UPPSALA UNIVERSITET

A CFT perspective on holographic correlators

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Dissertation presented at Uppsala University to be publicly examined in Sonja Lyttkens, Ångströmlaboratoriet, Lägerhyddsvägen 1, Uppsala, Thursday, 1 June 2023 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English. Faculty examiner: Professor Paul Heslop (Durham University, Department of Mathematical Sciences).

Abstract
Fardelli, G. 2023. A CFT perspective on holographic correlators. *Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology* 2255. 109 pp. Uppsala: Acta Universitatis Upsaliensis. ISBN 978-91-513-1774-8.

Conformal symmetry is ubiquitous in physics. It emerges in a variety of situations from critical phenomena, through condensed matter systems and jets at the LHC to string theory. Conformal field theories (CFTs) describe the universal behavior underlying these very different setups. An efficient and powerful way to study CFTs is represented by the conformal bootstrap. This non-perturbative technique allows constraining a CFT just relying on its own symmetries and a set of consistency conditions such as unitarity, crossing symmetry and the operator product expansion.In this thesis, we discuss applications of analytic bootstrap techniques to holographic superconformal field theories, which possess an additional space-time symmetry, known as supersymmetry, besides the conformal one. Through the AdS/CFT correspondence, these theories are dual to quantum gravity in AdS such that the analysis of CFT correlators gives access to gravitational amplitudes in curved space-time. We will see how the existence of a protected sector in such theories greatly simplifies the problem and allows us to bootstrap these observables.In this work, we devote our attention to the study of four-point functions in N=4 super Yang Mills (SYM) and in N=2 theories in four dimensions. In the first part of the thesis, we review the basics of superconformal algebra and superspace and then we introduce the main analytic bootstrap tool, the Lorentzian inversion formula. In the second part, after a brief description of the spectrum of N=4 SYM and its holographic realization in AdS, we focus on the correlator of four gravitons. We thoroughly analyze this four-point function in the supergravity approximation and as an expansion at large central charge. We conclude with a discussion of less supersymmetric correlators involving so-called quarter-BPS operators. In the last part instead, we change the setting and we study correlators of spinning operators belonging to the flavor current multiplet in N=2 superconformal theories. By using analytic superspace techniques, we build the four-point function of gluon superfields and, from that, we extract correlators of all component fields, including the four-current one. In the end, we comment on the existence of an AdS double copy connecting these gluon amplitudes with their gravitational counterpart in N=4 SYM.

Keywords: Superconformal field theories, Analytic bootstrap, Correlation functions, Quantum gravity

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*"Reserve your right to think, for even to think wrongly
is better than not to think at all."*
Hypatia of Alexandria

*Dedicated to all women in science,
with gratitude and admiration for the ones in the past
and great hopes for those in the future.*

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I   A. Bissi, G. Fardelli and A. Georgoudis, *Towards all loop supergravity amplitudes on $AdS_5 \times S^5$*, Phys. Rev. D **104** (2021) L041901 [2002.04604]

II   A. Bissi, G. Fardelli and A. Georgoudis, *All loop structures in supergravity amplitudes on $AdS_5 \times S^5$ from CFT*, J. Phys. A **54** (2021) 324002 [2010.12557]

III   A. Bissi, P. Dey and G. Fardelli, *Two Applications of the Analytic Conformal Bootstrap: A Quick Tour Guide*, Universe **7** (2021) 247 [2107.10097]

IV   A. Bissi, G. Fardelli and A. Manenti, *Rebooting quarter-BPS operators in $\mathcal{N} = 4$ super Yang-Mills*, JHEP **04** (2022) 016 [2111.06857]

V   A. Bissi, G. Fardelli, A. Manenti and X. Zhou, *Spinning correlators in $\mathcal{N} = 2$ SCFTs: Superspace and AdS amplitudes*, JHEP **01** (2023) 021 [2209.01204]

Papers not included in this thesis.

VI   A. Bombini and G. Fardelli, *Holographic entanglement entropy and complexity of microstate geometries*, JHEP **06** (2020) 181 [1910.01831]

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Contents

1. Introduction

Imagine you have a camera and you see the same image no matter how much you zoom in or zoom out. This situation seems quite unusual, and you could think that the object you are looking at would be rather special. In fact it is. If, now, we translate this observation in a quantum field theory (QFT) language, the object that you are observing is nothing more than a scale-invariant theory. Scale invariance is, in general, the hallmark of a bigger symmetry group, the *conformal group*.

In d spacetime dimensions, this is identified with $SO(d, 2)$ or $SO(d+1, 1)$, depending if we are in Lorentzian or Euclidean signature respectively. A QFT enjoying this property is called a *conformal field* theory, or CFT, for short. CFTs are ubiquitous in physics, from condensed matter to quantum gravity. They are defined by a set of local operators $\mathcal{O}_{\Delta,\ell}$, dubbed *primaries*, each of which is further classified by its conformal dimension Δ and Lorentz spin ℓ. Spacetime derivatives of these primaries generate operators with larger dimensions named *descendants*. Another extraordinary property of CFTs is that their operators have a convergent Operator Product Expansion (OPE). This means that when two operators, $\mathcal{O}_i$ and $\mathcal{O}_j$, approach one another, their product can be written as a sum of the primaries of the theory, $\mathcal{O}_k$, weighted with a coefficient, $\lambda_{\mathcal{O}_i \mathcal{O}_j \mathcal{O}_k}$, and this sum is well defined. Apart from this unknown number, consistency with conformal invariance fixes the form of the OPE. Consequently, three-point couplings of descendants are completely determined in terms of OPE coefficients of the respective primaries.

OPE coefficients, together with the dimensions and spins of the primary operators, represent the *OPE data* and they completely define a CFT; in the sense that they are all the information entering the correlation functions of local operators: the natural observables in a CFT. Conformal symmetry is powerful enough to completely fix the form of two- and three-point functions, hence the first dynamical objects to study are four-point correlators. Computing and analysing these four-point functions are the main focus of this thesis.

A very special class of conformal theories are those represented by *holographic CFTs*, which arise at the d-dimensional boundary of some $(d + 1)$-dimensional theory in Anti de Sitter (AdS) spacetime. The precise relation between the CFT and the bulk AdS theory is encoded in the AdS/CFT correspondence [1–3] or *gauge/gravity duality*: a duality

between a quantum gravity in AdS_{d+1} and the CFT_d living on its boundary. By dual we here mean that the two theories have identical Hilbert spaces and that they share the same symmetries.
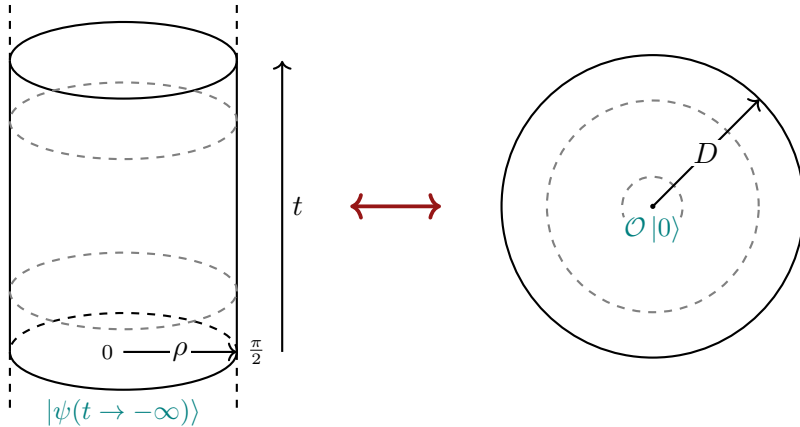
Figure 1.1. Euclidean AdS in global coordinates; the boundary is approached as $\rho \to \frac{\pi}{2}$. This is equivalent to a CFT in radial quantization. Time translation in the bulk translates to the action of dilation in the CFT, so that energies in AdS get mapped to conformal dimensions.

Let us briefly see how this correspondence is realized. As depicted in Fig 1.1, we can think of AdS as an infinite cylinder, where time runs vertically. In this picture, a specific time slice can be mapped to a fixed radius in the corresponding radially quantized CFT and an operator $\mathcal{O}$, at the origin in the CFT, can be taken to define a state in the infinite past of AdS. Since the action of the Hamiltonian in the bulk becomes the dilatation in the CFT, it is reasonable to expect that the mass, m, of an AdS state is related to the dimension, Δ, of the dual operator. Indeed

$$\Delta(\Delta - d) = m^2 R^2, \tag{1.1}$$

where R is the radius of AdS.
We can formulate the AdS/CFT correspondence in more precise terms as [3]

$$\langle e^{i \int_{\partial AdS} \bar{h}_i \mathcal{O}_i} \rangle_{\mathrm{CFT}} = \int_{\mathrm{AdS}} \mathcal{D}h_i \, e^{iS[h]} \Big|_{h_i|_{\partial AdS} = \bar{h}_i}. \tag{1.2}$$

That is as an equality between the generating functional of the CFT correlators and the path integral of the on-shell AdS action, evaluated on the field configurations that reduce to the sources $\bar{h}_i$ at the AdS boundary.

Given the constraints we are requiring, it is not obvious which CFTs are good to describe theories of quantum gravity and in the past decades

there has been tremendous progress in understanding what are the minimal requirements for a CFT in order to see the emergence of an AdS dual [4–11]. To explore these ideas, it is convenient to consider a family of theories characterisable by a small expansion parameter $\frac{1}{N}$, such that at $N \to \infty$ the theory is described by some *generalized free fields*.[1] In the case of the theory studied in Part II, this parameter will be related to the rank of the $SU(N)$ gauge group. Operators can then be classified as single- or multi-trace, depending on the number of traces they are made of under this gauge group. By analogy, yet with a slight abuse of notation, we will refer to single- and multi-trace operators also in the generic examples discussed below, even in the absence of a specified gauge group. In [6], it has been conjectured that any strongly coupled, *large N* CFTs, where all single-trace operators with spin greater than two have a parametrically large dimension $\Delta \sim \Delta_{\text{gap}} \gg 1$, are dual to a *local, weakly coupled, effective field theory* (EFT) of gravity in AdS.

Let us analyse this statement more carefully. First of all, a large N parameter in the CFT allows for a loop perturbative expansion on the gravity side and it permits us to identify single-trace operators with single-particle states. The requirement on the spin spectrum can be understood considering that all the known consistent, local effective theories of gravity have at most spin-two particles. Therefore, we expect that any other single-trace operator should decouple at low energies. This effect is obtained by having a large Δ_{gap}, which guarantees that massive states, with $m \sim \frac{\Delta_{\text{gap}}}{R}$ — see (1.1) — are separated from the rest of the spectrum. In Part II, we will identify these states with string excitations. Finally, let us comment that recently [13], this conjecture has been made more quantitative by deriving two-sided bounds on the EFT Wilson coefficients in terms of Δ_{gap}.

Examples of CFTs, that satisfy these conditions and provide a UV-complete theory of gravity, are typically realized in the string theory context. These cases require not only conformal symmetry but also supersymmetry and are, effectively, superconformal field theories (SCFT). To account for this additional set of symmetries, it is necessary to enlarge the AdS space by a compact internal manifold, whose size is the same as the AdS size. In this thesis, we will focus at first on the prototypical and original realization of AdS/CFT [1], namely the duality between $\mathcal{N} = 4$ Super Yang-Mills (SYM) and type IIB string theory on $\text{AdS}_5 \times \text{S}^5$. In particular, we will consider the regime in which the CFT is strongly coupled, thus dual to weakly coupled gravity on the AdS side. $\mathcal{N} = 4$ SYM is believed to be the sole theory, with at most spin-one

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[1]In [12], it has been shown that it is actually possible to break factorization of some correlators at large N and still have a holographic dual with now a strongly coupled matter sector.

particles, showing maximal supersymmetry in four dimensions and it is conformally invariant at the quantum level, *i.e.* its β-function does not receive any radiative corrections. Starting from $\mathcal{N} = 4$ SYM, we can add D-branes and break some supersymmetry, landing on a $\mathcal{N} = 2$ SCFT. In some regimes, this, together with other realizations, can be interpreted as a *gauge* theory in AdS, opening new possibilities for holographic investigations. Towards the end of the thesis, we will analyse this example and we will unveil connections to $\mathcal{N} = 4$ SYM.

Just as the main common interests of QFT in flat space — at least in high energy experiments — are S-matrix amplitudes, in holographic theories we will focus on CFT correlators. These are, in fact, the AdS analogue of flat-space amplitudes, to which they reduce in appropriate limits. This similarity can be used as inspiration to investigate whether or not analogous properties and simplifications, like those we observe in flat space, exist in the CFT. Questions one might ask are: can we use unitarity to relate loop amplitudes to tree-level ones and how? It is possible to construct amplitudes via dispersion relations? There exists a relation between gauge and gravity theories similar to flat-space color-kinematics duality and double copy? The answer is going to be that holographic CFTs enjoy most of these properties or adaptations thereof.

The strategy to explore these aspects will be using all the symmetries and consistency conditions of the theory, such as unitarity, crossing and causality (in Lorentzian configurations), to constrain observables and CFT data. This approach goes under the name of *conformal bootstrap*. The original bootstrap idea goes back to the 60s and early 70s, but after the original works [14, 15], it stayed quiescent until 2008, when the authors of [16] found an efficient numerical implementation of the *bootstrap equation*. The main motivation for their work was the phenomenological need to constrain technicolor-like theories, targeting a natural light Higgs, together with a dynamical explanation for flavour. This work renewed the interest in the conformal bootstrap and paved the way for the development of techniques to "solve" the bootstrap equation, both numerically [17] and analytically [18–20].

In this thesis, we will focus on the analytic side of the story, or the *light-cone bootstrap* [21–26]. The hope of the analytic approach is to eventually have full control and explicit results for theories at strong coupling, which are almost unattainable by other means. And, given the correspondence, perhaps someday, one could hope to understand quantum gravity based on the full non-perturbative comprehension of the boundary CFT.

1.1 An invitation: bootstrapping the quantum an-harmonic oscillator

To explain the ideas behind the bootstrap philosophy, let us look at an easy, yet powerful, application: the quantum oscillator with a quartic potential [27]. Its Hamiltonian is given by

$$H = p^2 + V(x), \qquad V(x) = x^2 + \lambda x^4, \tag{1.3}$$

where λ is the coupling constant, which should not necessarily be small, and x and p are the position and momentum operators satisfying the canonical commutation relation

$$[x, p] = i. \tag{1.4}$$

Let us denote by $|E\rangle$, the energy eigenstates such that

$$H |E\rangle = E |E\rangle, \quad E \in \mathbb{R}. \tag{1.5}$$

For simplicity let us define the expectation value on these energy eigenstates as $\langle E| \cdots |E\rangle \equiv \langle \cdots \rangle$. In this basis, it is straightforward to show that, for any operator $\mathcal{O}$:

$$\langle [\mathcal{O}, H] \rangle = 0, \tag{1.6}$$

and in particular

$$\langle [x^a, H] \rangle = 0, \tag{1.7a}$$

$$\langle [x^a p, H] \rangle = 0. \tag{1.7b}$$

Using the definition of H in (1.3), and after some simple commutator algebra, the relation (1.7a) gives

$$\langle x^a p \rangle = \frac{i}{2} a \langle x^{a-1} \rangle. \tag{1.8}$$

The second one is slightly more complicated

$$0 = \langle [x^a p, H] \rangle = 2ia \langle x^{a-1} p^2 \rangle + a(a-1) \underbrace{\langle x^{a-2} p \rangle}_{(1.8)} - i \langle x^a V'(x) \rangle$$

$$= \langle x^a V' \rangle + 2a (\langle x^{a-1} V \rangle - E \langle x^{a-1} \rangle) - \frac{(a-2)_3}{2} \langle x^{a-3} \rangle, \tag{1.9}$$

where we have used $p^2 = H - V(x)$ and $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ is the Pochhammer symbol. Substituting the explicit form of the potential as in (1.3), (1.9) gives a recurrence relation for the expectation value of powers of x in terms of the energy eigenvalue E

$$\lambda \langle x^a \rangle + \frac{a-2}{a-1} \langle x^{a-2} \rangle - E \frac{a-3}{a-1} \langle x^{a-4} \rangle - \frac{(a-5)_3}{4(a-1)} \langle x^{a-6} \rangle = 0. \tag{1.10}$$

13

Since we know that $\langle x^0 \rangle = 1$ and $\langle x^n \rangle = 0$ for n odd, any expectation value of x^n, with n even, can be computed from E and $\langle x^2 \rangle$ using the recursion (1.10). Now, inspired by the bootstrap idea, we aim to find an efficient way to bound these two unknown data, E and $\langle x^2 \rangle$, by means of symmetry constraints. In the usual case, we would require the four-point function to be crossing symmetric. In this simplified example, we will ask for the easier positivity condition

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0 \,. \tag{1.11}$$

With $\mathcal{O} = \sum_i c_a x^a$, $c_a \in \mathbb{C}$, this becomes

$$\sum_{a,b=0}^{N_{\max}} c_a^* c_b \mathrm{M}_{ab} \geq 0, \qquad \mathrm{M}_{ab} \equiv \langle x^{a+b} \rangle \,, \tag{1.12}$$

where $N_{\max}$ is the parameter controlling the size of the matrix and can be tuned in the numerical implementation. Putting all this together, our bootstrap problem reads

$$\left(E, \langle x^2 \rangle \right) \quad \text{allowed} \quad \Longleftrightarrow \quad \mathrm{M}_{a,b} \succcurlyeq 0 \,, \tag{1.13}$$

where imposing that the matrix is positive semi-definite is equivalent to (1.12). As one can see from the plots below, we were able in this way to carve out regions in parameter space and isolate islands of allowed values, smaller and smaller as $N_{\max}$ increases. From this very simple example we can already appreciate that by resorting to symmetries and consistency conditions, we can highly constrain our theory, no matter what the coupling is.

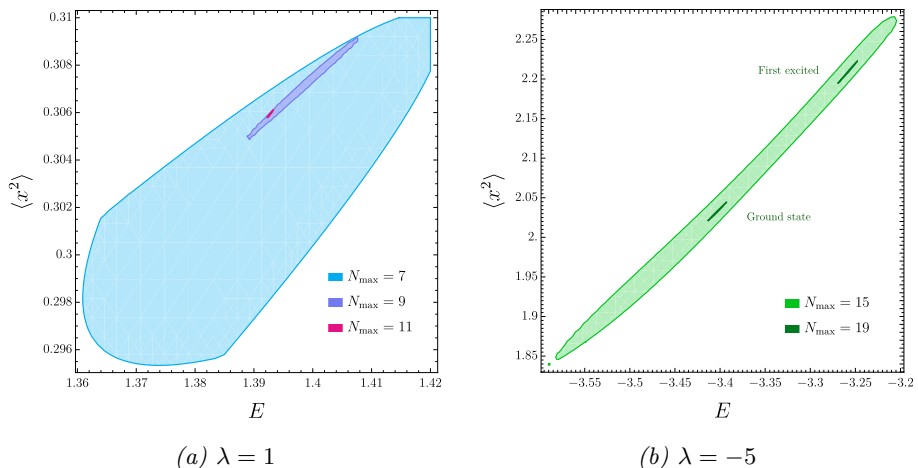(a) $\lambda = 1$ \qquad\qquad (b) $\lambda = -5$

Figure 1.2. $(E, \langle x^2 \rangle)$ parameter space with varying λ and matrix size. On the left, the island shrinks as $N_{\max}$ increases. On the right, the ground state and the first excited state can be resolved.

1.2 Analytic conformal bootstrap

The essence of the conformal bootstrap is represented by the bootstrap equation, which encodes the consequence of conformal invariance and crossing symmetry on correlation functions. In its easiest formulation, one can consider the four-point correlator of identical scalar fields ϕ.[2] Conformal symmetry fixes its functional form up to an unknown theory-dependant function,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{\mathcal{G}(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\,, \tag{1.14}$$

where Δ_ϕ is the conformal dimension of the operator, $x_{ij}^2 = (x_i - x_j)^2$ and we have defined the CFT cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad\qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{1.15}$$

Using the OPE, one can decompose $\mathcal{G}(u,v)$ as the sum over all possible exchanged primaries $\mathcal{O}$, with dimension Δ and spin ℓ,

$$\mathcal{G}(u,v) = \sum_{\mathcal{O}} a_{\Delta,\ell}\, g_{\Delta,\ell}(u,v)\,,$$
$$a_{\Delta,\ell} = \lambda_{\phi\phi\mathcal{O}}^2 > 0\,, \tag{1.16}$$

where $g_{\Delta,\ell}$ are the conformal blocks and an explicit expression is known in two and four dimensions [28]. Now, associativity of the OPE requires

$$\frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \sum a_{\Delta,\ell}\, g_{\Delta,\ell}(u,v) \quad = \quad \frac{1}{(x_{14}^2 x_{23}^2)^{\Delta_\phi}} \sum a_{\Delta,\ell}\, g_{\Delta,\ell}(v,u)\,.$$

This is the well-known bootstrap equation. To understand the power and the implication of this relation, let us consider a simple example: a free scalar in AdS. The corresponding CFT is a *mean field theory* (MFT) or

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[2]Here, and throughout the thesis, we are working in $d > 2$. $d = 2$ CFTs are special and deserve a completely separate treatment.

generalized free fields, a generalization of free theory in which correlators are given by the sum over all possible products of two-point functions. In this case,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \left(1 + u^{\Delta_\phi} + \frac{u^{\Delta_\phi}}{v^{\Delta_\phi}} \right). \quad (1.17)$$

By expanding in conformal blocks, it is possible to determine the spectrum and OPE coefficients of the exchanged operators [29]. They are the identity, $\Delta = 0$ and $\ell = 0$, and "double-trace" operators constructed out of the original field ϕ. They take the schematic form

$$[\phi\phi]_{n,\ell} = \phi\square^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \phi, \quad \text{with} \quad \Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell. \quad (1.18)$$

As shown above, crossing requires

$$\mathcal{G}(u,v) = \left(\frac{u}{v} \right)^{\Delta_\phi} \mathcal{G}(v,u). \quad (1.19)$$

Now expand the RHS in conformal blocks and substitute (1.17) in the LHS. What we get is

$$\frac{1}{u^{\Delta_\phi}} + 1 + \frac{1}{v^{\Delta_\phi}} = \frac{1}{v^{\Delta_\phi}} \left\{ 1 + \sum_{n,\ell} a_{n,\ell}\, g_{\Delta,\ell}(v,u) \right\}. \quad (1.20)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\mathbb{1} \qquad\qquad\qquad [\phi\phi]_{n,\ell}$$

The first thing to notice is that on the LHS, in the limit of $u \ll 1$, we have a power law divergence. This should somehow be reproduced by the sum on the right. However, in this limit the blocks behave as

$$g_{\Delta,\ell}(v,u) \xrightarrow{u \ll 1} \log u, \quad (1.21)$$

and this means that, to reproduce the $u^{-\Delta_\phi}$ divergence, we cannot have a finite number of operators in the sum, and, that the sum cannot converge uniformly. Let us better analyse these convergence issues. First of all, one can show that the sum over the twists, $\tau = \Delta - \ell = 2\Delta_\phi + 2n$, converges, so the problems should come from the spin sum. Indeed, by studying the large spin asymptotic in the regime $|u| \ll |v| \ll 1$, *i.e.* the *double lightcone limit*, one can prove that the sum over ℓ does not converge uniformly around $u \simeq 0$. Nevertheless, one can define the sum in a region where it is well defined and then analytically continue it. In this way, in [19] it was proved that the sum over the double-trace operators with large spin actually reproduces the $u^{-\Delta_\phi}$ divergence.

The take-home message of this example is that the divergent behaviour, as $u \to 0$, of the s-channel OPE is controlled by the sum over

operators at large ℓ in the t-channel OPE. In [19], the authors pointed out that this property applies more broadly, away from mean field theory, and has precise implications for the spectrum of a theory. Consider again the crossing equation (1.19), now for a generic scalar four-point function. By expanding in blocks on both sides, we obtain

$$1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\frac{\tau}{2}} \tilde{g}_{\tau,\ell}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \left\{ 1 + \sum_{\tau,\ell} a_{\tau,\ell} v^{\frac{\tau}{2}} \tilde{g}_{\tau,\ell}(v,u) \right\} , \quad (1.22)$$

where we have re-defined $g_{\Delta,\ell}(u,v) = u^{\frac{\tau}{2}} \tilde{g}_{\tau,\ell}(u,v)$, to make explicit the behaviour at $u \simeq 0$. In the small u limit, the infinite sum to the right should reproduce the identity contribution on the LHS. Remarkably, this requirement is enough to predict the existence of an infinite family of operators with twist $\tau = 2\Delta_\phi + 2n$ as $\ell \to \infty$ and OPE coefficients[3]

$$a_{\tau,\ell} = a_{n,\ell}^{\text{MFT}} + O\left(\frac{1}{\ell}\right) . \qquad (1.23)$$

Besides the MFT explicit example, to have an intuition of why we need this class of operators, observe that in (1.22), the identity contribution on the LHS does not have any v dependence. To reproduce it with the infinite sum, we expect that $v^{-\Delta_\phi + \frac{\tau}{2}} = 1$, *i.e.* $\tau = 2\Delta_\phi$. And a more careful analysis can show that the behaviour we should require is indeed $\tau = 2\Delta_\phi + 2n$, $n \in \mathbb{N}$. The details of the proof can be found in [19] — see also [18, 20] for similar results.

But one can say even more about the OPE data of the theory. In the case where, in the tower of double-trace operators, there is an operator for each spin, one can fix the dependence of the correction to the dimensions and OPE coefficients at large ℓ. The first corrections, turn out, in fact, to be controlled by the operator, in the OPE, with minimal twist $\tau_m \neq 0$, which can be for instance the stress-tensor, as

$$\begin{aligned}
\tau &\to 2\Delta_\phi + 2n + \frac{\gamma_n}{\ell^{\tau_m}} , \\
a_{\tau,\ell} &\to a_{n,\ell}^{\text{MFT}} + \frac{c_n}{\ell^{\tau_m}} .
\end{aligned} \qquad (1.24)$$

To conclude, we cannot refrain from noticing the similarity with scattering theory in flat space. In fact, in the CFT we have found that an infinite tower of operators with large spin in the t-channel OPE is needed to reproduce poles in the s-channel. But this is similar to what happens to a tree-level scattering amplitude. When decomposed in partial waves,

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[3]Notice that for this argument to hold true, there should exist a gap between the identity and the dimensions of the other operators. In $d > 2$, this is guaranteed by unitarity bounds.

only the poles in one channel are manifest and are controlled, in the other channel at high-energy and large angular momentum [30], by an infinite sum. Not by chance, the theory of complex-angular momenta and the Froissart-Gribov formula inspired another great advancement in the analytic bootstrap: the Lorentzian inversion formula [31].

1.3 Outline

The thesis is organized into three main parts. In Part I we introduce some general background material and the technical tools needed for the rest of the discussion. We start with a brief description of superconformal field theories, their algebra and representations, with a particular focus on $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry. Then we present a very convenient formulation of these theories in terms of superspaces. We conclude by reviewing the Lorentzian inversion formula: a key ingredient in the analytic bootstrap program and an essential element in the derivation of the results appearing in this thesis.

In Part II we specify to the maximally supersymmetric CFT in four dimensions: $\mathcal{N} = 4$ Super Yang-Mills. We first describe its spectrum and we fix the form of correlation functions of protected operators just relying on symmetries. Then we proceed to explore its connection to quantum gravity in AdS, we provide a few details on this holographic realization and the underlying dictionary. In Chapter 5 we study in depth the four-point function of the stress-tensor scalar superprimary, the supergraviton. We review the solution of the superconformal Ward Identities and we show how the knowledge of the protected spectrum is enough to fix the supergravity correlator in the large N and infinite 't Hooft coupling regime. Similar reasonings allow us to bootstrap the correlator at one loop (N^{-4}) and eventually to infer part of it at all loops. Following the results in Paper I and II — reviewed in Paper III — we interpret these findings as consequences of unitarity in AdS. The last part, based on Paper IV, contains a description of correlators of quarter-BPS operators.

Finally, in Part III, we devote our attention to the exploration of gauge theories in AdS by looking at the dual $\mathcal{N} = 2$ SCFTs, presenting the results in Paper V. Using superspace techniques, we construct the four-point function of flavour current superfields in terms of the one of the scalar superprimaries (supergluons), adapting a construction known for $\mathcal{N} = 4$. The Part ends with some comments on the existence of an AdS double copy relating supergluon correlators in $\mathcal{N} = 2$ and supergraviton ones in $\mathcal{N} = 4$ and on how to realize it for spinning correlators in position space.

A graphical summary and some concluding remarks, flashing possible future directions, are collected in the final chapter.

Part I:
Background material

Analytic bootstrap techniques have shown to be incredibly powerful when applied to superconformal field theories and they have led to tremendous progress in understanding and constraining these theories in various spacetime dimensions [32–40]. In this thesis we will focus in particular on four-dimensional SCFTs with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry.

To set conventions and properly introduce theories exhibiting superconformal symmetry, in this introductory part we will review the necessary background material. First of all, we will briefly explain what a superconformal algebra is and how to construct its representations. In doing so, we will get acquainted with specific protected operators, whose correlation functions are going to be the main focus of the rest of this thesis. To make symmetries manifest, we will then introduce a new language suitable to phrase our problems, the one of superspace.

In the second part, we will review the main analytic bootstrap tool we will use throughout our discussion: the Lorentzian inversion formula. This will give us a way to extract OPE data of operators exchanged in the four-point function, in terms of an easier object: the double discontinuity (dDisc) of the correlator.

2. Superconformal field theories

Given the vast amount of symmetries, superconformal field theories are the perfect playground to apply the bootstrap philosophy. This is, however, not an isolated example. SCFTs are, in fact, studied and constructed with lots of different methods and they appear in a variety of physics contexts. As we have mentioned before, they are connected to gravity through the AdS/CFT correspondence, they can describe theories with no Lagrangians [41–43], they can be studied with localizations [44–46] and integrability [47]. They can exhibit dualities, enhanced symmetries, like the existence of a chiral algebra — a protected subsector in $\mathcal{N} = 2$ theories [48] — and generalizations of usual symmetries, like invertible and non-invertible higher-form symmetries [49–54].

They are at the cross-road of diverse approaches and fields. This makes SCFTs very interesting to study, since we can look at the same problem from very different, and sometimes complementary, perspectives. Such a multilateral approach surely benefits from a knowledge of the theory just based on symmetries, which does not rely on any specific, theory-dependent, realization. In this section, we will focus exactly on that.

2.1 Superconformal algebra

In four dimensions, the conformal group is made of six Lorentz transformations $M^{\mu\nu}$, four translations P^{μ} and special conformal transformations K^{μ} and one dilatation D. It is thus identifiable with $\mathrm{SO}(5,1)$ in $\mathbb{R}^4$ and $SO(4,2)$ in $\mathbb{M}^4$. For convenience, let us introduce spinor notation for the Poincaré and special conformal generators

$$
\begin{aligned}
\mathrm{P}_{\alpha\dot\alpha} &= \sigma^{\mu}_{\alpha\dot\alpha} P_{\mu}\,, & \tilde{\mathrm{K}}^{\dot\alpha\alpha} &= \bar\sigma^{\mu\dot\alpha\alpha} K_{\mu}\,, \\
\mathrm{M}_{\alpha}{}^{\beta} &= -\frac{1}{4}i(\sigma^{\mu}\bar\sigma^{\nu})_{\alpha}{}^{\beta} M_{\mu\nu}\,, & \overline{\mathrm{M}}^{\dot\alpha}{}_{\dot\beta} &= -\frac{1}{4}i(\bar\sigma^{\mu}\sigma^{\nu})^{\dot\alpha}{}_{\dot\beta} M_{\mu\nu}\,,
\end{aligned} \tag{2.1}
$$

where the conventions for Pauli matrices are those of [55].

