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N³LO spin-orbit interaction via the EFT of spinning gravitating objects

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ABSTRACT: We present the derivation of the third subleading order (N³LO) spin-orbit interaction at the state of the art of post-Newtonian (PN) gravity via the EFT of spinning objects. The present sector contains the largest and most elaborate collection of Feynman graphs ever tackled to date in sectors with spin, and in all PN sectors up to third subleading order. Our computations are carried out via advanced multi-loop methods. Their most demanding aspect is the imperative transition to a generic dimension across the whole derivation, due to the emergence of dimensional-regularization poles across all loop orders as of the N³LO sectors. At this high order of sectors with spin, it is also critical to extend the formal procedure for the reduction of higher-order time derivatives of spin variables beyond linear order for the first time. This gives rise to a new unique contribution at the present sector. The full interaction potential in Lagrangian form and the general Hamiltonian are provided here for the first time. The consequent gravitational-wave (GW) gauge-invariant observables are also derived, including relations among the binding energy, angular momentum, and emitted frequency. Complete agreement is found between our results, and the binding energy of GW sources, and also with the extrapolated scattering angle in the scattering problem, derived via traditional GR. In contrast with the latter derivation, our framework is free-standing and generic, and has provided theory and results, which have been critical to establish the state of the art, and to push the precision frontier for the measurement of GWs.

KEYWORDS: Classical Theories of Gravity, Effective Field Theories, Renormalization and Regularization, Black Holes

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1 Introduction

The successful measurements in ground-based experiments of gravitational waves (GWs) from compact-binary inspirals and mergers, since the first breakthrough detection [1] by the Advanced LIGO [2] and Advanced VIRGO [3] collaboration, have been already providing abundant and rich data [4–6]. Moreover, there is a rapidly-growing worldwide network of GW detectors, including now also KAGRA in Japan [7]. The sources of these GWs, such as black-hole (BH) binaries, spend most of their evolution time in the inspiral phase, where the velocities of orbiting components are non-relativistic. Accordingly they have been studied analytically via the post-Newtonian (PN) approximation of General Relativity (GR) [8].

This PN description forms the basis for the effective-one-body (EOB) approach [9], which in turn enables to model complete theoretical waveforms. These are used to test our knowledge of gravity in the strong-field regime [10, 11], and also of QCD in extreme

conditions from neutron-star (NS) binaries [12] or mixed NS-BH binaries [13]. Since in reality all compact gravitating objects are rotating, it is essential to incorporate the effects of spin into any theoretical waveform models beyond the basic accuracy of the first PN (1PN) order [14]. For real precessing binaries with compact spinning objects the physics gets dramatically more intricate and intriguing.

The experimental breakthrough in the measurements of GWs has ushered in an era of booming activity in the analytical studies of the high-precision frontier of PN theory. In the conservative sector recent efforts to push the state of the art have peaked at the high perturbative order of 5PN in the point-mass sector via traditional GR [15–17], and via effective field theory (EFT) methods [18, 19]. The point-mass 5PN results in [15, 16] have been accomplished via a combined intricate deployment of traditional GR methods, with crucial assumptions from the EOB approach, and available results from self-force theory, see e.g. recent review in [20]. The work in [21, 22] then followed suit, and implemented the same approach of [15, 16] to the spin-orbit sector, that is linear in the spins, at the 4.5PN order.

Yet, at the present high orders of the precision frontier it is crucial to deploy various independent methodologies in order to carefully push and establish the state of the art. The EFT of spinning gravitating objects [23] constitutes a unique free-standing methodology, which has indeed enabled the completion of the state of the art at the 4PN order [24–27]. In the present paper we derive the complete next-to-next-to-next-to-leading order (N^3LO) spin-orbit interaction at the 4.5PN order via the EFT of spinning gravitating objects, and its automation introduced in the public code `EFTofPNG` [28], also building on [29–32]. This paper is part of a series, which has recently accomplished the completion all sectors up to the 5PN order [33–36]. In the radiative sector recent spin-dependent radiation effects that have been approached with EFT techniques reached at the NLO in [37, 38], and NNLO at [39], both at lower loop and spin orders.

In sections 2 and 3 of this paper, we present in full detail the formulation and evaluation of the Feynman diagrammatic expansion in this sector, which contains the largest and most elaborate collection of graphs ever tackled to date in sectors with spin, and in all PN sectors up to the third subleading order. Our computations are carried out via advanced multi-loop methods, and further development of the `EFTofPNG` code. The most significant computational leap that is taking place in the present sector, is the need to switch on a generic dimension and keep track of dimensional-regularization (DimReg) expansions, across the whole derivation. This is due to the emergence of DimReg poles across all loop orders in sectors as of the N^3LO in PN theory.

In section 4 we extend the formal procedure introduced in [40] for the reduction of higher-order time derivatives of spin variables via redefinitions, and present in full detail the intricate reduction process required in the present sector. We find that this sector uniquely requires to apply the redefinition of rotational variables beyond linear order for the first time, as previously the linear order has been sufficient for all sectors with spin at lower orders.

In sections 4.3 and 5, the full interaction potential in Lagrangian form and the general Hamiltonian of the sector, are provided here for the first time. The Lagrangian potential

obtained via our framework enables a direct derivation of the physical equations of motion (EOMs) for both the position and spin [23]. The full general Hamiltonian enables to explore possible EOB extensions for this sector, and test the performance of such variants of EOB Hamiltonians. The general Hamiltonian also enables to study the conserved integrals of motion, which form a representation of the Poincaré algebra on phase space. Indeed, in [35] we find the complete Poincaré algebra of the present sector, and thus verify the validity of the general Hamiltonian derived in the present paper.

In section 6, we proceed to derive the consequent GW gauge-invariant observables, namely the relations among binding energy, angular momentum, and emitted frequency. We find complete agreement with the binding energy of GW sources, and also with the extrapolated scattering angle in the scattering problem, derived via traditional GR [22]. Finally, all of the computations in our framework are also automated as extensions of the public `EFTofPNG` code. The independent generic derivation and results presented in this paper, and such development of the `EFTofPNG` code, have been essential to establish the state of the art, and to push the precision frontier for the measurement of GWs.

2 EFT of spinning gravitating objects

In order to derive the spin-orbit interaction at this high order, we should consider the EFT of spinning gravitating objects [23]. We start from an effective action of a two-particle system at the orbital scale, that captures a compact binary inspiral [18]:

$$S_{\text{eff}} = S_{\text{gr}}[g_{\mu\nu}] + \sum_{a=1}^2 S_{\text{pp}}(\lambda_a), \quad (2.1)$$

where S_{gr} is some bulk action of the gravitational field, and S_{pp} is a point-particle action for each component of the binary, that is localized on their worldline with the parameter λ_a for the a -th component.

For the bulk gravitational action, which is given in terms of the field modes at the orbital scale, $g_{\mu\nu}(x)$, we take the Einstein-Hilbert action of GR, which we gauge-fix with the fully-harmonic gauge:

$$S_{\text{gr}}[g_{\mu\nu}] = S_{\text{EH}} + S_{\text{GF}} = -\frac{1}{16\pi G_d} \int d^{d+1}x \sqrt{g} R + \frac{1}{32\pi G_d} \int d^{d+1}x \sqrt{g} g_{\mu\nu} \Gamma^\mu \Gamma^\nu, \quad (2.2)$$

where $\Gamma^\mu \equiv \Gamma_{\rho\sigma}^\mu g^{\rho\sigma}$. Note that at this high non-linear order in the gravitational self-interaction, the derivation must be kept in a generic dimension throughout, with d the number of spatial dimensions, and a modified minimal subtraction ($\overline{\text{MS}}$), see e.g. [41], for the d -dimensional gravitational constant:

$$G_d \equiv G_N \left(\sqrt{4\pi e^\gamma} R_0 \right)^{d-3}, \quad (2.3)$$

with $G_N \equiv G$ Newton's gravitational constant in 3-dimensional space, γ Euler's constant, and R_0 some renormalization scale. This is since dimensional regularization (DimReg) is used to evaluate the integrals, and poles in the dimensional parameter $\epsilon \equiv d - 3$ appear throughout, as shall be seen as of the next section.

