1-\frac{1}{y_{3,k}\dot{x}_j^+}}. \tag{13.1c}$$

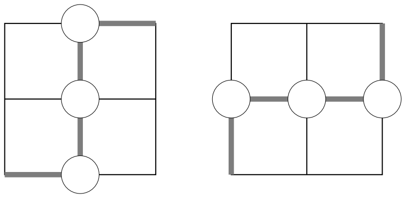


Figure 13.1. Grading used in the ABA for $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$.

The second set of equations, which we will call the dotted sector, is written in an $\mathfrak{sl}_2$ grading. The equations are

$$1=\prod_{j=1}^{K_{\dot 2}}\frac{y_{\dot 1,k}-\dot x_j^-}{y_{\dot 1,k}-\dot x_j^+}\prod_{j=1}^{K_2}\frac{1-\frac{1}{y_{\dot 1,k}x_j^+}}{1-\frac{1}{y_{\dot 1,k}x_j^-}}\,, \tag{13.2a}$$

$$\begin{aligned}\left(\frac{\dot x_k^+}{\dot x_k^-}\right)^L=&\prod_{\substack{j=1\\ j\neq k}}^{K_{\dot 2}}\frac{\dot x_k^- -\dot x_j^+}{\dot x_k^+ -\dot x_j^-}\frac{1-\frac{1}{\dot x_k^+\dot x_j^-}}{1-\frac{1}{\dot x_k^-\dot x_j^+}}\sigma^{\bullet\bullet}(\dot x_k,\dot x_j)^2\prod_{j=1}^{K_{\dot 1}}\frac{\dot x_k^- -y_{\dot 1,j}}{\dot x_k^+ -y_{\dot 1,k}}\prod^{K_{\dot 3}}\frac{\dot x_k^- -y_{\dot 3,j}}{\dot x_k^+ -y_{\dot 3,j}}\\ &\times\prod_{j=1}^{K_2}\frac{1-\frac{1}{\dot x_k^- x_j^-}}{1-\frac{1}{\dot x_k^+ x_j^+}}\frac{1-\frac{1}{\dot x_k^+ x_j^-}}{1-\frac{1}{\dot x_k^- x_j^+}}\hat\sigma^{\bullet\bullet}(\dot x_k,x_j)^2\prod_{j=1}^{K_1}\frac{1-\frac{1}{\dot x_k^+ y_{1,j}}}{1-\frac{1}{\dot x_k^- y_{1,j}}}\prod_{j=1}^{K_3}\frac{1-\frac{1}{\dot x_k^+ y_{3,j}}}{1-\frac{1}{\dot x_k^- y_{3,j}}}\,,\end{aligned} \tag{13.2b}$$

$$1=\prod_{j=1}^{K_{\dot 2}}\frac{y_{\dot 3,k}-\dot x_j^-}{y_{\dot 3,k}-\dot x_j^+}\prod\frac{1-\frac{1}{y_{\dot 3,k}x_j^+}}{1-\frac{1}{y_{\dot 3}x_j^-}}\,. \tag{13.2c}$$

While the equations are rather cumbersome in full generality they follow the expected group theoretical structure of $\mathfrak{psu}_{1,1|2}\oplus\mathfrak{psu}_{1,1|2}$, we illustrate this structure in Figure 13.1. Note in particular that if we were to send $g\to 0$ and assume that the dressing phases go to 1 the resulting Bethe equations are simply those of two rational $\mathfrak{psu}_{1,1|2}$ spin chains.

Finally, the anomalous dimension is computed from the standard formula

$$\gamma=2\mathtt{i}\,g\sum_{i=1}^{K_2}\left(\frac{1}{x_i^+}-\frac{1}{x_i^-}\right)+2\mathtt{i}\,g\sum_{i=1}^{K_{\dot 2}}\left(\frac{1}{\dot x_i^+}-\frac{1}{\dot x_i^-}\right)\,. \tag{13.3}$$

13.2 Monodromy Bootstrap

In Section 12.1.2 we described how the AdS_5 Quantum Spectral Curve can be found from a procedure we referred to as Monodromy Bootstrap. This idea was fully developed for $\mathfrak{gl}_{2|2}$ Q-systems in Paper IV. We will now review some

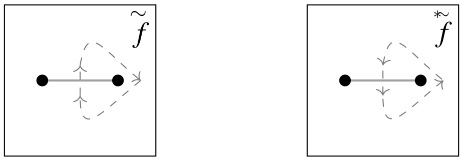


Figure 13.2. Encircling the branch point at $2g$ clockwise $\widetilde{f}$ or anti-clockwise $\overset{*}{\widetilde{f}}$ gives in general different results.

of the results of Paper IV. The method should extend naturally to theories of type $\mathfrak{gl}_{n|n}$. For models with symmetry $\mathfrak{gl}_{m|n}, m \neq n$ further investigations are needed, see [230–232].

Monodromy Bootstrap postulates that there exists an UHPA Q-system Q with $Q_{a|\emptyset}$ functions with short cuts and $Q_{\emptyset|i}$ functions with a long cut. While we do not require that $Q^{a|\emptyset}$ or $Q^{\emptyset|i}$ have a nice cut structure we do require that there exist a gauge transformation such that $Q^{a|\emptyset}$ have short cuts and $Q^{\emptyset|i}$ long cuts. To restore the broken symmetry between the upper half-plane and the lower half-plane we ask that there exist a symmetry transformation such that we can find a LHPA Q-system using either physical or mirror kinematics. We write schematically

$$Q^{\uparrow} = \omega_{\hat{h}} \cdot Q_1^{\downarrow} \quad (\text{physical kinematics})\,, \quad Q^{\uparrow} = \mu_{\check{h}} \cdot Q_2^{\downarrow} \quad (\text{mirror kinematics})\,. \tag{13.4}$$

where $\omega_{\hat{h}}$ and $\mu_{\check{h}}$ are shorthand notation for the application of an H-rotation ω/μ acting on fermionic/bosonic Q-functions and a gauge transformation $\hat{h}/\check{h}$.

There are now three options: $Q_1^{\downarrow} = Q_2^{\downarrow}, Q_1^{\downarrow} = (Q_2^{\downarrow})^{\star}$ or $Q_1^{\downarrow} \neq Q_2^{\downarrow}$ where $\star$ stands for Hodge duality. For the first two options this marks the end of the procedure. For the last option one is forced to repeat the same argumentation, introducing yet another UHPA Q-system and connect it to $Q_1^{\downarrow}, Q_2^{\downarrow}$. Once enough Q-systems have been introduced to close the procedure one now has to study the implications for the Q-systems and the symmetry transformations introduced. This was the main task of Paper IV.

A new technical complication not previously encountered in AdS_4 or AdS_5 is that Q-functions sometimes cannot have quadratic cuts, to account for this we will introduce notation $\widetilde{f}$ for clockwise analytic continuation around the branch point at $2g$ and $\overset{*}{\widetilde{f}}$ for anti-clockwise continuation around the same point. We illustrate this in Figure 13.2.

In Paper IV the option $Q_1^{\downarrow} = Q_2^{\downarrow}$ is called model A and $Q_1^{\downarrow} = (Q_2^{\downarrow})^{\star}$ model B. Schematically in these models analytic continuation of the Q-system reads

$$\widetilde{Q} = Q \quad (\text{Model A})\,, \qquad\qquad \widetilde{Q} = Q^{\star} \quad (\text{Model B})\,. \tag{13.5}$$

When $Q_1^{\downarrow} \neq Q_2^{\downarrow}$ two models were introduced. They were called model C and model D, they are schematically

$$\widetilde{Q} = \dot{Q}^{\star}\,, \widetilde{\dot{Q}} = Q^{\star} \quad (\text{Model C})\,, \qquad \widetilde{Q} = \dot{Q}\,, \widetilde{\dot{Q}} = Q^{\star} \quad (\text{Model D})\,, \tag{13.6}$$

with $\dot{Q}$ a new Q-system. When restricting attention to $\mathfrak{psu}_{1,1|2}$ model C and D coincides. It was conjectured in Paper IV that this model describes planar string theory on $\mathrm{AdS}_3{\times}\mathrm{S}^3{\times}\mathrm{T}^4$. We will spend the remaining part of this chapter detailing this curve.

13.3 A Conjectured QSC for $\mathrm{AdS}_3/\mathrm{CFT}_2$

In this section, we present the conjectured QSC for the spectral problem of strings on $\mathrm{AdS}_3{\times}\mathrm{S}^3{\times}\mathrm{T}^4$ in the presence of pure RR-flux explicitly. The presentation mirrors Section 12.1. We hope that this will ease the reading and clarify the similarities and differences between AdS_3 and AdS_5.

We will opt to mainly use the notation of Paper V, which agrees with [21]. The reason for this choice is that the notation of Paper IV uses an additional Hodge duality that is unnecessary to describe the AdS_3 QSC, it was present in Paper IV to make the comparison to the Hubbard model more natural. Furthermore, the author expects that future explorations of the AdS_3 curve will use the notation of Paper V, making this choice the most natural.

At this stage, we are also faced with the difficult task of naming the QSC. In the language of Paper IV the QSC is a type C/D curve, but this is not very snappy. We will in the following simply write the AdS_3 QSC with the understanding that it should be read as the Quantum Spectral Curve Conjectured to describe $\mathrm{AdS}^3{\times}\mathrm{S}^3{\times}\mathrm{T}^4$ with pure RR-flux in the planar limit. We emphasise that until a reliable comparison between the spectrum of $\mathrm{AdS}_3{\times}\mathrm{S}^3$ and the proposed QSC is performed it is fully possible that the curve must be modified, perhaps using twisting, or be fully discarded.

13.3.1 Formulation of the QSC

The AdS_3 QSC is an analytic $\mathfrak{psu}_{1,1|2} \oplus \mathfrak{psu}_{1,1|2}$ Q-system. To distinguish the two different Q-systems we will decorate one set of them with dotted indices. The full Q-system then consists of functions $Q_{A|K}, Q_{\dot{A}|\dot{K}}$ with $Q_{\emptyset|\emptyset} = Q_{\dot{\emptyset}|\dot{\emptyset}} = Q_{12|12} = Q_{\dot{1}\dot{2}|\dot{1}\dot{2}} = 1$. We will change the notation slightly from the $\mathrm{AdS}_5{\times}\mathrm{S}^5$ case and write [1]

$$\mathbf{P}_a = Q_{a|\emptyset}\,, \quad \mathbf{Q}_k = Q_{\emptyset|k}\,, \quad \mathbf{P}^a = -\varepsilon^{ab} Q_{b|12}\,, \quad \mathbf{Q}^k = -\varepsilon^{kl} Q_{\emptyset|l}\,, \tag{13.7}$$

[1] We will switch to using k instead of i in this section. The reason is that $\dot{i}$ is very hard to read.

as well as

$$Q^{a|k} = \epsilon^{ab}\epsilon^{kl}Q_{b|l}\,. \tag{13.8}$$

For future convenience, we note the most important QQ-relations

$$Q^{+}_{a|i} - Q^{-}_{a|i} = \mathbf{P}_a\mathbf{Q}_i\,, \qquad \mathbf{P}_a = Q^{\pm}_{a|k}\mathbf{Q}^k\,, \qquad \mathbf{Q}_k = Q^{\pm}_{a|k}\mathbf{P}^a\,, \tag{13.9a}$$

$$Q_{a|i}Q^{a|j} = \delta^i_j\,, \qquad Q_{a|k}Q^{b|k} = \delta^b_a\,, \qquad \mathbf{P}_a\mathbf{P}^a = \mathbf{Q}_i\mathbf{Q}^i = 0\,. \tag{13.9b}$$

