

RESEARCH ARTICLE

A singular Coxeter presentation

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Abstract

We enlarge a Coxeter group into a category, with one object for each finite parabolic subgroup, encoding the combinatorics of double cosets. This category, the singular Coxeter monoid, is connected to the geometry of partial flag varieties. Our main result is a presentation of this category by generators and relations. We also provide a new description of reduced expressions for double cosets. We describe all the braid relations between such reduced expressions, and prove an analogue of Matsumoto's theorem. This gives a proper development of ideas first introduced by Geordie Williamson. In type A , we also equip the singular Coxeter monoid with a diagrammatic presentation using webs.

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1 | INTRODUCTION

Weyl groups are ubiquitous in representation theory and geometry, and their Coxeter presentations are extremely useful for understanding the length function, the Bruhat order, and other features arising in the geometry of the flag variety. In this paper, we enlarge a Coxeter group into a category, with one object for each finite parabolic subgroup, encoding the combinatorics of double cosets. This category, the *singular*[†] *Coxeter monoid*, is connected to the geometry of partial flag varieties. Our main result is a presentation of this category by generators and relations. We also

[†] For more on what the confusing word 'singular' means, see Remark 1.32. For now, singular should connote 'relating to double cosets for parabolic subgroups in W , rather than to elements of W '. It also connotes a change in categorical type, similar to the change from a group to a groupoid or from a monoid to a category.

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provide a new description of *reduced expressions for a double coset*. We describe all the braid relations between such reduced expressions, and prove an analogue of Matsumoto’s theorem. This gives a proper development of ideas first introduced by Geordie Williamson in his PhD thesis [23, subsection 1.3].

This paper is primarily combinatorial, so we begin this introduction with the combinatorics. Readers who wish to be motivated first by geometry are welcome to skip ahead to Subsection 1.6, and return to the rest of the introduction afterward.

Notation 1.1. Let (W, S) be a Coxeter system. For $I \subset S$, we denote by W_I the subgroup of W generated by I . Then (W_I, I) is a Coxeter system, and W_I is called a (*standard*) *parabolic subgroup* of (W, S) . When W_I is finite, we say that I is *finitary*, and we write w_I for the longest element of W_I . The identity element of W is denoted by e .

Notation 1.2. Let I and J be finitary. It is well-known (see Lemma 2.12) that each double coset $p \in W_J \backslash W / W_I$ has a unique maximal element and a unique minimal element in the Bruhat order. Write \bar{p} for the maximal element and \underline{p} for the minimal element. We refer to double cosets in $W_J \backslash W / W_I$ as (J, I) -cosets.

Notation 1.3. To avoid needing to write set braces constantly for subsets of S , we use shorthand. We write s for $\{s\}$, st for $\{s, t\}$, Is for $I \sqcup \{s\}$, and $I \setminus s$ for $I \setminus \{s\}$. Whether st represents an element of W or a subset of S should be clear from context.

1.1 | Reduced expressions for double cosets: Definition and examples

So, we can jump into the fun, let us make a temporary definition that will be refined later.

Definition 1.4. An (*singular, multistep*) *expression* is the data of a sequence of finitary subsets of S of the form

$$[[I_0 \subset K_1 \supset I_1 \subset \dots \subset K_m \supset I_m]], \tag{1.1a}$$

often written as

$$[[I_0, K_1, I_1, \dots, K_m, I_m]].$$

It is *reduced* if

$$\ell(w_{K_1} w_{I_1}^{-1} w_{K_2} \dots w_{I_{m-1}}^{-1} w_{K_m}) = \ell(w_{K_1}) - \ell(w_{I_1}) + \ell(w_{K_2}) - \dots - \ell(w_{I_{m-1}}) + \ell(w_{K_m}) \tag{1.1b}$$

holds, where $\ell(w)$ denotes the Coxeter length of $w \in W$. In this case, it expresses the (I_0, I_m) -coset

$$p = W_{I_0} w_{K_1} w_{I_1}^{-1} w_{K_2} \dots w_{I_{m-1}}^{-1} w_{K_m} W_{I_m}. \tag{1.2}$$

By convention, if we write an expression as $[[K_1 \supset I_1 \subset \dots]]$, this means that $I_0 = K_1$, and similarly $[[\dots \subset K_m]]$ means that $I_m = K_m$. This does not affect (1.1b).

Note that if $[[I_0 \subset K_1 \supset I_1 \subset \dots \subset K_m \supset I_m]]$ is a reduced expression for p , then the maximal element \bar{p} satisfies

$$\bar{p} = w_{K_1} w_{I_1}^{-1} w_{K_2} \dots w_{I_{m-1}}^{-1} w_{K_m}. \tag{1.3}$$

Example 1.5. For an ordinary reduced expression $s_1 s_2 \dots s_m$ in (W, S) , we assign a singular reduced expression, which bounces between finitary subsets of size zero and one:

$$s_1 s_2 \dots s_m \rightsquigarrow [[\emptyset, s_1, \emptyset, s_2, \emptyset, \dots, \emptyset, s_m, \emptyset]]. \tag{1.4}$$

In this case, we have $w_{I_i} = e$ so that the condition (1.1b) becomes

$$\ell(s_1 s_2 \dots s_m) = \ell(s_1) + \ell(s_2) + \dots + \ell(s_m).$$

This agrees with the definition of a reduced expression in (W, S) , making the correspondence (1.4) a bijection. The ordinary expression expresses the element $w = s_1 \dots s_m$, while the singular expression expresses the (\emptyset, \emptyset) -coset $\{w\}$.

Example 1.6. The double coset $\{w\}$ may have singular reduced expressions which do not come from (1.4). For example, if $m_{st} = 3$, then $\{sts\}$ also has the reduced expression $[[\emptyset, st, \emptyset]]$.

Example 1.7. When S is finitary, there is a unique (\emptyset, S) -coset, containing all of W . Then $[[\emptyset, S]]$ is a reduced expression for this coset. However, $[[\emptyset, s, \emptyset, S]]$ is not a reduced expression for any $s \in S$.

Example 1.8. Suppose W has type A_3 with $S = \{s, t, u\}$ and $m_{su} = 2$. Then $[[st, s, su, s, st]]$ is a reduced expression for the (st, st) -coset containing the longest element w_S . This is because

$$w_S = stsuts = (sts) \cdot (s^{-1}) \cdot (su) \cdot (s^{-1}) \cdot (sts), \tag{1.5}$$

and the lengths add appropriately. Another reduced expression for this coset is $[[st, stu, st]]$. Meanwhile, we claim that $[[st, stu, tu]]$ is the unique reduced expression for the (st, tu) -coset containing w_S .

In this paper, we introduce the *(singular) braid relations*, and prove in Theorem 5.30 that any two singular reduced expressions for a double coset are related by the singular braid relations. However, while multistep expressions are the most convenient for defining reduced expressions, it makes the application of relations slightly annoying. To transform $[[\emptyset, st, \emptyset]]$ into $[[\emptyset, s, \emptyset, t, \emptyset, s, \emptyset]]$, we need to be able to go from \emptyset to st in two steps, first adding s and then adding t . For a variety of reasons, we prefer single-step expressions in this paper.

Definition 1.9. An *(singular, single-step) expression* $I_* = [I_0, I_1, \dots, I_d]$ is a sequence of finitary subsets of S , where for each $1 \leq i \leq d$ we have either $I_i = I_{i-1} \sqcup s$ or $I_i = I_{i-1} \setminus s$ for some $s \in S$. The number d is called the *width* of the expression[†]. A single-step expression is *reduced* if the corresponding multi-step expression is a reduced expression.

[†] The *length* of an expression will be defined in Subsection 3.4, and better reflects the length function in W .

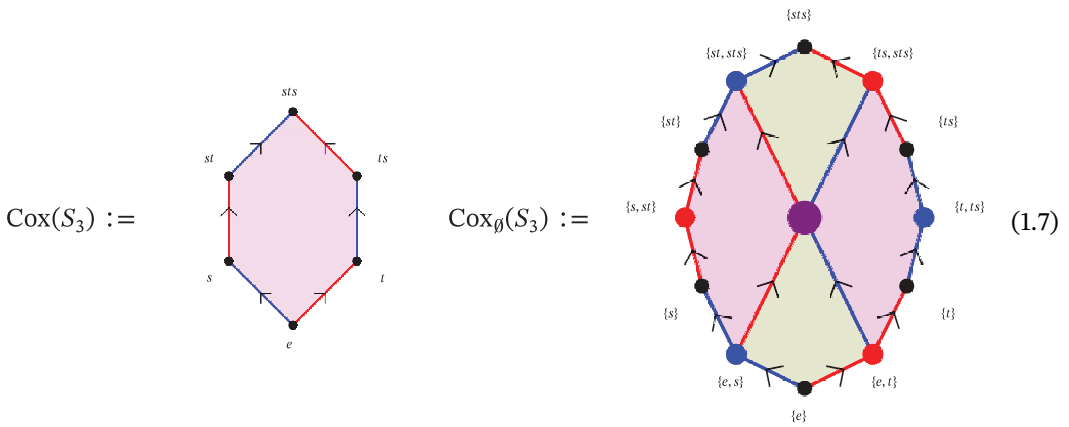
The singular reduced expressions from (1.4) are already single-step expressions. The multi-step expression $[[\emptyset, st, \emptyset]]$ corresponds to four distinct single-step expressions,

$$[\emptyset, s, st, s, \emptyset], \quad [\emptyset, t, st, s, \emptyset], \quad [\emptyset, s, st, t, \emptyset], \quad [\emptyset, t, st, t, \emptyset]. \tag{1.6}$$

From now on, all expressions are single-step unless stated otherwise.

Remark 1.10. An equivalent criterion for reducedness for single-step expressions is stated in Theorem 3.6. That these two versions are equivalent is proven in Lemma 3.8.

Example 1.11. Let $S = \{s, t\}$ with $m_{st} = 3$, so that W is the symmetric group S_3 . Let us contrast ordinary reduced expressions with singular reduced expressions. See Subsection 1.5 and Section 8 for more details on the constructions below.



The picture on the left is the (completed) dual Coxeter complex. Vertices correspond to elements of W . Oriented paths starting at the bottom vertex e are reduced expressions. Any two reduced expressions are related by the braid relation, seen here as a 2-cell.

The picture on the right is the *singular (dual) Coxeter complex* where the left parabolic subgroup is trivial. Vertices correspond to (\emptyset, I) -cosets for various finitary I , and the color of the vertex represents the choice of I . Oriented paths starting at the bottom vertex $\{e\}$ are reduced expressions. Any two reduced expressions are related by the braid relations, seen here as 2-cells. Altogether $\{sts\}$ has six reduced expressions: the two corresponding to sts and tst as in (1.4), and the four in (1.6).

To spell this out in symbols rather than pictures, let us describe the singular braid relations which transform sts into tst , or more precisely,

$$[\emptyset, s, \emptyset, t, \emptyset, s, \emptyset] \simeq [\emptyset, t, \emptyset, s, \emptyset, t, \emptyset]. \tag{1.8a}$$

Throughout this paper, we use \simeq to represent the equivalence of expressions. We start with the most interesting relation, our true analogue of the braid relation,

$$[s, \emptyset, t, \emptyset, s] \simeq [s, st, s], \tag{1.8b}$$

which is a relation between two reduced expressions for the (s, s) -coset containing sts . Applying (1.8b) within the left-hand side of (1.8a), we obtain

$$[\emptyset, s, \emptyset, t, \emptyset, s, \emptyset] \simeq [\emptyset, s, st, s, \emptyset].$$

Now we can apply two relatively simple relations, the *up-up* and *down-down relations*, namely,

$$[\emptyset, s, st] \simeq [\emptyset, t, st], \quad [st, s, \emptyset] \simeq [st, t, \emptyset] \tag{1.8c}$$

to obtain

$$[\emptyset, s, st, s, \emptyset] \simeq [\emptyset, t, st, t, \emptyset].$$

We conclude with the analogue of the braid relation again, this time in the form

$$[t, \emptyset, s, \emptyset, t] \simeq [t, st, t] \tag{1.8d}$$

to deduce (1.8a).

As seen in the previous example, singular expressions and their relations are more ‘local’ or ‘zoomed in’ than ordinary expressions. Each generator of W (multiplication by $s \in S$) is split into two smaller pieces (via $[\emptyset, s, \emptyset]$, a composition of $[\emptyset, s]$ with $[s, \emptyset]$). The braid relation $sts = tst$ follows from four smaller relations.

Remark 1.12. Fix $I \subset S$ finitary. The theory of reduced expressions for double cosets gives rise to a partial order on the set of all (I, J) -cosets as J varies, analogous to the weak Bruhat order. When $I = \emptyset$, the Hasse diagram of this partial order is the 1-skeleton of Cox_\emptyset above. After writing this paper we discovered (thanks to a tip from Nathan Reading) that this partial order (when $I = \emptyset$) was studied from a combinatorial perspective by Dermenjian–Hohlweg–Pilaud in [4], where it is called the *facial weak order*. They prove that the facial weak order is a lattice, and explicitly describe the meet and join [4, Theorem 3.19]. They investigate other properties of this lattice of interest to combinatorialists (such as the relationship with Cambrian lattices [17]). Even prettier pictures of the singular Coxeter complex Cox_\emptyset can be found in [4, pp. 7–8]. Also see Remark 1.31.

Remark 1.13. Singular (reduced) expressions of a very special form, like (1.4) but concatenated with $[[\emptyset, I]]$, are studied by Contou–Carrere and by Gaussent (see [13]), where they are also related to the geometry of Bott–Samelson resolutions.

1.2 | Reduced expressions for double cosets: Theory

All the braid relations we need will be generalizations of either (1.8b) or (1.8c). The up-up and down-down relations are easy to generalize:

$$[I, Is, Ist] \simeq [I, It, Ist], \quad [Ist, Is, I] \simeq [Ist, It, I] \tag{1.9}$$

whenever Ist is finitary. We call the generalizations of (1.8b) the *switchback relations*. They are described explicitly in Subsection 5.4 and Section 6, and are governed by relatively simple and interesting combinatorics. There is one switchback relation for each finite Coxeter system (W, S) and each pair $s \neq t \in S$ (though one can reduce to the case where W is irreducible).

Example 1.14. For a finite dihedral group, the regular braid relation is implied by four singular braid relations in much the same way as (1.8). For example, when $m_{st} = 4$ the switchback relation is

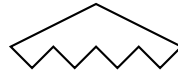
$$[s, \emptyset, t, \emptyset, s, \emptyset, t] \equiv [s, st, t]. \quad (1.10)$$

Example 1.15. In type E_8 , with simple reflections numbered as in Subsection 6.5.3, one switchback relation will be

$$[\widehat{3}, \widehat{32}, \widehat{2}, \widehat{27}, \widehat{7}, \widehat{72}, \widehat{2}, \widehat{23}, \widehat{3}, \widehat{38}, \widehat{8}] \equiv [\widehat{3}, S, \widehat{8}]. \quad (1.11)$$

Here \widehat{a} is shorthand for $S \setminus s_a$ and \widehat{ab} is shorthand for $S \setminus \{s_a, s_b\}$.

Remark 1.16. If vertical height represents the size of a subset of S , then the switchback relation states that the path going over the top of the mountain agrees with the zigzagging path below[†].



Remark 1.17. The sequence of simple reflections 327238 appearing in (1.11) is an example of a *rotation sequence*, which we define in Definition 5.10. Since this paper first appeared, we learned that rotation sequences and related concepts seem to appear in work in progress of Wemyss and Iyama (see [22, subsection 3.2 and references therein], where they study singularities using mutations and wall-crossings).

Remark 1.18. It is a wonderful feature of ordinary Coxeter theory that all relations in Coxeter groups come from finite rank 2 parabolic subgroups. This is in stark contrast to the singular Coxeter monoid, which has switchback relations in arbitrary rank! However, even the switchback relation is planar in some sense, see Subsection 8.5. Also, the generating relations in the singular Coxeter monoid satisfy the property that they involve expressions $[I_0, I_1, \dots, I_d]$ where the size of I_i lies between k and $k + 2$ for some k .

In Theorem 5.30, we prove that any two reduced expressions are related by the braid relations, which we call the *singular Matsumoto theorem*. Matsumoto's theorem lies tacitly beneath so much of Coxeter combinatorics that it is hard to point to a particular 'application' of the theorem, rather than to areas of study or general techniques (for example, positive lifts to braid groups, reduced expression graphs, and so on) which it makes possible.

[†] There were many other good suggestions for the name of this relation: the sawtooth or serriform relation, the waffle cone relation, the handkerchief relation, the iceberg relation, and so on. The Loch Lomond relation was a close contender, after 'you'll take the high road and I'll take the low road', the main issue being that, for us, the two roads reach Scotland at precisely the same time.

Remark 1.19. The *reduced expression graph* of an element $w \in W$ is the graph whose vertices are reduced expressions, and whose edges are braid relations. Matsumoto's theorem states that this graph is connected. The cycles in such a graph are classified; the most interesting ones are the Zamolodchikov cycles associated to the longest elements of rank 3 Coxeter groups. See [11] for more details. One can also define reduced expression graphs for any double coset p . Classifying the cycles in this graph is a very interesting open problem, related to finding the 3-cells in the singular Coxeter complex.

Remark 1.20. Matsumoto's theorem is used in a crucial way in the diagrammatic description of the Hecke category and its basis, the double leaves basis [10, 14]. The singular Matsumoto theorem was a crucial missing piece in a construction of singular double leaves for the singular Hecke category (see Subsection 1.6), which will appear in forthcoming work with Nicolas Libedinsky.

The original definition of a (reduced) singular expression is due to Williamson [23, Definition 1.3.2], and is reformulated as our Definition 2.24. He proves [23, Proposition 1.3.4] that every double coset has a reduced expression. This original definition has a very different flavor, is somewhat technical, and is not obviously equivalent to the definition above. Many basic and essential properties are obfuscated (and were not previously known). For example, it is not clear that $[I_0, \dots, I_d]$ being reduced implies that the contiguous subword $[I_k, \dots, I_\ell]$ is reduced for $0 \leq k < \ell \leq d$. This paper provides four equivalent criteria for an expression to be reduced (Definition 1.4, Theorem 3.6, Proposition 3.17, Proposition 8.11) which we have found to be more useful in practice. We give more details on the original definition in Subsection 2.4 and following.

In Sections 3 and 4, we prove many properties of singular reduced expressions, some of which are in analogy with properties of ordinary reduced expressions. In Subsection 3.3, we prove that reduced expressions are closed under taking contiguous subwords. In Subsection 3.4, we introduce a length function on expressions and on double cosets. We prove that reduced expressions minimize the length function among all expressions (see Subsection 1.3) for a given double coset. In Subsection 4.3, we prove that any reduced expression can be extended to a reduced expression for a coset containing the longest element (in a finite Coxeter group). In Subsection 4.10, we prove basic facts about products of Coxeter groups.

We also prove several results which have a new flavor. In Subsections 4.8 and 4.9, we examine the questions: for a given (J, I) -coset p , for which simple reflections $s \notin J$ admits a reduced expression for p starting with $[J, Js, \dots]$? How big can K_1 be in a multistep reduced expression $[[J \subset K_1 \supset \dots]]$ for p ? For which simple reflections $s \in J$ can a reduced expression start with $[J, J \setminus s, \dots]$? How small can I_1 be in $[[J \supset I_1 \subset \dots]]$? We give precise answers to these questions, proving the existence of reduced expressions which start by getting as big as possible, or as small as possible. These theorems are surprisingly useful.

1.3 | The singular Coxeter monoid

In ordinary Coxeter theory, expressions represent products of generating elements inside a group W . The singular Coxeter monoid is a category where singular expressions represent products of generating morphisms. This allows us to discuss expressions (not necessarily reduced) for a double coset, and relations between these. The singular Coxeter monoid is a singular version, not of the group W , but of the Coxeter monoid $(W, *, S)$.

Definition 1.21. Let (W, S) be a Coxeter system. The *Coxeter monoid* $(W, *, S)$ (sometimes called the *Coxeter star monoid* or the *0-Hecke monoid*[†]) has a presentation with generators S , and the following relations.

$$\text{The } * \text{-quadratic relation: } s * s = s \text{ for each } s \in S. \tag{1.12a}$$

$$\text{The braid relation: } \underbrace{s * t * \dots}_m = \underbrace{t * s * \dots}_m, \text{ for each } s, t \in S \text{ with } m = m_{st} < \infty. \tag{1.12b}$$

One can prove that the elements of $(W, *, S)$ are in natural bijection with elements of W , and in what follows we do not distinguish the former from the latter. Thus, an element $w \in W$ can be written as $w = s * t * \dots * u$ for some $s, t, \dots, u \in S$. The string $[s, t, \dots, u]$ is called a **-expression* of w . While the set of **-expressions* for w does not agree with the set of ordinary expressions for w , the reduced expressions (that is, expressions of shortest length) in both contexts agree.

Definition 1.22. The *singular Coxeter monoid* $SC = SC(W, S)$ is the category defined as follows.

- $\text{Ob}(SC)$ consists of the finitary subsets $I \subset S$.
- For $I, J \in \text{Ob}(SC)$, $\text{Hom}_{SC}(I, J)$ is the set of (J, I) -cosets, that is,

$$\text{Hom}_{SC}(I, J) := W_J \backslash W / W_I. \tag{1.13a}$$

- For $I, J, K \in \text{Ob}(SC)$, the composition ‘ $*$ ’ is given by

$$W_K x W_J * W_J y W_I := W_K (\bar{x} * \bar{y}) W_I, \tag{1.13b}$$

where \bar{x} and \bar{y} are the longest elements in $W_K x W_J$ and $W_J y W_I$, respectively.

In fact, the element $\bar{x} * \bar{y}$ from (1.13b) is the maximal element in its double coset, see Lemma 2.7. Thus, one can think about the morphisms in SC as being the maximal elements of double cosets, with composition given by $*$. This makes it easy to verify that SC is a well-defined category (details can be found in Subsection 2.3). The identity element of the object I is W_I itself, viewed as an (I, I) -coset. Note that $\text{End}_{SC}(\emptyset) \cong (W, *, S)$ as monoids.

Remark 1.23. By sending a double coset to its maximal element, one can construct a faithful functor from SC to $(W, *, S)$ (the latter being viewed as a category with one object). In fact, SC is obtained from $(W, *, S)$ as the idempotent completion or Karoubi envelope, see Theorem 2.11.

A given subset of W can appear as a double coset in multiple ways, for different choices of I and J . In this paper, whenever we speak about a double coset p , it is implicit that p carries with it the data of a pair (J, I) such that p is a (J, I) -coset, just as a morphism in a category implicitly carries the data of its source and target.

Example 1.24. If I is finitary and $s \in I$, then $t = w_I s w_I$ is another simple reflection (possibly equal to s) in I . Then $\{s w_I, w_I\} = \{w_I t, w_I\}$ is the underlying set of an (s, \emptyset) -coset p and an (\emptyset, t) -coset q and an (s, t) -coset r . We treat p, q , and r as distinct double cosets.

[†] The linearization of $(W, *, S)$ is the *0-Hecke algebra*. We do not linearize $(W, *, S)$ in this paper.

Remark 1.25. Double cosets and their enumeration were studied in [2], though (unlike us) they did not wish to distinguish between a subset $p \subset W$ viewed as a (J, I) -coset, or viewed as a (J', I') -coset, for different parabolic subgroups.

1.4 | Presenting the singular Coxeter monoid

Notation 1.26. Let $I, J \subset S$ be finitary. The *minimal* (J, I) -coset is the double coset containing the identity element e . If S is finitary, the *maximal* (J, I) -coset is the double coset containing the longest element w_S .

The current paper gives a presentation of SC by generators and relations, see Theorem 5.32. There is one pair of generators for each $s \notin I$ with Is finitary. Our generators are the morphisms

$$Is \rightarrow I, \quad I \rightarrow Is$$

given by the minimal (I, Is) - and (Is, I) -cosets, respectively. Both double cosets have the underlying set W_{Is} . Their maximal element is w_{Is} , and their minimal element is e .

Definition 1.27. To an expression $I. = [I_0, I_1, \dots, I_d]$ we can associate a composition

$$I_d \rightarrow \dots \rightarrow I_1 \rightarrow I_0$$

of generating morphisms in SC . If p is the (I_0, I_d) -coset given by this composition, we say that $[I_0, I_1, \dots, I_d]$ expresses p , and we write

$$[I_0, I_1, \dots, I_d] \simeq p.$$

We also write $I. \simeq I'$ when they express the same double coset. By convention, a width zero expression $[I]$ expresses the identity element of the object I , that is, the minimal (I, I) -coset.

If $[I_0, I_1, \dots, I_d] \simeq p$ and \bar{p} is the maximal element of p , then by (1.13b) we can describe \bar{p} as a star product of d elements. However, using the fact that

$$w_I * w_J = w_J = w_J * w_I \quad \text{when } I \subset J, \tag{1.14}$$

it is not hard to prove the simpler formula

$$\bar{p} = w_{I_0} * w_{I_1} * \dots * w_{I_d}. \tag{1.15}$$

One can think of p as the unique (I_0, I_d) -coset containing the element in (1.15).

Any $*$ -expression $[s, t, \dots, u]$ has a lift to SC as a singular expression $[\emptyset, s, \emptyset, t, \emptyset, \dots, \emptyset, u, \emptyset]$. It helps to spell this out in a small example.

Example 1.28. Consider the ordinary expression $[s, t, u]$ with $s, t, u \in S$ distinct, and its corresponding singular expression $[\emptyset, s, \emptyset, t, \emptyset, u, \emptyset]$. An ordinary expression gives rise to a sequence of elements via composition one index at a time, in this example giving e, s, st, stu . Similarly, a

singular expression gives rise to a sequence of double cosets[†] (whose maximal element is obtained by composing (1.15) one index at a time). In this example, we get the sequence

$$\{1\} \subset \{1, s\} \supset \{s\} \subset \{s, st\} \supset \{st\} \subset \{st, stu\} \supset \{stu\}. \quad (1.16)$$

Each of the singleton cosets above is an (\emptyset, \emptyset) -coset. The first doubleton is an (\emptyset, s) -coset, the second an (\emptyset, t) -coset, and the third an (\emptyset, u) -coset.

Now we discuss the relations in our presentation of SC . The quadratic relation $s * s = s$ in $(W, *, S)$ follows from a more local relation $[s, \emptyset, s] \Leftarrow [s]$ in SC . Our *general *-quadratic relation* is

$$[Is, I, Is] \Leftarrow [I], \quad (1.17)$$

whenever I is finitary.

The relations in SC are the braid relations (the up–up, down–down, and switchback relations), and the general *-quadratic relation. In Theorem 5.31, we prove an even stronger statement, that any expression can be transformed into a reduced expression using the braid relations and using the *-quadratic relation only in the length-reducing direction.

Remark 1.29. One could define a category by generators and relations similar to SC , but whose presentation consists only of the braid relations without the quadratic relation. Whether this is a reasonable singular analogue of the braid monoid is currently unclear.

Remark 1.30. The category SC is filtered by the length function. After linearizing, we can take the associated graded of SC , another variant with a similar presentation: the braid relations are unchanged, and the quadratic relation is replaced by $[Is, I, Is] = 0$. This is analogous to replacing the relation $s * s = s$ with the relation $\partial_s \circ \partial_s = 0$, a relation which holds for Demazure operators on the cohomology ring of the flag variety. In forthcoming work with Libedinsky, we interpret the associated graded category of SC using Demazure operators acting on various (equivariant) cohomologies.

