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A Kerr-full Study of Higher-Spin Amplitudes

An on-shell construction of higher-spin amplitudes for black holes and strings

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Abstract
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Recently, it was shown that Kerr black holes can be described via the classical infinite-spin limit of a special class of scattering amplitudes in a massive higher-spin quantum field theory. Although this approach has successfully obtained state-of-the-art results for spinning black-hole binaries, only the three-point amplitude that describes Kerr is known in full generality and a full understanding of the underlying Lagrangian is still missing. In particular, vertices at four points and beyond are necessary to perform higher-order calculations. Massive higher-spin Lagrangians are highly constrained by properties such as unitarity and degrees-of-freedom counting. A useful tool in building consistent theories is the introduction of a massive gauge symmetry. However, constructing gauge-invariant vertices beyond the cubic level is a daunting task, so far never attempted in the literature. We propose an alternative on-shell realisation of gauge invariance, in the form of novel massive Ward identities, which provides a significant simplification with respect to the traditional approach. We show that the amplitudes known to describe Kerr are the unique lowest-derivative solution to the Ward identities combined with a known high-energy unitarity constraint. Moreover, we apply the same methods to compute new four-point Compton amplitudes for higher-spin states and propose them as candidates to describe higher-order black-hole observables. In parallel, we study the amplitudes of leading Regge states in superstring theory, as another example of consistent massive higher-spin particles. Applying the classical-limit formalism, previously only studied in the context of black holes, we recover known classical string solutions. This provides important insights on the properties of the formalism. Moreover, it paves the way to studying more general string states and attempting to reproduce black holes from strings.

Keywords: scattering amplitudes, higher-spin theory, black holes, string theory, gravitational waves

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doctoral students at Uppsala University

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I M. Chiodaroli, H. Johansson and P. Pichini, *Compton black-hole scattering for $s \leq 5/2$*, *JHEP* **02** (2022) 156 [2107.14779].

II L. Cangemi and P. Pichini, *Classical Limit of Higher-Spin String Amplitudes*, 2207.03947.

III L. Cangemi, M. Chiodaroli, H. Johansson, A. Ochirov, P. Pichini and E. Skvortsov, *Kerr Black Holes Enjoy Massive Higher-Spin Gauge Symmetry*, 2212.06120.

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The following papers are not included in this thesis.

IV M. Diehl, J.R. Gaunt, P. Pichini and P. Plößl, *Double parton distributions out of bounds in colour space*, *Eur. Phys. J. C* **81** (2021) 1033 [2109.14304].

Contents

1. Introduction

In recent years, gravitational-wave experiments have collected a wealth of new data from binary systems of black holes and neutron stars [1]. In parallel, significant efforts have been made on the theory side, to provide high-precision analytic calculations in general relativity. An established approach to computing binary observables is to construct a worldline effective field theory and rely on the corresponding Feynman-diagrammatic expansion [2–9].

More recently, in a number of works, it has been shown that the same observables can be obtained from the classical limit of scattering amplitudes in an effective quantum field theory. In particular, Schwarzschild black holes are described by the remarkably simple theory of a scalar field minimally-coupled to gravity. This approach has provided many new state-of-the art results for conservative and dissipative observables in the *post-Minkowskian expansion*, the perturbative expansion in the gravitational coupling constant G [10–55].

Kerr black holes, on the other hand, require massive higher-spin fields but the underlying theoretical description is poorly understood [42, 56–107]. An exception is a special class of three-point amplitudes $\mathcal{M}(\Phi_1^s\,\bar{\Phi}_2^s\,h_3)$ with massive higher-spin particles Φ^s coupled to the graviton h was first studied in ref. [108],

$$\mathcal{M}(\Phi_1^s\,\bar{\Phi}_2^s\,h_3) = \mathcal{M}^0\frac{[\mathbf{12}]^{2s}}{m^{2s}}, \tag{1.1}$$

in terms of massive spinor-helicity variables. Note that the subscript on each field denotes the momentum label, and $\mathcal{M}^0 = \mathcal{M}(\Phi_1^0\,\bar{\Phi}_2^0\,h_3)$ is the scalar amplitude. The amplitudes (1.1) can be rewritten in terms of the angular momentum (Pauli-Lubanski) operator a^μ,

$$a^\mu = \frac{1}{2m^2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}, \tag{1.2}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor, P_μ is the momentum operator and $M_{\rho\sigma}$ are Lorentz generators. In the infinite-spin limit one obtains the classical amplitude

$$\mathcal{M}(\Phi_1^s\,\bar{\Phi}_2^s\,h_3) \xrightarrow{s\to\infty} (\varepsilon_3\cdot p_1)^2\exp(p_3\cdot a). \tag{1.3}$$

This result was shown to match the linearised energy-momentum tensor of a Kerr black hole, as given in ref. [109], and it was used to compute leading-order post-Minkowskian observables for a black-hole binary system [57–59, 63, 70]. Although the infinite-spin limit is in principle necessary to recover the complete classical result, the finite-spin amplitudes reproduce the same exponential, albeit truncated at $\mathscr{O}(a^{2s})$. In other words, the coefficients of each power of a^μ, known as *spin-multipole coefficients*, are independent of the spin s. This feature is known as *spin universality* [110, 111].

The next-to-leading post-Minkowskian observables require knowledge of the four-point Compton amplitude $\mathscr{M}(\Phi_1^s\,\bar{\Phi}_2^s\,h_3\,h_4)$. This can be obtained from the three-point amplitudes given above using Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion [112, 113], but only for states with spin $s \leq 2$. At spin $s > 2$, BCFW produces amplitudes with unphysical poles and attempts to cure them leave an ambiguity in the four-point contact term [62, 108]. One way to resolve this ambiguity is to study the classical amplitudes and identify patterns associated to black holes [90–92]. Another way is to study solutions to the Teukolsky equation, describing gravitational waves on a black-hole background, and match them to the Compton amplitude, as discussed in part IV [85, 103, 107]. A third way is to gain a better understanding of the quantum Lagrangians underlying the known three-point amplitudes, study their properties and attempt to understand the full theory [82, 102]. The last approach is the main subject of this thesis.

The Lagrangians that describe Kerr for $s \leq 2$ can be seen to arise from dimensional reduction of the massless theories, equivalent to spontaneous symmetry breaking in the $s = 1$ case [82,114]. However, for $s > 2$, it is well known that no consistent massless theory exists in flat space. A consequence of this is that amplitudes with massive higher-spin fields diverge in the high-energy limit, meaning that higher-spin particles must be either composite or effective descriptions. Nonetheless, the Kerr amplitudes can be obtained from special theories with an improved high-energy cutoff, as shown explicitly for $s = 5/2$ in ref. [82]. This supports the observation that black holes are extremely simple physical objects: by the no-hair theorem, they are fully defined by their mass, charge and spin and hence they resemble elementary particles [115].

A potential issue in introducing interactions in a higher-spin theory is that this can lead to violations of the correct number of degrees of freedom [116]. A systematic way to avoid this is to introduce a gauge symmetry for the higher-spin field, with the help of auxiliary degrees of freedom known as Stückelberg fields. However, constructing gauge-invariant Lagrangians is no simple task, and even the cubic interactions are known in the literature only in a few examples [117–119]. Four-point interactions require solving quadratic equations, and progress in this direction is largely absent in the current literature.

In this thesis, we introduce new on-shell Ward identities that implement gauge invariance at the level of scattering amplitudes and thus bypass the complexity of the underlying Lagrangians. Combining the Ward identities with known high-energy unitarity constraints [82, 120–122], we are able to derive the Kerr three-point amplitudes to any spin, as the unique lowest-derivative solution. Moreover, we apply the same methods to four-point amplitudes. We reproduce the known Kerr amplitudes for $s \leq 2$ and obtain new results for higher spins [123].

A fully consistent example of massive higher-spin theory is string theory. The study of string amplitudes has provided many important insights on the consistency of higher spins [124–128]. Moreover, the high-energy limit presents features that hint to a mechanism similar to spontaneous symmetry breaking [129–141]. Another goal of this thesis is to compare string amplitudes to those that describe Kerr. Explicit three-point amplitudes for massive string states coupled to a massless graviton have been studied in the case of the bosonic string and superstring [142, 143]. We focus on the superstring, since it is free of tachyons, and study states from the leading Regge trajectory. The amplitudes considered match eq. (1.1) up to $s = 4$. However, for higher spins, the two amplitudes show different behaviours.

The $s \to \infty$ limit of the superstring amplitudes is here shown to reproduce known classical string solution. Contrary to the black-hole case, the string amplitudes do not obey spin universality and classical physics can only be extracted from infinite-spin amplitudes. This illustrates that spin universality is a non-trivial feature of the three-point amplitudes (1.1). Moreover, the methods presented here can be applied to more general string states in the pursuit of a string-theory description of Kerr black holes.

This thesis is organised as follows: section I is a review of higher-spin quantum field theory. It discusses the construction of the free theory and the issues that can arise when introducing interactions, in the massless and massive case. The notion of massive gauge invariance is presented as a systematic way to circumvent such issues and a few examples are studied in detail. String theory is reviewed as an example of a consistent higher-spin theory.

Section II reviews the on-shell construction of higher-spin amplitudes via the massive spinor-helicity formalism [108]. In particular, the three-point amplitudes that describe Kerr black holes are presented and their extension to the four-point Compton case is discussed.

Section III presents the main results of this thesis. It begins with a discussion of a known high-energy unitarity constraint for higher-spin particles, referred to as the *current constraint*, and its relation to Kerr black holes. Then, it shows how to implement massive gauge invariance directly at the level of on-shell amplitudes by introducing *massive Ward identities*. Combining Ward identities and the current constraint, the Kerr three-point amplitudes are fixed uniquely as the lowest-derivative solution. The same methods are then applied to the Compton amplitude and new results for higher-spin states are displayed. Moreover, an even simpler realisation of gauge invariance is presented in the form of on-shell identities that require no knowledge of the underlying Lagrangians.

Section IV discusses the classical limit of quantum-field-theory amplitudes. The three-point amplitudes in ref. [108] are matched to the energy momentum tensor of a black hole and the result is used to compute the leading-order scattering angle in a binary system. Moreover, the relation between the Compton amplitude and the scattering of gravitational waves on a black-hole background is discussed. Finally, the same classical-limit framework is applied to superstring amplitudes. As a result, known classical string solutions are recovered and important differences to the black-hole case are discussed.

Part I:
Massive Higher-Spin Theory

2. Massive Spinning Particles

A massive spin-s particle is an irreducible representation of the Poincaré group, labelled by the eigenvalues of the two Casimir operators [144, 145]

$$
\begin{aligned}
C_1 &= P^\mu P_\mu, \\
C_2 &= W^\mu W_\mu
\end{aligned} \tag{2.1}
$$

where $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma$ is the Pauli-Lubanski pseudovector, P^μ generates translations and $M^{\mu\nu}$ Lorentz transformations. The eigenvalue of C_1 is m^2, where m is the *mass* of the particle, and the eigenvalue of C_2 is $m^2 s(s+1)$, where s is half-integer and it is known as the *spin quantum number*. Note that we work with a flat Minkowski metric $\eta_{\mu\nu}$ in mostly-minus signature.

In order to label the states belonging to such representation, we need to find a set of mutually-commuting generators. We start from the momentum P^μ, with four-momentum p^μ as eigenvalue. The other generators form the *little group*, a subgroup of Lorentz transformations that leave the momentum p^μ invariant. In the rest frame $p^\mu = (m, 0, \ldots, 0)$, these are all the spatial rotations J^{ij}, where $i, j = 1, \ldots, d-1$. Hence the little group is $SO(d-1)$. Some of its simplest irreducible representations are given by tensors $\varepsilon_{i_1 \ldots i_s}$ that satisfy

$$
\begin{aligned}
&\varepsilon_{i_1 \ldots i_s} = \varepsilon_{(i_1 \ldots i_s)}, \\
&\delta^{jk} \varepsilon_{jk i_3 \ldots i_s} = 0,
\end{aligned} \tag{2.2}
$$

where all Latin indices can take values $\{1, \ldots, d-1\}$ and δ^{jk} is the flat Euclidean metric. In words, we say that they are *fully-symmetric* and *traceless*. We can rewrite this in a Lorentz-invariant manner by defining polarisation tensor $\varepsilon_{\mu_1 \ldots \mu_s}(p)$, where all Greek indices are in the range $\{0, \ldots, d-1\}$ and $\varepsilon(p)$ satisfies

$$
\begin{aligned}
&\varepsilon_{\mu_1 \ldots \mu_s} = \varepsilon_{(\mu_1 \ldots \mu_s)}, \\
&\eta^{\rho\sigma} \varepsilon_{\rho\sigma\mu_3 \ldots \mu_s} = 0, \\
&p^\rho \varepsilon_{\rho \mu_2 \ldots \mu_s} = 0.
\end{aligned} \tag{2.3}
$$

The last condition is known as *transversality*.

Now let us try to write down a field theory for such a particle. This will be described by an action

$$
S = \int d^d x \mathscr{L}(\Phi_{\mu_1 \ldots \mu_s}) \tag{2.4}
$$

where the field $\Phi_{\mu_1\dots\mu_s}$ is an irreducible representation of the Lorentz group $SO(1,d-1)$, satisfying

$$\begin{aligned}\Phi_{\mu_1\dots\mu_s} &= \Phi_{(\mu_1\dots\mu_s)},\\ \eta^{\rho\sigma}\Phi_{\rho\sigma\mu_3\dots\mu_s} &= 0,\end{aligned} \tag{2.5}$$

matching the first two conditions in eq. (2.3). We say these are *off-shell conditions*, meaning that they are valid for arbitrary configurations of the field Φ. However, we want the field theory to describe an irreducible representation of the Poincaré group, as described above. The remaining conditions must be imposed as equations of motion, so we refer to them as *on-shell conditions*. In particular, given the momentum operator $P_\mu = -i\partial_\mu$, this implies

$$\begin{aligned}(\partial^2+m^2)\Phi_{\mu_1\dots\mu_s} &= 0,\\ \partial^\rho\Phi_{\rho\mu_2\dots\mu_s} &= 0.\end{aligned} \tag{2.6}$$

Now, the challenge is to find a free Lagrangian that reproduces the above equations of motion. We will see that the transversality condition will make this a rather non-trivial task. Moreover, the field obeying eq. (2.5) will need to be combined with additional auxiliary fields.

3. Free Lagrangians

3.1 Low-Spin Examples

Let us begin from the simplest case, a massive spin-0 particle. This is described by a scalar field ϕ and by the Klein-Gordon Lagrangian

$$\mathscr{L}^{s=0} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2. \tag{3.1}$$

Since ϕ does not carry any Lorentz indices, we do not need to worry about conditions (2.3). All we need is the Klein-Gordon equation of motion,

$$(\partial^2 + m^2)\phi = 0 \tag{3.2}$$

which follows from the above Lagrangian.

The next example is a massive spin-1 particle [1]. We can describe this via a vector field W_μ and via the Proca Lagrangian,

$$\begin{aligned}\mathscr{L}^{s=1} &= \partial^{[\mu}W^{\nu]}\partial_{[\mu}W_{\nu]} - \frac{m^2}{2}W^\mu W_\mu \\ &= \frac{1}{2}\partial^\mu W^\nu\partial_\mu W_\nu - \frac{m^2}{2}W^\mu W_\mu - \frac{1}{2}\partial^\mu W^\nu\partial_\nu W_\mu.\end{aligned} \tag{3.3}$$

Since W_μ only carries one Lorentz index, eq. (2.5) is satisfied trivially and we only need to reproduce the on-shell conditions (2.6). The equations of motion are

$$\mathscr{R}_\mu \equiv \partial^2 W_\mu - \partial_\mu\partial\cdot W + m^2 W_\mu = 0 \tag{3.4}$$

where we use the shorthand $\partial\cdot W = \partial_\mu W^\mu$. Taking the divergence $\partial\cdot\mathscr{R} = 0$ we can derive

$$\begin{aligned}&(\partial^2 + m^2)W_\mu = 0\\ &\partial\cdot W = 0.\end{aligned} \tag{3.5}$$

Note that eq. (3.3) is very close to a sum of d Klein-Gordon Lagrangians, one for each component of W_μ, where the difference lies in the last term $\partial^\mu W^\nu\partial_\nu W_\mu$. If we were to omit this term, the equations of motion would reduce to $(\partial^2 + m^2)W_\mu = 0$ and we would have d wave-like propagating *degrees of freedom*. However, a spin-1 particle is in the vector representation of

[1] In this section we focus on bosonic fields, but the same arguments apply to the fermionic case.

the little group $SO(d-1)$ and hence should have $d-1$ degrees of freedom. Therefore we need one constraint, the transversality condition, which can be derived as a low-derivative consequence of the equations of motion only if we add the term $\partial^\mu W^\nu \partial_\nu W_\mu$.

Next, we consider a massive spin-2 particle, with free Lagrangian

$$\mathscr{L}^{s=2} = \frac{1}{2}\partial_\mu \phi_{\nu\rho}\partial^\mu \phi^{\nu\rho} - \partial\cdot\phi_\mu \partial\cdot\phi^\mu - \frac{d-1}{2(d-2)}\partial_\mu \varphi \partial^\mu \varphi + \varphi \partial^\mu \partial^\nu \phi_{\mu\nu} \\ - \frac{m^2}{2}\left(\phi_{\mu\nu}\phi^{\mu\nu} - \frac{d(d-1)}{(d-2)^2}\varphi^2\right), \quad (3.6)$$

in terms of a symmetric traceless tensor $\phi_{\mu\nu}$ and an auxiliary scalar φ [145]. We use the short-hand notation $\partial\cdot\phi_\mu = \partial^\rho \phi_{\rho\mu}$. The equations of motion are

$$\mathscr{R}_{\mu\nu} = -\partial^2 \phi_{\mu\nu} + 2\partial_{(\mu}\partial\cdot\phi_{\nu)} + \partial_\mu\partial_\nu \varphi - m^2\phi_{\mu\nu} - \frac{2}{d}\eta_{\mu\nu}\partial\cdot\partial\cdot\phi - \frac{1}{d}\eta_{\mu\nu}\partial^2\varphi, \\ \tilde{\mathscr{R}} = (d-1)\partial^2\varphi + (d-2)\partial\cdot\partial\cdot\phi + m^2\frac{d(d-1)}{d-2}\varphi = 0, \quad (3.7)$$

obtained from varying $\phi_{\mu\nu}$ and φ respectively. We use $\partial\cdot\partial\cdot\phi = \partial^\mu\partial^\nu\phi_{\mu\nu}$. Combining $\partial\cdot\partial\cdot\mathscr{R}$ and $\tilde{\mathscr{R}}$, we can find $\varphi = \partial\cdot\partial\cdot\phi = 0$. Substituting this into $\partial\cdot\mathscr{R}_\mu$ yields $\partial\cdot\phi_\mu = 0$. Therefore, we obtain the following equations,

$$(\partial^2 + m^2)\phi_{\mu\nu} = 0, \\ \partial\cdot\phi_\mu = 0, \\ \varphi = 0, \quad (3.8)$$

matching all the conditions required for a spin-2 particle.

Here we see a common feature of most massive higher-spin Lagrangians: the correct theory cannot be written down in terms of a single (traceless) field $\phi_{\mu_1\ldots\mu_s}$, but we need to introduce *auxiliary fields*. In this case, both $\phi_{\mu\nu}$ and φ can be combined into a single traceful symmetric tensor $H_{\mu\nu}$, and the Lagrangian reduces to [146]

$$\mathscr{L}^{s=2} = \frac{1}{2}\partial_\mu H_{\nu\rho}\partial^\mu H^{\nu\rho} - \partial\cdot H_\mu \partial\cdot H^\mu - \frac{1}{2}\partial_\mu H \partial^\mu H + \partial\cdot H_\mu \partial^\mu H \\ - \frac{m^2}{2}(H_{\mu\nu}H^{\mu\nu} - H^2) \quad (3.9)$$

where $H \equiv H^\mu_\mu$. The equations of motion become $(\partial^2+m^2)H_{\mu\nu} = \partial\cdot H_\mu = H = 0$. Starting from spin-3, however, there is no conventional way to reabsorb all the auxiliary fields into a single tensor field.

3.2 Arbitrary Spin

Free Lagrangians for any massive spin-s particle in $d=4$ have been written down systematically in ref. [147], in terms of a tower of symmetric traceless fields $\Phi^{(k)}_{\mu_1\ldots\mu_k}$, where $k=0,\ldots,s-2,s$. Note that there is no Φ^{s-1} field. The Lagrangians can be written as

$$
\begin{aligned}
\mathscr{L}^s = & -\frac{1}{2}\Phi^{(s)}(\partial^2+m^2)\Phi^{(s)} - \frac{s}{2}(\partial\cdot\Phi^{(s)})^2 + \frac{s(s-1)^2}{2s-1}\Big\{\Phi^{(s-2)}\partial\cdot\partial\cdot\Phi^{(s)} \\
& + \frac{1}{2}\Phi^{(s-2)}(\partial^2+a_2m^2)\Phi^{(s-2)} - \frac{b_2}{2}(\partial\cdot\Phi^{(s-2)})^2 \\
& - \sum_{q=3}^{s}\left(\prod_{k=2}^{q-1}c_k\right)(-1)^q\Big[-\frac{1}{2}\Phi^{(s-q)}(\partial^2+a_qm^2)\Phi^{(s-q)} \\
& + \frac{b_q}{2}(\partial\cdot\Phi^{(s-q)})^2 + m\Phi^{(s-q)}\partial\cdot\Phi^{(s-q+1)}\Big]\Big\},
\end{aligned} \tag{3.10}
$$

where the Lorentz indices are omitted for simplicity, but they are all contracted between the two fields, unless contractions with derivatives are explicitly indicated. The coefficients a_q, b_q, c_q have values

$$
\begin{aligned}
a_q &= \frac{q(2s-q+1)(s-q+2)}{2(2s-2q+3)(s-q+1)}, \\
b_{q-1} &= \frac{(s-q+1)^2}{2s-2q+5}, \\
c_{q-1} &= \frac{(q-1)(s-q+1)^2(s-q+3)(2s-q+3)}{2(s-q+2)(2s-2q+3)(2s-2q+5)}.
\end{aligned} \tag{3.11}
$$

The Lagrangian (3.10) is not unique, since it is sensitive to field redefinitions. Despite this only describing free theory, the Lagrangian is rather involved. As we are about to see, interactions introduce new issues to take care of and therefore increase the complexity. One important goal of this thesis is to show how to achieve considerable simplifications by studying higher-spin particles from the point of view of *on-shell amplitudes*.