We make consistent use of a $d + 1$ non-relativistic (NR) decomposition of the gravitational field, similar to a Kaluza-Klein reduction over the time dimension [42, 43]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \equiv e^{2\phi} \left(dt - A_i dx^i \right)^2 - e^{-\frac{2}{d-2}\phi} \gamma_{ij} dx^i dx^j, \quad (2.4)$$

which defines the NR fields: ϕ , A_i , and $\gamma_{ij} \equiv \delta_{ij} + \sigma_{ij}$. This beneficial parametrization has been implemented first in sectors with spin in [44], and was incorporated all throughout the EFTofPNG code [28]. The propagators of such NR fields contain temporal delta functions, and the momenta integrals are then d -dimensional. These NR propagators get their relativistic corrections order by order via quadratic insertions with two time derivatives. In this third subleading sector the propagators get up to three such perturbative corrections. The d -dimensional propagators and their insertions, as well as the higher-point gravitational vertices for our sector are all provided in the supplementary material to this publication in human visual and machine-readable formats. They were all derived via the extension of the `FeynRul` module of the public code `EFTofPNG` [28].

Finally and importantly, the point-particle action S_{pp} should also be considered. In the present spin-orbit sector it is the effective action of a spinning particle which is minimally coupled to gravity, in the form [23, 45, 46]:

$$S_{\text{pp}}(\lambda) = \int d\lambda \left[-m\sqrt{u^2} - \frac{1}{2} \hat{S}_{\mu\nu} \hat{\Omega}^{\mu\nu} - \frac{\hat{S}^{\mu\nu} p_\nu}{p^2} \frac{Dp_\mu}{D\lambda} \right], \quad (2.5)$$

with m the mass, u^μ the 4-velocity, p_μ the linear-momentum, and $\hat{\Omega}^{\mu\nu}$, $\hat{S}_{\mu\nu}$, the generic angular velocity and spin variables of the particle, respectively. These generic rotational variables, together with the additional term in eq. (2.5) which contains the covariant derivative along the worldline, enable to switch the gauge of rotational variables. By contrast the action presented in [47–50] does not have generic rotational variables, nor does it include the generic last term in eq. (2.5).

Finite-size effects enter only at the 5PN order in the point-mass sector, and at the 6.5PN order in the spin-orbit sector of the effective action [30, 31]. Therefore from the action in eq. (2.5) one can extract all the required mass and spin worldline-couplings, where both play an important role in the spin-orbit interaction. Note that the spin couplings are given in terms of the local spatial components of the spin tensor in the generalized canonical gauge that we formulated in [23], so that the indices in all Feynman rules are Euclidean. All the corresponding Feynman rules of the worldline couplings are provided in the supplementary material to this publication in human visual and machine-readable formats.

3 Diagrammatic evaluation

We then generated the Feynman graphs that contribute to this sector using the input of Feynman rules within the `EFTofPNG` code. The graph distribution among the 4 orders in G of topologies is shown in table 1. All in all, there are 1305 graphs in the present sector, which is to date the largest collection of graphs ever tackled in PN sectors with

Order in G	1	2	3	4	Total
Number of graphs	11	204	702	388	1305

Table 1. The graph distribution among topology orders in G in the N^3LO spin-orbit sector.

spin, and also of all PN sectors up to third subleading order. The full list of graphs and their corresponding values can be found in the supplementary material to this publication in human visible and machine-readable formats. We present only the unique graphs with spin coupling on worldline “1”, where there is another copy of similar graphs from the exchange of worldline labels, $1 \leftrightarrow 2$. The graphs are listed in the files according to the enumeration (n_1, n_2, n_3) , where n_1 indicates order in G , n_2 serial number of topology, n_3 serial number of graph within topology, see [30, 46] for the list of topologies at each order in G . The highest-order topologies at G^4 , and their related integral structure, were analysed in detail [30], where the relevant 388 graphs were evaluated using advanced multi-loop methods.

Similar to [30], the analysis of the complete present sector also builds on the treatment of the N^2LO linear-in-spin sectors obtained in [25, 51] via EFT (also obtained in [52–54] via traditional GR). Beyond the exponentiated volume of graphs in this sector, the majority of graphs at order G^3 are also the most computationally demanding. These include 91 graphs in a 2-loop topology/integral (see figure 11 graph c3 in [46]), which requires reduction via integration-by-parts (IBP). Together with 219 nested 2-loop graphs, and noting that at this order in G the propagators also carry time insertions, these graphs were the most time-consuming to evaluate. Moreover, graphs at order G^1 entail the highest load of time derivatives, and require the maximal iteration ever implemented for the gauge of rotational variables, see e.g. [40] for such iteration required at lower orders. It was then necessary to further upgrade the automation of the diagrammatic evaluation in order to handle the full present sector. The upgrades that were developed upon the `EFTofPNG` code involved mainly projection methods due to the high-rank numerators of integrals [55–57], and IBP methods to reduce the resulting scalar integrals [58], via our adaption of Laporta’s algorithm [59]. We corroborated the values of graphs using parallel evaluations arising from independent development and implementation of codes.

The majority of graphs at each order of G concentrate on the highest-loop ones — as defined in [30] in the “worldline picture”. Moreover, similar to what was observed at order G^4 in [30], the evaluation of graphs yields DimReg poles in the dimensional parameter, $\epsilon_d \equiv d - 3$, in conjunction with logarithms in r/R_0 , for individual graphs. For example, out of the 204 graphs at order G^2 , where there are only 2 possible topologies, 165 are in the single 1-loop topology, which at this high subleading order yields the aforementioned DimReg poles with logarithms. Out of the 702 graphs at order G^3 , where there are topologies from 0- to 2-loop orders, 439 are of 2-loop order, and the majority of these 2-loop topologies yield such DimReg poles and logarithms. The emergence of DimReg poles across all loop orders in sectors as of the N^3LO , makes the transition from sectors up to the N^2LO — an especially critical one. This necessitates an overall switch to a generic dimension across

$n \backslash l$	(N ⁰)LO	N ⁽¹⁾ LO	N ² LO	N ³ LO
S ⁰			+	+
S ¹	+	++	++	++

Table 2. The notation (n, l) and our PN-counting formula for general sectors (n, l) was introduced in [29]. The overall sectors that have to be taken into account for the contributions to the present sector through redefinition of variables. Sectors with “+” introduce only position shifts, whereas sectors with “++” introduce redefinition of both position and rotational variables.

the whole evaluation, with an expansion in the dimensional parameter, ϵ_d , which stands out as the most computationally-demanding aspect in the evaluation of this higher-order sector. By contrast, in sectors up to the NNLO there is only a single 2-loop topology that yields very few graphs overall, which actually requires to keep the dimension generic, while otherwise the dimension can be practically specified already at the start of evaluation. Moreover, in sectors up to the NNLO even these few graphs that are somewhat more difficult to evaluate, never give rise to DimReg poles in their final result.

The systematic zero values in graphs of the 2 factorizable topologies at order G^3 , which were already observed at the NNLO due to contact interaction terms, are also found here [30, 51]. Moreover, similar to the analysis in [30], all the non-vanishing graphs in these 2 factorizable topologies are proportional to the transcendental number that is a Riemann zeta value, $\zeta(2) \equiv \pi^2/6$. The latter feature is also found in graphs of the single 2-rank topology at order G^3 , that through IBP reduction contains a linear combination of the factorizable and nested master integrals [30].

3.1 Unreduced action

Summing up all the graphs in this sector, we obtain the unreduced action. At this stage one gets a very bulky action, which contains several large pieces of higher-order time derivatives. The unreduced potential in the Lagrangian, $-V_{\text{N}^3\text{LO}}^{\text{SO}} \subset L$, can be expressed as:

$$V_{\text{N}^3\text{LO}}^{\text{SO}} = \sum_{i=0}^6 V_{3,1}^{(i)} + (1 \leftrightarrow 2), \quad (3.1)$$

where any piece $V^{(i)}$ contains only terms with a total of i higher-order time derivatives beyond the velocity and spin variables, and the indices n, l in the subscript correspond to the sector (n, l) , as in table 2. We present the full unreduced potential according to these pieces in appendix A, and in the supplementary material to this publication. Notice that all in all, there are now DimReg poles with logarithms at orders $G^2 - G^4$, and $\zeta(2)$ factors at orders G^3 and G^4 .