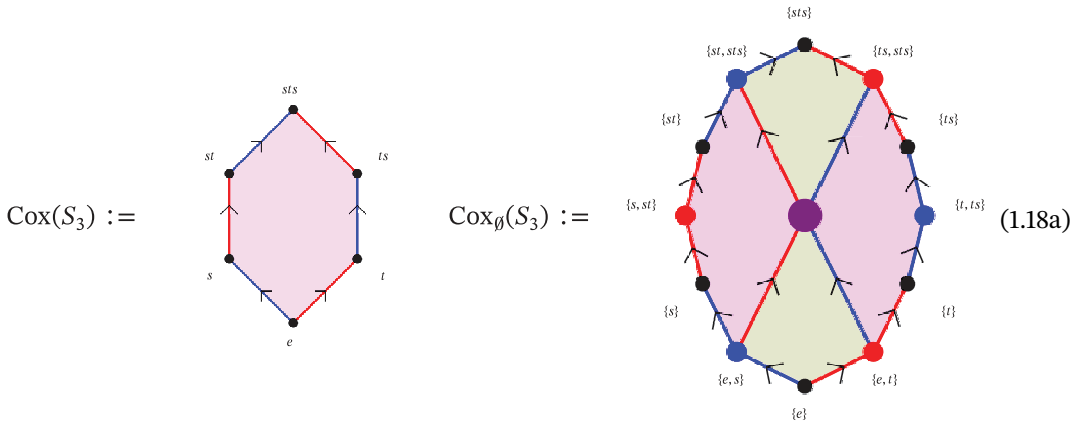
In type A , one can encode an expression diagrammatically as a *web*, a special kind of labeled trivalent graph. This idea is developed in Section 7, where we also discuss the connection between SC and the Cautis–Kamnitzer–Morrison web category [3].

1.5 | The singular Coxeter complex

The (completed) dual Coxeter complex $\text{Cox}(W)$ is a contractible CW complex with one k -cell for each coset in W/W_I , for each finitary $I \subset S$ of size k . Thus, the 0-cells correspond to elements of W , the 1-cells to right multiplication by a simple reflection, and the 2-cells to braid relations. For a comprehensive reference on the dual Coxeter complex see [1, subsection 12.3]. We repeat our

[†] In Definition 2.21, we call this the *forward path* of the expression.

picture from above in type A_2 .



For any Coxeter system (W, S) , we introduce a new 2-dimensional CW complex $\text{Cox}_\emptyset(W)$, constructed as follows. There will be one 0-cell for each (\emptyset, I) coset, for all finitary I . In other words, there is one 0-cell in $\text{Cox}_\emptyset(W)$ for each cell of any dimension in $\text{Cox}(W)$. The 1-cells in Cox_\emptyset correspond to composition with a generator in SC ; generators either add or subtract a simple reflection, which colors the edge. However, we only include as edges those compositions *which occur in a reduced expression* (this also explains the orientation). For example, there is no edge between the purple vertex and $\{t, ts\}$ above. Because of these missing edges W does not act on Cox_\emptyset on the left, as it does on Cox . The 2-cells in Cox_\emptyset come from the braid relations. Above, we have colored the up-up and down-down relations khaki, and the switchback relations fuchsia.

As we have drawn it, Cox_\emptyset appears to have a straight line through the purple vertex, including the blue vertex $v_1 = \{e, s\}$ and the red vertex $v_2 = \{ts, sts\}$. This is no accident! One can think of Cox_\emptyset as describing the Coxeter hyperplane arrangement in $\mathbb{R}^{|S|}$, with one vertex for each facet (an (\emptyset, \emptyset) coset for each chamber, an (\emptyset, s) coset for each s -wall, and so on). The purple vertex represents the origin in this example. Here is Cox_\emptyset with the Coxeter arrangement overlaid upon it.



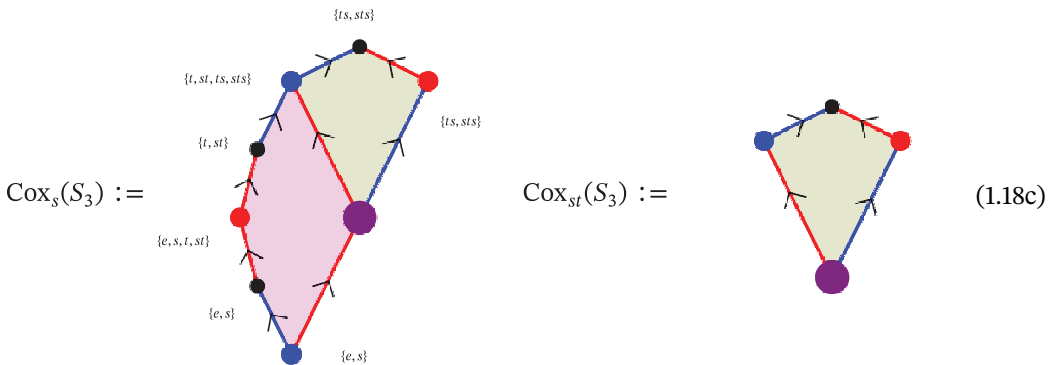
Each reflection hyperplane has a positive side (containing w_0 , when W is finite) and a negative side (containing e). In Proposition 8.11, we prove that an edge which goes onto a hyperplane is reduced if and only if it comes from the negative side, and an edge which goes off of a hyperplane is reduced if and only if it goes to the positive side. This principle explains all the arrows in (1.18a), and also the missing edges. For example, the edge (not drawn in (1.18a)) going from $\{t, ts\}$ to the

purple vertex enters two hyperplanes simultaneously, one from the positive side and one from the negative side.

Understanding the extra linear structure on singular Coxeter complexes is rather useful, because reduced expressions are close to geodesics! Recall that a *flat* of a hyperplane arrangement is a subspace obtained as the intersection of some of the hyperplanes. In a reduced expression, once you leave a flat you can never return (you enter from the negative side of some hyperplane, and leave from the positive side). So, if two vertices lie on the same flat F , then any reduced path between them is constrained to stay in F . For example, there is a unique reduced path from v_1 to v_2 , which passes through the origin, since both vertices lie on a 1-dimensional flat.

Our switchback relation may seem complicated, but it can be easily summarized. Take two distinct 1-dimensional flats, and the appropriate vertices v_1 and v_3 on each. For example, take the blue vertex $v_1 = \{e, s\}$ and the blue vertex $v_3 = \{st, sts\}$. Then there are two reduced paths from v_1 to v_3 : the path through the origin, and the path which avoids the origin but stays within the 2-dimensional span of v_1 and v_3 .

The switchback relation $[s, \emptyset, t, \emptyset, s] \Leftarrow [s, st, s]$ from (1.8b) is a relation between (s, s) -cosets, not between (\emptyset, s) -cosets. Of course, when (1.8b) is applied locally within a longer expression like (1.8a), it does produce a relation between paths in Cox_\emptyset . However, the reader may still want to visualize the switchback relation itself. Indeed, there is a CW-complex $\text{Cox}_s(S_3)$ whose vertices are (s, I) -cosets, drawn as below.



The switchback relation from (1.8b) is the fuchsia 2-cell above. One can also define Cox_J for any finitary J , whose vertices are (J, I) -cosets for various I . Also above is $\text{Cox}_{st}(S_3)$, where every vertex is the double coset with underlying set S_3 . We give examples for the symmetric group S_4 in Section 8.

The reader may note that Cox_J seems to embed inside Cox_\emptyset , preserving the orientations. We prove this in Proposition 8.10. One can think about Cox_J as a fundamental domain for the parabolic subgroup W_J inside Cox_\emptyset , even though technically there is no left action. Each vertex in Cox_J matches a vertex in Cox_\emptyset with the same maximal element. For example, the unique reduced path from v_1 to v_2 in Cox_\emptyset discussed above implies that there is a unique reduced path between the corresponding vertices in Cox_s , whence a unique reduced expression for the (s, t) -coset $\{ts, sts\}$. See Proposition 5.14(1) for a generalization.

We have not discussed k -cells for $k \geq 3$ inside the singular Coxeter complex. This is because we do not yet understand the combinatorics. One expects that what we have described above is only

the 2-skeleton of a polytopal CW complex $\text{Cox}_{\theta}(W)$, homeomorphic to and refining $\text{Cox}(W)$, but a description of the higher cells is currently missing. The 3-cells should correspond to the generating cycles in reduced expression graphs, discussed in Remark 1.19.

Remark 1.31. As noted in Remark 1.12, the 1-skeleton of the singular Coxeter complex Cox_{θ} also appears in work of Dermenjian–Hohlweg–Pilaud [4]. Although their pictures like [4, p. 8] appear to be part of a higher dimensional CW complex, there is no suggestion of higher cells in that paper, which only discusses the 1-skeleton. In particular, our braid relations (the 2-cells) are new, and the higher cells remain mysterious.

1.6 | Motivation from geometry

This section is designed to be readable even if the rest of the introduction was skipped.

Let (W, S) be a Coxeter system of a Weyl group associated to a semisimple Lie group G with Borel subgroup B . The Bruhat decomposition states that the B -orbits on the flag variety G/B are in bijection with W , and the closure relation corresponds to the Bruhat order. Thus, it is not surprising that many geometric features of flag varieties can be combinatorially encoded inside the poset (W, \leq) . As the classic example, Kazhdan–Lusztig polynomials encode the intersection cohomology of Schubert varieties, and have a combinatorial definition in terms of the Hecke algebra.

Geometric constructions can be associated not just to elements of W but to expressions as well. One effective tool to study intersection cohomology (and perverse sheaves) is to study the cohomology rings of resolutions of singularities. To any expression $\underline{w} = [s_1, \dots, s_d]$ (that is, any word in S) one has an associated Bott–Samelson variety $\text{BS}(\underline{w})$, equipped with a map to Fl . When \underline{w} is a reduced expression for $w \in W$, then $\text{BS}(\underline{w})$ is a resolution of singularities for the Schubert variety of w inside Fl . Many geometric features of $\text{BS}(\underline{w})$ are encoded inside the theory of reduced expressions and subexpressions. Just as flag varieties have an affine paving indexed by W , $\text{BS}(\underline{w})$ has an affine paving indexed by subexpressions of \underline{w} .

Ultimately, one is studying B -equivariant perverse sheaves on the flag variety, or $B \times B$ -equivariant perverse sheaves on G . This is a monoidal category under convolution, whose Grothendieck group is the Hecke algebra. It is this convolution structure which explains why multiplication in the Hecke algebra is useful, and it also explains why expressions appear frequently in the combinatorics; one can think of Bott–Samelson varieties as convolutions over sequences of simple reflections.

The complicated geometric category of perverse sheaves has a simpler algebraic model constructed by Soergel [19] using so-called *Soergel bimodules*. More recently, Soergel bimodules can be replaced by a combinatorially defined monoidal category, the *diagrammatic Hecke category* \mathcal{H} [7, 10]. The objects in \mathcal{H} are expressions, and one can construct a basis of morphisms between two expressions using the combinatorics of subexpressions, the *double leaves basis* [10, 14].

What about P_I -orbits in G/P_J , where P_I and P_J are parabolic subgroups of G associated to subsets $I, J \subset S$? As I and J vary, these orbits are parameterized by double cosets $W_I \backslash W / W_J$ for the parabolic subgroups $W_I, W_J \subset W$. What is the algebraic object encoding the intersection cohomologies of the closures of these orbits? What is the algebraic/combinatorial category encoding equivariant perverse sheaves? These questions were studied at length by Williamson [23] in his thesis, which was rewritten into the article [24].

Just as convolution turned $B \times B$ -equivariant perverse sheaves into a monoidal category, convolution turns parabolic-equivariant sheaves into a 2-category. There is one object for each subset $I \subset S$, and the morphism category from J to I is the category of $P_I \times P_J$ -equivariant perverse sheaves on G . To encode intersection cohomology, Williamson introduces the *Hecke algebroid*[†], a category whose objects are also in bijection with subsets $I \subset S$. Just as $B \times B$ -equivariant sheaves are a special case of $P_I \times P_J$ -equivariant sheaves corresponding to the case $I = J = \emptyset$, the endomorphism algebra of \emptyset inside the Hecke algebroid is precisely the Hecke algebra. More interestingly, all morphism spaces $\text{Hom}(I, J)$ inside the Hecke algebroid have a Kazhdan–Lusztig basis, in bijection with double cosets $W_I \backslash W / W_J$.

Williamson defines analogues for double cosets of the notion of (reduced) expressions[‡]. He uses reduced expressions to construct resolutions of singularities for general Schubert varieties[§]. Unlike the ordinary theory of reduced expressions which has an incredibly rich literature, Williamson's reduced expressions for double cosets have received very little attention, and many basic properties were unknown until now. We warn the reader that the article [24] omits some of the development on this theme which appeared in the original thesis [23].

Williamson also defines the analogue of Soergel bimodules, the 2-category of *singular Soergel bimodules* [23, section 3]. We mention at this point that (singular) Soergel bimodules can be defined not just for Weyl groups but for any Coxeter system [20]. They only have a connection to geometry for Weyl groups and other crystallographic groups, but many desirable properties which one can prove for geometric reasons in the Weyl group case can be proven in other ways for arbitrary Coxeter groups, for example, [9]. When working with infinite Coxeter groups, the objects in the Hecke algebroid or the 2-category of singular Soergel bimodules are parameterized by subsets $I \subset S$ whose parabolic subgroup W_I is finite, in which case we call I *finitary*. Singular Soergel bimodules and their geometric analogues (especially in affine type) have already played a major role in geometric and modular representation theory, see [6, 8, 18], for example.

The primary goal of this paper is to introduce the braid relations between expressions for double cosets, and to establish an analogue of Matsumoto's theorem, for arbitrary Coxeter groups. We also provide a presentation of the singular Coxeter monoid, an algebraic category underlying the Hecke algebroid in the same way that the Coxeter group underlies the Hecke algebra. The technology we develop is a crucial step toward the combinatorial study of the (diagrammatic) singular Hecke category.

Remark 1.32. In the literature, the (sometimes misleading) words *singular* and *parabolic* are used (not interchangeably) as adjectives to describe objects related to $W_I \backslash W / W_J$. Examples include singular or parabolic category \mathcal{O} , singular Soergel bimodules, and so on. Usually the adjective singular involves a transformation in categorical type, similar to the transformation from a monoid to a category. For example, ordinary Soergel bimodules are a monoidal category, whereas singular Soergel bimodules are a 2-category, with objects indexed by finitary subsets. The exception is the Hecke algebroid, which might have been called the 'singular Hecke algebra'. We prefer the 'singular' notation to the '-oid' notation in this paper to avoid conflicts with the literature, such as the unrelated concept of Weyl groupoids.

[†] It is called the Hecke category in [23, Definition 2.2.1] and called the Schur algebroid in [24].

[‡] They are called (*reduced*) *right translation sequences* in [23, Definition 1.3.2].

[§] This is implicit in the thesis, which does connect (without proof) singular Soergel bimodules to perverse sheaves.

1.7 | Outline of paper

In Section 2, we recall basic facts about $*$ -multiplication, double cosets, and Williamson’s theory of singular expressions. We tried to place all the previously known definitions and results in Section 2. In addition, Section 2 proves basic properties of SC , and relates them to singular expressions.

In Section 3, we give several new equivalent descriptions of singular reduced expressions. In Section 4, we use these new descriptions to prove many properties of singular reduced expressions.

In Section 5, we study relations between expressions. We construct the switchback relation and the other relations, and verify that they hold in SC . We prove the singular Matsumoto theorem in Subsection 5.7, and the section culminates with a proof that our presentation of SC is correct in Subsection 5.8. In Section 6, we explicitly elaborate on the switchback relation in all finite types.

In Section 7, we connect SC with webs in type A . We construct a monoidal category built from SC for all symmetric groups. Our main result states that this monoidal category and the Cautis–Kamnitzer–Morrison category of type A webs are isomorphic after taking the associated graded with respect to our length filtration.

In Section 8, we provide additional details about singular Coxeter complexes. Our intent is not to prove theorems, but mostly to showcase this new construction and its mysteries.

2 | SINGULAR EXPRESSIONS AND SC

2.1 | Properties of $*$ -multiplication

We recall some notation. Fix a Coxeter system (W, S) . We denote by \leq the Bruhat order on W and $\ell(w)$ the Coxeter length of $w \in W$. For $I \subset S$, the parabolic subgroup generated by I is denoted by W_I , and W^I (respectively, ${}^I W$) denotes the set of shortest representatives for the cosets W/W_I (respectively, $W_I \setminus W$). Even though we regularly use the $*$ -product, when we discuss cosets like xW_I we always refer to cosets in the group W .

Notation 2.1. We use the notation $w = x.y$ as a shorthand for the statement: $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$.

From the definition of $*$ -multiplication, we see that

$$x * s = x.s \text{ when } xs > x, \quad x * s = x \text{ when } xs < x. \tag{2.1}$$

We now state some basic and well-known results.

Lemma 2.2. *For any $x, y \in W$, $x * y = x.y'$ for some $y' \leq y$. In particular, if $y \in W_I$, then $x * y \in xW_I$.*

Proof. Choosing a reduced expression $y = s_1 s_2 \cdots s_d$, we know that $y = s_1 * s_2 * \cdots * s_d$. Also, $x * s$ is either equal to $x.s$ or to x . Thus,

$$x * s_1 * s_2 * \cdots * s_d = x.s_{i_1}.s_{i_2}. \cdots .s_{i_k} \tag{2.2}$$

for some $k \leq d$ and $1 \leq i_1 < i_2 < \dots < i_k \leq d$. Then $s_{i_1} s_{i_2} \dots s_{i_k}$ is a reduced expression for some $y' \in W$. Using the ‘subexpression definition’ of the Bruhat order, one sees that $y' \leq y$. \square

Lemma 2.3. *For $x, y \in W$ one has $xy = x.y$ if and only if $xy = x * y$.*

Proof. That $xy = x.y$ implies $x * y = xy$ is fairly well-trodden (being built into the implicit bijection between $(W, *, S)$ and W), so we leave it to the reader. For the converse, Lemma 2.2 says there is some y' such that $x * y = x.y'$. If $x * y = xy$, then $y = y'$ (since W is a group), so $xy = x * y = x.y$. \square

Notation 2.4. Let $LD(w) \subset S$ (respectively, $RD(w) \subset S$) denote the left (respectively, right) descent set of an element $w \in W$. Recall that descent sets are always finitary.

Lemma 2.5. *For any $w, x \in W$, $LD(w) \subset LD(w * x)$ and $RD(x) \subset RD(w * x)$.*

Proof. It is an immediate consequence of (2.1) that $x * s = x$ if and only if $s \in RD(x)$. Now the result follows from associativity:

$$s \in RD(x) \iff x * s = x \implies (w * x) * s = w * (x * s) = w * x \iff s \in RD(w * x). \quad (2.3)$$

The statement about left descents is proven similarly. \square

Lemma 2.6. *If $I \subset S$ is finitary, then $w * w_I = w$ if and only if $I \subset RD(w)$.*

Proof. By Lemma 2.5, the right descent set of $w * w_I$ contains $RD(w_I) = I$, which proves one direction. Conversely, if $I \subset RD(w)$, then

$$w * w_I = w * s_{i_1} * \dots * s_{i_d} = w.$$

Here, $s_{i_1} \dots s_{i_d}$ is a reduced expression for w_I , consisting only of simple reflections in I , and the second equality holds by (2.1). \square

2.2 | SC as Karoubi envelope

We now discuss maximal elements of double cosets in the context of star multiplication. In Lemma 2.12, we recall that each (J, I) -coset p has a unique (maximal) element \bar{p} satisfying the property that $J \subset LD(\bar{p})$ and $I \subset RD(\bar{p})$.

Lemma 2.7. *For any (I, J) -coset p and any (J, K) -coset q , $\bar{p} * \bar{q}$ is the maximal element in its (I, K) -coset.*

In the language of (1.13b), this would say that for any $x, y \in W$ one has $\overline{\bar{x} * \bar{y}} = \bar{x} * \bar{y}$.

Proof. By Lemma 2.5, $\bar{p} * \bar{q}$ has I in its left descent set and K in its right descent set. Thus, $\bar{p} * \bar{q}$ is maximal in its double coset. \square

Proposition 2.8. *The category SC is well-defined.*

Proof. Let I be finitary. If $I \subset \text{LD}(x)$, then $w_I * x = x$ by Lemma 2.6. Thus, the minimal (I, I) -coset, whose maximal element is w_I , serves as an identity element inside $\text{End}_{SC}(I)$.

Given double cosets $W_I x W_J$, $W_J y W_K$, and $W_K z W_L$, with maximal elements \bar{x} , \bar{y} , and \bar{z} , respectively, associativity is the statement that

$$W_I (\overline{\bar{x} * \bar{y}}) * \bar{z} W_K = W_I \bar{x} * (\overline{\bar{y} * \bar{z}}) W_K. \tag{2.4}$$

This follows from the associativity of $*$, since $\overline{(\bar{x} * \bar{y})} = \bar{x} * \bar{y}$ by the previous lemma. □

After this point we never use the notation \bar{x}, \bar{y} again, preferring the notation \bar{p} for a double coset p . For an element $x \in w$, the notation \bar{x} , understood as the maximal element of the (I, J) -coset containing x , will depend strongly on the choice of (I, J) . Meanwhile, p is a double coset already, and (I, J) are understood as part of its data.

There is a standard construction one can apply to any category, which formally adjoins the image of any idempotent as a new object. Applying this process to all idempotents in a category (typically preadditive, though not in our case), one obtains the so-called *Karoubi envelope*[†].

Lemma 2.9. *Let $w \in W$ satisfy $w * w = w$. Then $w = w_I$ for some finitary $I \subset S$.*

Proof. Let $I = \text{RD}(w)$. We know that w_I is the unique element of W_I whose right descent set contains I . Thus, $w = w_I$ if and only if $w \in W_I$. Pick a reduced expression $w = s_{i_1} \cdots s_{i_d}$. Following the proof of Lemma 2.2, one has

$$w * w = w * s_{i_1} * \cdots * s_{i_d}.$$

Using (2.1) repeatedly, $w * w = w$ if and only if $w * s_{i_k} = w$ for all $1 \leq k \leq d$, if and only if $s_{i_k} \in I$ for all $1 \leq k \leq d$, in which case $w \in W_I$. □

Thus, the Karoubi envelope of $(W, *, S)$ (viewed as a category with one object) has one object for each finitary subset. Let us denote this Karoubi envelope by $\text{Kar}(W)$, and denote the objects by I as in SC .

Definition 2.10. Let $F : SC \rightarrow \text{Kar}(W)$ be the functor which is defined on objects as $I \mapsto I$. On morphisms, a (J, I) -coset $p \in \text{Hom}_{SC}(I, J)$ is sent by F to the element $\bar{p} \in \text{Hom}_{\text{Kar}(W)}(I, J)$.

Theorem 2.11. *The functor F is well-defined, and is an equivalence of categories between SC and $\text{Kar}(W)$.*

Proof. By the Karoubi construction, $\text{Hom}_{\text{Kar}(W)}(I, J)$ is equal to those elements of W which are preserved by composition on the right with w_I and composition on the left with w_J . By Lemma 2.6, this is the same as those elements with I in their right descent set and J in their left descent set. Any such element is maximal in its (J, I) -coset, and each (J, I) -coset has a unique such element. Thus, F does send each double coset to a well-defined morphism, and F is a bijection on Hom sets.

[†] Standard online references, such as Wikipedia, are decent sources for learning about the Karoubi envelope.

Clearly, $\text{id}_J \in \text{End}_{SC}(J)$ is sent by F to $w_J \in \text{End}_{\text{Kar}(W)}(J)$, which is the identity element there. If p is a (J, I) -coset and q is a (K, J) -coset, then by Lemma 2.7, $F(q * p) = \overline{q} * \underline{p} = \overline{q} * \underline{p} = F(q) \circ F(p)$. Thus, F is a functor. \square

2.3 | Maximal and minimal elements in double cosets

Recall that W^I denotes the set of minimal representatives inside the right cosets W/W_I . If $I \subset J$, then $W^I \cap W_J$ has a longest element x which satisfies $x.w_I = w_J$. We often write

$$x = w_J w_I^{-1} \tag{2.5}$$

(even though $w_I^{-1} = w_I$) to emphasize the fact that the lengths add in the formula $x.w_I = w_J$. Similarly, $w_I^{-1} w_J$ is the longest element of ${}^I W \cap W_J$.

We record some useful properties of parabolic double cosets in W .

Lemma 2.12. *Let $J, I \subset S$ be finitary and let $p \in W_J \backslash W/W_I$.*

- (1) *The subposet $p \subset (W, \leq)$ is an interval, that is, there are elements $\overline{p}, \underline{p} \in p$ such that $p = \{x \in W \mid \underline{p} \leq x \leq \overline{p}\}$.*
- (2) *(Howlett’s theorem) Let $K = J \cap \underline{p} I \underline{p}^{-1}$ and $L = I \cap \underline{p}^{-1} J \underline{p}$. The following map is bijective.*

$$\begin{aligned} (W^K \cap W_J) \times W_I &\rightarrow p \\ (x, y) &\mapsto x.\underline{p}.y. \end{aligned} \tag{2.6}$$

Similarly, the following map is bijective.

$$\begin{aligned} W_J \times ({}^L W \cap W_I) &\rightarrow p \\ (x, y) &\mapsto x.\underline{p}.y. \end{aligned} \tag{2.7}$$

- (3) *One has*

$$\overline{p} = (w_J w_K^{-1}).\underline{p}.w_I = w_J.\underline{p}.(w_L^{-1} w_I). \tag{2.8}$$

- (4) *If z is any element of p , then there exist $x, x' \in W_J$ and $y, y' \in W_I$ such that*

$$z = x.\underline{p}.y, \tag{2.9a}$$

$$\overline{p} = x'.z.y'. \tag{2.9b}$$

- (5) *If $z \in p$, then*

$$z = \overline{p} \iff J \subset \text{LD}(z), I \subset \text{RD}(z), \tag{2.10a}$$

$$z = \underline{p} \iff J \cap \text{LD}(z) = \emptyset, I \cap \text{RD}(z) = \emptyset. \tag{2.10b}$$

Proof. See [12, Proposition 8.3]. To give the reader some additional context, we elaborate on the bijection in (2.6). If $w = x.p.y$, then the shortest element in the right coset wW_I is $x.p = wy^{-1}$, and the shortest element in the left coset $W_J(x.p)$ is \underline{p} . This gives an effective way to compute x and y from a given element w .

Since (2.6) (respectively, (2.7)) preserves length up to the constant $\ell(\underline{p})$, the longest element of p is the image of the (x, y) where x and y are the longest elements in their domain. This proves (2.8).

The bijection (2.6) immediately proves the existence of x and y such that (2.9a) holds. Moreover, we can assume that $x \in W^K \cap W_J$. By basic Coxeter theory, there exist x' and y' such that $x' . x = (w_J w_K^{-1})$ and $y . y' = w_I$. Then combining (2.9a) and (2.8), we obtain (2.9b).

It is clear that $J \subset LD(\underline{p})$ and $I \subset RD(\underline{p})$. Conversely, suppose $J \subset LD(z)$ and $I \subset RD(z)$. Writing $\underline{p} = x' . z . y'$ as in (2.9b), we see that $x' = y' = e$, or else the lengths will not add. The analogous statement for \underline{p} is straightforward. □

Remark 2.13. A final warning. A subset $p \subset W$ can be an (I, J) -coset in multiple different ways, as noted in Example 1.24. The elements \underline{p} and \overline{p} do not depend on the choice of (I, J) among such valid choices, though the elements x and y appearing in formulae like (2.6) will depend on this choice. Again, we never discuss a double coset p without the pair (I, J) being established by the context.