3.3 Gauge Invariance

Let us consider a generic Lagrangian $\mathscr{L}^s$ in terms of a rank-s symmetric tensor field Φ^s. The equations of motion are

$$
\mathscr{R}^{\mu_1\ldots\mu_s} \equiv \partial_\rho \frac{\delta\mathscr{L}^s}{\delta(\partial_\rho\Phi_{\mu_1\ldots\mu_s})} - \frac{\delta\mathscr{L}^s}{\delta\Phi_{\mu_1\ldots\mu_s}} = 0, \tag{3.12}
$$

where $\mathscr{R}^{\mu_1\ldots\mu_s}$ is a linear second-order differential equation. We know that in order to describe a spin-s particle, we need the constraint $\partial\cdot\Phi^{\mu_1\ldots\mu_{s-1}}=0$. In the examples above, we obtained this from the divergence $\partial\cdot\mathscr{R}^{\mu_1\ldots\mu_{s-1}}$ [2].

[2]Technically it is not enough to consider a single divergence, but we need to compute traces and higher derivatives of the equations of motion, such as $\partial\cdot\partial\cdot\mathscr{R}$ in the context of eq. (3.7). However, the divergence $\partial\cdot\mathscr{R}$ is often the first step, hence we focus on it in this analysis.

However, since $\mathscr{R}$ is a two-derivative expression, its divergence will (in general) give a three-derivative equation that cannot yield the required one-derivative constraint. The only way to solve this issue is to require the highest-derivative terms in $\partial \cdot \mathscr{R}$ to cancel, namely

$$\lim_{m\to 0} \partial \cdot \mathscr{R}^{\mu_1 \dots \mu_{s-1}} = 0, \tag{3.13}$$

where the limit $m \to 0$ extracts the highest-derivative part, as is clear from dimensional analysis.

We can look at a few examples. In the spin-1 case the equation of motion is as given in eq. (3.4), and we have

$$\lim_{m\to 0} \mathscr{R}_\nu = \partial^\mu F_{\mu\nu} = 0, \tag{3.14}$$

where $F_{\mu\nu} = 2\partial_{[\mu} W_{\nu]}$. These are just Maxwell's equations, describing a free massless spin-1 particle and obtained from the Lagrangian

$$\mathscr{L}^{s=1,m=0} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{3.15}$$

invariant under the gauge transformation $\delta W_\mu = \partial_\mu \lambda$, where λ is an arbitrary function. This invariance ensures the equations of motion are divergence-free, as required by eq. (3.13). In the spin-2 case, we have

$$\lim_{m\to 0} \mathscr{R}_{\mu\nu} = -\partial^2 (H_{\mu\nu} - \eta_{\mu\nu} H) - \partial_\mu \partial_\nu H - \eta_{\mu\nu} \partial \cdot \partial \cdot H + 2\partial_{(\mu} \partial \cdot H_{\nu)}. \tag{3.16}$$

This is just the Einstein tensor $G_{\mu\nu}$ for the metric $g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$, expanded to first order in $H_{\mu\nu}$. The Einstein tensor is divergence-free as a consequence of the diffeomorphism invariance of the Einstein-Hilbert action, hence eq. (3.13) follows.

Similarly, it can be shown that in the $m \to 0$ limit the Lagrangian (3.10) is invariant under the gauge transformations $\delta \phi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \dots \mu_s)}$, where $\xi_{\mu_1 \dots \mu_{s-1}}$ is a symmetric and traceless tensor field. This ensures that the highest-derivative component of $\mathscr{R}^{\mu_1 \cdots \mu_s}$ is divergence-free and $\partial \cdot \mathscr{R}$, combined with higher derivatives and traces of the equations of motion, will eventually lead to the constraints required to describe a massive higher-spin particle. In the next section we will see that gauge invariance can be extended to the full massive theory, by adding extra fields to the Lagrangian known as Stückelberg fields, and that this will make it easier to introduce interactions.

4. Gauge and Gravitational Interactions

So far we have discussed theories describing free higher-spin particles propagating in flat spacetime. These theories are perfectly healthy, even in the $m = 0$ case. However, turning on interactions introduces a range of known pathologies. For instance, we can consider an amplitude $\mathscr{A}(\Phi^s_q X_1 \ldots X_n)$ between a massless higher-spin particle Φ^s, with momentum $q^2 = 0$, and n generic particles X_i with momentum p_i. Weinberg showed that, in the soft limit $q \to 0$, we have [148]

$$\mathscr{A}(\Phi^s_q X_1 \ldots X_n) \xrightarrow{q\to 0} \sum_{i=1}^{n} g_i \frac{(p_i \cdot \varepsilon_q)^s}{2q \cdot p_i} \mathscr{A}(X_1 \ldots X_n) \tag{4.1}$$

where g_i is a constant parametrising the coupling between X_i and Φ^s. The spin-s particle is described by the polarisation tensor $\varepsilon_q^{\mu_1 \ldots \mu_s} = \varepsilon_q^{\mu_1} \ldots \varepsilon_q^{\mu_s}$. In a consistent theory, the unphysical modes $\varepsilon_q^{\mu_1 \ldots \mu_s} = q^{(\mu_1} \varepsilon_q^{\mu_2} \ldots \varepsilon_q^{\mu_s)}$ must decouple, leading to the constraint

$$\sum_{i=1}^{n} g_i p_i^{\mu_1} \ldots p_i^{\mu_{s-1}} = 0. \tag{4.2}$$

For $s = 1$, this reduces to charge conservation $\sum_i g_i = 0$. For $s = 2$, this is solved by momentum conservation, $\sum_i p_i = 0$, and by requiring the gravitational coupling constant to be universal, $g_i = \kappa$. For $s > 2$, there is no solution for generic momenta p_i, except the trivial solution $g_i = 0$, meaning the higher-spin particle cannot be interacting. This is only one of the many issues that arise for massless higher spins in flat space. Other examples include the Aragone-Deser problem, showing inconsistencies in the standard gravitational coupling of higher-spin particles, and the Weinberg-Witten theorem, showing that higher-spin theories cannot generate a conserved and gauge-invariant energy-momentum tensor [149, 150].

In the $m \neq 0$ case, the issues described above can be avoided and massive higher spin particles exist in nature as composite states. However, interactions can still introduce inconsistencies, for instance by modifying the number of degrees of freedom. For example, the Lagrangian (3.9) gives rise to the following equations of motion,

$$(\partial^2 + m^2) H_{\mu\nu} = 0, \quad \partial \cdot H_\mu = 0, \quad H = 0. \tag{4.3}$$

If we try to couple them to electromagnetism, via the minimal coupling prescription $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, we get

$$[D^\mu, D^2 + m^2]\phi_{\mu\nu} \simeq F^{\mu\rho} D_\mu \phi_{\rho\nu} = 0 \tag{4.4}$$

for a constant field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. This is an additional constraint equation which did not exist in the free theory, and hence it violates the correct number of degrees of freedom [145].

To get around this issue, we can introduce interactions at the level of the Lagrangian, and only then derive the equations of motion [146]. However, there can still be inconsistencies [116]. We start by complexifying eq. (3.9) to give it a $U(1)$ charge, and couple it minimally to electromagnetism via

$$\begin{aligned}\mathscr{L} = {}& D_\mu \overline{H}_{\nu\rho} D_\mu H_{\nu\rho} - D_\mu \overline{H}_{\nu\rho} D_\nu H_{\mu\rho} - D \cdot \overline{H}_\mu D \cdot H_\mu + D_\mu \overline{H} D \cdot H_\mu \\ & + D \cdot \overline{H}_\mu D_\mu H - D_\mu \overline{H} D_\mu H - m^2(\overline{H}_{\mu\nu} H_{\mu\nu} - \overline{H} H) + ie\alpha F_{\mu\nu} \overline{H}_{\mu\rho} H_{\nu\rho}\end{aligned} \tag{4.5}$$

where α is a free coefficient. The last term is introduced due to the ambiguity in the $\partial_\mu \to D_\mu$ prescription, since $[D_\mu, D_\nu] \neq 0$. The equations of motion are

$$\mathscr{R}_{\mu\nu} \equiv -(D^2 + m^2)(H_{\mu\nu} - \eta_{\mu\nu} H) - D_{(\mu} D_{\nu)} H - \eta_{\mu\nu} D \cdot D \cdot H \tag{4.6}$$

$$+ 2D_{(\mu} D \cdot H_{\nu)} - ie(\alpha + 1) F_{\rho(\mu} H_{\nu)\rho} \tag{4.7}$$

implying

$$D \cdot D \cdot \mathscr{R} - \frac{m^2}{2} \mathscr{R} = ie\alpha F_{\mu\nu} D_\mu D \cdot H_\nu - \frac{3}{2} m^4 H + \mathscr{O}(D_\mu F_{\nu\rho}, F_{\mu\nu}^2). \tag{4.8}$$

In the free-theory limit, $e = 0$, the equation above reduces to the constraint $H = 0$. For $e \neq 0$, the first term in eq. 4.8 contains the term $F_{0i}\ddot{H}_{0i}$, meaning that $F_{0i}H_{0i}$ becomes a new propagating degree of freedom, unless we set $\alpha = 0$. Note that we ignored derivatives of the field-strength $F_{\mu\nu}$ and terms quadratic in $F_{\mu\nu}$ for simplicity, since they can only appear with at most one derivative of the massive field $H_{\mu\nu}$ and hence do not affect the above argument.

The standard approach to counting the number of degrees of freedom is the Hamiltonian formalism [151]. However, this requires decomposing fields and derivatives into time and space components. In the next sections we will see an alternative covariant approach.

4.1 Massive Spin-1

A systematic way to preserve the correct number of degrees of freedom is to introduce a *massive gauge symmetry* [119]. Let us start from the free theory for a massive spin-1 field W^μ. We introduce an auxiliary scalar field ϕ, known as a *Stückelberg field*, and impose the linearised gauge transformations δ_0,

$$\delta_0 W_\mu = \partial_\mu \lambda, \qquad \delta_0 \phi = m\lambda. \tag{4.9}$$

The free Lagrangian $\mathscr{L}_0$ (with at most two derivatives) invariant under the above transformations is

$$\mathscr{L}_0 = 2\overline{\partial_{[\mu}W_{\nu]}}\partial^{[\mu}W^{\nu]} - (m\overline{W}_\mu - \partial_\mu\bar{\phi})(mW^\mu - \partial^\mu\phi). \tag{4.10}$$

The gauge choice $\phi = 0$ reproduces the complexified version of eq. (3.3), hence the theory correctly describes a massive spin-1 particle.

Now, we wish to couple this system to electromagnetism, described by a vector field A^μ and the Lagrangian (3.15). A standard way of doing so is the *minimal coupling* prescription, realised by the replacement $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, resulting in the Lagrangian

$$\mathscr{L}_0 = 2\overline{D_{[\mu}W_{\nu]}}D^{[\mu}W^{\nu]} - (m\overline{W}_\mu - D_\mu\bar{\phi})(mW^\mu - D^\mu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.11}$$

To ensure the interacting theory still has the correct number of degrees of freedom, we need to preserve the gauge invariance relating the fields W_μ and ϕ. However, the Lagrangian (4.11) is not invariant under $\delta_0 W_\mu = D_\mu\lambda, \delta_0\phi = m\lambda$. We have

$$\delta_0\mathscr{L}_0 = \overline{D^\mu W^\nu}[D_\mu, D_\nu]\lambda + \text{c.c.} = -ie\overline{D^\mu W^\nu}F_{\mu\nu}\lambda + \text{c.c.} \tag{4.12}$$

where we omit the complex-conjugate term for simplicity. To fix this issue we can add a non-minimal Lagrangian $\mathscr{L}_1$ and non-minimal gauge transformations δ_1. The lowest-derivative solution is

$$\begin{aligned} \mathscr{L}_1 &= ieF^{\mu\nu}\overline{W}_\mu W_\nu, \\ \delta_1 A_\mu &= ie(\overline{W}_\mu\lambda - \bar{\lambda}W_\mu). \end{aligned} \tag{4.13}$$

such that $(\delta_0 + \delta_1)(\mathscr{L}_0 + \mathscr{L}_1) = 0$. We will consistently neglect possible terms with more than two massive fields, since they do not contribute to the amplitudes discussed in this thesis. Note that the variation $\delta_1 A_\mu$ has no relation to the massless $U(1)$ gauge transformation, since λ is not the massless gauge parameter.

Setting $\phi = 0$ and omitting the kinetic term for A_μ, we obtain

$$\mathscr{L} = 2\overline{D_{[\mu}W_{\nu]}}D^{[\mu}W^{\nu]} - m^2\overline{W}_\mu W^\mu + ieF^{\mu\nu}\overline{W}_\mu W_\nu. \tag{4.14}$$

Remarkably, this theory is realised in nature, since it is precisely the interaction of W-bosons with the photon in the Standard Model. In that context, the field ϕ is the component of the Higgs field that gets absorbed by W_μ and $\phi = 0$ is known as *unitary gauge*. This Lagrangian yields the known three-point amplitude $\mathscr{A}(W_1\overline{W}_2A_3)$, where the subscripts denote the momentum label, between two massive vectors and a photon,

$$2\varepsilon_1\cdot\varepsilon_2\ \varepsilon_3\cdot p_1 + 2\varepsilon_2\cdot\varepsilon_3\ \varepsilon_1\cdot p_2 + 2\varepsilon_3\cdot\varepsilon_1\ \varepsilon_2\cdot p_3. \tag{4.15}$$

Before moving onto the next example, let us discuss another convenient gauge choice. We can add to the Lagrangian (4.10) the gauge-fixing term

$$\mathscr{L}_{\text{gf}} = (\overline{D\cdot W} + m\bar{\phi})(D\cdot W + m\phi) \tag{4.16}$$

without affecting any physical scattering amplitude [152,153]. The Lagrangian becomes diagonal,

$$\mathscr{L}_F = \mathscr{L}_0 + \mathscr{L}_{\text{gf}} = -\overline{W}^{\mu}(D^2+m^2)W_\mu + \bar{\phi}(D^2+m^2)\phi, \tag{4.17}$$

allowing us to write down simple propagators for the fields W_μ and ϕ and simplifying amplitude calculations. This is equivalent to the *Feynman gauge* in electrodynamics. We will see how to generalise this choice to higher spins below.

We can also couple a massive spin-1 field to gravity, via the minimal coupling prescription $\partial_\mu \to \nabla_\mu$, where ∇_μ is the covariant derivative with respect to the Levi-Civita connection. The Lagrangian is

$$\mathscr{L}_0 = \sqrt{-g}\left(2\overline{\nabla_{[\mu}W_{\nu]}}\nabla^{[\mu}W^{\nu]} - (m\overline{W}_\mu - \nabla_\mu\bar{\phi})(mW^\mu - \nabla^\mu\phi)\right). \tag{4.18}$$

The minimal gauge transformations are $\delta_0 W_\mu = \nabla_\mu\lambda, \delta_0\phi = m\lambda$. Similarly to the gauge-theory case in eq. (4.12), we get

$$\delta_0\mathscr{L}_0 = \sqrt{-g}\left(\overline{\nabla^\mu W^\nu}[\nabla_\mu,\nabla_\nu]\lambda + \text{c.c.}.\right). \tag{4.19}$$

However, here $[\nabla_\mu,\nabla_\nu]\lambda = 0$ since the Levi-Civita connection is torsion-free. Hence, minimal coupling to gravity does produce a healthy theory for a massive spin-1 particle. We can confirm this by realising that the above theory is just the Kaluza-Klein reduction of Maxwell theory in five dimensions. To see this, we start with the action

$$\mathscr{L}_0 = -\frac{1}{4}F_{MN}F^{MN}, \tag{4.20}$$

where $M,N = 0,\ldots,4$ and the theory is invariant under $\delta_0 A_M = \partial_M\lambda$. Then we compactify on the fifth dimension and only keep the first massive state, with $A_M = (W_\mu,\phi)$ and $\partial_M = (\partial_\mu, im)$. This recovers the Lagrangian (4.10) and the gauge transformations (4.9). Since eq. (4.20) can be minimally coupled to gravity, eq. (4.18) follows.

4.2 Massive Spin-2

We can repeat the analysis for a massive spin-2 field $H_{\mu\nu}$ [119]. In this case, we introduce two Stückelberg fields B_μ and ϕ. The minimally-coupled Lagrangian is

$$\mathscr{L}_0 = \mathscr{L}_{02} + \mathscr{L}_{01} + \mathscr{L}_{00} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.21}$$

where

$$
\begin{aligned}
\mathscr{L}_{02} =& \overline{D_\alpha H_{\mu\nu}} D_\alpha H_{\mu\nu} - \overline{D_\alpha H_{\mu\nu}} D_\mu H_{\nu\alpha} - \overline{D\cdot H_\mu} D\cdot H_\mu + \overline{D\cdot H_\nu} D_\nu H \\
&+ \overline{D_\nu H} D\cdot H_\nu - \overline{D_\mu H} D_\mu H - \overline{B}_{\mu\nu} B_{\mu\nu} + \frac{4(d-1)}{d-2} \overline{D_\alpha \phi} D_\alpha \phi, \\
\mathscr{L}_{01} =& 2m(\overline{H}_{\mu\nu} D_\mu B_\nu + \overline{D_\mu B_\nu} H_{\mu\nu}) - 2m(\overline{H} D\cdot B + \overline{D\cdot B} H) \\
&+ \frac{4m(d-1)}{d-2} (\overline{D\cdot B}\phi + \bar{\phi} D\cdot B), \\
\mathscr{L}_{00} =& -m^2(\overline{H}_{\mu\nu} H_{\mu\nu} - \overline{H} H) - \frac{2m^2(d-1)}{(d-2)} (\overline{H}\phi + \bar{\phi} H) + \frac{4m^2 d(d-1)}{(d-2)^2} \bar{\phi}\phi.
\end{aligned}
\tag{4.22}
$$

The second number in the subscript denotes the number of derivatives. When $e = 0$ and $D_\mu = \partial_\mu$, this is invariant under the gauge transformations

$$
\begin{aligned}
\delta_0 H_{\mu\nu} &= 2D_{(\mu}\xi_{\nu)} + \frac{2m}{d-2}\eta_{\mu\nu}\lambda, \\
\delta_0 B_\mu &= D_\mu \lambda + m\xi_\mu, \\
\delta_0 \phi &= m\lambda.
\end{aligned}
\tag{4.23}
$$

In the unitary gauge $B_\mu = \phi = 0$, the above reduces to the complexified version of eq. (3.9). When $e \neq 0$, the invariance is broken and must be restored via non-minimal terms. At cubic level, the lowest-derivative interesting solution is [1]

$$
\mathscr{L}_1 = \mathscr{L}_{13} + \mathscr{L}_{12} + \mathscr{L}_{11} \tag{4.24}
$$

[1] As discussed in ref. [119], there is another solution with at most two derivatives. However, the theory presented is more interesting because of its connection to massless higher-spin particles in curved spacetime. In addition, it satisfies the current constraint discussed in section III.

where

$$\begin{aligned}
\mathscr{L}_{13} = &- 2ia_0 F_{\mu\nu}\Big(\frac{1}{2}\overline{D_\mu H_{\alpha\beta}}D_\alpha H_{\beta\nu} - \frac{1}{2}\overline{D_\alpha H_{\beta\nu}}D_\mu H_{\alpha\beta} + \frac{1}{2}\overline{D_\alpha H_{\beta\mu}}D_\alpha H_{\beta\nu} \\
&- \overline{D_\alpha H_{\beta\mu}}D_\beta H_{\alpha\nu} - \frac{1}{2}\overline{D_\mu H_{\alpha\beta}}D_\nu H_{\alpha\beta} + \frac{1}{2}\overline{D_\mu H_{\nu\alpha}}D\cdot H_\alpha \\
&- \frac{1}{2}\overline{D\cdot H_\alpha}D_\mu H_{\nu\alpha} + \frac{1}{2}\overline{D\cdot H_\mu}D\cdot H_\nu - \frac{1}{2}\overline{D\cdot H_\mu}D_\nu H + \frac{1}{2}\overline{D_\nu H}D\cdot H_\mu \\
&- \frac{1}{2}\overline{D_\mu H_{\nu\alpha}}D_\alpha H + \frac{1}{2}\overline{D_\alpha H}D_\mu H_{\nu\alpha} + \frac{1}{2}\overline{D_\mu H}D_\nu H - \frac{d-4}{2(d-2)}\overline{B}_{\mu\alpha}B_{\alpha\nu}\Big), \\
\mathscr{L}_{12} = &- 2ia_0 F_{\mu\nu}\Big\{\frac{m}{d-2}\Big((d-4)\overline{D_\mu H_{\nu\alpha}}B_\alpha - (d-4)\overline{B}_\alpha D_\mu H_{\nu\alpha} \\
&+ (d-3)\overline{D\cdot H_\mu}B_\nu - (d-3)\overline{B}_\nu D\cdot H_\mu - (d-3)\overline{D_\mu H}B_\nu \\
&+ (d-3)\overline{B}_\nu D_\mu H\Big) - \frac{m(d-4)(d-1)}{2(d-2)^2}(\overline{B}_{\mu\nu}\phi - \bar{\phi}B_{\mu\nu})\Big\} \\
\mathscr{L}_{11} = &- 2ia_0 F_{\mu\nu}\Big(\frac{m^2}{2(d-2)}\overline{H}_{\mu\alpha}H_{\nu\alpha} + \frac{2m^2(d-3)}{d-2}\overline{B}_\mu B_\mu\Big),
\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
\delta_1 H_{\mu\nu} = &2ia_0\Big\{\frac{1}{2}(F_{\mu\alpha}D_{[\alpha}\xi_{\nu]} + F_{\nu\alpha}D_{[\alpha}\xi_{\mu]}) + \frac{1}{2(d-2)}\eta_{\mu\nu}F_{\alpha\beta}D_{[\alpha}\xi_{\beta]}\Big\}, \\
\delta_1 A_\mu = &2ia_0\Big\{\frac{2m}{d-2}(\overline{B}_\alpha D_{[\mu}\xi_{\alpha]} - \overline{D_{[\mu}\xi_{\alpha]}}B_\alpha) + \overline{D_\alpha H_{\beta\mu}}D_{[\alpha}\xi_{\beta]} \\
&- \overline{D_{[\alpha}\xi_{\beta]}}D_\alpha H_{\beta\mu} + \frac{m^2}{d-2}(\overline{H}_{\mu\alpha}\xi_\alpha - \bar{\xi}_\alpha H_{\mu\alpha}) \\
&- \frac{2m^2}{(d-2)^2}(\overline{B}_\mu\lambda - \bar{\lambda}B_\mu)\Big\}.
\end{aligned} \tag{4.26}$$

Here $a_0 \equiv -\frac{e(d-2)}{m^2(d-3)}$. As before, we have neglected terms with more than two massive fields. In addition, we only have $(\delta_0+\delta_1)(\mathscr{L}_0+\mathscr{L}_1) = 0 + \mathscr{O}(e^2)$. To restore gauge invariance at $\mathscr{O}(e^2)$ and beyond, we need to add higher-point contributions to the Lagrangian and the gauge transformations. Since these terms are complicated, we will not discuss them explicitly. However, we will show how to solve this problem from a simpler on-shell viewpoint in part III. We can also use the Lagrangian above to compute the three-point amplitude $\mathscr{A}(H_1\overline{H}_2A_3)$, obtaining

$$\begin{aligned}
&\frac{4}{m^2}(\varepsilon_1 p_2\varepsilon_1\varepsilon_3(\varepsilon_2\cdot p_1)^2 - \varepsilon_2 p_1\varepsilon_2\varepsilon_3(\varepsilon_1\cdot p_2)^2 - \varepsilon_1 p_2\varepsilon_2 p_1\varepsilon_1\varepsilon_2\varepsilon_3 p_1) \\
&4\varepsilon_1\varepsilon_2\varepsilon_1\varepsilon_3\varepsilon_2 p_1 - 4\varepsilon_1 p_2\varepsilon_1\varepsilon_2\varepsilon_2\varepsilon_3 - 2(\varepsilon_1\cdot\varepsilon_2)^2\varepsilon_3 p_1 = \frac{1}{x}\frac{[\mathbf{12}]^4}{m^3},
\end{aligned} \tag{4.27}$$

where the right-hand side is written in terms of the spinor-helicity variables of part II. This amplitude matches eq. (7.20) for $s = 2$ and we will discuss the significance of this in part III.