In the presence of supersymmetry, this algebra gets enlarged by the generators of supersymmetric transformations: Q^i_{α} and $\bar{Q}_{\dot\alpha i}$, where i runs from one to $\mathcal{N}$, amount of supersymmetry. In four dimensions, for non-gravitational theory, $\mathcal{N}$ ranges from 1 to 4. Q and $\bar{Q}$ have scaling dimensions $\frac{1}{2}$ and they have non-vanishing anticommutator

$$
\{Q^i_{\alpha}, \bar{Q}_{\dot\alpha j}\} = 2\delta^i{}_j \mathrm{P}_{\alpha\dot\alpha}\,. \tag{2.2}
$$

In order to close the superconformal algebra, one also needs to add conformal supercharges S_i^α and $\bar{S}^{\dot\alpha i}$, with conformal dimension $-\frac{1}{2}$, such that

$$\{\bar{S}^{\dot\alpha i}, S_j^\alpha\} = 2\delta^i{}_j \tilde{K}^{\dot\alpha\alpha}\,. \tag{2.3}$$

Both the supersymmetry and conformal supercharges transform under an additional symmetry, which is part of the algebra, the R-symmetry. It acts on the i, j indices and it coincides with $\mathfrak{u}(\mathcal{N})_R$ for $\mathcal{N} = 1, 2, 3$ and $\mathfrak{su}(\mathcal{N})_R$ for $\mathcal{N} = 4$.

All together, these generators form a Lie superalgebra, $\mathfrak{su}(2, 2|\mathcal{N})$ for $\mathcal{N} = 1, 2, 3$ and $\mathfrak{psu}(2, 2|4)$ for $\mathcal{N} = 4$. Representations can be labelled by the bosonic subalgebra

$$\underbrace{\mathfrak{so}(3, 1)}_{(2j, 2\bar{j})} \times \underbrace{\mathfrak{so}(1, 1)}_{\Delta} \times (\mathfrak{s})\mathfrak{u}(\mathcal{N})_R\,. \tag{2.4}$$

Consequently, we will denote a generic representation as $[2j, 2\bar{j}]_\Delta^R$. For the cases treated in this thesis, namely $\mathcal{N} = 4$ and $\mathcal{N} = 2$, we will specify the R-symmetry representations by the Dynkin labels $[q, p, \bar{q}]$ for the first case. While for the latter, we further split $\mathfrak{u}(2)_R = \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$ and we denote the representations with $(2j_R; r)$. In this notation and for the cases under consideration, the quantum numbers of the supercharges are

$$Q \in [1; 0]_{\frac{1}{2}}^{(1;-1)}\,, \qquad \bar{Q} \in [0; 1]_{\frac{1}{2}}^{(1;1)}\,, \qquad\qquad \text{for } \mathcal{N} = 2\,, \tag{2.5a}$$

$$Q \in [1; 0]_{\frac{1}{2}}^{[1,0,0]}\,, \qquad \bar{Q} \in [0; 1]_{\frac{1}{2}}^{[0,0,1]}\,, \qquad\qquad \text{for } \mathcal{N} = 4\,. \tag{2.5b}$$

2.2 Superconformal representations and unitarity bounds

Unitary representations are constructed starting from a *superprimary*, which is an operator satisfying

$$S_i^\alpha |\mathcal{O}\rangle = 0\,, \qquad \bar{S}^{\dot\alpha i} |\mathcal{O}\rangle = 0\,. \tag{2.6}$$

Notice that, given the commutation relation (2.3), a superprimary is in particular a *conformal primary*, *i.e.* it satisfies

$$K_{\alpha\dot\alpha} |\mathcal{O}\rangle = 0\,. \tag{2.7}$$

Superdescendants are obtained from superprimaries through the action of the Q and $\bar{Q}$ supercharges. Given the anticommutation relation (2.2), some of these superdescendants reduce, in particular, to usual conformal descendants, namely states obtained by the action of $P_{\alpha\dot\alpha}$ on a primary.

Moreover, due to the fermionic nature of the supercharges, in a given supermultiplet we will find a finite number of conformal primaries. In particular, if no other simplification occurs, a multiplet will generically contain $2^{4\mathcal{N}}$ states. These are called *long* multiplets.

What are these possible simplifications and where do they come from? We know that, for generic unitarity CFTs, requiring the norm of the states to be non-negative imposes strong constraints on the conformal dimensions of the operators. In four dimensions, these *unitarity bounds* read

$$
\begin{aligned}
\Delta &= 0 && \text{for identity}\,, \\
\Delta &\geq 1 && \text{for scalars}\,, \\
\Delta &\geq 1 + j && \text{for } j > 0,\ \bar{j} = 0\,, \\
\Delta &\geq 2 + j + \bar{j} && \text{for } j\bar{j} \neq 0\,,
\end{aligned}
\tag{2.8}
$$

where for symmetric traceless tensors ($j = \bar{j}$) we will identify $\ell = j + \bar{j}$. Operators saturating the bounds are null vectors and their dimension is protected from receiving quantum corrections. Examples of these operators are free scalars and fermions, conserved currents and the stress-tensor $T^{\mu\nu}$.

When we add supersymmetry, even stronger constraints appear. Suppose there exists a function of the quantum numbers $f(j, \bar{j}, \text{R-charges})$, such that

$$
\Delta \geq f(j, \bar{j}, \text{R-charges})\,,
\tag{2.9}
$$

where the precise form of this function depends on the amount of supersymmetry. The operators with dimensions strictly bigger than this function can be identified with the long operators mentioned above. Representations exactly saturating the inequality, contain some null vectors and they are referred to as operators *at threshold*. In the usual case without supersymmetry, the story would end here. In the case of SCFT, instead, one can still find other isolated allowed representations. These form *short multiplets* and they obey definite shortening conditions. Based on the combination of Q's and $\bar{Q}$'s annihilating the states, different types of shortenings exist. Since the action of the supercharges on a certain state gives zero, these multiplets are truncated and hence the name short. They are also often called half-, quarter-, ... BPS operators, depending on the number of Q's and $\bar{Q}$'s annihilating them. Among them, let us display two special ones [56, 57]: the stress-tensor multiplet in $\mathcal{N} = 4$ in Fig. 2.2 and the multiplet of conserved flavour (global) currents in $\mathcal{N} = 2$ in Fig. 2.1. The latter begins with the scalar $\mathcal{O}_2^{ab}$, where $a = 1, 2$ is the SU(2)$_R$ index. Then they follow two gluinos λ_α^a and $\bar{\lambda}_{\dot{\alpha}}^a$, the flavour current $\mathcal{J}_\mu$ and two complex scalars of opposite U(1)$_r$ R-charge $\mathcal{W}$ and $\overline{\mathcal{W}}$. The $\mathcal{N} = 4$ supermultiplet starts with a scalar superprimary $\mathcal{O}_2$ and it contains the SU(4)$_R$-symmetry current

$$\mathcal{O}_2 = [0;0]_2^{(2;0)}$$

$$\lambda_\alpha = [1;0]_{5/2}^{(1;-1)} \qquad \bar\lambda_{\dot\alpha} = [0;1]_{5/2}^{(1;1)}$$

$$\mathcal{W} = [0;0]_3^{(0;-2)} \qquad \mathcal{J}_\mu = [1;1]_3^{(0;0)} \qquad \overline{\mathcal{W}} = [0;0]_3^{(0;2)}$$
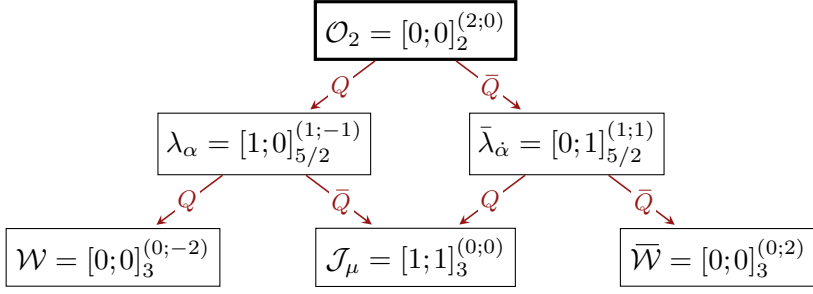
Figure 2.1. The conserved current multiplet in $\mathcal{N}=2$.

$$\mathcal{O}_2 = [0;0]_2^{[0,2,0]}$$

$$[1;0]_{5/2}^{[0,1,1]} \qquad [0;1]_{5/2}^{[1,1,0]}$$

$$[0;0]_3^{[0,0,2]} \quad [2;0]_3^{[0,1,0]} \qquad [1;1]_3^{[1,0,1]} \qquad [0;0]_3^{[2,0,0]} \quad [0;2]_3^{[0,1,0]}$$

$$[1;0]_{7/2}^{[0,0,1]} \qquad [2;1]_{7/2}^{[1,0,0]} \qquad [1;2]_{7/2}^{[0,0,1]} \qquad [0;1]_{7/2}^{[1,0,0]}$$

$$\mathcal{L} = [0;0]_4^{[0,0,0]} \qquad \mathcal{T}_{\mu\nu} = [2;2]_4^{[0,0,0]} \qquad \bar{\mathcal{L}} = [0;0]_4^{[0,0,0]}$$
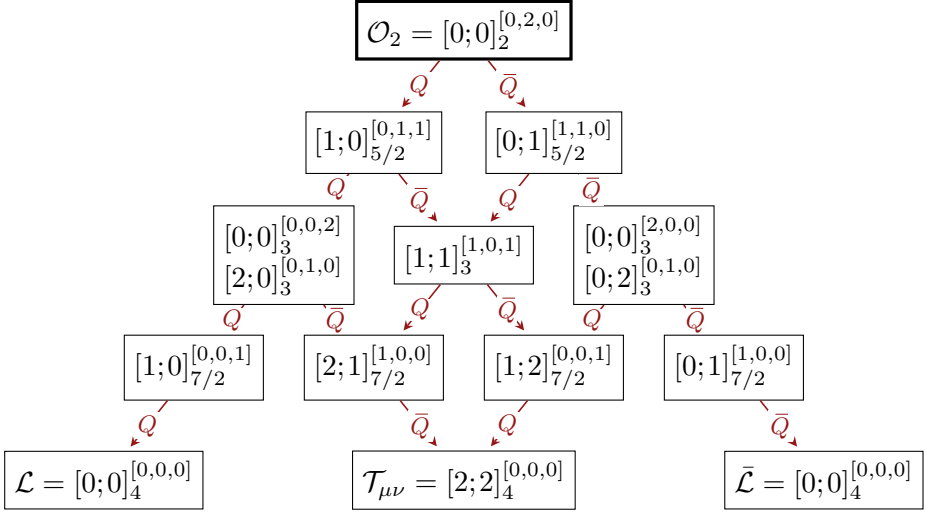
Figure 2.2. The stress-tensor multiplet in $\mathcal{N}=4$.

$\mathcal{J}_\mu^{\mathrm{SU}(4)} = [1;1]_3^{[1,0,1]}$ and the stress-tensor $\mathcal{T}_{\mu\nu}$. At the top, we also find the self-dual and anti-self-dual Lagrangians $\mathcal{L}$ and $\bar{\mathcal{L}}$, which carry opposite bonus $\mathrm{U}(1)_Y$ charge [58].

2.3 Superspace

When studying a quantum field theory, one would like to have a language in which symmetries are manifest. This can be obtained by the use of spaces in which a specific symmetry is realized geometrically through coordinate transformations [59]. As a very concrete example, consider Minkowski spacetime $\mathbb{M}^4$ with coordinates x^μ. Under a Poincaré transformation

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu \,, \tag{2.10}$$

invariant observables in the theory are explicitly left unchanged. A way to visualize that these are the "best" coordinates, is by considering the coset realization of Minkowski spacetime

$$\mathbb{M}^4 = \frac{\mathcal{P}}{\text{SO}(3,1)} = (x^\mu) \ . \tag{2.11}$$

In words, this means that $\mathbb{M}^4$ can be written as the quotient of the Poincaré group $\mathcal{P}$ by its Lorentz subgroup, such that the full space is covered starting from a point and applying a Poincaré transformation, with points, related by a Lorentz transformation, identified.

Similar coset constructions turn out to be particularly convenient when supersymmetry is introduced. To make supersymmetry manifest, one has to enlarge Minkowski spacetime by anticommuting Grassmann variables $\theta_{\alpha i}$ and $\bar{\theta}^i_{\dot{\alpha}}$, $i = 1, \cdots, \mathcal{N}$, suited coordinates for a new space, a *superspace*.

Then, similarly as before, one can construct various realizations of a superspace by identifying it with a coset of the corresponding supergroup quotiented by an appropriate subgroup. For instance, we can define

$$\mathbb{R}^{4|4\mathcal{N}} = \frac{\text{super}\mathcal{P}_\mathcal{N}}{\text{SO}(3,1) \times \text{SU}(\mathcal{N})} = \left(x^\mu, \theta_{\alpha j}, \bar{\theta}^j_{\dot{\alpha}}\right) \equiv \mathbf{z} \,, \tag{2.12}$$

the superspace obtained by quotienting the superPoincaré group with $\mathcal{N}$ supersymmetry by the product of the Lorentz and R-symmetry subgroup. An element of this coset can be parametrized by

$$\Omega(\mathbf{z}) = \exp\left[i\left(-x^\mu P_\mu + \theta_i^\alpha Q_\alpha^i + \bar{\theta}_{\dot{\alpha}}^i \bar{Q}_i^{\dot{\alpha}}\right)\right] \,. \tag{2.13}$$

In this supermanifold, we can introduce covariant derivatives

$$D_\alpha^i = \frac{\partial}{\partial \theta_i^\alpha} + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}i} \frac{\partial}{\partial x^\mu} \,, \tag{2.14a}$$

$$\bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}i}} - i\theta_i^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \frac{\partial}{\partial x^\mu} \,. \tag{2.14b}$$

A supersymmetric transformation is obtained by the action

$$X = \exp\left[i\varepsilon_i^\alpha Q_\alpha^i + i\bar{\varepsilon}_{\dot{\alpha}}^i \bar{Q}_i^{\dot{\alpha}}\right] \,, \tag{2.15}$$

and to derive the expression for Q and $\bar{Q}$ as differential operators acting on the supermanifold, we need to consider the infinitesimal variation $\delta\mathbf{z}$. This is determined by

$$\Omega^{-1}(\mathbf{z})X\,\Omega(\mathbf{z}) = \Omega^{-1}(\mathbf{z})\Omega(\mathbf{z} + \delta\mathbf{z}) \,, \tag{2.16}$$

which, by means of the Baker-Campbell-Hausdorff formula, returns

$$\delta x^\mu = -i\varepsilon_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i} - i\bar\varepsilon^{\dot\alpha i}\theta_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\,, \tag{2.17a}$$

$$\delta\theta_i^\alpha = \varepsilon_i^\alpha\,, \tag{2.17b}$$

$$\delta\bar\theta_{\dot\alpha}^i = \bar\varepsilon_{\dot\alpha}^i\,. \tag{2.17c}$$

With the variations, we can finally compute the generators[1]

$$Q_\alpha^i = \frac{\partial}{\partial\theta_i^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\frac{\partial}{\partial x^\mu}\,, \tag{2.18a}$$

$$\bar Q_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} + i\theta_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\,. \tag{2.18b}$$

It is easy to verify that this representation satisfies the anticommutator (2.2)

$$\{Q_\alpha^i, \bar Q_{\dot\alpha j}\} = 2i\delta_j^i \sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,,$$
$$P^\mu = i\frac{\partial}{\partial x^\mu}\,. \tag{2.19}$$

After having introduced coordinates and the action of transformations, the last piece we have to define is what is a field in superspace, or a *superfield*. It is an object depending on the superspace coordinates and it can be expanded in component fields using (2.13)

$$\mathcal{O}(x,\theta,\bar\theta) = \mathcal{O}(x) + i\theta_i^\alpha(Q_\alpha^i\mathcal{O})(x) + i\bar\theta_{\dot\alpha}^i(\bar Q_i^{\dot\alpha}\mathcal{O})(x)$$
$$+ \frac{1}{2}\theta_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha j}\left(Q^{\beta i}\sigma_{\mu\beta\dot\beta}\bar Q_j^{\dot\beta} + 2i\delta_j^i\partial_\mu\right)\mathcal{O}(x) + \cdots \tag{2.20}$$

Counting all the possible θ, $\bar\theta$ combinations, we can conclude that a generic superfield contains $2^{4\mathcal{N}}$ states and it corresponds to a long multiplet. As we have seen in the previous section, this number is reduced in the presence of shortening conditions. In this superspace language, shortening is imposed through the action of the covariant derivatives in (2.14). This is the correct choice because they are D and $\bar D$ that impose constraints as operator equations, whose action is equivalent to the application of Q and $\bar Q$ to all the component fields.[2] This should not be confused with the imposition of invariance under supersymmetry, which is instead realized just by requiring that a certain expression vanishes under the action of Q and $\bar Q$.[3]

---

[1]Notice the sign difference with respect to the covariant derivatives in (2.14).

[2]Moreover since covariant derivatives and supercharges anticommute, $\overset{(-)}{D}\mathcal{O}$ is a supersymmetric invariant constraint.

[3]I would like to thank Andrea Manenti to clarify this point to me.

As a simple example of how this shortening works, consider a $\mathcal{N} = 1$ supersymmetric theory. Here we can define a *chiral* superfield by requiring

$$\bar{D}_{\dot{\alpha}}\mathcal{O}(x, \theta, \bar{\theta}) = 0 \,. \tag{2.21}$$

It is clear that this constraint imposes relations among the various component fields, which become dependant, thus effectively reduced in number. However, dealing with a superspace with a constraint is not very convenient and one would like to have a formulation of a chiral theory with unconstrained fields. This is achievable by defining a new coordinate

$$z^{\mu} = x^{\mu} + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \,,$$
$$\bar{D}_{\dot{\alpha}}z^{\mu} = 0 \,, \tag{2.22}$$

so that by considering fields just depending on it, $\mathcal{O}(x, \theta, \bar{\theta}) \equiv \mathcal{O}(z, \theta)$, the constraint (2.21) is trivially satisfied. Notice that the $\bar{\theta}$ dependence is now hidden in the definition of z^{μ}, but there is no longer an explicit one. Since we have defined new coordinates, it makes sense to define a new superspace arranging for this additional symmetry. This is obtained by changing the subgroup in the quotient (2.12) to include half of the supertranslations

$$\mathbb{C}^{4|2} = \frac{\text{super}\mathcal{P}_1}{\{M_{\mu\nu}, \bar{Q}_{\dot{\alpha}}\}} \,, \tag{2.23}$$

this is called *chiral superspace*.

This is an emblematic example of how different superspaces can be better suited to describe a specific system based on the symmetries of the problem. Finding the most appropriate superspace is possibly more important when dealing with extended supersymmetry and when we add conformal symmetry. In particular, for SCFTs and certain classes of BPS operators, it can be shown that it is useful to adopt the so-called *harmonic* and *analytic* superspaces. In the next chapters, we are going to see a realization of these in the context of $\mathcal{N} = 4$ SYM and in $\mathcal{N} = 2$ SCFTs. In doing that, our focus will be on showing how the superspace formalism makes some computations achievable and some properties manifest. We refer to [59, 60] for a proper and detailed treatment.

3. Lorentzian inversion formula

As we have seen in the introduction, studying correlators in Lorentzian kinematics can make singular behaviours manifest and predict very general properties of CFTs. Another feature, which is obscured in Euclidean signature but emerges in Lorentzian one, is *analyticity in spin*. This is intimately tied to the non-vanishing of certain correlators at time-like separated points, which is evidently achievable only in Lorentzian kinematics.

For simplicity, let us consider the four-point function of four generic scalar operators of dimension Δ_i

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \mathcal{O}_{\Delta_4}(x_4) \rangle = \mathcal{K}_{\{\Delta_i\}}(x_i) \, \mathcal{G}(z, \bar{z}) \,, \qquad (3.1)$$

with

$$u = z\bar{z} \,, \qquad v = (1 - z)(1 - \bar{z}) \qquad\qquad\qquad (3.2\text{a})$$

$$\mathcal{K}_{\{\Delta_i\}} = \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1 + \Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3 + \Delta_4)}} \left(\frac{x_{14}^2}{x_{13}^2} \right)^{\frac{1}{2}\Delta_{34}} \left(\frac{x_{24}^2}{x_{14}^2} \right)^{\frac{1}{2}\Delta_{12}} \,, \tag{3.2b}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$. We will assume that these operators transform under an additional global symmetry — this is going to be relevant in Sec. 6. Therefore, we can further expand $\mathcal{G}$ in a basis of independent tensor structures $\mathbb{T}_k$, corresponding to the allowed representations exchanged in the OPE under this global symmetry,

$$\mathcal{G}(z, \bar{z}) = \sum_k \mathbb{T}_k \mathcal{G}_k(z, \bar{z}). \tag{3.3}$$

Apart from the conformal block expansion we have seen in (1.16), each $\mathcal{G}_k(z, \bar{z})$ admits a decomposition in a complete basis of single-valued functions $F_{\Delta, \ell}$, the principal series representation. These functions, often called conformal partial waves, have integer spin but complex dimension $\Delta = 2 + i\nu$ with $\nu \in \mathbb{R}$. This allows us to transform the sum over the dimensions of the exchanged operators into an integral

$$\mathcal{G}_k(z, \bar{z}) = \delta_{\Delta_1, \Delta_2} \delta_{\Delta_3, \Delta_4} + \sum_{\ell=0}^{\infty} \int_{2-i\infty}^{2+i\infty} \frac{d\Delta}{2\pi i} c_k(\Delta, \ell) F_{\Delta, \ell}(z, \bar{z}) \,, \tag{3.4}$$

where the first term represents the identity contribution, if present, and we assume to work in dimension $d = 4$. The partial wave function can be written as a combination of conformal blocks

$$F_{\Delta,\ell} = \frac{g_{\Delta,\ell}^{(\Delta_{12},\Delta_{34})}}{2} + \frac{\Gamma(4-\Delta-1)\Gamma(\Delta-2)}{2\Gamma(\Delta-1)\Gamma(2-\Delta)} \frac{\tilde{\kappa}_{4-\Delta+\ell}^{(\Delta_{12},\Delta_{34})}}{\tilde{\kappa}_{\Delta+\ell}^{(\Delta_{12},\Delta_{34})}} g_{4-\Delta,\ell}^{(\Delta_{12},\Delta_{34})}, \tag{3.5a}$$

$$\tilde{\kappa}_h^{(\Delta_{12},\Delta_{34})} = \frac{\Gamma\left(\frac{h+\Delta_{12}}{2}\right)\Gamma\left(\frac{h-\Delta_{12}}{2}\right)\Gamma\left(\frac{h+\Delta_{34}}{2}\right)\Gamma\left(\frac{h-\Delta_{34}}{2}\right)}{2\pi^2\Gamma(h)\Gamma(h-1)}, \tag{3.5b}$$

where $g_{\Delta,\ell}^{(\Delta_{12},\Delta_{34})}(z,\bar{z})$ represents the conformal block with generic external dimension. In four dimensions, it is given by

$$k_h^{(\Delta_{12},\Delta_{34})}(z) = z^h\, {}_2F_1\left(h - \tfrac{\Delta_{12}}{2}, h + \tfrac{\Delta_{34}}{2}; 2h; z\right), \tag{3.6a}$$

$$g_{\Delta,\ell}^{(\Delta_{12},\Delta_{34})}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(-\frac{1}{2}\right)^\ell\left(k_{\frac{\Delta+\ell}{2}}^{(\Delta_{12},\Delta_{34})}(z)\, k_{\frac{\Delta-\ell-2}{2}}^{(\Delta_{12},\Delta_{34})}(\bar{z}) - (z \leftrightarrow \bar{z})\right). \tag{3.6b}$$

The function $c_k(\Delta,\ell)$ in (3.4) can not be generic, since it should recover the usual OPE decomposition. This can be shown to be guaranteed if $c_k(\Delta,\ell)$ has the form

$$c_k(\Delta,\ell) \underset{\Delta\to\Delta_{\text{ex}}}{\sim} \frac{\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_{\text{ex}}}\lambda_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{\text{ex}}}}{\Delta_{\text{ex}} - \Delta}, \tag{3.7}$$

with poles at the location of physical exchanged operators $\mathcal{O}_{\text{ex}}$ and residues proportional to the OPE coefficient. Using the orthogonality of $F_{\Delta,\ell}(z,\bar{z})$, we can invert (3.4) to get

$$c_k(\Delta,\ell) = N(\Delta,\ell)\int d^2z \left|\frac{z-\bar{z}}{z\bar{z}}\right|^2 F_{\Delta,\ell}(z,\bar{z})\mathcal{G}_k(z,\bar{z}), \tag{3.8}$$

where notice that at this stage we are still Euclidean, the integration is over $\bar{z} = z^*$, and ℓ is an integer. This is sometimes called *Euclidean inversion formula*.

It is possible to show [31] that by a contour deformation, requiring analytic continuation to the Lorentzian region, similar to the Froissart-Gribov procedure in S-matrix, $c_k(\Delta,\ell)$ can be obtained "dispersively". Here, this means that $c_k(\Delta,\ell)$, corresponding to the s-channel OPE, can be obtained as the sum of some functions of the two other channels and these functions depend on a new dispersive object, the *double*

discontinuity:

$$c_k(\Delta, \ell) = c_k^t(\Delta, \ell) + (-1)^\ell c_k^u(\Delta, \ell)\,, \tag{3.9a}$$

$$c_k^t(\Delta, \ell) = \frac{\tilde{\kappa}_{\Delta+\ell}^{(\Delta_{12}, \Delta_{34})}(-2)^\ell}{2} \int_0^1 \frac{dz\, d\bar{z}}{z^2\, \bar{z}^2} \left[(1-z)(1-\bar{z})\right]^{\frac{\Delta_{34}-\Delta_{12}}{2}} \times \quad (3.9b)$$

$$\times\, k_{2-h}^{(\Delta_{12}, \Delta_{34})}(z)\, k_{h+\ell}^{(\Delta_{12}, \Delta_{34})}(\bar{z})\, \mathrm{dDisc}\left[\frac{\bar{z}-z}{z\bar{z}}\mathcal{G}_k(z, \bar{z})\right]\,,$$

with $h = \frac{\Delta-\ell}{2}$. The u-channel c_k^u can be obtained from c_k^t by replacing $\Delta_1 \leftrightarrow \Delta_2$ and $\mathcal{G}_k(z, \bar{z})$ with

$$\left((1-z)(1-\bar{z})\right)^{-\frac{\Delta_{34}}{2}} (\mathbb{M}_{1\leftrightarrow 2}^{\mathrm{T}})_{kk'} \left[\mathcal{G}_{k'}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right)\right]_{\Delta_1 \leftrightarrow \Delta_2}\,, \quad (3.10)$$

where $\mathbb{M}_{i\leftrightarrow j}$ is the change of basis matrix between the original tensor structures and the one with the indices i and j exchanged

$$\mathbb{T}_{k'}^{i\leftrightarrow j} = (\mathbb{M}_{i\leftrightarrow j})_{k'k}\, \mathbb{T}_k\,. \tag{3.11}$$

Moreover, differently from the Euclidean inversion formula, now both c_k^t and c_k^u are analytic in spin and the integration is over a Lorentzian domain, in which z and $\bar{z}$ are taken to be independent. This makes (3.9a) a *Lorentzian inversion formula*. Performing the contour deformation, one has to be careful in order to drop contributions at infinity. Regge boundness of the correlator allows us to do that and the specific scaling behaviour in the Regge limit is reflected in the fact that the inversion above is valid for $\ell > 1$.[1]

The main difference between the Euclidean and Lorentzian inversion formula is that the latter expresses OPE data in terms of the double discontinuity. This is defined as the difference between the Euclidean correlator and its two possible analytic continuations around $\bar{z} = 1$, namely

$$\mathrm{dDisc}\left[\mathcal{G}_k(z, \bar{z})\right] = \cos(\pi\alpha)\,\mathcal{G}_k - \frac{1}{2}e^{i\pi\alpha}\mathcal{G}_k^{\circlearrowleft} - \frac{1}{2}e^{-i\pi\alpha}\mathcal{G}_k^{\circlearrowright}\,,$$
$$\alpha = \frac{\Delta_{34} - \Delta_{12}}{2}\,. \tag{3.12}$$

One can show that this definition corresponds to the kinematic configuration in Fig. 3.1 and encoded in the double commutator [31, 61]

$$\mathrm{dDisc}[\mathcal{G}_k(z, \bar{z})] = -\frac{\langle\Omega|[\mathcal{O}_4(x_4), \mathcal{O}_1(x_1)]\,[\mathcal{O}_2(x_2), \mathcal{O}_3(x_3)]|\Omega\rangle}{2\,\mathcal{K}_{\Delta_i}(x_i)}\,. \quad (3.13)$$

The power of the Lorentzian inversion formula relies on the fact that the

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[1]Remark: higher orders in a large N expansion can suffer from worse ambiguities, thus reducing the regime of validity to a larger though finite number of low spins [34].
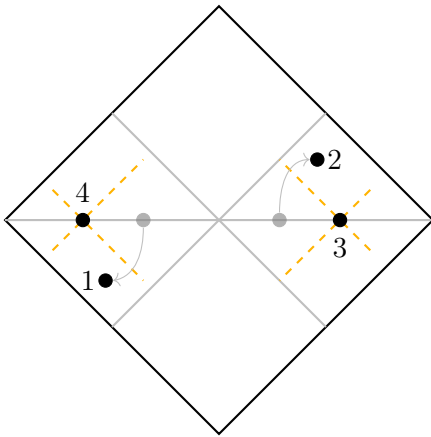
Figure 3.1. Lorentzian kinematic configuration reproducing the double commutator. We analytically continue from the Euclidean region (all x_{ij}^2 are spacelike) to the Lorentzian configuration where x_1 is in the past of x_4 and x_2 in the future of x_3. In this way, x_{14}^2 and x_{23}^2 become time-like, while all the other invariants stay space-like.

double discontinuity is an easier object to compute with the respect to the full correlator. This is because only certain contributions have non-vanishing dDisc and thus enter non-trivially in the inversion integrals. For instance, dDisc suppresses the contribution from exact double-trace operators (1.18). For theories at large N, this suppression remains true also at order $\frac{1}{N^2}$ and it implies that only information about single-trace operators is needed, at leading order, to define the four-point functions. The double commutator is simpler, yet it allows us to fully reconstruct the correlator, modulo low-spin ambiguities, through the position-space dispersion relation, worked out in [62]. There, the correlator is retrieved as an integral over double discontinuities over two different OPE channels.

As a final comment, let us mention that the Lorentzian inversion formula easily reproduces the large spin results we have seen in the introduction and, making the spin "analytic", it gives a concrete explanation of why operators lie on *Regge trajectories* — curves in the (Δ, ℓ)-plane.

3.1 Technical details

The efficient way to use the inversion formula, inspired by the original analytic bootstrap approaches, is to re-express the correlator $\mathcal{G}_k(z, \bar{z})$ (3.3)

in terms of its crossed-channel version [31, 34, 63]

$$\mathcal{G}_k(z,\bar{z}) = \frac{u^{\frac{\Delta_1+\Delta_2}{2}}}{v^{\frac{\Delta_2+\Delta_3}{2}}} (\mathbb{M}^T_{1\leftrightarrow 3})_{kk'} \left[\mathcal{G}_{k'}(1-z, 1-\bar{z}) \right]_{\Delta_1 \leftrightarrow \Delta_3}, \qquad (3.14)$$

with the same transformation performed in c_k^u.