4 Reduction of action

At this stage the EFT computation is done, and we now need to reduce the raw generalized action, which consists of several parts with higher-order time derivatives, to an action which

contains only the position, velocity, and spin variables. This reduction is carried out via a formal procedure of variable redefinitions, which was formulated to also handle rotational variables in [40]. To that end, it is useful to consider table 2 which summarizes the overall redefinitions that have to be taken into account from all the sectors relevant to the present one: we also need to take into account contributions to the present sector, which arise from the application of any redefinitions that were required at lower-order sectors. As can be seen in table 2, redefinitions at sectors without spin, which obviously consist only of position shifts, start only at the NNLO, that is at the 2PN sector. This is not the case for sectors with spin, in which position shifts are already required as of the LO at the 1.5PN order, and additional redefinitions of rotational variables are also required as of the NLO.

Position shifts, $\Delta\vec{x}$, scale in both PN and spin orders as the sector in which they were initially applied [40]. Thus simple power-counting shows that for the present sector position shifts from the sectors of both point-mass, i.e. without spin, and of spin-orbit, are only needed to be applied to linear order. This is in notable contrast to higher-spin sectors, starting with the NLO quadratic-in-spin sector, that already requires to go beyond the linear application of position shifts. Rotational redefinitions, ω^{ij} , scale as $v^{-1}S^{-1}$ with respect to the sector in which they were initially applied [40]. Yet since their formulation is inherently different than that of position shifts, it is not clear at this high order whether redefinitions of rotational variables are needed to be applied beyond the linear order, which has been sufficient for all lower-order sectors. To settle this question we extend the formal procedure for redefinition of rotational variables from [40] beyond linear order — in the following section.

4.1 Redefinition of rotational variables

We consider a redefinition of the Lorentz rotation matrix which relates local frames of the spin. Such a redefinition is also a rotation parametrized in a matrix exponential of a small anti-symmetric generator ω :

$$\Lambda^{ij} = \Lambda^{ik} (e^\omega)^{kj}, \quad (4.1)$$

and for the shift we get:

$$\Delta\Lambda^{ij} = \Lambda^{ik}\omega^{kj} + \frac{1}{2}\Lambda^{ik}\omega^{kl}\omega^{lj} + \mathcal{O}(\omega^3), \quad (4.2)$$

with $\Delta_1\Lambda^{ij} \equiv \Lambda^{ik}\omega^{kj}$ and $\Delta_2\Lambda^{ij} \equiv \frac{1}{2}\Lambda^{ik}\omega^{kl}\omega^{lj}$. Similar to [40] we take the redefinition of the spin ΔS^{ij} to be arbitrary, as it is fixed in the end from considering the effect of the redefinition of the Lorentz matrix on the rotational kinetic term in the Lagrangian:

$$L \supset -\frac{1}{2}S_a^{ij}\Omega_a^{ij} - V(\{S_a\}), \quad a \in \{1, 2\}, \quad (4.3)$$

where $\Omega^{ij} \equiv -\Lambda^{ki}\dot{\Lambda}^{kj}$, and we suppressed the dependence in position and velocity variables. Accordingly, in [40] we fixed ΔS to linear order in ω :

$$\Delta_1 S^{ij} = S^{ik}\omega^{kj} - S^{jk}\omega^{ki}, \quad (4.4)$$

and thus now we would like to fix $\Delta_2 S$ in the spin shift:

$$\Delta S^{ij} \equiv \Delta_1 S^{ij} + \Delta_2 S^{ij} + \mathcal{O}(\omega^3). \quad (4.5)$$

We remind that the interaction potentials do not contain any dependence in $\dot{\Lambda}$, but only in spin variables, and thus we should only study the effect of the Lorentz shift on the rotational kinetic term, and accordingly fix the spin shift which affects the potentials. We find the shift in the kinetic term (up to the prefactor $-1/2$ in the action) as:

$$\begin{aligned} \Delta(S^{ij}\Omega^{ij}) &= \Delta_1 S^{ij}\Omega^{ij} - S^{ij}\Delta_1\Lambda^{ki}\dot{\Lambda}^{kj} - S^{ij}\Lambda^{ki}\frac{d}{dt}(\Delta_1\Lambda^{kj}) + \Delta_2 S^{ij}\Omega^{ij} \\ &\quad - \Delta_1 S^{ij}\Delta_1\Lambda^{ki}\dot{\Lambda}^{kj} - \Delta_1 S^{ij}\Lambda^{ki}\frac{d}{dt}(\Delta_1\Lambda^{kj}) - S^{ij}\Delta_1\Lambda^{ki}\frac{d}{dt}(\Delta_1\Lambda^{kj}) \\ &\quad - S^{ij}\Delta_2\Lambda^{ki}\dot{\Lambda}^{kj} - S^{ij}\Lambda^{ki}\frac{d}{dt}(\Delta_2\Lambda^{kj}) + \mathcal{O}(\omega^3) \\ &= \dot{S}^{ij}\omega^{ij} + \Delta_2 S^{ij}\Omega^{ij} - (S^{il}\omega^{lj} - S^{jl}\omega^{li}) \left[\Lambda^{km}\omega^{mi}\dot{\Lambda}^{kj} + \Lambda^{ki}\frac{d}{dt}(\Lambda^{km}\omega^{mj}) \right] \\ &\quad - S^{ij}\Lambda^{kl}\omega^{li}\frac{d}{dt}(\Lambda^{km}\omega^{mj}) - \frac{1}{2}S^{ij} \left[\Lambda^{kl}\omega^{lm}\omega^{mi}\dot{\Lambda}^{kj} + \Lambda^{ki}\frac{d}{dt}(\Lambda^{kl}\omega^{lm}\omega^{mj}) \right] \\ &\quad + \mathcal{O}(\omega^3) \\ &= \dot{S}^{ij}\omega^{ij} + \Delta_2 S^{ij}\Omega^{ij} + (S^{il}\omega^{lj} - S^{jl}\omega^{li}) (\Omega^{kj}\omega^{ki} + \Omega^{ik}\omega^{kj}) \\ &\quad - (S^{ik}\omega^{kj} - S^{jk}\omega^{ki}) \dot{\omega}^{ij} + S^{ij}\Omega^{lm}\omega^{li}\omega^{mj} - S^{ij}\omega^{ki}\dot{\omega}^{kj} \\ &\quad + \frac{1}{2}S^{ij} (\Omega^{kj}\omega^{kl}\omega^{li} + \Omega^{ik}\omega^{kl}\omega^{lj}) - \frac{1}{2}S^{ij} (\dot{\omega}^{ik}\omega^{kj} + \omega^{ik}\dot{\omega}^{kj}) + \mathcal{O}(\omega^3) \\ &= \dot{S}^{ij}\omega^{ij} + S^{ij}\dot{\omega}^{ik}\omega^{kj} + (\Delta_2 S^{ij} + S^{ik}\omega^{jl}\omega^{kl} - S^{kl}\omega^{ik}\omega^{jl}) \Omega^{ij} + \mathcal{O}(\omega^3), \end{aligned} \quad (4.6)$$

where we used that $\Lambda^{ik}\Lambda^{jk} = \delta^{ij}$, and dropped total time derivatives. Therefore, to eliminate dependence in Ω^{ij} from the shift of the kinetic term, one should fix:

$$\Delta_2 S^{ij} = S^{kl}\omega^{ik}\omega^{jl} - \frac{1}{2}(S^{ik}\omega^{jl} - S^{jk}\omega^{il})\omega^{kl}. \quad (4.7)$$

Thus the new spin redefinition to quadratic order is also fixed from the Lorentz shift. Note however that here we have a new feature: beyond the spin shift in eq. (4.7) that affects spin-dependent potentials through the spin variables, we find in eq. (4.6) a new addition to the spin-dependent potentials, originating from the kinetic term:

$$\Delta V_{\dot{\omega}\omega} = \frac{1}{4}S^{ij}(\dot{\omega}^{ik}\omega^{kj} - \omega^{ik}\dot{\omega}^{kj}). \quad (4.8)$$

To recap, we obtained the generic redefinition of spin variables in eq. (4.7), and we discovered a new addition to the spin potentials in eq. (4.8), once one goes beyond linear order in the application of redefinition of rotational variables.