In the gravitational case, the lowest-derivative gauge-invariant theory is simply the minimally-coupled one, eq. (4.22) and eq. (4.23), with $D_\mu \to \nabla_\mu$. This can be checked explicitly, or derived from a Kaluza-Klein reduction of massless gravity in five dimensions [82]. We refer the reader to paper I for more details.

It is important to note that for $s > 2$ there is no consistent interacting massless theory in flat spacetime for a single massless field. Therefore, we cannot find massive higher-spin theories from a naive Kaluza-Klein reduction. Nevertheless, there are still strong ties between the massive theories we are studying and massless higher-spin fields in curved background, as discussed by ref. [119, 154].

4.3 Higher Spin

The approach outlined above can be applied to fields of arbitrary spin [117]. To describe a spin-s particle, we introduce a tower of symmetric tensor fields $\Phi^k \equiv \Phi^{\mu_1\cdots\mu_k}$, where $k = 0, 1, \ldots, s-1, s$. We can choose these fields to be double-traceless, $\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\Phi^{\mu_1\cdots\mu_k} = 0$, and we denote a single trace by $\tilde{\Phi}^k \equiv \eta_{\mu_{k-1}\mu_k}\Phi^{\mu_1\cdots\mu_k}$. We introduce the gauge transformation

$$\delta\Phi^k = \partial^{(1}\xi^{k-1)} + m\alpha_k\xi^k + m\beta_k\eta^{(2}\xi^{k-2)}\,, \tag{4.28}$$

in terms of traceless symmetric gauge parameters $\xi^k \equiv \xi^{\mu_1\cdots\mu_k}$, where $k = 0, 1, \ldots, s-1$. We use the notation $\partial^{(1}\xi^{k-1)} = \partial^{(\mu_1}\xi^{\mu_2\cdots\mu_k)}$ and $\eta^{(2}\xi^{k-2)} = \eta^{(\mu_1\mu_2}\xi^{\mu_3\cdots\mu_k)}$. The coefficients α_k and β_k are given by

$$\alpha_k = \frac{1}{k+1}\sqrt{\frac{(s-k)(s+k+1)}{2}}\,, \quad \beta_k = \frac{1}{2}\frac{k}{k-1}\alpha_{k-1}. \tag{4.29}$$

The free Lagrangian invariant under eq. (4.28) can be written as

$$\mathscr{L}_0 = \mathscr{L}_{\mathrm{F}} - \mathscr{L}_{\mathrm{gf}}, \tag{4.30}$$

where

$$\mathscr{L}_{\mathrm{F}} = \sum_{k=0}^{s}\frac{(-1)^k}{2}\left[\Phi^k(\Box + m^2)\Phi^k - \frac{k(k-1)}{4}\tilde{\Phi}^k(\Box + m^2)\tilde{\Phi}^k\right] \tag{4.31}$$

and

$$\mathscr{L}_{\mathrm{gf}} = -\frac{1}{2}\sum_{k=0}^{s-1}(-1)^k(k+1)G^kG^k\,, \tag{4.32}$$

with

$$G^k = \partial\cdot\Phi^{k+1} - \frac{k}{2}\partial^{(1}\tilde{\Phi}^{k+1)} + m\big(\alpha_k\Phi^k - \gamma_k\tilde{\Phi}^{k+2} - \delta_k\eta^{(2}\tilde{\Phi}^{k)}\big). \tag{4.33}$$

We use $\gamma_k = \frac{1}{2}(k+2)\alpha_{k+1}$ and $\delta_k = \frac{1}{4}(k-1)\alpha_k$. This provides a gauge invariant description of a free massive spin-s field. Using eq. (4.28) the traceless part of $\Phi^{k<s}$ can be set to zero, providing a higher-spin generalisation of the unitary gauge encountered in the spin-1 case. This choice reproduces the Lagrangian (3.9), up to field redefinitions. Alternatively, we can set $\mathscr{L}_0 = \mathscr{L}_F$ by adding the gauge-fixing term $\mathscr{L}_{\text{gf}}$ to the Lagrangian, generalising the spin-1 Feynman gauge.

The next step is to introduce interactions to the massless force carriers and restore gauge invariance by adding non-minimal terms, as discussed in the spin-1 and spin-2 cases. This is a cumbersome process and it is in general challenging to obtain closed-form expressions for any spin, so this will not be the goal of this thesis. However, in section III we will show how to circumvent this and make progress by using Ward identities.

4.4 High-Energy Limit

At the end of section I, we argued that the highest-derivative part of a free massive higher-spin Lagrangian must be invariant under the gauge transformations $\delta\Phi^{\mu_1\cdots\mu_s} = \partial^{(\mu_1}\xi^{\mu_2\cdots\mu_s)}$, where $\xi^{\mu_1\cdots\mu_{s-1}}$ is a symmetric traceless gauge parameter. We refer to this part of the theory as the *high-energy limit*. A similar statement can be made about interacting theories. For example, we consider the $s=2$ theory described by eq. (4.22) and eq. (4.25). The gauge variation $(\delta_0+\delta_1)(\mathscr{L}_0+\mathscr{L}_1)$ has at most four derivatives, where the highest-derivative cubic term is proportional to $1/m^2$ and gives

$$\delta_0\mathscr{L}_{13} + \delta_1\left(\mathscr{L}_{02} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = 0 + \mathscr{O}(m^{-1}). \tag{4.34}$$

The above equation only depends on the fields $H_{\mu\nu}$ and A_μ and the gauge parameter ξ_μ, up to $\mathscr{O}(m^{-1})$ terms. This is connected to the existence of cubic interaction vertices with massless higher-spin fields in (anti)-de Sitter spacetime, as discussed in ref. [117]. Cubic interactions for massless fields have been studied for an arbitrary spin-s particle, coupled to electromagnetism ($h=1$) or gravity ($h=2$), and the lowest-derivative solution for the three-point vertex is known to have $2s-h$ derivatives [155]. Hence, a massive higher-spin theory can be constructed starting from the $(2s-h)$-derivative massless vertex and adding lower-derivative terms to restore massive gauge invariance. This is consistent with the $s=1$ and $s=2$ examples discussed above. Note that there can be massive theories with even less derivatives, such as the two-derivative $s=2$ theory mentioned previously. However, these theories can be excluded by imposing additional constraints, such as eq. (9.2).

5. String Theory

The most notable example of consistent theory with massive higher-spin particles is string theory [156–158]. The bosonic string can be described by the Polyakov action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_a X^\mu \partial^a X_\mu, \tag{5.1}$$

in terms of the worldsheet coordinates σ and τ and the spacetime coordinates $X^\mu(\sigma,\tau)$. The Latin indices $a = \tau,\sigma$ are worldsheet indices, whereas the Greek indices $\mu = 0,\ldots,d-1$ belong to spacetime. The parameter α' is the inverse tension of the string.

In the case of *closed strings*, the spacetime coordinates X^μ are periodic functions of the spatial worldsheet coordinate $\sigma \in [0,2\pi]$ and their spectrum contains a massless graviton. If no periodicity is assumed, for instance by restricting $\sigma \in [0,\pi]$, we have an *open string* and the spectrum contains a massless vector boson (a photon or gluon). In addition, both open and closed strings give rise to an infinite tower of massive higher-spin particles. In both cases, however, the spectrum contains a tachyon, an unphysical particle with negative squared mass, $m^2 < 0$. To get around this problem, a new action was found by adding fermionic degrees of freedom to the Polyakov action and imposing worldsheet supersymmetry. The resulting theory is known as *superstring theory* and it is tachyon-free [159]. Paper II studies a special class of states in superstring theory, known as *leading Regge trajectory* states. In the open string case, they are defined by the relation

$$m^2 = \frac{s-1}{\alpha'} \tag{5.2}$$

where m is the mass of the state and s its spin.

Scattering amplitudes with leading Regge superstring states can be computed following the approach in ref. [143]. Each string state is associated to a *vertex operator* $\mathscr{V}_s(\varepsilon,p;z)$, where $z = e^{\tau+i\sigma}$ is a complexified worldsheet coordinate, p is the momentum of the particle, ε its polarisation tensor and s its spin. String amplitudes are correlators of such operators in the path integral defined by the action (5.1) or its supersymmetric extension. In the case of massive leading Regge states, the vertex operators are

$$\mathscr{V}_s^{(-1)}(\varepsilon,p;z) = \frac{1}{\sqrt{2\alpha'}^{s-1}} \varepsilon_{\mu_1\ldots\mu_s} :i\partial X^{\mu_1}\ldots i\partial X^{\mu_{s-1}}\psi^{\mu_s}e^{-\phi}e^{ip\cdot X}:, \tag{5.3}$$

where $\partial = \partial/\partial z$, ψ^μ is the worldsheet fermion field and ϕ is a superghost field needed to fix certain fermionic gauge symmetries. The polarisation tensor ε is symmetric, traceless and transverse. The colons $:(\dots):$ denote the normal-ordering prescription, and the superscript (-1) on the vertex operator denotes the ghost picture (we refer the reader to ref. [156, 159] for details).

Already the simplest string amplitudes necessitate vertex operators in alternative ghost pictures. For the massless vector bosons, the analogue of eq. (5.3) in the zero ghost picture is

$$\mathscr{V}_1^{(0)}(\varepsilon, p; z) = \frac{1}{\sqrt{2\alpha'}}\varepsilon_\mu : \left(2\alpha' p\cdot\psi\,\psi^\mu + i\partial X^\mu\right) e^{ip\cdot X} :, \tag{5.4}$$

where $p^2 = 0$ is the massless momentum and $\varepsilon^2 = \varepsilon\cdot p = 0$ is the polarisation vector.

The tree-level three-point amplitude $\mathscr{A}(\Phi_1^s\Phi_2^s A_3)$ between two massive spin-s states and one massless vector is given by the following correlator,

$$\left\langle c(z_1)\,\mathscr{V}_s^{(-1)}(\varepsilon_1, p_1; z_1)\,c(z_2)\,\mathscr{V}_s^{(-1)}(\varepsilon_2, p_2; z_2)\,c(z_3)\,\mathscr{V}_1^{(0)}(\varepsilon_3, p_3; z_3)\right\rangle, \tag{5.5}$$

where the ghost fields $c(z)$ are needed to fix the conformal invariance of the Polyakov action and render eq. (5.5) independent of z_1, z_2, z_3. This yields the amplitude

$$\begin{gathered}\mathscr{A}(\Phi_1^s\Phi_2^s A_3) = -g(2\alpha')^s(s-1)!\sum_{n=0}^{s}\frac{(-\varepsilon_1\cdot\varepsilon_2)^n}{(2\alpha')^n n![(s-n)!]^2}\times\\ \left(n(\varepsilon_3\cdot p_1)(-\varepsilon_1\cdot p_3\,\varepsilon_2\cdot p_3)^{s-n} - \frac{s(s-n)}{2\alpha'}\varepsilon_2\cdot f_3\cdot\varepsilon_1(-\varepsilon_1\cdot p_3\,\varepsilon_2\cdot p_3)^{s-n-1}\right)\end{gathered} \tag{5.6}$$

where the overall normalisation differs from ref. [143] and it is discussed in more detail in paper II.

In spacetime dimensions $d = 4$, we can rewrite this amplitude in the massive spinor variables discussed in part II, obtaining

$$\begin{gathered}\mathscr{A}(\Phi_1^s\Phi_2^s A_3^-) = -\frac{g(s-1)!}{m^{2s}}(\varepsilon_3^-\cdot p_1)\times\\ \sum_{n=0}^{s}\frac{(s-1)^{s-n-1}(\langle\mathbf{12}\rangle[\mathbf{12}])^n(\langle\mathbf{12}\rangle-[\mathbf{12}])^{2s-2n-1}(n(s-1)\langle\mathbf{12}\rangle-(s^2-n)[\mathbf{12}])}{n![(s-n)!]^2}\end{gathered} \tag{5.7}$$

in the case of a negative-helicity massless vector. Note that the amplitudes (5.7) can be thought of as a dimensional reduction of superstrings in $d = 10$.

In the closed superstring case, leading Regge trajectory states obey the relation

$$m^2 = \frac{2s-4}{\alpha'}. \tag{5.8}$$

The amplitudes $\mathscr{M}(\Phi_1^s\Phi_2^s h_3)$ between two leading Regge states Φ^s and the graviton h can be found via the Kawai-Lewellen-Tye (KLT) relation [160]

$$\mathscr{M}(\Phi_1^s\Phi_2^s h_3) = \left(\frac{1}{g}\mathscr{A}(\Phi_1^{s/2}\Phi_2^{s/2}A_3)\big|_{\alpha'\to\alpha'/4}\right)^2 \tag{5.9}$$

where $\mathscr{A}(\Phi_1^s\Phi_2^s A_3)$ is the open superstring amplitude given in eq. (5.6) and the transverse-traceless graviton polarisation tensor can be written as $\varepsilon_3^{\mu\nu} = \varepsilon_3^{\mu}\varepsilon_3^{\nu}$, since $\varepsilon_3^2 = \varepsilon_3 \cdot p_3 = 0$.

Part II:
Massive Higher-Spin Amplitudes

From here on we assume $d = 4$.

6. Massless Spinor-Helicity

In part I, we studied how to construct higher-spin Lagrangians that can describe states that are irreducible representations of the Poincaré group. In particular, we saw how requiring the correct number of degrees of freedom, for instance via the transversality condition $\partial \cdot \phi = 0$, leads to a massive gauge symmetry and a tower of auxiliary fields. However, if we are interested in on-shell observables, it is natural to wonder if one can find some variables that automatically encode the right degrees of freedom.

Massless theories give us reason to be optimistic. For instance, let us consider massless Yang-Mills theory

$$\mathscr{L} = -\frac{1}{4} F^{a\mu\nu} F^{a}_{\mu\nu}, \tag{6.1}$$

where $F^{a}_{\mu\nu} = 2\partial_{[\mu} A^{a}_{\nu]} + g/\sqrt{2} f^{abc} A^{b}_{\mu} A^{c}_{\nu}$, g is the coupling constant and f^{abc} are the structure constants for $SU(N)$. This Lagrangian is invariant under the gauge transformation $\delta A^{a}_{\mu} = \partial_\mu \lambda^a + g/\sqrt{2} f^{abc} A^b \lambda^c$. The color-ordered three-gluon amplitude $\mathscr{A}(A_1 A_2 A_3)$ is given by

$$\mathscr{A}(A_1 A_2 A_3) = \varepsilon_1 \cdot \varepsilon_2\, \varepsilon_3 \cdot p_1 + \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_1 \cdot p_2 + \varepsilon_3 \cdot \varepsilon_1\, \varepsilon_2 \cdot p_3, \tag{6.2}$$

up to normalisation factors. The subscript i on the field A_i denotes the label of the associated momentum and polarisation vector. The momenta obey the on-shell condition $p_i^2 = 0$ and the polarisation vectors ε_i obey $\varepsilon_i^2 = \varepsilon_i \cdot p_i = 0$. All momenta are assumed to be outgoing. All contractions are with respect to the flat Minkowski metric in mostly-minus signature, $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The amplitude (6.2) obeys the Ward identity

$$p_1 \cdot \frac{\partial}{\partial \varepsilon_1} \mathscr{A}(A_1 A_2 A_3) = 0, \tag{6.3}$$

and similar identities for legs 2 and 3. The operator $p_1 \cdot \partial/\partial \varepsilon_1$ implements the replacement $\varepsilon_1 \to p_1$. This type of identity arises because the vector ε_1^{μ} contains four components, but on-shell gluons in $d = 4$ only have two physical degrees of freedom, so there must be two constraints. One is the transversality condition $\varepsilon_1 \cdot p_1 = 0$, the other is the equivalence relation $\varepsilon_1 \sim \varepsilon_1 + \alpha p_1$ responsible for the Ward identity, where α is an arbitrary coefficient.

Alternatively, we can describe a gluon as the tensor product of two massless spin-$1/2$ fermions, as shown in eq. (6.6) [161]. We consider the two-component Weyl spinors $|p]^{\dot\alpha}$ and $|p\rangle_{\alpha}$, with positive and negative chirality respectively, obeying the Weyl equations

$$(p\cdot\bar{\sigma})^{\dot{\alpha}\beta}\,|p\rangle_{\beta} = (p\cdot\sigma)_{\alpha\dot{\beta}}|p]^{\dot{\beta}} = 0, \tag{6.4}$$

in terms of the Pauli Matrices $\sigma^{\mu} = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ in the Weyl representation, and their conjugates $\bar{\sigma}^{\mu} = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$. The indices $\alpha, \dot{\alpha} = 1, 2$ are left-handed and right-handed spinor indices. Since we have $\det p\cdot\sigma = \det p\cdot\bar{\sigma} = 0$, we can decompose the momentum in terms of the spinors,

$$\begin{aligned}(p\cdot\sigma)_{\alpha\dot{\beta}} &= |p\rangle_{\alpha}\,[p|_{\dot{\beta}},\\ (p\cdot\bar{\sigma})^{\dot{\alpha}\beta} &= |p]^{\dot{\alpha}}\,\langle p|^{\beta},\end{aligned} \tag{6.5}$$

where $[p|_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}|p]^{\dot{\beta}}$, $\langle p|^{\alpha} = \varepsilon^{\alpha\beta}\,|p\rangle_{\beta}$ and $\varepsilon^{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}$ are two-dimensional Levi-Civita tensors, normalised such that $\varepsilon^{12} = 1$. Note that eq. (6.5) is invariant under the transformation $|p] \to t|p], |p\rangle \to t^{-1}\,|p\rangle$, where t is a complex parameter. This is the action of the $U(1)$ little group for massless particles, i.e. the Lorentz transformations that leave the momentum invariant.

The polarisation vectors can also be expressed in terms of spinors, via the relations

$$\begin{aligned}\varepsilon_{+}^{\mu}(p) &= \frac{\langle q|\,\sigma^{\mu}\,|p]}{\sqrt{2}\langle qp\rangle},\\ \varepsilon_{-}^{\mu}(p) &= \frac{\langle p|\,\sigma^{\mu}\,|q]}{\sqrt{2}[pq]},\end{aligned} \tag{6.6}$$

describing a positive-helicity and negative-helicity gluon respectively. Note that $|q]$ and $|q\rangle$ are arbitrary reference spinors, parametrising the redundancy due to gauge invariance. Any gauge invariant observables, such as amplitudes, will not depend on the reference spinors and can hence be written only in terms of the Weyl spinors $|p]$ and $|p\rangle$. For instance, we can rewrite eq. (6.2) as

$$\mathscr{A}(A_1^- A_2^- A_3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \tag{6.7}$$

where we assumed helicities $h_1 = h_2 = -1$ and $h_3 = +1$. This is known as the three-point *Parke-Taylor formula* and it can be extended to higher-point amplitudes.

Note that the amplitude behaves correctly under little group transformations, namely

$$\mathscr{A}(A_1^{h_1}\dots) \xrightarrow[|1\rangle\to\frac{1}{t}|1\rangle]{|1]\to t|1]} t^{2h_1}\mathscr{A}(A_1^{h_1}\dots), \tag{6.8}$$

and similarly for the other particles $A_i^{h_i}$. Moreover, since the spinors $|p]$ and $|p\rangle$ only contain two on-shell degrees of freedom, there is no gauge redundancy and no need for identities similar to eq. (6.3).