In this setup, the terms with non-vanishing dDisc we will encounter are

$$\text{dDisc} \left[\left(\frac{1-\bar{z}}{\bar{z}} \right)^{\lambda} \right] = \left(\frac{1-\bar{z}}{\bar{z}} \right)^{\lambda} 2\sin(\pi\lambda)\sin(\pi(\lambda+\alpha)), \qquad (3.15a)$$

$$\text{dDisc} \left[\log^n (1-\bar{z}) \right] = 2\pi^2 n(n-1) \log^{n-2}(1-\bar{z})$$
$$+ \text{lower powers of } \log(1-\bar{z}), \qquad (3.15b)$$

where notice that $\log(1-\bar{z})$ has a vanishing dDisc. As a consequence, the relevant integrals (3.9b) are going to be of the form

$$\int_0^1 \frac{d\bar{z}}{\bar{z}^2} (1-\bar{z})^{\frac{\Delta_{34}-\Delta_{12}}{2}} k_{h+\ell}^{\Delta_{12},\Delta_{34}}(\bar{z}) \bar{z}^{-\frac{\Delta_{34}}{2}} \text{dDisc} \left[\tilde{f}(1-\bar{z}) \right] \times$$
$$\int_0^1 \frac{dz}{z^2} (1-z)^{\frac{\Delta_{34}-\Delta_{12}}{2}} k_{2-h}^{(\Delta_{12},\Delta_{34})}(z) f(\boldsymbol{x}(\tilde{\lambda}), \boldsymbol{z}(\tilde{\lambda}), \log z), \qquad (3.16)$$

with $\tilde{f}(1-\bar{z})$ as in (3.15), for some function f depending on the variables $\boldsymbol{x}(\tilde{\lambda}) = z^{-\frac{\Delta_{34}}{2}} \left(\frac{z}{1-z} \right)^{\tilde{\lambda}}$, $\boldsymbol{z}(\tilde{\lambda}) = z^{-\frac{\Delta_{34}}{2}} z^{\tilde{\lambda}}$ and possible $\log z$.

We will now report some results for the relevant integrals used in the following, more details can be found in Appendix D of Paper IV.

$$\overline{\mathcal{I}}^{\Delta_{12},\Delta_{34}}(\lambda) = \int_0^1 \frac{d\bar{z}}{\bar{z}^2} (1-\bar{z})^\alpha k_{h+\ell}^{(\Delta_{12},\Delta_{34})}(\bar{z}) \bar{z}^{-\frac{\Delta_{34}}{2}} \text{dDisc} \left[\left(\frac{1-\bar{z}}{\bar{z}} \right)^\lambda \right]$$

$$= \frac{2\Gamma(2(h+\ell))\Gamma\left(h+\ell-\frac{\Delta_{34}}{2}-\lambda-1\right)\sin(\pi\lambda)\Gamma(\lambda+1)}{\Gamma\left(h+\ell-\frac{\Delta_{12}}{2}\right)\Gamma\left(h+\ell-\frac{\Delta_{34}}{2}\right)\Gamma\left(h+\ell+\frac{\Delta_{34}}{2}+\lambda+1\right)}$$
$$\times \sin(\pi\alpha+\pi\lambda)\,\Gamma(\alpha+\lambda+1)$$

$$= \frac{2\pi^2\Gamma(2h+2\ell)\Gamma\left(h+\ell-\frac{\Delta_{34}}{2}-\lambda-1\right)}{\Gamma\left(h+\ell-\frac{\Delta_{12}}{2}\right)\Gamma\left(h+\ell-\frac{\Delta_{34}}{2}\right)\Gamma\left(h+\ell+\frac{\Delta_{34}}{2}+\lambda+1\right)}$$
$$\times \frac{1}{\Gamma(-\lambda)\Gamma(-\alpha-\lambda)}, \qquad (3.17)$$

where in the second line we have assumed $\lambda < 0$ and α defined in (3.12). The Γ functions are well defined for the considered values of λ and Δ_i. In $c_k(\Delta, \ell)$, this implies that the $\bar{z}$ integral just provides the spin dependence of the extracted OPE data and does not contain information about the twist of the possible exchange operators. This information is encoded

in the $z \to 0$ behaviour of the other integral. In the following, we will encounter three different types of z-integrals

$$\mathcal{I}_1^{\Delta_{12},\Delta_{34}}(\lambda) = \int_0^1 \frac{dz}{z^2}(1-z)^\alpha k_{2-h}^{(\Delta_{12},\Delta_{34})}(z)\left(\frac{z}{1-z}\right)^\lambda z^{-\frac{\Delta_{34}}{2}}, \quad (3.18a)$$

$$\mathcal{I}_2^{\Delta_{12},\Delta_{34}}(\lambda) = \int_0^1 \frac{dz}{z^2}(1-z)^\alpha k_{2-h}^{(\Delta_{12},\Delta_{34})}(z) z^\lambda z^{-\frac{\Delta_{34}}{2}}, \quad (3.18b)$$

$$\mathcal{I}_3^{\Delta_{12},\Delta_{34}}(\lambda) = \int_0^1 \frac{dz}{z^2}(1-z)^\alpha k_{2-h}^{(\Delta_{12},\Delta_{34})}(z) \frac{\log z}{z^{\frac{\Delta_{34}}{2}}}\left(\frac{z}{1-z}\right)^\lambda. \quad (3.18c)$$

In all these cases λ has to be considered positive and we are assuming $h \in \mathbb{N}$, $h \geq 1$.[2] Let us start from

$$\begin{aligned}
\mathcal{I}_1^{\Delta_{12},\Delta_{34}} = &-\frac{\pi r_{h,\lambda}^{\Delta_{12},\Delta_{34}} \sin\left(\pi\left(h + \lambda - \frac{\Delta_{34}}{2}\right)\right)}{\sin(2\pi h)\sin(\pi\lambda)\sin(\pi(\alpha+\lambda))} \\
&\times \frac{\sin\left(\pi\left(\frac{\Delta_{12}}{2}+h\right)\right)\sin\left(\pi\left(\frac{\Delta_{34}}{2}+h\right)\right)}{\sin\left(\pi\left(\frac{\Delta_{34}}{2}+h-\lambda\right)\right)},
\end{aligned} \quad (3.19)$$

where we have collected ratios of Γ's appearing throughout these computations in a single function

$$r_{h,\lambda}^{\Delta_{12},\Delta_{34}} = \frac{\Gamma\left(h + \frac{\Delta_{12}}{2} - 1\right)\Gamma\left(h + \frac{\Delta_{34}}{2} - 1\right)\Gamma\left(h - \frac{\Delta_{34}}{2} + \lambda - 2\right)}{\Gamma(2h-3)\Gamma(\lambda)\Gamma\left(\frac{\Delta_{12}}{2} - \frac{\Delta_{34}}{2} + \lambda\right)\Gamma\left(h + \frac{\Delta_{34}}{2} - \lambda\right)}. \quad (3.20)$$

This integral develops poles for certain values of h, depending on the external dimensions. For example for all equal operators, $\Delta_{12} = \Delta_{34} = 0$, we find simple poles for $h = \lambda + 1 + n$, $n \in \mathbb{N}$ with residues

$$\operatorname{Res}_{h=\lambda+1+n}\mathcal{I}_1^{0,0}(\lambda) = -r_{\lambda+n+1,\lambda}^{0,0}. \quad (3.21)$$

For the second kind of integral we get

$$\begin{aligned}
\mathcal{I}_2^{\Delta_{12},\Delta_{34}} = &\frac{\Gamma\left(\frac{1}{2}(-\Delta_{12}+\Delta_{34}+2)\right)\Gamma\left(-\frac{\Delta_{34}}{2}-h+\lambda+1\right)}{\Gamma\left(-\frac{\Delta_{12}}{2}-h+\lambda+2\right)} \\
&\times {}_3F_2\left(\begin{array}{c} -\frac{\Delta_{12}}{2}-h+2, \ \frac{\Delta_{34}}{2}-h+2, \ -\frac{\Delta_{34}}{2}-h+\lambda+1 \\ 4-2h, \ -\frac{\Delta_{12}}{2}-h+\lambda+2 \end{array}; 1\right).
\end{aligned} \quad (3.22)$$

By exploiting identities for the generalized hypergeometric function, we can locate the poles. Again for external equal dimensions, these are at $h = \lambda + n + 1$, $n \in \mathbb{N}$ with residues

$$\operatorname{Res}_{h=\lambda+n+1}\mathcal{I}_2^{0,0}(\lambda) = (-1)^{n+1}r_{\lambda+n+1,\lambda}^{0,0}. \quad (3.23)$$

---

[2]When solving these integrals spurious poles can appear at half-integer values of h. We will ignore them having in mind that they can be cancelled by adding a reflected block with $h \to 1 - h$ [31, 34].

The last integral is more challenging and we do not have a generic formula. Yet, for $\Delta_{12} = \Delta_{34} = 0$ with $\lambda > 0$ and $h \geq 1$

$$
\mathcal{I}_3^{0,0} = \frac{-\pi^2 r_{h,\lambda}^{0,0}}{\sin^2(\pi(\lambda - h))} + \frac{\pi \tan(\pi h) \sin(\pi(h + \lambda)))r_{h,\lambda}}{2\sin^2(\pi\lambda)\sin(\pi(\lambda - h)} \left(H_{h-\lambda} + H_{\lambda+h} \right.
$$
$$
\left. - 2H_h - \frac{1}{\lambda + h - 2} - \frac{1}{\lambda + h - 1} - \frac{1}{\lambda + h} + \frac{1}{\lambda - h} + \frac{2}{h - 1} + \frac{2}{h} \right),
$$
$$
(3.24)
$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number. Notice the appearance of double poles: these are the signatures of anomalous dimensions as we will be clearer when we apply these formulas to theories at large N.

Part II:
$\mathcal{N} = 4$ Super Yang-Mills

$\mathcal{N} = 4$ Super Yang-Mills (SYM) represents, so far, the only known example of a maximally supersymmetric conformal field theory in four dimensions. In the last decades, it attracted lots of attention for its connection to gravity and string theory. It represents in fact the original example of the AdS/CFT duality, as formulated in [1], and it is still one of the best grounds where to test its validity. In this Part, we will first introduce the theory by its Lagrangian and field content. In Sec. 4.2, we will briefly review the connection between $\mathcal{N} = 4$ SYM and type IIB string theory and eventually we will focus on the low energy regime of the gravity theory, type IIB supergravity. Once specialized to this approximation, we will study in detail four-graviton amplitudes by looking at the dual $\mathcal{O}_2$'s correlator. In the concluding section, we will instead look at less supersymmetric four-point functions by allowing for quarter-BPS scalars as external operators.

4. Generalities

Let us start with a very general description of $\mathcal{N} = 4$ SYM from the perspective discussed in Sec. 2.1 and 2.2. $\mathcal{N} = 4$ SYM admits a unique fundamental massless multiplet, the gauge or vector multiplet, which is composed of

	Field	Range	SU(4)$_R$
Real scalars	$\varphi^M(x)$	$M = 1, \cdots 6$	[0,1,0]
Weyl fermions	$\lambda_\alpha^m(x)$	$m = 1, \cdots 4$	[1,0,0]
Gauge field	$A^\mu(x)$		[0,0,0]

Given the presence of A^μ in the multiplet, the other elementary fields have to transform in the adjoint representation of the gauge group, which in this thesis, we identify with SU(N) — other realizations appear in the literature [64–67]. The corresponding central charge is

$$c = \frac{N^2 - 1}{4} . \tag{4.1}$$

In terms of the elementary fields, the $\mathcal{N} = 4$ SYM Lagrangian looks like [68–70]

$$\mathcal{L}_{\text{SYM}} = \text{tr}\Bigg\{ -\frac{1}{2g_{\text{YM}}^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_{\text{YM}}}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i\bar{\lambda}^m \bar{\sigma} D_\mu \lambda_m$$
$$+ g_{\text{YM}} \bar{\Sigma}^{Mmn} \lambda_m \left[\varphi^M, \lambda_n\right] + g_{\text{YM}} \Sigma_{mn}^M \bar{\lambda}^m \left[\varphi^M, \bar{\lambda}^n\right] \tag{4.2}$$
$$- D_\mu \varphi^M D^\mu \varphi^M + \frac{g_{\text{YM}}^2}{2} \sum_{M,N} \left[\varphi^M, \varphi^N\right]^2 \Bigg\} ,$$

where sums over repeated indices are understood and we have introduced the gauge covariant derivative $D_\mu = \partial_\mu + iA_\mu$, and Σ, $\bar{\Sigma}$ are six-dimensional Clifford matrices implementing the isomorphism SO(6)$_R \simeq$ SU(4)$_R$ — see Appendix A of Paper IV for our conventions. From the form of the Lagrangian, we can immediately see that there are no mass terms and in four dimensions all the couplings are dimensionless. This fact makes $\mathcal{N} = 4$ SYM a scale-invariant theory at least at the classical level. Remarkably, this property holds also at the full quantum level, *i.e.* the β-function vanishes exactly so that the full superconformal group SU(2, 2|4) is an exact quantum symmetry of the theory.

The last interesting feature we would like to stress about $\mathcal{N}{=}4$ SYM is that it enjoys S-duality [71–73]. Under this duality the theory with complexified coupling

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \, , \qquad (4.3)$$

gets mapped to the theory with gauge group $\text{SU}(N)/\mathbb{Z}_N$ and gauge coupling $\tau = -\frac{1}{\tau}$. Non-local observables are sensitive to the difference in the gauge group, while many other observables, like correlation functions of half-BPS operators, are invariant under this transformation. If we combine the S-duality transformation with the symmetry for $\theta_{\text{YM}} \to \theta_{\text{YM}}{+}2\pi$ (T-duality: $\tau \to \tau{+}1$) we get $\text{SL}(2,\mathbb{Z})$ transformations, defined as

$$\tau \to \frac{a\tau + b}{c\tau + d} \, , \qquad ad - bc = 1 \, , \quad a, b, c, d \in \mathbb{Z} \, . \qquad (4.4)$$

Recently, it has been possible to discover new properties and make interesting developments concerning a special class of *integrated correlators* of protected operators, exploiting their manifest invariance under these $\text{SL}(2,\mathbb{Z})$ transformations [74–79].

4.1 Spectrum of operators and correlation functions

As we have seen in Sec. 2.1, multiplets correspond to unitary irreducible representations of $\text{SU}(2,2|4)$. We can distinguish between short and non-protected (or long) multiplets. We will denote the latter by $\mathcal{A}^{\Delta}_{[q,p,\bar{q}],(j,\bar{j})}$ and their dimensions should satisfy the unitarity bound

$$\Delta \geq \max\left(2 + 2j + \frac{3}{2}q + p + \frac{\bar{q}}{2}, 2 + 2\bar{j} + \frac{3}{2}\bar{q} + p + \frac{q}{2}\right) \, . \qquad (4.5)$$

Among the protected multiplets we would like to spend a few words on half- and quarter-BPS operators, whose correlators we will study in the following. We will denote their supermultiplet as $\mathcal{B}_{[q,p,q]}$ [57] and the corresponding superconformal primaries $\mathcal{O}_{pq}$. These are scalars with protected dimension $\Delta = 2q + p$, transforming in the $[q,p,q]$ of $\text{SU}(4)_R$. When $q > 0$, the superprimary is annihilated by four supercharges and it corresponds to a quarter-BPS operator. When $q = 0$, four additional supercharges have vanishing action, thus giving a half-BPS shortening condition. For ease of notation, we will re-define these operators as $\mathcal{O}_{p0} \equiv \mathcal{O}_p$. In both cases, the superprimaries can be constructed out of the fundamental scalar fields φ^M. To do that it is convenient to use an index-free notation in which all R-symmetry indices are contracted with additional auxiliary vectors. Explicitly, all $\text{SO}(6)_R$ fundamental

indices are contracted with a six-dimensional complex vector y^M and all $\mathrm{SU}(4)_R$ (anti)fundamental indices with a four-vector S^m $(\bar{S}_m)$, where one can pass from one to the other by using the Dirac matrices appearing in (4.2)

$$\mathrm{y}_{mn} \equiv y_M \Sigma^M_{mn} \,, \qquad \bar{\mathrm{y}}^{mn} \equiv y_M \overline{\Sigma}^{M\,mn} \,.$$

To ensure the right transformation properties, these polarizations are not free, rather they have to satisfy

$$y \cdot y = 0 \,, \qquad S \cdot \bar{S} = 0 \,, \qquad \mathrm{y}\,S = 0 \,, \qquad \bar{\mathrm{y}}\,\bar{S} = 0 \,. \qquad (4.6)$$

Finally, if we denote with the collective variable $\mathbf{S} = (S, \bar{S}, y)$, a generic half- or quarter-BPS operator has the form

$$\mathcal{O}_{pq}(\mathbf{S}) \equiv (\mathcal{O}_{pq})^{m_1 \cdots m_q}_{n_1 \cdots n_q\,,\, M_1 \cdots M_p}\, \bar{S}_{m_1} \cdots \bar{S}_{m_q} S^{n_1} \cdots S^{n_q} y^{M_1} \cdots y^{M_p} \,. \quad (4.7)$$

It is possible to re-extract back the tensor form, by applying some differential operators, interior to the constraints. Examples can be found in [80–83]. To illustrate this formalism and the content in terms of φ^M, let us give the explicit form for the two operators we will consider in the following. The half-BPS operator with minimal dimension $\Delta = 2$

$$\mathcal{O}_2(x,y) = \sqrt{\frac{2}{N^2 - 1}}\,\mathrm{tr}\left(T^{I_1} T^{I_2}\right) \varphi^{M_1}_{I_1} \varphi^{M_2}_{I_2}\, y_{M_1} y_{M_2}\,, \qquad (4.8)$$

where we always fix the normalization in such a way that the two-point function is unit normalized and we denote with T^I the $\mathrm{SU}(N)$ generators. The quarter-BPS operator we will look at, which is also the easiest non-vanishing one, has a richer structure [84, 85]

$$\mathcal{O}_{02}(x,\mathbf{S}) = \sqrt{\frac{3}{3(N^2-4)(N^2-1)}}\left(\mathcal{O}^{\mathrm{dt}}_{02}(x,\mathbf{S}) + \frac{2}{N}\mathcal{O}^{\mathrm{st}}_{02}(x,\mathbf{S})\right)\,.$$
$$(4.9)$$

It is composed of two pieces, the first one, which is dominant at large N, is a double-trace operator — it can be written as the product of two $\mathrm{SU}(N)$ traces — the second term, instead, depends only on one single trace over the gauge group. Their exact form is given by

$$\mathcal{O}^{\mathrm{dt}}_{02} = \mathrm{tr}\!\left(\varphi^{M_1}\varphi^{M_2}\right)\mathrm{tr}\!\left(\varphi^{M_3}\varphi^{M_4}\right) S \cdot \Sigma_{M_1 M_3} \cdot \bar{S}\, S \cdot \Sigma_{M_2 M_4} \cdot \bar{S}$$
$$\mathcal{O}^{\mathrm{st}}_{02} = \mathrm{tr}\!\left(\varphi^{M_1}\varphi^{M_2}\varphi^{M_3}\varphi^{M_4}\right) S \cdot \Sigma_{M_1 M_2} \cdot \bar{S}\, S \cdot \Sigma_{M_3 M_4} \cdot \bar{S}\,. \qquad (4.10)$$

The advantage of using an index-free notation is that correlation functions can then be written in terms of a complete basis of tensor structures

made of elementary building blocks[1]

$$
\begin{aligned}
\mathsf{y}_{ij} &= y_i \cdot y_j \,, & \mathsf{S}_{ij} &= S_i \cdot \bar{S}_j \,, \\
\mathsf{Y}_{i_1 i_2 \cdots} &= \mathrm{tr}\,(y_{i_1} \bar{y}_{i_2} \cdots) \,, & \mathsf{J}^{ik}_{j_1 \cdots j_{2p}} &= S_i y_{j_1} \cdots \bar{y}_{j_{2p}} \bar{S}_k \,, \\
\mathcal{E}_{ijkl} &= \epsilon_{mnpq} S^m_i S^n_j S^p_k S^q_l \,, & \mathsf{K}^{ik}_{j_1 \cdots j_{2p+1}} &= S_i y_{j_1} \cdots y_{j_{2p+1}} S_k \,, \\
\bar{\mathcal{E}}_{ijkl} &= \mathcal{E}_{ijkl}|_{S \to \bar{S}} \,, & \bar{\mathsf{K}}^{ik}_{j_1 \cdots j_{2p+1}} &= \mathsf{K}^{ik}_{j_1 \cdots j_{2p+1}}\big|_{\substack{S \to \bar{S} \\ \mathsf{y} \to \bar{\mathsf{y}}}} \,.
\end{aligned}
\tag{4.11}
$$

Using this basis makes handling $SU(4)_R$ indices and big representations much easier and expressions more compact. For instance, the two-point function of generic scalars can be elegantly written as

$$
\langle \mathcal{O}^{[q,p,\bar{q}]}_{\Delta}(x_1, \mathbf{S}_1) \mathcal{O}^{[\bar{q},p,q]}_{\Delta}(x_2, \mathbf{S}_1) \rangle = \frac{(\mathsf{y}_{12})^p (\mathsf{S}_{12})^q (\mathsf{S}_{21})^{\bar{q}}}{(x_{12}^2)^{\Delta}} \,.
\tag{4.12}
$$

This formalism turns out to be very convenient also for higher-point functions. As in usual CFT, the form of three-point functions is completely determined, apart from a number, by conformal symmetry and, in this case, R-symmetry as well, which fixes the possible combinations of (4.11) appearing. The first correlator containing dynamical information is the four-point function.

Consider the correlator of four generic half- or quarter-BPS operators $\mathcal{O}_{p_i q_i}(x_i, \mathbf{S}_i)$. Since the external operators transform non trivially under R-symmetry, N_{str} different $SU(4)_R$ representations are going to be exchanged, *i.e.* the ones sitting in the intersection

$$
([q_1, p_1, q_1] \otimes [q_2, p_2, q_2]) \cap ([q_3, p_3, q_3] \otimes [q_4, p_4, q_4]) \,.
\tag{4.13}
$$

In our index-free formalism, this implies that the correlation function can be expanded in N_{str} independent tensor structures $\mathbb{T}_k$, built out of the building blocks in (4.11). In formulas, this reads

$$
\langle \mathcal{O}_{p_1 q_1} \mathcal{O}_{p_2 q_2} \mathcal{O}_{p_3 q_3} \mathcal{O}_{p_4 q_4} \rangle = \mathcal{K}_{2q_1 + p_1, \cdots, 2q_4 + p_4} \sum_{k}^{N_{\mathrm{str}}} \mathbb{T}_k \, \mathcal{G}_k(z, \bar{z}) \,,
\tag{4.14}
$$

where the $z, \bar{z}$ cross-ratios and the kinematic prefactor are defined in (3.2). Concretely, the explicit form of the tensor structures can be obtained by starting from an ansatz, built upon appropriate products of the monomials in (4.11), and then by rotating to a basis of eigenvectors of the $SU(4)_R$ Casimir operators with corresponding eigenvalues defining the representation they belong to. In our cases of interest, we considered the

---

[1]For six or more points, one also has to consider the Levi-Civita tensor $\epsilon_{MNPQRS} y^M_i y^N_j y^P_k y^Q_l y^R_m y^S_n$.

quadratic and the quartic Casimir, represented in terms of derivatives of **S** as [86]

$$\mathcal{C}_2(\partial_{\mathbf{S}}) = \frac{1}{2} L_{MN} L^{NM},$$

$$\mathcal{C}_4(\partial_{\mathbf{S}}) = \frac{1}{2} L_{MN} L^{NP} L_{PQ} L^{QM}, \tag{4.15}$$

with L_{MN}, generators of $\mathrm{SU}(4)_R$,

$$L_{MN} = L_{1,\,MN} + L_{2,\,MN},$$

$$L_{i,\,MN} = -\left(y_{iM} \frac{\partial}{\partial y_i^N} - y_{iN} \frac{\partial}{\partial y_i^M} \right) - S_i^m \Sigma_{MN\,m}{}^n \frac{\partial}{\partial S_i^n} \tag{4.16}$$

$$- \bar{S}_{im} \bar{\Sigma}_{MN}{}^m{}_n \frac{\partial}{\partial \bar{S}_{in}}.$$

Then we can define the $\mathbb{T}_k$ associated with the $[q, p, \bar{q}]$ representation exchanged in the (12)-OPE as the structure satisfying

$$\mathcal{C}_r(\partial_{\mathbf{S}_1}, \partial_{\mathbf{S}_2})\, \mathbb{T}_k(\mathbf{S}_1, \ldots, \mathbf{S}_4) = C_r\, \mathbb{T}_k(\mathbf{S}_1, \ldots, \mathbf{S}_4), \qquad r = 2, 4, \tag{4.17}$$

with eigenvalues [87]

$$C_2 = p\,(\bar{q} + q + 4) + \frac{1}{4}\left(3\bar{q}^2 + 2(q+6)\bar{q} + 3q(q+4)\right) + p^2,$$

$$C_4 = \frac{(\bar{q} + 2p + q)^2\,(\bar{q} + 2p + q + 8)^2}{16} - \frac{(q - \bar{q})^2\,(\bar{q} + q + 2)^2}{8} \tag{4.18}$$

$$+ \frac{3\,(\bar{q} + 2p + q)\,(\bar{q} + 2p + q + 8)}{2} + \frac{(\bar{q}\,(\bar{q}+2) + q(q+2))^2}{4}.$$

This concludes the general description of operators and correlation functions just based on superconformal symmetry. In the next section, we will see what else we can say when we resort to the AdS/CFT correspondence and we relate operators and correlators to the dual states and amplitudes in AdS.

4.2 The AdS/CFT correspondence

The duality between $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group and type IIB string theory on $AdS_5 \times S^5$ represents the original formulation of the correspondence [1]. This can be schematically summarized as

$4d$ $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group and g_{YM} gauge coupling	$\longleftrightarrow$	$10d$ type IIB string theory on $AdS_5 \times S^5$ with string coupling g_s and length $\sqrt{\alpha'}$

$$g_{\text{YM}}^2 \quad = \quad g_s \,,$$

$$\lambda = g_{\text{YM}}^2 N \quad = \quad \frac{R^4}{(\alpha')^2} \,.$$

We have denoted with $R = R_{AdS_5} = R_{S^5}$ and λ is called the 't Hooft coupling. As we have stressed in the introduction, the first thing to check for establishing a duality between two theories, is to look whether the symmetries on both sides match or not. We know from Sec. 2.1, that the bosonic subgroup of $SU(2,2|4)$ is made of the conformal group and the R-symmetry $SU(4)_R \simeq SO(6)_R$. On the gravity side, the first group can be identified with the isometries of AdS_5, while the second one represents exactly the symmetries of the 5-sphere. So the symmetries add up and, as we will soon see, also the spectrum of the two theories coincides.

In the scheme above, we have listed the various CFT parameters and how they are related to the string theory couplings. No matter what are their values, the correspondence is conjectured to be valid. Yet, most of the time, it is convenient to focus on specific limits. One can, for instance, take N large, but keep λ fixed. This is called the 't Hooft limit and in this regime the gravity theory becomes weakly coupled and admits a perturbative expansion in g_s, such that corrections in inverse even powers of N correspond to different genera on the string side. Starting from this limit, one can further take $\lambda \to \infty$, but N still the largest scale. This is often called the *supergravity approximation*, since it is equivalent to having a small string length, $\ell_s \sim \sqrt{\alpha'}$, compared to the AdS radius, such that strings become effectively point-like particles. In this limit, which is the one we will focus on, $\frac{1}{N^{2\kappa}}$ corrections can be interpreted as $(\kappa - 1)$-loops in the supergravity theory.

Now that we have introduced these limits, we can go back to discuss the spectrum and see which operators are relevant in the various regimes and how CFT operators are related to gravity fields in AdS. Let us start with protected single-trace operators

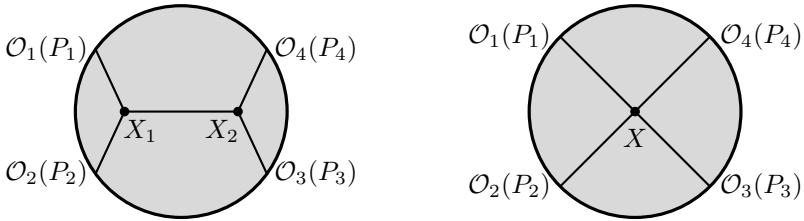$$\mathcal{O}_p(x, y) = \text{tr}(y \cdot \phi)^p \,, \tag{4.19}$$

Figure 4.1. Two examples of AdS tree-level Witten diagrams: an exchange (left) and a contact (right) diagram. The shaded gray area corresponds to the bulk of AdS, while the circle represents its boundary. A propagator connecting a point $P_i \in \partial$AdS and a point X_j in the bulk is called a bulk-to-boundary propagator. Lines between two bulk points define bulk-to-bulk propagators.

which are dual to supergravity fields [1, 88]. For $p = 2$, this operator is exactly the superprimary of the stress-tensor multiplet in Fig. 2.2, therefore is interpreted in AdS as a supergraviton. In the same multiplet, we can further identify the stress-tensor itself with the AdS metric fluctuations, the R-symmetry current with the gauge field, the combination $\mathcal{L} + \bar{\mathcal{L}}$ with the dilaton $\mathcal{O}_\phi$ and $\mathcal{L} - \bar{\mathcal{L}}$ with the axion $\mathcal{O}_C$. The operators with $p > 2$ can be identified as coming from the Kaluza Klein reduction on the S^5 of the $10d$ supergravity fields. Actually, in order to be truly dual to *single*-particle states in AdS, the single-trace operators (4.19) have to be "corrected" by a combination of multi-trace operators, suppressed at large N — see [89, 90] for details. BPS operators, which are instead genuinely multi-trace at large N, are composite states in AdS. They appear at the edge of the continuum spectrum and therefore they are sometimes called threshold-bound states. We will see an example in Sec. 6. Finally, long single-trace operators with $\Delta \sim \lambda^{\frac{1}{4}}$, like the Konishi $\sim \text{tr}\left(\varphi^M \varphi_M\right)$, correspond to type IIB massive string states. These operators acquire a parametric large dimension in the supergravity limit, so they decouple from the spectrum of the theory, leaving only a finite number of protected single-trace operators. This fact is going to be essential to bootstrap the correlation function of four $\mathcal{O}_2$'s in the next section.

Finally, let us comment on the relation between correlators and AdS "amplitudes". In the introduction, we have seen the defining relation of the AdS/CFT correspondence (1.2), *i.e.* the equality between the generating functional of CFT correlators and the AdS partition function, which, in the regime we are interested in, coincides with the action of type IIB supergravity reduced on the S^5. This identification gives us a recipe for how CFT correlation functions are related and can be computed starting from the various vertices appearing in the AdS action. This results in a set of rules, similar to Feynman ones, which can be summarized in *Witten diagrams* [3]. Two examples are shown in Fig. 4.1.

5. $\mathcal{O}_2$'s four-point function

In this chapter, we will retrace the main stages in the study of the correlators involving the stress-tensor superprimary. The first part of the discussion, concerning the protected contribution to the correlator, the solution of the Ward identities and the superspace analysis, does not rely on any dynamical information and it is valid for any regime of $\mathcal{N} = 4$ SYM, either at weak or strong coupling. Yet, we will always have in mind the connection to supergravity and eventually we will specialize in the large N, $\lambda \to \infty$ limit.