There exists a sheet, the defining sheet, on which the functions $\mathbf{P}_a, \mathbf{P}_{\dot{a}}, \mathbf{P}^a, \mathbf{P}^{\dot{a}}$ are analytic outside of a short cut in the u-plane stretching between $\pm 2g$. The $\mathbf{P}\mu$ system is

$$\tilde{\mathbf{P}}_a = \mathbf{P}_{\dot{a}}\,\mu^{\dot{a}}{}_a\,, \qquad \tilde{\mathbf{P}}^a = \mathbf{P}^{\dot{a}}\mu_{\dot{a}}{}^a\,, \qquad \overset{*}{\tilde{\mu}}_a{}^{\dot{b}} - \mu_a{}^{\dot{b}} = \mathbf{P}_a\tilde{\mathbf{P}}^{\dot{b}} - \overset{*}{\tilde{\mathbf{P}}}_a\mathbf{P}^{\dot{b}}\,, \tag{13.10}$$

with $\mu^a{}_{\dot{c}}\,\mu_b{}^{\dot{c}} = \delta^a_b$ and with another set of equations obtained by replacing $a \leftrightarrow \dot{a}$. In particular $\mu_a{}^{\dot{b}}$ is mirror-periodic, that is

$$\overset{*}{\tilde{\mu}}_a{}^{\dot{b}} = (\mu_a{}^{\dot{b}})^{[2]}\,. \tag{13.11}$$

The functions $\mathbf{Q}_k, \mathbf{Q}_{\dot{k}}, \mathbf{Q}^k, \mathbf{Q}^{\dot{k}}$ are functions with a long cut, they satisfy the $\mathbf{Q}\omega$ system

$$\overset{*}{\tilde{\mathbf{Q}}}_k = \omega_k{}^{\dot{l}}\mathbf{Q}_{\dot{l}}\,, \qquad \overset{*}{\tilde{\mathbf{Q}}}^k = \omega^k{}_{\dot{l}}\mathbf{Q}^{\dot{l}}\,, \qquad \tilde{\omega}_k{}^{\dot{l}} - \omega_k{}^{\dot{l}} = \mathbf{Q}_k\tilde{\mathbf{Q}}^{\dot{l}} - \overset{*}{\tilde{\mathbf{Q}}}_k\mathbf{Q}^{\dot{l}}\,, \tag{13.12}$$

where $\omega^i{}_{\dot{k}}\omega_j{}^{\dot{k}} = \delta^i_j$. There exists another set of equations obtained by sending $k \leftrightarrow \dot{k}$. ω is a periodic function, it is related to μ as

$$\mu_a{}^{\dot{b}} = Q^{-}_{a|k}\,\omega^k{}_{\dot{l}}\,(Q^{\dot{b}|\dot{l}})^{-}\,, \qquad \mu^a{}_{\dot{b}} = (Q^{a|k})^{-}\,\omega_k{}^{\dot{l}}Q^{-}_{\dot{a}|\dot{l}}\,. \tag{13.13}$$

13.3.2 Technical details

In this section, we comment on a few additional technical points.

Gauge transformations

We have not written the $\mathbf{P}\mu$, nor the $\mathbf{Q}\omega$-system, in its most general form. It should be clear that if one performs two separate gauge transformations for the dotted and undotted system this will in general not preserve the form of these systems. The remedy to this in Paper IV was to introduce a prefactor r in these systems, so that for example $\tilde{\mathbf{P}}_a = r\mathbf{P}_{\dot{a}}\mu^{\dot{a}}{}_a$. A way to avoid having to worry about gauge factors is to only consider gauge invariant combinations such as $\mathbf{P}_a\mathbf{P}^b$.

Spinning around the branch points
In AdS_5 we found that all branch cuts were quadratic. Since this is no longer the case it would be desirable to have a compact formula for how to perform analytic continuation around the branch point at $2g$ an arbitrary amount of times. Expressions for this were found in [21]. Let us introduce

$$(\mu^R)_a{}^{\dot b} = \mu_a{}^{\dot b} + \mathbf{P}_a \widetilde{\mathbf{P}}^{\dot b}\,, \qquad (\omega^R)_k{}^{\dot l} = \omega_k{}^{\dot l} - \overset{*\sim}{\mathbf{Q}}_k \mathbf{Q}^{\dot l}\,. \tag{13.14}$$

Note that these objects do not have cuts on the real axis by the $\mathbf{P}\mu$ and $\mathbf{Q}\omega$-systems. Let us then write

$$W_a{}^b = (\mu^R)_a{}^{\dot c} (\mu^R)_{\dot c}{}^b\,, \qquad U_k{}^l = (\omega^R)_k{}^{\dot m} (\omega^R)_{\dot m}{}^l\,. \tag{13.15}$$

Using these objects one finds

$$\overset{*\sim}{\overset{*\sim}{\mathbf{P}}}_a = W_a{}^b \mathbf{P}_b\,, \qquad \overset{*\sim}{\overset{*\sim}{\mathbf{Q}}}_k = U_k{}^l \mathbf{Q}_l\,. \tag{13.16}$$

13.3.3 Monodromy Bootstrap

Let us now outline how to reproduce the equations given in Setion 13.3.1 . We start by introducing 4 different Q-systems: $Q_{A|I}, Q_{\dot A|I}, Q_{A|\dot I}, Q_{\dot A|\dot I}$ where $Q_{A|I}$ and $Q_{\dot A|\dot I}$ are UHPA and $Q_{A|\dot I}, Q_{\dot A|I}$ LHPA. We require that there exist symmetry transformations $\mu, \dot\mu, \omega, \dot\omega$ such that

$$Q_{a|\emptyset} = Q_{a|\dot\emptyset}\,, \qquad Q_{\emptyset|k} = Q_{\emptyset|\dot l}\, \omega^{\dot l}{}_k\,, \quad Q_{\dot a|\dot\emptyset} = Q_{\dot a|\emptyset}\,, \qquad Q_{\dot\emptyset|\dot k} = Q_{\dot\emptyset|l}\, \omega^l{}_{\dot k}\,, \quad \text{(p.k)}, \tag{13.17a}$$

$$Q_{\dot a|\dot\emptyset} = \mu_{\dot a}{}^b Q_{b|\dot\emptyset}\,, \quad Q_{\dot\emptyset|\dot k} = Q_{\emptyset|\dot k}\,, \qquad Q_{a|\emptyset} = \mu_a{}^{\dot b} Q_{\dot b|\emptyset}\,, \quad Q_{\emptyset|k} = Q_{\dot\emptyset|k}\,, \qquad \text{(m.k)}. \tag{13.17b}$$

Where p.k is short for physical kinematics and m.k is short for mirror kinematics. We can now quickly verify the formulas in Section 13.3.1.

The expressions for $\widetilde{\mathbf{P}}_a$ and $\overset{*\sim}{\mathbf{Q}}_i$ follows immediately from (13.17). From $Q^-_{\dot a|k} = \mu^b{}_{\dot a} Q^-_{b|k} = \omega_k{}^{\dot l} Q^-_{\dot a|\dot l}$ we find (13.13). Finally, the discontinuity of ω follows from $\Delta(Q^-_{a|\dot k}) = 0$.

13.3.4 The spectral problem

It was proposed in Paper IV and [21] that for the spectral problem the asymptotic of all Q-functions should be powerlike

$$\mathbf{P}_a \simeq A_a u^{-\hat\lambda_a}\,, \quad \mathbf{P}_{\dot a} \simeq A_{\dot a} u^{-\hat\lambda_{\dot a}}\,, \quad \mathbf{Q}_k \sim B_k u^{-\hat\nu_k - 1}\,, \quad \mathbf{Q}_{\dot k} \sim B_{\dot k} u^{-\hat\nu_{\dot k} - 1}\,. \tag{13.18}$$

In terms of the magnon-numbers of (13.1) and (13.2) we find

$$\hat{\lambda}_a = \{\frac{L}{2} - K_2 + K_1 + 1, -\frac{L}{2} + K_2 - K_3\}, \tag{13.19a}$$

$$\hat{\lambda}_{\dot{a}} = \{\frac{L}{2} - K_{\dot{3}}, -\frac{L}{2} + K_{\dot{1}} + 1\}, \tag{13.19b}$$

$$\hat{\nu}_k = \{-\frac{\gamma}{2} - \frac{L}{2} - K_1 - 1, \frac{\gamma}{2} + \frac{L}{2} + K_3\}, \tag{13.19c}$$

$$\hat{\nu}_{\dot{k}} = \{-\frac{\gamma}{2} - \frac{L}{2} - K_{\dot{2}} + K_{\dot{3}}, \frac{\gamma}{2} + \frac{L}{2} + K_{\dot{2}} - K_{\dot{1}} - 1\}. \tag{13.19d}$$

We use the explicit labels appearing in the ABA for ease of comparison.

13.3.5 Large volume solution

To see if the structure proposed in the previous section have any chance of reproducing the spectrum of $AdS_3 \times S^3 \times T^4$ we must first verify that it can reproduce the Asymptotic Bethe Ansatz equations presented in Section 13.1. This exercise was carried out in Paper V and in [21]. We reproduce parts of the calculation here to highlight the new features found but will keep the discussion brief.

By large volume we mean taking $L \to \infty$ and using the assumption that Q-functions will scale as their asymptotics. The large volume derivation follows to a large extent that of AdS_5 presented in Section 12.2. Once again we introduces $\varepsilon = u^{-\frac{L}{2}}$ and find

$$Q_{a|k} \sim Q_{\dot{a}|\dot{k}} \sim \begin{pmatrix} 1 & \varepsilon^2 \\ \frac{1}{\varepsilon^2} & 1 \end{pmatrix}, \qquad \mathbf{P}_a, \mathbf{P}_{\dot{a}} \sim \{\varepsilon, \frac{1}{\varepsilon}\}, \tag{13.20}$$

and so

$$\mu_1{}^{\dot{2}} \sim Q^-_{1|1} Q^-_{\dot{1}|\dot{1}} \omega^1{}_{\dot{2}}. \tag{13.21}$$

Finding μ and ω

In the case of AdS_3 we need one more assumption: $\mu_1{}^{\dot{2}}$ should be a square-root cut function in the large volume limit. Then we can repeat the same calculation presented in Section 12.2.1. We write $\mathbb{Q}_{\text{tot}} = \mathbb{Q}_2 \mathbb{Q}_{\dot{2}}$ for the zeros of $\mu_1{}^{\dot{2}}$ and $\mu_{\dot{1}}{}^2$. Just as in AdS_5 we introduce

$$R_{(\pm),2} = \prod_{k=1}^{K_2} \sqrt{\frac{g}{x_k^{\mp}}}(x - x_k^{\mp}), \qquad R_{(\pm),\dot{2}} = \prod_{k=1}^{K_{\dot{2}}} \sqrt{\frac{g}{\dot{x}_k^{\mp}}}(x - \dot{x}_k^{\mp}), \tag{13.22}$$

and $B_{(\pm),k} = \widetilde{R}_{(\pm),k}$ and $\frac{f_k}{f_k^{[2]}} = \frac{B_{(+),k}}{B_{(-),k}}$. Then

$$\mu_1{}^{\dot{2}} \propto \mu_{\dot{1}}{}^2 \propto \mathbb{Q}_2^- \mathbb{Q}_{\dot{2}}^- f_{\text{tot}} \bar{f}_{\text{tot}}^{[-2]}, \quad \omega^1{}_{\dot{2}} \propto \omega^{\dot{1}}{}_2 \propto \frac{\bar{f}_{\text{tot}}^{[-2]}}{f_{\text{tot}}}, \quad f_{\text{tot}} = f_2 f_{\dot{2}}. \tag{13.23}$$

Using that $Q^-_{1|\dot{2}} \propto \omega^{\dot{1}}{}_2 \, Q^-_{1|1}$ we find $Q_{1|1}$ and $Q_{\dot{1}|\dot{1}}$ as

$$Q_{1|1} = \mathbb{Q}_2 \, f^+_{\rm tot}\,, \qquad\qquad Q_{\dot{1}|\dot{1}} = \mathbb{Q}_{\dot{2}} \, f^+_{\rm tot}\,. \tag{13.24}$$

We note the strong similarities between these results and those of AdS_5. We should also mention that very similar structures appear in ABJM, see [158].