7. Massive Spinor-Helicity

7.1 On-shell Construction of Amplitudes

In the previous section, we learned that massless particles can be described by on-shell spinors. These automatically encode the correct number of degrees of freedom, hence they are not sensitive to any gauge redundancy. Below we will see that the same can be done for massive particles. An on-shell massive spin-s particle can be thought of as a tensor product of $2s$ spin-$1/2$ particles. The latter can be described by Weyl spinors $|p^a]^{\dot\alpha}$ and $|p^a\rangle_\alpha$ for any given momentum $p^2 = m^2$, where $a = 1,2$ is the index associated to the massive little group $SU(2)$ [108]. The little group indices encode the $SU(2)$ spin degrees of freedom of the particles: $a = 1$ corresponds to spin up and $a = 2$ to spin down, for an appropriate choice of basis. We can define little-group polarisation variables z_a and bolded spinors as follows

$$|\boldsymbol{p}] = z_a|p^a]\,, \qquad |\boldsymbol{p}\rangle = z_a|p^a\rangle. \tag{7.1}$$

The Dirac spinors $\bar{u}(p)$ and $v(p)$, describing an outgoing particle and outgoing antiparticle respectively, can be written in terms of Weyl spinors as

$$\bar{u}(p) = \big(\langle\boldsymbol{p}|, -[\boldsymbol{p}|\big)\,, \qquad v(p) = \begin{pmatrix} |\boldsymbol{p}\rangle \\ |\boldsymbol{p}] \end{pmatrix}, \tag{7.2}$$

where $[\boldsymbol{p}|_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}|\boldsymbol{p}]^{\dot\beta}$ and $\langle\boldsymbol{p}|^\alpha = \varepsilon^{\alpha\beta}|\boldsymbol{p}\rangle_\beta$. The spinors can be related to standard on-shell polarisation vectors and momenta as follows

$$\varepsilon^\mu(p) = \frac{\langle\boldsymbol{p}|\sigma^\mu|\boldsymbol{p}]}{\sqrt{2}m}\,, \qquad p^\mu = \frac{1}{2}\langle p^a|\sigma^\mu|p_a], \tag{7.3}$$

where little group indices are lowered with the Levi-Civita tensor ε_{ab}. Therefore, any scattering amplitude can be written entirely in terms of spinors, or alternatively in terms of bold spinors and momenta. We will use the latter approach below.

Note that angle and square spinors are not independent, since they are related by the Dirac equation

$$p\cdot\sigma|\boldsymbol{p}] = m|\boldsymbol{p}\rangle\,, \qquad p\cdot\bar\sigma|\boldsymbol{p}\rangle = m|\boldsymbol{p}], \tag{7.4}$$

where as before σ^μ are Pauli matrices. Therefore, we can work with only square (or angle) spinors without loss of generality. A spin-s particle is described by the tensor product

$$|p^{(a_1}]\cdots|p^{a_{2s})}]. \tag{7.5}$$

The product of $2s$ spin-$1/2$ states produces states with spins $\{s, s-1, \ldots, 0\}$ and the symmetrisation in eq. (7.5) ensures only the spin-s state survives. The number of degrees of freedom is $2s+1$, as expected. Using bold spinors we write this simply as $|\boldsymbol{p}]^{2s}$, since the z_a variables guarantee symmetrisation. Any scattering amplitude $\mathscr{A}(\Phi^s(p)\ldots)$ involving a spin-s particle Φ^s must hence satisfy

$$\mathscr{A}(\Phi^s(p)\ldots) \propto |\boldsymbol{p}]^{2s}, \tag{7.6}$$

in order to transform correctly under the action of the little group. We can use this property to find the most general scattering amplitudes with massive spinning particles without the need for explicit Lagrangians.

As an example, we consider the construction of the most general tree-level amplitude $\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^-)$ between two equal-mass spin-s particles Φ^s and a negative-helicity massless vector A. Since the amplitude is a polynomial in polarisation vectors and momenta, it is also polynomial in bold spinors and momenta. Hence, all we need to do is identify a basis of monomials.

The dependence on the massless polarisation ε_3 can be factored out in terms of the structure

$$x^{\pm} = \frac{\sqrt{2}}{m}\varepsilon_3^{\pm} \cdot p_1, \tag{7.7}$$

we refer the reader to ref. [108] for details. The minus (plus) sign corresponds to a vector of helicity -1 $(+1)$. The only other independent objects that can appear the amplitude are the spinors $\{|\mathbf{1}], |\mathbf{2}]\}$ and two independent momenta, which we pick to be $\{p_1, p_3\}$. Contractions between momenta reduce to factors of the mass, since $p_1^2 = m^2$ and $p_3^2 = p_1 \cdot p_3 = 0$. The only nonzero contraction between spinors is $[\mathbf{12}]$, since $[\mathbf{11}] = [\mathbf{22}] = 0$. The last class of terms we need to consider are contractions involving both spinors and momenta. We only need the following set of building blocks,

$$\{[\mathbf{1}|p_1 p_3|\mathbf{2}], [\mathbf{1}|p_1 p_3|\mathbf{1}], [\mathbf{2}|p_1 p_3|\mathbf{2}]\}, \tag{7.8}$$

where the momenta are assumed to be contracted into the appropriate Pauli matrices. This is because only an even number of momenta can be placed in-between two square spinors, and if one momentum appears twice we can get rid of it via Clifford algebra identities. Moreover, $[\mathbf{1}|p_1 p_3|\mathbf{1}]$ and $[\mathbf{2}|p_1 p_3|\mathbf{2}]$ can only appear together, since there must be an equal number of spinors for particles 1 and 2. Since $[\mathbf{1}|p_1 p_3|\mathbf{1}][\mathbf{2}|p_1 p_3|\mathbf{2}] = [\mathbf{1}|p_1 p_3|\mathbf{2}]^2$, the only structure needed is the product $[\mathbf{1}|p_1 p_3|\mathbf{2}]$.

In short, we can write any amplitude as

$$\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^-) = \frac{m}{x} \sum_{k=0}^{2s} \frac{c_k}{m^{2s+2k}} [\mathbf{12}]^{2s-k} [\mathbf{1}|p_1 p_3|\mathbf{2}]^k, \tag{7.9}$$

where c_k are free coefficients and the factors of mass can be fixed by dimensional analysis. Using the relation $[\mathbf{1}|p_1 p_3|\mathbf{2}] = m(\langle\mathbf{12}\rangle + [\mathbf{12}])$ we can also write a formula democratic in left- and right-handed spinors,

$$\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^-) = \frac{m}{x} \sum_{k=0}^{2s} \frac{\tilde{c}_k}{m^{2s}} [\mathbf{12}]^{2s-k} \langle \mathbf{12} \rangle^k, \tag{7.10}$$

where $\tilde{c}_k$ are new free coefficients.

The same procedure outlined above can be repeated for the amplitude $\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^-)$ between two massive spin-s particles and one negative-helicity graviton h. The only difference is that the helicity of the massless particle is -2 instead of -1, which leads to the following expression,

$$\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^-) = \frac{m^2}{x^2} \sum_{k=0}^{2s} \frac{\tilde{c}_k}{m^{2s}} [\mathbf{12}]^{2s-k} \langle \mathbf{12} \rangle^k. \tag{7.11}$$

7.2 AHH Amplitudes

We can compare eq. (7.10) to three-point amplitudes in a few known theories [82]. The first example is a spin-1/2 fermion minimally-coupled to a massless vector, such as electrons in QED. The Lagrangian is

$$\mathscr{L}^{s=0} = \bar{\psi}(i\not{D} - m)\psi, \tag{7.12}$$

where $D_\mu = \partial_\mu - ieA_\mu$. The amplitude is

$$\mathscr{A}(\psi_1 \bar{\psi}_2^s A_3^-) = \frac{1}{x}[\mathbf{12}]. \tag{7.13}$$

Another example is a spin-1 massive particle, realised in nature by the W-boson coupling to the photon in the Standard Model. The Lagrangian is

$$\mathscr{L}^{s=1} = 2\overline{D_{[\mu} W_{\nu]}} D^{[\mu} W^{\nu]} - m^2 \overline{W}_\mu W^\mu + ieF_{\mu\nu} \overline{W}^\mu W^\nu. \tag{7.14}$$

The amplitude is

$$\mathscr{A}(W_1 \overline{W}_2^s A_3^-) = \frac{1}{mx}[\mathbf{12}]^2. \tag{7.15}$$

In the gravity case, we can consider minimally-coupled fields of spin $s = 1/2, 1, 3/2$, described by the Lagrangians

$$\begin{aligned}
\mathscr{L}^{s=1/2} &= \sqrt{-g}\,\bar{\psi}(i\not{\nabla} - m)\psi, \\
\mathscr{L}^{s=1} &= \sqrt{-g}\left(2\nabla_{[\mu} \overline{W}_{\nu]} \nabla^{[\mu} W^{\nu]} - m^2 \overline{W}_\mu W^\mu\right), \\
\mathscr{L}^{s=3/2} &= \sqrt{-g}\,\bar{\psi}_\mu \gamma^{\mu\nu\rho}\left(i\nabla_\nu - \frac{m}{2}\gamma_\nu\right)\psi_\rho,
\end{aligned} \tag{7.16}$$

where ∇_μ is the covariant derivative with respect to the Levi-Civita connection. All such theories result in a three-point amplitude of form

$$\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^-) = \frac{m^2}{x^2} \frac{[\mathbf{12}]^{2s}}{m^{2s}}. \tag{7.17}$$

Note that the positive-helicity amplitude $\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^+)$ is obtained by the replacements $x \to x^{-1}$ and $[\mathbf{12}] \to \langle \mathbf{12} \rangle$. The results above suggest that there is something special about the choice $\tilde{c}_{k>0} = 0$ in eq. (7.10) and (7.11). What all the theories discussed above have in common is that they can be obtained from a Kaluza-Klein reduction of the corresponding massless theories in higher dimensions, by truncating the tower of Kaluza-Klein states to the first massive state. As such, the amplitudes in these theories have a finite massless limit. This intuition is confirmed by studying the case of massive spin-2 particles $H_{\mu\nu}$ coupled to the graviton, where the amplitude

$$\mathscr{M}(H_1 \overline{H}_2 h_3^-) = \frac{1}{m^2 x^2} [\mathbf{12}]^4 \tag{7.18}$$

comes from a Kaluza-Klein reduction of the Einstein-Hilbert Lagrangian, where as shown in paper I the coupling of the massive particle to the graviton is given by [82]

$$\begin{aligned} \mathscr{L}^{s=2} = &\sqrt{-g}\,\big(\nabla_\mu \overline{H}_{\nu\rho} \nabla^\mu H^{\nu\rho} - 2\nabla_\nu \overline{H}^\nu_\mu \nabla^\rho H^\mu_\rho - \overline{H}^\rho_\rho \nabla_\mu \nabla_\nu H^{\mu\nu} \\ &- H^\rho_\rho \nabla_\mu \nabla_\nu \overline{H}^{\mu\nu} - \nabla_\mu \overline{H}^\nu_\nu \nabla^\mu H^\rho_\rho - m^2 \overline{H}_{\mu\nu} H^{\mu\nu} + m^2 \overline{H}^\mu_\mu H^\nu_\nu \\ &- 2R^{\mu\nu\rho\sigma} \overline{H}_{\mu\rho} H_{\nu\sigma}\big)\,. \end{aligned} \tag{7.19}$$

For $s > 2$ states coupled to gravitons, or $s > 1$ states coupled to massless vectors, there is no consistent massless theory in flat space, hence we cannot rely on Kaluza-Klein reduction. However, the amplitudes (7.17) naturally extend to all s, and so do their gauge-theory counterparts,

$$\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^-) = \frac{m}{x} \frac{[\mathbf{12}]^{2s}}{m^{2s}}. \tag{7.20}$$

We will refer to the amplitudes (7.20) and (7.17) as *AHH amplitudes*, since they were first discussed by the authors in ref. [108]. This brings us to one of the main questions discussed in this thesis: what are the higher-spin theories that produce these amplitudes, and what makes them special? This question became even more compelling after the work of ref. [58, 59], showing that eq. (7.17) is the amplitude for a Kerr black hole in the $s \to \infty$ limit, as we will show in more detail in part IV. Similarly, the $s \to \infty$ limit of eq. (7.20) reproduces the electromagnetic analog of the Kerr solution, known as *root-Kerr* [68, 162][1].

A first hint to the answer comes from the spin-2 electromagnetic theory in eq. (4.25), since its three-point amplitude $\mathscr{A}(H_1 \overline{H}_2 A_3^-)$ also matches eq. (7.20), for $s = 2$. This suggests that the AHH amplitudes may be the lowest-derivative solution to massive gauge invariance, even for values of s that are not compatible with naive Kaluza-Klein reduction. This idea is the main focus of paper III and it will be discussed in more detail in part III.

[1]The root-Kerr solution is the electromagnetic field sourced by a Kerr-Newman black hole, as discussed in ref. [67].

Before moving on, it is useful to rewrite the amplitudes (7.20) and (7.17) in covariant form, using polarisation vectors and momenta. We follow the discussion in paper I and present the result as a generating function. Let us use the shorthand $\mathscr{A}_{\phi\phi A} = \varepsilon_3 \cdot p_1$ for the $s = 0$ amplitude and $\mathscr{A}_{WWA} = \varepsilon_1 {\cdot} \varepsilon_2 \, \varepsilon_3 {\cdot} p_1 + \varepsilon_2 {\cdot} \varepsilon_3 \, \varepsilon_1 {\cdot} p_2 + \varepsilon_3 {\cdot} \varepsilon_1 \, \varepsilon_2 {\cdot} p_3$ for the $s = 1$ amplitude. Then we have, up to overall normalisation,

$$\sum_{s=0}^{\infty} \frac{m}{x} \frac{[\mathbf{12}]^{2s}}{m^{2s}} = \mathscr{A}_{\phi\phi A} + \frac{\mathscr{A}_{WWA} - (\varepsilon_1 \cdot \varepsilon_2)^2 \mathscr{A}_{\phi\phi A}}{(1 + \varepsilon_1 \cdot \varepsilon_2)^2 + \frac{2}{m^2} \varepsilon_1 \cdot p_2 \, \varepsilon_2 \cdot p_1}, \tag{7.21}$$

where the spin-s amplitude can be obtained by expanding the denominator and identifying the terms proportional to $\varepsilon_1^s \varepsilon_2^s$. In the gravity case, we have, up to overall normalisation,

$$\sum_{s=0}^{\infty} \frac{m^2}{x^2} \frac{[\mathbf{12}]^{2s}}{m^{2s}} = (\mathscr{A}_{\phi\phi A})^2 + \mathscr{A}_{\phi\phi A} \mathscr{A}_{WWA} + \frac{(\mathscr{A}_{WWA})^2 - (\varepsilon_1 \cdot \varepsilon_2)^2 \mathscr{A}_{\phi\phi A} \mathscr{A}_{WWA}}{(1 + \varepsilon_1 \cdot \varepsilon_2)^2 + \frac{2}{m^2} \varepsilon_1 \cdot p_2 \, \varepsilon_2 \cdot p_1}. \tag{7.22}$$

Note that the sum runs over integer values of s, but these formulae can be generalised to half-integer spins, as discussed in paper I.

7.3 Compton Amplitude

In part IV we will see that eq. (7.17) can be used to compute observables for a binary system of two spinning black holes, at leading order in the gravitational constant G. Going beyond leading order is essential to accurately predict gravitational waveforms, and in principle it requires knowledge of all amplitudes of form [19, 163, 164]

$$\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3 \dots h_n). \tag{7.23}$$

To compute $\mathscr{O}(G^2)$ observables, we only need the $n = 4$ case, the gravitational Compton amplitude $\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3 h_4)$ between two massive higher-spin particles and two gravitons. The gravitational Lagrangians discussed above produce the following results for $0 \leq s \leq 2$,

$$\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^+ h_4^+) = \frac{\langle \mathbf{12} \rangle^{2s} [34]^4}{m^{2s-4} s_{12} t_{13} t_{14}}, \tag{7.24a}$$

$$\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^- h_4^+) = \frac{[4|p_1|3\rangle^{4-2s} ([4\mathbf{1}]\langle 3\mathbf{2} \rangle + [4\mathbf{2}]\langle 3\mathbf{1} \rangle)^{2s}}{s_{12} t_{13} t_{14}}, \tag{7.24b}$$

where $s_{12} = (p_1 + p_2)^2$, $t_{13} = (p_1 + p_3)^2 - m^2$ and $t_{14} = (p_1 + p_4)^2 - m^2$. The other two helicity configurations can be obtained by conjugation. These amplitudes can also be obtained without any knowledge of the Lagrangians, by applying BCFW on-shell recursion to eq. (7.17) [62, 113]. Remarkably, they correctly reproduce Kerr observables at $\mathscr{O}(G^2)$ up to $\mathscr{O}(a^4)$ in the angular momentum a of the black hole, as we will discuss in part IV. In the gauge theory case, the story is similar. The Lagrangians discussed above produce the amplitudes for $0 \leq s \leq 1$,

$$\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^+ A_4^+) = \frac{\langle \mathbf{12} \rangle^{2s} [34]^2}{m^{2s-2} t_{13} t_{14}}, \tag{7.25a}$$

$$\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^- A_4^+) = \frac{[4|p_1|3\rangle^{2-2s} ([4\mathbf{1}]\langle 3\mathbf{2}\rangle + [4\mathbf{2}]\langle 3\mathbf{1}\rangle)^{2s}}{t_{13} t_{14}}, \tag{7.25b}$$

assuming the massless vector is a $U(1)$ photon. These can also be obtained by applying BCFW recursion to eq. (7.20).

However, when attempting to extend eq. (7.24) beyond $s = 2$, or eq. (7.25) beyond $s = 1$, one runs into trouble: the opposite-helicity amplitudes contain negative powers of the term $[4|p_1|3\rangle$. These are called *spurious poles*, since they do not correspond to particle exchange, nor can any local Lagrangian give rise to them. We interpret this as follows. Given helicities h_3 and h_4 for the massless particles, BCFW on-shell recursion relations rely on the assumption

$$\lim_{z \to \infty} z \mathscr{A}(\Phi^s(p_1), \bar{\Phi}^s(p_2), A^{h_3}(p_3 + zr), A^{h_4}(p_4 - zr)) < \infty, \tag{7.26}$$

where z is a complex coordinate and r^μ a four-vector such that $r^2 = r \cdot p_3 = r \cdot p_4 = 0$. This means the amplitude must go to zero in the special high-energy limit defined by eq. (7.26). However, for $s > 1$ particles coupling to photons or $s > 2$ particles coupling to gravitons, one expects the amplitude to diverge in the high-energy limit, since there are no consistent interacting massless higher-spin states in flat space. Hence it is not too surprising that BCFW recursion fails to produce valid amplitudes for massive higher-spin particles.

Since one cannot rely on on-shell recursion relations, we can write down Lagrangians that reproduce the amplitudes (7.17) and (7.20). However, we are free to add four-point (and higher-point) contact interactions without modifying the three-point amplitudes. The papers presented in this thesis show how to significantly reduce this freedom by imposing constraints from massive gauge invariance and requiring an improved high-energy behaviour.

Part III:
Higher-Spin Ward Identities

8. Overview

In section I we have seen how to construct massive higher-spin theories, highlighting the role of massive gauge invariance in ensuring the theory is healthy. This provides a framework to construct Lagrangians for any spin and any multiplicity, and easily identify interesting solutions, such as eq. (4.25). However, Lagrangians are highly redundant objects and computations quickly become intractable.

In section II we have seen a very different approach to constructing higher-spin theories. We have focussed directly on on-shell amplitudes, thus avoiding unphysical ambiguities, and introduced a set of variables that automatically encode the correct degrees of freedom. This allowed us to identify a special class of three-point amplitudes, shown in eq. (7.17) and eq. (7.20), and use it to derive the four-point Compton amplitudes (7.24) and (7.25). However, for $s > 2$ (or $s > 1$ in the electromagnetic case), the on-shell techniques used to derive the Compton amplitudes are no longer complete and a proper understanding of the theories underlying the AHH amplitudes is missing.

Papers I and III aim to find a common ground between the two approaches, using on-shell techniques that greatly simplify calculations without losing the connection to Lagrangians and massive gauge invariance. The main result is an all-spin understanding of the theories that give rise to eq. (7.17) and eq. (7.20), together with new results for higher-spin Compton amplitudes. This provides a new framework to study Kerr observables, as discussed in section IV, as well as a powerful way to construct higher-spin theories beyond what has been done via standard methods. There are two main tools that make all this possible. The first is a high-energy unitarity constraint, which we refer to as the *current constraint*. The second is an on-shell realisation of massive gauge invariance, in terms of higher-spin *massive Ward identities*. We will present both in more detail below.

9. Current Constraint

A first attempt to pin down the higher-spin theories that give rise to the AHH amplitudes is outlined in paper I. As we discussed earlier, for $s \leq 2$ in gravity and $s \leq 1$ in gauge theory the AHH amplitudes come from simple Kaluza-Klein reduction of massless theories in five dimensions. Let us consider one such theory, with a massless field $\phi^{s\leq 2}$ coupled to the graviton $h_{\mu\nu}$, or a massless field $\phi^{s\leq 1}$ coupled to a photon or gluon A_μ. We call $V(\phi_1^s \bar{\phi}_2^s A_3^h)$ the *off-shell* three-point vertex of this theory, where $h = 1,2$ denotes the helicity of the massless particle, $A^1 \equiv A_\mu$ is the massless vector and $A^2 \equiv h_{\mu\nu}$ the graviton. Since the underlying Lagrangian is gauge-invariant, the vertex will obey the *Ward identity*

$$p_1 \cdot \frac{\partial}{\partial \varepsilon_1} V(\phi_1^s \bar{\phi}_2^s A_3^h)\big|_{(2,3)} = 0, \tag{9.1}$$

where the subscript $(i, j, \ldots)$ means that legs $i, j, \ldots$ are subject to on-shell conditions, meaning $p_i^2 = \varepsilon_i^2 = \varepsilon_i \cdot p_i = 0$. The polarisation vector ε_1 is simply a placeholder, since leg 1 is off-shell, and the operator $p_1 \cdot (\partial/\partial\varepsilon_1)$ implements the replacement $\varepsilon_1^{\mu_1} \ldots \varepsilon_1^{\mu_s} \to p_1^{(\mu_1} \varepsilon_1^{\mu_1} \ldots \varepsilon_1^{\mu_s)}$. Upon dimensional reduction, the field ϕ^s contains a massive four-dimensional spin-s field Φ^s and eq. (9.1) becomes

$$p_1 \cdot \frac{\partial}{\partial \varepsilon_1} V(\Phi_1^s \bar{\Phi}_2^s A_3^h)\big|_{(2,3)} = \mathscr{O}(m), \tag{9.2}$$

meaning that violations to the Ward identity must be proportional to the mass, so that it is still valid in the $m \to 0$ limit. Note that in this case $p_2^2 = m^2$ on-shell. We call eq. (9.2) the *current constraint*. Let us see it in action in the example of a massive charged spin-1 field. We consider the Lagrangian

$$\mathscr{L} = 2\overline{D_{[\mu} W_{\nu]}} D^{[\mu} W^{\nu]} - m^2 \overline{W}_\mu W^\mu + ie\alpha F_{\mu\nu} \overline{W}^\mu W^\nu, \tag{9.3}$$

where α is a free parameter. We know $\alpha = 1$ is the value that follows from Kaluza-Klein reduction. Indeed, the three-point vertex is

$$\begin{aligned} V(W_1 \overline{W}_2 A_3)\big|_{(2,3)} &= \varepsilon_1 \cdot \varepsilon_2 \, \varepsilon_3 \cdot (p_2 - p_1) - \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_1 \cdot p_2 + \varepsilon_3 \cdot \varepsilon_1 \, \varepsilon_2 \cdot p_1 \\ &\quad - \alpha f_3^{\mu\nu} \varepsilon_{2\mu} \varepsilon_{1\nu}, \end{aligned} \tag{9.4}$$

where p_1 is off-shell and hence ε_1 is arbitrary. Here $f_3^{\mu\nu} = 2p_3^{[\mu} \varepsilon_3^{\nu]}$. It is easy to check that $p_1 \cdot (\partial/\partial\varepsilon_1) V(W_1 \overline{W}_2 A_3)\big|_{(2,3)} = \mathscr{O}(m)$ requires $\alpha = 1$, as expected.