5.1 Superconformal Ward identities

The superprimary of the stress-tensor multiplet defined in (4.8) is a scalar in the $[0, 2, 0]$ of $SU(4)_R$. Satisfying half-BPS shortening conditions, it has protected dimension $\Delta = 2$ and consequently protected two-point function. These non-renormalizability properties extend to higher-point functions [58, 91–95]. In particular three-point functions of two half-BPS operators with any another BPS operators are fixed by supersymmetry and are exact and independent of the coupling. Three-point functions of two half-BPS with a generic long and four-point function of four half-BPS operators are just partially non-renormalizable and they will depend only on one non-protected function. For the correlator of four $\mathcal{O}_2$'s this means

$$\langle \mathcal{O}_2(x_1, y_1)\mathcal{O}_2(x_2, y_2)\mathcal{O}_2(x_3, y_3)\mathcal{O}_2(x_4, y_4) \rangle = \frac{\mathsf{y}_{12}^2\mathsf{y}_{34}^2}{(x_{12}^2 x_{34}^2)^2}\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha}) \,,$$
(5.1)

where y_{ij} is defined in (4.11) and we have introduced the $SU(4)_R$ symmetry cross-ratios

$$\sigma = \alpha\bar{\alpha} = \frac{\mathsf{y}_{13}\mathsf{y}_{24}}{\mathsf{y}_{12}\mathsf{y}_{34}} \,, \qquad \tau = (1 - \alpha)(1 - \bar{\alpha}) = \frac{\mathsf{y}_{14}\mathsf{y}_{23}}{\mathsf{y}_{12}\mathsf{y}_{34}} \,.$$
(5.2)

The undetermined function $\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha})$ can be expanded in conformal blocks (3.6) and R-symmetry blocks as

$$\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha}) = \sum_{n,m} \sum_{\Delta,\ell} C(\Delta, \ell; m, n) \, Y_{n,m}(\alpha, \bar{\alpha}) \, g_{\Delta,\ell}^{(0,0)}(z, \bar{z}) \,.$$
(5.3)

Each tensor $Y_{n,m}(\alpha, \bar{\alpha})$ accounts for one of the six representations exchanged in $[0, 2, 0] \otimes [0, 2, 0]$

$$[n - m, 2m, n - m], \qquad n = 0, 1, 2, \qquad m = 0, 1, \ldots, n, \qquad (5.4)$$

and it is defined as

$$Y_{n,m}(\alpha, \bar{\alpha}) = (-1)^{n-m} g^{(0,0)}_{-n-m, n-m} \left(\frac{1}{\alpha}, \frac{1}{\bar{\alpha}} \right). \qquad (5.5)$$

The decomposition in (5.3) is not completely satisfying since it is not fully taking into account the underlying superconformal symmetry and the half-BPS nature of the operators. What we would prefer is a decomposition able to reflect these symmetries and in terms of a possible supersymmetric version of the blocks, some *superblocks* packing together the contributions of each superconformal multiplet exchanged in the OPE of $\mathcal{O}_2 \times \mathcal{O}_2$. Remarkably for $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$, and all half-BPS operators, such a decomposition exists and superblocks are known. Let us see how it works in detail. First of all, we know which multiplets are present in the OPE of two half-BPS stress-tensor multiplets [81, 95, 96]

$$\mathcal{B}_{[0,2,0]} \times \mathcal{B}_{[0,2,0]} \sim \mathbf{1} + \mathcal{B}_{[0,2,0]} + \mathcal{B}_{[0,4,0]} + \mathcal{B}_{[2,0,2]} + \sum_{\ell=0}^{\infty} \mathcal{C}_{[0,0,0],\ell}$$
$$+ \sum_{\ell=0}^{\infty} \mathcal{C}_{[0,2,0],\ell} + \sum_{\ell=0}^{\infty} \mathcal{C}_{[1,0,1],\ell} + \sum_{\Delta,\ell} \mathcal{A}^{\Delta}_{[0,0,0],\ell}, \qquad (5.6)$$

where apart from the identity, we have the half-BPS multiplet $\mathcal{B}_{[0,4,0]}$, with a double-trace superprimary $\mathcal{O}_4^{\mathrm{dt}} = \sqrt{\frac{2}{N^4-1}} \left[\mathrm{tr}(y \cdot \phi)^2 \right]^2$ and the quarter-BPS one in the $[2, 0, 2]$, we have introduced in (4.9). Then we find the semi-short multiplets $\mathcal{C}$ with different shortening conditions: $\mathcal{C}_{[0,0,0],\ell}$ has twist two and contains higher-spin currents while $\mathcal{C}_{[0,2,0],\ell}$ and $\mathcal{C}_{[1,0,1],\ell}$ contribute at twist four. Finally $\mathcal{A}$ are long, non-protected operators in the singlet of $\mathrm{SU}(4)_{\mathrm{R}}$.

In [81, 97], it has been found a solution of superconformal Ward Identities — the constraints imposed by superconformal invariance —

$$\mathcal{G}\left(\frac{1}{\alpha}, \frac{1}{\bar{\alpha}}; \alpha, \bar{\alpha} \right) = k, \qquad (5.7a)$$

$$\mathcal{G}\left(z, \frac{1}{\bar{\alpha}}; \alpha, \bar{\alpha} \right) = k + \left(\alpha - \frac{1}{z} \right) \hat{f}(z, \alpha). \qquad (5.7b)$$

These requirements can be trivially satisfied by parametrizing the four-point function as

$$\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha}) = k + \mathcal{G}^{\hat{f}}(z, \bar{z}; \alpha, \bar{\alpha}) + R(z, \bar{z}; \alpha, \bar{\alpha}) \mathcal{H}(z, \bar{z}), \qquad (5.8)$$

with

$$\mathcal{G}^{\hat{f}} = \frac{(\bar{\alpha}z-1)(\alpha\bar{z}-1)\left(\left(\alpha-\frac{1}{z}\right)\hat{f}(z,\alpha)+\left(\bar{\alpha}-\frac{1}{\bar{z}}\right)\hat{f}(\bar{z},\bar{\alpha})\right)}{(z-\bar{z})(\alpha-\bar{\alpha})} - \tag{5.9a}$$

$$\frac{(\alpha z-1)(\bar{\alpha}\bar{z}-1)\left(\left(\bar{\alpha}-\frac{1}{z}\right)\hat{f}(z,\bar{\alpha})+\left(\alpha-\frac{1}{\bar{z}}\right)\hat{f}(\bar{z},\alpha)\right)}{(z-\bar{z})(\alpha-\bar{\alpha})},$$

$$R = (z\alpha-1)(\bar{z}\alpha-1)(z\bar{\alpha}-1)(\bar{z}\bar{\alpha}-1). \tag{5.9b}$$

The constant k and the one-variable function $\hat{f}$ are protected and completely determined by their value at zero coupling. In particular, $\hat{f}(z)$ can be interpreted as the four-point function of a protected subsector of the theory described by a $2d$ chiral algebra [48] — see also Sec. 6.1.

Thus the interacting information is enclosed in the *reduced correlator* $\mathcal{H}(z,\bar{z})$. Notice that it does not depend on the R-symmetry cross-ratios, this is because the R prefactor effectively makes $\mathcal{H}$ a four-point function of operators of dimension $(2+2)$ and R-charge $(2-2)$, thus a singlet. Quite specially, in terms of the original decomposition (5.3) in R-symmetry representations, $\mathcal{H}$ is exactly proportional to the $[0,4,0]$ contribution

$$\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})|_{Y_{2,2}} = u^2\mathcal{H}(z,\bar{z}). \tag{5.10}$$

A price we pay to have the nice decomposition (5.8) is that $\mathcal{H}$ is no longer crossing symmetric on its own and it transforms as

$$\mathcal{H}(u,v) = \frac{u^2}{v^2}\mathcal{H}(v,u) + 1 - \frac{u^2}{v^2} + \frac{1}{c}\frac{v-u}{v^2}, \tag{5.11}$$

where the central charge c has been defined in (4.1). The additional pieces depend on the form of k and $\hat{f}$, which we recall is fixed by tree-level computations. By means of the Wick contractions

$$y_1 \cdot \overline{\phi^I(x_1)\, y_2 \cdot \phi^J}(x_2) = \frac{y_{12}}{x_{12}^2}\delta^{IJ}, \qquad I,J = 1,\dots,N^2-1, \tag{5.12}$$

we can indeed compute the free theory result

$$\mathcal{G}^{\text{free}}(u,v;\sigma,\tau) = 1 + u^2\left(\sigma^2 + \frac{\tau^2}{v^2}\right) + \frac{u}{c}\left(\sigma + \frac{\tau}{v} + \sigma\tau\frac{u}{v}\right), \tag{5.13}$$

and using (5.7), it is straightforward to get

$$k = 3\left(1 + \frac{1}{c}\right), \tag{5.14a}$$

$$\hat{f}(z, \alpha) = \frac{z\left(z^2 - 4z + 2\right)}{(z-1)^2} + \frac{z(2z - 3)}{c(z-1)} \tag{5.14b}$$

$$+ \alpha\left(\frac{z^2\left(z^2 - 2z + 2\right)}{(z-1)^2} - \frac{z^2}{c(z-1)}\right),$$

$$\mathcal{H}^{\text{free}}(z, \bar{z}) = 1 + \frac{1}{v^2} + \frac{1}{c}\frac{1}{v}. \tag{5.14c}$$

The functions appearing in (5.8) admit an expansion in superblocks [81]. For $\hat{f}$, in terms of one-variable blocks

$$\hat{f}(z, \alpha) = \sum_{n=0,1}\sum_{\ell=0}^{\infty} b_{n,\ell}\, y_n(\alpha)\, \mathfrak{g}_\ell^{(0,0)}(z), \tag{5.15}$$

with

$$y_n(\alpha) = k_{-n}^{(0,0)}\left(\frac{1}{\alpha}\right), \tag{5.16}$$

$$\mathfrak{g}_\ell^{(\Delta_{12},\Delta_{34})}(z) = z^{\frac{1}{2}\Delta_{34}} k_{\ell - \frac{1}{2}\Delta_{34}+1}^{(\Delta_{12},\Delta_{34})}(z).$$

While the reduced correlator can be expanded as

$$\mathcal{H}(z, \bar{z}) = \sum_{\Delta,\ell} a(\Delta, \ell) G_{\Delta,\ell}(z, \bar{z}), \tag{5.17}$$

where the superconformal block is defined as

$$G_{\Delta,\ell} = (z\bar{z})^{-2} g_{\Delta+4,\ell}^{(0,0)}(z, \bar{z}). \tag{5.18}$$

Before moving to the analysis of the coupling dependant and dynamical $\mathcal{H}$, we would like to pause for a second to comment on the superspace description of $\mathcal{N} = 4$ and in particular of the stress-tensor multiplet.

5.2 Interlude: $\mathcal{N} = 4$ Superspace

In Sec. 2.3, we have introduced superspaces with the idea that it is very powerful and useful to have a language which makes manifest the symmetries of a theory and the constraints on operators. In the case of $\mathcal{N} = 4$ SCFTs and for half-BPS operators, the most convenient description are *harmonic* and *analytic superspaces* [60,98–100]. Very similar to what we have seen happening with chiral fields and the chiral superspace (2.23)

in $\mathcal{N}=1$, a description of protected supermultiplets in terms of $\mathcal{N}=4$ superspace allows us to express them as some unconstrained fields depending on a reduced set of coordinates.

To apply this technology to the stress-tensor multiplet in Fig. 2.2, first of all we need to introduce harmonic coordinates

$$u_n^m = \left(\begin{array}{c|c} u_n^{+\mathsf{m}} & u_n^{-\mathsf{m}'} \end{array} \right) \tag{5.19}$$

and its barred counterpart, parametrizing the coset space

$$\frac{\mathrm{SU}(4)_R}{\mathrm{SU}(2) \times \mathrm{SU}(2)' \times \mathrm{U}(1)}\,, \tag{5.20}$$

where remember m and n are $\mathrm{SU}(4)_R$ indices and we use m for the first $\mathrm{SU}(2)$ and m' for the second one in the coset denominator. The θ and $\bar\theta$ split accordingly

$$\theta_\alpha^m = u_\mathsf{m}^{+\mathsf{m}}\theta_\alpha^m\,, \qquad \theta_\alpha^m = u_\mathsf{m}^{+\mathsf{m}}\theta_\alpha^m \tag{5.21}$$

and similarly

$$Q_\mathsf{m}^\alpha = \bar{u}_{+\mathsf{m}}^m Q_m^\alpha\,, \qquad Q_{\mathsf{m}'}^\alpha = \bar{u}_{-\mathsf{m}'}^m Q_m^\alpha\,. \tag{5.22}$$

For its BPS nature, the superprimary is annihilated by half of the supercharges. Under this constraint, we can further reduce the number of coordinates our superfield depends on by introducing analytic superspace. The stress-tensor superfield will then depend on a subset of the original superspace coordinates: the analytic ones

$$\mathbb{O}_2(x, \theta, \bar\theta) \equiv \mathbb{O}_2(\mathrm{x}_{\alpha\dot\alpha}, \theta_\alpha^\mathsf{m}, \bar\theta_{\mathsf{m}'}^{\dot\alpha}, y_{\mathsf{mm}'})\,, \tag{5.23}$$

where we have $\mathrm{x}_{\alpha\dot\alpha}$, spacetime coordinate, 4 chiral and 4 anti-chiral Grassmann variables, θ_α^m and $\bar\theta_{\mathsf{m}'}^{\dot\alpha}$, and finally $y_{\mathsf{mm}'}$. This last tensor is related to the original $\mathrm{SU}(4)_R$ polarization as

$$y_{mn} = \left(\begin{array}{c|c} \epsilon_{\mathsf{mn}} & -y_{\mathsf{mn}'} \\ \hline y_{\mathsf{nm}'} & \epsilon_{\mathsf{m}'\mathsf{n}'} \det ||y_{\mathsf{mm}'}|| \end{array} \right)\,. \tag{5.24}$$

The connection between harmonic and analytic coordinates is made explicit by the identification

$$\left(\begin{array}{c|c} u_m^{+\mathsf{m}} & u_m^{-\mathsf{m}'} \end{array} \right) = \left(\begin{array}{c|c} \delta_\mathsf{n}^\mathsf{m} & 0 \\ \hline y_\mathsf{n}^\mathsf{m} & \delta_{\mathsf{n}'}^{\mathsf{m}'} \end{array} \right)\,. \tag{5.25}$$

Given the superfield (5.23), we can recover the various component fields. Starting from the superprimary, this is obtained by setting all the thetas to zero

$$\mathbb{O}_2(\mathrm{x}_{\alpha\dot\alpha}, \theta_\alpha^\mathsf{m}, \bar\theta_{\mathsf{m}'}^{\dot\alpha}, y_{\mathsf{mm}'})|_{\theta=\bar\theta=0} = \mathcal{O}_2(x, y)\,. \tag{5.26}$$

Extracting the other components requires more work and one needs to apply appropriate differential operators, whose form can be found in [101, 102].

This formalism and this construction might seem a bit cumbersome at first. Yet, as we will soon see, it is powerful: it will allow us to build the four-point function of four $\mathbb{O}_2$ superfields in terms of just a scalar function [101, 102].

First of all, let us see how we can uplift the usual propagator to superspace

$$\frac{y_i \cdot y_j}{x_{ij}^2} \quad \rightarrow \quad \hat{g}_{ij} = \frac{\det ||\hat{y}_{ij}^{\mathsf{mm}'}||}{x_{ij}^2} \,, \tag{5.27}$$

where we have defined

$$\hat{y}_{ij}^{\mathsf{mm}'} = y_{ij}^{\mathsf{mm}'} - 4i \frac{\theta_{ij}^{\alpha\mathsf{m}}(\mathrm{x}_{ij})_{\alpha\dot\alpha}\bar\theta_{ij}^{\dot\alpha\mathsf{m}'}}{x_{ij}^2} \,, \tag{5.28}$$

with $y_{ij}^{\mathsf{mm}'} = y_i^{\mathsf{mm}'} - y_j^{\mathsf{mm}'}$ and similarly for the Grassmann variables. Then, starting from (5.1), we can think of promoting propagators to superpropagators $\hat{g}_{ij}$ and write

$$\langle \mathbb{O}_2 \mathbb{O}_2 \mathbb{O}_2 \mathbb{O}_2 \rangle = \hat{g}_{12}^2 \hat{g}_{34}^2 \left(\hat{\mathbb{G}}^{\mathrm{rational}} + \hat{\mathbb{G}}^{\mathrm{anom}} \right) \,, \tag{5.29}$$

where $\hat{\mathbb{G}}^{\mathrm{rational}}$ is a rational function of the superpropagators computed by means of Wick contractions. The interesting part is what we call $\hat{\mathbb{G}}^{\mathrm{anom}}$. Remarkably, this has been shown [101, 102] to be expressible in terms of a scalar function $F(x_1, x_2, x_3, x_4) \equiv F(x)$

$$\hat{\mathbb{G}}^{\mathrm{anom}} = Q^4 Q'^4 \bar S^4 \bar S'^4 \left[\theta_1^4 \theta_2^4 \theta_3^4 \theta_4^4 \frac{F(x)}{\hat{g}_{12}^2 \hat{g}_{34}^2} \right] \,, \tag{5.30}$$

where we have defined

$$\begin{aligned} Q^4 &= \frac{1}{12} Q_\mathsf{m}^\alpha Q_\alpha^\mathsf{n} Q_\mathsf{n}^\beta Q_\beta^\mathsf{m} \,, \\ Q'^4 &= \frac{1}{12} Q^{\alpha\mathsf{m}'} Q_{\alpha\mathsf{n}'} Q^{\beta\mathsf{n}'} Q_{\beta\mathsf{m}'} \end{aligned} \tag{5.31}$$

and similarly for θ, $\bar S$, $\bar S'$.[1] The scalar function depends only on the correlator of the scalar superprimaries. In fact, by requiring

$$\begin{aligned} \langle \mathbb{O}_2 \mathbb{O}_2 \mathbb{O}_2 \mathbb{O}_2 \rangle|_{\substack{\theta=0 \\ \bar\theta=0}} &= \mathsf{y}_{12}^2 \mathsf{y}_{34}^2 \left(\frac{\mathcal{G}^{\mathrm{free}}(z, \bar z, \alpha, \bar\alpha)}{(x_{12}^2 x_{34}^2)^2} + R(x_{13}^2 x_{24}^2)^2 F(x) \right) \\ &\overset{!}{=} \langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle \,, \end{aligned} \tag{5.32}$$

---

[1] The conformal supercharges $\bar S$, $\bar S'$ are defined exactly as (5.22).

where R is defined in (5.9b), we find

$$F(x) = \frac{\widetilde{\mathcal{H}}(z, \bar{z})}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2)^2} \,. \tag{5.33}$$

Here we have used an alternative decomposition with the respect to (5.8): namely

$$\begin{aligned} \mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha}) &= \mathcal{G}^{\text{free}}(z, \bar{z}; \alpha, \bar{\alpha}) + R\widetilde{\mathcal{H}}(z, \bar{z}) \,, \\ \widetilde{\mathcal{H}}(z, \bar{z}) &= \mathcal{H}(z, \bar{z}) - \mathcal{H}^{\text{free}}(z, \bar{z}) \,, \end{aligned} \tag{5.34}$$

with $\mathcal{G}^{\text{free}}$ and $\mathcal{H}^{\text{free}}$ respectively as in (5.13) and (5.14c).

With the form (5.30) and the identification (5.33), we can determine the correlation functions of all the fields composing the stress-tensor supermultiplet. In particular, the ones we will be interested in, in connection with the discussion in Part III, are correlators involving the stress-tensor. Despite the complexity of applying the supercharges and extracting the different components, the correlators of the various operators appearing in $\mathbb{O}_2$ can be expressed in a very nice and surprisingly compact form. For instance, the non-vanishing four-point functions involving the stress-tensor take the form

$$\langle \mathcal{T}\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle = \mathsf{y}_{23}\mathsf{y}_{34}\mathsf{y}_{24}\,\mathbb{D}_1^2 \left[(\eta_1 \mathsf{x}_{12}\mathsf{x}_{23}\mathsf{x}_{34}\mathsf{x}_{41}\eta_1)^2 F(x) \right], \tag{5.35a}$$

$$\langle \mathcal{T}\mathcal{T}\mathcal{O}_2\mathcal{O}_2 \rangle = \mathsf{y}_{34}^2\,(\mathbb{D}_1\mathbb{D}_2)^2 \left[(\eta_1 \mathsf{x}_{13}\mathsf{x}_{32}\eta_2)^2 (\eta_1 \mathsf{x}_{14}\mathsf{x}_{42}\eta_2)^2 F(x) \right], \tag{5.35b}$$

$$\langle \mathcal{T}\mathcal{T}\mathcal{T}\mathcal{T} \rangle = 4^4\,(\mathbb{D}_1\mathbb{D}_2\mathbb{D}_3\mathbb{D}_4)^2 \left[\Lambda(x, \eta)^2 F(x) \right], \tag{5.35c}$$

where for convenience we have contracted all the spinor indices with commuting auxiliary variables, such that $\mathcal{T} \equiv \eta^{\alpha_i}\bar{\eta}^{\dot{\alpha}_i}\eta^{\beta_i}\bar{\eta}^{\dot{\beta}_i}\mathcal{T}_{\alpha_i\beta_i,\dot{\alpha}_i\dot{\beta}_i}$. The structure $\Lambda(x, \eta)$ is defined as

$$\Lambda = \frac{[(\eta_1 \mathsf{x}_{12}\mathsf{x}_{23}\eta_3)(\eta_4 \mathsf{x}_{41}\mathsf{x}_{12}\eta_2) - (\eta_1 \mathsf{x}_{12}\mathsf{x}_{24}\eta_4)(\eta_3 \mathsf{x}_{31}\mathsf{x}_{12}\eta_2)]^2}{(x_{12}^2)^2} \,, \tag{5.36}$$

and we have introduced the differential operator

$$\mathbb{D}_i = \bar{\eta}_i^{\dot{\alpha}} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} \frac{\partial}{\partial \eta_{i\alpha}} \,, \tag{5.37}$$

which we will meet again in Part III.

Finally, we would like to comment on the correlator of the top component of the supermultiplet in Fig. 2.2, namely the self-dual and anti self-dual Lagrangian $\mathcal{L}$ and $\bar{\mathcal{L}}$. In the neutral channel — see [103] and Paper V — it reads

$$\begin{aligned} \langle \mathcal{L}(x_1)\bar{\mathcal{L}}(x_2)\mathcal{L}(x_3)\bar{\mathcal{L}}(x_4) \rangle &= \frac{1}{(x_{12}^2 x_{34}^2)^4} \mathcal{F}_{\mathcal{L}}^{\text{n}}(u, v) \,, \\ \mathcal{F}_{\mathcal{L}}^{\text{n}}(u, v) &= \Delta^{(8)}\widetilde{\mathcal{H}}(u, v) \,, \end{aligned} \tag{5.38}$$

where
$$\Delta^{(8)} = u^4 (\Delta^{(2)})^2 u^2 v^2 (\Delta^{(2)})^2 u^{-2} \,,$$
$$\Delta^{(2)} = u\partial_u^2 + v\partial_v^2 + (u + v - 1)\partial_u \partial_v + 2(\partial_u \partial_v) \,. \tag{5.39}$$

We can rewrite this in terms of the $z, \bar{z}$ coordinates, using the Casimir

$$\hat{\mathcal{D}}_x = \partial_x x (1 - x) \partial_x \,,$$
$$\hat{\mathcal{D}}_x k_h \left(\frac{1}{x} \right) = -h(h - 1) k_h \left(\frac{1}{x} \right) \,, \tag{5.40}$$

then

$$\Delta^{(8)} = \frac{(z\bar{z})^4}{z - \bar{z}} \hat{\mathcal{D}}_z (\hat{\mathcal{D}}_z - 2) \hat{\mathcal{D}}_{\bar{z}} (\hat{\mathcal{D}}_{\bar{z}} - 2) \left[\frac{z - \bar{z}}{z\bar{z}} (z\bar{z})^{-2} \right] \,. \tag{5.41}$$

This operator is going to play a very important role in Sec. 5.4.

5.3 Protected contribution and tree-level supergravity

Now that we have seen how to lift the four-point function of the stress-tensor superprimaries to a superspace correlator, we can get into the details of the decomposition (5.8). We will first determine the protected contributions to the correlator, the one in correspondence to the exchange of the short and semi-short representations in the OPE (5.6). At $\lambda \to \infty$, this information is going to be enough to completely fix the tree-level supergravity value of $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ by using the Lorentzian inversion formula.

First of all, let us expand the free-theory results (5.14) in conformal blocks (5.15) and (5.17).

$$b_{0,\ell} = -\frac{1 + (-1)^\ell}{2} \frac{\Gamma(\ell + 1)^2}{\Gamma(2\ell + 1)} \left((\ell - 1)(\ell + 2) - \frac{3}{c} \right) \,,$$
$$b_{1,\ell} = \frac{1 - (-1)^\ell}{2} \frac{\Gamma(\ell + 1)^2}{\Gamma(2\ell + 1)} \left(\ell(\ell + 1) + \frac{1}{c} \right) \,,$$
$$a^{\text{free}}(\tau, \ell) = 2^\ell (1 + (-1)^\ell) \frac{\Gamma\left(\frac{\tau}{2} + 1 \right)^2 \Gamma\left(\ell + \frac{\tau}{2} + 2 \right)^2}{\Gamma(\tau + 1)\Gamma(2\ell + \tau + 3)} \times \tag{5.42}$$
$$\left((\ell + 1)(\tau + \ell + 2) + \frac{(-1)^{\frac{\tau}{2}}}{c} \right) \,, \qquad \tau \geq 0 \,.$$

Problematically, $a^{\text{free}}(\tau, \ell)$ contains a tower of twist-zero operators, which are below the unitarity bounds in (2.8). Moreover, it contains infinite twist-two operators, which are exactly at threshold, which as we will explain in a second, we would like to remove. The appearance of these

operators is an artefact of the splitting we are using, in fact the full $\mathcal{G}$ does not contain any non-unitary contributions, as expected.

The origin of this behaviour relies on the fact that a short multiplet does not contribute only to $\hat{f}$ but enters $\mathcal{H}$ as well.

Let us consider a single protected multiplet and denote its contributions to $\hat{f}$ and $\mathcal{H}$ respectively as

$$\hat{f}(z,\alpha)\big|_{n,\ell} = b_{n,\ell}\, y_n(\alpha)\, \mathfrak{g}_\ell^{(0,0)}(z)\,, \tag{5.43a}$$

$$\mathcal{H}(z,\bar{z})\big|_{n,\ell} = a(\tau,\ell)G_{\tau+\ell,\ell}\,, \tag{5.43b}$$

for some coefficient $a(\tau,\ell)$ and twist τ. To make it clear the presence of this short protected subsector in $\mathcal{H}$ let us define

$$\mathcal{H}^{\mathrm{short}}(z,\bar{z}) = \sum_{\tau,\ell} a(\tau,\ell)G_{\tau+\ell,\ell}\,, \tag{5.44}$$

such that we can split

$$\mathcal{H}(z,\bar{z}) = \mathcal{H}^{\mathrm{short}}(z,\bar{z}) + \mathcal{H}^{\mathrm{long}}(z,\bar{z})\,, \tag{5.45}$$

where, as the name suggests, in $\mathcal{H}^{\mathrm{long}}(z,\bar{z})$ only long, non-protected operators are exchanged. The precise and more formal way of fixing $\mathcal{H}^{\mathrm{short}}$ involves a detailed analysis of how the various short multiplets contribute to k, $\mathcal{G}^{\hat{f}}$ and $\mathcal{H}$, how they are exchanged in the various representations and it requires to take into account possible multiplet recombination — when a long operator at threshold decomposes into a sum of semi-short ones. We refer to [81, 96, 97] for details on how this procedure should be carried out. Here, we prefer to resort to a shortcut for the determination of $\mathcal{H}^{\mathrm{short}}$, since this approach is similar in spirit to the strategy we will use in Ch. 6.

We will proceed in the following way. Starting from (5.43), we fix $a(\tau,\ell)$ in terms of $b_{n,\ell}$ such that when we combine together $\mathcal{G}^{\hat{f}}$ and $\mathcal{H}^{\mathrm{short}}$ inside the full correlator, all twist zeros and twist twos with $\ell > 2$ are absent. We can impose these requirements because we know that eventually, in the full $\mathcal{G}$, any given multiplet cannot yield any twist-zero — because non-physical — or twist-two operators — correspond to higher spin currents absent in a sensible interacting theory [104, 105]. Let us start from twist-zero contributions. These appear when we plug

$$\hat{f}\big|_{1,\ell} = \left(\alpha - \frac{1}{2}\right) b_{1,\ell+1}\, \mathfrak{g}_{\ell+1}^{(0,0)}(z)\,, \tag{5.46}$$

in $\mathcal{G}^{\hat{f}}$ and we expand in four-dimensional conformal blocks. To remove them, we allow for a twist-zero contribution in $\mathcal{H}^{\mathrm{short}}$ as well

$$\mathcal{H}^{\mathrm{short}}\big|_{1,\ell} = a(0,\ell)G_{\ell,\ell}\,, \tag{5.47}$$

where recall that we are expanding in superblocks, which are related to the usual $4d$ ones as $G_{\Delta,\ell} = u^{-2} g_{\Delta+4,\ell}$. Now we sum these two pieces together and expand according to the decomposition (5.3)

$$\mathcal{G}^{\hat{f}} + R\mathcal{H}^{\text{short}}\big|_{1,\ell} = \sum_{n,m} \sum_{\Delta,\ell} C'(\Delta, \ell; m, n)\, Y_{n,m}(\alpha, \bar{\alpha})\, g_{\Delta,\ell}\,. \qquad (5.48)$$

As you can see, we expect contributions from all six R-symmetry representations. To give an example, let us focus on the $n = m = 1$ case, *i.e.* the $[0, 2, 0]$ channel — one can check that the other components follow analogously. By using nice recurrence relation for the blocks, [106] and Appendix E of Paper IV, we can show that

$$\mathcal{G}^{\hat{f}} + R\mathcal{H}^{\text{short}}\big|_{1,\ell;[0,2,0]} = 4\left(a(0,\ell) - (-2)^{\ell} b_{1,\ell+1}\right) g_{\ell+2,\ell+2}$$

$$+ \left(a(0,\ell) - (-2)^{\ell} b_{1,\ell+1}\right)\left(g_{\ell+2,\ell} + \frac{(\ell+2)^2}{4\left(\ell+\frac{3}{2}\right)_2} g_{\ell+4,\ell+2}\right)$$

$$+ \left\{ \frac{5}{4} g_{\ell+2,\ell-2} + \frac{14\ell^2 + 42\ell + 25}{12(2\ell+1)(2\ell+5)} g_{\ell+4,\ell} + \frac{5(\ell+2)_2^2\, g_{\ell+6,\ell+2}}{32\left(\ell+\frac{3}{2}\right)_3 (2\ell+5)} \right.$$

$$+ \frac{1}{4}\left(g_{\ell+4,\ell-2} + \frac{(\ell+2)^2}{4\left(\ell+\frac{3}{2}\right)_2} g_{\ell+6,\ell}\right) + \frac{1}{60} g_{\ell+6,\ell-2} \bigg\} \frac{a(0,\ell)}{12}\,.$$

$$(5.49)$$

By looking at the first two lines, we can see that imposing

$$a(0,\ell) = (-2)^{\ell} b_{1,\ell+1}\,, \qquad (5.50)$$

we get rid of all twist zeros and the twist twos we have additionally introduced. The twists greater than two are not problematic and they will contribute to $\mathcal{G}$.

We can repeat a similar story to cancel twist-two operators. For example in the $[0, 2, 0]$ representation

$$\mathcal{G}^{\hat{f}} + R\mathcal{H}^{\text{short}}\big|_{0,\ell;[0,2,0]} = 4(a(2,\ell) + (-2)^{\ell} b_{0,\ell+2}) g_{\ell+2,\ell+4}$$

$$+ a(2,\ell)\bigg\{ \left(g_{\ell+4,\ell} + \frac{(\ell+3)^2}{4\left(\ell+\frac{5}{2}\right)_2} g_{\ell+6,\ell+2}\right) + \frac{1}{4} g_{\ell+4,\ell-2}$$

$$+ \frac{2\ell^2 + 10\ell + 11}{4(2\ell+3)(2\ell+7)} g_{\ell+6,\ell} + \frac{(\ell+3)_2^2\, g_{\ell+8,\ell+2}}{32\left(\ell+\frac{5}{2}\right)_3 (2\ell+7)}$$

$$+ \frac{1}{60}\left(g_{\ell+6,\ell-2} + \frac{(\ell+3)^2}{4\left(\ell+\frac{5}{2}\right)_2} g_{\ell+8,\ell}\right) + \frac{3}{2800} g_{\ell+8,\ell-2} \bigg\}\,,$$

$$(5.51)$$

and again we see that twist twos cancel whenever

$$a(2,\ell) = -(-2)^{\ell} b_{0,\ell+2}\,. \qquad (5.52)$$

Interpreted from the OPE perspective in (5.6), this condition is equivalent to cancel the $\mathcal{C}_{[0,0,0],\ell}$ multiplet with $\ell > 2$, which we recall contains higher-spin currents.