4.2 Redefinition of position and spin

As noted we now need to take into account all the contributions that arise via redefinitions applied in lower-order sectors, before we fix the new redefinition of variables that is

	to from	(0P)N
LO S ¹		$\Delta\vec{x}$

Table 3. Contribution to the LO spin-orbit sector from position shifts in a lower-order sector.

needed to reduce our newly computed action from section 3.1. The reduction is carried out consecutively over the sectors in table 2, according to their actual PN order, see [29] for our PN-counting formula for general (n, l) sectors. Tables 3–8 summarize the build-up of redefinitions that are applied at each of the 6 relevant sectors as noted in table 2, namely the 5 at lower orders, and the present one. Each of the tables 3–8 specifies from which sector the redefinition arose, and to which sector it is applied, so as to yield a contribution to the sector under consideration. Note that in all these tables only the left column involves new redefinitions that are newly defined in the sector under consideration, while all the rest are redefinitions that were already fixed in lower-order sectors.

In sectors where redefinitions of both the position and rotational variables are required, we present here the redefinitions, such that at each sector they were carried out first for the position, and then for the rotational variables. One can of course carry out the redefinition of rotational variables first, then of the position, in which case the redefinitions would be modified, albeit eventually leading to physically equivalent results. The redefinitions of position or rotational variables are fixed at each sector by iteratively eliminating terms with the most higher-order time derivatives at each step, until no terms with higher-order time derivatives are left [40]. After all the redefinitions for the sector are fixed, they are summed, and can all be applied at once at higher-order sectors. To recap, the reduction of higher-order time derivatives from the action is intricate. Yet once it is streamlined in an automated algorithm, its run-time over all sectors is quite rapid.

Let us then present according to the above procedure all redefinitions over the relevant sectors in order. First, for the LO spin-orbit sector at the 1.5PN order, our unreduced action (our unreduced potentials are all computed directly with the EFTofPNG code), and the position shift as noted in table 3, are identical to those we presented in [23]. Next, we proceed to the 2PN sector, where our unreduced potential can be expressed as:

$$V_{2\text{PN}} = \sum_{i=0}^2 V_{2,0}^{(i)}, \quad (4.9)$$

with n, l in the subscript as the sector's indices (n, l) , introduced in table 2, and with:

$$\begin{aligned} V_{2,0}^{(0)} = & -\frac{1}{16}m_1v_1^6 - \frac{1}{16}m_2v_2^6 \\ & + \frac{Gm_1m_2}{8r} \left[10v_1^2\vec{v}_1 \cdot \vec{v}_2 - 3v_1^2v_2^2 + 10\vec{v}_1 \cdot \vec{v}_2v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^2 - 7v_1^4 - 7v_2^4 \right. \\ & + 6\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} - 12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\ & \left. + v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 3(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2 \right] \\ & - \frac{G^2m_1m_2^2}{4r^2} \left[7v_1^2 - 14\vec{v}_1 \cdot \vec{v}_2 + 8v_2^2 + 2(\vec{v}_1 \cdot \vec{n})^2 \right] - \frac{G^2m_1^2m_2}{4r^2} \left[8v_1^2 - 14\vec{v}_1 \cdot \vec{v}_2 \right. \end{aligned}$$

	to from	(0P)N
2PN		$\Delta\vec{x}$

Table 4. Contribution to the 2PN sector from position shifts in a lower-order sector.

	to from	(0P)N	1PN
LO S ¹			$\Delta\vec{x}$
NLO S ¹	$\Delta\vec{x}, \Delta\vec{S}$		

Table 5. Contributions to the NLO spin-orbit sector from position shifts and spin redefinition in lower-order sectors.

$$+7v_2^2 + 2(\vec{v}_2 \cdot \vec{n})^2 \Big] \\ -\frac{G^3 m_1^3 m_2}{2r^3} - \frac{3G^3 m_1^2 m_2^2}{r^3} - \frac{G^3 m_1 m_2^3}{2r^3}, \quad (4.10)$$

where the kinetic term is included, and:

$$\overset{(1)}{V}_{2,0} = -\frac{1}{8} G m_1 m_2 \Big[12\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 14\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n} v_2^2 - v_1^2 \vec{a}_2 \cdot \vec{n} + 14\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\ - 12\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \Big], \quad (4.11)$$

$$\overset{(2)}{V}_{2,0} = -\frac{1}{8} G m_1 m_2 r \Big[15\vec{a}_1 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \Big]. \quad (4.12)$$

The position shift as noted in table 4 is then fixed as:

$$(\Delta\vec{x}_1)_{2\text{PN}} = \frac{1}{8} G m_2 \Big[v_2^2 \vec{n} + 12\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 14\vec{v}_2 \cdot \vec{n} \vec{v}_2 - (\vec{v}_2 \cdot \vec{n})^2 \vec{n} \Big] \\ + \frac{7G^2 m_1 m_2}{8r} \vec{n} \\ + \frac{1}{16} G m_2 r \Big[15\vec{a}_2 - \vec{a}_2 \cdot \vec{n} \vec{n} \Big]. \quad (4.13)$$

Next, we proceed to the NLO spin-orbit sector at the 2.5PN order, where our unreduced potential is identical to that we presented in [23]. As noted in table 5, in this sector new redefinitions of both position and rotational variables are required. The position shift is fixed here as:

$$(\Delta\vec{x}_1)_{\text{NLO}}^{\text{SO}} = \frac{1}{8m_1} v_1^2 \vec{S}_1 \times \vec{v}_1 \\ - \frac{G m_2}{2m_1 r} \Big[2\vec{S}_1 \times \vec{n} \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 + 6\vec{S}_1 \times \vec{v}_2 \Big] \\ - \frac{G}{4r} \Big[\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 4\vec{S}_2 \times \vec{n} \vec{v}_2 \cdot \vec{n} - 11\vec{S}_2 \times \vec{v}_2 \Big] \\ - G \dot{\vec{S}}_2 \times \vec{n}, \quad (4.14)$$

from \ to	(0P)N	1PN
2PN		$\Delta \vec{x}$
3PN	$\Delta \vec{x}$	

Table 6. Contributions to the 3PN sector from position shifts in lower-order sectors.

whereas the spin redefinition at this sector is identical to that we presented in [23]:

$$(\omega_1^{ij})_{\text{NLO}}^{\text{SO}} = -\frac{Gm_2}{r} \left[3v_2^i v_1^j + \vec{v}_2 \cdot \vec{n} n^i v_1^j - \vec{v}_2 \cdot \vec{n} n^i v_2^j - (i \leftrightarrow j) \right]. \quad (4.15)$$

We proceed to consider the 3PN sector. Our unreduced potential can be expressed as:

$$V_{\text{3PN}} = \sum_{i=0}^4 V_{3,0}^{(i)}, \quad (4.16)$$

where the explicit pieces are provided in appendix B. Notice there, that now DimReg poles with logarithms show up in pieces with and without higher-order time derivatives. Accordingly, the new redefinitions at this sector, as noted in table 6, are expected to also include such poles, and they should be applied on the Newtonian potential, including its piece that is linear in the DimReg zero, which reads:

$$\Delta V_N = -\epsilon \frac{Gm_1 m_2}{r} \left[\frac{1}{2} - \log \left(\frac{r}{R_0} \right) \right]. \quad (4.17)$$

Note that this extra piece also contains a logarithm. Now, we also add to the unreduced 3PN potential the following total time derivative (TTD):

$$\Delta V_{\text{3PN}}^{\text{TTD}} = \frac{d}{dt} \left[\frac{G^3 m_1^3 m_2}{3r^2} (3\vec{n} \cdot \vec{v}_1 - 2\vec{n} \cdot \vec{v}_2) \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \right] + (1 \leftrightarrow 2), \quad (4.18)$$

in order to ensure that we land on a reduced 3PN potential that contains neither poles nor logarithms, after the redefinition of position. The new one that we fix at this sector reads:

$$(\Delta \vec{x}_1)_{\text{3PN}} = \sum_{i=0}^3 \Delta \vec{x}_{1(3,0)}^{(i)}, \quad (4.19)$$

with the explicit pieces provided in appendix B.