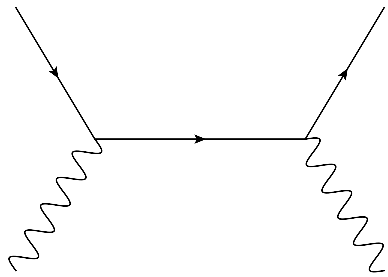

Figure 9.1. Feynman diagram contributing to the Compton amplitude $\mathscr{A}(\Phi^s \bar{\Phi}^s AA)$, with an exchange of the massive field.

To get some more intuition on the consequences of the current constraint, let us consider the diagram in figure 9.1, contributing to the Compton amplitude $\mathscr{A}(W_1 \overline{W}_2 A_3 A_4)$. The ingredients to compute this diagram are two vertices (9.4), relabelled appropriately, and the massive propagator

$$\Delta_{\mu\nu}(P) = \frac{1}{P^2 - m^2}\left(\eta_{\mu\nu} - \frac{P_\mu P_\nu}{m^2}\right), \tag{9.5}$$

where P is the off-shell exchanged momentum. Due to the m^2 factor at the denominator, the Compton amplitude is in general divergent in the $m \to 0$ limit. However, if the current constraint is satisfied, the mass divergence will disappear due to the factor $P^\mu P^\nu$ contracted into the vertices. This confirms that the value $\alpha = 1$ leads to a well-defined massless limit, for $s < 2$.

In the case of $s > 1$ charged particles, or $s > 2$ particles coupled to gravity, we do not expect the theory to have a well-defined massless limit. Nonetheless, the constraint (9.2) has been applied to higher-spin theories in the literature [120–122]. As an example, we consider a charged spin-$3/2$ particle [165]. The most general one-derivative Lagrangian is

$$\begin{aligned}\mathscr{L} =&\bar{\psi}^\mu \gamma_{\mu\nu\rho}\left(iD^\nu - \frac{1}{2}m\gamma^\nu\right)\psi^\rho - \frac{ie}{m}\Big(l_1 \bar{\psi}_\mu F^{\mu\nu}\psi_\nu + l_2 \bar{\psi}_\mu F_{\rho\sigma}\gamma^\rho\gamma^\sigma\psi^\mu \\ &+ l_3 F^{\mu\nu}(\bar{\psi}_\mu \gamma_\nu \gamma\cdot\psi + \bar{\psi}\cdot\gamma\gamma_\mu\psi_\nu) + l_4 \bar{\psi}\cdot\gamma F_{\rho\sigma}\gamma^\rho\gamma^\sigma\gamma\cdot\psi \\ &+ il_5 F^{\mu\nu}(\bar{\psi}_\mu\gamma_\nu\gamma\cdot\psi - \bar{\psi}\cdot\gamma\gamma_\mu\psi_\nu)\Big),\end{aligned} \tag{9.6}$$

where l_i are free parameters. The current constraint (9.2) fixes their value to be

$$l_1 = -2, l_2 = 1/2, l_3 = 1, l_5 = 0. \tag{9.7}$$

Note that l_4 does not appear since $\bar{\psi}\cdot\gamma = 0$ on-shell, but we can neglect it since it does not contribute to the three-point and four-point amplitudes studied in this thesis [1]. The Lagrangian following from eq. (9.7) can be rewritten as

$$\mathscr{L} = \mathscr{L}_{min} + \bar{\psi}_\mu \left(F^{\mu\nu} - \frac{i}{2}\gamma_5 \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \right) \psi_\nu, \tag{9.8}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\mathscr{L}_{min}$ is the minimally-coupled Lagrangian. This reproduces a truncation of $\mathscr{N} = 2$ gauged supergravity [166]. Moreover, the three-point amplitude is

$$\mathscr{A}(\psi_1\, \bar{\psi}_2 A_3^-) = \frac{1}{x}\frac{[\mathbf{12}]^3}{m^2}, \tag{9.9}$$

matching eq. (7.20) for $s = 3/2$.

Hence, the current constraint provides a promising avenue to understand the AHH amplitudes for higher-spin particles, and we can take this further by studying the Compton amplitude. The diagram in figure 9.1 now contains the massive propagator

$$\Delta^{\mu\nu}(P) \sim \left(\eta^{\mu\nu} - \frac{P^\mu P^\nu}{m^2}\right)(\not{P} + m) + \frac{1}{3}\left(\frac{P^\mu}{m} + \gamma^\mu\right)(\not{P} - m)\left(\frac{P^\nu}{m} + \gamma^\nu\right). \tag{9.10}$$

Similarly to the spin-1 case, this introduces mass divergences in the amplitude, but the current constraint removes them. However, the Compton amplitude does not have a finite massless limit, due to additional mass divergences in the Lagrangian (9.8). Nonetheless, we expect it to have an improved high-energy behaviour. The Compton amplitudes obtained are

$$\begin{aligned}
\mathscr{A}_4^{++} &= \frac{\langle \mathbf{12}\rangle^3 [34]^2}{m t_{13} t_{14}}, \\
\mathscr{A}_4^{-+} &= \frac{[4\mathbf{1}]\langle 3\mathbf{2}\rangle + [4\mathbf{2}]\langle 3\mathbf{1}\rangle}{[4|p_1|3\rangle} \left(\frac{([4\mathbf{1}]\langle 3\mathbf{2}\rangle + [4\mathbf{2}]\langle 3\mathbf{1}\rangle)^2}{t_{13}t_{14}} - \frac{[\mathbf{1}4][\mathbf{2}4]\langle \mathbf{1}3\rangle\langle \mathbf{2}3\rangle}{m^4} \right),
\end{aligned} \tag{9.11}$$

where $\mathscr{A}_4^{h_3h_4} \equiv \mathscr{A}(\psi_1\, \bar{\psi}_2 A_3^{h_3} A_4^{h_4})$. Remarkably, the same-helicity amplitude $\mathscr{A}_4^{++}$ matches eq. (7.25), meaning that eq. (7.26) is satisfied for this helicity configuration. In the opposite-helicity case, the amplitude $\mathscr{A}_4^{-+}$ differs from eq. (7.25) by an $\mathscr{O}(m^{-4})$ correction term. Although this is not manifest in eq. (9.11), there is no spurious pole $[4|p_1|3\rangle$ and the amplitude can be rewritten so that only physical poles appear, as presented in paper I. This amplitude is special since it has the smallest possible mass divergence compatible with the AHH amplitudes. Indeed, any other Compton amplitude that

[1] A massive spin-$(n+\frac{1}{2})$ on-shell particle can be described by a tensor-spinor $\psi^\alpha_{\mu_1\ldots\mu_n}$, where α is a Dirac spinor index, μ_i are Lorentz indices and n is an integer. The tensor-spinor is transverse, symmetric and traceless in the Lorentz indices. Moreover, it obeys the condition $\gamma^{\mu_1}_{\alpha\beta}\psi^\beta_{\mu_1\ldots\mu_n} = 0$, known as *gamma-tracelessness*.

matches eq. (7.20) on the factorisation channel must differ from eq. (9.11) by a contact term. Such contact term must contain three spinors for each massive particle and two for each massless particle, by representation theory. Since each spinor has mass dimension $1/2$ and the amplitude must have zero mass dimension, we conclude any contact term will be at least $\mathscr{O}(m^{-5})$. This shows that the current constraint singles out higher-spin effective field theories with an improved cutoff, since the resulting amplitudes have the tamest possible divergence in the high-energy limit [122].

A similar analysis can be carried out in the case of a massive spin-5/2 field coupled to gravity. The lowest-derivative solution of the current constraint is the Lagrangian

$$\mathscr{L} = \mathscr{L}_{min} + \bar{\psi}_{\mu\rho}\left(R^{\mu\nu\rho\sigma} - \frac{i}{2}\gamma_5 \varepsilon^{\rho\sigma\alpha\beta} R^{\mu\nu}{}_{\alpha\beta}\right)\psi_{\nu\sigma}. \tag{9.12}$$

The three-point amplitude is

$$\mathscr{A}(\psi_1\, \bar{\psi}_2\, h_3^-) = \frac{1}{x^2}\frac{[\mathbf{12}]^5}{m^3}, \tag{9.13}$$

matching eq. (7.17) for $s = 5/2$. The Compton amplitudes obtained are

$$\begin{aligned}
\mathscr{M}_4^{++} &= \frac{\langle \mathbf{12}\rangle^5 [34]^4}{m s_{12} t_{13} t_{14}}, \\
\mathscr{M}_4^{-+} &= \frac{[4\mathbf{1}]\langle 3\mathbf{2}\rangle + [4\mathbf{2}]\langle 3\mathbf{1}\rangle}{[4|p_1|3\rangle}\left(\frac{([4\mathbf{1}]\langle 3\mathbf{2}\rangle + [4\mathbf{2}]\langle 3\mathbf{1}\rangle)^4}{s_{12}t_{13}t_{14}} - \frac{([\mathbf{1}4][\mathbf{2}4]\langle \mathbf{1}3\rangle\langle \mathbf{2}3\rangle)^2}{m^6}\right),
\end{aligned} \tag{9.14}$$

where $\mathscr{M}_4^{h_3 h_4} \equiv \mathscr{M}(\psi_1\, \bar{\psi}_2\, h_3^{h_3}\, h_4^{h_4})$. Similarly to the spin-3/2 case above, this matches the same-helicity Compton amplitude (7.24) and it produces a new spurious-pole-free opposite-helicity Compton amplitude. Once again, the latter has the smallest possible mass divergence compatible with eq. (7.17), since any contact term is at least $\mathscr{O}(m^{-7})$.

Now we turn our attention to $s \geq 2$ in gauge theory and $s \geq 3$ in gravity. A consequence of eq. (9.2) is that the high-energy limit $p_i \gg m$ of $V(\Phi_1^s \bar{\Phi}_2^s A_3^h)$ satisfies a massless Ward identity similar to eq. (9.1). Namely,

$$p_1 \cdot \frac{\partial}{\partial \varepsilon_1} V(\Phi_1^s \bar{\Phi}_2^s A_3^h)\big|_{(2,3)} = 0 + \ldots \tag{9.15}$$

where the dots denote subleading terms. The lowest-derivative solution to eq. (9.15) has $2s - h$ derivatives, and the corresponding on-shell amplitude $\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^h) = V(\Phi_1^s \bar{\Phi}_2^s A_3^h)\big|_{(1,2,3)}$ is given by

$$\begin{aligned}
\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^h) =& \frac{(\varepsilon_1 \cdot p_2\, \varepsilon_2 \cdot p_1)^{s-h}}{m^{2s-2h}}(\varepsilon_1 \cdot \varepsilon_2\, \varepsilon_3 \cdot p_1 + \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_1 \cdot p_2 + \varepsilon_3 \cdot \varepsilon_1\, \varepsilon_2 \cdot p_3)^h \\
&+ \mathscr{O}(m^{-2s+2h+1}),
\end{aligned} \tag{9.16}$$

where we assumed all particles are on-shell for simplicity. This implies that the highest-derivative vertex originates from a massless higher-spin theory, as discussed at the end of section I. Also, eq. (9.16) correctly reproduces the highest-derivative part of the AHH amplitudes for any spin, which can be checked by comparing to eq. (7.21) and eq. (7.22). Although this is promising, the same highest-derivative term appears in the superstring amplitudes in eq. (5.6). Since the latter is not equal to the AHH amplitudes for generic spin s, we may expect the current constraint to not give a unique solution.

We can check this explicitly by constructing an ansatz with $2s-h$ derivatives and imposing eq. (9.2). For instance, the gauge theory case the solution yields

$$\mathscr{A}(\Phi_1^s\,\bar{\Phi}_2^s A_3^-) = \frac{1}{x}\frac{[\mathbf{12}]^3}{m^{2s-1}}\sum_{k=0}^{2s-3} c_k[\mathbf{12}]^{2s-3-k}\langle\mathbf{12}\rangle^k, \tag{9.17}$$

where c_k are free parameters. As anticipated, something is still missing to understand the AHH amplitudes for any spin. The same observation applies to the gravity case. In the next section we show this is related to the massive gauge invariance discussed in part I.

10. Massive Ward Identities

From eq. (9.17) we see that the current constraint is compatible with the AHH amplitude for a charged massive particle with spin $s \geq 2$, but it does not predict it uniquely. However, in eq. (4.27) we see that the $s = 2$ amplitude follows from imposing massive gauge invariance. Therefore, we are motivated to study gauge invariant Lagrangians for massive particles of higher spins. However, as we learned in section I this is not an easy task. In particular, it requires not only introducing non-minimal terms in the Lagrangian but also adding non-linear corrections to the gauge transformations. On the other hand, in massless theories gauge invariance can be imposed directly on scattering amplitudes in the form of the Ward identity (9.1), which only requires knowledge of the linear gauge transformation. This motivates the work of paper III, where analogous identities are derived to simplify the problem.

10.1 Low-Spin Examples

We can start from the charged massive spin-1 field described by the Lagrangian in eq. (4.11) and eq. (4.13). The three-point vertices obey the identity

$$ip_1^\mu \frac{\partial}{\partial \varepsilon_1^\mu} V(W_1 \overline{W}_2 A_3) - mV(\varphi_1 \overline{W}_2 A_3)\bigg|_{(2,3)} = 0. \tag{10.1}$$

In the $m \to 0$ limit this reduces to eq. (9.1), hence we refer to it as a *massive Ward identity*. Note that it follows directly from the linearised gauge transformations (4.9), meaning its form only depends on the free theory. This allows us to bypass the explicit construction of the interaction Lagrangian: once the free theory is known, we compute the minimally-coupled part of $V(W_1\bar{W}_2A_3)$ and $V(\varphi_1\bar{W}_2A_3)$, we make an ansatz for the non-minimal part and we fix it via the identity (10.1). The lowest-derivative solution to eq. (10.1) is

$$\begin{aligned} V(W_1\bar{W}_2A_3) &= \varepsilon_1\cdot\varepsilon_2\;\varepsilon_3\cdot(p_1-p_2)+\varepsilon_2\cdot\varepsilon_3\;\varepsilon_1\cdot(p_2-p_3)+\varepsilon_3\cdot\varepsilon_1\;\varepsilon_2\cdot(p_3-p_1), \\ V(\varphi_1\bar{W}_2A_3) &= im\,\varepsilon_2\cdot\varepsilon_3. \end{aligned} \tag{10.2}$$

This reproduces the three-point amplitude (4.15) previously derived from the full Lagrangian.

We can apply the same method to a charged massive spin-2 particle. We start with the minimally-coupled Lagrangian (4.22), we make an ansatz for the non-minimal terms and impose the identities

$$
\begin{aligned}
&ip_1\cdot\frac{\partial}{\partial\varepsilon_1}V(H_1,\overline{H}_2,A_3)+mV(B_1,\overline{H}_2,A_3)\big|_{(2,3)}=0,\\
&m\frac{\partial}{\partial\varepsilon_1}\cdot\frac{\partial}{\partial\varepsilon_1}V(H_1,\overline{H}_2,A_3)+ip_1\cdot\frac{\partial}{\partial\varepsilon_1}V(B_1,\overline{H}_2,A_3)+mV(\phi_1,\overline{H}_2,A_3)\big|_{(2,3)}=0,
\end{aligned}
\tag{10.3}
$$

following the gauge transformations (4.23). The operator $(\partial/\partial\varepsilon_1)\cdot(\partial/\partial\varepsilon_1)$ is equivalent to the replacement $\varepsilon_{1\mu}\varepsilon_{1\nu}\to 2\eta_{\mu\nu}$. We expect the lowest-derivative solution to agree with the Lagrangian (4.25), which contains vertices with at most three derivatives, so we construct an ansatz with up to three powers of momenta. The general solution gives the following on-shell three-point amplitude,

$$
\begin{aligned}
\mathscr{A}(H_1,\overline{H}_2,A_3)=-\left(\frac{\alpha}{m^2}\varepsilon_1\cdot p_2\,\varepsilon_2\cdot p_1+\frac{4+\alpha}{2}\varepsilon_1\cdot\varepsilon_2\right)&(\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot p_1\\
+\varepsilon_2\cdot\varepsilon_3\,\varepsilon_1\cdot p_2+\varepsilon_3\cdot\varepsilon_1\,\varepsilon_2\cdot p_3)&+\frac{\alpha}{2}(\varepsilon_1\cdot\varepsilon_2)^2\,\varepsilon_3\cdot p_1,
\end{aligned}
\tag{10.4}
$$

up to overall normalisation, where α is a free parameter. Further imposing the current constraint (9.2) fixes $\alpha=4$, recovering eq. (4.27) and hence the AHH amplitude. Note that the choice $\alpha=0$ recovers the lower-derivative solution mentioned in section I and ref. [119], which was discarded as less interesting. Now we can make this statement more precise, since that solution is ruled out by the current constraint.

10.2 Arbitrary Spin

As shown above, for charged massive higher-spin particles, if we combine the massive Ward identities and the current constraint, the AHH amplitudes (7.20) are the unique lowest-derivative solution up to $s=2$. This is an improvement on eq. (9.17). Now we want to see if the same methods can be used to fix the amplitudes for $s>2$.

As discussed, we expect the lowest-derivative cubic vertex to have $2s-h$ derivatives, where s is the spin of the massive particle and $h=1,2$ the helicity of the massless one. We can improve on this by examining eq. (4.11) and eq. (4.25). There the vertices $V(\Phi_1^{s_1}\overline{\Phi}_2^{s_2}A_3)$ have at most s_1+s_2-1 derivatives, where s_i is the rank of the tensor Φ_i. Moreover, $V(\Phi_1^{s_1}\overline{\Phi}_2^{s_2}A_3)$ is zero if $|s_1-s_2|>1$. We can generalise this via the conditions

$$V(\Phi_1^{s_1}\overline{\Phi}_2^{s_2}A_3^h)\sim\partial^{s_1+s_2-h}, \tag{10.5a}$$

$$V(\Phi_1^{s_1}\overline{\Phi}_2^{s_1-k}A_3^h)\big|_{k>h}=0, \tag{10.5b}$$

where h is the helicity of the massless particle, such that $A^{\pm 1}$ is a massless vector A_μ and $A^{\pm 2}$ is a graviton $h_{\mu\nu}$.

The Ward identities for arbitrary spin can be extracted from the free Lagrangian (4.30) and the transformations (4.28). We define the gauge-transformed vertex $V(\xi_1^k \overline{\Phi}_2^s A_3^h)$ as

$$V(\xi_1^k \overline{\Phi}_2^s A_3^h) \equiv m\alpha_k V(\Phi_1^k \overline{\Phi}_2^s A_3^h) - \frac{ip_1}{k+1} \cdot \frac{\partial}{\partial \varepsilon_1} V(\Phi_1^{k+1} \overline{\Phi}_2^s A_3^h) + \frac{m}{2}\beta_{k+2}\left(\frac{\partial}{\partial \varepsilon_1}\right)^2 V(\Phi_1^{k+2} \overline{\Phi}_2^s A_3^h), \tag{10.6}$$

obtained from the gauge variation of the three-point vertex with respect to the gauge parameter ξ_1^k and proportional to the field $\overline{\Phi}_2$. Then the massive Ward identity reads

$$V(\xi_1^k \overline{\Phi}_2^s A_3^h)\big|_{(2,3)} = 0. \tag{10.7}$$

In order to solve eq. (10.7), we begin by computing the three-point vertices that follow from the minimal coupling of the Lagrangian (4.30). Then, we make an ansatz for the non-minimal vertices, which must be proportional to the massless field strength $F_{\mu\nu}$ (or the Riemann tensor $R_{\mu\nu\rho\sigma}$ in the gravity case) due to gauge invariance. In constructing the ansatz, we assume eq. (10.5a). Combining minimal and non-minimal contributions, we obtain the vertices $V(\Phi_1^k \overline{\Phi}_2^s A_3^h)$ and compute the gauge transformations $V(\xi_1^k \overline{\Phi}_2^s A_3^h)$ via eq. (10.6). Finally, we impose eq. (10.7). Assuming $\{s \leq 1, h = 1\}$ or $\{s \leq 2, h = 2\}$, this procedure yields a unique solution matching the amplitudes (7.20) and (7.17). If we have $h = 1$ and general spin s, we get

$$\mathscr{A}(\Phi_1^s \bar{\Phi}_2^s A_3^-) = \mathscr{A}_0 \frac{[\mathbf{12}]^{2s}}{m^{2s}} \left\{ 1 + \sum_{k=1}^{s-1} c_k \left(\frac{\langle \mathbf{12} \rangle^k}{[\mathbf{12}]^k} - 1 \right) \right\}, \tag{10.8}$$

where c_k are free parameters and $\mathscr{A}_0 = \mathscr{A}(\Phi_1^0 \bar{\Phi}_2^0 A_3^-)$ is the scalar amplitude. Imposing eq. (9.2) fixes $\sum_k c_k = 0$. Imposing both eq. (9.2) and eq. (10.5b) fixes $c_k = 0$. In the $h = 2$ case, the Ward identity (10.7) and eq. (10.5a) give

$$\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3^-) = \mathscr{M}_0 \frac{[\mathbf{12}]^{2s}}{m^{2s}} \left\{ 1 + \left(1 - \frac{\langle \mathbf{12} \rangle}{[\mathbf{12}]} \right)^2 \sum_{k=0}^{s-4} c'_k \frac{\langle \mathbf{12} \rangle^k}{[\mathbf{12}]^k} \right\}, \tag{10.9}$$

where c'_k are free parameters and $\mathscr{M}_0 = \mathscr{M}(\Phi_1^0 \bar{\Phi}_2^0 h_3^-)$ is the scalar amplitude. In this case, eq. (9.2) alone is sufficient to yield $c'_k = 0$. The condition (10.5b) is compatible with this solution. Note that eq. (10.8) and eq. (10.9) were checked explicitly up to $s = 6$, since the size of the ansatz increases very rapidly with spin. Nonetheless, we believe this pattern to be valid for any spin, since we are not aware of any new features starting from $s > 6$.