Dispersion relation

We can immediately reproduce the dispersion relation (13.3) using the same arguments as in AdS_5. From

$$Q_{1|1} Q_{\dot{1}|\dot{1}} \simeq u^{\gamma + K_2 + K_{\dot{2}}}\,, \tag{13.25}$$

and from the large u limit of $f_{\rm tot}$ we obtain

$$\gamma = 2\,\mathtt{i}\, g \left(\sum_{k=1}^{K_2} \Big(\frac{1}{x^+_k} - \frac{1}{x^-_k}\Big) + \sum_{k=1}^{K_{\dot{2}}} \Big(\frac{1}{\dot{x}^+_k} - \frac{1}{\dot{x}^-_k}\Big) \right)\,. \tag{13.26}$$

Finding **P**

In the large volume limit the $\mathbf{P}\mu$-system implies

$$\tilde{\mathbf{P}}_{\dot{1}} \simeq \mathbf{Q}^2 \omega_2{}^{\dot{1}} Q^-_{\dot{1}|\dot{1}}\,. \tag{13.27}$$

This equation connects $\mathbf{P}$ from one system to $\mathbf{Q}$ in the other. This is how the QSC encodes the preferred grading used in (13.1) and (13.2). Using also expressions from (13.17) we deduce that $\mathbf{P}_a / f^{[2]}_{\rm tot} \overline{f}^{[-2]}_{\rm tot}$ is a function with one cut on the real axis. To handle the factor of $f^{[2]}_{\rm tot} \overline{f}^{[-2]}_{\rm tot}$ we once again introduce a function $\sigma_{\rm tot}$ such that

$$\tilde{\sigma}_{\rm tot} \sigma_{\rm tot} = f^{[2]}_{\rm tot} \overline{f}^{[-2]}_{\rm tot}\,. \tag{13.28}$$

We recognize by now $\sigma_{\rm tot}$ as the basic building block of the BES phase. All other parts of $\mathbf{P}_a$ can only have a cut on the real axis. At this stage in the AdS_5 derivation we could appeal to the quadratic cut nature of $\mathbf{P}_a$ to essentially finish the computation. This is not possible in AdS_3. To see that $\mathbf{P}_{\dot{1}}$ cannot have a quadratic cut we compute

$$\tilde{\tilde{\mathbf{P}}}_{\dot{1}} \simeq Q^+_{\dot{1}|\dot{1}} \omega^{\dot{1}}{}_2 \mathbf{Q}^2\,. \tag{13.29}$$

We see that the shift of $Q_{\dot{1}|\dot{1}}$ differs from (13.27), this is the reason the cut is no longer quadratic.

At this time follows a little work to find the remaining Q-functions relevant at large volume. We will refer to Paper IV and [21] for details and quote the results in terms of the gauge invariant combinations $\mathbf{P}_1\mathbf{P}^2$ and $\mathbf{Q}_1\mathbf{Q}^2$ in the case when all auxiliary roots are turned off, $K_1 = K_3 = K_{\dot 1} = K_{\dot 3} = 0$,

$$\mathbf{P}_1\mathbf{P}^2 \propto x^{-L} B_{(+),2} B_{(-),2} \Sigma_2^2 \hat{\Sigma}_{\dot 2}^2 \sigma_{\text{tot}}^2 \,, \tag{13.30a}$$

$$\mathbf{P}_{\dot 1}\mathbf{P}^{\dot 2} \propto x^{-L} B_{(+),\dot 2} B_{(-),\dot 2} \Sigma_{\dot 2}^2 \hat{\Sigma}_2^2 \sigma_{\text{tot}}^2 \,, \tag{13.30b}$$

$$\mathbf{Q}_1\mathbf{Q}^2 \propto x^L \frac{B_{(+),2}}{B_{(-),2}} \frac{1}{B_{(-),\dot 2}^2} \frac{(f_{\text{tot}}^{[2]})^2}{\sigma_{\text{tot}}^2 \Sigma_2^2 \hat{\Sigma}_{\dot 2}^2} \,, \tag{13.30c}$$

$$\mathbf{Q}_{\dot 1}\mathbf{Q}^{\dot 2} \propto x^L \frac{B_{(+),\dot 2}}{B_{(-),\dot 2}} \frac{1}{B_{(-),2}^2} \frac{(f_{\text{tot}}^{[2]})^2}{\sigma_{\text{tot}}^2 \Sigma_{\dot 2}^2 \hat{\Sigma}_2^2} \,. \tag{13.30d}$$

The additional phases Σ and $\hat{\Sigma}$ are constrained by the following crossing equations

$$\widetilde{\Sigma}^2 \hat{\Sigma}^2 \propto \frac{R_{(-)}}{R_{(+)}} \,, \qquad \widetilde{\hat{\Sigma}}^2 \Sigma^2 \propto \frac{B_{(+)}}{B_{(-)}} \,. \tag{13.31}$$

Finally, let us compare against the AdS_3 ABA. Written in terms of Q-functions the two momentum-carrying equations are

$$\left. \frac{Q_{1|1}^{[2]}}{Q_{1|1}^{[2]}} \frac{(\mathbf{Q}_1\mathbf{Q}^2)^-}{(\mathbf{Q}_1\mathbf{Q}^2)^+} \right|_{Q_{1|1}=0} = -1 \,, \qquad \left. \frac{Q_{\dot 1|\dot 1}^{[2]}}{Q_{\dot 1|\dot 1}^{[2]}} \frac{(\mathbf{P}_{\dot 1}\mathbf{P}^{\dot 2})^-}{(\mathbf{P}_{\dot 1}\mathbf{P}^{\dot 2})^+} \right|_{Q_{\dot 1|\dot 1}=0} = -1 \,. \tag{13.32}$$

Comparing against (13.1) and (13.2) we find the identifications

$$(\sigma^{\bullet\bullet})^2 \propto \frac{(\Sigma^2)^+}{(\Sigma^2)^-} \frac{(\sigma^2)^+}{(\sigma^2)^-} \,, \qquad (\hat{\sigma}^{\bullet\bullet})^2 \propto \frac{(\hat{\Sigma}^2)^+}{(\hat{\Sigma}^2)^-} \frac{(\sigma^2)^+}{(\sigma^2)^-} \,. \tag{13.33}$$

We will verify that this identification correctly solves the crossing equations in Section 13.5.

13.4 Solving the Curve

In Paper V the AdS_3 QSC was solved at weak coupling. In particular explicit results were presented for operators with a similar structure to the operators $\mathrm{tr} Z \mathscr{D}^N Z$ in $\mathcal{N} = 4$. In this section, we explain the methods of Paper V and the results obtained.

13.4.1 The $\mathfrak{sl}_2$ operators

The operators under consideration in Paper V are the AdS_3 cousins of twist-two operators in AdS_5. They are obtained in the QSC by setting $K_1 = K_3 =$

$K_{\dot{1}} = K_{\dot{3}} = K_2 = 0$ and fixing $K_{\dot{2}} = S$. We will refer to this choice as being in the $\mathfrak{sl}_2$ sector. The asymptotic of $\mathbf{P}_a$ and $\mathbf{P}_{\dot{a}}$ are then given by

$$\mathbf{P}_a \simeq \{\frac{A_1}{u^2}, A_2\, u\}\,, \qquad \mathbf{P}_{\dot{a}} \simeq \{A_{\dot{1}}\,\frac{1}{u}, A_{\dot{2}}\}\,. \tag{13.34}$$

In this sector we also can pick a gauge in which

$$\mathbf{P}^a = -\varepsilon^{ab}\mathbf{P}_b\,, \qquad \mathbf{Q}^k = \varepsilon^{kl}\mathbf{Q}_l\,. \tag{13.35}$$

Furthermore, the states also exhibit parity symmetry. This means that the **P**-functions satisfy

$$\mathbf{P}_a(-u) = \mathbb{g}_a^b\mathbf{P}_b(u)\,, \qquad \mathbf{P}_{\dot{a}}(-u) = \mathbb{g}_{\dot{a}}^{\dot{b}}\mathbf{P}_{\dot{b}}(u)\,. \tag{13.36}$$

with $\mathbb{g}, \mathbb{g}$ constant diagonal matrices. As explained in Section 12.3.4 it is then possible to obtain a set of gluing equations that relates $\overset{*\sim}{\mathbf{Q}}_k$ to $\mathbf{Q}_{\dot{k}}(-u)$. For AdS_3 the gluing takes the form

$$\overset{*\sim}{\mathbf{Q}}_k(u) = N_k{}^{\dot{l}}\mathbf{Q}_{\dot{l}}(-u)\,. \tag{13.37}$$

We will henceforth require that N is strictly off-diagonal. This assumption was argued from symmetry in Paper V, but it was never proven to be true. As such, it remains an assumption.

Using the QQ-relations (13.9) one can use the gluing condition (13.37) to deduce the form of μ^R. With a little bit of algebra one finds

$$(\mu^R)_a{}^{\dot{b}} = -Q_{a|k}(u+\frac{\mathtt{i}}{2})\varepsilon^{kl}N_l{}^{\dot{l}}Q_{\dot{c}|\dot{l}}(-u+\frac{\mathtt{i}}{2})\varepsilon^{\dot{c}\dot{d}}\mathbb{g}_{\dot{d}}^{\dot{b}}\,. \tag{13.38}$$

13.4.2 The Q-system at zero-coupling

The Q-system at $g = 0$ is equivalent to two rational $\mathfrak{psu}_{1,1|2}$ Q-systems. We tackled solving these chains in Section 10.2. In particular, the undotted Q-system is in a vacuum state. That is, all Q-functions are in this case simply polynomials. The dotted system is instead in the $\mathfrak{sl}_2$ sector which we solved in Section 10.2.

13.4.3 Weak coupling solution

With the QSC solved at zero coupling we now need to understand how to develop perturbation theory. The approach of Paper V is to try and once again build Q-functions with quadratic cuts. To accomplish this we defined

$$\mathbb{P}_a = (W^\ell)_a{}^b\mathbf{P}_b\,, \qquad \ell = \frac{\mathtt{i}}{2\pi}\log\frac{x-1}{x+1}\,. \tag{13.39}$$

Where $\mathbb{P}$ has a quadratic cut on the real axis. It thus follows that $\mathbb{P}$ can be written as a series in x. However, it does not need to be bounded either from below or from above. Thus in general

$$\mathbb{P}_a = \sum_{n=-\infty}^{\infty} \frac{d_{a,n}}{x^n}\,. \tag{13.40}$$

By inverting (13.39) one obtains the following identity

$$\mathbf{P}_a = \sum_{n=0}^{\infty} \frac{(-\ell)^n}{n!} \log^n(W)_a{}^b \mathbb{P}_b\,. \tag{13.41}$$

This implies that we can use W and $\mathbb{P}$ to parameterise $\mathbf{P}_a$. Furthermore, given $(\mu^R)_{\dot{a}}{}^b$ it is possible to also construct $\mathbf{P}_{\dot{a}}$. Since both μ and W are functions without cuts on the real axis they can be reliably parameterised as polynomials in $v \equiv \frac{u}{g}$ close to the real axis.