We proceed to the N²LO spin-orbit sector at the 3.5PN order. Our unreduced potential can be expressed as:

$$V_{\text{N}^2\text{LO}}^{\text{SO}} = \sum_{i=0}^4 V_{2,1}^{(i)} + (1 \leftrightarrow 2), \quad (4.20)$$

with the explicit pieces provided in appendix B. As noted in table 7, the new redefinitions in the N²LO spin-orbit sector for both the position and spin variables can be expressed as:

$$(\Delta \vec{x}_1)_{\text{N}^2\text{LO}}^{\text{SO}} = \sum_{i=0}^3 \Delta \vec{x}_{1(2,1)}^{(i)}, \quad (4.21)$$

from \ to	(0P)N	1PN	LO S ¹	2PN
LO S ¹				$\Delta\vec{x}$
2PN			$\Delta\vec{x}$	
NLO S ¹		$\Delta\vec{x}$	$\Delta\vec{S}$	
N ² LO S ¹	$\Delta\vec{x}, \Delta\vec{S}$			

Table 7. Contributions to the N²LO spin-orbit sector from position shifts and spin redefinitions in lower-order sectors.

from \ to	(0P)N	1PN	LO S ¹	2PN	NLO S ¹	3PN
LO S ¹						$\Delta\vec{x}$
2PN					$\Delta\vec{x}$	
NLO S ¹	$(\Delta\Lambda)^2$			$\Delta\vec{x}$	$\Delta\vec{S}$	
3PN			$\Delta\vec{x}$			
N ² LO S ¹		$\Delta\vec{x}$	$\Delta\vec{S}$			
N ³ LO S ¹	$\Delta\vec{x}, \Delta\vec{S}$					

Table 8. Contributions to the N³LO spin-orbit sector from position shifts and spin redefinitions in lower-order sectors.

$$(\omega^{ij})_{\text{N}^2\text{LO}}^{\text{SO}} = \sum_{k=0}^1 \omega_{1(2,1)}^{ij} - (i \leftrightarrow j), \quad (4.22)$$

with the explicit pieces provided in appendix B.

Finally, we arrive at the present N³LO spin-orbit sector at the 4.5PN order. As we already noted in our unreduced action in section 3.1, there are DimReg poles with logarithms that show up in our unreduced action in pieces with and without higher-order time derivatives. Accordingly, the new redefinitions at the present sector, as noted in table 8, are also expected to include such poles, and thus similar to the 3PN sector (without spins), these redefinitions should be applied on the LO spin-orbit potential, including its piece that is linear in the DimReg zero, which also contains a logarithm, that reads:

$$\Delta V_{\text{LO}}^{\text{SO}} = \epsilon \left[\frac{2Gm_2(\hat{n} \times (\vec{v}_1 - \vec{v}_2)) \cdot \vec{S}_1}{r^2} \left(1 - \log \left(\frac{r}{R_0} \right) \right) + (1 \leftrightarrow 2) \right]. \quad (4.23)$$

Also similar to the 3PN sector, we also add in advance to the present unreduced potential a TTD:

$$\begin{aligned} \Delta V_{\text{N}^3\text{LO:SO}}^{\text{TTD}} = & \frac{d}{dt} \left[\left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left(-\frac{2G^3 m_2^3}{15r^3} [3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{v}_1 \cdot \vec{n} - 12\vec{v}_2 \cdot \vec{n}) \right. \right. \\ & \left. \left. + 24\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} + 7\vec{v}_2 \cdot \vec{n}) - 20\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2] \right) \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{G^3 m_1^2 m_2}{30 r^3} \left[3 \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (157 \vec{v}_1 \cdot \vec{n} + 79 \vec{v}_2 \cdot \vec{n}) \right. \\
& \quad \left. - 12 \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (19 \vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}) + 155 \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] \Big) \\
& + (1 \leftrightarrow 2), \tag{4.24}
\end{aligned}$$

which leads — after the redefinition of variables that we fix at this sector — to a reduced potential that is free of poles and logarithms.

Now, as noted in table 8, we also need to recall the result of section 4.1, where we extended the redefinition of rotational variables beyond linear order, and found a new addition to the spin potentials in eq. (4.8), which scales as $S\dot{\omega}\omega$. Examining the leading redefinition with ω , that appears in eq. (4.15), we see that it scales as v^4 , and thus we find that it gives rise to a new addition in the present sector, which contributes the following piece to the unreduced potential:

$$\begin{aligned}
\Delta V_{N^3LO:SO}^{\dot{\omega}\omega} = & -\frac{3G^2 m_2^2}{2r^3} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \vec{v}_2 v_2^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 - v_2^4 + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right. \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 2v_2^2 (\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 \vec{v}_1 \cdot \vec{v}_2 \\
& - v_1^2 v_2^2 + \vec{v}_1 \cdot \vec{v}_2 v_2^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 - 2\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& + 2v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 2\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - \vec{v}_1 \cdot \vec{n} v_2^2 + \vec{v}_2 \cdot \vec{n} v_2^2 - 2\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 4\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - 2(\vec{v}_2 \cdot \vec{n})^3) \Big] \\
& + \frac{G^2 m_2^2}{2r^2} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 3v_2^2 \vec{a}_2 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \right. \\
& + \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 3\vec{v}_2 \cdot \vec{n} v_2^2 \\
& + \vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - (\vec{v}_2 \cdot \vec{n})^3) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (9v_2^2 + 7(\vec{v}_2 \cdot \vec{n})^2) \\
& - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (3\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 3v_1^2 \vec{a}_2 \cdot \vec{n} - 3\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& + \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (3\vec{a}_1 \cdot \vec{v}_2 - 3\vec{v}_1 \cdot \vec{a}_2 \\
& + \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (9\vec{v}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 4(\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (3v_1^2 \vec{v}_2 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - (\vec{v}_2 \cdot \vec{n})^3) \\
& - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (9\vec{v}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 4(\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 (9v_1^2 \\
& \left. + 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + (\vec{v}_2 \cdot \vec{n})^2) \right] + (1 \leftrightarrow 2). \tag{4.25}
\end{aligned}$$

Finally, as noted in table 8, the new redefinitions for both the position and spin variables, that are fixed in the present sector, can be written as:

$$(\Delta \vec{x}_1)_{N^3LO}^{SO} = \sum_{i=0}^5 \Delta \vec{x}_{1(3,1)}^{(i)}, \tag{4.26}$$

$$(\omega^{ij})_{N^3LO}^{SO} = \sum_{k=0}^2 \omega_{1(3,1)}^{ij(k)} - (i \leftrightarrow j), \tag{4.27}$$

where the explicit pieces in these redefinitions are provided in appendix B. Notice there that both the position shifts and the spin redefinitions in the present sector contain pieces

that are proportional to the DimReg poles. All of the ingredients of the reduction process, that was detailed in this section, are also provided in the supplementary material to this publication in machine-readable format.

4.3 Generic action

After we carry out the full procedure of reduction described above, we get a generic action, that is an action with no higher-order time derivatives beyond velocity and spin. It is provided in appendix C, and in machine-readable format in the supplementary material to this publication. Clearly this reduced action is much more compact than that which was initially obtained from the EFT computation, see section 3.1 and appendix A. This final action still displays a new important feature in the sectors with spin: the occurrence of a transcendental number — the Riemann zeta value $\zeta(2) \propto \pi^2$ — at orders G^3 and G^4 .

In contrast, we reached an action which is free from the DimReg poles and logarithms that showed up in intermediate stages. This is thanks to the 2 TTDs in eqs. (4.18), (4.24), that we added in advance to the unreduced actions of the 2 relevant sectors at the N^3LO , which start exhibiting such poles. Thus the DimReg poles, which are clearly unphysical, can be removed at the level of the action by adding TTDs to the unreduced (or reduced) action, as we illustrated here prior to the reduction. Alternatively, one can just apply the full reduction procedure, and then use — as is — the reduced action with the unphysical DimReg poles (which would have appeared there at orders G^3 and G^4), for the extraction of physical observables. The latter would obviously not contain such poles nor logarithms, that drop out of physical observables.

As explained in [23] due to our generalized canonical gauge, the equations of motion (EOMs) for both the position and spin are now readily obtained via a variation of the action in eq. (C.1). Specifically for the spin, the generic form of the EOMs from the potentials derived via our EFT of spinning objects [23] was provided there as:

$$\dot{S}_a^{ij} = -4S_a^{k[i}\delta^{j]l}\frac{\delta\int dt V}{\delta S_a^{kl}} = -4S_a^{k[i}\delta^{j]l}\left[\frac{\partial V}{\partial S_a^{kl}} - \frac{d}{dt}\frac{\partial V}{\partial \dot{S}_a^{kl}}\right], \quad (4.28)$$

such that only a variation of the spin potentials with respect to the spin variables is required in order to get the precession equations.