In short, the AHH amplitudes eq. (7.20) and eq. (7.17) are the unique solutions to the massive Ward identities and the current constraint, assuming the lowest-derivative prescriptions eq. (10.5b) and eq. (10.5a). This is remarkable since it provides a complete Lagrangian understanding of the AHH amplitudes and it highlights the physical properties that underlie them. Moreover, the methods outlined above can be extended to higher-point amplitudes. Below we will apply them to the four-point Compton amplitudes and see how they improve on the BCFW results of eq. (7.25) and eq. (7.24).

10.3 Compton Amplitudes

In order to use a gauge-invariant Lagrangian to compute the four-point amplitude in figure 12.2, we need to fix the gauge. The first gauge choice we encountered in part I is the unitary-like gauge, where the traceless part of the Stückelberg fields $\Phi^{k<s}$ is set to zero. The second gauge choice we encountered is the Feynman gauge, in which all the fields $\Phi^{k<s}$ are still present. Since the explicit dependence on the Stückelberg fields was important in deriving the Ward identities (10.7), we choose to work in Feynman gauge.

The propagators $(\Delta^{(k)})^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_k}$ for a rank-k field Φ^k can be computed from the gauge-fixed Lagrangian (4.31), which can be written schematically as

$$\mathscr{L}_{\mathrm{F}} = \sum_{k=0}^{s} \Phi^k \cdot K^{(k)} \cdot \Phi^k, \tag{10.10}$$

where $(K^{(k)})^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_k}$ is the two-derivative kinetic operator. Since the fields Φ^k are double-traceless, we can impose $\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}(K^{(k)})^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_k} = 0$ and similarly $\eta^{\nu_1\nu_2}\eta^{\nu_3\nu_4}(K^{(k)})^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_k} = 0$. Therefore, the propagator $\Delta^{(k)}$ is defined as the inverse of $K^{(k)}$ within the subspace of double-traceless symmetric tensors.

In the $s=0$ and $s=1$ case, we have $K^{(0)} = (\Box + m^2)$ and $(K^{(1)})^{\mu}_{\nu} = -\delta^{\mu}_{\nu}(\Box + m^2)$, yielding $\Delta^{(0)} = 1/(p^2-m^2)$ and $(\Delta^{(1)})^{\mu}_{\nu} = -\delta^{\mu}_{\nu}/(p^2-m^2)$ in momentum space. For $s=2$ we have

$$(K^{(2)})^{\mu_1\mu_2}_{\nu_1\nu_2} = \left(\delta^{(\mu_1}_{(\nu_1}\delta^{\mu_2)}_{\nu_2)} - \frac{1}{2}\eta^{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right)(\Box + m^2), \tag{10.11}$$

resulting in

$$(\Delta^{(2)})^{\mu_1\mu_2}_{\nu_1\nu_2} = \frac{\delta^{(\mu_1}_{(\nu_1}\delta^{\mu_2)}_{\nu_2)} - \frac{1}{2}\eta^{\mu_1\mu_2}\eta_{\nu_1\nu_2}}{p^2-m^2}. \tag{10.12}$$

To solve the general case, we write an ansatz for the tensor $\Delta^{(k)}$ and impose the following equation,

$$K^{(k)} \cdot \Delta^{(k)} \cdot K^{(k)} = K^{(k)}. \tag{10.13}$$

Note that we cannot write $\Delta^{(k)} = (K^{(k)})^{-1}$, because the operator $K^{(k)}$ is only invertible when acting on double-traceless tensors. In addition, we impose

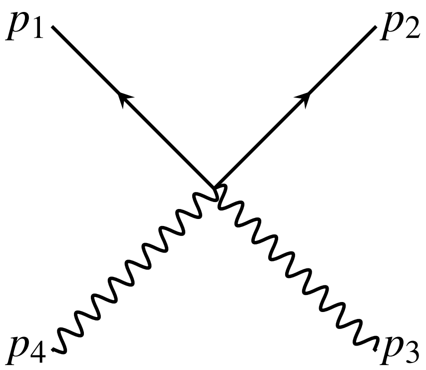

Figure 10.1. Four-point contact interaction vertex between two massive particles 1 and 2 and two massless bosons 3 and 4. All momenta are assumed to be outgoing.

$\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}(\Delta^{(k)})^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_k} = 0$ and $\eta^{\nu_1\nu_2}\eta^{\nu_3\nu_4}(\Delta^{(k)})^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_k} = 0$ to ensure the propagator is restricted to the correct subspace. This yields a unique solution that can be written as a generating function,

$$\sum_{k=0}^{\infty}(\varepsilon)^s\cdot\Delta^{(k)}\cdot(\bar{\varepsilon})^s = \frac{1}{p^2-m^2}\frac{1-\frac{1}{4}\varepsilon^2\bar{\varepsilon}^2}{1+\varepsilon\cdot\bar{\varepsilon}+\frac{1}{4}\varepsilon^2\bar{\varepsilon}^2}, \tag{10.14}$$

where ε and $\bar{\varepsilon}$ are arbitrary reference vectors and eq. (10.14) was checked explicitly up to $k = 10$.

The propagators can be used, together with the three-point vertices, to compute the massive exchange diagrams shown in figure 9.1. Then we need the four-point contact terms $V(\Phi_1^k\overline{\Phi}_2^s A_3^h A_4^h)$ in figure 10.1. One contribution comes from the expansion of the covariant derivatives in the quadratic and cubic Lagrangians, for instance in eq. (4.22) and eq. (4.25). Another contribution comes from new quartic vertices quadratic in the field strength $F_{\mu\nu}$, or the Riemann tensor $R_{\mu\nu\rho\sigma}$ in the gravity case, and it is accounted for by an ansatz. All the above ingredients can be used to compute the *off-shell* Compton amplitude $\mathscr{A}_{\text{off}}(\Phi_1^k\overline{\Phi}_2^s A_3^h A_4^h)$, such that the physical on-shell amplitude is $\mathscr{A}(\Phi_1^s\overline{\Phi}_2^s A_3^h A_4^h) = \mathscr{A}_{\text{off}}(\Phi_1^s\overline{\Phi}_2^s A_3^h A_4^h)\big|_{(1,2,3,4)}$ [1]. We define the gauge-transformed amplitude $\mathscr{A}_{\text{off}}(\xi_1^k\overline{\Phi}_2^s A_3^h A_4^h)$ in the same fashion as eq. (10.6) and impose the Ward identities

$$\mathscr{A}_{\text{off}}(\xi_1^k\overline{\Phi}_2^s A_3^h A_4^h)\big|_{(2,3,4)} = 0, \tag{10.15}$$

for all the gauge parameters ξ^k.

For $s \leq 1$ with $h = 1$, and $s \leq 2$ with $h = 2$, the lowest-derivative solution to eq. (10.15) yields a unique on-shell amplitude $\mathscr{A}(\Phi_1^s\overline{\Phi}_2^s A_3^h A_4^h)$ and it matches

[1] The off-shell amplitude is to be understood as a four-point correlator stripped of the propagators for the external legs.

eq. (7.25) and eq. (7.24). In the $h=1$ case, the opposite-helicity amplitudes $\mathscr{A}(\Phi_1^s\Phi_2^s A_3^- A_4^+)$ can be rewritten in a convenient form,

$$\mathscr{A}(\Phi_1^s\,\Phi_2^s A_3^-\,A_4^+) = \frac{\langle 3|1|4]^2(U+V)^{2s}}{m^{4s}t_{13}t_{14}} + \frac{\langle 3|1|4]\langle \mathbf{1}3\rangle[\mathbf{2}4]P_{2s}}{m^{4s}t_{13}} + \langle \mathbf{1}3\rangle\langle 3\mathbf{2}\rangle[\mathbf{1}4][4\mathbf{2}]\frac{P_{2s-1}}{m^{4s}} + C_s, \tag{10.16}$$

where C_s is a contact term, obeying $C_{s\leq 3/2}=0$. For integer k, $P_k \equiv (2V)^{-1}\{(U+V)^k-(U-V)^k\}$ is a polynomial in the variables $V=\frac{1}{2}\big(\langle\mathbf{1}|4|\mathbf{2}]+\langle\mathbf{2}|4|\mathbf{1}]\big)$ and $U=\frac{1}{2}\big(\langle\mathbf{1}|4|\mathbf{2}]-\langle\mathbf{2}|4|\mathbf{1}]\big)-m[\mathbf{12}]$. We use the shorthand $\langle a|\,i|b]\equiv\langle a|\,p_i|b]$. Note that, in the $s=3/2$ case, eq. (10.16) reproduces the amplitude $\mathscr{A}_4^{-+}$ in eq. (9.11).

For $s=2$ and $h=1$, we get a new opposite-helicity Compton amplitude, given by eq. (10.16) with the contact term

$$\begin{aligned} C_2 = \frac{a}{m^8}\Big\{&\langle 3|\mathbf{1}|4]\langle\mathbf{12}\rangle(\langle\mathbf{12}\rangle+[\mathbf{12}])(\langle\mathbf{1}|3|\mathbf{2}]\langle\mathbf{2}3\rangle[\mathbf{1}4]+\langle\mathbf{2}|3|\mathbf{1}]\langle\mathbf{1}3\rangle[4\mathbf{2}]) \\ &+\langle 3|\mathbf{1}|4]^2[\mathbf{12}]\langle\mathbf{12}\rangle^3+s_{12}\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{1}3\rangle\langle 3\mathbf{2}\rangle[\mathbf{1}4][4\mathbf{2}]\Big\} \\ &+\frac{b}{m^6}(\langle\mathbf{12}\rangle^2+[\mathbf{12}]^2)\langle\mathbf{1}3\rangle\langle 3\mathbf{2}\rangle[\mathbf{1}4][4\mathbf{2}]+\frac{c}{m^6}\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{1}3\rangle\langle 3\mathbf{2}\rangle[\mathbf{1}4][4\mathbf{2}], \end{aligned} \tag{10.17}$$

where a, b and c are free parameters. Paper III discusses the classical limit of eq. (10.16) via the techniques in section IV.

This result leaves a number of open questions. First, as mentioned in part II, the amplitudes (7.20) reproduce the electromagnetic counterpart of the Kerr solution, known as root-Kerr, at the linearised level. Hence, it would be interesting to check if any choice of free parameters in eq. (10.17) can reproduce root-Kerr observables at the next order in perturbation theory, as discussed in part IV. However, there is currently no classical computation that this result can be compared to. Another option is to repeat the analysis in the case of a spin-3 particle in gravity, which can be directly compared to the classical results for Kerr black holes in ref. [103]. Last but not least, in the three-point case the current constraint (9.2) was crucial to fix the AHH amplitudes uniquely. Its generalisation to the four-point case is not yet known, but it may help further constrain the amplitude (10.17). We leave these questions to future work.

11. On-Shell Massive Ward Identities

The massive Ward identities defined above have made it possible to study cubic vertices for particles of arbitrary spin, and find the theories that underlie the AHH amplitudes. This is a major improvement on the methods in section I, where even the cubic Lagrangian (4.25) for a spin-2 charged particle required significant computational effort. Moreover, we were able to study four-point vertices, a notoriously hard problem in the higher-spin literature, and obtain explicit formulae for higher-spin Compton amplitudes.

The Ward identities are simpler than off-shell Lagrangians because they are not sensitive to non-linear gauge transformations and because, since all but one of the external legs are on-shell, they are less sensitive to redundancies due to field redefinitions. Nonetheless, they still require knowledge of the free higher-spin Lagrangians (4.30) and they depend on the definition of the Stückelberg fields. For example, the normalisation of the fields is responsible for the cumbersome factors α_k and β_k appearing in eq. (10.7). Such factors are unphysical and do not affect on-shell scattering amplitudes, therefore there should be a simpler on-shell framework that is independent of them.

In this section, we propose a candidate framework that reproduces the results obtained via eq. (10.7) but it requires no knowledge of explicit higher-spin Lagrangians. More details will be given in upcoming work.

11.1 Free Theory

Let us start again from the free theory for a massive particle of spin s. In this setting, the only non-trivial correlator is the two-point function, i.e. the propagator $(\Delta^{(s)}(p))^{\mu_1\ldots\mu_s}_{\nu_1\ldots\nu_s}$. For an on-shell momentum $p^2=m^2$, this is proportional to the state sum

$$(\Delta^{(s)}(p))^{\mu_1\ldots\mu_s}_{\nu_1\ldots\nu_s}\Big|_{p^2=m^2} \propto \sum_{I=1}^{2s+1} \bar{\varepsilon}^I(p)^{\mu_1\ldots\mu_s}\varepsilon^I(p)_{\nu_1\ldots\nu_s} \equiv (P_{(s)}(p))^{\mu_1\ldots\mu_s}_{\nu_1\ldots\nu_s}, \quad (11.1)$$

where $\varepsilon^I(p)$ is a spin-s on-shell polarisation tensor, $\bar{\varepsilon}^I(p)$ is its complex conjugate and I is the little-group index in the spin-s representation. The tensor $P_{(s)}(p)$ is known as the spin-s *projector* and it is traceless and transverse. This simple property can be used to bootstrap $P_{(s)}$ to any spin. We can write an ansatz for $P_{(s)}(\overline{\mathscr{E}},\mathscr{E}) \equiv \overline{\mathscr{E}}\cdot P_{(s)}\cdot\mathscr{E}$, where $\mathscr{E}^{\mu_1\ldots\mu_s}=\mathscr{E}^{\mu_1}\ldots\mathscr{E}^{\mu_s}$ and $\overline{\mathscr{E}}^{\mu_1\ldots\mu_s}=\overline{\mathscr{E}}^{\mu_1}\ldots\overline{\mathscr{E}}^{\mu_s}$ are arbitrary symmetric reference tensors. Note that, in previous sections, we used the symbol ε to denote on-shell polarisations, as well as arbitrary tensors associated to off-shell momenta. In this section, we consider arbitrary tensors associated to on-shell momenta, hence we use the new notation $\mathscr{E}$. We impose the conditions

$$
\begin{aligned}
p \cdot \frac{\partial}{\partial \mathscr{E}} P_{(s)}(\overline{\mathscr{E}}, \mathscr{E}) &= 0, \\
\frac{\partial}{\partial \mathscr{E}} \cdot \frac{\partial}{\partial \mathscr{E}} P_{(s)}(\overline{\mathscr{E}}, \mathscr{E}) &= 0,
\end{aligned} \tag{11.2}
$$

and we find the unique solution

$$
P_{(s)}(\overline{\mathscr{E}}, \mathscr{E}) = \frac{s!}{(2s)!} \left(\frac{\partial}{\partial P_{(1)}(\overline{\mathscr{E}}, \mathscr{E})} \right)^s \left(P_{(1)}(\overline{\mathscr{E}}, \mathscr{E})^2 - P_{(1)}(\mathscr{E}, \mathscr{E}) P_{(1)}(\overline{\mathscr{E}}, \overline{\mathscr{E}}) \right)^s, \tag{11.3}
$$

where $P_{(1)}^{\mu\nu} = \eta^{\mu\nu} - p^\mu p^\nu / m^2$ is the spin-1 projector. Different representations of eq. (11.3) are discussed in paper I. Note that we assumed $p^2 = m^2$ throughout the calculation.

The conditions (11.2) imply that the longitudinal and trace polarisations are projected out by the free higher-spin theory. This is exactly what gauge invariance achieves in the Lagrangian (4.30). The correspondence between off-shell gauge invariance and on-shell decoupling of unphysical states can be extended to the interacting theory, as we will show below.

11.2 Interactions

Consider a scattering amplitude $\mathscr{A}(\widehat{\Phi}_1^{s_1} \Phi_2^{s_2} \ldots \Phi_n^{s_n})$ between n particles with spin s_i and momenta $p_i^2 = m_i^2$. Let the particles $\Phi_{i \geq 2}$ be described by transverse-traceless polarisation tensors $\varepsilon_i^{\mu_1 \ldots \mu_{s_i}} = \varepsilon_i^{\mu_1} \ldots \varepsilon_i^{\mu_{s_i}}$, where $\varepsilon_i^2 = \varepsilon_i \cdot p_i = 0$ as usual. On the other hand, we want the particle Φ_1 to be described by an arbitrary polarisation tensor $\mathscr{E}_i^{\mu_1 \ldots \mu_{s_1}}$, and we keep track of this via the hatted notation $\widehat{\Phi}_1^{s_1}$. We propose that if the amplitude comes from a theory that satisfies massive gauge invariance, it will obey the relations

$$
\begin{aligned}
p_1 \cdot \frac{\partial}{\partial \mathscr{E}_1} \mathscr{A}(\widehat{\Phi}_1^{s_1} \Phi_2^{s_2} \ldots \Phi_n^{s_n}) &= 0, \\
\frac{\partial}{\partial \mathscr{E}_1} \cdot \frac{\partial}{\partial \mathscr{E}_1} \mathscr{A}(\widehat{\Phi}_1^{s_1} \Phi_2^{s_2} \ldots \Phi_n^{s_n}) &= 0.
\end{aligned} \tag{11.4}
$$

We refer to eq. (11.4) as *on-shell massive Ward identities*. They are simply the generalisation of eq. (11.2) to the interacting theory and they suggest that massive gauge invariance is equivalent to the decoupling of unphysical states from scattering amplitudes [1]. Note that each particle is described by a single field

[1] In this context, we call "unphysical states" anything outside the $2s+1$ transverse-traceless polarisations that describe a physical spin-s particle.

$\Phi_i^{s_i}$, and there are no auxiliary Stückelberg fields. Once $\mathscr{A}(\widehat{\Phi}_1^{s_1}\Phi_2^{s_2}\ldots\Phi_n^{s_n})$ is fixed, the physical scattering amplitude can be recovered via

$$\mathscr{A}(\Phi_1^{s_1}\Phi_2^{s_2}\ldots\Phi_n^{s_n}) = \mathscr{A}(\widehat{\Phi}_1^{s_1}\Phi_2^{s_2}\ldots\Phi_n^{s_n})\big|_{\mathscr{E}_1=\varepsilon_1}, \tag{11.5}$$

where ε_1 is the usual spin-s transverse-traceless polarisation tensor. There is currently no proof for eq. (11.4), but we will present extensive evidence for its validity below.

Let us apply the methods above to a few simple examples. We start with the amplitude $\mathscr{A}(W_1\overline{W}_2A_3)$ between two massive spin-1 particles and a photon. There is one unphysical polarisation, given by

$$\mathscr{E}_\mu = \frac{p_\mu}{m}. \tag{11.6}$$

We construct an ansatz for $\mathscr{A}(\widehat{W}_1\overline{W}_2A_3)$ and impose eq. (11.4), to ensure the unphysical state decouples. The lowest-derivative solution to eq. (11.4) yields

$$\mathscr{A}(W_1\overline{W}_2A_3) = 2\varepsilon_1\cdot\varepsilon_2\;\varepsilon_3\cdot p_1 + 2\varepsilon_2\cdot\varepsilon_3\;\varepsilon_1\cdot p_2 + 2\varepsilon_3\cdot\varepsilon_1\;\varepsilon_2\cdot p_3, \tag{11.7}$$

matching eq. (4.15). We can repeat the analysis for the amplitude $\mathscr{A}(H_1\overline{H}_2A_3)$, now in terms of two massive spin-2 particles $H_{\mu\nu}$. The unphysical states are

$$\mathscr{E}_{\mu\nu} = \left\{\frac{p_{(\mu}\varepsilon_{\nu)}}{m}, \frac{p_\mu p_\nu}{m^2}, \eta_{\mu\nu}\right\}. \tag{11.8}$$

The lowest-derivative solution to eq. (11.4) recovers eq. (10.4), previously obtained from Ward identities.

In general, if we consider the amplitudes $\mathscr{A}(\Phi_1^s\overline{\Phi}_2^sA_3^h)$, the lowest-derivative solution matches eq. (10.8) and eq. (10.9), previously obtained by imposing eq. (10.7) and eq. (10.5a). This is remarkable, since eq. (10.5a) was first introduced as an assumption on the Stückelberg fields, and now we have derived it from a framework where such fields are not present.

Furthermore, we can apply eq. (11.4) to the Compton amplitude. We study the amplitudes $\mathscr{A}(\widehat{\Phi}_1^s\overline{\Phi}_2^sA_3A_4)$ with two massive spin-s fields and two photons. As before, the polarisation of the field $\widehat{\Phi}_1^s$ is left arbitrary. In the $s=1$ case, we find a unique solution reproducing the amplitudes in eq. (7.25). In the $s=2$ case, we assume $\alpha=4$ in eq. (10.4) and construct the diagrams in figure 9.1 by sewing together two on-shell three-point amplitudes. Explicitly, assuming an exchanged momentum $P=p_2+p_3$, we have a first diagram

$$\frac{1}{P^2-m^2}\sum_{H_P}\mathscr{A}(\widehat{H}_1\overline{H}_PA_4)\mathscr{A}(H_{-P}\overline{H}_2A_3), \tag{11.9}$$

where the sum runs over all polarisations of the exchanged particle H_P. Note that, in principle, we should only sum over the physical transverse-traceless tensors ε_P. However, since the amplitudes satisfy eq. (11.4), we can include arbitrary tensors $\mathscr{E}_P$ and the unphysical ones will automatically decouple. This amounts to contracting the Lorentz indices of $\overline{H}_P$ and H_{-P} between the two amplitudes, without using any intermediate projector. The second diagram is obtained from eq. (11.9) via the operation $3 \leftrightarrow 4$. Once the diagrams are computed, we make an ansatz for the contact term in figure 10.1 and sum up all contributions into the amplitude $\mathscr{A}(\widehat{H}_1 H_2 A_3 A_4)$. Remarkably, imposing eq. (11.4) recovers exactly eq. (10.17).