Notice that $a(0,\ell) = a^{\text{free}}(0,\ell)$ in (5.42), as expected since the free theory should not have twist zero contributions apart from the identity. Whereas $a(2,\ell) \neq a^{\text{free}}(2,\ell)$, since twist twos are admissible in the non-interacting theory.

We can resum the $a(0,\ell)$ and $a(2,\ell)$ to a full expression for $\mathcal{H}^{\text{short}}$

$$\mathcal{H}^{\text{short}} = \sum_{\ell=0}^{\infty} (-2)^{\ell} b_{1,\ell+1} G_{\ell,\ell}(z,\bar{z}) - (-2)^{\ell} b_{0,\ell+2} G_{\ell+2,\ell}(z,\bar{z}), \qquad (5.53)$$

$$= \frac{\left(2z\left(2\bar{z}^3 - \bar{z}^2 - 8\bar{z} + 6\right) - 6\left(\bar{z}^3 - 6\bar{z} + 4\right)\right)\log(1-z)}{z^2(\bar{z}-1)^2(z-\bar{z})}$$

$$- \frac{24\log(1-\bar{z})}{z\bar{z}(z-\bar{z})} + \frac{12(z-2)\log(1-z)\log(1-\bar{z})}{z^2\bar{z}(z-\bar{z})} - \frac{6\left(\bar{z}^3 - 6\bar{z} + 4\right)}{z(\bar{z}-1)^2(z-\bar{z})}$$

$$+ \frac{1}{c}\left(-\frac{36\log(1-\bar{z})}{z\bar{z}(z-\bar{z})} + \frac{2(4z\bar{z} - 9z - 9\bar{z} + 18)\log(1-z)}{z^2(\bar{z}-1)(z-\bar{z})}\right.$$

$$\left.+ \frac{18(z-2)\log(1-z)\log(1-\bar{z})}{z^2\bar{z}(z-\bar{z})} - \frac{18(\bar{z}-2)}{z(\bar{z}-1)(z-\bar{z})}\right) + z \leftrightarrow \bar{z}\,.$$

This agrees with the result in [96] and it is $\frac{1}{c}$ exact. Notice that by summing ℓ from 0 and because of the shift in the second term ($\ell \to \ell+2$), we are not including the coefficient $b_{0,0}$. So let us consider its role separately: it contributes at twist zero in the singlet representation and it nicely combines with k in (5.14a) to give, in the convention of (5.3),

$$C(0,0;0,0) = k - b_{0,0} = 1\,, \qquad (5.54)$$

which is precisely the normalization we require for the identity operator. Then, it appears at twist two in the following combination

$$C(2,0;1,1) = b_{0,0} - b_{1,1} = \frac{2}{c} = \lambda^2_{\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2}\,, \qquad (5.55)$$

which is exactly the protected three-point coefficient of $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$. With the expression for $\mathcal{H}^{\text{short}}(z,\bar{z})$ we can also find the OPE coefficient for other protected operators inside the $\mathcal{O}_2$ multiplet in Fig. 2.2, namely the stress-tensor $T_{\mu\nu}$, and the R-symmetry current $J_\mu^{\text{SU}(4)}$

$$\lambda^2_{\mathcal{O}_2\mathcal{O}_2 T_{\mu\nu}} = \frac{2}{45c} = \frac{1}{5}(\lambda^{\text{free}}_{\mathcal{O}_2\mathcal{O}_2 \text{``}T_{\mu\nu}\text{''}})^2\,,$$

$$\lambda^2_{\mathcal{O}_2\mathcal{O}_2 J_\mu^{\text{SU}(4)}} = \frac{2}{3c} = \frac{1}{3}(\lambda^{\text{free}}_{\mathcal{O}_2\mathcal{O}_2 \text{``}J_\mu^{\text{SU}(4)}\text{''}})^2\,.$$

In the expressions above, we have stressed the fact that the exact OPE coefficients differ from the ones that one can naively obtain from the

free theory in (5.13). The reason is that the free-theory coefficients receive contributions from superdescendants of long operators [107], like the Konishi.

Once identified the protected sector in $\mathcal{G}$, we are left with $\mathcal{H}^{\text{long}}$, which contains only long operators and has a conformal block expansion

$$\mathcal{H}^{\text{long}}(z, \bar{z}) = \sum_{\Delta, \ell} a_{\Delta, \ell} G_{\Delta, \ell}, \qquad (5.56)$$

with three-point coefficients real and positive, in a reflection positive configuration. Moreover at large central charge and in the supergravity regime, long contributions can come only from multi-trace operators, constructed from the protected single-trace operators $\mathcal{O}_p$. In fact, as discussed in Sec. 4.2, the other long single-trace operators, like the Konishi, acquire a large dimension ($\Delta \sim \lambda^{\frac{1}{4}}$) and decouple from the spectrum. Up to order $\frac{1}{c}$, actually only double-trace operators appear. Their dimension and OPE coefficients admit a perturbative expansion in $1/c$ as

$$\Delta_{n,\ell} = 4 + 2n + \ell + \sum_{\kappa=1}^{\infty} \frac{\gamma_{n,\ell}^{(\kappa)}}{c^{\kappa}}, \qquad (5.57a)$$

$$a_{n,\ell} = \sum_{\kappa=0}^{\infty} \frac{a_{n,\ell}^{(\kappa)}}{c^{\kappa}}, \qquad (5.57b)$$

where $n = 0, 1, \ldots$ and $\gamma_{n,\ell}^{(\kappa)}$ are anomalous dimensions. Notice the twist gap: the tower of double-trace operators starts at $\tau = 4$. These operators take the schematic form[2]

$$[\mathcal{O}_2 \mathcal{O}_2]_{n,\ell} = \mathcal{O}_2 \square^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}_2. \qquad (5.58)$$

$\mathcal{H}^{\text{long}}$ has a similar expansion

$$\mathcal{H}^{\text{long}} = \sum_{\kappa=0}^{\infty} \frac{\mathcal{H}^{(\kappa)}}{c^{\kappa}}. \qquad (5.59)$$

The conformal block expansion for the first few orders reads

$$\mathcal{H}^{(0)} = u^{n+2} a_{n,\ell}^{(0)} \tilde{G}_{n,\ell},$$

$$\mathcal{H}^{(1)} = u^{n+2} \left(\frac{\log u}{2} a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)} + a_{n,\ell}^{(1)} + a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)} \partial_{\Delta} \right) \tilde{G}_{n,\ell},$$

$$\mathcal{H}^{(2)} = u^{n+2} \left(\frac{\log^2 u}{8} a_{n,\ell}^{(0)} (\gamma_{n,\ell}^{(1)})^2 + \frac{\log u}{2} (a_{n,\ell}^{(1)} \gamma_{n,\ell}^{(1)} + a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(2)} + \right.$$

$$(5.60)$$

---

[2]We will see later on that actually more operators with the same classical twist and spin mix among each other.

$$a_{n,\ell}^{(0)}(\gamma_{n,\ell}^{(1)})^2\partial_\Delta) + a_{n,\ell}^{(2)} + (a_{n,\ell}^{(1)}\gamma_{n,\ell}^{(1)} + a_{n,\ell}^{(0)}\gamma_{n,\ell}^{(2)})\partial_\Delta + \frac{a_{n,\ell}^{(0)}(\gamma_{n,\ell}^{(1)})^2}{2}\partial_\Delta^2\Big)\tilde{G}_{n,\ell}\,,$$

where we have denoted $G_{4+2n+\ell,\ell}(z,\bar{z}) = (u)^{\frac{\tau}{2}}\tilde{G}_{n,\ell}$ and the sums over n and ℓ are understood. From these few terms, we can easily guess that the leading logarithmic part of the correlator always depends only on $a_{n,\ell}^{(0)}$ and $\gamma_{n,\ell}^{(1)}$ of the double-trace operators in the combination

$$\mathcal{H}^\kappa|_{\log^\kappa u} = \sum_{n,\ell} u^{n+2}\frac{a_{n,\ell}^{(0)}\left(\gamma_{n,\ell}^{(1)}\right)^\kappa}{2^\kappa \kappa!}\tilde{G}_{n,\ell}\,. \tag{5.61}$$

5.3.1 Tree-level supergravity from the inversion formula

In Ch. 3, we have seen how the OPE data are encoded in the function $c_k(\Delta,\ell)$ and how this can be written dispersively in terms of dDisc. At large c, given the form of the corrections (5.57) that the exchanged operators receive, $c_k(\Delta,\ell)$ inherits the expansion

$$c_k(\Delta,\ell) \sim -\frac{1}{2}\Big\langle\frac{a^{(0)}}{h - \frac{\tau^{(0)}}{2}}\Big\rangle - \frac{1}{c}\Big(\frac{1}{4}\Big\langle\frac{a^{(0)}\gamma^{(1)}}{(h - \frac{\tau^{(0)}}{2})^2}\Big\rangle + \frac{1}{2}\Big\langle\frac{a^{(1)}}{h - \frac{\tau^{(0)}}{2}}\Big\rangle\Big)$$
$$-\frac{1}{c^2}\Big(\frac{1}{8}\Big\langle\frac{a^{(0)}(\gamma^{(1)})^2}{(h - \frac{\tau^{(0)}}{2})^3}\Big\rangle + \frac{1}{4}\Big\langle\frac{a^{(0)}\gamma^{(2)} + a^{(1)}\gamma^{(1)}}{(h - \frac{\tau^{(0)}}{2})^2}\Big\rangle$$
$$+\frac{1}{2}\Big\langle\frac{a^{(2)}}{h - \frac{\tau^{(0)}}{2}}\Big\rangle\Big) + \mathcal{O}(c^{-3})\,, \tag{5.62}$$

where the brackets stand for averages over possible degenerate operators with the same twist $\tau^{(0)} = \Delta^{(0)} - \ell$ and spin. From this formula, it is clear that at order c^0 we expect simple poles for the $h = \frac{\Delta - \ell}{2}$ of the exchanged operators, whose OPE coefficients should recover the free theory results computed at $c \to \infty$. At the next order, double poles arise in correspondence of those operators developing an anomalous dimension, while simple poles take into account the correction to the OPE coefficients. Studying order by order the analytic structure of $c_k(\Delta,\ell)$, we can extract more and more corrections.

To apply the inversion integral directly to $\mathcal{H}$, it is necessary to slightly adapt the expression (3.9) to account for the appearance of superblocks. Fortunately, it is enough to shift $h \to h+2$ and multiply by u^2, so that the final expression reduces to

$$c(\Delta,\ell) = \frac{1+(-1)^\ell}{2}(-2)^\ell\tilde{\kappa}_{\Delta+4+\ell}^{(0,0)}\int_0^1\frac{dz}{z^2}\frac{d\bar{z}}{\bar{z}^2}k_{-h}^{(0,0)}(z)k_{h+\ell+2}^{(0,0)}(\bar{z})\times$$
$$\text{dDisc}\left[(\bar{z}-z)z\bar{z}\mathcal{H}(z,\bar{z})\right]\,, \tag{5.63}$$

where we have used that for identical operators the t- and u- channels are the same.

As we have explained in Sec. 3.1, the prescription to compute the double discontinuity is to first pass to the crossed channel. Because of the presence of k and $\mathcal{G}^{\hat{f}}$, $\mathcal{H}$ is not crossing symmetric but rather transforms as (5.11). So at order $c^{-\kappa}$ crossing reads

$$
\mathcal{H}^{\mathrm{short}}(u,v) + \sum_{\kappa=0}^{1} \frac{\mathcal{H}^{(\kappa)}(u,v)}{c^{\kappa}} = \frac{u^2}{v^2} \mathcal{H}^{\mathrm{short}}(v,u)
$$
$$
+ \frac{u^2}{v^2} \sum_{\kappa=0}^{1} \frac{\mathcal{H}^{(\kappa)}(u,v)}{c^{\kappa}} + 1 - \frac{u^2}{v^2} + \frac{1}{c} \frac{v-u}{v^2} \,, \qquad\qquad \kappa = 0,1\,, \quad (5.64a)
$$
$$
\mathcal{H}^{(\kappa)}(u,v) = \frac{u^2}{v^2} \mathcal{H}^{(\kappa)}(v,u)\,, \qquad\qquad\qquad\qquad \kappa \geq 2\,. \qquad (5.64b)
$$

Now let us focus on the first crossing equation and let us try to understand which terms contribute to the inversion formula. Remember that the terms with non-vanishing dDisc are either negative powers of v or $(\log v)^{\kappa \geq 2}$ — see (3.15). These are present only in $\mathcal{H}^{\mathrm{short}}$ and in the last two rational terms. Both $\mathcal{H}^{(0)}(v,u)$ and $\mathcal{H}^{(1)}(v,u)$, in fact, start at order v^2 as we can easily see from (5.60). So schematically

$$
\mathrm{dDisc}\left[\mathcal{H}(u,v)\right]_{\frac{1}{c}} \sim \mathrm{dDisc}\left[\frac{u^2}{v^2}\mathcal{H}^{\mathrm{short}}(v,u) - \frac{u^2}{v^2} + \frac{1}{c}\frac{v-u}{v^2}\right]\,. \qquad (5.65)
$$

Remarkably, this means that the protected contributions completely fix the full tree-level (up to $\frac{1}{c}$) supergravity correlator. In other words, the singular behaviour comes solely from the single-trace protected sector. This is the same as what happens for amplitudes in flat space, where cuts of single-particle states allow reconstructing the tree-level amplitude. By analogy, we can represent dDisc as some cut operator[3] acting on the Witten diagram with four $\mathcal{O}_2$ external operators such that it is split in two three-point functions, as depicted in Fig. 5.1.

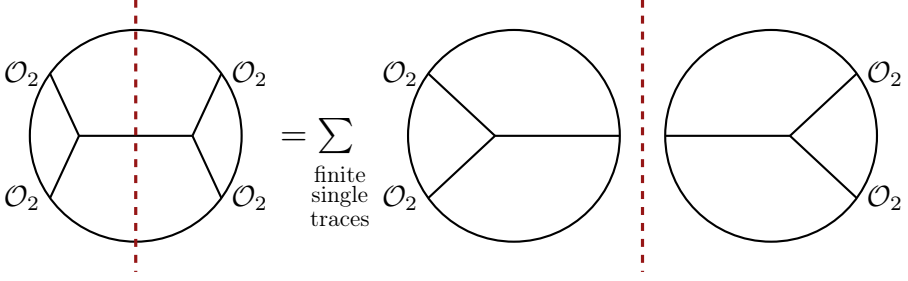[3]Similar ideas are explored in [108].

Figure 5.1. The full tree-level supergravity correlator can be reconstructed through a dispersion relation starting from the discontinuity, denoted with the red cut line, produced by single-trace protected operators.

The precise form of (5.65) is[4]

$$
\begin{aligned}
\mathrm{dDisc}\left[(\bar{z}-z)z\bar{z}\,\mathcal{H}(z,\bar{z})\right] &= \mathrm{dDisc}\left[\frac{\bar{z}^2}{(1-\bar{z})^2}\right]\frac{z}{1-z} \\
&\quad - \mathrm{dDisc}\left[\frac{\bar{z}}{(1-\bar{z})}\right]\frac{z^2}{(1-z)^2} + \frac{1}{c}\mathrm{dDisc}\left[\frac{\bar{z}}{(1-\bar{z})}\right]\left(\frac{z}{1-z}\right. \\
&\quad \left. - 2\frac{z^2}{(1-z)^2} - 2\log z\frac{z^3}{(1-z)^3}\right) \\
&= \mathrm{dDisc}\left[(\bar{z}-z)z\bar{z}\left(\mathcal{H}^{\mathrm{free}}(z,\bar{z}) - \frac{1}{c}(z\bar{z})^2\,\overline{D}_{2422}(z,\bar{z})\right)\right],
\end{aligned}
\tag{5.66}
$$

and we can thus identify

$$
\mathcal{H}(z,\bar{z})\Big|_{\frac{1}{c}} = \mathcal{H}^{\mathrm{free}}(z,\bar{z}) - \frac{1}{c}(z\bar{z})^2\,\overline{D}_{2422}(z,\bar{z})\,.
\tag{5.67}
$$

The appearance of the $\overline{D}$-function[5] $\overline{D}_{2422}$ agrees with previous supergravity results [109] and it is compatible with the expected dimensions and crossing-symmetry properties

$$
\overline{D}_{2422}(u,v) = \overline{D}_{2422}(v,u)\,.
\tag{5.68}
$$

Concretely it can be computed from a seed function as

$$
\begin{aligned}
\overline{D}_{2422} &= (3 + u\partial_u + v\partial_v)\partial_v\partial_u\overline{D}_{1111}\,, \\
\overline{D}_{1111} &= \frac{1}{z-\bar{z}}\left(2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right)\,.
\end{aligned}
\tag{5.69}
$$

Since we are reconstructing the result from a dispersion relation, one can worry about potential ambiguities with a finite support in low spins.

---

[4]One should remember to expand also $\log\bar{z} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{1-\bar{z}}{\bar{z}}\right)^k$.
[5]See also Sec. 8.1.

At order $1/c$, one can prove convergence of the inversion formula for $\ell > -2$ [31], but the situation might change and get worse proceeding in the loop expansion. As a non-trivial check, we can extract the OPE data as in (5.62) by plugging (5.66) into (5.63). Let us start from the disconnected contributions[6]

$$c^{(0)}(\Delta,\ell) = (-2)^\ell \tilde{\kappa}^{(0,0)}_{\Delta+4+\ell} \left(\bar{\mathcal{I}}^{(0,0)}_{-2} \mathcal{I}^{(0,0)}_1(1) - \bar{\mathcal{I}}^{(0,0)}_{-1} \mathcal{I}^{(0,0)}_1(2) \right), \quad (5.70)$$

where the definitions and the values of the integrals can be found in Sec. 3.1. The OPE coefficients of the double-trace operators can be read out as residues at the poles with twist $4 + 2n$

$$\begin{aligned}
\langle a^{(0)}_{n,\ell} \rangle &= -2\mathrm{Res}_{h=2+n}\, c^{(0)}(\Delta,\ell) \\
&= \frac{2^{\ell+1}(\ell+1)(2n+\ell+6)\Gamma(n+3)^2\Gamma(n+\ell+4)^2}{\Gamma(2n+5)\Gamma(2n+2\ell+7)},
\end{aligned} \quad (5.71)$$

and this exactly coincides with (5.42). Now let us pass to the first order in $1/c$, where we expect contributions from the anomalous dimensions. These are encoded in the double poles coming from the $\log z$ integration.

$$\begin{aligned}
c^{(1)}(\Delta,\ell) &= (-2)^\ell \tilde{\kappa}^{(0,0)}_{\Delta+4+\ell} \bar{\mathcal{I}}^{(0,0)}_{-1} \left(\mathcal{I}^{(0,0)}_1(1) - 2\mathcal{I}^{(0,0)}_1(2) - 2\mathcal{I}^{(0,0)}_3(3) \right) \\
&= \frac{\pi^2(h-1)h(h+1)(h+2)2^{\ell-1}\Gamma(h+1)^2\Gamma(h+\ell+2)^2}{\sin(\pi h)^2\Gamma(2h+1)\Gamma(2h+2\ell+3)},
\end{aligned}$$
$$(5.72)$$

where, quite remarkably, the simple poles cancel among each others and we have just the double poles coming from the $\sin(\pi h)^2$. From (5.62), it is straightforward to see

$$\begin{aligned}
\langle a^{(0)}_{n,\ell}\gamma^{(1)}_{n,\ell} \rangle &= -\frac{(n+1)(n+2)(n+3)(n+4)}{(\ell+1)(2n+\ell+6)} \langle a^{(0)}_{n,\ell} \rangle, \\
\langle a^{(1)}_{n,\ell} \rangle &= \frac{1}{2}\partial_n \langle a^{(0)}_{n,\ell}\gamma^{(1)}_{n,\ell} \rangle.
\end{aligned} \quad (5.73)$$

5.4 Double-trace degeneracy and leading logs

Notice that in the expressions (5.73) for the anomalous dimensions and OPE coefficients of the double-trace operators we have kept the average. This is not by accident. In fact, apart from the $n = 0$ case, for fixed n and ℓ, there exists more than one superconformal primary, singlet under $\mathrm{SU}(4)_\mathrm{R}$. These degenerate operators, having the same classical

---

[6]Here and in the following, we will suppress the prefactor $\frac{1+(-1)^\ell}{2}$ and it will be understood that only even spins are exchanged.

twist ($\tau = 4 + 2n$), mix among each other at tree level[7] and they give rise to the so-called *mixing problem*. We can label them with an index $i = 1, \ldots, n+1$

$$\{[\mathcal{O}_2\mathcal{O}_2]_{n,\ell}, [\mathcal{O}_3\mathcal{O}_3]_{n-1,\ell}, \ldots, [\mathcal{O}_{n+2}\mathcal{O}_{n+2}]_{0,\ell}\} \equiv \mathcal{K}_{n,\ell,i}. \tag{5.74}$$

To isolate single operators in the averages, which is essential if we would like to go beyond tree level and take powers of the anomalous dimensions, it is necessary to disentangle the various contributions. A solution for this mixing problem is known at order $1/c$ and it has been worked out in a series of papers [33, 35, 110–112] by using correlators of the form $\langle \mathcal{O}_p\mathcal{O}_p\mathcal{O}_q\mathcal{O}_q \rangle$. In particular, it is possible to show

$$\lambda^2_{\mathcal{O}_2\mathcal{O}_2\mathcal{K}} = a^{(0)}_{n,\ell} R_{n,\ell,i} A_{n,i}, \tag{5.75a}$$

$$R_{n,\ell,i} = \frac{(4i + 2\ell + 3)(n + \ell + 6)_{i-1}(i + \ell + 1)^{\mathrm{sgn}(-i+n+1)}_{-i+n+1}}{3 \cdot 4^{n+1}\left(i + \ell + \frac{5}{2}\right)_{n+1}} \tag{5.75b}$$

$$A_{n,i} = \frac{(2i + 2)!n!(-2i + 2n + 6)!}{(i-1)!(i+1)!(n+4)(-i+n+1)!(-i+n+3)!}, \tag{5.75c}$$

$$\gamma^{(1)}_{n,\ell,i} = -\frac{(n+1)_4(n + \ell + 2)_4}{(\ell + 2i - 1)_6}, \tag{5.75d}$$

where $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$ is the Pochhammer symbol.

Later on, these results have been interpreted as emerging from a hidden $10d$ conformal symmetry [63] such that $10d$ blocks diagonalize the mixing problem. Before seeing how this symmetry works and what are its consequences, we need some preliminary definitions and observations. The first remark we would like to make is that the numerator of the anomalous dimensions (5.75d) is proportional to the eigenvalue of the eight-order Casimir operator $\Delta^{(8)}$, we encountered in Sec. 5.2, acting on $\mathcal{N} = 4$ superblocks[8]

$$\Delta^{(8)} = \frac{z\bar{z}}{z - \bar{z}}\mathcal{D}_z(\mathcal{D}_z - 2)\mathcal{D}_{\bar{z}}(\mathcal{D}_{\bar{z}} - 2)\left[\frac{z - \bar{z}}{z\bar{z}}(z\bar{z})^2\right], \tag{5.76a}$$

$$\Delta^{(8)}G_{4+2n+\ell,\ell}(z, \bar{z}) = (n+1)_4(n + \ell + 2)_4 G_{4+2n+\ell,\ell}(z, \bar{z}), \tag{5.76b}$$

with

$$\mathcal{D}_x = x^2\partial_x(1 - x)\partial_x,$$
$$\mathcal{D}_x k_h(x) = h(h - 1)k_h(x). \tag{5.77}$$

The careful reader might object that this is different from the $\Delta^{(8)}$ in (5.39). This apparent contradiction is solved if one considers, instead

---

[7]Mixing with higher multi-trace operators would occur only at higher-loop order.
[8]The same operator can be generalized for generic external dimensions [63].

of the correlator in (5.38), the charged Lagrangian four-point function $\langle \mathcal{L}\mathcal{L}\bar{\mathcal{L}}\bar{\mathcal{L}} \rangle$, which is the one where $\Delta^{(8)}$ as in (5.76a) naturally acts. To actually prove that the two definitions, (5.76a) and (5.41), are the same it is enough to recall that under a $2 \leftrightarrow 3$ crossing

$$\mathcal{F}_{\mathcal{L}}^{\text{ch}}(z, \bar{z}) = (z\bar{z})^4 \mathcal{F}_{\mathcal{L}}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right),$$

$$\widetilde{\mathcal{H}}(z, \bar{z}) = \widetilde{\mathcal{H}}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right).$$

(5.78)

A hand-wavy explanation of why this version of $\Delta^{(8)}$ appears rather than the other is that in the (12)-OPE of the charged channel the only exchanged operator is something like $Q^8 \mathcal{O}_{\text{exc}}$. This means that all the protected multiplets appearing in $\mathcal{O}_2 \times \mathcal{O}_2$ are killed and we select only the non-protected operators.

The second ingredient we need to introduce before moving to the main discussion are ten-dimensional blocks. A generic d-dimensional block, for equal external dimensions, is defined through the Casimir equation [113]

$$\left(\mathcal{D}_z + (d-2)\frac{z\bar{z}}{z-\bar{z}}(1-z)\partial_z + z \leftrightarrow \bar{z}\right) g_{\Delta,\ell}^{(d)}(z, \bar{z}) =$$
$$\left(\frac{1}{2}\tau(\tau-d) + \ell^2 + (\tau-1)\ell\right) g_{\Delta,\ell}^{(d)}(z, \bar{z}).$$

(5.79)

In [63], a closed form has been found for $10d$ conformal blocks at the unitarity bound ($\tau = d - 2$)

$$g_{\ell+8,\ell}^{(d=10)}(z, \bar{z}) = \mathcal{D}_z^{(3)} \varrho_\ell(z) + \mathcal{D}_{\bar{z}}^{(3)} \varrho_\ell(\bar{z}),$$

(5.80)

where we have defined

$$\mathcal{D}_z^{(3)} = \left(\frac{z\bar{z}}{\bar{z}-z}\right)^7 + \left(\frac{z\bar{z}}{\bar{z}-z}\right)^6 \frac{z^2}{2}\partial_z + \left(\frac{z\bar{z}}{\bar{z}-z}\right)^5 \frac{z^3}{10}\partial_z^2 z$$
$$+ \left(\frac{z\bar{z}}{\bar{z}-z}\right)^4 \frac{z^4}{120}\partial_z^3 z^2,$$

(5.81)

$$\varrho_\ell(z) = \frac{120}{(\ell+1)_3} z^{\ell+1} {}_2F_1(\ell+1, \ell+4; 2\ell+8; z).$$

Now that we have all the ingredients we are ready to see how this hidden $10d$ symmetry appears and how it greatly simplifies the resolution of the mixing problem. Details together with other important interesting applications can be found [63, 89], see also [114–118] for generalizations. Inspired by the similarity between the anomalous dimensions and the partial-wave coefficients of $2 \to 2$ scattering of axi-dilatons in flat space $10d$ IIB supergravity, one can imagine that the theory enjoys a $SO(10, 2)$

conformal symmetry. Then the rough idea is that one can define a 10-dimensional object from which all tree-level correlators are extracted. As explained in [63], at the disconnected order the object one has to consider is

$$\mathcal{G}^{(0)}_{(d=10)}(u,v) = \Delta^{(8)}\mathcal{H}^{\text{free}}(u,v) = 144\left(u^4 + \frac{u^4}{v^4}\right), \qquad (5.82)$$

which can be expanded in $10d$ blocks (5.80)

$$\mathcal{G}^{(0)}_{(d=10)}(z,\bar{z}) = \sum_{\substack{\ell=0 \\ \ell \text{ even}}} \tilde{a}^{(0)}_\ell g^{(d=10)}_{\ell,\ell+8}(z,\bar{z}),$$

$$\tilde{a}^{(0)}_\ell = \frac{8\Gamma(\ell+4)^2}{\Gamma(2\ell+7)}(\ell+1)_6. \qquad (5.83)$$

At order $\frac{1}{c}$, instead,

$$\mathcal{G}^{(1)}_{(d=10)}(u,v) = -u^4\overline{D}_{2422}. \qquad (5.84)$$

If now we focus on its $\log u$ terms, which are the ones contributing to the anomalous dimensions, we find, quite incredibly, that they can also be expanded in the same basis

$$\mathcal{G}^{(1)}_{(d=10)}(u,v)|_{\log u} = \frac{1}{2}\sum_{\substack{\ell=0 \\ \ell \text{ even}}} \tilde{a}^{(0)}_\ell \tilde{\gamma}^{(1)}_\ell g^{(d=10)}_{\ell,\ell+8}(z,\bar{z}),$$

$$\tilde{\gamma}^{(1)}_\ell = -\frac{1}{(\ell+1)_6}, \qquad (5.85)$$

where the relevant point is that for every spin ℓ only one block appears. In this sense, $10d$ blocks manage to "diagonalize" the mixing problem and they act somehow as projectors of $SO(10,2)$ to $SO(4,2) \times SO(6)$. Since this interpretation gives us a way to solve the mixing problem at tree level, it allows us to compute all the leading logarithmic terms in (5.61), since they depend only on this information. Remember that these terms are of the form

$$\frac{\log^\kappa u}{2^\kappa \kappa!}\sum_{n,\ell,i} u^{n+2}a^{(0)}_{n,\ell,i}\left(\gamma^{(1)}_{n,\ell,i}\right)^\kappa \tilde{G}_{n,\ell}(u,v), \qquad (5.86)$$

where we have included explicitly the sum over the mixing index $i = 1,\ldots,n+1$. Now given the result in (5.85), together with the observation (5.76b) that $\Delta^{(8)}$ on a superblock gives the numerator of the anomalous dimensions, we can reduce all the sums in (5.86) to a simpler

problem

$$
\mathcal{H}^\kappa |_{\log^\kappa u} = \left[\Delta^{(8)} \right]^{\kappa-1} \sum_{\substack{\ell=0 \\ \ell \text{ even}}} \frac{1}{(\ell+1)_6^{\kappa-1}} \frac{\tilde{a}_\ell^{(0)} \tilde{\gamma}_\ell^{(1)}}{(-2)^\kappa \kappa!} g_{\ell,\ell+8}^{(d=10)}(z,\bar{z})
$$

$$
= \left[\Delta^{(8)} \right]^{\kappa-1} \cdot \left(\mathcal{D}_z^{(3)} h^{(\kappa)}(z) + z \leftrightarrow \bar{z} \right) .
$$
(5.87)

In Paper I and II we have thoroughly analysed the function

$$
h^{(\kappa)} = \frac{1}{(-2)^\kappa \kappa!} \sum_{\substack{\ell=0 \\ \ell \text{ even}}} \frac{1}{(\ell+1)_6^{\kappa-1}} \frac{960 \Gamma(\ell+1) \Gamma(\ell+4)}{\Gamma(2\ell+7)} \times
$$

$$
z^{\ell+1} {}_2F_1(\ell+1, \ell+4; 2\ell+8; z) .
$$
(5.88)

By studying explicit examples, we were able to derive the general structure[9]

$$
h^{(\kappa)}(z) = j_0(z) + j_1(z) H_1(z) + \left(j_2(z) K_2(z) + j_2 \left(\frac{z}{z-1} \right) I_2(z) \right)
$$

$$
+ \left(j_3 \left(\frac{z}{z-1} \right) K_3(z) + j_3(z) I_3(z) \right) + \cdots
$$

$$
+ \begin{cases} \left(j_\kappa(z) K_\kappa(z) + j_\kappa \left(\frac{z}{z-1} \right) I_\kappa(z) \right) & \text{for } \kappa \text{ even} \\ \left(j_\kappa \left(\frac{z}{z-1} \right) K_\kappa(z) + j_\kappa(z) I_\kappa(z) \right) & \text{for } \kappa \text{ odd} \end{cases} ,
$$
(5.89)

where j_i are generic function, I_n and K_n are iterative integrals defined as

$$
I_n(z) = \int_0^z \frac{dz'}{z'(1-z')^{(n-1)\bmod 2}} I_{n-1}(z'), \quad I_1(z) \equiv H_1(z),
$$

$$
K_n(z) = \int_0^z \frac{dz'}{z'(1-z')^{n\bmod 2}} K_{n-1}(z'), \qquad K_1(z) \equiv H_1(z).
$$
(5.90)

We have denoted with $H_{...}(x)$ the Harmonic Polylogarithms (HPL) [120–122]. Notice that from this general expression, we can directly infer that at every κ^{th}-order, the leading logarithmic term $\mathcal{H}^{(\kappa)} |_{\log^\kappa u}$ has at most a $\log^2 v$. For the leading (L) and next-to-leading (NL) transcendental pieces we also found an explicit form for the generic polynomials $j_n(z)$[10]

---

[9]Recently, it has been shown a connection between this form and a particular class of four-dimensional loop integrals dubbed *Zigzag integrals* [119].