Let p be a (J, I) -coset. The subset $K = J \cap pI\underline{p}^{-1}$ associated to $p \in W_J \backslash W / W_I$ will appear many times in this paper, and we call it the (*left*) *redundancy*. The parabolic subgroup $W_K \subset W_J$ accounts for the redundancy between left multiplication by W_J and right multiplication by W_I on \underline{p} , as seen in Howlett’s theorem (2.6) above. Similarly, $L = I \cap \underline{p}^{-1}Jp$ is called the *right redundancy*. The following lemma helps explain why the redundancy (between left and right multiplication) is a standard parabolic subgroup, that is, a subgroup generated by the simple reflections it contains.

Lemma 2.14 (Kilmoyer’s theorem). *Let $I, J \subset S$ be finitary and $p \in W_J \backslash W / W_I$. Then*

$$\underline{p}I\underline{p}^{-1} \cap W_J = \underline{p}I\underline{p}^{-1} \cap J = \underline{p}W_I\underline{p}^{-1} \cap J. \tag{2.11}$$

Proof. We prove the first equality, and leave the second to the reader. Let $s \in I$ be such that $x := \underline{ps}\underline{p}^{-1} \in W_J$. Since $\underline{p} < \underline{p}.s$, the element $\underline{ps} = x\underline{p}$ has length $\ell(\underline{p}) + 1$. Since $x \in W_J$, $x\underline{p} = x.p$, so $x\underline{p}$ has length $\ell(\underline{p}) + \ell(x)$. It follows that $\ell(x) = 1$, so $x \in J$. □

One major use of (2.6) will be to compare minimal and maximal elements of different double cosets when one is contained inside the other. The moral of the next lemma is: when we change the parabolic subgroup on the right, then we can get between minimal (respectively, maximal) elements only using multiplication on the right.

Lemma 2.15. *Let J, I, I' be finitary with $I \subset I'$. Suppose that p is a (J, I) -coset and q is a (J, I') -coset and $p \subset q$. Then there exist $y, y' \in W_{I'}$ such that $\underline{p} = \underline{q}.y$ and $\overline{p}.y' = \overline{q}$.*

Proof. By (2.9a), we know that $\underline{p} = x.q.y$ for some $x \in W_J$ and $y \in W_{I'}$. However, left multiplication by any non-identity element of W_J will increase the length of \underline{p} , a contradiction if $x \neq e$ (since multiplication by x^{-1} will both increase and decrease length).

By (2.9b) we know that $\bar{q} = x' \cdot \bar{p} \cdot y'$ for some $x' \in W_J$ and $y' \in W_{I'}$. However, $J \subset \text{LD}(\bar{p})$, so $x' = e$. □

2.4 | Expressions and paths

Most of the definitions[†] in this section were made in [23, chapter 2, subsection 1.3].

Definition 2.16. An (*singular*) *expression* is a sequence of finitary subsets $I_* = [I_0, \dots, I_d]$ in S such that I_i and I_{i-1} differ by one element (that is, either $I_{i-1} = I_i \sqcup \{s\}$ or $I_{i-1} \sqcup \{s\} = I_i$ for some $s \in S$) for each i . An expression with $I_0 = J$ will be called a *J-expression*.

As noted in the introduction, an expression represents a composition of generating morphisms in the category SC . We will temporarily ignore this fact, and first focus on the analogue of subexpressions.

Definition 2.17. Let I_* be a J -expression. A *path* subordinate to I_* , is a sequence of double cosets $p_* = [p_0, \dots, p_d]$, where $p_i \in W_J \setminus W/W_{I_i}$, satisfying two conditions:

- $e \in p_0$,
- $p_{i-1} \cap p_i \neq \emptyset$ for all $1 \leq i \leq d$.

The *endpoint* of a path $[p_0, \dots, p_d]$ is the (J, I_d) -coset p_d . The (*left*) *redundancy sequence* of the path is the sequence (K_0, \dots, K_d) where $K_i = J \cap \underline{p_i I_i p_i^{-1}}$.

There is a dichotomy which runs throughout this paper: in an expression, either $I_i = I_{i-1} \sqcup s$ or $I_i = I_{i-1} \setminus s$ (for some $s \in S$). If $s \notin I_i$, then a (J, I_i) -coset is either entirely contained in or disjoint from any (J, I_{i-1}) -coset. Because of this, the condition that $p_{i-1} \cap p_i \neq \emptyset$ in a path can be rephrased more clearly as follows.

- If $I_i = I_{i-1} \sqcup s$, then $p_{i-1} \subset p_i$. Note that p_i is uniquely determined as the double coset containing p_{i-1} .
- If $I_i = I_{i-1} \setminus s$, then $p_i \subset p_{i-1}$. There may be many choices of p_i , depending on how p_{i-1} splits into double cosets upon restriction from $W_{I_{i-1}}$ to W_{I_i} .

We give several examples. In each, we point out one special path, the forward path, see Definition 2.21.

Example 2.18. Let $m_{st} = 3$ and consider the expression $[\emptyset, s, \emptyset, t, \emptyset, s, \emptyset]$. There are eight paths subordinate to this expression, corresponding to the eight subexpressions of sts . Here are two paths, and the subexpressions they correspond to.

$$s^1 t^0 s^0 : (\{e\}, \{e, s\}, \{s\}, \{s, st\}, \{s\}, \{e, s\}, \{s\})$$

$$s^0 t^1 s^1 : (\{e\}, \{e, s\}, \{e\}, \{e, t\}, \{t\}, \{t, ts\}, \{ts\}).$$

[†] Only the names have been changed to protect the innocent.

Note that every path is a sequence of (\emptyset, I) -cosets for various I , so the redundancy path is constantly the empty set. This last example is the forward path.

$$s^1 t^1 s^1 : \quad (\{e\}, \{e, s\}, \{s\}, \{s, st\}, \{st\}, \{st, sts\}, \{sts\}).$$

Example 2.19. Let $m_{st} = 3$ and consider the expression $[s, \emptyset, s, st, t, \emptyset, s]$. There are three paths subordinate to this expression. Note that every path is a sequence of (s, I) -cosets for various I . We also record their redundancy sequences.

$$\begin{aligned} (\{e, s\}, \{e, s\}, \{e, s\}, W_{st}, \{e, s, t, st\}, \{e, s\}, \{e, s\}) \quad \text{redundancy} &= (s, \emptyset, s, s, \emptyset, \emptyset, s), \\ (\{e, s\}, \{e, s\}, \{e, s\}, W_{st}, \{e, s, t, st\}, \{t, st\}, \{t, ts, st, sts\}) \quad \text{redundancy} &= (s, \emptyset, s, s, \emptyset, \emptyset, \emptyset), \\ (\{e, s\}, \{e, s\}, \{e, s\}, W_{st}, \{ts, sts\}, \{ts, sts\}, \{t, ts, st, sts\}) \quad \text{redundancy} &= (s, \emptyset, s, s, s, \emptyset, \emptyset). \end{aligned}$$

We write $W_{st} = \{e, s, t, st, ts, sts\}$ for the entire parabolic subgroup, as a double coset. The third path above is the forward path.

Example 2.20. Let $m_{st} = 3$ and consider the expression $[\emptyset, s, st, s, \emptyset]$. There are six paths subordinate to this expression, one for each element w of the parabolic subgroup W_{st} . The corresponding path is

$$(\{e\}, \{e, s\}, W_{st}, \{w, ws\}, \{w\}).$$

The path associated to $w = sts$ is the forward path of the expression.

Definition 2.21. A path p_i is said to be *forward* if for each i with $I_{i+1} \subset I_i$, the double coset p_{i+1} is *maximal* in p_i , meaning that $\overline{p_{i+1}} = \overline{p_i}$ (see Lemma 2.12 (2.12)). An expression has a unique forward path. We say that p is the *endpoint of an expression I* , if the forward path of I has the endpoint p . We say I *expresses p* and write $I \rightleftharpoons p$.

A singular expression is the analogue of a $*$ -expression. A path is the analogue of a subexpression[†]. The forward path is the analogue of the unique subexpression which is the expression itself. If the singular expression represents a sequence of generating morphisms in SC , then the endpoint of the forward path is the overall composition, as we now prove. Recall that a double coset is called *minimal* if it contains the identity element.

Proposition 2.22. Let I be a J -expression, with forward path p_* . For each i , let q_i be the minimal (I_{i-1}, I_i) -coset. Let id_J be the minimal (J, J) -coset. Then for all $0 \leq i \leq d$, we have the following equality of morphisms in SC :

$$p_i = \text{id}_J * q_1 * \dots * q_i. \tag{2.12}$$

[†] Perhaps it is better to say that a path is the analogue of the Bruhat stroll of a subexpression, see [10, subsection 2.4]. In the combinatorics of the Hecke category, the primary use of subexpressions is based on their Bruhat strolls, see, for example, the Deodhar defect formula, or the light leaf construction [10, subsections 2.4 and 6.1].

Proof. By definition of a path, $p_0 = \text{id}_J$. We need only confirm that $p_i = p_{i-1} * q_i$ for each $1 \leq i \leq d$. To show that two double cosets are equal, we need only show they intersect nontrivially. There are two cases to check: when $I_{i-1} \subset I_i$ and when $I_{i-1} \supset I_i$.

If $I_{i-1} \subset I_i$, then $\overline{q_i} = w_{I_i}$. By definition, $p_{i-1} * q_i$ is the coset containing $\overline{p_{i-1}} * w_{I_i}$, and this element is contained in $\overline{p_{i-1}}W_{I_i}$ by Lemma 2.2. Thus, $p_{i-1} * q_i$ contains $\overline{p_{i-1}}$. Since p_i is the unique double coset containing p_{i-1} , it also contains $\overline{p_{i-1}}$, and thus equals $\overline{p_{i-1}} * q_i$.

If $I_{i-1} \supset I_i$, then p_i contains $\overline{p_{i-1}}$ by definition of a forward path. Meanwhile, $\overline{q_i} = w_{I_{i-1}}$, and $\overline{p_{i-1}}$ already has I_{i-1} in its right descent set. Thus, $\overline{p_{i-1}} * w_{I_{i-1}} = \overline{p_{i-1}}$. Again, $p_{i-1} * q_i = p_i$, as desired. □

Corollary 2.23. *Let $I_* = [I_0, \dots, I_d]$ and $I_* \Leftarrow p$. Then*

$$\overline{p} = w_{I_0} * w_{I_1} * \dots * w_{I_d}. \tag{2.13}$$

Proof. This is just a more explicit unraveling of Proposition 2.22, rephrased using Theorem 2.11. Using the notation of Proposition 2.22, it is easy to see that $\overline{q_i} = w_{I_{i-1}} * w_{I_i}$, and $\overline{\text{id}_J} = w_{I_0}$. By Theorem 2.11, $\overline{q * q'} = \overline{q} * \overline{q'}$ for two composable double cosets. Thus,

$$\overline{p} = w_{I_0} * (w_{I_0} * w_{I_1}) * (w_{I_1} * w_{I_2}) * \dots * (w_{I_{d-1}} * w_{I_d}) = w_{I_0} * w_{I_1} * \dots * w_{I_d}$$

as desired. □

2.5 | Williamson’s definition of reduced expressions

Any $p \in W_J \setminus W/W_I$ has an expression. In fact, any p has a reduced expression in the following sense (see Proposition 2.31).

Definition 2.24. Let I_* be a J -expression, let p_* be a subordinate path, and let K_* be the redundancy sequence of p_* . Then p_* is *reduced* if, for each i with $I_i \subset I_{i+1}$, one has

$$\text{the double coset } p_i \text{ is minimal in } p_{i+1}, \text{ meaning } \underline{p_i} = \underline{p_{i+1}}, \tag{2.14a}$$

$$K_i = K_{i+1}. \tag{2.14b}$$

We say that I_* is *reduced* if its forward path is *reduced*.

The reader may wish to recall Lemma 2.14 and the preceding discussion for more on redundancy. We will give several other ways to think about the mysterious condition (2.14b) below. The reader may also wish to skip immediately to Lemma 3.3, which gives an efficient condition for checking whether a path is reduced.

If $I_{i+1} = I_i \sqcup s$ and (2.14a) holds, then (2.14b) is equivalent to $\underline{p_i s p_i^{-1}} \notin J$. This is because $\underline{p_i s p_i^{-1}}$ is the only possible element in $K_{i+1} \setminus K_i$. Note that when $J = \emptyset$ one need only worry about the first condition (2.14a).

Example 2.25. The expression $[\emptyset, s, \emptyset, t, st]$ is not reduced. Its forward path is

$$(\{e\}, \{e, s\}, \{s\}, \{s, st\}, W_{st}),$$

and $\{s, st\}$ is not the minimal (\emptyset, t) -coset inside the (\emptyset, st) -coset W_{st} .

Example 2.26. When $m_{st} = 3$ there are six reduced expressions for the (\emptyset, \emptyset) -coset $\{sts\}$:

$$[\emptyset, s, \emptyset, t, \emptyset, s, \emptyset], \quad [\emptyset, t, \emptyset, s, \emptyset, t, \emptyset],$$

$$[\emptyset, s, st, s, \emptyset], \quad [\emptyset, s, st, t, \emptyset], \quad [\emptyset, t, st, s, \emptyset], \quad [\emptyset, t, st, t, \emptyset].$$

Example 2.27. The expression $[\emptyset, s_1, \emptyset, s_2, \dots, \emptyset, s_d]$ is reduced if and only if $s_1 \dots s_d$ is a reduced expression in W .

Example 2.28. As in the previous example, the expression $[\emptyset, s, \emptyset, s]$ is not reduced. Its forward path is $(\{e\}, \{e, s\}, \{s\}, \{e, s\})$, and reducedness fails because of (2.14a): $\{s\}$ is not minimal inside $\{e, s\}$. Meanwhile, the expression $[s, \emptyset, s]$ is also not reduced. Its forward path is $(\{e, s\}, \{e, s\}, \{e, s\})$, and this time the reducedness fails because of (2.14b): the redundancy increases from $K_1 = \emptyset$ to $K_2 = s$. See Remark 2.34 for more commentary on this phenomenon.

When examining whether an expression is reduced or not, it helps to examine conditions (2.14) one index at a time.

Definition 2.29. Let $J, I, I' \subset S$ be finitary, where $I \subset I'$ or $I' \subset I$. Let p be a (J, I) -coset and q be a (J, I') -coset with $p \cap q \neq \emptyset$. Let $K = J \cap \underline{p}I\underline{p}^{-1}$ and $K' = J \cap \underline{q}I'\underline{q}^{-1}$. We say that $[p, q]$ is *reduced* if either

- $I \supset I'$ and $\bar{p} = \bar{q}$, or
- $I \subset I'$ and $K = K'$ and $\underline{p} = \underline{q}$.

Now use the notation of Definition 2.24. We say that p_\bullet is *reduced at index i* if $[p_i, p_{i+1}]$ is reduced. When p_\bullet is the forward path of I_\bullet , we also say that I_\bullet is *reduced at index i* .

Example 2.30. The expression $[s, \emptyset, s, st, t, \emptyset, s]$ from Example 2.19 is not reduced. It is not reduced at index 1 (the first instance of $[\emptyset, s]$) because $K_1 = \emptyset$ and $K_2 = s$. It is not reduced at index 5 (the second instance of $[\emptyset, s]$) because $\underline{p}_5 = ts$ and $\underline{p}_6 = t$. It is reduced at all other indices.

The existence of reduced expressions for arbitrary double cosets was proven by Williamson.

Proposition 2.31. *Each parabolic double coset in W has a reduced expression.*

Proof. Our definition of a reduced expression agrees with that of a reduced right translation sequence in [23, chapter 2, subsection 1.3]. The desired result is proven in [23, chapter 2, Proposition 1.3.4]. □

The following recap may help the reader keep track of various definitions. An expression I_\bullet may have many subordinate paths p_\bullet .

When $I_i \supset I_{i+1}$, then p_{i+1} is one of the many (J, I_{i+1}) -cosets inside the (J, I_i) -coset p_i , a non-deterministic choice. However, a forward path forces a single deterministic choice for p_{i+1} , the maximal such double coset. Conversely, p_i is uniquely determined by p_{i+1} .

When $I_i \subset I_{i+1}$, then p_{i+1} is uniquely determined by p_i , but p_i is not determined by p_{i+1} . For a reduced path, (2.14a) says that p_i is uniquely determined by p_{i+1} , as the minimal double coset. So, a reduced forward path is deterministic in both directions: p_i and p_{i+1} determine each other uniquely. However, reducedness also has second requirement (2.14b). One use of (2.14b) is to prevent infinite loops like $[s, \emptyset, s, \emptyset, s, \emptyset, \dots]$, where the underlying sets of the double cosets in the path p_i are constant, see Example 2.28.

Lemma 2.32. *Let I_i be a J -expression. If $I_i \subset I_{i+1}$ and $[p_i, p_{i+1}]$ is reduced then $p_i \subsetneq p_{i+1}$.*

Proof. Let $I_{i+1} = I_i \sqcup s$. We claim that $\underline{p_i}s \in p_{i+1} \setminus p_i$. If $\underline{p_i}s$ were in p_i , then the length-preserving bijection (2.6) would imply that either $\underline{p_i}s = \underline{p_i}t$ for some $t \in I_i$, or $\underline{p_i}s = u\underline{p_i}$ for some $u \in J$. The first option is evidently impossible, while the second option would imply that $s \in \underline{p_i}^{-1}J\underline{p_i} \cap I_{i+1}$, the right redundancy of p_{i+1} . Since p_i and p_{i+1} have the same right redundancy, we also have $s \in \underline{p_i}^{-1}J\underline{p_i} \cap I_i$. But this implies $s \in I_i$, a contradiction. \square

Thus, part of being reduced is the statement (about the forward path) that ‘when a double coset is supposed to get bigger, it actually does get bigger’. However, this condition is strictly weaker than reducedness.

Example 2.33. Let $m_{st} = 3$ and consider the expression $[s, \emptyset, t, st]$ with

$$p_* = (\{e, s\}, \{e, s\}, \{e, s, t, st\}, W_{st}), \quad K_* = (s, \emptyset, \emptyset, s).$$

This expression is not reduced at index 2 because the redundancy increases, though $\{e, s, t, st\} \subsetneq W_{st}$. Note that the expression is reduced at index 0, even though $\{e, s\} = \{e, s\}$; in a reduced expression, ‘when a double coset is supposed to get smaller, it is permitted to stay the same size’.

Remark 2.34. If one places \emptyset at the start of the expression in Example 2.33, one obtains the non-reduced expression $[\emptyset, s, \emptyset, t, st]$ studied in Example 2.25. These two non-reduced expressions are both non-reduced at the ‘same’ index $[t, st]$ but for different reasons, one because of (2.14b) and one because of (2.14a). One can think that (2.14a) is the primal condition for reducedness, while (2.14b) exists to ensure that reduced expressions are closed under precomposition, including J -expressions inside \emptyset -expressions. In Proposition 4.13, we will prove that a J -expression is reduced if and only if it is reduced after precomposition with $[\emptyset, s_1, s_1s_2, \dots, J]$ for some enumeration $J = \{s_1, \dots, s_{|J|}\}$.

Remark 2.35. Paths for J -expressions are strolls in the real hyperplane arrangement of the Coxeter group, staying within the fundamental domain for W_J (equivalently, strolls in the J -singular dual Coxeter complex Cox_J). Reduced expressions will be oriented strolls. Condition (2.14b) can be interpreted as the statement that one never returns to a reflection hyperplane after leaving it. For more on this geometric intuition, see Subsection 8.4.

3 | NEW DESCRIPTIONS OF REDUCED EXPRESSIONS

Studying a singular expression in terms of its forward path makes it harder to study contiguous subwords and other structures. For example, the forward path of $[I_0, \dots, I_d]$ is a sequence of (I_0, I_i) -cosets, while the forward path of a contiguous subword $[I_k, \dots, I_\ell]$ is a sequence of (I_k, I_i) -cosets. Comparing the apples with the oranges takes some work. This is one of the main reasons to find alternatives for Williamson’s definition of reduced expressions.

3.1 | Another criterion for one-step reducedness

First, we give two easy lemmas.

Lemma 3.1. *Let $J, I, I' \subset S$ be finitary with $I \subset I'$. For a (J, I) -coset p and (J, I') -coset q with $p \subset q$, we have*

$$K = J \cap \underline{pI} \underline{p}^{-1} \subset K' = J \cap \underline{qI'} \underline{q}^{-1}. \tag{3.1}$$

Proof. By Lemma 2.15 we have $\underline{p}W_{I'} = \underline{q}W_{I'}$. By (2.11), we have

$$K = J \cap \underline{p}W_I \underline{p}^{-1} \subset J \cap \underline{p}W_{I'} \underline{p}^{-1} = J \cap \underline{q}W_{I'} \underline{q}^{-1} = K'. \tag{3.2}$$

□

Lemma 3.2. *Whenever I and J are finitary with $I \subset J$ or $J \subset I$, one has*

$$w_I.(w_{I \cap J}^{-1} w_J) = (w_I w_{I \cap J}^{-1}).w_{I \cap J}.(w_{I \cap J}^{-1} w_J) = (w_I w_{I \cap J}^{-1}).w_J. \tag{3.3}$$

Proof. This is obvious. Note also that either $(w_{I \cap J}^{-1} w_J)$ or $(w_I w_{I \cap J}^{-1})$ is the identity. □

Here is a useful condition on maximal elements which is equivalent to being reduced. Like in Lemma 2.15, when we change the parabolic subgroup on the right, we should be able to compare cosets using only right multiplication. Recall that Definition 2.29 stated what it means for $[p, q]$ to be reduced.

Lemma 3.3. *Let $J, I, I' \subset S$ be finitary, with $I \subset I'$. For a (J, I) -coset p and (J, I') -coset q with $p \subset q$, the following are equivalent.*

- (1) $[p, q]$ is reduced.
- (2) $\bar{q} = \bar{p}(w_I^{-1} w_{I'})$.
- (3) $\bar{p}(w_I^{-1} w_{I'}) = \bar{p}.(w_I^{-1} w_{I'})$, that is, the lengths add in this product.

Proof. Let $K = J \cap \underline{pI} \underline{p}^{-1}$ and $K' = J \cap \underline{qI'} \underline{q}^{-1}$. By (2.8) we have $\bar{q} = x'.\underline{q}.w_{I'}$ and $\bar{p} = x.\underline{p}.w_I$, where $x = w_J w_K^{-1}$ and $x' = w_J w_{K'}^{-1}$.

We first prove that (1) implies (2) and (3). If $[p, q]$ is reduced, then $K = K'$ and $\underline{p} = \underline{q}$. Thus, $x = x'$ and

$$\bar{q} = (x.\underline{p}).w_{I'} = (\bar{p}w_I^{-1}).w_{I'} = \bar{p}.(w_I^{-1} w_{I'}),$$

as desired[†]. The remainder of the proof will show that (2) implies (1), and (3) implies (2).

[†]Note the crucial use of the mysterious condition (2.14b), which essentially implied the left multiplication played no significant role when comparing \bar{p} and \bar{q} .

Suppose that $\bar{q} = \bar{p}w_I^{-1}w_{I'}$. Then $x'q_{w_{I'}} = xp_{w_{I'}}$, so $x'.q = x.p$. But $\bar{q} \leq \bar{p}$ because of the containment of cosets, and $x' \leq x$ because $K \subset K'$ by Lemma 3.1. By comparing lengths, we see that $\bar{p} = \bar{q}$ and $x = x'$, thus $K = K'$. So, $[p, q]$ is reduced. Thus, (2) implies (1).

Now suppose that the lengths add in the expression $\bar{p}.(w_I^{-1}w_{I'})$. Since I is in the right descent set of \bar{p} , we can use (3.3) to show that

$$\bar{p}.(w_I^{-1}w_{I'}) = (\bar{p}w_I^{-1}).w_{I'}.(w_I^{-1}w_{I'}) = (\bar{p}w_I^{-1}).w_{I'}. \tag{3.4}$$

Thus, the lengths also add in $(\bar{p}w_I^{-1}).w_{I'}$. By Lemma 2.3, this means that $\bar{p}w_I^{-1}w_{I'} = (\bar{p}w_I^{-1}) * w_{I'}$. By Lemma 2.5 this means that $J \subset \text{LD}(\bar{p}w_I^{-1}w_{I'})$ and $I' \subset \text{RD}(\bar{p}w_I^{-1}w_{I'})$. Since $\bar{p}w_I^{-1}w_{I'}$ is clearly in q , Lemma 2.10a implies that it is equal to \bar{q} . Thus, (3) implies (2). \square

We can summarize the above lemma as

$$[p, q] \text{ is reduced} \iff \bar{q} = \bar{p}.(w_I^{-1}w_{I'}) = (\bar{p}w_I^{-1}).w_{I'}. \tag{3.5}$$

Corollary 3.4. *Let I_i be an expression and let p_i be its forward path. Then I_i is reduced if and only if for $0 \leq i < d$ we have*

$$\overline{p_{i+1}} = \overline{p_i} \text{ when } I_{i+1} \subset I_i, \quad \overline{p_{i+1}} = \overline{p_i}.w_{I_i}^{-1}w_{I_{i+1}} \text{ when } I_i \subset I_{i+1}. \tag{3.6}$$

3.2 | Another criterion for reducedness

We first give a reinterpretation of Corollary 2.23, giving a formula for \bar{p} associated to any reading of the expression.

Proposition 3.5. *Let $I_i = [I_0, \dots, I_d]$ be a singular expression with $I_i \rightleftharpoons p$. Define elements y_i and z_i in W by the formulae*

$$y_i = w_{I_i} w_{I_i \cap I_{i+1}}^{-1}, \quad z_i = w_{I_{i-1} \cap I_i}^{-1} w_{I_i}. \tag{3.7}$$

For $0 \leq k \leq d$, define $a_k \in W$ as

$$a_k := y_0 * y_1 * \dots * y_{k-1} * w_{I_k} * z_{k+1} * \dots * z_{d-1} * z_d. \tag{3.8}$$

Then for any $0 \leq k \leq d$ we have

$$\bar{p} = a_k. \tag{3.9}$$

Note that when $I_{i-1} \supset I_i$ we have $z_i = e$, and when $I_i \subset I_{i+1}$ we have $y_i = e$.

Proof. For any $1 \leq i \leq d$, we claim that

$$w_{I_{i-1}} * z_i = w_{I_{i-1}} * w_{I_i} = y_{i-1} * w_{I_i}. \tag{3.10}$$

This is easy, and left to the reader; see also (3.3).

Let us show that (3.8) agrees with (2.13). Starting at w_{I_k} and working outward, we have

$$\begin{aligned} \cdots * y_{k-2} * y_{k-1} * w_{I_k} * z_{k+1} * z_{k+2} * \cdots &= \cdots y_{k-2} * w_{I_{k-1}} * w_{I_k} * w_{I_{k+1}} * z_{k+2} * \\ \cdots &= \dots = w_{I_0} * \cdots * w_{I_d}, \end{aligned}$$

as desired. □

When we use (3.9) for a given value of k , we say that we are using the expression for \bar{p} which is centered at k . We now give an formulation of reducedness which is adapted to any reading of the expression.

Theorem 3.6. *Continue the notation of Proposition 3.5. We say that the lengths add in (3.8) if*

$$a_k = y_0 \cdot y_1 \cdots \cdot y_{k-1} \cdot w_{I_k} \cdot z_{k+1} \cdots \cdot z_{d-1} \cdot z_d. \tag{3.11}$$

The following are equivalent.

- (1) I_* is reduced.
- (2) The lengths add in (3.8) for some $0 \leq k \leq d$.
- (3) The lengths add in (3.8) for all $0 \leq k \leq d$.