5 General Hamiltonian

From the generic action in eq. (C.1), one can proceed to derive the full general Hamiltonian in a straightforward manner, where one should make a Legendre transform only with respect to the position variables. For this Legendre transform, we need to take into account all the reduced actions resulting from the reduction procedure detailed in section 4.2: in the point-mass sector from the Newtonian up to the 3PN order, and in the spin-orbit sector up to the N^3LO . We then obtain the general Hamiltonian for the present sector, which is provided in appendix D, and in machine-readable format in the supplementary material to this publication.

5.1 Simplified Hamiltonian

Let us start by recalling some standard conventions useful to express simplified Hamiltonians, and subsequently various gauge-invariant relations, see also [40] for more details. First we consider some quantities related with the masses of the binary, namely the total mass $m \equiv m_1 + m_2$, the mass ratio $q \equiv m_1/m_2$, the reduced mass $\mu \equiv m_1 m_2/m$, and the symmetric mass ratio $\nu \equiv m_1 m_2/m^2 = \mu/m = q/(1+q)^2$.

To simplify the general Hamiltonian it is customary to first specify to the center-of-mass (COM) frame where $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$. The COM specification actually results in the major simplification of general Hamiltonians. We then rescale all variables to become dimensionless:

$$\tilde{H} \equiv \frac{H}{\mu}, \quad \tilde{r} \equiv \frac{r}{Gm}, \quad \tilde{L} \equiv \frac{L}{Gm\mu}, \quad \tilde{S}_a \equiv \frac{S_a}{Gm\mu}, \quad (5.1)$$

with the orbital angular momentum $\vec{L} \equiv r\vec{n} \times \vec{p}$. The simplified Hamiltonian in the COM frame then reads:

$$\begin{aligned} \tilde{H}_{\text{N}^3\text{LO}}^{\text{SO}} = & \frac{\nu}{\tilde{r}^6} \tilde{L} \cdot \left[\left(\tilde{S}_1 + \tilde{S}_2 \right) \left(\left(-\frac{79867}{1200} - \frac{27\pi^2}{2} \right) \nu^2 + \left(\frac{479\pi^2}{24} - \frac{752881}{3600} \right) \nu - \frac{1497}{50} \right. \right. \\ & + \frac{\tilde{L}^2}{\tilde{r}} \left(\left(\frac{199109}{2400} + \frac{27\pi^2}{2} \right) \nu^2 + \left(\frac{254077}{1200} - \frac{675\pi^2}{64} \right) \nu - \frac{3}{50} \right) \\ & + \frac{\tilde{L}^4}{\tilde{r}^2} \left(-\frac{29\nu^3}{32} - \frac{263\nu^2}{16} + \frac{83\nu}{32} \right) + \frac{\tilde{L}^6}{\tilde{r}^3} \left(\frac{151\nu^3}{128} - \frac{101\nu^2}{32} + \frac{109\nu}{128} \right) \\ & + \tilde{p}_r^2 \tilde{r} \left(\left(-\frac{635261}{2400} - 54\pi^2 \right) \nu^2 + \left(\frac{675\pi^2}{16} - \frac{296233}{1200} \right) \nu + \frac{6}{25} \right. \\ & + \frac{\tilde{L}^2}{\tilde{r}} \left(-\frac{17\nu^3}{4} - \frac{1695\nu^2}{32} + \frac{815\nu}{32} \right) + \frac{\tilde{L}^4}{\tilde{r}^2} \left(\frac{777\nu^3}{128} - \frac{387\nu^2}{32} + \frac{471\nu}{128} \right) \\ & + \tilde{p}_r^4 \tilde{r}^2 \left(-\frac{263\nu^3}{32} - \frac{705\nu^2}{8} + \frac{1091\nu}{32} + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{1671\nu^3}{128} - \frac{471\nu^2}{32} + \frac{375\nu}{128} \right) \right) \\ & + \tilde{p}_r^6 \tilde{r}^3 \left(\frac{2025\nu^3}{128} - \frac{325\nu^2}{32} + \frac{153\nu}{128} \right) \Big) \\ & + \left(\tilde{S}_1/q + \tilde{S}_2 q \right) \left(\left(-\frac{79867}{1200} - \frac{27\pi^2}{2} \right) \nu^2 + \left(\frac{859\pi^2}{64} - \frac{62191}{480} \right) \nu - \frac{303}{8} \right. \\ & + \frac{\tilde{L}^2}{\tilde{r}} \left(\left(\frac{188609}{2400} + \frac{27\pi^2}{2} \right) \nu^2 + \left(\frac{20887}{160} - \frac{771\pi^2}{128} \right) \nu + \frac{63}{16} \right) \\ & + \frac{\tilde{L}^4}{\tilde{r}^2} \left(-\frac{29\nu^3}{32} - \frac{297\nu^2}{32} + \frac{39\nu}{16} - \frac{41}{16} \right) + \frac{\tilde{L}^6}{\tilde{r}^3} \left(\frac{29\nu^3}{32} - \frac{123\nu^2}{32} + \frac{151\nu}{64} - \frac{45}{128} \right) \\ & + \tilde{p}_r^2 \tilde{r} \left(\left(-\frac{20248}{75} - 54\pi^2 \right) \nu^2 + \left(\frac{771\pi^2}{32} - \frac{5683}{160} \right) \nu - \frac{1001}{16} \right. \\ & \left. \left. + \frac{\tilde{L}^2}{\tilde{r}} \left(-\frac{17\nu^3}{4} - \frac{377\nu^2}{16} + \frac{1843\nu}{32} - \frac{63}{8} \right) + \frac{\tilde{L}^4}{\tilde{r}^2} \left(\frac{153\nu^3}{32} - \frac{531\nu^2}{32} + \frac{525\nu}{64} - \frac{135}{128} \right) \right) \right) \end{aligned}$$

$$\begin{aligned}
& + \tilde{p}_r^4 \tilde{r}^2 \left(-\frac{263\nu^3}{32} - \frac{1401\nu^2}{16} + \frac{2391\nu}{32} - \frac{85}{16} \right. \\
& + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{339\nu^3}{32} - \frac{1491\nu^2}{64} + \frac{597\nu}{64} - \frac{135}{128} \right) \Big) \\
& \left. + \tilde{p}_r^6 \tilde{r}^3 \left(\frac{425\nu^3}{32} - \frac{745\nu^2}{64} + \frac{223\nu}{64} - \frac{45}{128} \right) \right] . \tag{5.2}
\end{aligned}$$

Note that the new addition to the potential from eq. (4.8) vanishes in the COM frame, and thus does not affect this simplified Hamiltonian.

Second, it is also common to make the simplifying assumption that the spins are aligned with the orbital angular momentum, namely that the conditions $\vec{S}_a \cdot \vec{n} = \vec{S}_a \cdot \vec{p} = 0$ hold for both spins. Yet for the spin-orbit sector there is only a single spin in the interaction, which only appears in the Hamiltonian in the overall coupling $S_a \cdot L$ as can be seen in eq. (5.2). Therefore the aligned-spins constraints do not affect the spin-orbit COM Hamiltonian, and yield no further loss of information for generic spin orientations. For this reason, in the simple spin-orbit sector, results obtained for the aligned-spins configuration can be trivially extended to generic spin orientations.

At this point we remark that a simplified EOB Hamiltonian was reconstructed in [22] via some ansatz, assumptions from the EOB approach, and available results from self-force theory. That EOB Hamiltonian is similar to our simplified Hamiltonian in eq. (5.2) in that they are both specified to the COM frame, and trivially satisfy the aligned-spins constraints, by virtue of the simplicity of the spin-orbit coupling. Yet, the simplified EOB Hamiltonian in [22] differs from ours: it is based on some EOB ansatz, namely it is not fixed from theory, and it is specified to the so-called “quasi-isotropic” gauge, where dependence in factors of L^2 is hidden, and which is not clearly well-defined.

Finally, an additional common simplification of the Hamiltonian in eq. (5.2) is to specify it to circular orbits, with the necessary condition: $p_r \equiv \vec{p} \cdot \vec{n} = 0 \Rightarrow p^2 = p_r^2 + L^2/r^2 \rightarrow L^2/r^2$. Notice that this assumption of circular orbits, namely of a constant orbital separation, makes such a simplified Hamiltonian particularly useful in the inspiral phase, but not as useful when the binary is approaching merger, and the PN approximation breaks down.