[10]We checked that the expression for the highest transcendental polynomial agreed with the results in [112].

$$j_\kappa^{\mathrm{L}}(z) = \frac{4^{3-2\kappa}15}{(-15)^\kappa \kappa!}\left(\frac{10^\kappa - 6\cdot 5^\kappa + 50}{z^3} + \frac{3}{z^4}\left(5^\kappa - 25\right) + \frac{30}{z^5}\right), \quad (5.91\mathrm{a})$$

$$j_{\kappa-1}^{\mathrm{NL}}(z) = \frac{4^{2-2\kappa}5}{(-15)^\kappa \kappa!}\left\{\frac{12}{z}\left(3\cdot 5^\kappa - 10^\kappa - 5\right) - \frac{3}{z^2}\left(18\cdot 5^\kappa + 10 - 10^{\kappa+1}\right)\right.$$

$$+ \frac{1}{z^3}\left[6\left(17\cdot 5^\kappa + 10^\kappa + 265\right) - 2\kappa\left(39\cdot 5^\kappa + 2\cdot 10^\kappa + 685\right)\right]$$

$$\left.+ \frac{3}{z^4}\left[\left(13\cdot 5^\kappa + 685\right)\kappa - 20\left(5^\kappa + 43\right)\right] + \frac{6}{z^5}\left(184 - 137\kappa\right)\right\}.$$

$$(5.91\mathrm{b})$$

For concreteness let us report the results for the first few $h^{(\kappa)}(z)$

$$h^{(1)} = j_1^{\mathrm{L}}(z)H_1 + \frac{10\left(z^3 + 4z^2 - 18z + 12\right)}{z^3}, \quad (5.92\mathrm{a})$$

$$h^{(2)} = j_2^{\mathrm{L}}(z)H_{01} + j_2^{\mathrm{L}}\left(\frac{z}{z-1}\right)(H_{01} + H_{11}) + j_2^{\mathrm{NL}}(z)H_1$$

$$+ j_2^{\mathrm{NL}}\left(\frac{z}{z-1}\right)H_1 + \frac{(z-2)}{z^3}\left(\frac{235}{576}z^2 - z + 1\right), \quad (5.92\mathrm{b})$$

$$h^{(3)} = j_3^{\mathrm{L}}\left(\frac{z}{z-1}\right)(H_{001} + H_{101}) + j_3^{\mathrm{L}}(z)(H_{001} + H_{011}) + j_3^{\mathrm{NL}}(z)H_{01}$$

$$+ j_3^{\mathrm{NL}}\left(\frac{z}{z-1}\right)(H_{01} + H_{11}) + \frac{(z-1)\left(1258z^2 - 2903(z-1)\right)}{172800z^4}H_1$$

$$- \frac{(z-2)\left(217855z^2 - 249714(z-1)\right)}{12441600z^3}. \quad (5.92\mathrm{c})$$

where the HPLs are evaluated in z.

5.5 From one loop to all loops

Let us go back to the main discussion, namely constructing or at least constraining the correlator of four $\mathcal{O}_2$'s at higher orders in $\frac{1}{c}$. We have seen that dDisc completely fixes the tree-level result, so it makes sense to ask if this generalises to higher loops. At order $\frac{1}{c^2}$, the only contribution to the four-point function comes from the non-protected sector $\mathcal{H}^{(2)}$, which admits the expansion in (5.60). From this explicit form, it is evident that the only term contributing to dDisc is the leading logarithmic term. In fact under crossing, the $\log^2 u$ gets mapped to $\log^2(1 - \bar{z})$, which is the only term with non-vanishing double discontinuity. From the discussion above, we know that this depends only on the anomalous dimensions we have unmixed. Therefore we can completely determine it

from (5.87)

$$\text{dDisc}\left[\mathcal{H}^{(2)}(z,\bar{z})\right] = 4\pi^2 \frac{u^2}{v^2}\left[\Delta^{(8)}\left(\mathcal{D}_z^{(3)} h^{(2)}(z) + z \leftrightarrow \bar{z}\right)\right]_{u \leftrightarrow v},$$
$$= \mathcal{D}_z(\mathcal{D}_z - 2)\mathcal{D}_{\bar{z}}(\mathcal{D}_{\bar{z}} - 2)\mathcal{H}^{(2)'}(z,\bar{z}). \tag{5.93}$$

The precise form of $\mathcal{H}^{(2)'}(z,\bar{z})$ will not be relevant for our discussion and can be found in [34, Appendix A]. The only important observation we want to make is that its functional form precisely produces the pole structure we expect in $c(\Delta,\ell)$ at one loop (5.62).
Schematically

$$\mathcal{H}^{(2)'} \sim r_1(z,\bar{z})\log^2 z + r_2(z,\bar{z})\log z + r_3(z,\bar{z})$$

<div align="center">

↑   ↑   ↑

triple poles   double poles   simple poles

↓   ↓   ↓

$a^{(0)}(\gamma^{(1)})^2$ $a^{(0)}\gamma^{(2)} + a^{(1)}\gamma^{(1)}$ $a^{(2)}$

</div>

To summarise, at one loop the double discontinuity is completely determined by the leading logarithmic term in $\mathcal{H}^{(2)}$, given by (5.87). When we plug its expression in the inversion integral (5.63), the pole structure of the result encodes the new OPE data at one loop, namely $a_{n,\ell}^{(2)}$ and $\gamma_{n,\ell}^{(2)}$ — see also [33,35]. In other words, the tree-level data $a_{n,\ell}^{(0)}$ and $\gamma_{n,\ell}^{(1)}$, once unmixed, completely fix the correlator at one loop. Pictorially, this can be seen in Fig. 5.2.

For the moment, we have just sketched how to get the OPE data at order $\frac{1}{c^2}$. Finding the explicit expression for the full $\mathcal{H}^{(2)}$ requires further non-trivial steps. One can think of constructing it directly from dDisc
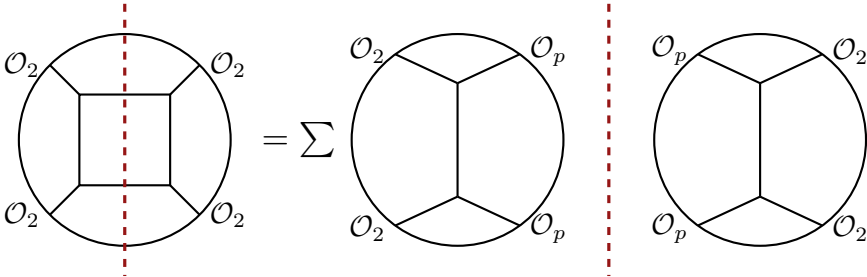

Figure 5.2. Pictorial representation of the OPE needed to reconstruct the one-loop result. This picture gives also an intuition of why we need to consider correlators with different external dimensions to solve the mixing and in order to take the proper "square" of the anomalous dimensions.

through a dispersive integral with the methods in [62]. Alternatively, once fixed the $\log^2 u$ term, one can complete it to a full crossing symmetric function as in [35]. The result can be written in a very suggestive way [123]

$$\mathcal{H}^{(2)} = \frac{1}{u^2} \Delta^{(8)} \mathcal{L}^{(2)} + \mathcal{H}^{(1)} \,, \tag{5.94}$$

where $\mathcal{L}^{(2)}$ is an easier function, dubbed "pre-amplitude".

As we have tried to emphasise in Fig. 5.1 and 5.2, dDisc allows us to fix the tree-level correlator from the protected single-trace exchanges and one loop from tree-level data. This generalises to higher orders as well: in general, the κth correlator can be constructed from the information at $(\kappa - 1)$-loop. This shares lots of similarities with unitarity methods and reconstructions of amplitude through dispersion relations appearing in the flat space S-matrix theory. Pushing the analogy, this suggests to interpret the dDisc of a correlator similarly to the discontinuity of an amplitude. But CFT correlators are dual to AdS amplitudes, therefore one can think of dDisc and the Lorentzian inversion formula as the right way of formulating unitarity in AdS [108, 124, 125]. The analogy can be made more precise and concrete by studying the results, we have obtained so far, in the flat space limit and compare them with flat-space $10d$ supergravity amplitudes. Starting from the CFT correlator in position space, flat space physics is probed considering the *bulk-point limit* [6, 126, 127]. Consider sufficiently localized wave-packets in AdS, such that they focus on a bulk point, whose neighbourhood can be approximated as flat space. Then one can show that, in this configuration, bulk S-matrix elements are reproduced by correlators in the boundary CFT with a prescribed singular behaviour as z approaches $\bar{z}$. Let us stress that this type of singularity makes sense only in Lorentzian kinematics, where z and $\bar{z}$ are real, independent variables. Therefore an analytic continuation of the Euclidean correlator to imaginary times is necessary. The existence and the behaviour of this singularity is the real hallmark of the existence of a local gravity dual. In [34], the bulk-point limit is performed directly on the inversion integral. This allowed the authors to find a precise connection between the large n limit of a specific quantity in the CFT, where n labels the double-trace spectrum, and the $5d$ partial wave coefficients $b_\ell(s)$ of the dual amplitude

$$\lim_{n \to \infty} \frac{\langle a\, e^{-i\pi\gamma} \rangle_{n,\ell}}{\langle a^{(0)} \rangle_{n,\ell}} = b_\ell(s)\,, \quad R\sqrt{s} = 2n\,. \tag{5.95}$$

Without entering into details, this relation translates directly to a statement relating the flat space limit of dDisc and the discontinuity of the graviton amplitude in $10d$ flat space

$$\lim_{x \to 0} \text{dDisc} \left[\mathcal{H}^{(\kappa)}(z^{\circlearrowright}, \bar{z}) \right] \leftrightarrow \text{disc}_s \left[\mathcal{A}^{10d}_\kappa(s, \cos\theta) \right] \,. \tag{5.96}$$

66

Let us unpack this formula. On the LHS, the bulk-point limit is reached by taking the analytic continuation $z^{\circlearrowleft}$ ($z \to z e^{-2\pi i}$) and the limit $z \to \bar{z}$ is parametrized as $z = \bar{z} + 2x\bar{z}\sqrt{1-\bar{z}}$. Once we have taken the limit, the final CFT object we obtain is a function of $\bar{z}$ only. This can now be interpreted from the amplitude perspective as a function of the Mandelstam invariants $\frac{1}{\bar{z}} = 1 + \frac{t}{s} = \frac{1+\cos\theta}{2}$, where θ is the center of mass angle of the $2 \to 2$ scattering. $\mathcal{A}_\kappa^{10d}$ is the flat space amplitude of gravitons, in a specific polarization configuration [111], in type IIB supergravity at order $(8\pi G_N)^\kappa$. Recall that the Newton constant is related to the central charge through $8\pi G_N = \frac{\pi^5 R^8}{c}$. To extract the $5d$ partial wave coefficient $b_\ell(s)$, one should simply divide $\mathcal{A}_\kappa^{10d}$ by the S^5 volume. The relation (5.96) has been explicitly checked at one loop in [34]. In Paper II, we explored the consequences of this connection at all loops for the leading logarithmic term in (5.86). What we found, is that this term is related no longer to a single discontinuity of the amplitude, but rather to a multiple s-channel cut. Pictorially, indicating with a dashed red line a cut,

$$\sum_{n,\ell,i} u^{n+2} a_{n,\ell,i}^{(0)} (\gamma_{n,\ell,i}^{(1)})^\kappa \tilde{G}_{n,\ell} \xleftarrow[\text{limit}]{\text{bulk-point}} \quad \text{(5.97)}$$

The κ-rungs ladder diagram is the only contribution to the amplitude, at $(\kappa - 1)$-loop, admitting this type of multiple discontinuities. At one loop, in particular, this is the sole diagram appearing, hence the perfect agreement between dDisc and the full discontinuity on the gravity side. At higher loops, more topologies contribute to the amplitude, as we will see in more detail below in the two-loop example. Proving the relation (5.97) at all loops is possible because (5.87) greatly simplifies in the bulk-point limit — see Paper II. Since we are interested in the leading singularity as $z \to \bar{z}$, we were able to show that

$$\frac{\text{dDisc}\left[(\bar{z} - z)z\bar{z}\mathcal{H}^{(\kappa)}|_{\log^\kappa u}\right]}{4\pi^2} \xrightarrow[\text{limit}]{\text{bulk-pt}} \frac{\Gamma(8\kappa - 2)}{(2x)^{8\kappa - 2}} \log^{\kappa-2}(1 - \bar{z})$$

$$\underbrace{\frac{(1 - \bar{z})^{4\kappa-1}}{240\bar{z}^{4\kappa-4}} \kappa(\kappa - 1) \left[h^{(\kappa)}(1 - z^{\circlearrowleft}) - h^{(\kappa)}(1 - \bar{z})\right]}_{g^{(\kappa)}(\bar{z})}\Bigg|_{z=\bar{z}} \quad \text{(5.98)}$$

where we have used (3.15b) and we restricted to the leading log coming from the double discontinuity. The powers of x are in one-to-one correspondence to powers of n and the correct ones to give the right scaling dimension for the amplitudes — remember that $n \sim \sqrt{s}$ through (5.95). It is exactly the function $g^{(\kappa)}(\bar{z})$ that has to be matched with the multiple

discontinuity

$$g^{(\kappa)}(\bar{z}) \propto \quad \text{(diagram)} \quad + \text{ crossed}$$

$$\propto \int \prod_{i=1}^{\kappa-1} \mathrm{d}^{10}k_i \frac{\delta^{(+)}(q_1^2)\delta^{(+)}(q_2^2)\cdots\delta^{(+)}(q_{2\kappa-2}^2)}{q_{2\kappa-1}^2 \cdots q_{3\kappa-2}^2} \equiv \mathcal{R}_\kappa(s,t)\,,$$

(5.99)

where a cut propagator is put on-shell by substituting it with a one-dimensional delta function $\delta^{(+)}(p^2)$ and q_i^μ is a specific combination of external and loop momenta k_i^μ. The integral in (5.99) can be performed explicitly in a recursive way as

$$\mathcal{R}_\kappa(s,t) = \int_0^{\frac{t+s}{s}} \mathrm{d}v\, \mathcal{R}_{\kappa-1}(s,-s(1-v))\mathcal{K}^+(v,s,t)$$

$$+ \int_{\frac{t+s}{s}}^1 \mathrm{d}v\, \mathcal{R}_{\kappa-1}(s,-s(1-v))\mathcal{K}^-(v,s,t)\,,$$

(5.100)

where $\mathcal{K}^\pm(v,s,t) = \mathcal{K}(v,s,t)_{|x|=\pm x}$ is a kernel defined as

$$\mathcal{K} = \frac{\pi t^{-3}}{(s+t)^3}\Big\{ s(s+t-sv)^4 \left|\frac{t}{s} - v + 1\right| - (s+t)^5 + 5sv(s+t)^4$$

$$- 5v^4(s+t)\left(s^4 + 8st^3 + 6t^4\right) + 10v^3(s+t)^2\left(s^3 + 2t^3\right) \quad \text{(5.101)}$$

$$- 10s^2v^2(s+t)^3 + v^5\left(20s^2t^3 + s^5 + 30st^4 + 12t^5\right) \Big\}\,,$$

where the proportionality factor is just some numbers times power of s, which can be reintroduced by dimensional analysis. Notice that in establishing the connection in (5.97), we did not worry about UV divergences. This is motivated by the fact that the multiple cuts are not divergent. Whereas if one were to consider the full amplitude, one should take into account this issue and introduce appropriate counterterms. See [34] and [119] for an exhaustive discussion of this problem respectively at one and two loops.

What it is remarkable about this discussion, is that it is possible to establish a one-to-one correspondence between the leading logarithmic terms of the correlator, which can be isolated through multiple dDisc, and a specific object defined on the dual amplitude. Unfortunately, the information obtained in this way, it is not enough to fully fix the correlator in the same way as it is known that multiple discontinuities can not reconstruct the full amplitude. On the CFT side, this impossibility to get the complete answer is rooted in two interconnected facts. First of all, at $\frac{1}{c^\kappa}$, all the OPE data up to order $(\kappa - 1)$ enter dDisc, so $a^{(0)}$ and $\gamma^{(1)}$ are not enough any more. Moreover, starting at order $\frac{1}{c^3}$, these

OPE data start to depend not only on double-trace operators but on more generic multi-trace ones.

Let us explain this last point in a concrete example, the two-loop case described in Paper I. We know that the knowledge of the leading $\log u$ term, as in (5.87), does not exhaust dDisc. At order $\frac{1}{c^3}$, in fact, also $\log^2 u$ contribute. This piece will depend on one-loop OPE data of double-trace operators, whose complete unmixed form is unfortunately not known. But this is not enough: one should also consider the appearance of triple-trace operators. In our work, we argued for their presence by studying the discontinuity of the dual flat space amplitude.[11] Let us go back to the original relation in (5.96). By virtue of this, we argue that the information needed to construct the two-loop correlator is the same needed to determine the two-loop supergravity amplitude in ten dimensions. At two loops this is given by [128]

$$\frac{\mathcal{A}_3^{10d}}{(8\pi G_N)^3 s^4} = s^2 \left\{ \quad + \quad \right. \tag{5.102}$$

$$\left. + \quad + \quad \right\} + \text{crossed},$$

and in our work we have explicitly computed the finite part of both the planar — first line — and non-planar — second line — double box. The s-discontinuity of this amplitude can be evaluated, for example, employing Cutkosky rules [129]. The discontinuity in a given channel is given by the sum over all cuts where the corresponding moment is flowing. In our example,

$$\text{disc}_s \mathcal{A}_3^{10d} \sim \quad + \quad + \quad + \tag{5.103}$$

$$+ \quad + \text{crossed}.$$

This picture gives an intuition of how triple-trace operators enter at two loops. The first two and the last types of diagrams, where two

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[11]Other diagrammatic arguments in AdS can be found in [108]. Their presence is also subsidised by the results in [119], where the full two-loop correlator is computed and analysed.

propagators are put on-shell, represent two-particle cuts. These can be interpreted, from the CFT perspective, as the information related to double-trace operators. Notice that, in these cases, a single cut divides a diagram in a product of a tree-level times a one-loop diagram. This agrees with the CFT expectation that one-loop double-trace OPE data should enter dDisc at order $\frac{1}{c^3}$. The remaining diagrams, instead, contain a three-particle cut. We interpret it as a sign of the appearance of triple-trace operators in the CFT four-point function.

This example makes it clear the need for information about multi-trace operators in order to bootstrap four-point functions of half-BPS operators beyond one loop. Moving away from four-point functions and allowing for more external operators, these operators will also start to appear in specific kinematic limits of five- or higher-point correlators. The need to find new ways to constrain the spectrum of this class of operators motivated the work in Paper IV. There, we focused on correlators of the quarter-BPS operator $\mathcal{O}_{02}$. As one can see from its definition in (4.9), this operator is genuinely double-trace at large N. As a consequence, by studying its OPE, one can try to access information about higher-trace data. In the following section, we will discuss our analysis in more detail.

6. Quarter-BPS operators

In Paper IV, we started a systematic study of these quarter-BPS operators. We developed a technology to deal with the increasing number of R-symmetry tensor structures and we found a way of employing the underlying chiral algebra to solve the superconformal Ward Identities (SWI) in the case of less supersymmetric objects. Apart from the proliferation of the $SU(4)_R$ representations, one of the big challenges in dealing with quarter-BPS operators is the absence of known superconformal blocks. In the case of four-point functions of half-BPS operators, instead, superconformal blocks have been computed and their knowledge allows us to properly identify superprimary and repack the contribution of the whole supermultiplet in just one object. This made it possible to solve the SWI and turned out in the nice decomposition in (5.8) and it allowed finding the exact form of $\mathcal{H}^{\text{short}}$. As we will soon see, the situation for correlators involving quarter-BPS operators is going to be more involved and will eventually leave us with a few unfixed ambiguities. Nonetheless, we will be able to almost completely fix the protected part in some concrete examples of correlators involving the easiest quarter-BPS operators, $\mathcal{O}_{02}$ in (4.9). Then, following similar reasoning as in Sec. 5.3.1, in Paper IV we used the Lorentzian inversion formula to extract the leading order OPE data in the large N supergravity.

6.1 How to use the chiral algebra

As anticipated before, we will use the chiral algebra of [48,96] to solve the SWI satisfied by the four-point functions. The chiral algebra describes the protected subsector of a $\mathcal{N} = 2$ SCFT in four dimensions. Thus, to apply this description to the $\mathcal{N} = 4$ case, we have, first of all, to decompose our operators in $\mathcal{N} = 2$ submultiplets. In particular, the R-symmetry breaking is realized as

$$SU(4) \rightarrow SU(2)_R \times U(1)_r \times SU(2)_F \,. \qquad (6.1)$$

When we apply this decomposition to $\mathcal{O}_{pq}$, we see scalar operators with $U(1)_r$ charge $r = 0$, $SU(2)_R$ charge $2p + q$ and flavour charge p. These are rather special operators, dubbed *Schur operators*, and they belong to $\mathcal{N} = 2$ half-BPS multiplets. Their presence allows us to exploit the chiral algebra for solving the WIs.

We refer to the original references [48, 130, 131] for exhaustive explanations of what the chiral algebra is and how it appears in SCFTs. Here we simply state the key facts necessary to constrain the four-point correlators under consideration.

Let us call χ the chiral algebra map, such that when applied to a four-point function of quarter-BPS operators

$$\chi\left[\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle\right] = \mathfrak{K}\mathfrak{f}(z, \tilde{\eta})\,, \tag{6.2}$$

where $\mathfrak{K}$ is a kinematic prefactor and $\mathfrak{f}$ is a holomorphic function of z and depends on the $\mathcal{N} = 2$ polarization $\tilde{\eta}$.[1] Importantly $\mathfrak{f}$ is protected and thus is completely determined by free theory computation. In the half-BPS case, it is related to $\hat{f}(z)$.

Now recall that a generic four-point function can be expanded in the R-symmetry tensor structures as

$$\mathcal{G}(z, \bar{z}, \mathbf{S}_1, \ldots, \mathbf{S}_4) = \sum_{k=1}^{N_{\mathrm{str}}} \mathbb{T}_k(\mathbf{S}_1, \ldots, \mathbf{S}_4)\mathcal{G}_k(z, \bar{z})\,. \tag{6.3}$$

Using the chiral algebra map, we can split each $\mathcal{G}_k$ in a protected part and an non-protected piece, $\mathcal{H}_m$, as follows

$$\mathcal{G}_k(z, \bar{z}) = w_k(z, \bar{z}) + \sum_{m=1}^{N_{\mathrm{u}}} \mathcal{H}_m(z, \bar{z})\, v_k^{(m)}(z, \bar{z})\,,$$

$$\chi\left[\sum_{k=1}^{N_{\mathrm{str}}} \mathbb{T}_k w_k\right] = \mathfrak{f}(z, \tilde{\eta})\,, \tag{6.4}$$

$$\chi\left[\sum_{k=1}^{N_{\mathrm{str}}} \mathbb{T}_k v_k^{(m)}\right] = 0\,, \qquad \text{for } m = 1, \ldots, \dim(\ker \chi) \equiv N_{\mathrm{u}}\,.$$

From the definition, this decomposition naturally contains an ambiguity

$$w_k(z, \bar{z}) \sim w_k(z, \bar{z}) + \sum_{m=1}^{N_{\mathrm{u}}} \mathcal{A}_m(z, \bar{z})\, v_k^{(m)}(z, \bar{z})\,. \tag{6.5}$$

In our work, we tried to fix $\mathcal{A}_m$ as much as possible based on some reasonable assumptions. We tested this procedure on $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ and then we applied it to $\langle \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_2 \mathcal{O}_2 \rangle$ and to the four $\mathcal{O}_{02}$'s correlator. Let us briefly outline our strategy. First of all we determine $\mathfrak{f}$ starting from the free-theory results, computed by means of Wick contractions.[2] With

---

[1] We contract all R-symmetry indices with a complex two-vector η^a and all flavour indices with another two-vector $\tilde{\eta}^{a'}$. Indices are raised and lowered with the Levi-Civita tensors ϵ_{ab} and $\epsilon_{a'b'}$.

[2] Alternatively, $\mathfrak{f}$ can be computed directly in the $2d$ theory.

that, we can specify the functions $w_k(z, \bar{z})$ up to the ambiguities $\mathcal{A}_m$. To fix these, we impose a series of constraints. Firstly, we require the identity and $\mathcal{O}_2$ to be exchanged in the OPE with the right coefficients. Then we make sure that the disconnected contribution is the same as the one in the free theory. Finally, the strongest condition comes from enforcing that the correlator cannot exchange twist-two operators with spin higher than two, for the same reasons explained in Sec. 5.3, We showed that for the four-point function of $\mathcal{O}_2$, this procedure is equivalent to the one described in Sec. 5.3, modulo identifying $\mathcal{A} = u^2 \mathcal{H}^{\text{short}}$. One of the main differences, is that in this new approach, we did not need the form of the superconformal blocks.

6.2 $\langle \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_2 \mathcal{O}_2 \rangle$

As an explanatory example, let us report a few details about the easiest correlator we have considered[3]

$$
\langle \mathcal{O}_{02}(x_1, \mathbf{S}_1) \mathcal{O}_{02}(x_2, \mathbf{S}_2) \mathcal{O}_2(x_3, y_3) \mathcal{O}_2(x_4, y_4) \rangle = \mathcal{K}_{4422} \sum_{k=1}^{10} \mathbb{T}_k \, \mathcal{G}_k(z, \bar{z}) \,,
\tag{6.6}
$$

where the precise form of the R-symmetry tensors can be found in (4.14) of Paper IV.

In this case, the chiral algebra map has dimension 8, so we have 8 possible ambiguities $\mathcal{A}_m$. By imposing the requirement explained above, we were able to fix them to

$$
\mathcal{A}_5 = \mathcal{A}_6 = \frac{\mathcal{A}_4}{\lambda} = \frac{N^2 - 4}{(N^2 - 1)(\kappa - 2)} \tilde{\mathcal{A}}_5 - \frac{2(N^2 - 3)}{(\kappa - 2)(N^2 - 1)} g_{4,0} \,, \quad \text{(6.7a)}
$$

$$
\mathcal{A}_7 = -\frac{40}{N^2 - 1} g_{2,0} \,,
\tag{6.7b}
$$

where the constant λ parametrizes our inability to fix the twist-four contributions and the other $\mathcal{A}_m$ are left unconstrained by our arguments. The precise form of $\tilde{\mathcal{A}}_5$ is given by

$$
\tilde{\mathcal{A}}_5(z, \bar{z}) = a(z, \bar{z}) \log(1 - \bar{z}) - 12 \log(1 - z) \log(1 - \bar{z}) + z \leftrightarrow \bar{z} \,, \quad \text{(6.8)}
$$

with

$$
a(z, \bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \left(\hat{a}(z) + \hat{a}\left(\frac{z}{z - 1} \right) \right) \,, \qquad \hat{a}(z) = 12 + z^2 \,.
\tag{6.9}
$$

For the crossed channel, namely $\langle \mathcal{O}_2 \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_2 \rangle$, instead, we have not enough constraints to find a specific form for the ambiguities.

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[3]The correlator $\langle \mathcal{O}_{02} \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ is protected since it is "next to extremal" [90, 132, 133].

Even if the information that we have is only partial, we can play the game described in Sec. 5.3.1, plug the results in the inversion formula and obtain the OPE data in $\mathcal{O}_2 \times \mathcal{O}_{02}$ by looking at $c_k(\Delta, \ell)$ of $\langle \mathcal{O}_2 \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_2 \rangle$. By the $(1 \leftrightarrow 2)$ crossing, required by the inversion integral (3.14), the ambiguities (6.7), will enter. Remarkably, only the identity and twist-two terms in there contribute, so that the final result does not depend on the unconstrained constant λ. Similarly as described in the all half-BPS case, we were able to compute the anomalous dimensions by isolating the double-poles in $c_k(\Delta, \ell)$. Among them, one $c_k(\Delta, \ell)$ caught our attention for its simplicity, the one corresponding to the $[1, 2, 1]$ representation. This observation made us speculate, that perhaps the $[1, 2, 1]$ representation plays for this correlator the same role as the $[0, 4, 0]$ in $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$, as explained around (5.10), and it can be expanded in superconformal blocks, which are written as a *single* conformal block, possibly with shifted quantum numbers.[4]

Besides very concrete examples, in Paper IV we have tried in general to revive the study of quarter-BPS operators in $\mathcal{N} = 4$ SYM and we have set the ground for a study of their correlators through analytic bootstrap techniques. Further improvements would benefit a lot from an explicit form of the superblocks and from supplementary explicit computations coming perhaps from higher-point functions. For instance, if the correlator of six $\mathcal{O}_2$'s were known, one would be able to extract $\langle \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_2 \mathcal{O}_2 \rangle$ simply by taking two coincident limits of the six-point function.

A parallel line of attack can come from combining position space results with the study of AdS Mellin amplitude of the dual operators. As we have seen, quarter-BPS operators are in fact dual to threshold bound states in AdS. A characterization of multi-particle scattering has been initiated in [134]. Here the authors find a very peculiar behaviour of the tree-level Mellin amplitudes suggesting to enlarge the basis of functions, away from the usual $\bar{D}$, for the position space four-point functions involving multi-trace operators — similar evidence has been seen in a different context in [135].

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[4]Similar reasoning applies to the $[4, 0, 4]$ in $\langle \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_{02} \mathcal{O}_{02} \rangle$.

Part III:
Gauge theories in AdS and Supergluons

In this concluding part, we abandon maximal supersymmetry and we explore holographic theories preserving only eight supercharges. These four-dimensional $\mathcal{N} = 2$ SCFTs have been recently at the center of investigation for their connection with gauge theories in AdS. Correlators of four scalar *supergluons*, gauge analogue of the supergravitons encountered before, have been computed at tree-level [38], one [136] and two loops [137] and also their five-point function has been determined in [138]. Moreover, in Mellin space the four-point function has been shown to enjoy color-kinematics duality [38,115] and a double copy construction [139], AdS versions of the well-known properties of gauge and gravity amplitudes in flat space [140,141]. In this Part, based on Paper V, we will further explore the existence of an AdS double copy in position space for four-point functions of spinning operators in the $\mathcal{N} = 2$ flavour current multiplet and we will study their connection to the $\mathcal{N} = 4$ SYM counterparts in Part II. To derive the correlator of currents, in Ch. 7 we introduce analytic and harmonic $\mathcal{N} = 2$ superspaces. In this setup, we will determine the correlator of four superfields by revisiting the results of [101, 102], derived for $\mathcal{N} = 4$ SYM. In Ch. 8, we consider the supergluon four-point function in the holographic limit and we will provide evidence for the existence of an AdS double copy relating correlator of currents in $\mathcal{N} = 2$ and stress-tensors in $\mathcal{N} = 4$ SYM. Although computed in supersymmetric theories, these correlators coincide with their version in bosonic Yang-Mills and Einstein gravity theories. Thus, finding a duality for these spinning components would exactly reproduce the original works in flat space.