Proof. It is an immediate consequence of (3.3) that

$$w_{I_k} \cdot z_{k+1} = y_k \cdot w_{I_{k+1}}, \tag{3.12}$$

and in particular, these lengths always add. Since a_k and a_{k+1} have the same expression outside of (3.12), this proves that $a_k = a_{k+1}$, and that the lengths add in a_k if and only if they add in a_{k+1} . Hence, it suffices to prove that I_* is reduced if and only if the lengths add in (3.11) for $k = 0$. We prove this statement by induction on the width d . When $d = 0$, I_* is automatically reduced, and $\bar{p} = w_{I_0} = a_0$. Let us now fix an expression I_* of width d and assume the result for all expressions of width $d - 1$.

Suppose that I_* is reduced. Since the forward path of $[I_0, \dots, I_{d-1}]$ agrees with the first $d - 1$ steps of the forward path for I_* , this subsequence is also reduced. By induction, the lengths add in the following expression for $\overline{p_{d-1}}$.

$$\overline{p_{d-1}} = w_{I_0} \cdot z_1 \cdots \cdot z_{d-1}. \tag{3.13}$$

By (3.6), the fact that $[p_{d-1}, p_d]$ is reduced implies that

$$\overline{p_d} = \overline{p_{d-1}} \cdot z_d. \tag{3.14}$$

Combining these, we deduce that $\overline{p_d} = a_0$ and the lengths add, as desired.

Conversely, suppose that the lengths add in the expression for a_0 in (3.11). By induction, this implies that $[I_0, \dots, I_{d-1}]$ is reduced, and (3.13) holds. Moreover, the lengths add in $\overline{p_{d-1}} \cdot z_d$, which by Lemma 3.3 implies that $[p_{d-1}, p_d]$ is reduced. Thus, I_* is reduced. □

Remark 3.7. Recall the discussion from Subsection 2.4, where we observed that the forward path is associated to the left reading of a singular expression. In contrast, one should think that the element a_k is associated with any reading which starts at I_k and moves outward contiguously.

Because Theorem 3.6 is adapted to all readings of an expression, we think of it as an intrinsic definition of reducedness. It should be no surprise that a_0 played a special role in the proof, since it corresponds to the left reading, and is thus most obviously connected with the definition of reduced expressions.

Lemma 3.8. *The criterion for reducedness given in Theorem 3.6 is equivalent to the criterion for reducedness stated in the introduction in Equation (1.1).*

Proof. Consider any singular expression of the form $[I_0 \subset I_1 \subset \dots \subset I_d]$. Then it is easy to verify that $z_1.z_2.\dots.z_d = w_{I_0}^{-1}w_{I_d}$. Similarly, consider any singular expression of the form $[I_0 \supset I_1 \supset \dots \supset I_d]$. It is easy to verify that $z_1.z_2.\dots.z_d = e$.

Now let I_* be arbitrary. Combining these observations, it is straightforward to match the expression for a_0 from (3.11) with the expression for \bar{p} in (1.3). That the lengths add in (3.11) is equivalent to (1.1b). □

3.3 | Locality of (reduced) expressions

Definition 3.9. If I_* and I'_* are two J -expressions with the same endpoint, we say that I_* and I'_* are *equivalent*, and write $I_* \simeq I'_*$.

We identify equivalence classes of expressions with double cosets, and freely use ‘transitivity’ when examining statements like $I_* \simeq p \simeq I'_*$.

Notation 3.10. We use interval notation for contiguous subwords of singular expressions: if $I_* = [I_0, \dots, I_d]$ is an expression, and $0 \leq m \leq n \leq d$, then we write

$$I_{[m,n]} := [I_m, \dots, I_n].$$

The following statement is the *locality of expressions*.

Proposition 3.11. *Let $I_* = [I_0, \dots, I_d]$ be an expression, and fix $0 \leq m \leq n \leq d$. If $H_* = [I_m = H_0, H_1, \dots, H_{r-1}, H_r = I_n]$ is another I_m -expression satisfying $H_* \simeq I_{[m,n]}$, then*

$$I_* \simeq I'_* := [I_0, \dots, I_m, H_1, \dots, H_{r-1}, I_n, \dots, I_d]. \tag{3.15}$$

Proof. By Proposition 2.22, the double coset expressed by I_* is just a composition in SC of generators. If $[I_0, \dots, I_m] \simeq p$ and $[I_m, \dots, I_n] \simeq p'$ and $[I_n, \dots, I_d] \simeq p''$, then $I_* \simeq p * p' * p''$. But $H_* \simeq I_{[m,n]} \simeq p'$ as well, so $I'_* \simeq p * p' * p''$. □

The following statement is the *locality of reduced expressions*.

Proposition 3.12. *Use the same setup and notation as in Proposition 3.11. If I_* is reduced, then $I_{[m,n]}$ is reduced. If I_* and H_* are reduced, then I'_* is reduced.*

Proof. Let I_* be reduced, and use the notation of Theorem 3.6. We will compare I_* and $I_{[m,n]}$ by using (3.9) centered at m . By Theorem 3.6, the lengths add in (3.11) for $k = m$. This product (3.11)

has a subword where the lengths add:

$$a := w_{I_m} \cdot z_{m+1} \cdots z_n. \tag{3.16}$$

Again by Theorem 3.6, this implies that $I_{[m,n]}$ is reduced. Moreover, $I_{[m,n]} \Leftrightarrow p'$ where $\overline{p'} = a$.

Now let z'_i and y'_i and a'_k be the elements of W associated to the expression I'_\bullet , and z''_i, y''_i be associated with H_\bullet . Note that $z''_i = z'_{m+i}$. If H_\bullet is also reduced, then we have another expression for a , as

$$a = w_{I_m} \cdot z''_1 \cdots z''_r = w_{I_m} \cdot z'_{m+1} \cdots z'_{m+r}. \tag{3.17}$$

Substituting (3.17) in place of (3.16) inside the product formula (3.11) for a_m , one obtains the product formula (3.11) for a'_m . Thus, the lengths add in the formula for a'_m , and Theorem 3.6 implies that I'_\bullet is reduced. □

Remark 3.13. In a previous version of this manuscript, we proved this result directly from Williamson’s definition of reduced expressions, by studying forward paths in depth. That proof took an additional four pages. Again, we recall the discussion from the end of Subsection 2.4: intrinsic constructions involving singular expressions (for example, multiplication in SC , or the criterion for reducedness from Theorem 3.6) are more useful for proving statements about subwords than constructions involving the left reading (for example, forward paths).

3.4 | The length function(s)

The (weak left) Bruhat graph of a Coxeter system (W, S) is a ranked poset, where each $w \in W$ is ranked by its length. A length function is an easy way to detect whether a given edge goes up or down in the Bruhat order. In other words, for $s \in S$ and $w \in W$, we have $ws > w$ if and only if $\ell(ws) > \ell(w)$, and otherwise $ws < w$. In addition to the length function on elements of W , one also has a length function on (ordinary) expressions. The length of an expression for w is always at least the length of w , and equality holds if and only if the expression is reduced.

We transport these ideas to the setting of double cosets, defining the length of a double coset, and the length of a (singular) expression. Three crucial features our length function will satisfy are the following.

- The length of an expression for p is always at least the length of p , with equality if and only if the expression is reduced.
- The length of an expression will strictly increase with the addition of each index.
- The length of an identity expression (that is, a width zero expression) is zero.

These constraints imply that the length of an (I, J) -coset p cannot be determined by the underlying set of p , but also needs the data of (I, J) . For example, the (s, s) -coset $p = \{e, s\}$ is the identity element in SC of the object s , and has a length zero expression $[s]$. The (s, \emptyset) -coset $q = \{e, s\}$ has the longer reduced expression $[s, \emptyset]$, so we should have $\ell(q) > \ell(p)$ even though they have the same underlying set.

Definition 3.14. Henceforth, we write ℓ_W for the usual length function on W . If J is finitary we write $\ell(J)$ for $\ell_W(w_J)$.

For $p \in W_J \setminus W/W_I$, we let

$$\ell^+(p) = \ell_W(\overline{p}) - \ell(J), \quad \ell^-(p) = \ell_W(\underline{p}) + \ell(J) - \ell(J \cap \underline{pI\underline{p}^{-1}}), \quad \ell(p) = \ell^+(p) + \ell^-(p). \tag{3.18}$$

We refer to these as the $+$ -length, the $-$ -length, and the length of p , respectively.

Given an expression $I_\bullet = [I_0, \dots, I_n]$ with the forward path p_\bullet , we let $d_0^+ = d_0^- = 0$. We let

$$d_i^\pm = \begin{cases} \ell(I_i) - \ell(I_{i-1}) & \text{if } I_{i-1} \subset I_i \\ 0 & \text{if } I_{i-1} \supset I_i \end{cases}, \quad d_i^\pm = \begin{cases} 0 & \text{if } I_{i-1} \subset I_i \\ \ell(I_{i-1}) - \ell(I_i) & \text{if } I_{i-1} \supset I_i \end{cases}, \tag{3.19}$$

for $i \geq 1$. We define the (\pm) -lengths of I_\bullet as

$$\ell^\pm(I_\bullet) = \sum_{i=0}^n d_i^\pm, \quad \text{and } \ell(I_\bullet) = \ell^+(I_\bullet) + \ell^-(I_\bullet). \tag{3.20}$$

Be warned! If $w \in W$ and we let $p = \{w\}$ be an (\emptyset, \emptyset) -coset, then $\ell(p) = 2\ell_W(w)$!

Example 3.15. Let $I_\bullet = [\emptyset, \{s_1\}, \{s_1, s_2\}, \dots, \{s_1, \dots, s_d\} = I_d]$. Then $I_\bullet \rightleftharpoons p$ where p is the (\emptyset, I_d) -coset containing the identity. We have

$$\ell^+(p) = \ell(I_d) = \ell^+(I_\bullet), \quad \ell^-(p) = 0 = \ell^-(I_\bullet). \tag{3.21}$$

Example 3.16. Let I_\bullet be an expression of the form $[\emptyset, s_1, \emptyset, s_2, \emptyset, \dots, s_d, \emptyset]$. Then $\ell^+(I_\bullet) = \ell^-(I_\bullet) = d$. Meanwhile, if p is the endpoint of the expression, that is, $p = \{w\}$ where $w = s_1 \dots s_d$, then $\ell_W(w) \leq d$ with equality if and only if I_\bullet is reduced. Thus, $\ell^+(I_\bullet) = \ell^+(p)$ if and only if I_\bullet is reduced if and only if $\ell^-(I_\bullet) = \ell^-(p)$.

Proposition 3.17. Let $I_\bullet = [I_0, \dots, I_n]$ be an expression and let p be its endpoint. Then $\ell^+(I_\bullet) \geq \ell^+(p)$ and $\ell^-(I_\bullet) \geq \ell^-(p)$. Moreover, the following are equivalent.

- I_\bullet is reduced.
- $\ell^+(I_\bullet) = \ell^+(p)$ and $\ell^-(I_\bullet) = \ell^-(p)$.
- $\ell(I_\bullet) = \ell(p)$.

Proof. Let p_\bullet be the forward path of I_\bullet . We will prove (for $1 \leq i \leq n$) that $d_i^+ \geq \ell^+(p_i) - \ell^+(p_{i-1})$ and $d_i^- \geq \ell^-(p_i) - \ell^-(p_{i-1})$, and that $[p_{i-1}, p_i]$ is reduced if and only if both inequalities are equalities. Since $d_0^+ = \ell^+(p_0)$ and $d_0^- = \ell^-(p_0)$ by definition, we deduce that

$$\sum_{i=0}^k d_i^+ \geq \ell^+(p_k), \quad \sum_{i=0}^k d_i^- \geq \ell^-(p_k), \quad \text{with both equalities if and only if } I_{[0,k]} \text{ is reduced.} \tag{3.22}$$

Setting $k = n$ we get the desired result.

Let $K = J \cap \underline{p_{i-1}}I_{i-1}\underline{p_{i-1}}^{-1}$ and $K' = J \cap \underline{p_i}I_i\underline{p_i}^{-1}$ for the rest of this proof.

When $I_{i-1} \supset I_i$, then $[p_{i-1}, p_i]$ is always reduced. By definition of a forward path, $\ell^+(p_i) - \ell^+(p_{i-1}) = 0 = d_i^+$. By (2.8), we have

$$(w_J w_K^{-1}) \cdot \underline{p}_{i-1} \cdot w_{I_{i-1}} = \overline{p_{i-1}} = \overline{p_i} = (w_J w_{K'}^{-1}) \cdot \underline{p}_i \cdot w_{I_i}. \tag{3.23}$$

Thus,

$$\begin{aligned} \ell^-(p_i) + \ell(I_i) &= \ell_W(\underline{p}_i) + \ell(I_i) + \ell(J) - \ell(K') \\ &= \ell_W(\underline{p}_{i-1}) + \ell(I_{i-1}) + \ell(J) - \ell(K) = \ell^-(p_{i-1}) + \ell(I_{i-1}), \end{aligned} \tag{3.24}$$

from which we conclude that

$$d_i^- = \ell(I_{i-1}) - \ell(I_i) = \ell^-(p_i) - \ell^-(p_{i-1}). \tag{3.25}$$

If $I_{i-1} \subset I_i$, then Lemma 3.1 implies that $K \subset K'$. Since $p_{i-1} \subset p_i$ we have $\underline{p}_i \leq \underline{p}_{i-1}$, with equality if and only if p_{i-1} is minimal in p_i (by definition). Thus,

$$\ell_W(\underline{p}_i) - \ell_W(\underline{p}_{i-1}) \leq 0, \text{ with equality if and only if } p_{i-1} \text{ is minimal in } p_i, \tag{3.26a}$$

$$\ell(K) - \ell(K') \leq 0, \text{ with equality if and only if } K = K'. \tag{3.26b}$$

Combining these, we have

$$d_i^- = 0 \geq (\ell_W(\underline{p}_i) - \ell_W(\underline{p}_{i-1})) + (\ell(K) - \ell(K')) = \ell^-(p_i) - \ell^-(p_{i-1}), \tag{3.26c}$$

with equality if and only if $[p_{i-1}, p_i]$ is reduced.

Meanwhile, we know from Proposition 2.22 that $\overline{p_i} = \overline{p_{i-1}} * w_{I_i}$, and since I_{i-1} is in the right descent set of $\overline{p_{i-1}}$, we also have

$$\overline{p_i} = \overline{p_{i-1}} * (w_{I_{i-1}}^{-1} w_{I_i}). \tag{3.27}$$

In particular, $\ell_W(\overline{p_i}) - \ell_W(\overline{p_{i-1}}) \leq d_i^+ = \ell(I_i) - \ell(I_{i-1})$. Moreover, Corollary 3.4 states that $[p_{i-1}, p_i]$ is reduced if and only if $\overline{p_i} = \overline{p_{i-1}} \cdot (w_{I_{i-1}}^{-1} w_{I_i})$, in which case $\ell_W(\overline{p_i}) - \ell_W(\overline{p_{i-1}}) = d_i^+$. □

4 | ADDITIONAL PROPERTIES OF REDUCED EXPRESSIONS

4.1 | Comparing forward paths with subexpressions

Proposition 4.1. *Let $I = [I_0, \dots, I_d]$ be a reduced expression, and pick $0 \leq m \leq n \leq d$. The forward paths p of I , and q of $I_{[m,n]}$ satisfy*

$$\overline{p_i} = (\overline{p_m} w_{I_m}^{-1}) \cdot \overline{q_i} = \overline{p_m} \cdot (w_{I_m}^{-1} \overline{q_i}) \tag{4.1}$$

for $m \leq i \leq n$.

Proof. By Proposition 3.12, $I_{[m,n]}$ is reduced, and the same for any other interval in I_* . Pick i such that $m \leq i \leq n$. Because $I_{[0,i]}$ is a reduced expression for p_i , (3.9) centered at m gives the formula

$$\overline{p}_i = y_0 \cdots y_{m_1} \cdot w_{I_m} \cdot z_{m+1} \cdots z_i, \tag{4.2}$$

from which we deduce

$$\overline{p}_i = \overline{p}_m \cdot z_{m+1} \cdots z_i.$$

Meanwhile, (3.9) for $I_{[m,i]}$ centered at m gives

$$\overline{q}_i = w_{I_m} \cdot z_{m+1} \cdots z_i.$$

From these observations, (4.1) follows easily. □

Remark 4.2. One should not expect nice formulae which compare the forward path of a non-reduced expression with its contiguous subwords. For example, when S is finitary, take any expression $I_* = [I_0, \dots, I_d]$ and extend it to the non-reduced expression $[[S, I_0, \dots, I_d, S]]$. Every double coset in the forward path has the longest element w_S as its maximal element, so one loses effectively all the information in I_* .

4.2 | Concatenation

Proposition 4.3. *Let $I_* = [I_0, \dots, I_m] \Leftarrow p$ and $K_* = [K_0, \dots, K_n] \Leftarrow q$ be reduced expressions with $K_0 = I_m =: J$. Let*

$$I_* \circ K_* := [I_0, \dots, I_m = K_0, \dots, K_n] \tag{4.3}$$

denote the concatenation. Then

$$I_* \circ K_* \text{ is reduced} \iff \overline{p}w_J^{-1}\overline{q} = (\overline{p}w_J^{-1})\cdot\overline{q} = \overline{p}\cdot(w_J^{-1}\overline{q}) \iff \overline{p}w_J^{-1}\overline{q} = \overline{p} * \overline{q}. \tag{4.4}$$

Proof. Let us use Theorem 3.6 to prove the first implication. Write $z_i = w_{I_{i-1} \cap I_i}^{-1} w_{I_i}$ and $z'_i = w_{K_{i-1} \cap K_i}^{-1} w_{K_i}$, and use (3.9) centered at $k = 0$ to describe both \overline{p} and \overline{q} . Then

$$\overline{p} * (w_J^{-1}\overline{q}) = (w_{I_0} \cdot z_1 \cdots z_m) * (w_J^{-1} w_J \cdot z'_1 \cdots z'_n) = (w_{I_0} \cdot z_1 \cdots z_m) * (z'_1 \cdots z'_n),$$

which is exactly the product in (3.8) for $I_* \circ K_*$. Thus, the lengths add if and only if $I_* \circ K_*$ is reduced.

Now we prove the second implication. By Proposition 2.22, we know that $\overline{p} * \overline{q} = \overline{p} \cdot \overline{q}$. Since J is in the right descent set of \overline{p} and in the left descent set of \overline{q} , we have

$$(\overline{p}w_J^{-1}) * \overline{q} = \overline{p} * \overline{q} = \overline{p} * (w_J^{-1}\overline{q}).$$

Finally, Lemma 2.3 tells us that $\overline{p}w_J^{-1}\overline{q} = (\overline{p}w_J^{-1}) * \overline{q}$ if and only if $\overline{p}w_J^{-1}\overline{q} = (\overline{p}w_J^{-1})\cdot\overline{q}$. □

4.3 | Extension to the longest element

In ordinary Coxeter theory, a reduced expression for $w \in W$ can be extended to a longer reduced expression if and only if w is not the longest element. In particular, a reduced expression for w can always be extended to a reduced expression for the longest element. We prove the analogous statements here.

Proposition 4.4. *Let (W, S) be a finite Coxeter group with longest element w_S . Let $I_\bullet = [I_0, \dots, I_m = J] \rightleftharpoons p$ be reduced. Then there exists an expression $K_\bullet = [J = K_0, \dots, K_n]$ such that $I_\bullet \circ K_\bullet$ is a reduced expression for the (I_0, K_n) -coset containing w_S . In fact, we can find such a reduced expression where $K_n = K'$ if and only if $K' \subset K := \text{RD}(w_J \bar{p}^{-1} w_S)$.*

Proof. Suppose that $K_\bullet \rightleftharpoons q$ and $p * q$ has maximal element w_S . By Proposition 4.3, $I_\bullet \circ K_\bullet$ is reduced if and only if $\bar{p} w_J^{-1} \bar{q} = w_S$. In particular, $\bar{q} = w := w_J \bar{p}^{-1} w_S$. Let $K = \text{RD}(w)$. If $K' \not\subset K$, then w cannot be the maximal element of any (I_0, K) coset, so no such q exists.

Now suppose $K' \subset \text{RD}(w)$. We claim that w is maximal in some (J, K') -coset q . By (2.10a), it is equivalent to prove that $J \subset \text{LD}(w)$ and $K' \subset \text{RD}(w)$. The latter holds by assumption. We prove that $J \subset \text{LD}(w)$ using two standard facts about descent sets.

The first fact is that, for any $x \in W$ one has

$$\text{LD}(x) \sqcup \text{LD}(xw_S) = S. \tag{4.5}$$

In other words, the left descent sets of x and xw_S are complementary. This is because multiplication by w_S reverses the Bruhat order. Thus, $\text{LD}(w)$ is complementary to $\text{LD}(w_J \bar{p}^{-1})$.

The second fact is that $\text{LD}(x) = \text{RD}(x^{-1})$. Thus, $\text{LD}(w_J \bar{p}^{-1}) = \text{RD}(\bar{p} w_J^{-1})$. Since \bar{p} has J in its right descent set, $\bar{p} w_J^{-1}$ is minimal in its right W_J coset, so $J \cap \text{RD}(\bar{p} w_J^{-1}) = \emptyset$. Thus, the complement $\text{LD}(w)$ contains J .

So, let q be the (J, K') -coset such that $\bar{q} = w$. By Proposition 2.31, there is some reduced expression K_\bullet for q . It may help to write $w = (\bar{p} w_J^{-1})^{-1} w_S$, in that this accurately reflects the fact that lengths add in the expression $(\bar{p} w_J^{-1}) \cdot w = w_S$. Because the lengths add, (4.4) states that the concatenation is reduced, and that $p * q$ has maximal element w_S . □

Corollary 4.5. *Let (W, S) , I_\bullet , J , and p be as in Proposition 4.4. Then there is a reduced expression extending I_\bullet which passes through the parabolic subgroup S if and only if $\bar{p} = w_J$.*

Proof. If there is a reduced expression passing through S , then by truncating (see Proposition 3.12), we obtain a reduced expression ending in $K' = S$. By necessity, the maximal element of its endpoint is w_S . Proposition 4.4 says that such a reduced expression exists if and only if $K' \subset K := \text{RD}(w_J \bar{p}^{-1} w_S)$. But then $K = S$, so $w_J \bar{p}^{-1} w_S = w_S$, implying $\bar{p} = w_J$. □

Example 4.6. Suppose that $S = \{s, t\}$ and $m_{st} = 3$. The reduced expression $[\emptyset, s, \emptyset] \rightleftharpoons \{s\}$ can be extended to $[\emptyset, s, \emptyset, t, \emptyset, s] \rightleftharpoons \{st, sts\}$ and further to $[\emptyset, s, \emptyset, t, \emptyset, s, \emptyset] \rightleftharpoons \{sts\}$. However, $[\emptyset, s, \emptyset]$ can not be extended to any reduced expression containing the finitary subset st . Meanwhile, $[\emptyset, s] \rightleftharpoons \{e, s\}$ can be extended to the reduced expression $[\emptyset, s, st]$.

4.4 | Reversing the word

In this section, we use the fact that $(W, *, S)$ has an antiinvolution which sends $w \mapsto w^{-1}$. The notation w^{-1} represents the inverse element in the group W ; the Coxeter monoid $(W, *, S)$ does not have inverses.

Given an expression $I_* = [I_0, \dots, I_d]$, the sequence $[I_d, \dots, I_0]$ is an expression and is denoted by I_*^{-1} . If p is a (J, I) -coset, then the unique (I, J) -coset with maximal element $\overline{p^{-1}}$ agrees, as a set, with $p^{-1} := \{w^{-1} \mid w \in p\}$.

Proposition 4.7. *Let $I_* \Leftarrow p$. Then $I_*^{-1} \Leftarrow p^{-1}$, and I_*^{-1} is reduced if and only if I_* is reduced.*

Proof. From the formula (2.13), it is clear that $I_*^{-1} \Leftarrow p^{-1}$. Applying the antiinvolution $w \mapsto w^{-1}$ to the expression a_0 for \overline{p} in (3.8) (centered at the first index), we get the expression for $\overline{p^{-1}}$ (centered at the last index). The lengths add in a_0 if and only if they add after applying the anti-involution, so Theorem 3.6 implies the result. □

4.5 | Addable elements

Let I_* be a reduced expression. We call a simple reflection $s \notin I_*$ *addable* if $I_* \circ [I, Is]$ is still reduced. The next proposition states that if both s and t are addable individually, then they are addable together.

Proposition 4.8. *Suppose that $I_* := [J = I_0, I_1, \dots, I_d = I]$ is a reduced expression, and $s, t \notin I_*$ with Ist finitary. Then the following are equivalent.*

- (1) $I_* \circ [I, Is, Ist]$ is reduced.
- (2) $I_* \circ [I, Is]$ is reduced and $I_* \circ [I, It]$ is reduced.

Proof. Suppose that $I_* \circ [I, Is, Ist]$ is reduced. By Proposition 3.12, $I_* \circ [I, Is]$ is reduced. Note that $[I, Is, Ist] \Leftarrow [I, It, Ist]$ and both are reduced expressions. Then by Proposition 3.12, both $I_* \circ [I, It, Ist]$ and $I_* \circ [I, It]$ are reduced as well.

Let $I_* \Leftarrow p$. Now suppose that $I_* \circ [I, Is] \Leftarrow q$ and $I_* \circ [I, It] \Leftarrow r$ are both reduced. Note that $\underline{p} = \underline{q} = \underline{r}$. Let m be the (J, Ist) -coset containing \underline{p} . By (2.10b) applied to q and r , $LD(\underline{p}) \cap J = \emptyset$ and $RD(\underline{p}) \cap Ist = \emptyset$. Thus, (again by (2.10b)), $\underline{p} = \underline{m}$. By (2.14b) applied to $I_* \circ [I, Is]$ and $I_* \circ [I, It]$, respectively, we see that $s \notin \underline{q}^{-1}J\underline{q}$ and $t \notin \underline{r}^{-1}J\underline{r}$. Thus, the redundancy of m agrees with the redundancy of p (and q and r). Hence, $[q, m]$ is reduced, so $I_* \circ [I, Is, Ist]$ is reduced. □

4.6 | Going in

Let $I_* = [I_0, \dots, I_d]$ be an expression. We think of *going in* as the statement that, if $I_0 \subset I_1$ (respectively, $I_d \subset I_{d-1}$), then removing I_0 (respectively, I_d) does not change the maximal elements in the forward path, nor does it alter whether the expression is reduced. We state this in two separate lemmas, mostly for notational reasons.

Lemma 4.9. *Let $I_* = [I_0, \dots, I_d]$ be an expression such that $I_d \subset I_{d-1}$. Let p_* be its forward path. Let $I_{[0,d-1]} := [I_0, \dots, I_{d-1}]$, with forward path $q_* = [q_0, \dots, q_{d-1}]$. Then $p_i = q_i$ for all $0 \leq i \leq d-1$, and $\overline{p_d} = \overline{q_{d-1}}$. Moreover, $I_{[0,d-1]}$ is reduced if and only if I_* is reduced.*

Proof. The statement that $p_i = q_i$ is trivial, because paths are read from left to right. Because p_* is a forward path we know that $\overline{p_d} = \overline{p_{d-1}}$. The ‘moreover’ statement holds because there is no condition on reducedness of $[I_{d-1}, I_d]$ when $I_{d-1} \supset I_d$. □

Lemma 4.10. *Let $I_* = [I_0, \dots, I_d]$ be an expression such that $I_0 \subset I_1$. Let p_* be its forward path. Let $I_{[1,d]} := [I_1, \dots, I_d]$, with forward path $q_* = [q_1, \dots, q_d]$. Then*

$$\overline{p_i} = \overline{q_i} \tag{4.6}$$

for each $1 \leq i \leq d$. In other words, p_i is the maximal (I_0, I_i) -coset inside the (I_1, I_i) -coset q_i , for all $1 \leq i \leq d$. Moreover, $I_{[1,d]}$ is reduced if and only if I_ is reduced.*

Proof. This follows immediately from Proposition 3.5 and Theorem 3.6 because of the simple observation that $y_0 = e$. □

Remark 4.11. One could have also proven Lemma 4.9 using the observation that $z_d = e$, using a rocket launcher to pop a balloon. However, proving Lemma 4.10 directly from Definition 2.24 is more difficult.