Previously the amplitude (10.17) was obtained from eq. (10.7) after fixing the gauge in the Lagrangian, deriving the propagators (10.14), extracting contributions to the vertices due to covariant derivatives, assuming a rank-dependent derivative counting for the Stückelberg fields. Now this result was reproduced only using on-shell ansätze, without any need for auxiliary fields.

11.3 Open Questions

The on-shell Ward identities (11.4) provide a considerable simplification compared to eq. (10.7). Moreover, it is remarkable that the intricate structure of gauge-invariant higher-spin Lagrangians can be condensed into such simple formulae. However, there are still some open questions. For instance, in the $s=1$ case the Feynman rules from eq. (4.14) produce the following amplitude,

$$\mathscr{A}(\widehat{W}_1 \overline{W}_2 A_3) = -2\varepsilon_3 \cdot p_1\, \mathscr{E}_1 \cdot \varepsilon_2 + 2\mathscr{E}_1 \cdot \varepsilon_3\, \varepsilon_2 \cdot p_1 + \varepsilon_2 \cdot \varepsilon_3\, \mathscr{E}_1 \cdot (p_3 - p_2). \quad (11.10)$$

The unphysical polarisation $\mathscr{E}_1 = p_1/m$ then yields

$$\mathscr{A}(\widehat{W}_1 \overline{W}_2 A_3)\big|_{\mathscr{E}_1 = p_1/m} = m^2 \varepsilon_2 \cdot \varepsilon_3, \quad (11.11)$$

violating eq. (11.4). However, we can add to eq. (11.10) the term

$$-\varepsilon_2 \cdot \varepsilon_3\, \mathscr{E}_1 \cdot p_1, \quad (11.12)$$

without changing the physical scattering amplitude $\mathscr{A}(W_1 \overline{W}_2 A_3)$. This will cancel the right-hand side of eq. (11.11) and thus satisfy the on-shell Ward identity. This suggests some degree of incompatibility between the gauge-invariant off-shell Lagrangians and the idenitities (11.4), resolved only at the level of physical on-shell observables. Although this may just be a healthy feature of the formalism, it is worth trying to understand it further.

A related issue is the current constraint (9.2). As we have discussed above, this constraint is important to fix the AHH amplitudes uniquely, for instance by setting $\alpha = 4$ in eq. (10.4). However, the vertices that satisfy it are obtained from a higher-spin Lagrangian, such as in eq. (11.10), and hence they may be incompatible with eq. (11.4). If we cannot rely on eq. (9.2), we have to find an equivalent constraint that is compatible with the on-shell Ward identities.

This may be related to requiring the tamest possible high-energy divergence in the Compton amplitude, as discussed previously. This problem will be studied more in detail in upcoming work.

Part IV:
Classical Limits

In this section, we review the formalism required to extract classical observables from scattering amplitudes, focusing on the treatment of spin. We will see how the three-point amplitudes discussed in previous parts can be matched to the energy-momentum tensor for a black-hole, and can be used to compute binary observables at leading order in the post-Minkowskian expansion. We will also discuss how the Compton amplitude reproduces the classical cross-section for gravitational-wave scattering on a black-hole background.

12. Schwarzschild Black Holes

We begin by reviewing the non-spinning case. From a distance, Schwarzschild black holes can be described by the point-particle action S_{pp},

$$S_{pp} = m \int d^4x d\lambda \delta^4(x - z(\lambda)) \sqrt{-g_{\mu\nu}(x)\dot{z}^\mu \dot{z}^\nu}, \tag{12.1}$$

where $z(\lambda)$ is the worldline of the black-hole, $\dot{z} = dz/d\lambda$ [3, 8] and λ is a worldline parameter. Using this as a source in Einstein's equations, we can study binary systems of two non-spinning black holes and extract observables such as the gravitational potential and waveform. It has been shown that the same observables can be obtained from the action S for a massive scalar field minimally-coupled to gravity [167, 168],

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{m^2}{2} \phi^2 \right). \tag{12.2}$$

This is true in an appropriate kinematic limit, known as the *classical limit*, where the momenta of the gravitons are much smaller than the momenta of the massive scalars [10]. To see how this works, we consider a few simple examples.

12.1 Energy-Momentum Tensor

Consider $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, $h_{\mu\nu}$ is a small perturbation and $\kappa = \sqrt{32\pi G}$ is the gravitational coupling. Then the above action can be expanded as

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu} + \mathscr{O}(h_{\mu\nu}^2) \right), \tag{12.3}$$

where

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \left(\frac{1}{2} \partial_\rho \phi \partial^\rho \phi - \frac{m^2}{2} \phi^2 \right) \tag{12.4}$$

is the linearised energy-momentum tensor. We can use it to compute the three-point amplitude between two massive scalars and a graviton, given by

$$\mathscr{M}(\phi_1 \phi_2 h_3) = \kappa (\varepsilon_3 \cdot p_1)^2 \tag{12.5}$$

up to overall normalisation. Note that we have $p_1^2 = p_2^2 = m^2$, $p_3^2 = 0$, $\varepsilon_3^{\mu\nu} = \varepsilon_3^\mu \varepsilon_3^\nu$ is the graviton polarisation tensor and $\varepsilon_3 \cdot p_3 = \varepsilon_3^2 = 0$.

On the other hand, the energy-momentum tensor of a Schwarzschild black hole is, in momentum space [58],

$$T^{\mu\nu}_{Schw}(-k) = 2\pi\delta(p\cdot k)p^\mu p^\nu \tag{12.6}$$

where $p^2 = m^2$ is the black-hole four-momentum. We can contract it with an on-shell graviton $h_{\mu\nu}(k) = 2\pi\delta(k^2)\varepsilon_{k\mu}\varepsilon_{k\nu}$ to get

$$\kappa h_{\mu\nu}(k)T^{\mu\nu}_{Schw}(-k) = \kappa(2\pi)^2\delta(k^2)\delta(p\cdot k)(\varepsilon_k \cdot p)^2. \tag{12.7}$$

We see that, up to normalisation and delta functions, eq. (12.7) and eq. (12.5) match upon identifying $p \equiv p_1$ and $k \equiv p_3$.

12.2 Scattering Angle

Consider the scattering problem between two non-spinning black holes, with momenta p_1 and p_2 and masses m_1 and m_2, following the approach of ref. [169]. We assume that their Schwarzschild radii $r_s^{(i)} = Gm_i$ are much smaller than the impact parameter b. This leads to a perturbative expansion in $\frac{Gm_i}{b}$ known as the *post-Minkowskian expansion*. At leading order, the scattering angle is known to be

$$\theta = \frac{4GE}{b}\frac{\hat{E}^4 - 2m_1^2m_2^2}{\hat{E}^4 - 4m_1^2m_2^2} + \mathscr{O}(G^2) \tag{12.8}$$

where $E^2 = (p_1+p_2)^2$, $\hat{E}^2 = E^2 - m_1^2 - m_2^2$ and we work in the centre-of-mass frame $p_1 + p_2 = (E,0,0,0)$. Below we will show how to reproduce this result from the action (12.2).

We consider the $2 \to 2$ scattering amplitude $\mathscr{M}(q) \equiv \mathscr{M}(\phi_1\phi_{1'}\varphi_2\varphi_{2'})$ between two distinct scalar particles ϕ and φ, shown in figure 12.1. The incoming momenta are $-p_1$ and $-p_2$ and the outgoing ones are p_1' and p_2'. This yields

$$\mathscr{M}(q) = \frac{-16\pi G}{q^2}\left(m_1^2m_2^2 - 2(p_1\cdot p_2)^2 - (p_1\cdot p_2)q^2\right) \tag{12.9}$$

where $q = -p_1 - p_1'$ is the momentum of the exchanged graviton. Classical physics is reproduced in the limit $q \ll p_i, m_i$, so the last term in the brackets can be neglected. A way to understand this intuitively is that classical physics corresponds to the $\hbar \to 0$ limit of quantum observables. However, the graviton is classically a wave with macroscopic wavenumber $\bar{q} = q/\hbar$. Therefore, we need $q \to 0$ to ensure $\bar{q}$ remains finite. On the other hand, the masses m_i and momenta p_i describe black holes, so we expect them to be finite macroscopic quantities.

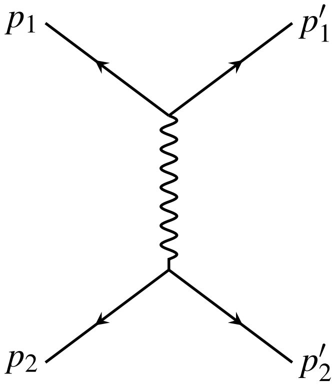

Figure 12.1. Tree-level $2 \to 2$ scattering of two scalar particles, mediated by graviton exchange.

In the limit described above, the G-expansion of the amplitude can be resummed in impact parameter space [170],

$$\mathscr{M}(b) \equiv \int d^2q e^{-iq\cdot b}\mathscr{M}(q) = 2\sqrt{\hat{E}^4 - 4m_1^2m_2^2}(e^{\chi(b)} - 1), \qquad (12.10)$$

where $\chi(b)$ is known as the *eikonal phase*. The vectors b and q lie in the scattering plane, orthogonal to p_1 and p_2, and hence belong to a two-dimensional subspace. At leading order in G, we have

$$\chi(b) = \frac{1}{2\sqrt{\hat{E}^4 - 4m_1^2m_2^2}}\int\frac{d^2q}{(2\pi)^2}e^{-iq\cdot b}\mathscr{M}(q) = -2G\frac{\hat{E}^4 - 2m_1^2m_2^2}{\sqrt{\hat{E}^4 - 4m_1^2m_2^2}}\log b + \ldots \qquad (12.11)$$

where the terms omitted are independent of b. The scattering angle can be obtained from the eikonal, as explained in ref. [171], via

$$\theta = \frac{-2E}{2\sqrt{\hat{E}^4 - 4m_1^2m_2^2}}\frac{\partial\chi}{\partial b} \qquad (12.12)$$

which recovers eq. (12.8).

If we had included the last term in eq. (12.9), we would have an additional contribution to $\chi(b)$ proportional to $\delta^2(b)$. The delta function makes it obvious that this is a short-range effect, irrelevant to long-range classical computations [172].

12.3 Wave Scattering

Another interesting application is the case of a gravitational wave scattering onto a black hole, as discussed in ref. [85, 103]. This process is described

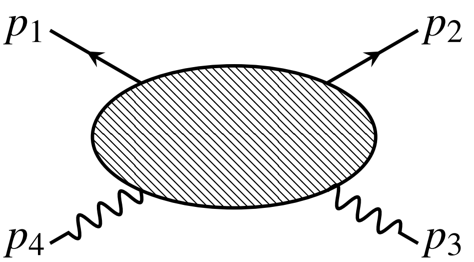

Figure 12.2. Compton amplitude describing the scattering of a graviton and a massive particle or black hole. Legs 1 and 2 are massive states, legs 3 and 4 are massless gravitons. All momenta are assumed to be outgoing.

by the Compton amplitude in figure 12.2 and it is important to compute next-to-leading order post-Minkowskian binary observables in the amplitude-based approach (see for instance ref. [11, 12, 18, 173]). Using the methods of black hole perturbation theory, we consider an incoming plane wave $\psi_{in}(x) = e^{-ik\cdot x}$ and find the scattered wave $\psi_{out}(x) \propto e^{ik'\cdot x}$. We can then compute the differential cross-section $d\sigma/d\Omega$ for the scattered wave. For a Schwarzschild black hole, this is known to be [174–177]

$$\frac{d\sigma}{d\Omega} = \frac{G^2m^2}{\sin^4\frac{\theta}{2}}\left(\cos^8\frac{\theta}{2} + \sin^8\frac{\theta}{2}\right) \tag{12.13}$$

where we work in the rest frame of the black hole and θ is the angle between the incoming and outgoing wave momenta k and k'.

In quantum field theory, this process is described by the Compton amplitude in figure 12.2. From the Lagrangian (12.2) one can work out the compact formulae

$$\mathscr{M}(\phi_1\bar{\phi}_2h_3^+h_4^+) = \kappa^2\frac{m^4[34]^4}{s_{12}t_{13}t_{14}}, \tag{12.14a}$$

$$\mathscr{M}(\phi_1\bar{\phi}_2h_3^-h_4^+) = \kappa^2\frac{[4|p_1|3\rangle^4}{s_{12}t_{13}t_{14}}, \tag{12.14b}$$

in terms of the spinor-helicity variables discussed in part II, matching eq. (7.24) for $s = 0$. As before we have $s_{12} = (p_1+p_2)^2$, $t_{13} = (p_1+p_3)^2 - m$ and $t_{14} = (p_1+p_4)^2 - m^2$. The other two helicity configurations can be obtained from complex conjugation. Using the shorthand $\mathscr{M}_{l_3l_4} \equiv \mathscr{M}(\phi_1\bar{\phi}_2h_3^{l_3}h_4^{l_4})$, the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2m^2}\left(\mathscr{M}_{++}\mathscr{M}_{--} + \mathscr{M}_{+-}\mathscr{M}_{-+}\right). \tag{12.15}$$

To compare to the black-hole case, we identify $p = -p_1$, $k = -p_4$ and $k' = p_3$ and use the parametrisation

$$\begin{aligned}
&p^\mu = (m, 0, 0, 0), \\
&k = \omega(1, 0, 0, 1), \\
&k' = \omega(1, \sin\theta, 0, \cos\theta) + \mathscr{O}\left(\frac{\omega}{m}\right)
\end{aligned} \tag{12.16}$$

where we take $\omega \ll p, m$, required for the classical limit. Then we have

$$\begin{aligned}
\mathscr{M}_{++}\mathscr{M}_{--} &= \frac{\kappa^4 m^8 s_{12}^4}{s_{12}^2 t_{13}^2 t_{14}^2} = \frac{\kappa^4 m^4 \sin^8 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}}, \\
\mathscr{M}_{+-}\mathscr{M}_{-+} &= \frac{\kappa^4 (t_{13} t_{14} - m^2 s_{12})^4}{s_{12}^2 t_{13}^2 t_{14}^2} = \frac{\kappa^4 m^4 \cos^8 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}},
\end{aligned} \tag{12.17}$$

using $[34]\langle 34\rangle = -s_{12}$ and $[4|p_1|3\rangle\,[3|p_1|4\rangle = t_{13}t_{14} - m^2 s_{12}^2$. Using the above, it is easy to see that eq. (12.15) and eq. (12.13) are equal.

13. Kerr Black Holes

A Kerr black hole can be described by its worldline position $z(\lambda)$ and its angular momentum $a^{\mu}(\lambda)$, where λ is a worldline parameter such as the proper time. We choose $a^{\mu}(\lambda)$ to have dimensions of length, such that its magnitude is equal to the Kerr *ring radius*. The Kerr worldline action, a spinning generalisation of eq. (12.1), is known up to linear order in the Riemann tensor $R_{\mu\nu\rho\sigma}$ and it can be fixed by matching to the known Kerr energy-momentum tensor $T^{\mu\nu}_{Kerr}$ [6, 109]. Similarly to the non-spinning case, we will show how to reproduce $T^{\mu\nu}_{Kerr}$ from three-point scattering amplitudes of form $\mathscr{M}(\Phi^s_1\bar{\Phi}^s_2 h_3)$, where Φ^s is a massive spin-s field. In particular, we will see that amplitudes involving a spin-s particle can be used to extract Kerr observables up to $\mathscr{O}(a^{2s})$ in the spin-multipole expansion.

However, starting from $\mathscr{O}(R^2_{\mu\nu\rho\sigma})$ there are additional worldline operators that can be added to the action. Their Wilson coefficients have been fixed up to $\mathscr{O}(a^4)$, but they are still unknown at higher spin order [103]. In the language of scattering amplitudes, the four-point Compton amplitude $\mathscr{M}(\Phi^s_1\bar{\Phi}^s_2 h_3 h_4)$ that reproduces Kerr observables is known up to $s \leq 2$, but for higher spin fields there is a contact term ambiguity [58, 59, 82, 102, 108].

Possible solutions to this issue are discussed in more detail in parts II and III. In this section, we focus on the three-point and low-spin four-point cases and review how to extract classical observables from scattering amplitudes with spin.

13.1 Spin Vector in Quantum Field Theory

Classical spin degrees of freedom are encoded in the angular momentum four-vector a^{μ}. On the other hand, spin degrees of freedom of massive quantum particles are described by polarisation tensors $\varepsilon^{\mu_1\cdots\mu_s} = \varepsilon^{\mu_1}\ldots\varepsilon^{\mu_s}$. In order to compare the two pictures, we need a more suitable choice of variables. In a quantum theory, classical quantities can be reproduced from expectation values of the corresponding operator. The covariant form of the angular momentum operator is the Pauli-Lubanski pseudovector, given in eq. (2.1). In order to match the dimensions of the classical vector a^{μ}, we define the *ring-radius operator* $a^{\mu}_{(s)}$,

$$a^{\mu}_{(s)} = \frac{1}{2m^2}\varepsilon^{\mu\nu\rho\sigma}p_{\nu}M_{(s)\rho\sigma}, \tag{13.1}$$

given in the spin-s representation. The spin-s Lorentz generators are

$$(M_{(s)\rho\sigma})^{\mu(s)}{}_{\nu(s)} = 2is\,\delta^{(\mu_1}{}_{[\rho}\eta_{\sigma](\nu_1}\delta^{\mu_2}_{\nu_2}\dots\delta^{\mu_s)}_{\nu_s)}, \tag{13.2}$$

where we use a multi-index notation $\mu(s) \equiv (\mu_1 \dots \mu_s)$ with fully-symmetrised Lorentz indices.

Given a particle described by the polarisation tensor $\varepsilon(p)^{\mu_1\cdots\mu_s}$, we can write down symmetrised expectation values of the operator (13.1),

$$\langle a_{(s)}^{(\mu_1}\dots a_{(s)}^{\mu_n)}\rangle = \frac{1}{(\bar{\varepsilon}\cdot\varepsilon)^s}\bar{\varepsilon}_{\rho(s)}(a_{(s)}^{(\mu_1}a_{(s)}^{\mu_2}\dots a_{(s)}^{\mu_n)})^{\rho(s)}{}_{\nu(s)}\varepsilon^{\nu(s)} \tag{13.3}$$

where $\bar{\varepsilon}$ is the complex-conjugate polarisation. As we are about to show, we can rewrite any scattering amplitude in terms of the variables in eq. (13.3) and the simple replacement $\langle a_{(s)}^{(\mu_1}\dots a_{(s)}^{\mu_n)}\rangle \to a^{\mu_1}\dots a^{\mu_s}$ will reproduce the correct classical observables [178]. Note that we consider symmetrised products in eq. (13.3), since by the Lorentz algebra we have $[a^{\mu}_{(s)}, a^{\nu}_{(s)}] \sim \varepsilon^{\mu\nu\rho\sigma}p_\rho a_{(s)\sigma}$ and hence antisymmetric contributions reduce to lower-spin effects. As discussed in paper II, we can also write eq. (13.3) in terms of the massive spinor variables in part II,

$$\langle a_{(s)}^{(\mu_1}\dots a_{(s)}^{\mu_n)}\rangle = \frac{1}{\langle\bar{\mathbf{1}}\mathbf{1}\rangle^{2s}}\langle\bar{\mathbf{1}}|^{\alpha(2s)}(a_{(s)}^{(\mu_1}a_{(s)}^{\mu_2}\dots a_{(s)}^{\mu_n)})^{\alpha(s)}{}_{\beta(s)}|\mathbf{1}\rangle_{\beta(2s)} \tag{13.4}$$

where $\alpha(2s) = (\alpha_1 \dots \alpha_{2s})$ is a symmetrised spinor multi-index. This expression is also valid for half-integer spins.

13.2 Energy-Momentum Tensor

The linearised energy-momentum tensor of a Kerr black hole is, in momentum space [58],

$$T^{\mu\nu}(-k) = 2\pi\delta(p\cdot k)p^{(\mu}\exp(ia*k)_\rho^{\nu)}p^\rho \tag{13.5}$$

where $p^2 = m^2$ is the black-hole momentum, $a = S/m$ the (rescaled) angular momentum vector and $(a*k)^\mu_\nu = \varepsilon^\mu{}_{\nu\rho\sigma}a^\rho k^\sigma$. We can contract it with an on-shell graviton $h_{\mu\nu}(k) = 2\pi\delta(k^2)\varepsilon_{k\mu}\varepsilon_{k\nu}$ to obtain a spinning generalisation of eq. (12.7),

$$\kappa h_{\mu\nu}(k)T^{\mu\nu}_{Kerr}(-k) = \kappa(2\pi)^2\delta(k^2)\delta(p\cdot k)(\varepsilon_k\cdot p)^2\exp(\mp k\cdot a) \tag{13.6}$$

where we used the identity $i\varepsilon^{\mu\nu\rho\sigma}k_\mu\varepsilon^{\pm}_{k\nu}p_\rho a_\sigma = \mp k\cdot a\varepsilon^{\pm}_k\cdot p$ for a graviton $\varepsilon^{\pm}_k$ with helicity ± 2 [82]. If we want to describe Kerr physics from scattering amplitudes, we need to find a quantum field theory that reproduces the above result.

A puzzle, in trying to match a quantum-field-theory amplitude $\mathcal{M}(\Phi_1^s \bar{\Phi}_2^s h_3)$ to eq. (13.6), is that the classical black hole has a definite momentum p, whereas the three-point amplitude contains two massive momenta p_1 and p_2. However, the classical limit implies $p_3 \ll p_1, p_2$, meaning that the two momenta only differ by an infinitesimally small amount. Hence we make the identification $p \equiv -p_1$ and $p_2 = p + \mathcal{O}(\hbar)$. We can then write the polarisation vector ε_2 as a boost of ε_1 as follows,

$$\varepsilon_2^\mu = \bar{\varepsilon}_1^\mu - \frac{p_3 \cdot \bar{\varepsilon}_1}{m_1^2}\left(p_1^\mu + \frac{1}{2}p_3^\mu\right), \tag{13.7}$$

preserving the on-shell conditions $\varepsilon_2^2 = \varepsilon_2 \cdot p_2 = 0$. This formula is valid in the case of $p_3^2 = 0$, satisfied by the three-point on-shell kinematics. This allows us to identify a unique polarisation vector $\varepsilon \equiv \varepsilon_1$ to describe the black hole.