The following strategy was then implemented: An ansatz for $\mathbb{P}_a$, μ^R and W was taken. From this $\mathbf{P}_a, \mathbf{P}_{\dot{a}}$ was reconstructed and re-expanded as functions of u. Using the algorithm of Section 12.3.4 it is then possible to compute $Q_{a|k}$ and $Q_{\dot{a}|\dot{k}}$. From this data it is possible to once again find $(\mu^R)_a{}^{\dot{b}}$ using (13.38). Requiring that the reconstructed μ^R agrees with the initial ansatz eventually closes the equations. For some explicit examples of all functions see the Appendix of Paper V.

13.4.4 Numerical solution

To develop a numerical algorithm one can use the fact that $\tilde{\mathbb{P}}_a$ can be computed both from $x \to \frac{1}{x}$ but also from

$$(W^{\tilde{l}})_a{}^b (\mu^R)_b{}^{\dot{c}} \mathbf{P}_{\dot{c}} = \sum_{n=-\infty}^{\infty} \frac{\tilde{d}_{a,n}}{x^n}\,, \tag{13.42}$$

assuming one stays close enough to the cut on the real axis.

One then follows the same steps as in Section 12.3.6 to build $Q^+_{a|k}, Q^+_{\dot{a},\dot{k}}$ close to the cut. From these objects one constructs μ and uses a minimisation algorithm to find where the differences $d_{a,k} - \tilde{d}_{a,-k}$ vanishes exactly.

13.4.5 Results

Studying $S = 2,4,6,8$ the following all S guess was produced in Paper V

$$\gamma = 8\,g^2 H_1 + g^3\,\frac{384}{35\pi}\,H_1^2 + g^4\left(\Delta_2^{\mathcal{N}=4} - \frac{512}{21\pi^2}H_1^3\right) + \mathcal{O}(g^5)\,. \tag{13.43}$$

These results were obtained with analytical methods and agree with high accuracy numerics.

The proposal (13.43) has intriguing, and perhaps, unpleasant properties. The first observation is that we find the $\mathcal{N}=4$ in the slightly unorthodox limit $\pi \to \infty$. It is tempting to believe that what we are observing is a deformation of the $\mathcal{N}=4$ result by wrapping effects coming from new massless particles not previously present.

Assuming we can trust (13.43) we are faced with the puzzling statement that there is no good cusp limit due to the additional $\log(S)$ coming in at higher orders. This is problematic since in the strong coupling regime it is expected that $\gamma \simeq \log(S)$. From our discussion of twisted QSC in Section 12.4 it does not seem easy to fix this by implementing twist. This issue surely requires further study.

13.5 The Massive Dressing Phases

A topic that has received interest lately is what dressing phases should appear in the ABA for $AdS_3 \times S^3 \times T^4$. The main constraint on the dressing phases is crossing equations. For the phases appearing in the $AdS_3 \times S^3 \times T^4$ massive ABA the crossing equations were obtained in [224]. Let us define two new phases by stripping away a BES factor

$$\sigma^{\bullet\bullet} = \Sigma^{\bullet\bullet}\sigma_{\text{BES}}\,, \qquad\qquad \hat{\sigma}^{\bullet\bullet} = \hat{\Sigma}^{\bullet\bullet}\sigma_{\text{BES}}\,. \tag{13.44}$$

Then the crossing equations read

$$((\Sigma^{\bullet\bullet})^2)^{\gamma_c}(\hat{\Sigma}^{\bullet\bullet})^2 = \frac{R^+_{(-)}}{R^+_{(+)}}\frac{R^-_{(+)}}{R^-_{(-)}}\,, \quad ((\hat{\Sigma}^{\bullet\bullet})^2)^{\gamma_c}(\Sigma^{\bullet\bullet})^2 = \frac{B^+_{(+)}B^-_{(-)}}{B^+_{(-)}B^-_{(+)}}\,. \tag{13.45}$$

It is now an easy task to see that the QSC reproduces these crossing relations. Recall from Section 13.3.5 that $\Sigma^{\bullet\bullet} = \frac{\Sigma^+}{\Sigma^-}, \hat{\Sigma}^{\bullet\bullet} = \frac{\hat{\Sigma}^+}{\hat{\Sigma}^-}$ and then using (13.33) we reproduce (13.45). It should be noted that the QSC crossing equations are *stronger* than (13.45). This is so because they postulate that upon descending from the defining sheet we arrive at a new sheet with only one branch cut. This is very different from how for example σ_{tot} behaves under crossing, see (13.28). Indeed, the absence of additional cuts on the second sheet is reminiscent of what happens in the strong-coupling expansion of AdS_5.

13.5.1 Old and new solutions of the dressing phases

By studying quantum corrections to classical string theory in the same vain as in AdS_5 [188,233–235] an expansion of the AdS_3 dressing phase at strong coupling was made in [236]. Another proposal, different from the first, was made in [237] by solving the crossing equations. This proposal was recently argued to be inconsistent and subsequently modified in [238]. The QSC construction is not compatible with the dressing phases of [237] nor [186], while [238] satisfy all necessary analytic requirements. It remains to verify if the QSC predicts the phases of [238]. Such an analysis should be complemented with explicit solutions of large L states, something that has not yet been attempted.

13.6 Outlook

There exist many interesting directions related to the AdS_3 QSC that deserves further study. The most pressing questions are how to include massless modes and to see if the curve agrees or not with the TBA. For massless mode there now exist interesting results from TBA [239]. It would be highly desirable to make contact with CFT, see [240]. An important generalisation would be to allow for NSNS-flux [241]. It would be very interesting to study the Hagedorn temperature following [242].

The superalgebra $\mathfrak{psu}_{1,1|2}$ also makes an appearance in the AdS_2/CFT_1 correspondence. Here an ABA has been formulated and investigated [243, 244]. There also exists an S-matrix [245, 246] and progress in understanding massless modes [247, 248] but it is still unclear if all these methods give the same results. It is natural to investigate if it is possible to conjecture a QSC for AdS_2/CFT_1 following the ideas in this section.

Svensk Sammanfattning

Mänskligheten har vid tillkomsten av denna avhandling observerat fyra fundamentala naturkrafter: elektromagnetisk växelverkan, svag växelverkan, stark växelverkan samt gravitation. De tre första kan beskrivas med hjälp av kvantfältsteori, en teori kompatibel med både speciell relativitetsteori och kvantmekanik. Gravitation på stora skalor hanteras med allmän relativitesteori, men denna formalism bryter samman vid de skalor där kvanteffekter inte längre kan ignoreras. Att finna den teori som korrekt beskriver både gravitation samt kvantfältsteori i vår värld är den moderna teoretiska fysikens största utmaning.

Vid nuvarande tidpunkt är strängteori den främsta kandidaten. Strängteori beskriver både kvantfältsteori samt gravitation, men det är ännu inte etablerat om den beskriver vårt universum då vi saknar experimentella bevis. Strängteori är därmed förpassad till en matematisk modell vid skrivande stund. Lyckligtvis är det en fascinerande modell, det är väl värt att studera strängteori och dess konsekvenser såsom supersymmetri oberoende av huruvida detta beskriver världen som omger oss eller inte. Att inte försöka belysa varje hörn av en teori kompatibel med både gravitation och kvantmekanik vore tjänstefel.

En av de största upptäckterna inom matematisk fysik i modern tid är att strängteori och kvantfältsteori är intimt besläktade. AdS/CFT dualititen lär oss att strängteori i Anti-de Sitter, en speciell krökt geometri, är ekvivalent med en konform fältteori i en dimension lägre. Det mest välstuderade exemplet av denna dualitet är mellan strängteori på $\mathrm{AdS}_5 \times \mathrm{S}^5$ och fyra-dimensionell $\mathcal{N} = 4$ Super-Yang-Mills, en ytterst symmetrisk kvantfältsteori.

Att studera strängteori och kvantfältsteori är dock utmanande. Nuvarande tekniker tillåter oss endast i allmänhet att utforska dessa teorier i speciella gränser. Lyckligtvis är det så att $\mathcal{N} = 4$ och dess duala strängteori uppvisar integrabilitet. Med hjälp av metoder och redskap grundade i integrabilitet utvecklades runt 2010 en kraftfull ny formalism: Quantum Spectral Curve (QSC). Med QSC är det möjligt för vem som helst med en dator och tillgång till Mathematica att beräkna det exakta energispektrat av $\mathcal{N} = 4$. Det är anmärkningsvärt att QSC beskriver aspekter av både strängteori samt kvantfältsteori perfekt. QSC har sedan dess formulering även använts i flera andra sammanhang. För att nämna några få applikationer har QSC varit av användning vid studier av defekter, deformationer och vid beräkningar av så-kallade strukturkonstanter.

Denna avhandling syftar till att öka vår förståelse för den bakomliggande formalismen till QSC samt att försöka applicera formalismen i nya teorier, framförallt för strängteori på $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. QSC bygger på analytiska Q-system, detta är en samling av så kallade Q-funktioner som beror på en komplex variabel och är relaterad via en samling av QQ-relationer. Q-funktioner är ett mycket allmänt koncept; dessa funktioner har studerats i vitt skilda sammanhang. Dessa funktioner är viktiga inte bara inom strängteori utan även inom statisk fysik och i ren matematik. Det som skiljer Q-funktioner i AdS/CFT från sina kusiner är att de har mycket unika analytiska egenskaper; de är funktioner som lever på en komplicerad tvådimensionell yta. I de flesta andra sammanhang där Q-funktioner studeras beskrivs de vanligtvis av mer välkända funktioner som polynom eller trigonometriska funktioner. Syftet med denna avhandling är att bättre förstå hur Q-funktioner kan generaliseras och hur de Q-funktioner som är relevanta för AdS/CFT är relaterade till de mer typiska Q-funktioner som är av intresse inom andra områden. Genom att studera detta problem förväntas framsteg inom både matematik och fysik. En fullständig teori kring Q-funktioner för godtyckliga superalgebror kommer innebär att större förståelse för i vilka sammanhang integrabilitet kan förväntas inom AdS/CFT samt även belysa hur objekt såsom Yangians kan generaliseras bortom de välkända modeller som studerats intensivt under de senast årtionden.

I de bifogade artiklarna generaliseras både analytiska samt algebraiska aspekter av Q-system och nya modeller för låg-dimensionell AdS/CFT föreslås. I Artikel I samt Artikel II undersökes hur Q-system kan appliceras i integrerbara system som uppvisar en symmetrialgebra av typ $\mathfrak{so}_{2r}$. I dessa fall kan Q-systemet parameteriseras av spinorer med anmärkningsvärda egenskaper. Med de tekniker som utarbetas i dessa artiklar är det möjligt att finna spektrat för mycket långa spinkedjor. Detta är användbart i många olika delar av matematisk fysik.