By combining Lemmas 4.10 and 4.9, we can effectively embed any (reduced) expression inside a (reduced) expression from \emptyset to \emptyset , without changing the maximal elements of the cosets in the forward path.

Definition 4.12. Let $I_* = [J = I_0, I_1, \dots, I_d]$ be a J -expression, and fix an enumeration $J = \{s_1, \dots, s_m\}$ of J . Define the \emptyset -expression $\iota_J I_*$ to be

$$\iota_J I_* := [\emptyset, s_1, s_1 s_2, \dots, J, I_1, \dots, I_d]. \tag{4.7}$$

We think of ι_J as a function from J -expressions to \emptyset -expressions. Note that ι_J depends on the enumeration of J , though this is suppressed from the notation.

Similarly, let $I_* = [I_0, I_1, \dots, I_d = L]$ be an expression ending in L , and fix an enumeration $L = \{t_1, \dots, t_n\}$ of L . Define an expression $\kappa_L I_*$ ending in \emptyset to be

$$\kappa_L I_* := [I_0, \dots, I_{d-1}, L, \dots, t_1 t_2, t_1, \emptyset]. \tag{4.8}$$

Proposition 4.13. *Let $J, L \subset S$ be finitary, and pick enumerations of J and L . For any expression $I_* = [J = I_0, I_1, \dots, I_{d-1}, I_d = L]$, one has*

$$I_* \text{ reduced} \iff \iota_J I_* \text{ reduced} \iff \kappa_L I_* \text{ reduced} \iff \iota_J \kappa_L I_* \text{ reduced}. \tag{4.9}$$

Furthermore, the maximal elements in the forward paths of these four expressions agree where they overlap, regardless of whether they are reduced or not. Finally, the endpoints of all four expressions all have the same maximal element (while being four different kinds of double cosets).

Proof. Apply Lemma 4.10 $|J|$ times and/or Lemma 4.9 $|L|$ times. □

The upshot of Proposition 4.13 is that the J -singular Coxeter complex Cox_J will embed into the \emptyset -singular Coxeter complex Cox_\emptyset , see Proposition 8.10.

Remark 4.14. Let I_\bullet be a J -expression, and let $\iota_J I_\bullet$ and $\bar{\iota}_J I_\bullet$ be \emptyset -expressions arising from two different enumerations of J . Then $\iota_J I_\bullet$ and $\bar{\iota}_J I_\bullet$ are related by the up–up and down–down (braid) relations.

In an earlier proof of Lemma 4.10, we used the following result. Currently we do not use the result in this paper, but it seems useful so we record it for posterity.

Lemma 4.15. *Let $I, I', J, J' \subset S$ be finitary, such that $I \subset I'$ and $J \subset J'$. Let q be a (J', I) -coset. Let q' be the (J', I') -coset containing q . Now let r be the maximal (J, I) -coset contained in q , and let r' be the maximal (J, I') -coset contained in q' . Then $r \subset r'$.*

Proof. To restate the lemma, set $r = W_J \bar{q} W_I$ and $r' = W_J \bar{q}' W_{I'}$. To show that $r \subset r'$, it is equivalent to show that

$$W_J \bar{r} W_{I'} = W_J \bar{r}' W_{I'}.$$

Since $\bar{r} = \bar{q}$ and $\bar{r}' = \bar{q}'$ by assumption, we need only show that

$$W_J \bar{q} W_{I'} = W_J \bar{q}' W_{I'}. \tag{4.10}$$

From Lemma 2.15, we know that $\bar{q} W_{I'} = \bar{q}' W_{I'}$. It follows that

$$r = W_J \bar{r} W_I \subset W_J \bar{r} W_{I'} = W_J \bar{q} W_{I'} = W_J \bar{q}' W_{I'} = W_J \bar{r}' W_{I'} = r'. \tag{4.11}$$

4.7 | Going out

Let $I_\bullet = [I_0, \dots, I_d]$ be an expression. *Going out* explores the relationship between I_\bullet and $I_{[1,d]}$ when $I_0 \supset I_1$ (respectively, between I_\bullet and $I_{[0,d-1]}$ when $I_{d-1} \subset I_d$). We state this in two separate lemmas, mostly for notational reasons.

Lemma 4.16. *Let $I_\bullet = [I_0, \dots, I_d]$ be a reduced expression such that $I_0 \supset I_1$. Let p_\bullet be its forward path and K_\bullet its (left) redundancy sequence. Let $I_{[1,d]} := [I_1, \dots, I_d]$, with forward path $[q_1, \dots, q_d]$ and (left) redundancy sequence $[L_1, \dots, L_d]$. Then the expression $I_{[1,d]} = [I_1, \dots, I_d]$ is reduced. Moreover,*

$$\underline{p}_i = \underline{q}_i, \tag{4.11a}$$

$$\bar{p}_i = (w_{I_0} w_{I_1}^{-1}) \cdot \bar{q}_i, \tag{4.11b}$$

and

$$K_i = L_i \tag{4.11c}$$

for each $1 \leq i \leq d$.

There are major differences between Lemmas 4.16 and 4.10. We only have a one-sided implication for reducedness; $I_{[1,d]}$ being reduced does not imply that I_* is reduced. Also, the formulae (4.11a) and (4.11b) only hold when I_* is reduced, whereas in Lemma 4.10 we gave a formula (4.6) which held regardless of whether I_* was reduced or not. Before giving the proof, we give a counterexample when I_* is not reduced.

Example 4.17. Fix $s \in S$. The expression $I_* = [s, \emptyset, s, \emptyset]$ is not reduced, with forward path $(p_0 = \{e, s\}, \{e, s\}, \{e, s\}, p_3 = \{e, s\})$. The expression $I_{[1,3]} = [\emptyset, s, \emptyset]$ is reduced, with forward path $(q_1 = \{e\}, \{e, s\}, q_3 = \{s\})$. Note that $\underline{p}_3 = e \neq s = \underline{q}_3$, and $\overline{p}_2 = s \neq ss = w_{I_0} w_{I_1}^{-1} \overline{q}_2$. Also note that $K_2 = s$ while $L_2 = \emptyset$.

Proof of Lemma 4.16. That $I_{[1,d]}$ is reduced follows from Proposition 3.12, and (4.11b) follows from Theorem 3.6 and the observation that $y_0 = w_{I_0} w_{I_1}^{-1}$.

By (2.8), we know that

$$\overline{p}_i = (w_{I_0} w_{K_i}^{-1}) \cdot \underline{p}_i \cdot w_{I_i}, \quad \overline{q}_i = (w_{I_1} w_{L_i}^{-1}) \cdot \underline{q}_i \cdot w_{I_i}.$$

Using (4.11b) and the fact that $w_{I_0} = (w_{I_0} w_{I_1}^{-1}) \cdot w_{I_1}$, we see that

$$(w_{I_0} w_{K_i}^{-1}) \cdot \underline{p}_i \cdot w_{I_i} = \overline{p}_i = (w_{I_0} w_{I_1}^{-1}) \cdot \overline{q}_i = (w_{I_0} w_{L_i}^{-1}) \cdot \underline{q}_i \cdot w_{I_i}.$$

Dividing both sides by w_{I_i} on the right, we get

$$(w_{I_0} w_{K_i}^{-1}) \cdot \underline{p}_i = (w_{I_0} w_{L_i}^{-1}) \cdot \underline{q}_i. \tag{4.12}$$

Now we use the same method as in Lemma 3.3. Since \underline{p}_i and \underline{q}_i are in the same left W_{I_0} coset we have $q_i \subset p_i$, and thus $\underline{p}_i \leq \underline{q}_i$ in the Bruhat order. By Lemma 3.1 (applied on the left rather than the right), we have $L_i \subset K_i$, and thus $(w_{I_0} w_{K_i}^{-1}) \leq (w_{I_0} w_{L_i}^{-1})$ in the Bruhat order. But then the equality in (4.12) implies that $\underline{p}_i = \underline{q}_i$ and $L_i = K_i$, as desired. \square

As before, the analogous statement for the tail of an expression is trivial.

Lemma 4.18. *Let $I_* = [I_0, \dots, I_d]$ be a reduced expression such that $I_{d-1} \subset I_d$. Then the expression $I_{[0,d-1]} = [I_0, \dots, I_{d-1}]$ is reduced. It has the same forward path and the same redundancy sequence, for indices between 0 and $d - 1$.*

Proof. This is trivial, because paths are constructed using the left reading of an expression. \square

Once again, the converse of Lemma 4.18 fails. For example, $I_* = [\emptyset, s, \emptyset, s]$ is not reduced but $I_{[0,2]} = [\emptyset, s, \emptyset]$ is reduced.

4.8 | Our favorite reduced expressions: The high road

We now investigate what can be said about finding a reduced expression for a particular double coset p . We first explore reduced expressions which begin by adding as many simple reflections as possible.

Lemma 4.19. *Let I_* be a (reduced) expression with forward path p_* . Then $\text{LD}(\overline{p_i})$ is weakly increasing.*

Proof. By (3.6), $\overline{p_{i+1}} = \overline{p_i} * x$ for some x . By Lemma 2.5, $\text{LD}(\overline{p_i}) \subset \text{LD}(\overline{p_{i+1}})$. □

Now let p be any (J, I) -coset. We know that $J \subset \text{LD}(\overline{p})$, but this left descent set can be larger still.

Corollary 4.20. *Let p be a (J, I) -coset and K be finitary with $J \subset K$. If there is a reduced expression for p of the form $[[J, K]] \circ M_*$, for any M_* , then $K \subset \text{LD}(\overline{p})$.*

Proof. Note that $[[J, K]]$ expresses a double coset with maximal element w_K . Thus, by the previous lemma, the left descent set of p must contain K . □

So, there is no reduced expression for p which begins by adding any simple reflection not in $\text{LD}(\overline{p})$. By the next result, there is a reduced expression which does begin with $[[J, \text{LD}(\overline{p})]]$.

Proposition 4.21. *Let p be a (J, I) -coset, with $M = \text{LD}(\overline{p})$ and $N = \text{RD}(\overline{p})$. Let q be the (M, N) -coset with $\overline{q} = \overline{p}$. If I_* is any reduced expression for q , then $[[J, M]] \circ I_* \circ [[N, I]]$ is a reduced expression for p .*

Proof. First, observe that there does exist an (M, N) -coset q with $\overline{q} = \overline{p}$, since $M \subset \text{LD}(\overline{p})$ and $N \subset \text{RD}(\overline{p})$. Now we apply Theorem 3.6 to deduce that $[[J, M]] \circ I_* \circ [[N, I]]$ is reduced and expresses p . Every y_i inside $[[J, M]]$ is equal to e , and every z_i inside $[[N, I]]$ is equal to e , so this is straightforward. □

This theorem states that p has a reduced expression which gets as large as possible, maintains the same left descent set until it reaches the correct maximal element, and then descends. Combining Theorem 4.21 with Corollary 4.20, we see that the simple reflections which can be added first in a reduced expression for p are precisely $\text{LD}(\overline{p}) \setminus J$.

4.9 | Our favorite reduced expressions: The low road

Now we explore reduced expressions which begin by subtracting as many simple reflections as possible. Recall the definition of the (left) redundancy sequence from Definition 2.17.

Lemma 4.22. *Let I_* be a reduced expression with left redundancy sequence K_* . Then K_* is weakly decreasing.*

Proof. If $I_i \subset I_{i+1}$, then $K_i = K_{i+1}$, by definition of reduced, see (2.14b). If $I_i \supset I_{i+1}$, then $K_i \supset K_{i+1}$ (not necessarily strictly), which follows from Lemma 3.1. □

Corollary 4.23. *Let $I_* = [I_0, I_1, \dots, I_d]$ be a reduced expression, with left redundancy sequence K_* . If $I_0 \supset I_1 = I_0 \setminus s$, then $s \notin K_d$.*

Proof. If $I_0 \supset I_1$, then $K_1 = I_1$. Since $K_d \subset K_1$, we see that $s \notin K_d$. □

Definition 4.24. Let p be a (J, I) -coset. We say that p has *full redundancy* if $J = \underline{pI\underline{p}^{-1}}$.

Lemma 4.25. Let p be a (J, I) -coset with full redundancy. Let I_\bullet be a reduced expression for p . Then the redundancy sequence (see Definition 2.17) of I_\bullet is constant.

Proof. If p has full redundancy and I_\bullet is a reduced expression for p of width d , then $K_0 = I = K_d$. Since the redundancy sequence is weakly decreasing, it must be constant. □

Now we introduce another double coset related to a given double coset p , which in some sense contains the ‘interesting part’ of p .

Definition 4.26. Let p be a (J, I) -coset, with left redundancy $K = J \cap \underline{pI\underline{p}^{-1}}$ and right redundancy $L = I \cap \underline{p^{-1}J\underline{p}}$. The *core* of p is the (K, L) -coset p^{core} with minimal element $\underline{p^{\text{core}}} = \underline{p}$.

Lemma 4.27. The core is well-defined. The core has full redundancy.

Proof. We know that $\text{LD}(\underline{p}) \cap J = \emptyset$, which implies that $\text{LD}(\underline{p}) \cap K = \emptyset$. Similar statements hold for the right descent set. So, \underline{p} is the minimal element of its (K, L) -coset, by (2.10b). Since $\underline{pL\underline{p}^{-1}} = K$, we see that K is the left redundancy of p^{core} as well, so it has full redundancy. □

Proposition 4.28. Let p be a (J, I) -coset, with left redundancy $K = J \cap \underline{pI\underline{p}^{-1}}$ and right redundancy $L = I \cap \underline{p^{-1}J\underline{p}}$. Let $[[J, K]]$ denote any expression which removes one reflection at a time to get from J to K . Let $[[L, I]]$ denote any expression which adds one reflection at a time to get from L to I . If M_\bullet is any reduced expression for p^{core} , then

$$[[J, K]] \circ M_\bullet \circ [[L, I]] \tag{4.13}$$

is a reduced expression for p .

Proof. By (2.8), we know that $\overline{p^{\text{core}}} = \underline{p^{\text{core}}}w_L = \underline{p}w_L$. By (2.8) again, we have

$$\overline{p} = (w_Jw_K^{-1}).(\underline{p}w_L).(w_L^{-1}w_I) = (w_Jw_K^{-1}).\overline{p^{\text{core}}}.(w_L^{-1}w_I). \tag{4.14}$$

Now the result follows immediately from Proposition 4.3. □

This theorem states that p has a reduced expression which gets as small as possible, maintains the same left redundancy until it reaches the correct minimal element, and then ascends. Combining Theorem 4.28 with Corollary 4.23, we see that the simple reflections which can be removed first in a reduced expression for p are precisely $J \setminus \underline{pI\underline{p}^{-1}}$.

4.10 | Products of Coxeter groups

Every Coxeter group is a product of irreducible Coxeter groups. Thankfully, the properties of reduced expressions (as well as braid relations, and so on) really depend only on each irreducible factor.

Definition 4.29. Let (W, S) and (W', S') be two Coxeter groups. Let $I_\bullet = [I_0, I_1, \dots, I_d]$ be any expression where $I_i \subset S$ for all $0 \leq i \leq d$. Let $J \subset S'$ be finitary. We denote by $I_\bullet^{(+J)}$ the corresponding expression

$$I_\bullet^{(+J)} = [I_0 \sqcup J, I_1 \sqcup J, \dots, I_d \sqcup J] \tag{4.15}$$

inside the Coxeter group $(W \times W', S \sqcup S')$. Similarly, let p be an (I_0, I_d) -coset, where $I_0, I_d \subset S$. The $(I_0 \sqcup J, I_d \sqcup J)$ -coset inside $W \times W'$ containing $\bar{p}.w_J$ will be denoted $p^{(+J)}$.

Proposition 4.30. *Continue the notation of Definition 4.29. The following properties hold.*

- (1) $\overline{p^{(+J)}} = \bar{p}.w_J$ and $\underline{p^{(+J)}} = \underline{p}$.
- (2) $\ell^+(p^{(+J)}) = \ell^+(p)$ and $\ell^-(p^{(+J)}) = \ell^-(p)$.
- (3) $\ell^+(I_\bullet^{(+J)}) = \ell^+(I_\bullet)$ and $\ell^-(I_\bullet^{(+J)}) = \ell^-(I_\bullet)$.
- (4) $I_\bullet \preceq p$ if and only if $I_\bullet^{(+J)} \preceq p^{(+J)}$.
- (5) I_\bullet is reduced if and only if $I_\bullet^{(+J)}$ is reduced.

Proof. The basic point is that $\ell(I \sqcup J) = \ell(I) + \ell(J)$, for any $I \subset S$. One can also check that the left redundancy of $p^{(+J)}$ is just the disjoint union of J and the left redundancy of p . This immediately implies all the statements above about lengths. The statement about reduced expressions follows from Proposition 3.17. The statement that $I_\bullet \preceq p$ if and only if $I_\bullet^{(+J)} \preceq p^{(+J)}$ is straightforward from Proposition 3.5 and (3.8) in particular: the values of y_i and z_i do not differ between I_\bullet and $I_\bullet^{(+J)}$, the only difference is w_{I_k} versus $w_{I_k \sqcup J} = w_{I_k}.w_J$. □

We think of Proposition 4.30 as a another form of locality. Propositions 3.11 and 3.12 imply that, whenever $I_\bullet \preceq I'_\bullet$, then $H_\bullet \circ I_\bullet \circ K_\bullet \preceq H_\bullet \circ I'_\bullet \circ K_\bullet$, and this operation plays nicely with reducedness. Meanwhile, Proposition 4.30 implies that $I_\bullet^{(+J)} \preceq (I'_\bullet)^{(+J)}$, and this operation plays nicely with reducedness.

Remark 4.31. One can glue the categories $SC(W, S)$ for all Coxeter systems together into a large monoidal category. The objects are triples (W, S, I) where (W, S) is a Coxeter system and $I \subset S$ is finitary. There are no morphisms between (W, S, I) and (W', S', I') unless $(W, S) = (W', S')$, in which case the Hom spaces are governed by $SC(W, S)$. We place a monoidal structure on objects where $(W, S, I) \otimes (W', S', I') := (W \times W', S \sqcup S', I \sqcup I')$. On morphisms, the operation $p \mapsto p^{(+J)}$ corresponds to taking the tensor product with the identity morphisms of (W', S', J) . The second kind of locality discussed in the previous paragraph is locality for the horizontal (tensor) product, rather than for the vertical (composition) product[†].

[†] We have chosen to be imprecise in this remark because being precise obfuscates the point. Asking that $(W, S) = (W', S')$ for morphisms to exist is unusual, demanding an equality of sets $S = S'$, whereas it would be more appropriate to demand and keep track of the data of an isomorphism $(W, S) \rightarrow (W', S')$. However, this would produce additional automorphisms of a Coxeter system with Dynkin diagram symmetries. Perhaps the best way to make this construction precise is to let the objects be triples $(W, S = \{s_1, \dots, s_r\}, I)$ where S is equipped with a total order, and to declare that there are no morphisms between (W, S, I) and (W', S', I') unless the unique ordered map $S \rightarrow S'$ induces an isomorphism $W \rightarrow W'$, which we use to identify W and W' . For this construction, the monoidal structure will not be symmetric, which is perhaps for the best.

Remark 4.32. For a different style of monoidal category built from $SC(W, S)$ as the Coxeter system varies, see Section 7. It is still Proposition 4.30 which implies that the monoidal structure is well-defined.

5 | RELATING EXPRESSIONS

Recall that we write $I_* \rightleftharpoons p$ if the composition of these generating morphisms is the double coset p , see Proposition 2.22. We write $I_* \rightleftharpoons I'_*$ if I_* and I'_* express the same double coset p . Clearly, \rightleftharpoons is an equivalence relation on expressions. Recall from Proposition 3.12 that \rightleftharpoons is a *monoidal* relation, meaning that it is preserved by enlarging an expression to the left or right (that is, it can be applied to contiguous subwords). We seek a minimal set of relations which, applied locally, imply the equivalence relation \rightleftharpoons on expressions. By Proposition 2.22, this will also be a list of relations in a presentation of SC .

Before embarking on the study of relations, we provide yet more convenient shorthand for J -expressions.

Notation 5.1. If $J \subset S$ is finitary, we may write

$$[J + s - t + u - s] := [J, Js, Js \setminus t, Js u \setminus t, Ju \setminus t].$$

For this to make sense (if s, t , and u are distinct), we must have $t \in J$ and $s, u \notin J$.

Formally, we denote an expression $I_* = [J, I_1, \dots, I_d]$ by $[J \varepsilon_1 t_1 \varepsilon_2 t_2 \dots \varepsilon_d t_d]$, where $t_i \in S$ and $\varepsilon_i \in \{+, -\}$. Then $+t_i$ encodes the statement $I_i = I_{i-1} \sqcup \{t_i\}$, and $-t_i$ encodes $I_i = I_{i-1} \setminus \{t_i\}$.

Example 5.2. The expressions in Examples 3.15 and 3.16 can be denoted as

$$[\emptyset, \{s_1\}, \{s_1, s_2\}, \dots, \{s_1, \dots, s_d\}] = [\emptyset + s_1 + s_2 + \dots + s_d]$$

$$[\emptyset, s_1, \emptyset, s_2, \emptyset, \dots, s_d, \emptyset] = [\emptyset + s_1 - s_1 + s_2 - s_2 + \dots + s_d - s_d].$$

5.1 | The quadratic relation

Lemma 5.3. For any $J \subset S$ finitary and $s \in J$ we have the $*$ -quadratic relation

$$[J - s + s] \rightleftharpoons [J]. \tag{5.1}$$

The left-hand expression is not reduced, and the right-hand expression has strictly smaller length.

Proof. Every coset in the forward path of $[J - s + s]$ has underlying set equal to W_J itself. Thus, the two expressions both express the identity (J, J) -coset. The right-hand expression is obviously reduced, so it has minimal length among all expressions. The left-hand expression is not reduced at the step $+s$ since the redundancy increases. \square

In fact, the $*$ -quadratic relation will be the only non-braid relation we need to present SC , as we prove in Subsection 5.8.

5.2 | Easy braid relations

Lemma 5.4. For $s, t \in S$, the following relations hold for all J for which they make sense, and preserve length:

$$[J + s + t] \simeq [J + t + s], \tag{5.2}$$

$$[J - s - t] \simeq [J - t - s]. \tag{5.3}$$

Proof. This is obvious. □

We call (5.2) the *up-up* or *+associativity relation*, and we call (5.3) the *down-down* or *--associativity relation*.

We also have braid relations analogous to the $m_{su} = 2$ commutation relation in ordinary Coxeter theory.

Lemma 5.5. Let $I = I_1 \sqcup I_2$, where $m_{su} = 2$ for all $s \in I_1$ and $u \in I_2$. Pick some $s \in I_1$ and $u \in I_2$, and let $J = I \setminus s$. Then

$$[J + s - u] \simeq [J - u + s]. \tag{5.4}$$

This relation preserves length.

Proof. The left-hand expression in (5.4) is a reduced expression for the double coset p with maximal element w_I . Because $W_I = W_{I_1} \times W_{I_2}$ we have $w_I = w_{I_1} \cdot w_{I_2}$. We also have

$$w_{I \setminus s} = w_{I_1 \setminus s} \cdot w_{I_2},$$

and similar equations for $w_{I \setminus u}$ and $w_{I \setminus s, u}$. So,

$$w_{I \setminus s} w_{I \setminus s, u}^{-1} w_{I \setminus u} = (w_{I_1 \setminus s} w_{I_1 \setminus s}^{-1} w_{I_1}) \cdot (w_{I_2} w_{I_2}^{-1} w_{I_2 \setminus u}) = w_{I_1} \cdot w_{I_2} = w_I. \tag{5.5}$$

The lengths also add in this expression, since $\ell(I \setminus s) - \ell(I \setminus s, u) + \ell(I \setminus u) = \ell(I)$. Thus, by Theorem 3.6, the right-hand side of (5.4) is a reduced expression for p as well. □

Lemma 5.5 is the easy case of Proposition 5.14, which explores reduced expressions equivalent to $[J + s - t]$ in general.

5.3 | Two easy lemmas

If $s \notin J$ and Js is finitary, it is very easy to confirm that $[J + s]$ is a reduced expression.

Lemma 5.6. Let Js be finitary with $u \in J$ and $s \notin J$. Then $[J - u + s]$ is a reduced expression.

In the proofs in this paper, we repeatedly use the fact that an element x is minimal in its right W_I coset if and only if its right descent set is disjoint from I , if and only if $xw_I = x \cdot w_I$.

Proof. We need only check that the sequence is reduced at the $+s$ step. Let $x = w_J w_{J \setminus u}^{-1}$. By (3.5) it is enough to check $xw_{J \setminus u} = x \cdot w_{J \setminus u}$, which holds if and only if $\text{RD}(x) \cap (J \setminus u) = \emptyset$. Since $x \in W_J$, $\text{RD}(x) \subset J$. Since x is minimal in its right $W_{J \setminus u}$ -coset, $\text{RD}(x) \cap (J \setminus u) = \emptyset$. Combining these we deduce the result. \square

The second lemma is a callback to Example 2.33.

Lemma 5.7. *If $s \notin J$ and J_s is finitary, then $[J - u + s + u]$ is not reduced for all $u \in J$.*

Proof. We use $+$ -associativity and the $*$ -quadratic relation:

$$[J - u + s + u] \rightleftharpoons [J - u + u + s] \rightleftharpoons [J + s]. \tag{5.6}$$

The length decreases when the $*$ -quadratic relation is applied. \square

5.4 | Switchback relations

Throughout this section, we write w_0 for the longest element in a finite Coxeter group, when the choice of Coxeter group is understood.

When $s \notin J$ and J_s is finitary, the expression $[J + s - t] \rightleftharpoons p$ is always reduced for any $t \in J_s$ (including $t = s$). Note that $\bar{p} = w_{J_s}$ is the longest element of the parabolic subgroup W_{J_s} . There may or may not be another reduced expression for p which only factors through proper subsets of J_s . In Proposition 5.14, we give a precise criterion for when such a reduced expression of p exists, and an explicit construction of one when it does. Our *switchback* (braid) relation will relate $[J + s - t]$ with this other reduced expression. We make the switchback relation even more explicit for each finite Coxeter group (types A through H) in Section 6. Here we illustrate the dihedral case, which is nicely visualized in the singular Coxeter complex, see (1.18a) and (8.1).

Example 5.8. Let $S = \{s, t\}$ and $m_{st} = 3$ and $J = t$, so that $J_s = S$ has maximal element $w_0 = sts$. Then $[J + s - t]$ is the unique reduced expression for the (t, s) -coset containing w_0 . However, there are two reduced expressions for the (t, t) -coset containing w_0 ,

$$[t, st, t] = [J + s - s] \rightleftharpoons [J - t + s - s + t] = [t, \emptyset, s, \emptyset, t], \tag{5.7}$$

and this is an example of our switchback relation.