7. Flavour current multiplet in $4d$ $\mathcal{N} = 2$ SCFTs

In four dimensions $\mathcal{N} = 2$ superconformal field theories preserve half of the maximum allowed number of supersymmetry. As we have seen in Sec. 2.1, their symmetry algebra coincides with $\mathfrak{su}(2,2|2)$, where the R-symmetry subalgebra can be further split in

$$\mathfrak{u}(2)_R \simeq \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r \, . \tag{7.1}$$

Shortening conditions and unitarity representations have been studied and classified in [56,57,142]. In this thesis, we will focus our attention on a specific multiplet, the one containing the conserved current $\mathcal{J}_\mu$ of some global flavour group G_F. This corresponds to the half-BPS multiplet depicted in Fig. 2.1. As expected, the current is the top component, it transforms as a vector and it is exactly at the unitarity bound (2.8) with protected dimension $\Delta = 3$. The corresponding superprimary is a scalar of protected dimension $\Delta = 2$, neutral under $\mathrm{U}(1)_r$ and in the adjoint of $\mathrm{SU}(2)_R$. For the reasons we will explain in Ch. 8, we refer to it as a *supergluon* and we will denote it[1,2]

$$\mathcal{O}_2^I(x,\xi) = \xi_{a_1}\xi_{a_2}\, \mathcal{O}_2^{I;a_1 a_2}(x) \, . \tag{7.2}$$

The index $I = 1, ..., \dim(G_F)$ is the color index in the adjoint representation of the flavour group and we have contracted the a_i $\mathrm{SU}(2)_R$ indices with auxiliary (commuting) polarization spinors ξ_{a_i}, implementing the correct symmetrization properties.

Retracing similar steps as in our discussion for $\mathcal{N} = 4$ SYM, in this chapter we will study the four-point function of $\mathcal{O}_2^I$ just imposing the constraints of superconformal invariance. Then we will proceed by lifting this scalar correlator to its counterpart in superspace and we will see how to extract the different components. Remarkably, in this way we will get an explicit expression for the correlator of four flavour currents.

---

[1]We use the same notation $\mathcal{O}_2$ both for the current superprimary in $\mathcal{N} = 2$ and for the stress-tensor superprimary in $\mathcal{N} = 4$ SYM. Whenever it is not clear from the context what we refer to, we will add a superscript gl or gr respectively for $\mathcal{N} = 2$ and $\mathcal{N} = 4$.

[2]In the literature it is also sometimes called the moment map operator [143].

7.1 Scalar four-point function

The correlator of four $\mathcal{O}_2^I$'s can be written as

$$\langle \mathcal{O}_2^{I_1}(x_1, \xi_1) \cdots \mathcal{O}_2^{I_4}(x_4, \xi_4) \rangle = \frac{\xi_{12}^2 \xi_{34}^2}{(x_{12}^2 x_{34}^2)^2} \mathcal{G}^{I_1 I_2 I_3 I_4}(z, \bar{z}; \alpha), \qquad (7.3)$$

where $\xi_{ij} \equiv \xi_i^a \xi_j^b \epsilon_{ab}$ and we have introduced the $SU(2)_R$ cross-ratio

$$\alpha = \frac{\xi_{13} \xi_{24}}{\xi_{12} \xi_{34}}, \qquad \alpha - 1 = \frac{\xi_{14} \xi_{23}}{\xi_{12} \xi_{34}}. \qquad (7.4)$$

Similarly to the $\mathcal{N} = 4$ case, the four-point function satisfies superconformal Ward identities [81, 106] such that the solution looks like

$$\mathcal{G}^{I_1 \cdots I_4}(z, \bar{z}; \alpha) = \frac{(z\alpha - 1)\bar{z} \, \hat{f}^{I_1 \cdots I_4}(z) - (\bar{z}\alpha - 1)z \, \hat{f}^{I_1 \cdots I_4}(\bar{z})}{z - \bar{z}} \qquad (7.5)$$
$$+ (z\alpha - 1)(\bar{z}\alpha - 1)\mathcal{H}^{I_1 \cdots I_4}(z, \bar{z}).$$

Another alternative decomposition, the one that we will mainly use, reads

$$\mathcal{G}^{I_1 \cdots I_4}(z, \bar{z}; \alpha) = \mathcal{G}_{\text{rational}}^{I_1 \cdots I_4}(z, \bar{z}; \alpha) + (z\alpha - 1)(\bar{z}\alpha - 1)\widetilde{\mathcal{H}}^{I_1 \cdots I_4}(z, \bar{z}). \qquad (7.6)$$

We identify the rational part $\mathcal{G}_{\text{rational}}^{I_1 \cdots I_4}$ with [136]

$$\mathcal{G}_{\text{rational}}^{I_1 \cdots I_4} = \delta^{I_1 I_2} \delta^{I_3 I_4} + \delta^{I_1 I_3} \delta^{I_2 I_4} \alpha^2 u^2 + \delta^{I_1 I_4} \delta^{I_2 I_3} (1 - \alpha)^2 \frac{u^2}{v^2}$$
$$+ \frac{(C_{2,2,2})^2}{3} \frac{\alpha u}{v} \left((c_t - c_s) \frac{(1 - \alpha)}{\alpha} + (c_s - c_u)v + (c_t - c_u)(1 - \alpha)u \right), \qquad (7.7)$$

where we have introduced the $c_{s,t,u}$ color structures in terms of the f^{IJK} flavour structure constants

$$c_s = f^{I_1 I_2 J} f^{J I_3 I_4}, \quad c_t = f^{I_1 I_4 J} f^{J I_2 I_3}, \quad c_u = f^{I_1 I_3 J} f^{J I_4 I_2}, \quad (7.8a)$$
$$c_s + c_t + c_u = 0. \qquad (7.8b)$$

In the expression above, it appears the constant $C_{2,2,2}$, which can be expressed in terms of the flavour central charge

$$C_{2,2,2}^2 = \frac{6}{C_\mathcal{J}}. \qquad (7.9)$$

Relying on the results of [144], we will see how we can lift (7.7) to a full $\mathcal{N} = 2$ superspace expression by mimicking the $\mathcal{N} = 4$ construction of [101, 102] and reviewed in Sec. 5.2.

7.2 Flavour current four-point function in $\mathcal{N} = 2$ superspaces

In this section, we will analyse the half-BPS flavour current multiplet from the perspective of $\mathcal{N} = 2$ harmonic [59, 145] and analytic superspace [144] — see also [60] for a review. In this setup, we will derive the expressions for the supercharges and we will construct the four-point function of current superfields.

7.2.1 $\mathcal{N} = 2$ Superspaces

Let us start with the construction of the superspace. As we have seen in Sec. 2.3, the first step is to enlarge the original space by a set of Grassmann coordinates. In this case, we have eight new spinor variables $\theta_{\alpha a}$ and $\bar{\theta}_{\dot{\alpha}}^a$, where the index a transforms respectively under the fundamental and anti-fundamental representations of $SU(2)_R$. Then we define a superfield in terms of these new coordinates such that when all the θ and $\bar{\theta}$ are set to zero we recover (7.2)

$$
\begin{aligned}
\mathbb{J}(x, \theta_{\alpha a}, \bar{\theta}_{\dot{\alpha}}^a, \xi) &= \xi_{a_1} \xi_{a_2} \mathbb{J}^{a_1 a_2}(x, \theta_{\alpha a}, \bar{\theta}_{\dot{\alpha}}^a) \,, \\
\mathbb{J}^{a_1 a_2}(x, 0, 0) &= \mathcal{O}_2^{a_1 a_2}(x) \,,
\end{aligned}
\tag{7.10}
$$

where we have suppressed the flavour index I to avoid cluttering. Notice that we have included the polarization ξ among the superfield variables. Soon this is going to be interpreted as coordinates of the supercoset as well. In fact, differently from the example done at the end of Sec. 2.3, when we consider superconformal symmetry in the superspace construction it is necessary to allow for an additional compact internal manifold in order to accommodate for the R-symmetry and realize the entire superconformal group as the superspace isometries, as we have seen in (5.20).

To impose the shortening condition we should impose that the covariant derivatives (2.14) act on the superfield such that

$$
\xi_a D_\alpha^a \mathbb{J}(x, \theta_{\alpha c}, \bar{\theta}_{\dot{\alpha}}^c, \xi) = \epsilon^{ba} \xi_a \bar{D}_{\dot{\alpha} b} \mathbb{J}(x, \theta_{\alpha c}, \bar{\theta}_{\dot{\alpha}}^c, \xi) = 0 \,.
\tag{7.11}
$$

Similarly to what we have done for a chiral field in $\mathcal{N} = 1$ theories in Sec. 2.3, we would like to reduce the number and redefine the coordinates in order to satisfy these constraints automatically. For doing that, we need to rotate the original Grassmann coordinates to a new basis

$$
\begin{aligned}
\theta_\alpha^+ &= u_a^+ \theta_\alpha^a \,, & \bar{\theta}_{\dot{\alpha}}^+ &= u^{+a} \bar{\theta}_{\dot{\alpha} a} \,, \\
\theta_\alpha^- &= u^{-a} \theta_{\alpha a} \,, & \bar{\theta}_{\dot{\alpha}}^- &= u_a^- \bar{\theta}_{\dot{\alpha}}^a \,, \\
z^\mu &= x^\mu + i \theta^a \sigma^\mu \bar{\theta}^b (u_a^+ u_b^- + u_b^+ u_a^-) \,.
\end{aligned}
\tag{7.12}
$$

Now we would like the differentials constraints in (7.11) to act just as derivatives with the respect to θ^- and $\bar{\theta}^-$, so that we can just declare

that $\mathbb{J}$ does not depend on those θ's and cut by half the number of Grassmann coordinates. This is achieved by imposing

$$\xi_a D^a \theta^+ = \xi_a \epsilon^{ba} u_a^+ \overset{!}{=} 0\,, \qquad\qquad \Rightarrow \quad u_a^+ = \xi_a\,, \qquad (7.13a)$$

$$\xi_a D^a \theta^- = \xi_a u^{-a} \overset{!}{=} 1\,, \qquad\qquad \Rightarrow \quad u_a^+ u^{-a} = 1 \qquad (7.13b)$$

where spinor indices are understood. These conditions together with the definitions (7.12) guarantee that the analogous barred constraints are also satisfied and that

$$\xi_a D^a z^\mu = 0 = \epsilon^{ba} \xi_a \bar{D}_b z^\mu\,. \qquad (7.14)$$

Notice that if we further demand $(u_a^+)^* = u^{-a}$ as a matching condition, all the constraints can be summarized in a matrix

$$\mathcal{U} = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} \in \mathrm{SU}(2)\,, \qquad \det \mathcal{U} \overset{(7.13b)}{=} 1\,, \qquad (7.15)$$

and this set of coordinates exactly defines the harmonic superspace [59, 145].[3]

We can verify that under this change of coordinates the covariant derivatives become

$$D_\alpha^a = -u^{+a} \frac{\partial}{\partial \theta^{+\alpha}} - 2i\,u^{+a} \sigma_{\alpha\dot\alpha}^\mu \bar{\theta}^{-\dot\alpha} \frac{\partial}{\partial z^\mu} + u^{-a} \frac{\partial}{\partial \theta^{-\alpha}}\,, \qquad (7.16a)$$

$$\bar{D}_{\dot\alpha a} = u_a^+ \frac{\partial}{\partial \bar{\theta}^{+\dot\alpha}} + 2i\,u_a^+ \theta^{-\alpha} \sigma_{\alpha\dot\alpha}^\mu \frac{\partial}{\partial z^\mu} - u_a^- \frac{\partial}{\partial \bar{\theta}^{-\dot\alpha}}\,, \qquad (7.16b)$$

such that

$$\xi_a D_\alpha^a = \frac{\partial}{\partial \theta^{-\alpha}}\,, \qquad \epsilon^{ba} \xi_a \bar{D}_{\dot\alpha b} = \frac{\partial}{\partial \bar{\theta}^{-\dot\alpha}}\,. \qquad (7.17)$$

Finally the shortening conditions (7.11) are trivially realized declaring that the multiplet depends only on the other half coordinates

$$\mathbb{J}(z, \theta^+, \bar{\theta}^+, \xi) \equiv \mathbb{J}(\mathbf{z})\,. \qquad (7.18)$$

An equivalent formulation can be given also in terms of the analytic superspace [144]. One has to complexify the $\mathcal{U}$ matrix and identify the polarizations as

$$\xi_a = u_a^+ = \begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix}\,, \qquad u_a^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \qquad (7.19)$$

With harmonic and analytic coordinates at hand, we can derive the differential representations of the supercharges in terms of that. For this

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[3]Further details and observations supporting this identification can be found in Paper V using the action of the $\mathrm{SU}(2)_R$ generators on the highest weight state.

purpose, we need to determine supersymmetric infinitesimal variations following the procedure described in Sec. 2.3 with the additional complication of including the R-symmetry generators. Details can be found in Paper V, here we simply quote the results in harmonic[4]

$$Q_\alpha^a = -u^{+a}\frac{\partial}{\partial\theta^\alpha} + 2i\,u^{-a}\bar\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial z^\mu}\,, \tag{7.20a}$$

$$\bar Q_{\dot\alpha a} = u_a^+\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i\,u_a^-\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial z^\mu}\,, \tag{7.20b}$$

$$S_a^\alpha = -4u_a^+u_b^-\theta^\alpha\frac{\partial}{\partial u_b^+} + 2u_a^-\theta^2\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} + i\,u_a^+\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$$
$$\qquad - 2u_a^-\theta^\beta\sigma^\mu_{\beta\dot\alpha}\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial z^\mu}\,, \tag{7.20c}$$

$$\bar S^{\dot\alpha a} = 4u^{+a}u_b^-\bar\theta^{\dot\alpha}\frac{\partial}{\partial u_b^+} + i\,u^{+a}\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial\theta^\alpha} + 2u^{-a}\bar\theta^2\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}}$$
$$\qquad + 2u^{-a}\tilde z^{\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial z^\mu}\,, \tag{7.20d}$$

and analytic superspace, where we define

$$Q_\alpha^1 = Q_\alpha^+\,, \qquad Q_\alpha^2 = Q_\alpha^-\,, \qquad \bar Q_{\dot\alpha 1} = \bar Q_{\dot\alpha}^-\,, \qquad \bar Q_{\dot\alpha 2} = -\bar Q_{\dot\alpha}^+\,,$$
$$S_1^\alpha = S^{\alpha-}\,, \qquad S_2^\alpha = -S^{\alpha+}\,, \qquad \bar S^{\dot\alpha 1} = \bar S^{\dot\alpha+}\,, \qquad \bar S^{\dot\alpha 2} = \bar S^{\dot\alpha-}\,, \tag{7.21}$$

$$Q_\alpha^+ = -\tilde y\frac{\partial}{\partial\theta^\alpha} + 2i\sigma_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial z^\mu}\,, \qquad Q_\alpha^- = \frac{\partial}{\partial\theta^\alpha}\,, \tag{7.22a}$$

$$\bar Q_{\dot\alpha}^+ = -\tilde y\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i\theta^\alpha\sigma_{\alpha\dot\alpha}\frac{\partial}{\partial z^\mu}\,, \qquad \bar Q_{\dot\alpha}^- = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\,, \tag{7.22b}$$

$$S^{\alpha+} = 2\theta^2\frac{\partial}{\partial\theta_\alpha} + 4\tilde y\theta^\alpha\frac{\partial}{\partial\tilde y} - i\tilde y\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot\alpha}\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial z^\mu}\,, \tag{7.22c}$$

$$S^{\alpha-} = -4\theta^\alpha\frac{\partial}{\partial\tilde y} + i\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\,, \tag{7.22d}$$

$$\bar S^{\dot\alpha+} = -2\bar\theta^2\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + 4\tilde y\bar\theta^{\dot\alpha}\frac{\partial}{\partial\tilde y} + i\tilde y\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial\theta^\alpha} + 2\tilde z^{\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial z^\mu}\,, \tag{7.22e}$$

$$\bar S^{\dot\alpha-} = -4\bar\theta^{\dot\alpha}\frac{\partial}{\partial\tilde y} - i\tilde z^{\dot\alpha\alpha}\frac{\partial}{\partial\theta^\alpha}\,. \tag{7.22f}$$

---

[4]In our convention $\tilde z^{\dot\alpha\alpha} = \bar\sigma^{\mu\,\dot\alpha\alpha}z_\mu$ and $\theta^2 = \theta^\alpha\theta_\alpha$, $\bar\theta^2 = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$.

7.2.2 Superfield correlators

Let us start defining the "superpropagator": the superspace version of the scalar propagator $\frac{\xi_{ij}}{x_{ij}^2}$

$$g_{ij} \equiv \frac{1}{z_{ij}^2} \left(\xi_{ij} - 4i \frac{\theta_{ij}^{+\alpha}(z_{ij})_{\alpha\dot\alpha}\bar\theta_{ij}^{+\dot\alpha}}{z_{ij}^2} \right) , \qquad (7.23)$$

with $\theta_{ij}^+ = \theta_i^+ - \theta_j^+$, similarly for $\bar\theta_{ij}^+$ and $\xi_{ij} = \tilde y_i - \tilde y_j \equiv \tilde y_{ij}$. As a non-trivial consistency check, we can verify that it behaves covariantly under superconformal transformations

$$
\begin{aligned}
(Q_i^\pm + Q_j^\pm)g_{ij} &= (\bar Q_i^\pm + \bar Q_j^\pm)g_{ij} = 0 , \\
(S_i^- + S_j^-)g_{ij} &= (\bar S_i^- + \bar S_j^-)g_{ij} = 0 , \\
(S_i^{\alpha+} + S_j^{\alpha+})g_{ij} &= -4(\theta_i^\alpha + \theta_j^\alpha)g_{ij} , \\
(\bar S_i^{\dot\alpha+} + \bar S_j^{\dot\alpha+})g_{ij} &= 4(\bar\theta_i^{\dot\alpha} + \bar\theta_j^{\dot\alpha})g_{ij} .
\end{aligned}
\qquad (7.24)
$$

The two-point function of (7.18) can be then simply written in terms of the superpropagator as

$$\langle \mathbb{J}^{I_1}(\mathbf{z}_1) \mathbb{J}^{I_2}(\mathbf{z}_2) \rangle = \delta^{I_1 I_2} g_{12}^2 . \qquad (7.25)$$

It reduces to the scalar correlator of two $\mathcal{O}_2^I$'s when we set all the thetas to zero and can be used to fix the differential operators extracting the various component of the multiplet as depicted in Fig. 2.1.[5] In particular, if we focus on the last line, we can determine the differential operators extracting the R-symmetry current

$$\mathcal{D}_{k,\alpha\dot\alpha} = \frac{1}{4\sqrt{3}} \left(i \frac{\partial}{\partial\theta_k^{+\alpha}} \frac{\partial}{\partial\bar\theta_k^{+\dot\alpha}} + \frac{\partial}{\partial\tilde y_k} \sigma_{\alpha\dot\alpha}^\mu \frac{\partial}{\partial z_k^\mu} \right) , \qquad (7.26)$$

such that

$$
\begin{aligned}
\mathcal{D}_1 \mathcal{D}_2 \langle \mathbb{J}^{I_1}(\mathbf{z}_1) \mathbb{J}^{I_2}(\mathbf{z}_2) \rangle \big|_{\theta_i^+ = \bar\theta_i^+ = 0} &= -\delta^{I_1 I_2} \frac{\eta_1 \mathrm{x}_{12} \bar\eta_2 \, \eta_2 \mathrm{x}_{12} \bar\eta_1}{(x_{12}^2)^4} \\
&= \frac{\pi^2}{C_{\mathcal{J}}} \langle \mathcal{J}^{I_1}(x_1) \mathcal{J}^{I_2}(x_2) \rangle ,
\end{aligned}
\qquad (7.27)
$$

where we have defined $\mathcal{D}_i \equiv \eta_i^\alpha \bar\eta_i^{\dot\alpha} \mathcal{D}_{i,\alpha\dot\alpha}$ and $\mathcal{J}(x_i) \equiv \eta_i^\alpha \bar\eta_i^{\dot\alpha} \mathcal{J}_{\alpha\dot\alpha}(x_i)$, contracting the spinor indices with commuting auxiliary variables. More-

---

[5]We explicitly checked that the operators so extracted are conformal primaries and transform as expected.

over, we can identify the other two scalar top components

$$\mathcal{W}(x) = \frac{1}{8\sqrt{2}} \frac{\partial}{\partial \theta_\alpha^+} \frac{\partial}{\partial \theta^{+\alpha}} \mathbb{J}(\mathbf{z})\big|_{\theta^+=\bar\theta^+=0} \equiv \mathcal{D}^{\mathcal{W}} \mathbb{J}(\mathbf{z}), \qquad (7.28\text{a})$$

$$\overline{\mathcal{W}}(x) = \frac{1}{8\sqrt{2}} \frac{\partial}{\partial \bar\theta^{+\dot\alpha}} \frac{\partial}{\partial \bar\theta_{\dot\alpha}^+} \mathbb{J}(\mathbf{z})\big|_{\theta^+=\bar\theta^+=0} \equiv \mathcal{D}^{\overline{\mathcal{W}}} \mathbb{J}(\mathbf{z}), \qquad (7.28\text{b})$$

since they are charged under $\mathrm{U}(1)_r$, the only non vanishing two-point function is

$$\mathcal{D}_1^{\mathcal{W}} \mathcal{D}_2^{\overline{\mathcal{W}}} \langle \mathbb{J}^{I_1}(\mathbf{z}_1) \mathbb{J}^{I_2}(\mathbf{z}_2) \rangle = \frac{\delta^{I_1 I_2}}{(x_{12}^2)^3} = \langle \mathcal{W}(x_1) \overline{\mathcal{W}}(x_2) \rangle \qquad (7.29)$$

To construct the four-point function of current superfields we will follow closely the procedure adopted for $\mathcal{N} = 4$ SYM and reviewed in Sec. 5.2. For doing that, it is fundamental the assumption, proved in [144], that this correlator depends only on one scalar function, which can be fixed in terms of the superprimaries component. Inspired by the form (7.6), we write

$$\langle \mathbb{J}^{I_1}(\mathbf{z}_1) \cdots \mathbb{J}^{I_4}(\mathbf{z}_4) \rangle = g_{12}^2 g_{34}^2 \left(\mathbb{G}_{\text{rational}}^{I_1 I_2 I_3 I_4} + \mathbb{G}_{\text{anom}}^{I_1 I_2 I_3 I_4} \right),$$

$$\mathbb{G}_{\text{rational}}^{I_1 I_2 I_3 I_4} = \mathcal{G}_{\text{rational}}^{I_1 I_2 I_3 I_4}\Big|_{\frac{\xi_{ij}}{x_{ij}^2} \to g_{ij}}. \qquad (7.30)$$

The interesting piece is represented by the anomalous part, for which we propose an ansatz in terms of the supercharges

$$\mathbb{G}_{\text{anom}}^{I_1 I_2 I_3 I_4} = (Q^-)^2 (Q^+)^2 (\bar S^-)^2 (\bar S^+)^2 \left[\theta_1^2 \theta_2^2 \theta_3^2 \theta_4^2 \frac{F(x)}{g_{12}^2 g_{34}^2} \right], \qquad (7.31)$$

where x stands for the collection (x_1, x_2, x_3, x_4) and the flavour indices are implicit in $F(x)$. We define

$$\theta_i^2 = \theta_i^{+\alpha} \theta_{i\alpha}^+, \qquad \bar\theta_i^2 = \bar\theta_{i\dot\alpha}^+ \bar\theta_i^{+\dot\alpha}, \qquad (7.32\text{a})$$

$$(Q^\pm)^2 = \left(\sum_{i=1}^4 Q_i^{\alpha\pm} \right) \left(\sum_{i=1}^4 Q_{i\alpha}^\pm \right), \qquad (7.32\text{b})$$

$$(\bar S^\pm)^2 = \left(\sum_{i=1}^4 \bar S_{i\dot\alpha}^\pm \right) \left(\sum_{i=1}^4 \bar S_i^{\dot\alpha\pm} \right). \qquad (7.32\text{c})$$

Written as in (7.31) and in virtue of the nilpotency of the Q and $\bar S$, it is evident that this expression is annihilated by half of the supercharges, but remarkably, it satisfies also the other half of the Ward Identities. This expression can be further simplified by introducing

$$\tilde S^{\dot\alpha} \equiv g_{12}^2 g_{34}^2 \left(\sum_i \bar S_i^{\dot\alpha+} \right) g_{12}^{-2} g_{34}^{-2} = \sum_{i=1}^4 \bar S_i^{\dot\alpha+} - 8 \bar\theta_i^{\dot\alpha}, \qquad (7.33)$$

and (7.31) can be re-expressed as

$$\mathbb{G}_{\text{anom}}^{I_1 I_2 I_3 I_4} = \frac{1}{g_{12}^2 g_{34}^2} (Q^+)^2 \tilde{S}^2 \left[x_{12}^2 x_{13}^2 x_{14}^2 \Theta(x) F(x) \right] ,$$

$$\Theta(x) = \left(\frac{\theta_{13} x_{13}}{x_{13}^2} - \frac{\theta_{12} x_{12}}{x_{12}^2} \right)^2 \left(\frac{\theta_{14} x_{14}}{x_{14}^2} - \frac{\theta_{12} x_{12}}{x_{12}^2} \right)^2 . \tag{7.34}$$

This expression is much more manageable and in Paper V we were able to explicitly compute the action of Q^+ and $\tilde{S}$. By setting all the θ's and $\bar{\theta}$'s to zero, we can fix $F(x)$ in term of the $\mathcal{O}_2^I$'s correlator (7.6)

$$\mathbb{G}_{\text{anom}}^{I_1 I_2 I_3 I_4} \bigg|_{\theta_i = 0 = \bar{\theta}_i} \overset{!}{=} (z\alpha - 1)(\bar{z}\alpha - 1) \widetilde{\mathcal{H}}^{I_1 \cdots I_4}(z, \bar{z}) , \quad \Rightarrow$$

$$F(x) = \frac{1}{2^4} \frac{\widetilde{\mathcal{H}}^{I_1 \cdots I_4}(z, \bar{z})}{(x_{12}^2 x_{34}^2)^2 x_{13}^2 x_{24}^2} . \tag{7.35}$$

We can extract other components by applying the differential operators (7.26) and (7.28). In the next chapter, we will report the form of the correlators involving the flavour current $\mathcal{J}$. Here we will focus on four-point functions of $\mathcal{W}$ and $\overline{\mathcal{W}}$. The non-vanishing ones are permutation of

$$\langle \mathcal{W}(x_1) \overline{\mathcal{W}}(x_2) \mathcal{W}(x_3) \overline{\mathcal{W}}(x_4) \rangle = \mathcal{D}_1^{\mathcal{W}} \mathcal{D}_2^{\overline{\mathcal{W}}} \mathcal{D}_3^{\mathcal{W}} \mathcal{D}_4^{\overline{\mathcal{W}}} \mathbb{G}_{\text{anom}}^{I_1 I_2 I_3 I_4}$$

$$= \frac{1}{(x_{12}^2 x_{34}^2)^3} \mathcal{G}_{\mathcal{W}}(z, \bar{z}) , \tag{7.36}$$

with

$$\mathcal{G}_{\mathcal{W}}(u, v) = u^3 \, \Delta^{(2)} \, uv \, \Delta^{(2)} \, u^{-2} \widetilde{\mathcal{H}}(u, v) \equiv \Delta^{(4)} \widetilde{\mathcal{H}}(u, v) , \tag{7.37}$$

and $\Delta^{(2)}$ defined in (5.39). Similarly as we have seen for $\Delta^{(8)}$ in Part II, we can express $\Delta^{(4)}$ in terms of the Casimir operator (5.40)

$$\mathcal{G}_{\mathcal{W}}(z, \bar{z}) = \frac{(z\bar{z})^3}{z - \bar{z}} \hat{\mathcal{D}}_z \hat{\mathcal{D}}_{\bar{z}} (z - \bar{z})(z\bar{z})^{-2} \widetilde{\mathcal{H}}(z, \bar{z}) . \tag{7.38}$$

Equivalently we can study the charged channel

$$\langle \mathcal{W}(x_1) \mathcal{W}(x_2) \overline{\mathcal{W}}(x_3) \overline{\mathcal{W}}(x_4) \rangle = \frac{\mathcal{G}_{\mathcal{W}}^u(z, \bar{z})}{(x_{12}^2 x_{34}^2)^3} , \tag{7.39}$$

and write this function in terms of the other Casimir operator in (5.77)

$$\mathcal{G}_{\mathcal{W}}^u(z, \bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \mathcal{D}_z \mathcal{D}_{\bar{z}} (z - \bar{z}) \widetilde{\mathcal{H}}(z, \bar{z}) . \tag{7.40}$$

Analogously to our discussion in Sec. 5.4, this second form of $\Delta^{(4)}$ can be used to solve and diagonalize a tree-level "mixing" problem [115], this time in $\text{AdS}_5 \times \text{S}^3$, at strong coupling — see Ch. 8 for details on this holographic realization — relying on a hidden $8d$ symmetry [38]. In [137], these considerations were used to derive all leading logs in a large N loop expansion in strong analogy with the results in (5.87) and (5.88).

8. Holographic realizations and hints for an AdS double-copy

In the previous chapter we have described four-dimensional $\mathcal{N} = 2$ SCFTs quite generally and we have assumed only the existence of an unspecified flavour group G_F. From the holographic perspective, this flavour group derives from a gauge group in AdS. There exist various string theory constructions of gauge theories in AdS. Here we are interested in those giving rise to a dual $\mathcal{N} = 2$ SCFT on the boundary, recently analysed in a series of papers [38, 115, 136–139] and at the center of Paper V.

8.1 Supergluons

While we will not enter into the details of holographic constructions preserving eight supercharges, explicit examples can be found in [146–148], we will just comment on the common characteristics relevant to our discussion. In all these cases, the near horizon geometry includes an $AdS_5 \times S^3$ subspace, where there lives an $\mathcal{N} = 1$ SYM theory with gauge group G_F. When reduced on the 3-sphere, the $\mathcal{N} = 1$ vector multiplet gives rise to a tower of Kaluza-Klein modes, which belong to half-BPS $\mathcal{N} = 2$ multiplets. The superprimaries are scalars of the form

$$\mathcal{O}_R^I(x, \xi, \xi') = \xi_{a_1} \cdots \xi_{a_R} \xi'_{a'_1} \cdots \xi'_{a'_{R-2}} \mathcal{O}_R^{I; a_1 \cdots a_R; a'_1 \cdots a'_{R-2}}(x) , \qquad (8.1)$$

where as before I is the adjoint flavour index, a_i is the $SU(2)_R$ R-symmetry index while a'_i is an additional $SU(2)_L$ global symmetry index. These last two groups can be interpreted as the S^3 isometries $SO(4) \simeq SU(2)_R \times SU(2)_L$. The dimension of $\mathcal{O}_R^I$ is fixed by symmetries to be $\Delta = R$ and the superprimary with the lowest possible value is exactly $\mathcal{O}_2^I$ in (7.2), which we named supergluon on account of the fact it contains exactly the conserved current associated to G_F.

Importantly in all these models, regardless of their actual realizations, there is a parameter N such that in the $N \to \infty$ limit the gluon degrees of freedom decouple from the graviton ones living in the full bulk theory. This derives from a hierarchy between the graviton coupling $C_{\mathcal{T}} \sim N^2$ and the gluon one $C_{\mathcal{J}} \sim N$, such that the graviton self-interaction and the one of two gluons and one graviton are parametrically suppressed [38]. Therefore at tree-level, leading order in $\frac{1}{N}$, we are

left with an effective $8d$ theory of supergluons in AdS with no dynamical gravity. In this setup, in Paper V we studied the four-point function of scalar supergluons and by using the superspace lift in (7.30) and (7.31) we were able to explicitly compute the four-current correlator.