Alternatively, we can rewrite the amplitude in terms of massive spinor-helicity variables via eq. (7.3), and use

$$\begin{aligned} |\mathbf{2}\rangle &= |\bar{\mathbf{1}}\rangle + \frac{1}{2m}(p_3 \cdot \sigma)|\bar{\mathbf{1}}], \\ |\mathbf{2}] &= -|\bar{\mathbf{1}}] - \frac{1}{2m}(p_3 \cdot \bar{\sigma})|\bar{\mathbf{1}}\rangle, \end{aligned} \tag{13.8}$$

where $|\bar{\mathbf{1}}] = \bar{z}_a|1^a]$, $|\bar{\mathbf{1}}\rangle = \bar{z}_a|1^a\rangle$ and $\bar{z}_a$ is the complex-conjugate of the little-group polarisation variables z_a, discussed in part II.

Let us see how this works in a simple example. We consider two massive spin-1 fields W_μ minimally coupled to a negative-helicity graviton $h_{\mu\nu}$, to see if they bear any connection to Kerr. The Lagrangian is given in eq. (7.16). The three-point amplitude is

$$\mathcal{M}(W_1 \overline{W}_2 h_3^-) = \varepsilon_3 \cdot p_1 (\varepsilon_1 \cdot \varepsilon_2\, \varepsilon_3 \cdot p_1 + \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_1 \cdot p_2 + \varepsilon_3 \cdot \varepsilon_1\, \varepsilon_2 \cdot p_3) = \frac{1}{x^2}[\mathbf{12}]^2. \tag{13.9}$$

Using eq. (13.7) and the following relation,

$$\bar{\varepsilon}_1^\mu \varepsilon_1^\nu = -m^2 \langle a_{(1)}^{(\mu} a_{(1)}^{\nu)} \rangle + \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma} p_{1\rho} \langle a_{(1)\sigma} \rangle - \left(\eta^{\mu\nu} - \frac{p_1^\mu p_1^\nu}{m^2}\right), \tag{13.10}$$

we find

$$\mathcal{M}(W_1 \overline{W}_2 h_3^-) = (\varepsilon_3 \cdot p_1)^2 \left(1 + p_3 \cdot a + \frac{(p_3 \cdot a)^2}{2}\right). \tag{13.11}$$

Remarkably, the amplitude (13.11) matches the spin-multipole expansion of eq. (13.6) up to $\mathcal{O}(a^2)$, upon identifying $k \equiv p_3$. Higher-order terms do not appear because $(M_{(s)}^{\mu\nu})^{2s+1}$ can always be rewritten in terms of $(M_{(s)}^{\mu\nu})^{k \leq 2s}$, by the properties of finite-spin Lorentz generators. Therefore, to describe Kerr we need to study the infinite-spin limit.

In part II we discussed three-point amplitudes of form $\mathscr{M}(\Phi_1^s \bar{\Phi}_2^s h_3)$ for any spin, and identified the AHH amplitudes (7.17) as a special choice. Using eq. (13.4) and eq. (13.8) we can rewrite them in terms of spin vectors as

$$\frac{m^2}{x^2}\frac{[\mathbf{12}]^{2s}}{m^{2s}} = (\varepsilon_3 \cdot p_1)^2 \sum_{n=0}^{2s} \frac{1}{n!}(p_3 \cdot a)^n \xrightarrow{s\to\infty} (\varepsilon_3 \cdot p_1)^2 \exp(p_3 \cdot a), \qquad (13.12)$$

reproducing eq. (13.6). We know from part III that the AHH amplitudes are the lowest-derivative solution to constraints from gauge invariance and high-energy unitarity. The fact that they also describe Kerr is remarkable and it points to a deep connection between black-holes and fundamental particles. Note that, as mentioned in part II, the Kerr solution has an electromagnetic analog known as root-Kerr. The root-Kerr three-point amplitude is proportional to $(\varepsilon_3 \cdot p_1)\exp(p_3 \cdot a)$ and this is reproduced by the amplitudes (7.20) in the $s \to \infty$ limit.

13.3 Classical Observables

In this section we briefly discuss how to compute classical observables in the case of non-zero spin. The first observable we consider is the leading-order scattering angle. The method is the same as the non-spinning case, so we need to compute the amplitude in fig. 12.1, where this time the external states have masses m_1, m_2 and spins $s_1, s_2 \to \infty$. In the classical limit, $q \ll p_1, p_2$ and hence $q^2 = 0 + \mathscr{O}(\hbar^2)$. This implies the exchanged graviton is effectively on-shell and the answer can be written as a product of three-point amplitudes,

$$\mathscr{M}(q) \xrightarrow{\hbar\to 0} \frac{1}{q^2}\sum_{\pm} \mathscr{M}(\Phi_1^\infty \bar{\Phi}_{1'}^\infty h_q^\mp)\mathscr{M}(\Phi_2^\infty \bar{\Phi}_{2'}^\infty h_{-q}^\pm). \qquad (13.13)$$

Using the amplitudes in eq. (13.12) we obtain [63, 69, 81, 106]

$$\mathscr{M}(q) = \frac{\kappa^2}{4}\frac{m_1^2 m_2^2 \sigma^2}{q^2}\sum_{\pm}(1\pm v)^2 \exp\left(\pm i \frac{\varepsilon(p_1, p_2, q, a)}{m_1 m_2 \sigma v}\right), \qquad (13.14)$$

where $\sigma = 1/\sqrt{1-v^2} = (p_1 \cdot p_2)/(m_1 m_2)$ is the relativistic Lorentz factor, and $a^\mu = a_1^\mu + a_2^\mu$. We used the relation $m_1 m_2 \sigma v\, q \cdot a = i\varepsilon(p_1, p_2, q, a)$, where $\varepsilon(p_1, p_2, q, a)$ denotes a contraction with the Levi-Civita tensor, to obtain a parity-even expression, since a^μ is parity-odd. Following the same steps as in the non-spinning case, we reproduce the known scattering angle [58, 109],

$$\theta = \frac{GE}{v^2}\sum_{\pm}\frac{(1\pm v)^2}{b \pm (a_1 + a_2)} + \mathscr{O}(G^2), \qquad (13.15)$$

where we assumed the spins a_1 and a_2 are aligned and $E^2 = (p_1 + p_2)^2$ as before. If they are not aligned, the motion of the two black holes is not confined to a plane and hence the angle θ is ill-defined. In that case, we can compute covariant observables such as the total change in momentum Δp^{μ} (see ref. [63] for details).

Next, we consider wave scattering on a Kerr black hole background. We use the shorthand $\mathscr{M}_4^{l_3 l_4} = \mathscr{M}(\Phi^{\infty}\bar{\Phi}^{\infty} h_3^{l_3} h_4^{l_4})$ for the amplitudes (7.24). In the classical limit, each helicity sector simplifies to

$$\begin{aligned}\mathscr{M}_4^{-+} &= \mathscr{M}_4^{0,-+} e^{(2w-p_4+p_3)\cdot a},\\ \mathscr{M}_4^{++} &= \mathscr{M}_4^{0,++} e^{(-p_4-p_3)\cdot a},\end{aligned} \tag{13.16}$$

where $w^{\mu} = [4|\bar{\sigma}^{\mu}|3\rangle(t_{13}-t_{14})/(4[4|p_1|3\rangle)$ [69, 91, 92, 103]. This has been explicitly compared against solutions to the Teukolsky equations, governing wave scattering on Kerr background, up to $\mathscr{O}(a^6)$ in ref. [103]. Perfect matching was found up to $\mathscr{O}(a^4)$, where the amplitudes (13.16) are free of spurious poles. However, at $\mathscr{O}(a^5)$ and $\mathscr{O}(a^6)$, there are additional terms in the Teukolsky result for $\mathscr{M}_4^{-+}$, which are not yet understood from the quantum field theory side. This is a compelling motivation to study the structure of massive higher-spin theories and find new higher-spin Compton amplitudes, as discussed in part III.

14. Classical Strings

The methods outlined so far to compute classical observables with spin have been almost exclusively applied to the case of black holes and the amplitudes (7.17). Paper II discusses another example that is instructive to better understand the formalism: relativistic strings. As discussed in the context of eq. (13.16), the Compton (and higher-point) amplitudes that describe Kerr black holes are not known to all orders in spin. String theory could hold the key to resolving this problem. If one found a string state (or superposition of string states) that reproduces the Kerr three-point amplitude (13.6), all higher-point amplitudes involving such a state could be computed unambiguously via the string path integral. This provides a strong motivation to study the classical limit of string amplitudes.

We start from the three-point amplitude between a photon and two leading Regge trajectory states of the open superstring, given by eq. (5.6). Rewriting it in terms of spin vectors, we obtain

$$\mathscr{A}(\Phi_1^s \Phi_2^s A_3^-) = g\varepsilon_3^- \cdot p_1 \sum_{n=0}^{2s} c_n^{(s)} (p_3 \cdot a)^n \tag{14.1}$$

where

$$c_0^{(s)} = c_1^{(s)} = 1, \quad c_2^{(s)} = \frac{4s^2 - 7s + 4}{2s(2s-1)}, \quad c_3^{(s)} = \frac{2s-3}{2(2s-1)}, \quad \dots \tag{14.2}$$

and we omit $c_{n>3}^{(s)}$ for simplicity (more details provided in paper II). To extract classical observables, we consider the $s \to \infty$ limit, yielding

$$\lim_{s\to\infty} \mathscr{A}(\Phi_1^s \Phi_2^s A_3^-) = g\varepsilon_3^- \cdot p_1 \left(I_0(2p_3 \cdot a) + I_1(2p_3 \cdot a)\right) \tag{14.3}$$

where $I_0(x)$ and $I_1(x)$ are modified Bessel functions of the first kind [1]. Now all that is left is to find a classical system that reproduces this result. We expect it to be a classical solution of the string theory equations of motion, which follow from the action (5.1). A well-known solution is that of a rigid string

[1] The Bessel functions are defined by $I_0(2x) = \sum_{k=0}^{\infty}(1/(k!)^2)x^{2k}$ and $I_1(2x) = \sum_{k=0}^{\infty} 1/(k!(k+1)!)x^{2k+1}$.

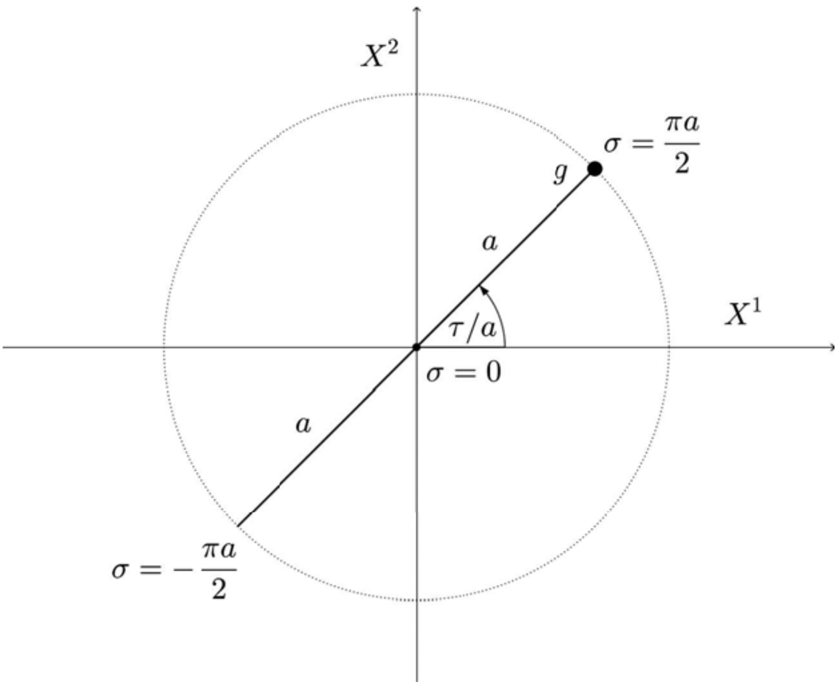

Figure 14.1. Rigid open string rotating about its centre. Total length $2a$ and angular velocity $1/a$. Electromagnetic coupling realised by placing a charge g at one endpoint.

rotating about its centre, as shown in fig. 14.1, given by [179–181]

$$\begin{aligned}
X^0 &= \tau, \\
X^1 &= a\cos\frac{\tau}{a}\sin\frac{\sigma}{a}, \\
X^2 &= a\sin\frac{\tau}{a}\sin\frac{\sigma}{a}, \\
X^3 &= 0,
\end{aligned} \tag{14.4}$$

where $\sigma \in [-\frac{\pi a}{2}, \frac{\pi a}{2}]$ and $\tau \in (-\infty, \infty)$ parametrise the string worldsheet. The three-point amplitude (14.3) encodes the linear coupling of the massive string to the photon, classically described by the electromagnetic current $j^\mu(x)$. The above solution can be coupled to the electromagnetic field by placing a charge g at the endpoint, as discussed in ref. [182], such that

$$j^\mu(x) = (\rho(x), \mathbf{j}(x))^\mu = \frac{g}{a}\delta(r-a)\delta(\phi - t/a)\delta(z)n^\mu, \tag{14.5}$$

where $n^\mu = (1, -\sin(t/a), \cos(t/a), 0)$ in the coordinates (X^0, X^1, X^2, X^3). We use $r = \sqrt{(X^1)^2 + (X^2)^2}$, $\phi = \arctan(X^2/X^1)$ and $z = X^3$. To find the three-point amplitude, we Fourier transform $j^\mu(x)$ and contract it with an on-shell photon state ε_k,

$$\begin{aligned}\varepsilon_k \cdot j(-k) &= \frac{1}{2\pi a}\int d^4x\, e^{ik\cdot x}\,\frac{g}{a}\,\varepsilon_k \cdot n\,\delta(r-a)\delta(\phi - t/a)\delta(z) \\ &= g\frac{\varepsilon_k \cdot p}{m}\left[I_0(2k\cdot a) + I_1(2k\cdot a)\right],\end{aligned} \tag{14.6}$$

where $p^\mu = (m,0,0,0)$ is the total four-momentum of the string and $a^\mu = (0,0,0,a)$ its angular momentum. Identifying $p \equiv p_1$ and $k \equiv p_3$, we find that the classical limit of leading Regge superstring amplitudes in eq. (14.3) matches the string solution (14.4).

The gravitational case, also studied in paper II, is analogous to its electromagnetic counterpart. The closed string amplitude (5.9) can be rewritten in terms of spin vectors as

$$\mathscr{M}(\Phi_1^s \Phi_2^s h_3^-) = (\varepsilon_3^- \cdot p_1)^2 \sum_{n=0}^{2s} c_n^{(s)} (p_3 \cdot a)^n, \tag{14.7}$$

where

$$c_0^{(s)} = c_1^{(s)} = 1, \quad c_2^{(s)} = \frac{3s^2 - 7s + 8}{2s(2s-1)}, \quad c_3^{(s)} = \frac{3s^2 - 12s + 14}{2(2s-1)(2s-2)}, \quad \ldots \tag{14.8}$$

and we omit $c_{n>3}^{(s)}$ for simplicity. In the $s \to \infty$ limit we have

$$\lim_{s\to\infty} \mathscr{M}(\Phi_1^s \Phi_2^s h_3^-) = (\varepsilon_3^- \cdot p_1)^2 \left(I_0(p_3\cdot a) + I_1(p_3 \cdot a)\right)^2. \tag{14.9}$$

The relevant string solution is identical to eq. (14.4), but now $\sigma \in [-\pi a, \pi a]$ describing a rigid folded closed string. The energy-momentum tensor of the solution is

$$T^{\mu\nu} = \frac{1}{\pi\alpha'}\frac{a\gamma(r)}{r} n^{\mu\nu}(r,\phi)\delta(z)\left[\delta(t - a\phi) + \delta(t - a\phi - a\pi)\right]\Theta(a - r), \tag{14.10}$$

where $\gamma(r) = \left(1 - \frac{r^2}{a^2}\right)^{-1/2}$, and

$$n^{\mu\nu}(r,\phi) = \begin{pmatrix} 1 & -\frac{r}{a}\sin\phi & \frac{r}{a}\cos\phi & 0 \\ -\frac{r}{a}\sin\phi & \frac{r^2}{a^2} - \cos^2\phi & -\frac{1}{2}\sin 2\phi & 0 \\ \frac{r}{a}\cos\phi & -\frac{1}{2}\sin 2\phi & \frac{r^2}{a^2} - \sin^2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14.11}$$

Contracting this with an on-shell graviton state $\varepsilon_{k\mu\nu} = \varepsilon_{k\mu}\varepsilon_{k\nu}$ we get

$$\begin{aligned}\varepsilon_k \cdot T(-k) \cdot \varepsilon_k =& \frac{1}{2\pi^2\alpha'}\int d^4x\, e^{ik\cdot x}\frac{\gamma(r)}{r}\varepsilon_k{\cdot}n{\cdot}\varepsilon_k\,\delta(z)\\ &\times\big(\delta(t-a\phi)+\delta(t-a\phi-a\pi)\big)\Theta(a-r)\\ =& \frac{(\varepsilon_k\cdot p)^2}{m}\left[I_0(k\cdot a)+I_1(k\cdot a)\right]^2\end{aligned} \tag{14.12}$$

reproducing eq. (14.9). The function $\Theta(x)$ is the Heaviside step function, equal to 1 when $x \geq 0$ and zero otherwise.

The classical limit of the string amplitudes discussed above has one important difference to that of black holes and the AHH amplitudes. In eq. (14.2) and eq. (14.8) we can see that the spin-multipole coefficients $c_n^{(s)}$ depend on the spin s, and the classical values can only be found after taking the $s \to \infty$ limit. On the other hand, the spin multipoles obtained from the amplitude (7.17) are independent of s, namely we have $c_n^{(s)} = 1/n!$, for instance in eq. (13.11). As a consequence, the classical result can be read off from a finite-spin amplitude, up to $\mathcal{O}(a^{2s})$, without an explicit $s \to \infty$ limit. This special property of the black-hole amplitudes is referred to as *spin universality*. This also holds for the Compton amplitudes (7.24) for $s \leq 2$ and hence up to $\mathcal{O}(a^4)$. Nonetheless, as shown in the case of strings, it is not a general feature of the classical-limit framework and there is no guarantee it will hold for the $s > 2$ Compton amplitudes.

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16. Svensk Sammanfattning

Vi lever i en era av precisionsmätningar av gravitationsvågor. Det är nu viktigare än någonsin att kunna producera precisa teoretiska förutsägelser för att analysera den data som samlas in via gravitationsinterferometrar.

Binära svarta hålssystem är viktiga källor till gravitationsstrålning. Ett fascinerande nytt sätt att modellera sådana system är att behandla de svarta hålen som partiklar i en kvantfältsteori och studera deras växelverkan med hjälp av spridningsamplituder.

Svarta hål av Schwarzschild-typ kan beskrivas av massiva skalarfält minimalt kopplade till gravitation. I synnerhet beskrivs gravitationspotentialen mellan två svarta hål av fyrapunktsamplituder med externa skalärer. Fempunktsamplituder med fyra skalärer och en emitterad graviton beskriver gravitationsstrålningen från ett binärt system. Comptonamplituder med två skalärer och två gravitoner kan användas för att studera spridningen av gravitationsvågor mot ett svart hål. Varje observabel extraheras från en speciell kinematisk gräns av spridningsamplituderna, den så kallade klassiska gränsen. Amplituderna beräknas perturbativt, i potenser av Newtons gravitationskonstant G. Detta reproducerar den post-Minkowskiska expansionen av klassiska observabler. Detta tillvägagångssätt har redan gett ett flertal resultat av vikt, t.ex. den konservativa gravitationspotentialen $\mathscr{O}(G^4)$ eller vågformen $\mathscr{O}(G^3)$.

Svarta hål av Kerr-typ beskrivs av massiva högre spinn kvantfältsteorier i en klassisk gräns med oändligt spinn. För Kerr är endast trepunktsamplituden känd i sin helhet och en fullständig förståelse av den underliggande lagrangianen saknas fortfarande. Framförallt är lokala interaktioner mellan fyra eller fler fält ej kända, men de är nödvändiga för att utföra beräkningar av högre ordningar i den post-Minkowskiska expansionen. Denna avhandling innehåller en djupgående studie av de högre spinn-lagrangianer som ligger till grund för de kända Kerr-amplituderna, och den presenterar ett recept för att inskränka den form interaktioner kan ta.

Massiva Lagrangianer med högre spinn är begränsade av t.ex. unitaritet och antalet tillgängliga frihetsgrader. Ett användbart verktyg för att bygga konsistenta teorier är att införa massiv gaugesymmetri genom hjälpfält som kallas Stückelbergfält. Explicita trepunktsinteraktioner har beräknats i litteraturen. Mer komplicerade interaktioner har dock fått mindre uppmärksamhet då dessa leder till komplicerade system av kvadratiska ekvationer.

För att kringgå dessa problem visar den här avhandlingen att gaugeinvarians kan införas direkt för amplituder genom en ny uppsättning *massiva Ward-identiteter*. Detta innebär att endast linjära ekvationer behövs lösas, en betydande förenkling jämtemot andra metoder baserade på langrangianen. Dessutom är amplituder mindre känsliga för redundanser som kan uppkomma genom omdefinitioner av fält i lagrangianen. Vid tre punkter reproducerar Ward-identiteterna, i kombination med ett känt unitaritetsvillkor, de amplituder som används för att beskriva Kerr. Vid fyra punkter tillämpas samma metoder för att beräkna nya Compton-amplituder för tillstånd med högre spinn och föreslås som kandidater för högre ordningars observabler för svarta hål.

Parallellt med detta studeras amplituder i supersträngteori. Strängteorin är ett exempel på en konsekvent teori med högre spinn, och därför är det lärorikt att jämföra dess amplituder med de amplituder som beskriver Kerr. Genom att tillämpa den klassiska gränsen, som tidigare endast har studerats i samband med svarta hål, visas att supersträngamplituder reproducerar kända klassiska stränglösningar. Detta ger viktiga insikter om formalismens egenskaper. Dessutom banar det väg för studier av mer allmänna strängtillstånd och försök att reproducera Kerr-svarta hål från strängteori.

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