Example 5.9. More generally, let $S = \{s, t\}$ and $m_{st} = m < \infty$ and $J = t$. If m is odd let $u = s$ and $v = t$, and if m is even let $u = t$ and $v = s$. More abstractly, we have $u = w_0 t w_0$ and $v = w_0 s w_0$, so that $S = \{u, v\}$. Then $[J + s - v]$ is the unique reduced expression for the (t, u) -coset containing w_0 . However, there are two reduced expressions for the (t, v) -coset containing w_0 ,

$$[t, st, v] = [J + s - u] \rightleftharpoons [J - t + s - s + t - t + s \cdots - u + v] = [t, \emptyset, s, \emptyset, t, \emptyset, \dots, \emptyset, v], \tag{5.8}$$

where a total of $m - 1$ operations ‘subtract-then-add’ appear in the right-hand expression.

Many times in this section, we use the observation that, for a finite Coxeter system (W, S) , conjugation by w_0 preserves S , on which it acts by an involution.

Definition 5.10. Fix the data of $s \notin J \subset S$ with J s finitary, and $t \in Js$ (we allow $t = s$), such that $s \neq w_{Js}tw_{Js}$. Define elements $u_0, u_{-1} \in S$ and finitary subsets $I_0, I_{-1} \subset S$ by the formulae

$$u_0 = s, \quad u_{-1} = w_{Js}tw_{Js}, \quad I_0 = J, \quad I_{-1} = Js \setminus u_{-1}. \tag{5.9}$$

Now define $u_i \in S$ and $I_i \subset S$ for all $i \in \mathbb{Z}$ recursively by the formulae

$$u_{i+1} = w_{I_i}u_{i-1}w_{I_i}, \quad I_i = Js \setminus u_i. \tag{5.10}$$

We call $(u_i)_{i \in \mathbb{Z}}$ the *u-sequence* or *rotation sequence* associated to the triple (J, s, t) .

Motivation for the rotation sequence comes from the reflection representation of (W, S) , see Subsection 8.5. It is not immediately obvious that the elements u_i are actually simple reflections, or that they live in Js so that I_i is well-defined.

Lemma 5.11. *Definition 5.10 makes sense, giving a well-defined sequence (u_i) of elements of Js .*

Proof. We claim that $u_{i-1}, u_{i+1} \in I_i \subset Js$, which we prove by induction. This suffices to show that u_i and I_i are well-defined. Let us first treat the case of $i \geq 0$. Note that $u_{-1} \in Js$ since it is the conjugation by w_{Js} of a simple reflection inside Js . Since $u_{-1} \neq s$ by assumption, we must have $u_{-1} \in J = I_0$. Now, if $u_{i-1} \in I_i$, then conjugation by w_{I_i} produces another element of I_i , namely, u_{i+1} . Since $I_i = Js \setminus u_i$ we deduce that $u_{i+1} \neq u_i$. Since $I_{i+1} = Js \setminus u_{i+1}$ and $u_i \in Js$ we deduce that $u_i \in I_{i+1}$, completing the induction step.

For $i < 0$ we use the formula $u_{i-1} = w_{I_i}u_{i+1}w_{I_i}$ which is equivalent to (5.10). Working backward (that is, using decreasing induction on $i \leq -1$), the same argument will work. □

Because S is finite, eventually the pair (u_i, u_{i+1}) is equal to (u_k, u_{k+1}) for some $i \neq k$, implying that the rotation sequence is periodic.

Example 5.12. Use the same notation from Example 5.9. There is no rotation sequence associated to (J, s, v) since $w_0vw_0 = s$. The rotation sequence associated to (J, s, u) is merely the alternating sequence

$$\dots, s, t = u_{-1}, s = u_0, t = u_1, s, \dots \tag{5.11}$$

We have $u_{i+1} = u_{i-1}$ since $I_i = \{u_{i-1}\}$.

Example 5.13. Let $W = S_n$ and $S = \{s_1, \dots, s_{n-1}\}$ be the Coxeter group of type A_{n-1} . Let $s = s_i$ and $J = S \setminus s$. There is no rotation sequence associated to (J, s_i, s_{n-i}) since $w_0s_iw_0 = s_{n-i}$. The rotation sequence associated to (J, s_i, s_{n-j}) for $j \neq i$ is actually six-periodic (for all i, j, n), and $u_{i+3} = w_0u_iw_0$. We leave this as an exercise to the reader.

For example, when $n = 10$ and $i = 3$ and $j = 4$, we have sequence

$$\dots, s_4 = u_{-1}, s_3 = u_0, s_9, s_6, s_7, s_1, s_4 = u_5, s_3 = u_6, \dots \tag{5.12}$$

Moreover, $u_{i+3} = w_0u_iw_0$.

Proposition 5.14 (Switchback relations). *Let $s \notin J \subset S$ with J s finitary, and $t \in Js$. Let $I = Js \setminus t$. Let p be the (J, I) -coset containing w_{Js} .*

- (1) If $s = w_{J_s}tw_{J_s}$, then $[J + s - t]$ is the unique reduced expression of p .
- (2) If $s \neq w_{J_s}tw_{J_s}$, then p has exactly two reduced expressions, given in (5.14).

To elaborate in case (2), let (u_i) be the rotation sequence associated with (J, s, t) , see Definition 5.10. Let $I_i = Js \setminus u_i$ as in Definition 5.10, and $L_i = Js \setminus \{u_i, u_{i-1}\}$. Let δ be the largest integer such that

$$[I_0, L_1, I_1, \dots, L_k, I_k] \tag{5.13}$$

is reduced for all $1 \leq k \leq \delta$. Then $u_\delta = t$ and

$$\begin{aligned} [J + s - t] &\Leftrightarrow [J - u_1 + s - u_2 + u_1 - u_3 + u_2 - u_4 \cdots - u_{\delta-1} + u_{\delta-2} - t + u_{\delta-1}] \\ &= [J = I_0, L_1, I_1, L_2, I_2, L_3, I_3, L_4, \dots, L_\delta, I_\delta = I]. \end{aligned} \tag{5.14}$$

Moreover, $u_{\delta+1} = w_{J_s}sw_{J_s}$.

We prove the result in the next section, after some remarks and easy lemmas. First we discuss the number δ associated to the triple (J, s, t) .

Remark 5.15. Note that $u_\delta = t = w_{J_s}u_{-1}w_{J_s}$, and $u_{\delta+1} = w_{J_s}u_0w_{J_s}$. Using the recursive definition of the u -sequence we see that

$$u_{i+\delta+1} = w_{J_s}u_iw_{J_s} \tag{5.15}$$

for all $i \in \mathbb{Z}$. Since w_{J_s} is an involution, (5.15) implies that the u -sequence is $2(\delta + 1)$ -periodic. When w_{J_s} is central in W_{J_s} , then the u -sequence must actually be $(\delta + 1)$ -periodic. However, the period of the u -sequence can be smaller still, as we already saw in dihedral type where it was 2-periodic. This also happens sporadically in type EFH . In particular, δ is *not* the smallest index for which $u_\delta = t$, or for which (5.15) holds.

Remark 5.16. In Section 6, we give explicit formulae for δ and the sequence $(u_1, u_2, \dots, u_\delta)$, when J_s is an irreducible finite Coxeter group. In classical types, the value of δ is relatively small. For example, $\delta = 2$ in type A , and $2 \leq \delta \leq 3$ for types BCD .

Remark 5.17. Lemma 5.7 gives an example where $\delta = 1$. Conversely, every example where $\delta = 1$ will be a consequence of Lemma 5.7, as we justify shortly.

Remark 5.18. Within the reduced expression in (5.14), one has contiguous subwords of the form

$$[L_k, I_k, L_{k+1}] = [L_k + u_{k-1} - u_{k+1}]. \tag{5.16}$$

While also being of the form $[J + s - t]$ for a different triple (J, s, t) , this subword will always fall into the regime of Proposition 5.14(1), because $u_{k+1} = w_{I_k}u_{k-1}w_{I_k}$ by construction. In particular, we cannot apply the switchback relation to subwords of the right-hand expression in (5.14).

Remark 5.19. It sometimes helps to think about the switchback relation as being associated not with the triple (J, s, t) where $s \neq w_{J_s}tw_{J_s}$, but with the triple (J, s, u_1) where $s \notin J$ and $u_1 \in J$ (so $s \neq u_1$). If we start at J one might begin a reduced expression by adding s or by removing u_1 . The switchback relation explains how to extend these beginnings to reduced expressions for the same coset.

Now let us continue Example 5.13.

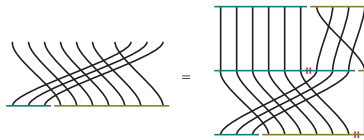
Example 5.20. Recall that when $n = 10$ and $s = s_3$ and $t = s_6$ we have $u_1 = s_9$, $u_2 = t$. The switchback relation is

$$[J + s_3 - s_6] \Leftrightarrow [J - s_9 + s_3 - s_6 + s_9]. \tag{5.17}$$

The right-hand side is a reduced expression if and only if $w_0 = w_{\hat{3}}.(w_{\hat{39}}^{-1}w_{\hat{9}}).(w_{\hat{69}}^{-1}w_{\hat{6}})$, or equivalently, if and only if

$$w_{\hat{3}}^{-1}w_0 = (w_{\hat{39}}^{-1}w_{\hat{9}}).(w_{\hat{69}}^{-1}w_{\hat{6}}). \tag{5.18}$$

Here is the picture in S_{10} which justifies this equality.



The next lemma will simplify our notation.

Lemma 5.21. Suppose that p is a parabolic double coset in (W, S) and $M \subset S$ is finitary such that $\bar{p} \in W_M$. Then in any expression $I_* = [I_0, \dots, I_d]$ for p , $I_i \subset M$ for all $0 \leq i \leq d$.

Proof. If $x \leq y$ in the Bruhat order on W , and the simple reflection s appears in a reduced expression for x , then it also appears in any reduced expression for y . Letting p_i be the forward path of I_* , we know that $\overline{p_{i-1}} \leq \overline{p_i}$ in the Bruhat order for all i , and thus $\overline{p_i} \leq \overline{p} = \overline{p_d}$ for all i . If $I_i \not\subset M$, then $\overline{p_i}$ has some $s \notin M$ in its right descent set, and this s must appear in a reduced expression for \overline{p} , contradicting the assumption $\overline{p} \in W_M$. \square

By Lemma 5.21 with $M = Js$, it suffices when studying Proposition 5.14 to restrict to the parabolic subgroup W_{Js} . Below we adopt the convention that $S = Js$ and W is finite and $\bar{p} = w_0$. Thus, $J = S \setminus s$ and $I = S \setminus t$. Using the results of Subsection 5.6 we could reduce further to the case when W is irreducible, though the assumption of irreducibility does not simplify the proof so we do not make it here.

5.5 | Proof of the switchback relations

Before embarking on the proof, one more lemma.

Lemma 5.22. Let $s, t \in S$ for S finitary (possibly $s = t$). For $J = S \setminus s$ and $I = S \setminus t$ and $p = W_J w_0 W_I$, we have $J \cap \underline{pI\underline{p}}^{-1} = J$ if and only if $\underline{p} = w_0 w_t^{-1}$ if and only if $s = w_0 t w_0$.

Proof. Clearly, $w_0 = \bar{p}$. The formula (2.8) implies that $\underline{p}w_t = \bar{p}$ if and only if $J \cap \underline{pI\underline{p}}^{-1} = J$. If $\underline{p}w_t = w_0$, then $J \cap w_0 I w_0 = J \cap \underline{pI\underline{p}}^{-1} = J$, which implies that $w_0 t w_0 = s$. Conversely, if

$w_0tw_0 = s$, then $w_0Jw_0 = I$. Thus, there is an equality of sets $W_Jw_0 = w_0W_I$, and $p = w_0W_I$. Consequently $\underline{p} = w_0w_I^{-1}$. □

Proof of Proposition 5.14(1). By Lemma 5.21, we can assume $Js = S$. We write $p = W_Jw_0W_I$ and note that $[J + s - t] \asymp p$.

Suppose that there exists a reduced expression $J_\bullet = [J = J_0, \dots, J_{m-1} = K, J_m = I]$ for p ending in $+u$, that is, satisfying $I = Ku$. The fact that $[K, I]$ is reduced implies that

$$J \cap \underline{pK\underline{p}}^{-1} = J \cap \underline{pI\underline{p}}^{-1}. \tag{5.19}$$

The size of K is one less than the size of I , which equals the size of J . Thus, $J \cap \underline{pK\underline{p}}^{-1}$ is necessarily a proper subset of J . By Lemma 5.22, this implies that $s \neq w_0tw_0$. Taking the contrapositive, if $s = w_0tw_0$, then there is no reduced expression for p ending in $+u$ for any $u \in S$.

To finish proving (1), we need to prove that p has no reduced expressions aside from $[J + s - t]$ which end in $-u$ for some $u \in S$. Of course, since $I = S \setminus t$ one must have $u = t$, and the penultimate parabolic in the expression is S itself. Preceding this is a reduced expression for the unique (J, S) -coset q . There are a number of easy ways to argue that $[J + s]$ is the unique reduced expression for q ; here is one. We know that $\underline{q} = e$, whereas the minimum element \underline{p}_i strictly increases whenever a simple reflection is removed in a reduced expression. So, we can only add simple reflections and the only possibility is $[J + s]$. □

Now we introduce more lemmas on route to proving Proposition 5.14(2). The first lemma helps to justify the importance of the first term u_{+1} in the rotation sequence.

Lemma 5.23. *Let S be finitary, with $s, t \in S$ (possibly equal) satisfying $w_0tw_0 \neq s$. Let $J = S \setminus s$ and $I = S \setminus t$. Let p be the (J, I) -coset containing the longest element w_0 of (W, S) . Then the left redundancy of p is $J \setminus u$, where $u = w_Jw_0tw_0w_J$.*

Proof. Let $u_{-1} = w_0tw_0$, and $u = u_{+1} = w_Ju_{-1}w_J$. Note that $u_{+1} \in J$, that is, $u_{+1} \neq s$. This is because $u_{-1} \neq s$ by assumption, so $u_{-1} \in J$, and conjugation by w_J preserves J .

Let L be the right redundancy of p . Then (2.8) states that $\bar{p} = w_J\underline{p}w_Lw_I$. Since $\bar{p} = w_0$, we deduce that

$$\underline{p} = w_Jw_0w_Iw_L. \tag{5.20}$$

By Kilmoyer’s theorem (Lemma 2.14), the left redundancy K is equal to $J \cap \underline{p}W_I\underline{p}^{-1}$. Since $L \subset I$ we have

$$K = J \cap w_Jw_0W_Iw_0w_J. \tag{5.21}$$

Now $I = S \setminus t$, so $w_0W_Iw_0 = W_{S \setminus u_{-1}}$. Also, $w_J(J \setminus u_{-1})w_J = J \setminus u_{+1}$. Thus,

$$J \setminus u_{+1} \subset J \cap w_JW_{S \setminus u_{-1}}w_J = K. \tag{5.22}$$

By Lemma 5.22, we can not have $J = K$, so we must have $J \setminus u_{+1} = K$. □

Corollary 5.24. *Continue the notation of Lemma 5.23. For any $v \in J$, there is a reduced expression for p of the form $[J - v] \circ M$, if and only if $v = u$.*

Proof. This follows from Theorem 4.28 and Corollary 4.23. □

To help cement these ideas in the reader’s mind, let us give another proof of Corollary 5.24, this time using Proposition 4.4 instead.

Second proof of Corollary 5.24. Proposition 4.4 states that $[J - v]$ can be extended to a reduced expression for p if and only if I is contained in the right descent set of $x := w_{J \setminus v} w_J w_0$. In other words, we must show that $t \notin \text{RD}(x)$, or equivalently $xt > x$. Right multiplication by w_0 reverses the Bruhat order, so

$$w_{J \setminus v} w_J u_{-1} = w_{J \setminus v} w_J w_0 t w_0 < w_{J \setminus v} w_J.$$

Both sides of this inequality live in W_J , and right multiplication by w_J reverses the Bruhat order on W_J , so

$$w_{J \setminus v} u_{+1} = w_{J \setminus v} w_J u_{-1} w_J > w_{J \setminus v}.$$

This means that $u_{+1} \notin J \setminus v$. Since $u_{+1} \in J$, we deduce that $v = u_{+1}$. □

The second lemma introduces a ‘rotation’ operation on reduced expressions for certain double cosets containing w_0 .

Lemma 5.25. *Let S be finitary, containing elements s, t with $w_0 t w_0 = s' \neq s$ and $w_0 s w_0 = t' \neq t$. Let $J = S \setminus s$ and $I = S \setminus t$. Let $u = w_J w_0 t w_0 w_J$ as in Lemma 5.23. If $[J - u + s] \circ M_\bullet$ is a reduced expression for p , the (J, I) -coset containing w_0 , then $M_\bullet \circ [I - t' + t]$ is a reduced expression for p' , the $(S \setminus u, S \setminus t')$ -coset containing w_0 .*

Proof. By locality, $M_\bullet \Leftarrow r$ and $[J - u + s]$ are both reduced expressions. Let $J' = S \setminus u$ and $L = S \setminus \{s, u\}$. We have

$$w_0 = \bar{p} \stackrel{(4.4)}{=} w_J w_L^{-1} \bar{r}. \tag{5.23}$$

Moreover, the lengths add, in that $\ell(w_0) = \ell(J) - \ell(L) + \ell(\bar{r})$. Now $L = J \setminus u$ and, by Lemma 5.23, we have $w_J u w_J = s'$. Hence, $w_L w_J = w_J w_{L'}$ where $L' = S \setminus \{s, s'\}$. So

$$\bar{r} = w_L w_J w_0 = w_J w_{L'} w_0 = w_0 w_{w_0 J w_0} w_{w_0 L' w_0}. \tag{5.24}$$

Letting $I' = w_0 J w_0 = S \setminus t'$ and $L'' = w_0 L' w_0 = S \setminus \{t, t'\}$ we get

$$w_0 = \bar{r} w_{L''}^{-1} w_{I'}, \tag{5.25}$$

and the lengths add in that $\ell(w_0) = \ell(\bar{r}) - \ell(L'') + \ell(I')$. By Proposition 4.3, this implies that $M_\bullet \circ [I - t' + t]$ is reduced. □

Now we prove the switchback relation.

Proof of Proposition 5.14(2). As before, by Lemma 5.21 we can assume $J s = S$. Suppose $s \neq w_0 t w_0 =: u_{-1}$. We leave it to the reader to verify that $[J + s - t]$ is the unique reduced expression for p which extends $[J + s]$. The other reduced expressions begin with $[J - v]$ for some $v \in J$. By

Corollary 5.24, we must have $v = u_1$. Moreover, by Corollary 5.24, there exists a reduced expression for p which extends $[J - u_1]$. In the following paragraphs, we give an algorithm to extend $[J - u_1]$ toward a reduced expression of p .

By Lemma 5.23 and Corollary 4.23, there is no reduced expression for p which extends $[J - u_1 - v]$ for any $v \in J$. So, the reduced expression must continue by adding some simple reflection not in $J \setminus u_1$, and there are only two choices: s and u_1 . Also, $[J - u_1 + u_1]$ is not reduced by Lemma 5.3. Thus, p must have a reduced expression of the form $[J - u_1 + s] \circ M_*$ for some M_* .

Lemma 5.6 verifies that $[J - u_1 + s]$ is reduced. It is possible that $[J - u_1 + s]$ is a reduced expression for p , at which point M_* has width zero and our algorithm terminates. Thus, assume that M_* has positive width. We note that M_* cannot begin by adding a simple reflection, since the only available choice is u_1 , and $[J - u_1 + s + u_1]$ is not reduced by Lemma 5.7. So, M_* begins $[I_1 - v]$ for some $v \in I_1 = S \setminus u_1$.

By Lemma 5.25 (and using the notation therein), $M_* \circ [I - t' + t]$ is a reduced expression for p' , the $(I_1, S \setminus t')$ -coset containing w_0 . Since $w_0 t' w_0 = s \neq u_1$, we are back exactly in the situation of Proposition 5.14(2) again! This time u_1 is playing the role of s , and t' plays the role of t . Thus, the new u_0 is the old u_1 , and the new $u_{-1} = w_0 t' w_0 = s$ is the old u_0 .

Again, M_* begins $[I_1 - v]$ for some v . Repeating the arguments used already in this proof (citing Corollary 5.24, and so on) we see that M_* necessarily has the form $[I_1 - u_2 + u_1] \circ N_*$ for some N_* . So,

$$[J - u_1 + s - u_2 + u_1] \circ N_* \rightleftharpoons p, \quad [I_1 - u_2 + u_1] \circ N_* \circ [I - t' + t] \rightleftharpoons p'.$$

Rotating again with Lemma 5.25, we see that

$$N_* \circ [I - t' + t - t'' + t']$$

is a reduced expression for a double coset containing w_0 , and $t'' = w_0 u_1 w_0$.

When N_* is width zero our algorithm terminates. Otherwise we can continue, repeating until we have exhausted the reduced expression. Note that the width of M_* is two less than the original reduced expression, the width of N_* is two smaller still, and so on. This process will eventually terminate (let δ be the number of steps required), resulting in a reduced expression for p of the form (5.14). Our choices were constrained at every step, so this is the unique reduced expression for p which extends $[J - u_1]$.

Necessarily we have $u_\delta = t$. Furthermore, we know that the rotation

$$[I_1 - u_2 + u_1 - \dots - t + u_{\delta-1} - t' + t]$$

is a reduced expression for p' , yielding $u_{\delta+1} = t' = w_0 s w_0$. □

5.6 | Switchback and products of Coxeter systems

Lemma 5.26. *Let (W, S) and (W', S') be finite Coxeter systems, with $J = S \sqcup S' \setminus s$ and $I = S \sqcup S' \setminus t$ inside the Coxeter system $(W \times W', S \sqcup S')$. If $s, t \in S$, then the rotation sequence associated to (J, s, t) agrees with the rotation sequence associated to $(J \cap S, s, t)$. Moreover, the relation (5.14) for (J, s, t) follows from the relation (5.14) for $(J \cap S, s, t)$ via the operation $(-)^{(+S')}$, see Proposition 4.30.*

Proof. This is a straightforward exercise in the definitions. □

Lemma 5.27. *Let (W, S) and (W', S') be finite Coxeter systems, with $J = S \sqcup S' \setminus s$ and $I = S \sqcup S' \setminus t$ inside the Coxeter system $(W \times W', S \sqcup S')$. If $s \in S$ and $t \in S'$, then the rotation sequence associated to (J, s, t) is*

$$\dots, u_{-1} = w_{S'} t w_{S'}, \quad u_0 = s, \quad u_1 = t, \quad u_2 = w_S s w_S, \quad \dots \tag{5.26}$$

and $u_{i+2} = w_{S \sqcup S'} u_i w_{S \sqcup S'}$. The relation (5.14) holds with $\delta = 1$, because it agrees with (5.4).

Proof. Again, this is straightforward. □

Lemma 5.28. *Let (W, S) be a finitary Coxeter system, and $s, t \in S$. The following are equivalent.*

- (1) *There is a switchback relation with $\delta = 1$ for the triple $(S \setminus s, s, t)$.*
- (2) $w_S = w_{S \setminus s} \cdot (w_{S \setminus s, t}^{-1} w_{S \setminus t})$.
- (3) $\ell(S) = \ell(S \setminus s) - \ell(S \setminus s, t) + \ell(S \setminus t)$.
- (4) *s and t belong in different irreducible components of S , that is, we are in the setting of Lemma 5.27.*

Proof. Let $J = S \setminus s$. Lemma 5.27 says that (4) implies (1). Meanwhile, (1) says that $[J - t + s]$ is a reduced expression for the double coset containing w_S , which by Theorem 3.6 implies (2). By taking lengths, (2) implies (3). It remains to prove that (3) implies (4).

If s and t were in the same irreducible component of S (call it L) and $S = L \sqcup L'$, then $\ell(S) = \ell(L) + \ell(L')$, $\ell(S \setminus s) = \ell(L \setminus s) + \ell(L')$, and so on. Thus, (3) implies that

$$\ell(L) = \ell(L \setminus s) - \ell(L \setminus s, t) + \ell(L \setminus t), \tag{5.27}$$

which reduces to the case when $W = W_L$ and the Coxeter group is irreducible. One can now check all cases to verify that (5.27) never occurs. □

Remark 5.29. Relatedly, in Section 6, we provide explicitly (by checking all cases) the switchback relations for finitary irreducible Coxeter systems. One always has $\delta \geq 2$. The previous two lemmas reduce the switchback relation to irreducible Coxeter systems, so we can directly confirm that (1) and (4) are equivalent.

5.7 | Matsumoto’s theorem

We prove the singular Matsumoto theorem by repeated use of the techniques developed in Subsections 4.8 and 4.9.

Theorem 5.30. *Let (W, S) be a Coxeter group and let p be a double coset. The braid relations from Lemma 5.4 and Proposition 5.14 generate all relations between reduced expressions for p .*

Proof. Suppose that p is a (J, I) -coset. Let $K_\bullet = [J = K_0, K_1, \dots, K_{d-1}, K_d = I]$ and $L_\bullet = [J = L_0, L_1, \dots, L_{c-1}, L_c = I]$ be two reduced expression for p . We prove by induction on $\ell(p)$ (not on the widths d and c) that K_\bullet and L_\bullet are related by the braid relations. The base case is trivial: the only expressions of length zero are identity expressions $[J]$. For the rest of this proof we write $K_\bullet \sim L_\bullet$ to indicate that K_\bullet and L_\bullet are related by the braid relations.

Let $K'_\bullet = K_{[1,d]} \Leftarrow q$ and $L'_\bullet = L_{[1,c]} \Leftarrow r$; these are both reduced expressions by locality. If $K_1 = L_1$, then $q = r$, since Proposition 4.3 implies

$$w_J w_{J \cap K_1}^{-1} \bar{q} = \bar{p} = w_J w_{J \cap L_1}^{-1} \bar{r}.$$

The double coset q also has shorter length than p , since extending a reduced expression increases length. By induction, $K'_\bullet \sim L'_\bullet$, hence $K_\bullet \sim L_\bullet$.

We now examine three cases where $K_1 \neq L_1$. In the first case, $K_1 = J \setminus s$ and $L_1 = J \setminus t$ for $s \neq t \in J$. By Corollary 4.23, both s and t are in $J \setminus \underline{pI\underline{p}^{-1}}$. But Proposition 4.28 implies the existence of a reduced expression $M_\bullet \Leftarrow p$ starting with $[\underline{J} \bar{s} - t] = [J, K_1, K_1 \cap L_1]$. Applying the down-down relation, there is also a reduced expression $N_\bullet \Leftarrow p$ starting with $[J - t - s] = [J, L_1, K_1 \cap L_1]$. By induction (and the previous paragraph), we have

$$K_\bullet \sim M_\bullet \sim N_\bullet \sim L_\bullet.$$

In the second case, $K_1 = Js$ and $L_1 = Jt$ for $s, t \notin J$ with $s \neq t$. We argue similarly, this time using Corollary 4.20 and Proposition 4.21. Since both s and t live inside $\text{LD}(\bar{p}) \setminus J$, we are permitted to add both and interpolate between K_\bullet and L_\bullet .

In the final case, $K_1 = J \setminus u$ and $L_1 = Js$ for some $u \in J$ and $s \notin J$. Let $t \in J$ be such that the rotation sequence of (J, s, t) has $u_1 = u$. If we can prove that $[J + s - t] \circ M_\bullet$ is a reduced expression for p for some M_\bullet , then we will obtain a chain

$$L_\bullet \sim [J + s - t] \circ M_\bullet \sim [J - u + s - u_2 + u - \dots - t + u_{\delta-1}] \circ M_\bullet \sim K_\bullet.$$

The middle relation is the switchback relation, while the other two relations follow by induction.