Let us start from the correlator of the four scalar superprimaries in Sec. 7.1, which we report here for convenience

$$\langle \mathcal{O}_2^{I_1} \mathcal{O}_2^{I_2} \mathcal{O}_2^{I_3} \mathcal{O}_2^{I_4} \rangle = \frac{\xi_{12}^2 \xi_{34}^2}{(x_{12}^2 x_{34}^2)^2} \mathcal{G}^{I_1 I_2 I_3 I_4}(z, \bar{z}; \alpha),$$

$$\mathcal{G}^{I_1 \cdots I_4}(z, \bar{z}; \alpha) = \mathcal{G}_{\text{rational}}^{I_1 \cdots I_4} + (z\alpha - 1)(\bar{z}\alpha - 1) \widetilde{\mathcal{H}}_{\text{gl}}^{I_1 \cdots I_4}(z, \bar{z}). \tag{8.2}$$

In the holographic approximation[1]

$$\widetilde{\mathcal{H}}_{\text{gl}}(z, \bar{z}) \equiv \widetilde{\mathcal{H}}_{\text{gl}}^{I_1 I_2 I_3 I_4}(z, \bar{z}) = \frac{6}{C_{\mathcal{J}}} \left(\mathsf{c}_{\mathsf{s}} \mathcal{H}_s + \mathsf{c}_{\mathsf{t}} \mathcal{H}_t + \mathsf{c}_{\mathsf{u}} \mathcal{H}_u \right),$$

$$\mathcal{H}_s = \frac{1}{3}(z\bar{z})^2 \left(\bar{D}_{2321} - \bar{D}_{3221} \right),$$

$$\mathcal{H}_t = \frac{1}{3}(z\bar{z})^2 \left(\bar{D}_{2231} - \bar{D}_{2321} \right), \tag{8.3}$$

$$\mathcal{H}_u = \frac{1}{3}(z\bar{z})^2 \left(\bar{D}_{3221} - \bar{D}_{2231} \right),$$

where the color factors are defined in (7.8a) and the $\bar{D}$-functions already appeared in (5.69). These are defined starting from an integral over AdS_5

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = \int \frac{dz_0 d^4 z}{z_0^5} \prod_{i=1}^4 \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x}_i)^2} \right)^{\Delta_i},$$

$$\bar{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \frac{(x_{13}^2)^{\Sigma - \Delta_4}(x_{24}^2)^{\Delta_2}}{(x_{14}^2)^{\Sigma - \Delta_1 - \Delta_4}(x_{34}^2)^{\Sigma - \Delta_3 - \Delta_4}} \times \tag{8.4}$$

$$\times \frac{2 \prod_i^4 \Gamma(\Delta_i)}{\pi^2 \Gamma(\Sigma - 2)} D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4},$$

with $\Sigma = \frac{1}{2} \sum_{i=1}^4 \Delta_i$. D-functions of higher external dimensions can be constructed iteratively from lower ones using the very nice property

$$D_{\cdots (\Delta_i + 1) \cdots (\Delta_j + 1) \cdots} = \frac{2 - \Sigma}{\Delta_i \Delta_j} \frac{\partial}{\partial x_{ij}^2} D_{\cdots \Delta_i \cdots \Delta_j \cdots}. \tag{8.5}$$

8.2 AdS double-copy

It is a fairly well-established fact that flat-space scattering amplitudes of various QFTs enjoy color-kinematics duality and satisfy a double copy

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[1] We have added the subscript gl to distinguish the gluon correlator and the graviton one $\widetilde{\mathcal{H}}_{\text{gr}} = -\frac{4}{N^2} u^2 \bar{D}_{2422}$.

construction — see [149] for a recent review. Roughly, the duality states that scattering amplitudes in some gauge theories can be written in such a way that the kinematic factors satisfy the same algebra as their color factors, in particular, they satisfy the same Jacobi identities as in (7.8b). Then substituting, in the gauge amplitude, each color factor by the corresponding kinematic one gives a gravity amplitude, schematically realizing

$$(\text{gauge}) \otimes (\text{gauge}) \sim \text{gravity}. \tag{8.6}$$

To give a concrete example consider the tree-level four-point amplitude of gluons in a $4d$ flat-space Yang-Mills theory with gauge coupling g. In momentum space, its form can be arranged as

$$\text{color factors: } \mathsf{c}_s + \mathsf{c}_t + \mathsf{c}_u = 0$$

$$i\mathcal{A}_{\text{YM}}^{\text{tree}} = g^2 \left(\frac{\mathsf{c}_s\,\mathsf{n}_s}{s} + \frac{\mathsf{c}_t\,\mathsf{n}_t}{t} + \frac{\mathsf{c}_u\,\mathsf{n}_u}{u} \right) \quad \text{numerators} \tag{8.7}$$

$$\text{propagators} \qquad\qquad \mathsf{n}_s + \mathsf{n}_t + \mathsf{n}_u = 0$$

Starting from this expression, one can get the graviton amplitude in Einstein-Hilbert gravity trough

$$i\mathcal{A}_{\text{YM}}^{\text{tree}}\Big|_{\substack{\mathsf{c}_{s,t,u}\to\mathsf{n}_{s,t,u} \\ g\to\sqrt{8\pi G_N}}} = 8\pi G_N \left(\frac{\mathsf{n}_s^2}{s} + \frac{\mathsf{n}_t^2}{t} + \frac{\mathsf{n}_u^2}{u} \right) = i\mathcal{A}_{\text{EH}}^{\text{tree}}. \tag{8.8}$$

The existence of a double copy has led to various important advancements in the amplitude context and has provided a unified framework where to study gauge and gravity theories. Finding an AdS analogue is therefore incredibly interesting and it can lead to new important discoveries.[2] In [139], an AdS double copy has been found in Mellin space for the four-point functions of scalar superprimaries $\mathcal{O}_2^I$ in $\mathcal{N} = 2$ and $\mathcal{O}_2$ in $\mathcal{N} = 4$ SYM, together with their Kaluza-Klein generalizations, such that

$$\mathcal{N} = 2 \otimes \mathcal{N} = 2 \quad \underset{\text{color} \leftrightarrow \text{kinematics}}{\longleftrightarrow} \quad \mathcal{N} = 4$$

which reproduces (8.6). In Paper V, we explored the existence of this connection in position space in an attempt to lift it to the correlators of four flavour currents and four stress-tensors. Studying the correlators of the spinning components is paramount if we want to truly test the double copy as originally formulated in flat space, namely a duality between gluon and graviton amplitudes. In fact, even though we will compute four-point functions in supersymmetric theories, at tree level they coincide with amplitudes in purely bosonic Yang-Mills and Einstein-Hilbert gravity theories in AdS.

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[2]Related works attempting to define a double copy for AdS scalars are [150–155].

Table 8.1. *Conventions for superfield correlators in $\mathcal{N} = 2$ and $\mathcal{N} = 4$.*

	$\mathcal{N} = 2$	$\mathcal{N} = 4$
$\mathbb{G}_{\text{anom}}$	$Q^4 \bar{S}^4 \left[\theta_1^2 \theta_2^2 \theta_3^2 \theta_4^2 \frac{F_{\text{gl}}}{g_{12}^2 g_{34}^2} \right]$	$Q^8 \bar{S}^8 \left[\theta_1^4 \theta_2^4 \theta_3^4 \theta_4^4 \frac{F_{\text{gr}}}{\hat{g}_{12}^2 \hat{g}_{34}^2} \right]$
$F(x)$	$\frac{1}{2^4} \frac{\widetilde{\mathcal{H}}_{\text{gl}}(z,\bar{z})}{(x_{12}^2 x_{34}^2)^2 x_{13}^2 x_{24}^2}$	$\frac{\widetilde{\mathcal{H}}_{\text{gr}}(z,\bar{z})}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2)^2}$
$\widetilde{\mathcal{H}}$	(8.3)	$-u^2 \bar{D}_{2422}$

Let us start from the supergluon correlator, for convenience in Tab. 8.1 we have collected all the conventions we have used. In particular the function F_{gl} appearing in (7.34) can be rewritten as

$$F_{\text{gl}} \propto c_s \, \mathbb{N}_s W_s + c_t \, \mathbb{N}_t W_t + c_u \, \mathbb{N}_s W_u \,, \tag{8.9}$$

with

$$W_s = x_{12}^{-2} D_{1122} \,, \quad W_t = x_{14}^{-2} D_{1221} \,, \quad W_u = x_{13}^{-2} D_{1212} \,, \tag{8.10}$$

exchange Witten diagrams in the s-, t-, u-channels and they can be interpreted as the analogue of flat space propagators in (8.7). The role of the numerators instead is played by $\mathbb{N}_{s,t,u}$, which are written as

$$\mathbb{N}_s = \mathbb{D}_t - \mathbb{D}_u \,, \quad \mathbb{N}_t = \mathbb{D}_u - \mathbb{D}_s \,, \quad \mathbb{N}_u = \mathbb{D}_s - \mathbb{D}_t \,, \tag{8.11}$$

where

$$\mathbb{D}_s = \frac{1}{x_{12}^2} \frac{\partial}{\partial x_{34}^2} \,, \quad \mathbb{D}_t = \frac{1}{x_{14}^2} \frac{\partial}{\partial x_{23}^2} \,, \quad \mathbb{D}_u = \frac{1}{x_{13}^2} \frac{\partial}{\partial x_{24}^2} \,. \tag{8.12}$$

We can verify that expression (8.9) coincides with the original one in (8.3) by using the relation (8.5).

Clearly the kinematic numerators satisfy the Jacobi identity

$$\mathbb{N}_s + \mathbb{N}_t + \mathbb{N}_u = 0 \,. \tag{8.13}$$

Now, following the same steps as in the flat space example, we expect that by replacing $c_{s,t,u}$ with $\mathbb{N}_{s,t,u}$, we would get the four-point function of the supergravitons at tree level (5.67) — see Tab. 8.1. However, if we perform this replacement we get

$$\mathbb{N}_s^2 W_s + \mathbb{N}_t^2 W_t + \mathbb{N}_u^2 W_u = \frac{9\pi^2}{2} F_{\text{gr}} + \mathsf{R} \,, \tag{8.14}$$

which, apart from prefactors, is almost the supergraviton correlator except for a remainder[3]

$$\mathtt{R} = \frac{1}{(x_{12}^2 x_{34}^2)^4} \frac{u^2(u + v + uv)}{2v^2} \ . \tag{8.15}$$

We have then found a position space prescription for an AdS double copy, up to rational terms. Now we want to see what this implies for the spinning correlators. For doing that, we have to go back to the expressions obtained from superspace. In the table below we report the relevant results for correlators involving one, two or four spinning insertions. For $\mathcal{N} = 2$ theories, the spinning operator we are considering is the R-symmetry current $\mathcal{J} \equiv \eta^\alpha \bar{\eta}^{\dot\alpha} \mathcal{J}_{\alpha\dot\alpha}$, while in $\mathcal{N} = 4$ SYM we are focusing on the stress-energy tensor $\mathcal{T} \equiv \eta^{\alpha_i} \bar{\eta}^{\dot\alpha_i} \eta^{\beta_i} \bar{\eta}^{\dot\beta_i} \mathcal{T}_{\alpha_i\beta_i,\dot\alpha_i\dot\beta_i}$.

# spinning insertions	$\mathcal{N} = 2$	$\mathcal{N} = 4$
1	$\tilde{y}_{23}\tilde{y}_{34}\tilde{y}_{24}\, \mathbb{D}_1 \Lambda_1 F_{\mathrm{gl}}$	$\mathsf{y}_{23}\mathsf{y}_{34}\mathsf{y}_{24}\, \mathbb{D}_1^2 \Lambda_1^2 F_{\mathrm{gr}}$
2	$\tilde{y}_{34}^2\, \mathbb{D}_1 \mathbb{D}_2 \Lambda_2 F_{\mathrm{gl}}$	$\mathsf{y}_{34}^2\, \mathbb{D}_1^2 \mathbb{D}_2^2 \Lambda_2^2 F_{\mathrm{gr}}$
4	$\mathbb{D}_1 \mathbb{D}_2 \mathbb{D}_3 \mathbb{D}_4 \Lambda_3 F_{\mathrm{gl}}$	$\mathbb{D}_1^2 \mathbb{D}_2^2 \mathbb{D}_3^2 \mathbb{D}_4^2 \Lambda_3^2 F_{\mathrm{gr}}$

The differential operator $\mathbb{D}_i$ is defined in (5.37) and we have introduced

$$\begin{aligned}
\Lambda_1 &= (\eta_1 \mathsf{x}_{12}\mathsf{x}_{23}\mathsf{x}_{34}\mathsf{x}_{41}\eta_1)\,, \\
\Lambda_2 &= (\eta_1 \mathsf{x}_{13}\mathsf{x}_{32}\eta_2)(\eta_1 \mathsf{x}_{14}\mathsf{x}_{42}\eta_2)\,, \\
\Lambda_3 &= \Lambda(x, \eta) \text{ in (5.35c)}\,.
\end{aligned} \tag{8.16}$$

Looking at the table above, the first thing that catches the eye is that the $\mathcal{N} = 4$ expressions differ from the $\mathcal{N} = 2$ ones just for the powers of $\mathbb{D}_i$ and Λ_i. Combine this fact, with (8.9) and (8.14), a trivial double copy would be satisfied, if we were to have a "squaring" of the form

$$(\mathbb{D}_1 \mathbb{D}_2 \mathbb{D}_3 \mathbb{D}_4 \Lambda \mathbb{N}_{s,t,u})^2 \overset{?}{=} \mathbb{D}_1^2 \mathbb{D}_2^2 \mathbb{D}_3^2 \mathbb{D}_4^2 \Lambda^2 \mathbb{N}_{s,t,u}^2\,, \tag{8.17}$$

and similarly for the correlators with fewer spinning insertions. However, we are dealing with differential operators and the expression in (8.17) does not automatically hold since taking the square of an operator means applying it twice. So differently from the scalar component, it seems that we can not obtain a double copy simply by substituting $\mathsf{c}_{s,t,u}$ with the new numerators. The better formal way of obtaining it is instead

$$\mathsf{c}_{s,t,u} \longrightarrow \mathbb{D}_1^2 \mathbb{D}_2^2 \mathbb{D}_3^2 \mathbb{D}_4^2 \Lambda(x, \eta)^2 \, \mathbb{N}_{s,t,u} \, (\mathbb{D}_1 \mathbb{D}_2 \mathbb{D}_3 \mathbb{D}_4 \Lambda(x, \eta))^{-1}\,, \tag{8.18}$$

---

[3]This result does not contradict the exact double copy found in Mellin space in [139], because rational crossing symmetric terms as the one appearing in $\mathtt{R}$ have a vanishing Mellin transform.

in such a way that we obtain $\langle \mathcal{TTTT} \rangle$ starting from the current correlators. We do reckon that the expression in (8.18) is formal and it requires further study to make it explicit, especially given its non-local form. Since we have access to explicit expressions, though in a specific polarization configuration, we also tried to construct numerators, whose squares could give a double copy for the four-point functions listed above. Unfortunately, we were not able to get a result similar to (8.13) and (8.14) and we leave this analysis to future work. A good starting point could be exploring the connections with the recent works [156, 157].

9. Concluding Remarks
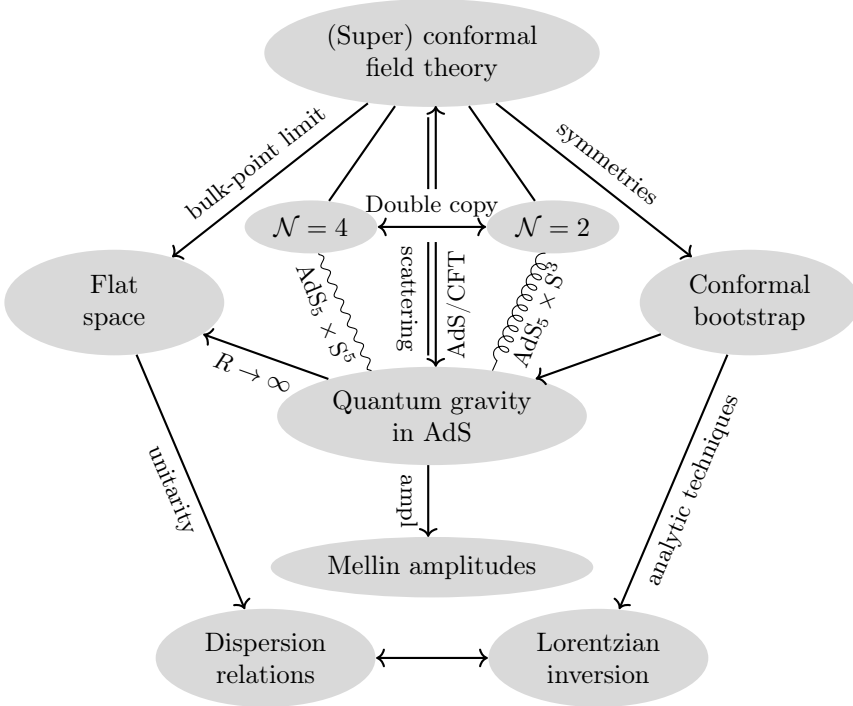
9.1 Outlook and future directions

In this thesis we have reviewed various topics in superconformal field theories: their connection to quantum gravity, their description in superspace and their analysis through analytic bootstrap techniques. As we have tried to emphasize in the scheme above, all these different aspects are deeply interconnected with one or more degrees of separation. A fundamental piece of this diagram, which is missing from this thesis, but it is present in Paper II–V, is Mellin amplitudes. We will make a few comments regarding them at the end of this section.

Throughout the various chapters, we have spent some time trying to stress the connection between CFT correlators and flat-space physics and what we can learn from it. This analogy is yet to be fully explored

and it would be interesting to further analyse the consequences of unitarity to constrain loop amplitudes in $\mathcal{N} = 4$ SYM. In Part II, we have seen how iterated s-channel discontinuities are mapped to leading logarithmic terms of the correlator of four $\mathcal{O}_2$'s. Thus a natural question to ask is whether there is a way to plug multiple discontinuities in some generalized dispersive representations such that we can reconstruct at least part of the full correlator. A good candidate to look at is the corresponding Mellin amplitudes, mentioned above. Here we can hope to use some of the results known in the amplitudes and S-matrix literature, like the recent approach proposed in [158]. Instead of looking at flat space, a complementary approach to bootstrap higher-genus correlators could be to find inspiration from string theory amplitudes. Recently, great progress has been made to compute them by combining old-fashioned methods with insights from modular graph forms and the closed/open string relations [159–165].

In the two-loop example in Sec. 5.5, we have shown how the triple-trace cut, interpreted as the contribution from triple-trace operators, enters the single discontinuity of the flat space amplitude. How does it manifest in the corresponding position space result of [119, 166]?

The need for constraining the multi-trace spectrum of $\mathcal{N} = 4$ SYM was among the motivations for the discussion at the end of Part II. Here we have studied correlators of quarter-BPS operators and in particular of $\mathcal{O}_{02}$, a genuinely double-trace operator at leading large N. Given the presence of Schur-type operators in its $\mathcal{N} = 2$ decomposition, we constructed the protected part of four-point functions involving $\mathcal{O}_{02}$ and $\mathcal{O}_2$ using the underlying chiral algebra. We mentioned that with this information we could extract corrections at large N to the averaged OPE data of operators appearing in $\mathcal{O}_2 \times \mathcal{O}_{02}$ and $\mathcal{O}_{02} \times \mathcal{O}_{02}$ by means of the Lorentzian inversion formula. It would be nice to explore this mixed correlator system using numerical bootstrap techniques.

Moreover, when studying these correlators, one of the problems we encountered is the absence of explicitly known superblocks. Being eigenfunctions of the superconformal group Casimir one could think of employing a superspace approach to solve this eigenvalue problem.

Finally, in Part III, we have reviewed $\mathcal{N} = 2$ harmonic and analytic superspace and, in this setup, we have reported explicit results for the differential representations of the supercharges. We have used them to construct the four-point function of the flavour current supermultiplet starting from the correlator of the scalar superprimaries. Then we looked for the existence of a position space AdS double-copy between the $\mathcal{N} = 2$ current correlators and the stress-tensor ones in $\mathcal{N} = 4$ SYM. The identification of an "almost" double copy for the bottom scalar component together with the form of the correlator in superspace strongly suggest an extension of the squaring $(\mathcal{N} = 2 \otimes \mathcal{N} = 2) \simeq \mathcal{N} = 4$ to spinning

correlators. Unfortunately, we did not manage to extract a precise double copy prescription, so there is room for further investigation. Since we obtained very explicit results for the spinning correlators in position space, it will be important to better study their structures, especially in comparison with their flat space analogue. This can be a promising way to find the exact form of numerators and denominators and to establish a precise form for the AdS double copy. Finally, it would be interesting to generalize our results and especially our superspace analysis to higher Kaluza Klein modes and perhaps to different dimensions.

9.2 Mellin amplitudes

Mellin amplitudes are defined as integral transforms of position-space correlators. They were introduced by Mack in [167, 168] and from that moment on the interest around them saw a very rapid growth, especially in the holographic context.

The Mellin representation turns out, in fact, to be a very natural language to describe CFTs with weakly coupled duals [169] and they are the closest analogue to a scattering amplitude we can define in AdS. As evidence for that, four-point Mellin amplitudes are functions of two variables, which can be identified with the usual Mandelstam s and t. They have a very nice and clean analytic structure, they enjoy factorization properties in terms of lower-point amplitudes at propagator poles and it is possible to write diagrammatic rules mimicking Feynman ones [169, 170]. Moreover, some of the simplifications that we have seen at large N are more evident in Mellin space. For example, the statement that only single-trace exchanges should be enough to reconstruct tree-level correlators is directly incorporated in the Mellin amplitude definition. In fact, the Mellin integral factorizes in a ratio of gamma functions times the proper Mellin amplitude. At leading large N, these gamma functions alone, having poles at the location of the double-trace dimensions, correctly take into account their contribution, in such a way that the actual Mellin amplitude retains only poles corresponding to true single-trace exchanges [171]. This property has been essential to bootstrap all tree-level Mellin amplitudes in $AdS_5 \times S^5$ [36, 172], in $AdS_5 \times S^3$ [38] and in $AdS_7 \times S^4$ [173] and higher-point functions [138, 174, 175] in similar setups. Other simplifications occur in Mellin representations because of their manifest and straightforward crossing properties like writing dispersion relations [176, 177] or deriving the original formulation of the AdS double-copy [38, 139].

As the last remark, AdS Mellin amplitudes truly reduce to the corresponding flat space counterparts when the Mellin variables are taken to be large [171] and this is yet another way to take the flat space limit and

check for consistencies. In this case as well as in our discussion about the bulk-point limit, we have always considered the flat space limit as the place where to look for agreement and inspiration. It would be interesting to reverse this reasoning and perhaps to learn about flat space starting from AdS and in particular from the well-defined boundary CFT, related ideas are presented in [13, 178].

Acknowledgements

First of all I would like to thank my supervisor Agnese. You have been a source of inspiration and a great mentor for me. Thanks for believing in me, for your constant support and for always having the exact words I needed to hear. I am really grateful for all the academic experiences you allowed and pushed me to try, they made me grow as a researcher. I want to thank you for everything you taught me, not only physics-wise but also for trying to prepare me for this world, which — we should admit — sometimes can be a bit senseless and unfair. You are a very special person and I am really thankful for having you as my advisor.

This thesis (and most probably my PhD) would not have been possible without my two other "co-supervisors" Alessandro and Andrea. You mean a lot of different things to me: you are my collaborators, my friends, my big brothers. Thanks for every moment spent together. I would like to thank Alessandro for taking me under his wing from the very moment I arrived in the Physics Department. Thanks for being there on the sad Sundays spent in the division solving cuts, for your advice and support every time I was feeling insecure and for your positive enthusiasm during our discussion. I promise one day we are going to finally understand how to make the eikonal work in our computations. I would like to thank Andrea for his endless patience with me, for the hours you spent teaching me things about CFTs and many more and for being there whenever I needed a friend. You had a great influence on my approach to research and for sure you improved the quality of my Mathematica notebooks. Thanks for always having your door open for me, even when I was rushing into your office several times a day. I am really lucky to have met you both, I hope you will always be part of my life because you make me a better physicist and a better person.

My work would not have been possible without the help of my other collaborators, Parijat and Xinan. I am truly in debt to you.

During my PhD I have met incredible people and, sadly, I had to say goodbye to some of them. I want to thank all the PhD students that welcomed me in the division when I first arrived in Uppsala: Anastasios, Gregor, Konstantina, Lorenzo, Matt, Rebecca, Robin, Suvendu.

A special mention then goes to Giuseppe, thanks for all the Italian dinners, which made me feel at home even kilometres away.

Some people, whose presence made my life much better inside and outside the division, are Konstantina, Lorenzo and Matt. I really missed

you these last times. Konstantina, you are one of the kindest and most generous person I have ever met in my life. You have this natural gift of always making people feel at ease and comfortable. You have always taken care of me, especially during the aerial silk classes, I can not count how many times you prevented me from falling or getting injured. Thanks for having never given up on me and for having pushed me to try new things, especially when it came to food. Lorenzo, you are a great office mate and a great landlord, but most importantly a dear friend. I will never forget all the time spent together gossiping, playing crosswords with Alessandro and discussing about Italian politics. When the problems in my life and in the PhD seemed too big, a beer at Palermo's with you, Matt and Robin always managed to cheer me up. Matt, before thanking you, I think I owe you an apology for every time I was complaining for nothing. At every coffee break, you were there, patiently listening and supporting me. Thank you for having taught me what a 2-category is and for understanding me without the need for words.

A special thank is also due to my PhD fellows, Lucia and Paolo. We started this journey together five years ago and we have shared all its good and bad times. Sometimes, the PhD can be tough and having someone who can understand you is really of help. Your presence made me not to feel alone, so thank you for your friendship.

During these years in Uppsala, I enjoyed a lot my time in the division, the discussions, the Journal Clubs and all the parties. So thanks to all the senior members of the division: Dima, Henrik, Joe, Lisa, Marco, Maxim, Michele, Oliver, Pietro, Ulf. A special thank to Guido for being my co-supervisor and Magdalena, for kindly helping me with my Svensk Sammanfattning.

Thanks to all the old and new Post Docs: Bram, Ingrid, Kays, Martijn, Max, Sourav, Tobias, Vlad; the Italian crew, Carlo, Jacopo, Luca, Nicolò, Rodolfo and finally Paul, who made the division a more lively and happy place.

Thanks to all the PhD students for creating a friendly and diverse environment: Alexander, Carlos, Charles, Daniel, Lucile, Maor, Roman, Yoann and in particular to Simon, for sharing all the stress of the thesis deadlines.

Outside of academia, I am grateful for my family, for accepting and loving me despite my very strong opinions and difficult character and especially to my little nephew Alessandro, you brought the purest joy in my life. A special mention for my parents, the boxes full of Italian food were the real reasons why I survived the PhD in Sweden.

I also want to thank my friends in Italy. Starting from my "Physics friends": Dario, Giada, Lorenzo, Pietro and Simone. If it was not for you who made Physics so much fun, I most probably would have never

started a PhD. I am also grateful for my best friends in Arezzo: Ire, Ludo, Mary, Sara and Vale. You were (and you are always) there any time I was coming back home, ready to hang out and party. You are the breath of fresh air and light-heartedness I need in my life. I miss you more than I usually admit.

Finally, I must thank the author of the stylish plots in the introduction and the person who makes my life special and happy every day: Lorenzo. It is just thanks to you that I kept my sanity in the hardest moments of this PhD, thanks for answering my physics questions even just awake in the morning and most importantly thanks for always being there for me. Thanks for being the companion of my adventures, for always believing and seeing the best in me.

Svensk Sammanfattning

En konform fältteori (CFT) är en speciell typ av kvantfältteori (QFT), som är symmetrisk under konforma transformationer. Det betyder att teorin inte förändras när vi tillämpar en transformation som bevarar vinklar. Konforma symmetriska teorier är särskilt invarianta under omskalning, så de förändras inte oavsett hur långt eller nära vi tittar på dem.

CFTs förekommer i olika fysikaliska sammanhang: de beskriver andra ordningens fasövergångar i statistisk mekanik och kvantkritiska punkter i kondenserade materiens teori. Förutom deras roll i att förklara kritiska fenomen är CFTs en nyckelingrediens för att förstå och klassificera QFTs. Faktum är att CFTs kan ses som start och slutpunkten för ett flöde, kallat renormeringsgruppsflöde, som renormeringsgruppsflöde olika QFTs. Slutligen, via AdS/CFT-korrespondensen ger CFTs en icke-perturbativ definition av kvantgravitation. AdS/CFT-korrespondensen definierar en dualitet mellan en gravitationsteori i en krökt Anti de Sitter rum-tid och en CFT som lever på randen av AdS rum-tiden. Dualiteten är sådan att fysiken som kodas in i den $d+1$-dimensionella bulken är densamma som den som beskrivs av den d-dimensionella CFT. Med tanke på att sambandet gäller mellan olika dimensioner kallas dessa teorier holografiska.

Strävan efter en teori för kvantgravitation är ett problem med lång historia teoretisk fysik. Faktum är att det är svårt att hitta en beskrivning som på samma gång kvantmekaniken och Einsteins allmänna relativitetsteori, av de få passar konsistenta förslagen är strängteorin. Genom AdS/CFT-korrespondensen kan CFT faktiskt tillhandahålla en ram för att hantera detta problem. Med tanke på deras rika matematiska struktur och den mycket exakta formuleringen i termer av axiom kan CFTs representera det rätta sättet att se på observerbara kvantgravitationsobjekt. Samtidigt är det viktigt att notera att inte alla CFT är lämpliga teorier för kvantgravitation. Ändå är det möjligt att visa att teorier med ett stort antal frihetsgrader och ett stort gap i dimensionen av operatörer ger en konsekvent, lokal lågenergiteori för gravitation i bulken.

I denna avhandling kommer vi huvudsakligen att fokusera på studiet av dessa holografiska CFT. Speciellt kommer vi att titta på den supersymmetriska versionen av konforma teorier, dvs superkonforma fältteorier eller SCFT. Supersymmetri kan förstås som en extra symmetri hos teorin, vilken är väldigt speciell eftersom det kan relatera bosoner till fermioner, som är fält med olika statistik. Med tanke på mängden symmetrier som

dessa teorier besitter, är ett mycket kraftfullt verktyg vi kan använda för att analysera dem den analytiska konforma "bootstrap"-metoden. Denna metod är en icke-perturbativ teknik designad för att studera CFTs baserade på de principerna: kausalitet, korsningssymmetri och associativitet för operatorproduktsutvecklingen (OPE). I denna avhandling tillämpar vi dessa tekniker för att studera fyrapunktsfunktioner i två exempel av SCFT: $\mathcal{N} = 4$ Super Yang-Mills (SYM), det prototypiska exemplet på holografisk CFT som är en maximalt supersymmetrisk teori i fyra dimensioner, samt en klass av $\mathcal{N} = 2$ teorier.

I Artiklar I and II studerade vi fyrpunktsfunktionen hos de superkonforma primära operatorerna av stress-tensormultipleten i $\mathcal{N} = 4$ SYM,double till gravitonamplituder i AdS, vid stark koppling och vi begränsade en del av den. Vi kopplade också våra resultat till intressanta egenskaper hos amplituder i platt rymd. I samma sammanhang analyserar vi i Artikel IV fyrpunktsfunktionen hos en annan typ av operatorer, kallad kvarts-BPS. Vi fixerade en del av deras korrelatorer med hjälp av teorins underliggande symmetri och vi tillämpade analytiska bootstrap-tekniker för att fixera korrigering till dimensionerna och OPE-koefficienterna för de operatorer som finns i spektrumet. Vi tar ett annat perspektiv i Artikel V. Här studerade vi fyrpunktsfunktioner hos spinnoperatorer i $\mathcal{N} = 2$ SCFTs och vi bevisade för ett samband mellan korrelatorer i dessa teorier och de i $\mathcal{N} = 4$. På detta sätt försökte vi etablera en form av "AdS dubbelkopia": Med avstamp av en amplitud i en gauge teori ($\mathcal{N} = 2$) kan man konstruera spridningsamplituder i en gravitationsteori ($\mathcal{N} = 4$). Artikel III är en översiktsartikel av analytiska bootstrap-tekniker för konforma och superkonforma fältteorier och innehåller några av de resultat som tidigare visats i de andra artiklarna.

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