I Artikel IV samt Artikel V konstrueras Q-system som potentiellt kan beskriva strängteori på $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. Detta mål uppnås genom att studera hur Q-system med komplicerade analytiska egenskaper kan kopplas samman med hjälp av symmetriargument. En algoritm för att lösa dessa Q-system i en svagkopplingsgräns utvecklas och appliceras, både anaytiskt och numeriskt. Från denna algoritm finner vi information som förhoppningsvis kommer vara användbart för att effektivt kunna beskriva den CFT som är dual till strängteori på $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$.

I Artikel III används algebraiska tekniker för att studera spektrat hos differentialoperatorer på en deformerad sju-sfär. Detta är intressant då dessa operatorer beskriver massan hos partiklar på AdS_4, en effektiv teori som uppkommer efter kompaktifiering av supergravitation i elva dimensioner.

Acknowledgments

I have been very fortunate to find a great supervisor in Dima. Thank you for all the time and energy you have invested in me and our projects. I am very grateful for our many discussions.

During my PhD I have had the pleasure of collaborating with Andrea, Kolya, Joseph, Bengt, Paul, Hongfei and Charles. Thank you Charles for being an excellent office mate during these last years, Paul for answering a thousand questions and Bengt for introducing me to theoretical physics.

My time at Uppsala has been very pleasant thanks to the many great people at the institute. Thank you Agnese, Henrik, Joe, Maxim, Magdalena, Pietro, Oliver and Marco for creating a nice atmosphere. I want to thank all my fellow PhD students Alexander, Anastasios, Carlos, Charles, Daniel, Filippo, Giulia, Gregor, Konstantina, Lorenzo, Lucia, Lucile, Maor, Matt, Paolo, Rebecca, Robin, Roman and Yoann for making my time in Uppsala very pleasant. Thank you Giulia for being so helpful during the writing of this thesis. Thank you Roman for saving me from teaching every day. I am very happy to have started together with Carlos, Charles, Maor and Lucile, thanks for the fish and the canoes. I also want to thank all the post-docs I had the opportunity to interact with Andrea, Antonio, Arash, Bram, Carlo, Fei, Gustav, Ingrid, Jacopo, Kays, Luca, Marjorie, Martijn, Max, Nicolo, Parijat, Paul, Rodolfo, Sourav, Tobias, Usman and Vladimir. I truly enjoyed playing frisbee with many of the Uni people and also Emma and friends. I am especially grateful to the running group at the institute; Joe, Oliver, Charles, Paul, Rodolfo, Gregor, Konstantina and Andrea.

A warm thank you to all the people who took time out of their day to discuss or teach me something. A severely incomplete list includes Alessandro Georgoudis, Alessandro Sfondrini, Michelangelo Preti, Bogdan Stefanski, Dennis Le Plat, Rob Klabbers, Jules Lamers, Olof Ohlsson Sax and Andrii Liashyk.

During my fourth year, I had the luck of being able to travel to many places. I am very grateful for the invitation to spend two weeks in London in the group of Nikolay Gromov. Thank you, Kolya, Andrea, Michelangelo, Paul, Nicolo, Julius and Nika for the hospitality during these weeks. Thank you to Arash for the great bagels at Stony.

References

[1] J. A. Minahan and K. Zarembo, *The Bethe ansatz for N=4 superYang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[2] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum Spectral Curve for Planar $\mathcal{N}=4$ Super-Yang-Mills Theory*, *Phys. Rev. Lett.* **112** (2014) 011602 [1305.1939].

[3] C. Marboe and D. Volin, *The full spectrum of AdS$_5$/CFT$_4$ II: Weak coupling expansion via the quantum spectral curve*, *J. Phys. A* **54** (2021) 055201 [1812.09238].

[4] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *Quantum spectral curve and the numerical solution of the spectral problem in AdS 5 /CF t 4*, *Journal of High Energy Physics* **2016** (2016) .

[5] N. Gromov and F. Levkovich-Maslyuk, *Quantum Spectral Curve for a cusped Wilson line in $\mathcal{N}=4$ SYM*, *JHEP* **04** (2016) 134 [1510.02098].

[6] A. Cavaglià, N. Gromov, J. Julius and M. Preti, *Integrability and conformal bootstrap: One dimensional defect conformal field theory*, *Phys. Rev. D* **105** (2022) L021902 [2107.08510].

[7] B. Basso, A. Georgoudis and A. K. Sueiro, *Structure constants of short operators in planar $\mathcal{N}=4$ SYM theory*, 2207.01315.

[8] T. Harmark and M. Wilhelm, *Solving the Hagedorn temperature of AdS$_5$/CFT$_4$ via the Quantum Spectral Curve: chemical potentials and deformations*, *JHEP* **07** (2022) 136 [2109.09761].

[9] A. Cavaglià, D. Fioravanti, N. Gromov and R. Tateo, *Quantum Spectral Curve of the $\mathcal{N}=6$ Supersymmetric Chern-Simons Theory*, *Phys. Rev. Lett.* **113** (2014) 021601 [1403.1859].

[10] R. J. Baxter, *Partition function of the eight-vertex lattice model*, *Annals of Physics* **70** (1972) 193.

[11] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of W-algebras*, math/9810055.

[12] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz*, *Commun. Math. Phys.* **177** (1996) 381 [hep-th/9412229].

[13] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory. 2. Q operator and DDV equation*, *Commun. Math. Phys.* **190** (1997) 247 [hep-th/9604044].

[14] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory. 3. The Yang-Baxter relation*, *Commun. Math. Phys.* **200** (1999) 297 [hep-th/9805008].

[15] P. Dorey and R. Tateo, *Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations*, *J. Phys. A* **32** (1999) L419 [hep-th/9812211].

[16] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable quantum field theories in finite volume: Excited state energies*, *Nucl. Phys. B* **489** (1997) 487 [hep-th/9607099].
[17] G. Ferrando, R. Frassek and V. Kazakov, *QQ-system and Weyl-type transfer matrices in integrable SO(2r) spin chains*, *JHEP* **02** (2021) 193 [2008.04336].
[18] J. Balog and A. Hegedus, *TBA equations for the mass gap in the O(2r) non-linear sigma-models*, *Nucl. Phys. B* **725** (2005) 531 [hep-th/0504186].
[19] V. Kazakov and E. Olivucci, *Biscalar Integrable Conformal Field Theories in Any Dimension*, *Phys. Rev. Lett.* **121** (2018) 131601 [1801.09844].
[20] B. Basso, G. Ferrando, V. Kazakov and D.-l. Zhong, *Thermodynamic Bethe Ansatz for Biscalar Conformal Field Theories in any Dimension*, *Phys. Rev. Lett.* **125** (2020) 091601 [1911.10213].
[21] A. Cavaglià, N. Gromov, B. Stefański, Jr., Jr. and A. Torrielli, *Quantum Spectral Curve for AdS_3/CFT_2: a proposal*, *JHEP* **12** (2021) 048 [2109.05500].
[22] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*. Cambridge University Press, 2003.
[23] W. Fulton and J. Harris, *Representation theory : a first course*. Springer, 2004.
[24] V. G. Kac, *Lie superalgebras*, *Advances in mathematics* **26** (1977) 8.
[25] L. Frappat, P. Sorba and A. Sciarrino, *Dictionary on Lie superalgebras*, hep-th/9607161.
[26] J. M. Drummond, J. M. Henn and J. Plefka, *Yangian symmetry of scattering amplitudes in N=4 super Yang-Mills theory*, *JHEP* **05** (2009) 046 [0902.2987].
[27] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, *Sov. Math. Dokl.* **32** (1985) 254.
[28] V. G. Drinfeld, *A New realization of Yangians and quantized affine algebras*, *Sov. Math. Dokl.* **36** (1988) 212.
[29] V. Chari and A. Pressley, *A guide to quantum groups*. 1994.
[30] F. Loebbert, *Lectures on Yangian Symmetry*, *J. Phys. A* **49** (2016) 323002 [1606.02947].
[31] N. J. MacKay, *Introduction to Yangian symmetry in integrable field theory*, *Int. J. Mod. Phys. A* **20** (2005) 7189 [hep-th/0409183].
[32] L. D. Faddeev and G. P. Korchemsky, *High-energy QCD as a completely integrable model*, *Phys. Lett. B* **342** (1995) 311 [hep-th/9404173].
[33] A. Molev, M. Nazarov and G. Olshansky, *Yangians and classical Lie algebras*, *Russ. Math. Surveys* **51** (1996) 205 [hep-th/9409025].
[34] N. Guay, V. Regelskis and C. Wendlandt, *Equivalences between three presentations of orthogonal and symplectic yangians*, *Letters in Mathematical Physics* **109** (2018) 327.
[35] M. L. NAZAROV, *Quantum berezinian and the classical capelli identity*, *Letters in mathematical physics* **21** (1991) 123.
[36] D. Arnaudon, J. Avan, N. Crampé , L. Frappat and E. Ragoucy, *R-matrix presentation for super-yangians $y(osp(m|2n))$*, *Journal of Mathematical Physics* **44** (2003) 302.
[37] H. Knight, *Spectra of tensor products of finite dimensional representations of yangians*, *Journal of Algebra* **174** (1995) 187.

[38] D. Arnaudon, A. Molev and E. Ragoucy, *On the r-matrix realization of yangians and their representations*, *Annales Henri Poincaré* **7** (2006) 1269.

[39] E. K. Sklyanin, *Separation of variables in the quantum integrable models related to the Yangian Y[sl(3)]*, *Zap. Nauchn. Semin.* **205** (1993) 166 [hep-th/9212076].

[40] N. Y. Reshetikhin, *A Method Of Functional Equations In The Theory Of Exactly Solvable Quantum Systems*, *Lett. Math. Phys.* **7** (1983) 205.

[41] A. Kuniba, T. Nakanishi and J. Suzuki, *T-systems and y-systems in integrable systems*, *Journal of Physics A: Mathematical and Theoretical* **44** (2011) 103001.

[42] E. Mukhin, V. Tarasov and A. Varchenko, *Bethe algebra of homogeneous xxx heisenberg model has simple spectrum*, 2012.

[43] E. Mukhin, V. Tarasov and A. Varchenko, *Spaces of quasi-exponentials and representations of the yangian $y(gl_n)$*, 2013.

[44] C. Huang, K. Lu and E. Mukhin, *Solutions of $\mathfrak{gl}_{m|n}$ xxx bethe ansatz equations and rational difference operators*, *Journal of Physics A: Mathematical and Theoretical* **52** (2019) 375204.

[45] K. Lu and E. Mukhin, *On the supersymmetric xxx spin chains associated to* $\mathfrak{gl}_{1|1}$, 2019.

[46] D. Chernyak, S. Leurent and D. Volin, *Completeness of Wronskian Bethe Equations for Rational $\mathfrak{gl}_{\mathsf{m}|\mathsf{n}}$ Spin Chains*, *Commun. Math. Phys.* **391** (2022) 969 [2004.02865].

[47] M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Thea n /(1) face models*, *Communications in mathematical physics* **119** (1988) 543.

[48] E. Date, M. Jimbo, T. Miwa, M. Okado and A. Kuniba, *Exactly solvable sos models ii: Proof of the star-triangle relation and combinatorial identities*, in *Conformal Field Theory and Solvable Lattice Models*, vol. 16, (United States), pp. 17–122, Elsevier Inc, (1988).

[49] A. Kirillov and N. Reshetikhin, *Representations of yangians and multiplicities of occurrence of the irreducible components of the tensor product of representations of simple lie algebras*, *Journal of Soviet Mathematics* (1990) .