The formula for t is straightforward from Definition 5.10, we have $t = w_{L_1} w_J u w_J w_{L_1}$. However, we must rule out the possibility that $w_J u w_J = s$, so that no rotation sequence exists. This is easy, since by assumption $u \in J$ (so $w_J u w_J \in J$) while $s \notin J$.

As before let $L'_\bullet = L_{[1,d]}$ express the (L_1, I) -coset r . If $t \in L_1 \setminus \underline{rI\underline{r}^{-1}}$, then there is some reduced expression for r which starts with $[L_1 - t]$, by Proposition 4.28. Composing this with $[J, L_1]$, we get the desired reduced expression for p starting with $[J + s - t]$. Clearly, $w_J u w_J \in J \subset L_1$ so $t = w_{L_1} w_J u w_J w_{L_1} \in L_1$. We need only prove that $t \notin \underline{rI\underline{r}^{-1}}$. The other thing we know, from Corollary 4.23, is that $u \in J \setminus \underline{pI\underline{p}^{-1}}$. The rest of the proof is tricks with conjugation.

By Lemma 2.14, it suffices to examine $J \cap \underline{pW_I\underline{p}^{-1}}$ and $L_1 \cap \underline{rW_I\underline{r}^{-1}}$. By (2.8), the right W_I coset of \underline{p} agrees with the right W_I coset of $w_J^{-1}\bar{p}$, and similarly for r . Also, by Theorem 3.6, $\bar{r} = \bar{p}$ (since $w_J w_{J \cap L_1}^{-1} = e$). So

$$\begin{aligned} u \notin \underline{pW_I\underline{p}^{-1}} &\iff u \notin w_J \bar{p} W_I \bar{p}^{-1} w_J \iff u \notin w_J \bar{r} W_I \bar{r}^{-1} w_J \\ &\iff w_J u w_J \notin \bar{r} W_I \bar{r}^{-1} \iff t = w_{L_1} w_J u w_J w_{L_1} \notin w_{L_1} \bar{r} W_I \bar{r}^{-1} w_{L_1} \iff t \notin \underline{rW_I\underline{r}^{-1}}. \end{aligned} \tag{5.28}$$

□

5.8 | Presentation theorem

Let us recall the list of relations. We have the \pm -associativity relations from Lemma 5.4

$$[J + s + t] \rightleftharpoons [J + t + s], \quad [J - s - t] \rightleftharpoons [J - t - s], \tag{5.29a}$$

the switchback relation (details are in Proposition 5.14)

$$[J + s - t] \rightleftharpoons [J - u_1 + s - u_2 + u_1 - \dots - t + u_{\delta-1}], \tag{5.29b}$$

and the $*$ -quadratic relation

$$[J - s + s] \rightleftharpoons [J]. \tag{5.29c}$$

The \pm -associativity and switchback relations are called braid relations.

Theorem 5.31. *Let I_\cdot be a non-reduced expression. By applying the braid relations, and applying the $*$ -quadratic relation in the length-reducing direction, one can relate I_\cdot to a reduced expression of strictly smaller length.*

Proof. Let $I_\cdot = [I_0, \dots, I_d]$. We prove the result by induction on the length $\ell := \ell(I_\cdot)$ of I_\cdot . Any expression of length zero also has width zero and is reduced. This handles the base case.

Induction on length allows us to assume that all proper subexpressions of I_\cdot are reduced. Suppose that $I_{[m,n]}$ is not reduced for a proper interval $[m, n]$ (meaning $m \neq 0$ or $n \neq d$). Because $\ell(I_{[m,n]}) < \ell$, we can apply relations to $I_{[m,n]}$ to obtain a reduced expression of smaller length. Applying the same relations within I_\cdot , produces an expression I'_\cdot with smaller length. By induction we can also reduce I'_\cdot , so we win.

Now suppose $I_{[0,d-1]}$ is a reduced expression for some (J, I_{d-1}) -coset q where $J = I_0$. Note that I_\cdot is reduced unless $I_{d-1}t = I_d$ for some $t \in I_d$. We can also assume that $d \geq 2$, since any expression of width $d \leq 1$ is reduced.

If the right redundancy $L = \underline{q}^{-1}J\underline{q} \cap I_{d-1}$ of q is a proper subset of I_{d-1} , then Proposition 4.28 implies that q has a reduced expression ending in $+s$ for some $s \in I_{d-1} \setminus L$. By Theorem 5.30, we can get from $I_{[0,d-1]}$ to this expression, say $[J_\cdot + s]$, using braid relations. By Proposition 3.17, we also have $\ell(I_{[0,d-1]}) = \ell([J_\cdot + s])$ and thus $\ell = \ell([J_\cdot + s + t])$. So, we can assume that $I_\cdot = [J_\cdot + s + t]$ for some reduced expression J_\cdot . The braid relation (5.2) produces $I'_\cdot := [J_\cdot + t + s]$, an expression of length ℓ . If $I'_{[0,d-1]}$ is not reduced, then we conclude by induction as above. If $[J_\cdot + s]$ and $[J_\cdot + t]$ are both reduced, then by Proposition 4.8 so is I_\cdot , a contradiction.

So, we assume that $L = I_{d-1}$ henceforth. In this case,

$$\bar{q} = w_J \cdot \underline{q} \tag{5.30}$$

by (2.8). In particular, (5.30) implies $\text{RD}(\underline{q}) \subset \text{RD}(\bar{q})$, see Lemma 2.5.

Suppose that $t \in \text{RD}(\bar{q})$. Then by Proposition 4.21, there is a reduced expression for q ending in $-t$. Again, by Theorem 5.30 we can use braid relations to assume that $I_\cdot = [J_\cdot - t + t]$ for some reduced expression J_\cdot . Now the $*$ -quadratic relation reduces I_\cdot to a reduced expression.

So, suppose that $t \notin \text{RD}(\bar{q})$ and $L = I_{d-1}$. We claim that I_\cdot is reduced, a contradiction which concludes the proof. If $I_\cdot \rightleftharpoons p$, then p is the (J, It) -coset containing q . But $t \notin \text{RD}(\underline{q})$, so by (2.10b) we know that $\underline{q} = \underline{p}$. Moreover, $\underline{q}^{-1}W_J\underline{q} = \bar{q}^{-1}W_J\bar{q}$ by (5.30). If $t \in \bar{q}^{-1}W_J\bar{q}$, then $\bar{q}t = x\bar{q}$ for

some $x \in W_J$. But $x\bar{q} \leq \bar{q} < \bar{q}t$, a contradiction. So, $t \notin \bar{q}^{-1}W_J\bar{q}$, and the redundancy of p equals that of q . Hence, $[p, q]$ is reduced. □

We warn the reader that, in the process of replacing a non-reduced expression I_* with a reduced expression of shorter length, one may increase the width of the expression. This is because the switchback relations do not change the length of an expression, but do change the width.

Theorem 5.32. *The relations from (5.29) give a presentation of the singular Coxeter monoid.*

Proof. We need only prove that two expressions for the same double coset are related. By Theorem 5.31, each of them can be related to a reduced expression. By Theorem 5.30, any two reduced expressions are related. □

6 | SWITCHBACK RELATIONS

In this section, we describe explicitly all relations guaranteed by and classified in Proposition 5.14. By Lemmas 5.26 and 5.27, we can reduce to the case where (W, S) is a finite irreducible Coxeter group.

In all cases, we know that a switchback relation must exist. We explicitly state the relevant part of the rotation sequence. The interesting thing to explain is what δ is, that is, how far along the rotation sequence to travel. The algorithm from Subsection 5.5 states that we should continue working along the rotation sequence until one has expressed a double coset containing w_0 . We can thus determine if we have gone the right distance merely by computing the length of the purported right-hand side of (5.14). We did this by computer in types EFH . We give as many details in the computations below as we find interesting.

The following lemma cuts in half the amount of work needed.

Lemma 6.1. *Suppose that $s, t \in S$ and $w_0tw_0 \neq s$. If the switchback relation for the triple $(S \setminus s, s, t)$ has the form*

$$[(S \setminus s) + s - t] \rightleftharpoons [(S \setminus s) - c_1 + s - c_2 + c_1 - \dots - t + c_{\delta-1}], \tag{6.1a}$$

then the switchback relation for the triple $(S \setminus t, t, s)$ has the form

$$[(S \setminus t) + t - s] \rightleftharpoons [(S \setminus t) - c_{\delta-1} + t \dots - c_1 + c_2 - s + c_1]. \tag{6.1b}$$

In other words, the rotation sequences in these two cases are reversed.

Proof. Writing the expressions in (6.1) using the notation $[I_0, I_1, \dots, I_d]$ rather than the notation of addition and subtraction, we see that the left-hand side of (6.1a) is just the reverse of the left-hand side of (6.1b), and similarly with the right-hand side. Now the result follows from Proposition 4.7. □

One can also deduce many switchback relations from others using Lemma 5.25. One can find a detailed discussion of some examples of both Lemmas 5.25 and 6.1 in type E_7 below.

6.1 | Switchback in type A

Let (W, S) be of type A_n , with $S = \{s_1, \dots, s_n\}$ in the usual numbering. We have for each $1 \leq a, b \leq n$ such that $a + b \neq n + 1$ (that is, $s_a \neq w_0 s_b w_0$),

$$[J - s_c + s_a - s_b + s_c] \rightleftharpoons [J + s_a - s_b] \quad (6.2)$$

with $c = a + b$ (if $a + b \leq n$) or $c = a + b - n - 1$ (if $a + b > n + 1$), where $J = S \setminus \{s_a\}$.

Confirming the value of c is straightforward, so we verify that the lengths of both sides of (6.2) agree. By symmetry, we need only treat the case $a + b \leq n + 1$.

Since $W_S = S_{n+1}$ and $W_J = S_a \times S_{n+1-a}$, we have

$$\ell(S) - \ell(J) = a(n + 1 - a),$$

the length of the relative longest element. Similarly, $W_{S \setminus s_c} = S_{a+b} \times S_{n+1-a-b}$, and $W_{S \setminus s_a, s_c} = S_a \times S_b \times S_{n+1-a-b}$. The relative longest element agrees with the relative longest element of S_{a+b} over $S_a \times S_b$ so we have

$$\ell(S \setminus s_c) - \ell(S \setminus s_a, s_c) = ab.$$

Finally, $W_{S \setminus s_b} = S_b \times S_{n+1-b}$ and $W_{S \setminus s_b, s_c} = S_b \times S_a \times S_{n+1-a-b}$, so

$$\ell(S \setminus s_b) - \ell(S \setminus s_a, s_c) = a(n + 1 - a - b).$$

That the lengths of the two expressions in (6.2) are equal is equivalent to

$$a(n + 1 - a) = ab + a(n + 1 - a - b).$$

Remark 6.2. If one knew that $\delta = 2$ and $c > a, b$, one could have solved for c using the above argument. One requires

$$a(n + 1 - a) = a(c - a) + (n + 1 - c)(c - b).$$

The solutions to this quadratic equation are $c = a + b$ and the nonsensical $c = n + 1$.

6.2 | Switchback in dihedral types

Let (W, S) be of rank 2 with $S = \{s, t\}$. We have for $J = \{s\}$

$$[J + k(-s + t - t + s)] \rightleftharpoons [J + t - t]$$

if $m_{st} = 2k + 1$ is odd, where the left-hand side denotes $[J - s + t - t + s \cdots - s + t - t + s]$ where $(-s + t - t + s)$ appears k times, and

$$[J + (k - 1)(-s + t - t + s) - s + t] \rightleftharpoons [J + t - s]$$

if $m_{st} = 2k$ is even. Note that $m_{st} = \infty$ cannot be the case, since S is then not finitary.

Via the embedding to \emptyset on both sides (Proposition 4.13), the above relation is exactly the braid relation in the Coxeter group.

To obtain the case where $1 \leq b < a \leq n$ merely apply Lemma 6.1. The arguments here are analogous to the case of type B , only with more cases to consider.

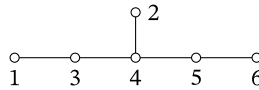
6.5 | Switchback in types EFH

For each Coxeter type, we index the elements in S by integers $1, \dots, n$, and record for each $1 \leq a \leq b \leq n$ (such that $s_a \neq w_0 s_b w_0$) the sequence $c_{a,b}$ which satisfies the relation

$$[J + s_a - s_b] \Leftarrow [J - s_{c_1} + s_a - s_{c_2} + s_{c_1} - s_{c_3} + \dots - s_{c_{\delta-1}} + s_{c_{\delta-2}} - s_b + s_{c_{\delta-1}}]$$

where $J = S \setminus \{s_a\}$. The size of $c_{a,b}$ is $\delta - 1$. To obtain the switchback when $1 \leq b \leq a \leq n$, merely apply Lemma 6.1. The result was obtained by computer computation.

6.5.1 | Type E_6



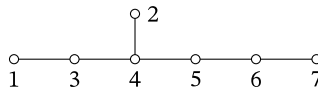
The following is a full list $(a, b : c_{a,b})$ where $c_{a,b}$ are displayed without commas.

a	b	$c_{a,b}$
1	1	6
1	2	31
1	3	52
1	4	435
1	5	26
6	6	1

a	b	$c_{a,b}$
2	3	36
2	4	4242
2	5	51
2	6	65
5	5	464
5	6	23

a	b	$c_{a,b}$
3	3	414
3	4	546
3	6	12
4	5	143
4	6	354

6.5.2 | Type E_7



a	b	$c_{a,b}$
1	2	27
1	3	3131
1	4	4363
1	5	5242
1	6	61
1	7	76
6	7	71

a	b	$c_{a,b}$
2	3	657
2	4	5152
2	5	4251
2	6	375
2	7	12
5	6	732
5	7	623

a	b	$c_{a,b}$
3	4	6341
3	5	5474
3	6	4143
3	7	265
4	5	7453
4	6	6464
4	7	5354

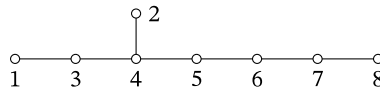
Type E_7 gives many opportunities to see Lemma 5.25 in action.

Example 6.3. The entry $a = 1$ and $b = 5$ gives the rotation sequence $(1,5,2,4,2,5)$ which ends up being 6-periodic because conjugation by w_0 is trivial, see (5.15). The entry $a = 2$ and $b = 5$ gives the rotation sequence $(2,4,2,5,1,5)$, which is the rotation of the previous. The entry $a = 2$ and $b = 4$ gives the third rotation, $(2,5,1,5,2,4)$. There are three other rotations as well, though they are not pictured in the table above because $a > b$. For example, the rotation sequence for $a = 5$ and $b = 2$ is $(5,1,5,2,4,2)$, which is both the reverse of the sequence for $a = 2$ and $b = 5$, and its rotation.

Example 6.4. The rotation sequence for $a = 2$ and $b = 3$ is $(2,6,5,7,3)$, and is 5-periodic. When $a = 5$ and $b = 6$ we get the rotation $(5,7,3,2,6)$. Another rotation comes from $a = 6$ and $b = 2$, which is not pictured in the table above. Applying Lemma 6.1 to the case of $a = 2$ and $b = 6$, we see that the case of $a = 6$ and $b = 2$ has rotation sequence $(6,5,7,3,2)$ as expected. The five rotations of $a = 2$ and $b = 6$ are related to the five rotations of $a = 6$ and $b = 2$ by taking the reversed sequence.

Example 6.5. Finally, the rotation sequence of $a = 4$ and $b = 6$ is $(4,6,4,6,4,6)$, which is 2-periodic. It both rotates and flips to give the rotation sequence of $a = 6$ and $b = 4$. This gives an example where the periodicity is strictly less than $\delta + 1$.

6.5.3 | Type E_8



a	b	$c.$
1	2	3128
1	3	2821
1	4	437573
1	5	525152
1	6	6161
1	7	7686
1	8	81

a	b	$c.$
2	3	7238
2	4	658562
2	5	515251
2	6	426585
2	7	3832
2	8	1312
7	8	8787

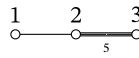
a	b	$c.$
3	4	757341
3	5	635484
3	6	548453
3	7	414375
3	8	2723
6	7	8671
6	8	7176

a	b	$c.$
4	5	845363
4	6	746474
4	7	647464
4	8	536354
5	6	856242
5	7	734143
5	8	624265

6.5.4 | Type F_4

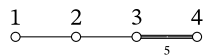
a	b	$c.$
1	2	2121
1	3	3242
1	4	41
2	3	4231
2	4	3132
3	4	4343

6.5.5 | Type H_3



a	b	$c.$
1	2	3231
1	3	2132
2	3	3121

6.5.6 | Type H_4



a	b	$c.$
1	2	21212121
1	3	3242313242
1	4	4341434143
2	3	4231324231
2	4	3132423132
3	4	4143414341

7 | TYPE A: SC AND WEBS

7.1 | Definition of the diagrammatic category

In type A , one can give a diagrammatic presentation for SC similar to the presentation of Type A webs from [3]. More precisely, we get a presentation of the monoidal category obtained by gluing $SC(S_n)$ together for all $n \geq 0$.

Definition 7.1. Let $SC(S_\star)$ be the monoidal category defined as follows. The objects are sequences $\underline{n} = (n_1, n_2, \dots, n_k)$ of non-negative integers, and the monoidal structure on objects is concatenation. Let $\sum \underline{n} := n_1 + \dots + n_k$. There are no morphisms from \underline{n} to \underline{m} if $\sum \underline{n} \neq \sum \underline{m}$. If $\sum \underline{n} = \sum \underline{m} = N$, then we view \underline{n} as a subset in S consisting of the simple generators of the parabolic subgroup $S_{\underline{n}} := S_{n_1} \times S_{n_2} \times \dots \times S_{n_k}$ inside S_N , and similarly for \underline{m} . Then

$$\text{Hom}_{SC(S_\star)}(\underline{n}, \underline{m}) = \text{Hom}_{SC(S_N)}(\underline{n}, \underline{m}) = S_{\underline{m}} \setminus S_N / S_{\underline{n}},$$

with $SC(S_N)$ controlling composition as well. The monoidal structure on morphisms is given by the product of parabolic double cosets

$$S_{\underline{m}} \setminus S_N / S_{\underline{n}} \times S_{\underline{m}'} \setminus S_{N'} / S_{\underline{n}'} \rightarrow S_{\underline{m}} \times S_{\underline{m}'} \setminus S_N \times S_{N'} / S_{\underline{n}} \times S_{\underline{n}'} \cong S_{\underline{m} \underline{m}'} \setminus S_{N+N'} / S_{\underline{n} \underline{n}'}$$

$$(p, q) \mapsto p \times q$$

Lemma 7.2. *The monoidal structure is well-defined.*

Proof. We need only verify the interchange law. It follows from

$$(\overline{p} \times \overline{p'}) * (\overline{q} \times \overline{q'}) = (\overline{p} * \overline{q}) \times (\overline{p'} * \overline{q'}).$$

□

Square :

if $a + b > N$, (7.2e)

It is standard practice to also permit webs where edges are labeled with 0. The object 0 is treated as the monoidal identity, and can be erased from a diagram freely. For example, the generating trivalent vertex $m \otimes 0 \rightarrow m$ is equal to the identity map of m , and the pictures become the same after the 0-labeled edge is erased. With this convention, one can also make sense of various relations when they have zero labels. However, no new relations are introduced in this fashion. For example,

(7.3)

is already true by definition.

Example 7.4. To give another example where edges are labeled 0, consider (7.2d) and (7.2e) where $a + b = N$. Both sides are already equal by definition. We call these the *trivial* square relations.

Let us match up the relations in (7.2) with the relations in SC . The bigon relation matches the $*$ -quadratic relation $[I - s + s] \simeq [I]$. The associativity relation (7.2b) matches the up-up relation $[I + s + t] \simeq [I + t + s]$, and coassociativity (7.2c) matches the down-down relation. Most interesting are the square relations (7.2d) and (7.2e), which match the switchback relation from (6.2).

Remark 7.5. There are also other instances of the up-up, down-down, and switchback relations which correspond to equalities of diagrams which hold purely from the interchange law. For example,

is an instance of up-up, and

is a switchback relation as in Lemma 5.5.

Proposition 7.6. *The categories $SC(S_*)$ and SC_{webs} are monoidally equivalent.*

Proof. As noted above, the relations given in Definition 7.3 agree with those in (5.29) (for the restriction to the objects \underline{n} with $\sum \underline{n} = N$, for each N). This gives a well-defined full functor $SC_{\text{Webs}} \rightarrow SC(S_\star)$ which is the identity on objects and maps ‘ $\pm s$ ’ to the generators (7.1). It is clearly monoidal (see Remark 7.5 and Lemma 5.5) and is faithful by Theorem 5.32. The claim follows. \square

Remark 7.7. We can assign a degree to each diagram, matching the length function of an expression from Subsection 3.4. We have

$$\deg \left(\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right) = ab, \quad \deg \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = ab. \tag{7.4}$$

With the exception of (7.2a), the relations of (7.2) preserve degree.

7.2 | More on squares

Suppose that $a + b < N$, so that (7.2d) holds for a, b . Let $a' = N - a$ and $b' = N - b$, so that $a' + b' > N$ and (7.2e) holds for a', b' . One can verify easily that (7.2e) for a', b' is none other than the horizontal flip of (7.2d) for a, b . As a consequence, the relations (7.2) are closed under taking the horizontal flip.

For the rest of this section, we let $x = N - a$ and $y = N - b$. Given a diagram of the following shape

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ | \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \tag{7.5}$$

when is it the right-hand side of a (nontrivial) square relation? Precisely when $a + b \neq N$, or equivalently, $b \neq x$. If $b < x$, then we are in the regime of (7.2d), and if $b > x$, then we are in the regime of (7.2e). When $b = x$ we have the picture

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ | \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \tag{7.6}$$

which is the unique reduced expression for its double coset, see Proposition 5.14(1). Note that (7.6) appears as a subdiagram inside the left-hand side of (7.2d)! This corresponds to the observation in Remark 5.18.

Given a diagram of the following shape

(7.7)

when is it the left-hand side of a square relation? Certainly $f = b \iff g = a$ and both imply $a + b \leq N$, because $f + a = g + b = j \leq N$. If $f = a$ and $a + b = N$, then we have a trivial square relation, see Remark 7.4. If $f = a$ and $a + b < N$, then we have (7.2d).

Now let us examine diagrams of the form (7.7) which are not part of a square relation.

In the first case, $f > b$ and $g > a$. Note that $a + b < N$. In this case, we claim that the result is a non-reduced expression for the maximal coset. That is, there is an equality in SC_{Webs} of the form

(7.8)

where only the right-hand side is reduced. We call this the *non-reduced square relation*. In particular, we should be able to use the relations (7.2), with at least one use of (7.2a), to deduce the non-reduced square relation. The reason this works is the opposite to the observation above. Above we said that the subdiagram (7.6) inside left-hand side of (7.2d) can not be reduced further when $f = b$ and $g = a$. In this case, since $f > b$ and $g > a$, it can be reduced further.

Lemma 7.8. *The equality (7.8) is a consequence of (7.2).*

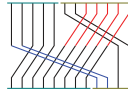
Proof. Continuing the notation of (7.7), we assume $f > b$ and $g > a$.

(7.9)

The first equality is (7.2d) with bottom boundary (a, f) instead of (a, x) . The second equality follows from (7.2b) and (7.2c). □

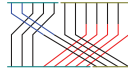
In the second case, $f < b$ and $g < a$. It is possible that $a + b < N$ or $a + b > N$ but it will not matter in the end. We claim that the result is a reduced expression, for some non-maximal coset q . One can verify this by showing that the lengths add in a product of relative longest elements. An example is demonstrative.

Example 7.9. Let $a = 8, b = 5, f = 2, g = 5,$ and $N = 12.$ Starting with the longest element of $S_8 \times S_4,$ we multiply by the element below to get $\bar{q}.$



First, f blue strands are pulled left across the initial a strands. Then g red strands are pulled right across $x - f$ remaining strands.

However, this coset q also has another reduced expression!



First one pulls g red strands right across the final x strands. Then f blue strands are pulled left across $a - g$ remaining strands.

More generally, there is an equality between two reduced expressions which we might call the *rung swap*, which holds whenever $f < b$:

$$(7.10)$$

The square relations themselves can be viewed as special instances of the rung swap, when $f = b$ and $g = a.$

Lemma 7.10. *The equality (7.10) is a consequence of (7.2).*

Proof. Both sides of (7.10) are equal to

$$(7.11)$$

Starting from the left-hand side of (7.10), and noting that $b > f,$ we can apply (7.2e) with bottom inputs (a, f) to get (7.11). Similarly, we can apply (7.2d) with bottom inputs (g, x) to the right-hand side of (7.10). □

7.3 | Relationship to ordinary type A webs

This section is written for the reader familiar with type A webs.

In [3], Cautis–Kamnitzer–Morrison construct a $\mathbb{Z}[q, q^{-1}]$ -linear monoidal category $\text{Webs}(\mathfrak{sl}_n)$ which describes morphisms between tensor products of fundamental representations of \mathfrak{sl}_n . Variants on their construction include a category for \mathfrak{gl}_n as well, and a category Webs^∞ which is the limit as $n \rightarrow \infty$ of $\text{Webs}(\mathfrak{gl}_n)$ (see [16, 21]). The objects in Webs^∞ agree with the objects of $\mathcal{SC}_{\text{Webs}}$, where 1 represents the standard representation V . By Schur–Weyl duality, the endomorphism ring of $(\mathbb{C}^n)^{\otimes m}$ has dimension equal to $|S_m|$ when $m \leq n$. In Webs^∞ , the endomorphism ring of $(1, 1, \dots, 1)$, having m copies of 1 , always has dimension $|S_m|$. Meanwhile, in $\mathcal{SC}_{\text{Webs}}$, the endomorphism algebra of the object $(1, 1, \dots, 1)$ is a set, having the same size as S_m . By the same token, other morphism spaces in Webs^∞ have dimensions equal to the size of the corresponding morphism spaces in $\mathcal{SC}_{\text{Webs}}$.

The degree function on webs which we introduce in (7.4) can be used to equip $\text{Webs}(\mathfrak{gl}_n)$ or Webs^∞ or (the \mathbb{Z} -linearization[†] of) $\mathcal{SC}_{\text{Webs}}$ with a filtration. A morphism is in degree $\leq k$ if it can be represented by a linear combination of webs whose degrees are at most k . Since the dimension of an associated graded algebra agrees with the original dimension, the following result might be expected from the discussion of dimensions above.

Theorem 7.11. *The $\mathbb{Z}[q, q^{-1}]$ -linearization of $\mathcal{SC}_{\text{Webs}}$ and Webs^∞ have isomorphic associated graded categories.*

Proof. Given a category presented by generators and relations, one obtains a presentation of the associated graded category by taking only the top degree terms in each relation. For example, both (7.2a) and the corresponding bigon relation [3, (2.4)] descend to the relation



$$= 0 \tag{7.12}$$

in the associated graded. All the remaining (braid) relations of $\mathcal{SC}_{\text{Webs}}$ are homogeneous, and descend verbatim to the associated graded. The associativity and coassociativity relations [3, (2.6)] are homogeneous as well.