6 Gauge-invariant relations

Even when we implement the aforementioned necessary condition for circular orbits, the resulting simplified Hamiltonians are still gauge-dependent since they depend on the radial coordinate. To eliminate the latter, we solve for the additional condition for circular orbits: $\dot{p}_r = -\partial \tilde{H}(\tilde{r}, \tilde{L})/\partial \tilde{r} = 0$, to obtain $\tilde{r}(\tilde{L})$. We can then substitute this relation back in the Hamiltonian to get the gauge-invariant binding energy $e \equiv \tilde{H}$ in terms of the gauge-invariant angular momentum. In this manner, we obtain a gauge-invariant relation

between the binding energy and the angular momentum, due to the present sector:

$$(e)_{\text{SO}}^{\text{N}^3\text{LO}}(\tilde{L}) = \frac{\nu}{\tilde{L}^{11}} \left[\left(\frac{35\nu^3}{128} + \frac{369\nu^2}{64} + \left(\frac{3193\pi^2}{192} - \frac{1104095}{1152} \right) \nu + 1512 \right) (\tilde{S}_1 + \tilde{S}_2) \right. \\ \left. + \left(6\nu^2 + \left(\frac{205\pi^2}{16} - \frac{71021}{96} \right) \nu^1 + \frac{123363}{128} \right) (\tilde{S}_1/q + \tilde{S}_2q) \right]. \quad (6.1)$$

This relation is a critical tool for comparing and evaluating the performance of different analytical and numerical descriptions of the binary dynamics.

We can proceed to compute the angular momentum as a function of the orbital frequency. From the latter it is useful to define the gauge-invariant PN parameter $x \equiv \tilde{\omega}^{2/3}$, inferred from Kepler's third law in the Newtonian limit, such that x^{-1} measures the orbital separation. Yet, since the latter is now expressed in terms of the observable emitted frequency, it is gauge-invariant. We can compute this frequency from Hamilton's equation for the orbital phase: $d\phi/d\tilde{t} \equiv \tilde{\omega} = \partial\tilde{H}(\tilde{r}, \tilde{L})/\partial\tilde{L} = 0$. Then, using the definition of the parameter x , and the relation we already found, $\tilde{r}(\tilde{L})$, we obtain $x(\tilde{L})$, and inverting it we get the relation between the angular momentum and the frequency, due to the contribution of the present sector:

$$\frac{1}{\tilde{L}^2} \supset x^{11/2}\nu \left[\left(-\frac{12439\nu^3}{15552} + \frac{17735\nu^2}{288} + \left(\frac{416569}{576} - \frac{3599\pi^2}{96} \right) \nu \right) (\tilde{S}_1 + \tilde{S}_2) \right. \\ \left. + \left(-\frac{70\nu^3}{81} + \frac{521\nu^2}{12} + \left(\frac{13891}{24} - \frac{205\pi^2}{8} \right) \nu + \frac{243}{64} \right) (\tilde{S}_1/q + \tilde{S}_2q) \right]. \quad (6.2)$$

The final gauge-invariant relation that we derive here is the energy as a function of the frequency. This binding energy, in conjunction with the energy flux as functions of the frequency of the GWs, is used to derive the change in the frequency of the GWs over time, namely the phasing of GWs. Plugging the relation $\tilde{L}(x)$ in eq. (6.2) to the relation $e(\tilde{L})$ in eq. (6.1), where the point-mass contributions up to 3PN should also be taken into account, leads to the gauge-invariant relation, commonly referred to as the binding energy, for the present sector:

$$(e)_{\text{SO}}^{\text{N}^3\text{LO}}(\tilde{x}) = x^{11/2}\nu \left[\left(-\frac{265\nu^3}{3888} - \frac{1979\nu^2}{36} + \left(\frac{19679}{144} + \frac{29\pi^2}{24} \right) \nu - 45 \right) (\tilde{S}_1 + \tilde{S}_2) \right. \\ \left. + \left(-\frac{25\nu^3}{324} - \frac{1109\nu^2}{24} + \frac{565\nu}{8} - \frac{135}{16} \right) (\tilde{S}_1/q + \tilde{S}_2q) \right]. \quad (6.3)$$

This binding energy is in complete agreement with that derived via traditional GR methods in [22], which assumed input and results from the EOB approach, and from self-force theory.

As noted in section 5.1, due to the simplicity of the spin-orbit coupling, one can just take the COM Hamiltonian in eq. (5.2), assume that the binding energy that it stands for can be extended to a kinetic energy of scattering, and compute the scattering angle, which assumes aligned spins [22, 60]. As detailed in [34], we carried out this computation, and our consequent scattering angle is in complete agreement with that derived in [22] via traditional GR methods, that assumed input and results from the EOB approach, and from self-force theory.

7 Conclusions

In this paper we derived the complete N^3LO spin-orbit interaction at the 4.5PN order via the EFT of spinning gravitating objects [23]. At the present high orders of loop and of spin at the precision frontier, it is crucial to deploy various independent methodologies, in order to push and carefully establish the state of the art of PN theory. The EFT of spinning gravitating objects uniquely constitutes such a free-standing framework, including its automation in the public code `EFTofPNG` [28], which has enabled the completion of the state of the art at the 4PN order [24–27]. Together with the present paper, the EFT of spinning gravitating objects has also uniquely enabled the recent completion of the new precision frontier at the 5PN order in [33–36].

The sector studied in this paper contains the largest and most elaborate collection of Feynman graphs ever tackled to date in sectors with spins, and in all PN sectors up to the third subleading order. Our computations were carried out via advanced multi-loop methods, and required further code development mainly for the projection and IBP methods. The most significant computational leap, that took place in the present sector, is the need to switch on a generic dimension, and keep track of DimReg expansions, across the whole derivation. This is due to the emergence of DimReg poles across all loop orders in sectors as of the N^3LO in PN theory. This transition constituted the most computationally-demanding aspect of the EFT computation in this work.

At this high order of sectors with spins, it was also critical to extend the formal procedure, which was introduced in [40] for the reduction of higher-order time derivatives from spin variables via redefinitions. We found that the present sector uniquely requires to apply the redefinition of rotational variables beyond linear order, which has been sufficient for all other sectors with spins at lower-orders, and even up to the 5PN order. As a consequence, we found a new unique addition to the spin potentials, originating from the rotational kinetic term. Though the reduction process for the present high-order sector with spins is intricate, we found that once we streamlined it in an automated algorithm, it was executed efficiently and rapidly.

In this paper we provided the full interaction potential in Lagrangian form, and the general Hamiltonian of the sector for the first time. The Lagrangian potential obtained via our framework enables a direct derivation of the physical EOMs for both the position and spin [23]. The general Hamiltonian enables to explore possible EOB applications extended to this sector, and test the performance of such variants of EOB Hamiltonians. The general Hamiltonian also enables to study the conserved integrals of motion, which form a representation of the Poincaré algebra on phase space. Indeed, in [35] we found the complete Poincaré algebra of the present sector, and thus verified the validity of the general Hamiltonian derived in the present paper.

We also derived consequent GW gauge-invariant observables, namely relations among the binding energy, angular momentum, and orbital frequency. We found complete agreement with the binding energy of GW sources, as well as with the extrapolated scattering angle for aligned spins in the scattering problem, derived via traditional GR methods [22]. Yet, the approach implemented in [22] for the spin-orbit sector, following [15, 16], is an ad-

hoc approach, which is limited to this specific sector. It relied on some EOB assumptions, and available high-order results taken from self-force theory, and even required results at the 4PN order (including the tail), which go beyond the N³LO, namely the subleading order that was being targeted [22].

Moreover, the EOB Hamiltonian reconstruction that was carried out in [22] cannot be extended beyond the spin-orbit sector, namely to higher orders in spin, not even to the simple sector that is bilinear in the spins of the binary. This is since the derivation in [22] assumes the aligned-spins simplification, whose results can be trivially extended to generic spin orientations — only in the spin-orbit sector, due to the unique simplicity of the spin-orbit coupling. Such an extension to generic spin orientations, which give rise to rich physics that is present for real precessing binaries, and manifests as modulations of the gravitational waveform, is impossible for all other sectors with spin.

By contrast, our framework is completely independent and generic: it provides a conceptual methodology to tackle the various generic sectors that are needed to complete a certain PN accuracy. It thus constitutes a unique elementary methodology to study PN theory. Our framework is free-standing, and provides self-contained theory, derivations, results, and technology, which are critical to push the precision frontier at the present high orders. Finally, all of the computations in our framework are also automated as extensions of the public `EFTofPNG` code. The independent generic derivation and results presented in this paper, and such development of the `EFTofPNG` code, have been essential to establish the state of the art, and to push the precision frontier for the measurement of GWs.