[50] A. Kuniba, T. Nakanishi and J. Suzuki, *Functional relations in solvable lattice models. 1: Functional relations and representation theory*, *Int. J. Mod. Phys. A* **9** (1994) 5215 [hep-th/9309137].

[51] V. Bazhanov and N. Reshetikhin, *Restricted Solid on Solid Models Connected With Simply Based Algebras and Conformal Field Theory*, *J. Phys. A* **23** (1990) 1477.

[52] A. Kuniba, S. Nakamura and R. Hirota, *Pfaffian and determinant solutions to a discretized Toda equation for B(r), C(r) and D(r)*, *J. Phys. A* **29** (1996) 1759 [hep-th/9509039].

[53] E. Ogievetsky and P. Wiegmann, *Factorized S Matrix and the Bethe Ansatz for Simple Lie Groups*, *Phys. Lett. B* **168** (1986) 360.

[54] V. V. Bazhanov and N. Reshetikhin, *Restricted solid-on-solid models connected with simply laced algebras and conformal field theory*, *Journal of physics. A, Mathematical and general* **23** (1990) 1477.

[55] A. Kuniba, *Thermodynamics of the uq(xr(1)) bethe ansatz system with q a root of unity*, *Nuclear Physics B* **389** (1993) 209.

[56] N. Y. Reshetikhin, *Integrable models of quantum one-dimensional magnets with o (n) and sp (2k) symmetry*, *Theor. Math. Phys.;(United States)* **63** (1985) .

[57] A. Kuniba, Y. Ohta and J. Suzuki, *Quantum Jacobi-Trudi and Giambelli formulae for U-q(B(r)(1)) from analytic Bethe ansatz*, *J. Phys. A* **28** (1995) 6211 [hep-th/9506167].

[58] A. Kuniba and J. Suzuki, *Analytic Bethe Ansatz for fundamental representations of Yangians*, *Commun. Math. Phys.* **173** (1995) 225 [hep-th/9406180].

[59] W. Nakai and T. Nakanishi, *Paths and tableaux descriptions of jacobi-trudi determinant associated with quantum affine algebra of type d_n*, 2007.

[60] W. Nakai and T. Nakanishi, *Paths and tableaux descriptions of jacobi-trudi determinant associated with quantum affine algebra of type c_n*, 2007.

[61] V. V. Bazhanov, R. Frassek, T. Lukowski, C. Meneghelli and M. Staudacher, *Baxter Q-Operators and Representations of Yangians*, *Nucl. Phys. B* **850** (2011) 148 [1010.3699].

[62] R. Frassek, T. Lukowski, C. Meneghelli and M. Staudacher, *Baxter Operators and Hamiltonians for 'nearly all' Integrable Closed $\mathfrak{gl}(n)$ Spin Chains*, *Nucl. Phys. B* **874** (2013) 620 [1112.3600].

[63] R. Frassek, *Oscillator realisations associated to the D-type Yangian: Towards the operatorial Q-system of orthogonal spin chains*, *Nucl. Phys. B* **956** (2020) 115063 [2001.06825].

[64] R. Frassek and A. Tsymbaliuk, *Rational Lax Matrices from Antidominantly Shifted Extended Yangians: BCD Types*, *Commun. Math. Phys.* **392** (2022) 545 [2104.14518].

[65] K. Costello, E. Witten and M. Yamazaki, *Gauge Theory and Integrability, I*, *ICCM Not.* **06** (2018) 46 [1709.09993].

[66] K. Costello, D. Gaiotto and J. Yagi, *Q-operators are 't Hooft lines*, 2103.01835.

[67] Y. Boujakhrout and E. H. Saidi, *Minuscule ABCDE Lax operators from 4D Chern-Simons theory*, *Nucl. Phys. B* **981** (2022) 115859 [2207.14777].

[68] Y. Boujakhrout and E. H. Saidi, *On exceptional 't Hooft lines in 4D-Chern-Simons theory*, *Nucl. Phys. B* **980** (2022) 115795 [2204.12424].

[69] A. B. Zamolodchikov, *Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-yang Models*, *Nucl. Phys. B* **342** (1990) 695.

[70] A. Zamolodchikov, *On the thermodynamic bethe ansatz equations for reflectionless ade scattering theories*, *Physics Letters B* **253** (1991) 391.

[71] F. Ravanini, R. Tateo and A. Valleriani, *Dynkin TBAs*, *Int. J. Mod. Phys. A* **8** (1993) 1707 [hep-th/9207040].

[72] D. Gaiotto, G. W. Moore and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun. Math. Phys.* **299** (2010) 163 [0807.4723].

[73] S. Cecotti and M. Del Zotto, *Y systems, Q systems, and 4D $\mathcal{N}=2$ supersymmetric QFT*, *J. Phys. A* **47** (2014) 474001 [1403.7613].

[74] P. Dorey, C. Dunning, D. Masoero, J. Suzuki and R. Tateo, *Pseudo-differential equations, and the Bethe ansatz for the classical Lie algebras*, *Nucl. Phys. B* **772** (2007) 249 [hep-th/0612298].

[75] J. Sun, *Polynomial relations for q-characters via the ODE/IM correspondence*,

Symmetry, Integrability and Geometry: Methods and Applications (2012) .

[76] D. Masoero, A. Raimondo and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case*, *Commun. Math. Phys.* **344** (2016) 719 [1501.07421].

[77] D. Masoero, A. Raimondo and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie algebra–Valued Connections II: The Non Simply–Laced Case*, *Commun. Math. Phys.* **349** (2017) 1063 [1511.00895].

[78] P. Dorey, C. Dunning and R. Tateo, *The ODE/IM Correspondence*, *J. Phys. A* **40** (2007) R205 [hep-th/0703066].

[79] P. Dorey, C. Dunning, S. Negro and R. Tateo, *Geometric aspects of the ODE/IM correspondence*, *J. Phys. A* **53** (2020) 223001 [1911.13290].

[80] B. Feigin and E. Frenkel, *Quantization of soliton systems and Langlands duality*, 0705.2486.

[81] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum integrable models and discrete classical hirota equations*, *Communications in Mathematical Physics* (1997) .

[82] A. Zabrodin, *Discrete Hirota's equation in quantum integrable models*, *Int. J. Mod. Phys. B* **11** (1997) 3125 [hep-th/9610039].

[83] V. Kazakov, S. Leurent and D. Volin, *T-system on T-hook: Grassmannian Solution and Twisted Quantum Spectral Curve*, *JHEP* **12** (2016) 044 [1510.02100].

[84] C. Marboe and D. Volin, *Fast analytic solver of rational Bethe equations*, *J. Phys. A* **50** (2017) 204002 [1608.06504].

[85] E. J. Cartan, *The theory of spinors*, Dover books on mathematics. Hermann, Paris, 1966.

[86] A. Kuniba, M. Okado, J. Suzuki and Y. Yamada, *Difference l operators related to q-characters*, *Journal of Physics A: Mathematical and General* **35** (2002) 1415.

[87] I. Buric, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, *From Gaudin Integrable Models to d-dimensional Multipoint Conformal Blocks*, *Phys. Rev. Lett.* **126** (2021) 021602 [2009.11882].

[88] O. Gürdoğan and V. Kazakov, *New Integrable 4D Quantum Field Theories from Strongly Deformed Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **117** (2016) 201602 [1512.06704].

[89] J. Caetano, O. Gürdoğan and V. Kazakov, *Chiral limit of $\mathcal{N} = 4$ SYM and ABJM and integrable Feynman graphs*, *JHEP* **03** (2018) 077 [1612.05895].

[90] N. Gromov and A. Sever, *The holographic dual of strongly γ-deformed $\mathcal{N} = 4$ SYM theory: derivation, generalization, integrability and discrete reparametrization symmetry*, *JHEP* **02** (2020) 035 [1908.10379].

[91] N. Gromov and A. Sever, *Quantum fishchain in AdS_5*, *JHEP* **10** (2019) 085 [1907.01001].

[92] N. Gromov and A. Sever, *Derivation of the Holographic Dual of a Planar Conformal Field Theory in 4D*, *Phys. Rev. Lett.* **123** (2019) 081602 [1903.10508].

[93] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *New Construction of Eigenstates and Separation of Variables for SU(N) Quantum Spin Chains*, *JHEP* **09** (2017) 111 [1610.08032].

[94] P. Ryan and D. Volin, *Separated variables and wave functions for rational gl(N) spin chains in the companion twist frame*, *J. Math. Phys.* **60** (2019) 032701 [1810.10996].
[95] N. Gromov, F. Levkovich-Maslyuk, P. Ryan and D. Volin, *Dual Separated Variables and Scalar Products*, *Phys. Lett. B* **806** (2020) 135494 [1910.13442].
[96] P. G. O. Freund and M. A. Rubin, *Dynamics of Dimensional Reduction*, *Phys. Lett. B* **97** (1980) 233.
[97] M. Duff, B. Nilsson and C. Pope, *Kaluza-klein supergravity*, *Physics Reports* **130** (1986) 1.
[98] E. Cremmer, B. Julia and J. Scherk, *Supergravity in theory in 11 dimensions*, *Physics Letters B* **76** (1978) 409.
[99] A. Salam and J. Strathdee, *On kaluza-klein theory*, *Annals of Physics* **141** (1982) 316.
[100] F. Bais, H. Nicolai and P. Van Nieuwenhuizen, *Geometry of coset spaces and massless modes of the squashed seven-sphere in supergravity*, *Nuclear Physics B* **228** (1983) 333.
[101] K. Yamagishi, *Mass Spectra of Vector Particles in $D = 11$ Supergravity Compactified on Squashed Seven Sphere*, *Phys. Lett. B* **137** (1984) 165.
[102] B. E. W. Nilsson, A. Padellaro and C. N. Pope, *The role of singletons in S^7 compactifications*, *JHEP* **07** (2019) 124 [1811.06228].
[103] B. Duboeuf, E. Malek and H. Samtleben, *Kaluza-Klein Spectrometry beyond Consistent Truncations: The Squashed S^7*, 2212.01135.
[104] J. Karlsson, *Compactifications of String/M-Theory and the Swampland: A Study of the AdS_4 Mass Spectrum of Eleven-Dimensional Supergravity on the Squashed Seven-Sphere*, Master's thesis, Chalmers U. Tech., 2021.
[105] B. E. W. Nilsson and C. N. Pope, *De-Higgsing In Eleven-Dimensional Supergravity On The Squashed S^7*, 2302.03842.
[106] P. P. Kulish and E. K. Sklyanin, *Solutions of the yang-baxter equation*, *Journal of Soviet Mathematics* **19** (1982) 1596.
[107] P. P. Kulish, *Integrable graded magnets*, *Journal of Soviet Mathematics* **35** (1986) 2648.
[108] V. Kazakov, A. S. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, *Nucl. Phys. B* **790** (2008) 345 [hep-th/0703147].
[109] Z. Tsuboi, *Solutions of the T-system and Baxter equations for supersymmetric spin chains*, *Nucl. Phys. B* **826** (2010) 399 [0906.2039].
[110] Z. Tsuboi, *Analytic Bethe Ansatz And Functional Equations Associated With Any Simple Root Systems Of The Lie Superalgebra $sl(r+1|s+1)$*, *Physica A* **252** (1998) 565 [0911.5387].
[111] C. Marboe and V. Velizhanin, *Twist-2 at seven loops in planar $\mathcal{N} = 4$ SYM theory: full result and analytic properties*, *JHEP* **11** (2016) 013 [1607.06047].
[112] M. Günaydin and D. Volin, *The Complete Unitary Dual of Non-compact Lie Superalgebra $\mathfrak{su}(\mathrm{p},\mathrm{q}|\mathrm{m})$ via the Generalised Oscillator Formalism, and Non-compact Young Diagrams*, *Commun. Math. Phys.* **367** (2019) 873 [1712.01811].