In the associated graded, (7.8) becomes

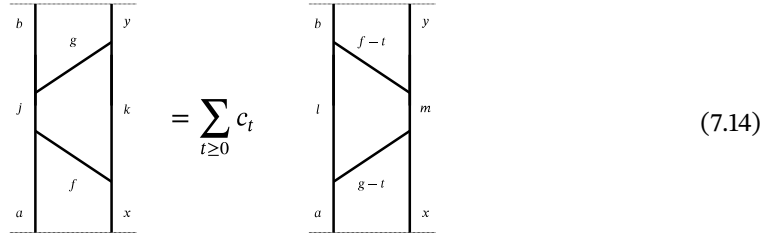


$$= 0, \tag{7.13}$$

and (7.10) is unchanged. We will conclude by proving that the associate graded of the only remaining relations from Cautis–Kamnitzer–Morrison, the square flop relations [3, (2.10)], is either (7.13) or the rung swap relation (7.10). With our indexing conventions, the square flop relation has the

[†] To define the associated graded of a category, one must first linearize it. This introduces a zero morphism.

form



for some coefficients c_t .

It is easy to verify that the degree of the left-hand side of (7.14) is $f(N - f) + g(N - g)$. The reader should verify that $N + t - f - g = k + l$. Thus,

$$f(N - f) + g(N - g) - [(f - t)(N - (f - t)) + (g - t)(N - (g - t))] = 2t(k + l) \geq 0. \tag{7.15}$$

In particular, the degree of the left-hand side of (7.14) is at least as large as any term on the right-hand side, and only those terms with $2t(k + l) = 0$ contribute to the associated graded.

If $f > b$, then the right-hand side of (7.14) has no $t = 0$ term. If $k \neq 0$, then there is no term with $2t(k + l) = 0$, and in the associated graded we get (7.13). If $k = 0$, then there is a unique term where $2t(k + l) = 0$, namely $t = f - b$ and $l = 0$. In the associated graded we get the trivial square relation.

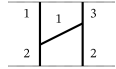
If $f < b$, then the right-hand side of (7.14) has a $t = 0$ term, which matches the degree of the left-hand side. Moreover, there are no terms where $l = 0$, so no other terms with $2t(k + l) = 0$. Thus, in the associated graded we get the relation (7.10), but with some coefficient c_0 . Thankfully, $c_0 = 1$, so we get (7.10) with no scalars. □

This associated graded category is the singular analogue of the nil-Coxeter algebra, where $s^2 = 0$ (rather than $s^2 = 1$ as in W , or $s^2 = s$ as in $(W, *, S)$). For the rest of this section we paint with broad brush strokes, in order to state a rough conjecture.

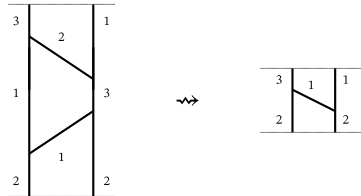
In [5, Definition 2.37], a basis for morphisms in Webs^∞ is constructed, the so-called *double ladders basis*. We wish to conjecture that double ladders are reduced expressions, and provide an enumeration of the morphisms in SC_{Webs} , though this statement requires some clarification. Double ladders are constructed by gluing together elementary light ladders (see [5, (2.17)]) and neutral ladders (see [5, p. 22]), and these neutral ladders can be placed willy-nilly. In Webs^∞ , neutral ladders are isomorphisms modulo ‘lower weights’, and composing with them will alter the basis by an upper triangular change of basis matrix. In SC_{Webs} however, neutral ladders have positive degree and cannot be placed arbitrarily within a reduced expression.

The construction [5, (2.17)] explains how to construct an elementary light ladder when the input sequences are sorted (that is, when the condition $y_1 < x_1 < \dots$ is satisfied). When the input sequences are not sorted, the procedure from [5] is to apply neutral ladders to sort the sequences (and also to sort any extraneous strands out of the picture), and then apply the elementary light ladder for sorted sequences. However, the resulting web can often be simplified modulo lower terms, and this simplified web will have lower degree. We give an example below. The combinatorics of this minimal-degree representative of the light ladder (modulo lower terms) has not yet been developed. We conjecture that, once they are well-defined, the minimal-degree double ladders will be reduced expressions.

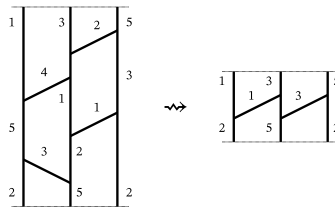
Example 7.12. The (sorted) elementary light ladder giving a morphism $(2, 2) \rightarrow (1, 3)$ is



The construction in [5] gives the (unsorted) light ladder $(2, 2) \rightarrow (3, 1)$ below, but it can be simplified modulo lower weights to a lower degree diagram, which is reduced.



The construction in [5] gives the (unsorted) light ladder $(2, 5, 2) \rightarrow (1, 3, 5)$ below, but it can be simplified modulo lower weights to a lower degree diagram, which is reduced.



8 | SINGULAR COXETER COMPLEXES

The dual Coxeter complex is an important combinatorial CW complex, and we would like to suggest the existence of a singular analogue. We are able to construct the 2-skeleton, but do not know the combinatorial structure of the higher cells.

8.1 | Completed dual Coxeter complexes

We will be brief in recalling the classical definitions, as [1] is an excellent reference (see Section 3 and Subsection 12.3). Fix a Coxeter system (W, S) where n is the size of S . There is a well-known $(n - 1)$ -dimensional simplicial complex called the Coxeter complex. Its dual is a CW complex or polytope known as the dual Coxeter complex. By adding in an n -cell when W is finite, one obtains the (completed) dual Coxeter complex $\text{Cox}(W)$, which is always contractible. The completed dual Coxeter complex in type A is called the (solid) permutohedron, which the reader can easily find pictures of online.

The complex $\text{Cox}(W)$ has one 0-cell for each $w \in W$, each of which is attached to n 1-cells. The 0-cell associated to w is connected to the 0-cell for ws by a 1-cell labeled by s . More generally, for each $0 \leq k \leq n$ there is a k -cell associated to each coset in W/W_I for each finitary subset $I \subset S$ of size k . The 0-cells in the closure of this k -cell correspond to the elements in this coset.

Let \mathfrak{h} denote the reflection representation of W over \mathbb{R} , which has a hyperplane H_α for each root α , and a chamber (that is, a connected component in the complement of $\bigcup_\alpha H_\alpha$) for each $w \in W$. When W is finite, the Coxeter complex can be obtained by intersecting this hyperplane arrangement with the unit sphere in \mathfrak{h} . For example, a chamber intersects the unit sphere in a simplex, while a wall intersects the unit sphere in a face of a simplex. The hyperplane interpretation equips the Coxeter complex and its dual with extra structure. For example, we know what it means for two facets to lie in the same flat (see Subsection 1.5) of the hyperplane arrangement. Viewing $\text{Cox}(W)$ as a purely topological structure forgets this inherent ‘linear’ structure, which will be significant for us.

The 1-skeleton of $\text{Cox}(W)$ is the covering diagram of the weak right Bruhat order. In particular, one can orient each 1-cell, so that $w \rightarrow ws$ when $\ell(w) < \ell(ws)$. This orientation is a phenomenon special[†] to the 1-skeleton. Oriented paths from the source to w can be interpreted as reduced expressions for w , and any two such paths are related by the braid relations. In fact, the 2-cells precisely correspond to such braid relations.

8.2 | The singular 2-skeleton: One-sided cosets

Let us construct an oriented graph, which we wish to consider as the 1-skeleton of a singular analogue of the completed dual Coxeter complex. For lack of a better name, we will call it the *singular Coxeter complex*. The example of type A_2 was given in Subsection 1.5. Before we introduce it, let us discuss what we will *not* do.

Definition 8.1. Let Cox_{cube} denote the CW complex constructed in [1, Proposition 12.63]. We only describe the 1-skeleton. There is one vertex for each (\emptyset, I) -coset p . Whenever $I' = Is$ and the (\emptyset, I) -coset p is contained in the (\emptyset, I') -coset q , there is an edge from p to q .

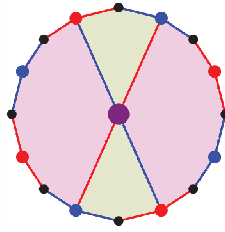
The complex Cox_{cube} is a refinement of $\text{Cox}(W)$ and is homeomorphic to it. The complex is *cubical* (see [1, Proposition 12.63]) in that the closure of each top-dimensional cell is a cube with its standard CW structure. One can interpret Cox_{cube} within \mathfrak{h} as follows. There is one point on each facet of the hyperplane arrangement for which the stabilizer is a finite parabolic subgroup. All edges occur between facets which differ in dimension by 1, where one is contained in the closure of the other.

We think of Cox_{cube} as being relatively boring. The 2-cells are unrelated to the braid relations. Instead, we throw away those edges not involved in any reduced expression, and get a different CW complex, where the 2-cells correspond to braid relations.

Definition 8.2. Let $J = \emptyset$. Let $\text{Cox}_\emptyset^1(W, S)$ denote the oriented graph defined as follows. There is one vertex for each (J, I) -coset p , for all finitary $I \subset S$. There is an edge $p \rightarrow q$ if $[p, q]$ is reduced, see Definition 2.29. This can be made into a graded graph, where the length of each double coset is defined as in Definition 3.14.

[†] In type A one can place semiorientations on the higher cells, related to the higher Bruhat orders of Manin–Schechtmann [15].

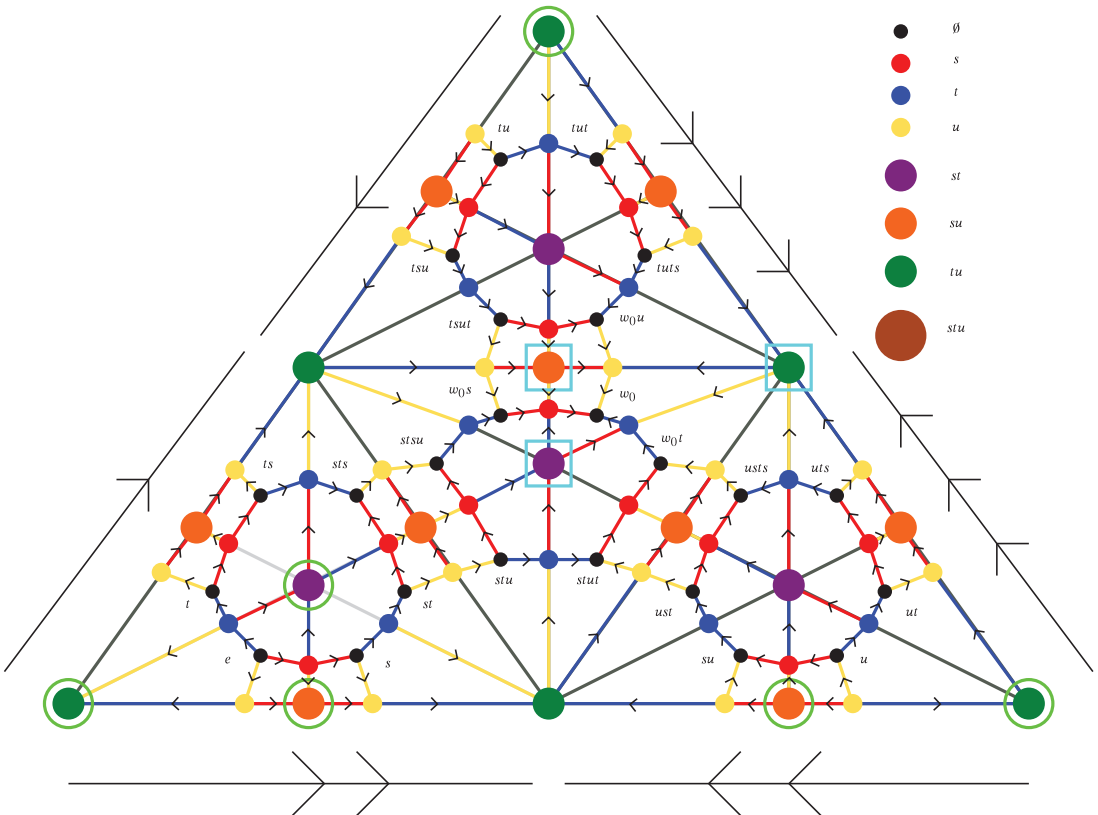
Example 8.3. In type B_2 , Cox_\emptyset^1 is the 1-skeleton of the following diagram. The orientation goes from bottom to top.



(8.1)

(With the 2-cells, the diagram represents Cox_\emptyset^2 , defined below.)

Example 8.4. Let us describe Cox_\emptyset^1 in type A_3 †. Recall that the ordinary Coxeter complex of S_4 is a barycentrically subdivided tetrahedron. Cut this tetrahedron along all the edges adjacent to one vertex, and unfold it to make it flat. The result is a large equilateral triangle built from four smaller equilateral triangles (the original faces of the tetrahedron), which is what we draw below.



(8.2a)

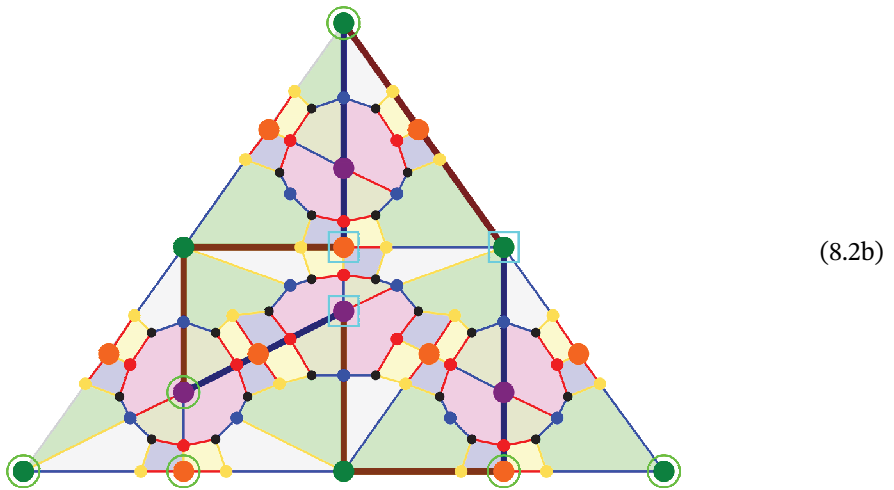
† This nice depiction is due to Janelle Currey.

One can recover the original tetrahedron by gluing back together the cut edges as indicated (one arrow to one arrow, two arrows to two arrows, etc.). We label only the black vertices, corresponding to (\emptyset, \emptyset) -cosets. The labels on every other coset are determined by which black vertices are nearby. The singular Coxeter complex also has a vertex corresponding to the coset W/W , the *generic point*. We do not draw the generic point, which fills in the tetrahedron, but we circle the three vertices with edges to the generic point, and square the three vertices with edges from the generic point. The gray edges[†] should be deleted, as they correspond to edges in Cox_{cube} which do not appear in Cox_{\emptyset} . For the 2-cells, see Example 8.6.

If p is an (\emptyset, I) -double coset, then a reduced expression for p is the same data as an oriented path to p from the (\emptyset, \emptyset) -coset $\{e\}$. Any two such reduced expressions are related by the braid relations (Theorem 5.30), and we can use these as our 2-cells.

Definition 8.5 (Loosely worded). Let $\text{Cox}_{\emptyset}^2(W, S)$ be the 2-dimensional CW complex defined by adding 2-cells to $\text{Cox}_{\emptyset}^1(W, S)$. These 2-cells are attached along the braid relations, for any application of a braid relation within any reduced expression.

Example 8.6. Let us continue Example 8.4 by indicating the 2-cells. The duller colors (gray, tan, slate) represent \pm -associativity relations. The brighter colors (fuchsia, lemon, lime) represent switchback relations. There are six 2-cells we could not draw, the switchback relations involving the generic point. These 2-cells are attached to the pictured region along a path of length four, from a circled vertex to a squared vertex. We've indicated these six paths with thick brown or blue lines.



Remark 8.7. Because Cox_{\emptyset}^1 is obtained from $\text{Cox}_{\text{cube}}^1$ by throwing away some edges, it seems plausible that there is some formal way to adjoin higher cells (gluing together various cells of Cox_{cube}) to construct a CW complex Cox_{\emptyset} which is homeomorphic to Cox , but we do not know how. Formal constructions aside, initial explorations reveal that there is interesting combinatorics involved in describing the attaching maps of the higher cells explicitly.

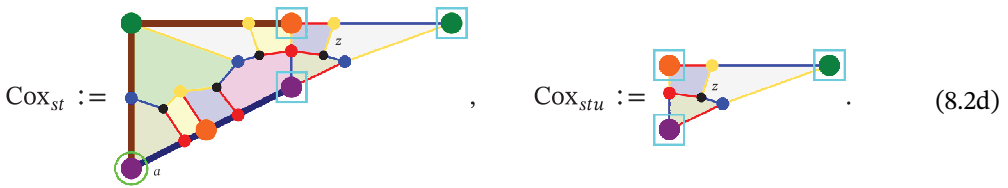
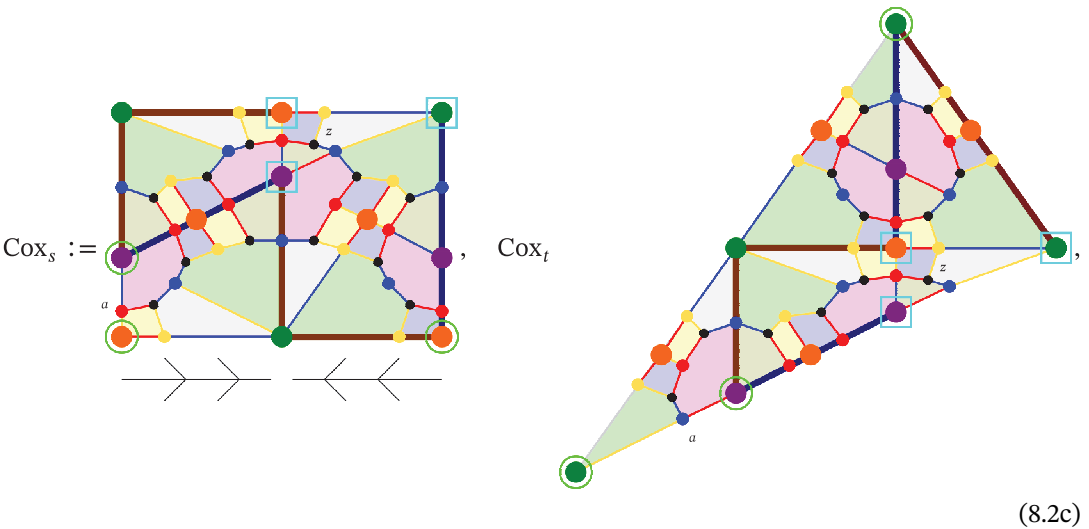
[†] The reader can observe that, with the gray edges included, every pictured face is a square. Including the generic point and the gray edges to it, every 3-cell is a cube, as expected in Cox_{cube} .

8.3 | The singular 2-skeleton: General case

Let us construct the analogous 2-dimensional complex for an arbitrary finitary subset $J \subset S$.

Definition 8.8. Let $J \subset S$ be finitary. Let $\text{Cox}_J^1(W, S)$ denote the oriented graph defined as follows. There is one vertex for each (J, I) -coset p , for all finitary $I \subset S$. There is an edge $p \rightarrow q$ if $[p, q]$ is reduced, see Definition 2.29. This can be made into a graded graph, where the length of each double coset is defined as in Definition 3.14. Let $\text{Cox}_J^2(W, S)$ be the 2-dimensional CW complex defined by adding 2-cells to $\text{Cox}_J^1(W, S)$. These 2-cells are attached along the braid relations, for any application of a braid relation within any reduced expression.

Example 8.9. Here are the singular Coxeter complexes in type A_3 with J equal to $s, t,$ and $st,$ respectively. We have labeled the source a (the (J, J) -coset containing e) and the sink z (the (J, \emptyset) -coset containing w_0); the orientation of arrows is the same as in (8.2a). In Cox_{stu} the source a is the undrawn generic vertex.



All these examples above have one thing in common: the complex $\text{Cox}_J^2(W, S)$ looks a portion of $\text{Cox}_\emptyset^2(W, S)$, though with the labels changed! Inside \mathfrak{h} , we interpret Cox_J as the portion of Cox_\emptyset living within a fundamental domain for the left action of W_J .

Proposition 8.10. Let $X \subset \text{Cox}_\emptyset^2(W, S)$ denote the sub-CW-complex defined as follows. It contains all the 0-cells associated to (\emptyset, I) -cosets p such that $J \subset \text{LD}(\bar{p})$. It contains all the k -cells whose closure only contains these 0-cells. Then there is a cell-preserving homeomorphism $X \rightarrow \text{Cox}_J^2(W, S)$, which on 0-cells sends each (\emptyset, I) -coset p to the (J, I) -coset q with $\bar{p} = \bar{q}$.

Proof. That there is some (unique) q with $\bar{q} = \bar{p}$ is immediate from (2.10a). By Proposition 4.13, we can take any reduced expression of q and compose it with $[[\emptyset, J]]$, to obtain a reduced expression of p . An implication is that $[p, p']$ is reduced if and only if $[q, q']$ is reduced. Said another way, every edge in Cox_J^1 appears along some reduced expression, and we can send it to the corresponding edge in the corresponding reduced expression in Cox_\emptyset^1 . We can match up the 2-cells (braid relations between reduced expressions) using the same technique. \square

8.4 | Reduced expressions and hyperplanes

Let H be a root hyperplane in \mathfrak{h} . Then H divides the real vector space \mathfrak{h} into two halves, the *negative half* H_- containing the chamber associated to e , and the *positive half* H_+ . It is well-known that ordinary reduced expressions are paths between chambers which always cross walls from the negative side to the positive side. Each wall that is crossed is an ‘inversion’ which contributes to the length of the element being expressed. If x and xs are the chambers on opposite sides of a facet inside H , and $x < xs$, then $x \in H_-$ and $xs \in H_+$. Now we reinterpret these ideas in Cox_\emptyset , where we do not just walk between chambers but also walk on the walls. A given double coset will either lie within H_- , within H_+ , or on H itself.

Proposition 8.11. *Let $I' = Is$ and J be finitary subsets of S . Let p be a (J, I) -coset and q a (J, I') -coset with $p \subset q$. Then $[p, q]$ is reduced if and only if $p \notin H_+$ for every root hyperplane H containing q . Similarly, $[q, p]$ is reduced if and only if $p \notin H_-$ for every root hyperplane H containing q .*

Proof. If we can prove this result for $J = \emptyset$, then it will hold for arbitrary J using Proposition 8.10. So, assume $J = \emptyset$. Now $[p, q]$ is reduced if and only if $\underline{p} = \underline{q}$.

Suppose $\underline{p} = \underline{q}$. Since p is a facet adjacent to the chamber containing \underline{p} , it is contained in the closure of that chamber. If q lies on H , then the chamber of \underline{q} lies in H_- (because $\text{RD}(q) \cap I' = \emptyset$, and this facet of H corresponds to right multiplication by some $t \in I'$). Hence, p is in the closure of H_- , which is disjoint from H_+ .

Conversely, suppose that $p \notin H_+$ for any H containing q . Then $\underline{p} \in H_-$, and hence $\text{RD}(\underline{p}) \cap I' = \emptyset$. Thus, by (2.10b), $\underline{p} = \underline{q}$.

Similarly, $[q, p]$ is reduced if and only if $\bar{q} = \bar{p}$. The arguments in this case are nearly identical. \square

Now let us give some intuition why Proposition 8.11 should match with Definition 2.29 when $J \neq \emptyset$. The geometric version of the condition (2.14b) states that a reduced expression will never return to the flat determined by J after leaving it. For example, consider the complex Cox_t from (8.2c), which takes the half of Cox_\emptyset living on one side of the reflection hyperplane of t . The edges can be oriented precisely as a subgraph of Cox_\emptyset , see (8.2a). The important thing to note is that no edges point into the reflection hyperplane of t ! Once you leave this fixed hyperplane, you can never return.

Let us justify this statement about (2.14b). This mostly involves trying to better understand the connection between the label on a facet and its stabilizer.

Let F be a facet in \mathfrak{h} adjacent to the dominant chamber. If the stabilizer of F is W_I (because F lies on the intersection of the reflecting hyperplanes of $s \in I$), then F is labeled with the minimal (\emptyset, I) -coset in Cox_\emptyset . In Cox_I , F would be labeled by the minimal (I, I) -coset instead.

If F is a facet adjacent to the dominant chamber, then $w \cdot F$ is a facet adjacent to the w chamber. If $I \subset S$ is the stabilizer of F , then the non-standard parabolic subgroup wW_Iw^{-1} is the stabilizer of $w \cdot F$. Note that $w \cdot F$ still corresponds to an (\emptyset, I) -coset, because right multiplication by W_I is intertwined with left multiplication by ww_Iw^{-1} . In fact, $w \cdot F$ is the unique (\emptyset, I) -coset containing w . Working backward, suppose that a facet F is labeled by the (\emptyset, I) -coset q , and $w \in q$. Then the stabilizer of F is wW_Iw^{-1} .

An edge which goes onto the s -hyperplane would add s to the set wW_Iw^{-1} . This increases the redundancy (of the coset corresponding to the facet) if and only if $s \in J$. In conclusion, violating (2.14b) means that one returned to an s -hyperplane for some $s \in J$.

8.5 | Geometric intuition for the switchback relation

Let (W, S) be a finite Coxeter system, acting on the left on its reflection representation \mathfrak{h} . Then S generates the stabilizer of the origin in \mathfrak{h} . For any $s \in S$, $\widehat{s} := S \setminus s$ generates the stabilizer of a line in \mathfrak{h} , the intersection of all the root hyperplanes for $t \neq s$. This line splits into two rays R and R' , where R is adjacent to the dominant chamber, and R' to the antidominant chamber. The ray R is labeled with the minimal (\emptyset, \widehat{s}) -coset p in Cox_{\emptyset} . Note that R' need not equal $w_0 \cdot R$, as the former has stabilizer $W_{\widehat{s}}$ and the latter has stabilizer $W_{w_0\widehat{s}w_0}$. For any $t \in S$, we are interested in describing reduced paths from p to q , where q is the maximal (\emptyset, \widehat{t}) -coset.

When $t = w_0sw_0$, then q is the label on R' , which is colinear with R . A reduced expression can never leave this line by Proposition 8.11, so one expects a single reduced expression from p to q , which passes through the origin. This matches up with Proposition 5.14(1).

If $t \neq w_0sw_0$, then q is the label on some other ray T . The rays R and T span a 2-dimensional subspace P of \mathfrak{h} , and identify a ‘quadrant’ of this plane. By Proposition 8.11, a reduced path from p to q stays within (the closure of) this quadrant. The intersection of the hyperplane arrangement with P is a collection of lines, so as one passes from R to T one crosses a number of these lines, which we call $T_1, T_2, \dots, T_{\delta} = T$. The picture is as below.



The ray opposite T , which we call T_{-1} , has stabilizer $\widehat{u_{-1}}$ where $u_{-1} = w_0tw_0$. Visually, it appears that T_1 is the reflection (in the plane) of T_{-1} over the ray R . The longest element $w_{\widehat{s}}$ does not usually act on \mathfrak{h} as a reflection, but it does act by a reflection on the plane P , since it is an involution whose fixed points form a line. This reflection sends T_{-1} to T_1 , so the stabilizer of T_1 is $\widehat{u_1}$ where $u_1 = w_{\widehat{s}}u_{-1}w_{\widehat{s}}$. To obtain the stabilizer of T_2 , we reflect R across T_1 . Similar procedures indicate that the stabilizer of T_i is $\widehat{u_i}$, where (u_i) is the rotation sequence. That this intuition holds true is justified by Proposition 5.14(2).

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