[113] N. Gromov, V. Kazakov, S. Leurent and Z. Tsuboi, *Wronskian Solution for AdS/CFT Y-system*, *JHEP* **01** (2011) 155 [1010.2720].

[114] N. Beisert and B. I. Zwiebel, *On Symmetry Enhancement in the psu(1,1|2) Sector of N=4 SYM*, *JHEP* **10** (2007) 031 [0707.1031].

[115] O. Ohlsson Sax and B. Stefanski, Jr., *Integrability, spin-chains and the AdS3/CFT2 correspondence*, *JHEP* **08** (2011) 029 [1106.2558].

[116] G. Korchemsky, *Bethe ansatz for qcd pomeron*, *Nuclear Physics B* **443** (1995) 255.

[117] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021 [hep-th/0610251].

[118] C. Marboe and D. Volin, *The full spectrum of AdS5/CFT4 I: Representation theory and one-loop Q-system*, *J. Phys. A* **51** (2018) 165401 [1701.03704].

[119] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[120] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[121] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109].

[122] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [1012.3982].

[123] S. Mandelstam, *Light-cone superspace and the ultraviolet finiteness of the n=4 model*, *Nuclear Physics B* **213** (1983) 149.

[124] L. Brink, O. Lindgren and B. E.W. Nilsson, *N = 4 yang-mills theory on the light cone*, *Nuclear Physics B* **212** (1983) 401.

[125] L. Brink, J. H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills Theories*, *Nucl. Phys. B* **121** (1977) 77.

[126] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *BMN gauge theory as a quantum mechanical system*, *Phys. Lett. B* **558** (2003) 229 [hep-th/0212269].

[127] N. Beisert, C. Kristjansen and M. Staudacher, *The Dilatation operator of conformal N=4 superYang-Mills theory*, *Nucl. Phys. B* **664** (2003) 131 [hep-th/0303060].

[128] N. Beisert, *The complete one loop dilatation operator of N=4 superYang-Mills theory*, *Nucl. Phys. B* **676** (2004) 3 [hep-th/0307015].

[129] N. Beisert and M. Staudacher, *The N=4 SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [hep-th/0307042].

[130] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in AdS(5) x S**5 background*, *Nucl. Phys. B* **533** (1998) 109 [hep-th/9805028].

[131] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the AdS(5) x S**5 superstring*, *Phys. Rev. D* **69** (2004) 046002 [hep-th/0305116].

[132] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, *Strings in flat space and pp waves from N=4 superYang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[133] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[134] V. A. Kazakov and K. Zarembo, *Classical / quantum integrability in non-compact sector of AdS/CFT*, *JHEP* **10** (2004) 060 [hep-th/0410105].
[135] N. Beisert and M. Staudacher, *Long-range psu(2,2|4) Bethe Ansatze for gauge theory and strings*, *Nucl. Phys. B* **727** (2005) 1 [hep-th/0504190].
[136] N. Beisert, *The SU(2|2) dynamic S-matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [hep-th/0511082].
[137] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2|2) Symmetry*, *J. Stat. Mech.* **0701** (2007) P01017 [nlin/0610017].
[138] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, *The Off-shell Symmetry Algebra of the Light-cone AdS(5) x S**5 Superstring*, *J. Phys. A* **40** (2007) 3583 [hep-th/0609157].
[139] G. Arutyunov, S. Frolov and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for AdS(5) x S**5 superstring*, *JHEP* **04** (2007) 002 [hep-th/0612229].
[140] J. Ambjorn, R. A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, *Nucl. Phys. B* **736** (2006) 288 [hep-th/0510171].
[141] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher and V. N. Velizhanin, *Dressing and wrapping*, *J. Stat. Mech.* **0710** (2007) P10003 [0704.3586].
[142] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Wrapping at four loops in N=4 SYM*, *Phys. Lett. B* **666** (2008) 100 [0712.3522].
[143] Z. Bajnok and R. A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys. B* **807** (2009) 625 [0807.0399].
[144] G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the AdS(5) x S(5) Mirror Model*, *JHEP* **05** (2009) 068 [0903.0141].
[145] D. Bombardelli, D. Fioravanti and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT: A Proposal*, *J. Phys. A* **42** (2009) 375401 [0902.3930].
[146] N. Gromov, V. Kazakov and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **103** (2009) 131601 [0901.3753].
[147] G. Arutyunov and S. Frolov, *On String S-matrix, Bound States and TBA*, *JHEP* **12** (2007) 024 [0710.1568].
[148] S. J. van Tongeren, *Introduction to the thermodynamic Bethe ansatz*, *J. Phys. A* **49** (2016) 323005 [1606.02951].
[149] Z. Tsuboi, *Solutions of the T-system and Baxter equations for supersymmetric spin chains*, *Nucl. Phys. B* **826** (2010) 399 [0906.2039].
[150] N. Gromov, V. Kazakov and Z. Tsuboi, *PSU(2,2|4) Character of Quasiclassical AdS/CFT*, *JHEP* **07** (2010) 097 [1002.3981].
[151] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Solving the AdS/CFT Y-system*, *JHEP* **07** (2012) 023 [1110.0562].
[152] R. A. Janik, *The AdS(5) x S**5 superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev. D* **73** (2006) 086006 [hep-th/0603038].
[153] P. Vieira and D. Volin, *Review of AdS/CFT Integrability, Chapter III.3: The Dressing factor*, *Lett. Math. Phys.* **99** (2012) 231 [1012.3992].
[154] K. Zarembo, *Strings on Semisymmetric Superspaces*, *JHEP* **05** (2010) 002 [1003.0465].

[155] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [0806.1218].

[156] J. A. Minahan and K. Zarembo, *The Bethe ansatz for superconformal Chern-Simons*, *JHEP* **09** (2008) 040 [0806.3951].

[157] D. Bombardelli, D. Fioravanti and R. Tateo, *TBA and Y-system for planar AdS(4)/CFT(3)*, *Nucl. Phys. B* **834** (2010) 543 [0912.4715].

[158] D. Bombardelli, A. Cavaglià, D. Fioravanti, N. Gromov and R. Tateo, *The full Quantum Spectral Curve for AdS_4/CFT_3*, *JHEP* **09** (2017) 140 [1701.00473].

[159] N. Gromov, *Introduction to the Spectrum of $N = 4$ SYM and the Quantum Spectral Curve*, 1708.03648.

[160] V. Kazakov, *Quantum Spectral Curve of γ-twisted $\mathcal{N} = 4$ SYM theory and fishnet CFT*, 1802.02160.

[161] F. Levkovich-Maslyuk, *A review of the AdS/CFT Quantum Spectral Curve*, *J. Phys. A* **53** (2020) 283004 [1911.13065].

[162] M. Alfimov, N. Gromov and V. Kazakov, *QCD Pomeron from AdS/CFT Quantum Spectral Curve*, *JHEP* **07** (2015) 164 [1408.2530].

[163] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4*, *JHEP* **09** (2015) 187 [1405.4857].

[164] D. Volin, *Minimal solution of the AdS/CFT crossing equation*, *J. Phys. A* **42** (2009) 372001 [0904.4929].

[165] C. Marboe and D. Volin, *Quantum spectral curve as a tool for a perturbative quantum field theory*, *Nucl. Phys. B* **899** (2015) 810 [1411.4758].

[166] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **115** (2015) 251601 [1507.04010].

[167] S. Leurent and D. Volin, *Multiple zeta functions and double wrapping in planar $N = 4$ SYM*, *Nucl. Phys. B* **875** (2013) 757 [1302.1135].

[168] M. Alfimov, N. Gromov and G. Sizov, *BFKL spectrum of $\mathcal{N}$ = 4: non-zero conformal spin*, *JHEP* **07** (2018) 181 [1802.06908].

[169] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, *Three loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model*, *Phys. Lett. B* **595** (2004) 521 [hep-th/0404092].

[170] J. A. M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, *Int. J. Mod. Phys. A* **14** (1999) 2037 [hep-ph/9806280].

[171] C. Marboe, V. Velizhanin and D. Volin, *Six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory*, *JHEP* **07** (2015) 084 [1412.4762].

[172] A. I. Onishchenko, *Anomalous dimensions of twist 2 operators and $\mathcal{N} = 4$ SYM quantum spectral curve*, 2105.03945.

[173] L. Freyhult, *Review of AdS/CFT Integrability, Chapter III.4: Twist States and the cusp Anomalous Dimension*, *Lett. Math. Phys.* **99** (2012) 255 [1012.3993].

[174] C. Marboe and E. Widén, *The fate of the Konishi multiplet in the β-deformed Quantum Spectral Curve*, *JHEP* **01** (2020) 026 [1902.01248].

[175] F. Levkovich-Maslyuk and M. Preti, *Exploring the ground state spectrum of γ-deformed N = 4 SYM*, *JHEP* **06** (2022) 146 [2003.05811].

[176] N. Gromov, V. Kazakov, G. Korchemsky, S. Negro and G. Sizov, *Integrability of Conformal Fishnet Theory*, *JHEP* **01** (2018) 095 [1706.04167].
[177] E. Pomoni, R. Rabe and K. Zoubos, *Dynamical spin chains in 4D $\mathcal{N}$ = 2 SCFTs*, *JHEP* **08** (2021) 127 [2106.08449].
[178] E. Pomoni, *4D $\mathcal{N} = 2$ SCFTs and spin chains*, *J. Phys. A* **53** (2020) 283005 [1912.00870].
[179] T. Skrzypek, *Integrability treatment of AdS/CFT orbifolds*, 2211.03806.
[180] N. Beisert and R. Roiban, *Beauty and the twist: The Bethe ansatz for twisted N=4 SYM*, *JHEP* **08** (2005) 039 [hep-th/0505187].
[181] N. Beisert and R. Roiban, *The Bethe ansatz for Z(S) orbifolds of N=4 super Yang-Mills theory*, *JHEP* **11** (2005) 037 [hep-th/0510209].
[182] G. Arutyunov, M. de Leeuw and S. J. van Tongeren, *Twisting the Mirror TBA*, *JHEP* **02** (2011) 025 [1009.4118].
[183] M. de Leeuw and S. J. van Tongeren, *Orbifolded Konishi from the Mirror TBA*, *J. Phys. A* **44** (2011) 325404 [1103.5853].
[184] M. de Leeuw and T. Lukowski, *Twist operators in N=4 beta-deformed theory*, *JHEP* **04** (2011) 084 [1012.3725].
[185] N. Gromov and F. Levkovich-Maslyuk, *Y-system and β-deformed N=4 Super-Yang-Mills*, *J. Phys. A* **44** (2011) 015402 [1006.5438].
[186] M. Beccaria and G. Macorini, *Y-system for Z_S Orbifolds of N=4 SYM*, *JHEP* **06** (2011) 004 [1104.0883].
[187] E. Pomoni and C. Sieg, *From N=4 gauge theory to N=2 conformal QCD: three-loop mixing of scalar composite operators*, 1105.3487.
[188] N. Beisert and A. A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett. B* **629** (2005) 102 [hep-th/0509084].
[189] N. Gromov, F. Levkovich-Maslyuk, G. Sizov and S. Valatka, *Quantum spectral curve at work: from small spin to strong coupling in $\mathcal{N}$ = 4 SYM*, *JHEP* **07** (2014) 156 [1402.0871].
[190] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **115** (2015) 251601 [1507.04010].
[191] M. Alfimov, N. Gromov and G. Sizov, *BFKL spectrum of $\mathcal{N}$ = 4: non-zero conformal spin*, *JHEP* **07** (2018) 181 [1802.06908].
[192] M. Alfimov, N. Gromov and V. Kazakov, *Chapter 13: $\mathcal{N} = 4$ SYM Quantum Spectral Curve in BFKL Regime*, 2003.03536.
[193] N. Gromov and F. Levkovich-Maslyuk, *Quark-anti-quark potential in $\mathcal{N} = 4$ SYM*, *JHEP* **12** (2016) 122 [1601.05679].
[194] T. Harmark and M. Wilhelm, *The Hagedorn temperature of AdS_5/CFT_4 at finite coupling via the Quantum Spectral Curve*, *Phys. Lett. B* **786** (2018) 53 [1803.04416].
[195] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *Quantum spectral curve and structure constants in $\mathcal{N} = 4$ SYM: cusps in the ladder limit*, *JHEP* **10** (2018) 060 [1802.04237].
[196] J. McGovern, *Scalar insertions in cusped Wilson loops in the ladders limit of planar $\mathcal{N}$ = 4 SYM*, *JHEP* **05** (2020) 062 [1912.00499].
[197] A. Cavaglia, D. Grabner, N. Gromov and A. Sever, *Colour-twist operators. Part I. Spectrum and wave functions*, *JHEP* **06** (2020) 092 [2001.07259].

[198] C. Bercini, A. Homrich and P. Vieira, *Structure Constants in $\mathcal{N} = 4$ SYM and Separation of Variables*, 2210.04923.
[199] M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point Functions in Defect CFT and Integrability*, *JHEP* **08** (2015) 098 [1506.06958].
[200] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen and M. Wilhelm, *Asymptotic One-Point Functions in Gauge-String Duality with Defects*, *Phys. Rev. Lett.* **119** (2017) 261604 [1704.07386].
[201] C. Kristjansen, D. Müller and K. Zarembo, *Integrable boundary states in D3-D5 dCFT: beyond scalars*, *JHEP* **08** (2020) 103 [2005.01392].
[202] C. Kristjansen, D. Müller and K. Zarembo, *Overlaps and fermionic dualities for integrable super spin chains*, *JHEP* **03** (2021) 100 [2011.12192].
[203] C. Kristjansen, D. Müller and K. Zarembo, *Duality relations for overlaps of integrable boundary states in AdS/dCFT*, *JHEP* **09** (2021) 004 [2106.08116].
[204] M. de Leeuw, A. C. Ipsen, C. Kristjansen and M. Wilhelm, *Introduction to integrability and one-point functions in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and its defect cousin*, 1708.02525.
[205] M. de Leeuw, *One-point functions in AdS/dCFT*, *J. Phys. A* **53** (2020) 283001 [1908.03444].
[206] Y. Jiang, S. Komatsu and E. Vescovi, *Structure constants in $\mathcal{N} = 4$ SYM at finite coupling as worldsheet g-function*, *JHEP* **07** (2020) 037 [1906.07733].
[207] M. Guica, F. Levkovich-Maslyuk and K. Zarembo, *Integrability in dipole-deformed $\mathcal{N} = 4$ super Yang–Mills*, *J. Phys. A* **50** (2017) 39 [1706.07957].
[208] T. Meier and S. J. van Tongeren, *Quadratic twist-noncommutative gauge theory*, 2301.08757.
[209] S. Frolov and A. Sfondrini, *Mirror thermodynamic Bethe ansatz for AdS3/CFT2*, *JHEP* **03** (2022) 138 [2112.08898].
[210] A. Sfondrini, *Towards integrability for* $\mathrm{AdS_3/CFT_2}$, *J. Phys. A* **48** (2015) 023001 [1406.2971].
[211] I. Pesando, *The GS type IIB superstring action on AdS(3) x S(3) x T**4*, *JHEP* **02** (1999) 007 [hep-th/9809145].
[212] J. Rahmfeld and A. Rajaraman, *The GS string action on AdS(3) x S(3) with Ramond-Ramond charge*, *Phys. Rev. D* **60** (1999) 064014 [hep-th/9809164].
[213] J. Park and S.-J. Rey, *Green-Schwarz superstring on AdS(3) x S**3*, *JHEP* **01** (1999) 001 [hep-th/9812062].
[214] R. R. Metsaev and A. A. Tseytlin, *Superparticle and superstring in AdS(3) x S**3 Ramond-Ramond background in light cone gauge*, *J. Math. Phys.* **42** (2001) 2987 [hep-th/0011191].
[215] A. Babichenko, B. Stefanski, Jr. and K. Zarembo, *Integrability and the AdS(3)/CFT(2) correspondence*, *JHEP* **03** (2010) 058 [0912.1723].
[216] P. Sundin and L. Wulff, *Classical integrability and quantum aspects of the AdS 3 × s 3 × s 3 × s 1 superstring*, *Journal of High Energy Physics* **2012** (2012) .
[217] R. Borsato, O. Ohlsson Sax and A. Sfondrini, *A dynamic $\mathfrak{su}(1|1)^2$ S-matrix for AdS_3/CFT_2*, *JHEP* **04** (2013) 113 [1211.5119].
[218] J. R. David and B. Sahoo, *Giant magnons in the D1-D5 system*, *JHEP* **07** (2008) 033 [0804.3267].
[219] R. Borsato, O. Ohlsson Sax and A. Sfondrini, *All-loop Bethe ansatz equations*

for AdS3/CFT2, *JHEP* **04** (2013) 116 [1212.0505].

[220] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański and A. Torrielli, *The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector*, *JHEP* **08** (2013) 043 [1303.5995].

[221] O. Ohlsson Sax, B. Stefanski, jr. and A. Torrielli, *On the massless modes of the AdS3/CFT2 integrable systems*, *JHEP* **03** (2013) 109 [1211.1952].

[222] T. Lloyd and B. Stefański, Jr., *AdS_3/CFT_2, finite-gap equations and massless modes*, *JHEP* **04** (2014) 179 [1312.3268].

[223] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *Towards the All-Loop Worldsheet S Matrix for $AdS_3 \times S^3 \times T^4$*, *Phys. Rev. Lett.* **113** (2014) 131601 [1403.4543].

[224] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *The complete $AdS_3\times S^3\times T^4$ worldsheet S matrix*, *JHEP* **10** (2014) 066 [1406.0453].

[225] T. Lloyd, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *The complete worldsheet S matrix of superstrings on $AdS_3\times S^3\times T^4$ with mixed three-form flux*, *Nucl. Phys. B* **891** (2015) 570 [1410.0866].

[226] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, *The* $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ *worldsheet S matrix*, *J. Phys. A* **48** (2015) 415401 [1506.00218].

[227] B. Hoare and A. A. Tseytlin, *On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: tree-level S-matrix*, *Nucl. Phys. B* **873** (2013) 682 [1303.1037].

[228] B. Hoare and A. A. Tseytlin, *Massive S-matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux*, *Nucl. Phys. B* **873** (2013) 395 [1304.4099].

[229] C. Ahn and D. Bombardelli, *Exact S-matrices for AdS_3/CFT_2*, *Int. J. Mod. Phys. A* **28** (2013) 1350168 [1211.4512].

[230] N. Gromov, V. Kazakov and P. Vieira, *Finite Volume Spectrum of 2D Field Theories from Hirota Dynamics*, *JHEP* **12** (2009) 060 [0812.5091].

[231] V. Kazakov and S. Leurent, *Finite size spectrum of $SU(N)$ principal chiral field from discrete Hirota dynamics*, *Nucl. Phys. B* **902** (2016) 354 [1007.1770].

[232] S. Leurent and E. Sobko, *Exact Zero Vacuum Energy in twisted SU(N) Principal Chiral Field*, 1511.08491.

[233] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [hep-th/0603204].

[234] L. Freyhult and C. Kristjansen, *A Universality test of the quantum string Bethe ansatz*, *Phys. Lett. B* **638** (2006) 258 [hep-th/0604069].

[235] N. Gromov and P. Vieira, *Constructing the AdS/CFT dressing factor*, *Nucl. Phys. B* **790** (2008) 72 [hep-th/0703266].

[236] M. Beccaria, F. Levkovich-Maslyuk, G. Macorini and A. A. Tseytlin, *Quantum corrections to spinning superstrings in $AdS_3 \times S^3 \times M^4$: determining the dressing phase*, *JHEP* **04** (2013) 006 [1211.6090].

[237] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefanski, Jr. and A. Torrielli, *Dressing phases of AdS3/CFT2*, *Phys. Rev. D* **88** (2013) 066004 [1306.2512].

[238] S. Frolov and A. Sfondrini, *New dressing factors for AdS3/CFT2*, *JHEP* **04** (2022) 162 [2112.08896].

[239] A. Brollo, D. le Plat, A. Sfondrini and R. Suzuki, *The Tensionless Limit of Pure-Ramond-Ramond AdS3/CFT2*, 2303.02120.

[240] O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *Integrability and the*

Conformal Field Theory of the Higgs branch, *JHEP* **06** (2015) 103 [1411.3676].

[241] A. Cagnazzo and K. Zarembo, *B-field in AdS(3)/CFT(2) Correspondence and Integrability*, *JHEP* **11** (2012) 133 [1209.4049].

[242] E. Y. Urbach, *The black hole/string transition in* AdS_3 *and confining backgrounds*, 2303.09567.

[243] D. Sorokin, A. Tseytlin, L. Wulff and K. Zarembo, *Superstrings in AdS(2)xS(2)xT(6)*, *J. Phys. A* **44** (2011) 275401 [1104.1793].

[244] J. Murugan, P. Sundin and L. Wulff, *Classical and quantum integrability in* AdS_2/CFT_1, *JHEP* **01** (2013) 047 [1209.6062].

[245] B. Hoare, A. Pittelli and A. Torrielli, *Integrable S-matrices, massive and massless modes and the* $AdS_2 * S2$ *superstring*, *JHEP* **11** (2014) 051 [1407.0303].

[246] B. Hoare, A. Pittelli and A. Torrielli, *S-matrix algebra of the* $AdS_2 \times S^2$ *superstring*, *Phys. Rev. D* **93** (2016) 066006 [1509.07587].

[247] A. Fontanella and A. Torrielli, *Massless* AdS_2 *scattering and Bethe ansatz*, *JHEP* **09** (2017) 075 [1706.02634].

[248] A. Fontanella and A. Torrielli, *Geometry of Massless Scattering in Integrable Superstring*, *JHEP* **06** (2019) 116 [1903.10759].

Acta Universitatis Upsaliensis

Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology 2254

Editor: The Dean of the Faculty of Science and Technology

A doctoral dissertation from the Faculty of Science and Technology, Uppsala University, is usually a summary of a number of papers. A few copies of the complete dissertation are kept at major Swedish research libraries, while the summary alone is distributed internationally through the series Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology. (Prior to January, 2005, the series was published under the title "Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology".)

ACTA
UNIVERSITATIS
UPSALIENSIS
UPPSALA
2023

Distribution: publications.uu.se
urn:nbn:se:uu:diva-499273