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A Unreduced action

As noted in section 3.1 the unreduced potential can be written as:

$$V_{\text{N}^3\text{LO}}^{\text{SO}} = \sum_{i=0}^6 V_{3,1}^{(i)} + (1 \leftrightarrow 2), \quad (\text{A.1})$$

where any piece $V^{(i)}$ contains only terms with a total of i higher-order time derivatives, namely beyond the velocity and spin variables, and the indices n, l in the subscript correspond to the sector (n, l) as indicated in table 2.

First, we have a piece without higher-order time derivatives:

$$\begin{aligned} {}^{(0)}V_{3,1} = & -\frac{Gm_2}{8r^2} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5v_1^2 \vec{v}_1 \cdot \vec{v}_2 v_2^2 - v_1^2 (\vec{v}_1 \cdot \vec{v}_2)^2 - 5v_1^6 + 8\vec{v}_1 \cdot \vec{v}_2 v_1^4 - 4v_2^2 v_1^4 \right. \\ & \left. - 4v_1^2 v_2^4 + 6\vec{v}_1 \cdot \vec{v}_2 v_2^4 - 5v_2^6 - 9\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 15\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 \right] \end{aligned}$$

$$\begin{aligned}
& -6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 v_2^2 + 3v_1^2 v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 + 3v_1^2 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 9\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_1^4 + 3(\vec{v}_2 \cdot \vec{n})^2 v_1^4 + 3(\vec{v}_1 \cdot \vec{n})^2 v_2^4 \\
& + 9\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^4 - 15\vec{v}_2 \cdot \vec{n}v_2^2 (\vec{v}_1 \cdot \vec{n})^3 - 15v_1^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 15\vec{v}_1 \cdot \vec{n}v_1^2 (\vec{v}_2 \cdot \vec{n})^3 - 15v_2^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 35(\vec{v}_1 \cdot \vec{n})^3 (\vec{v}_2 \cdot \vec{n})^3 \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 \vec{v}_1 \cdot \vec{v}_2 v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^3 - 2v_2^2 (\vec{v}_1 \cdot \vec{v}_2)^2 + v_1^2 v_2^4 - 3\vec{v}_1 \cdot \vec{v}_2 v_2^4 \\
& + 5v_2^6 - 9\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n}v_2^2 - 3\vec{v}_1 \cdot \vec{v}_2 v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2 \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 - 3v_1^2 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 3(\vec{v}_1 \cdot \vec{n})^2 v_2^4 - 9\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^4 \\
& + 15\vec{v}_2 \cdot \vec{n}v_2^2 (\vec{v}_1 \cdot \vec{n})^3 + 15\vec{v}_1 \cdot \vec{n}v_1^2 (\vec{v}_2 \cdot \vec{n})^3 + 15\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 15v_2^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 - 35(\vec{v}_1 \cdot \vec{n})^3 (\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (3\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_1 \cdot \vec{v}_2 \\
& + 2v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n}v_1^2 v_2^2 - 3v_1^2 \vec{v}_2 \cdot \vec{n}v_2^2 + 2\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 - 2\vec{v}_1 \cdot \vec{n}v_1^4 \\
& - \vec{v}_2 \cdot \vec{n}v_1^4 - 2\vec{v}_1 \cdot \vec{n}v_2^4 - 3\vec{v}_2 \cdot \vec{n}v_2^4 + 3v_1^2 \vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& + 9\vec{v}_2 \cdot \vec{n}v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 3v_1^2 (\vec{v}_2 \cdot \vec{n})^3 + 6\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 6\vec{v}_1 \cdot \vec{n}v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 15(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^3) \\
& + \frac{G^2 m_1 m_2}{3r^3} [\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (36v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 4v_1^2 v_2^2 - 3\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 20(\vec{v}_1 \cdot \vec{v}_2)^2 - 30v_1^4 \\
& + 21v_2^4 + 184\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 252\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 144v_1^2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - 132\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 + 148v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 80v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 12v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 96(\vec{v}_1 \cdot \vec{n})^4 - 456\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 + 264(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 96\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 \\
& + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (21v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 10v_1^2 v_2^2 + 15\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 14(\vec{v}_1 \cdot \vec{v}_2)^2 - 3v_1^4 \\
& - 9v_2^4 - 36\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} + 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 36\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 \\
& - 36\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 - 8v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 28v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 24(\vec{v}_1 \cdot \vec{n})^4 \\
& + 120\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 96(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (6\vec{v}_1 \cdot \vec{n}v_1^2 + 22v_1^2 \vec{v}_2 \cdot \vec{n} \\
& - 106\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 26\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 52\vec{v}_1 \cdot \vec{n}v_2^2 + 27\vec{v}_2 \cdot \vec{n}v_2^2 + 8(\vec{v}_1 \cdot \vec{n})^3 \\
& + 92\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 136\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) - \frac{G^2 m_2^2}{48r^3} [\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (903v_1^2 \vec{v}_1 \cdot \vec{v}_2 \\
& - 472v_1^2 v_2^2 - 1488\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 140(\vec{v}_1 \cdot \vec{v}_2)^2 - 267v_1^4 + 1464v_2^4 + 672\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} \\
& - 3680\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 8256\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - 48v_1^2 (\vec{v}_1 \cdot \vec{n})^2 - 336\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& + 448v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 296v_1^2 (\vec{v}_2 \cdot \vec{n})^2 + 3000\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 8064v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 384\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 960(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 768\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 - 120(\vec{v}_2 \cdot \vec{n})^4) \\
& - 8\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (9v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 34v_1^2 v_2^2 + 78\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 122(\vec{v}_1 \cdot \vec{v}_2)^2 + 69v_2^4 \\
& + 24\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} + 568\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 72\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 12\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& + 136v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 112v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 606\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 324v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 48\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 672(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 912\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 \\
& - 183(\vec{v}_2 \cdot \vec{n})^4) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (423\vec{v}_1 \cdot \vec{n}v_1^2 - 748v_1^2 \vec{v}_2 \cdot \vec{n} - 2372\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& + 4448\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 2320\vec{v}_1 \cdot \vec{n}v_2^2 - 3504\vec{v}_2 \cdot \vec{n}v_2^2 + 112(\vec{v}_1 \cdot \vec{n})^3 \\
& + 928\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 3304\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 + 1520(\vec{v}_2 \cdot \vec{n})^3)
\end{aligned}$$

$$\begin{aligned}
& + \frac{G^3 m_1^2 m_2}{150 r^4} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2917 v_1^2 - 2438 \vec{v}_1 \cdot \vec{v}_2 - 479 v_2^2 - 1175 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right. \\
& - 13400 (\vec{v}_1 \cdot \vec{n})^2 + 14200 (\vec{v}_2 \cdot \vec{n})^2) - 3 \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (907 v_1^2 - 1279 \vec{v}_1 \cdot \vec{v}_2 + 372 v_2^2 \\
& + 3625 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 3875 (\vec{v}_1 \cdot \vec{n})^2 + 125 (\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (2404 \vec{v}_1 \cdot \vec{n} \\
& \left. - 4045 \vec{v}_2 \cdot \vec{n}) \right] + \frac{G^3 m_2^3}{150 r^4} \left[3 \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (725 v_1^2 - 1916 \vec{v}_1 \cdot \vec{v}_2 + 1191 v_2^2 \right. \\
& + 5500 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 2200 (\vec{v}_1 \cdot \vec{n})^2 - 3400 (\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (827 v_1^2 \\
& + 3752 \vec{v}_1 \cdot \vec{v}_2 - 4579 v_2^2 - 21700 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 5800 (\vec{v}_1 \cdot \vec{n})^2 + 27800 (\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (5977 \vec{v}_1 \cdot \vec{n} - 2044 \vec{v}_2 \cdot \vec{n}) \Big] \\
& + \frac{G^3 m_1^2 m_2}{5 r^4} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (51 v_1^2 + 16 \vec{v}_1 \cdot \vec{v}_2 - 67 v_2^2 - 80 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right. \\
& - 255 (\vec{v}_1 \cdot \vec{n})^2 + 335 (\vec{v}_2 \cdot \vec{n})^2) - 2 \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (19 v_1^2 - 18 \vec{v}_1 \cdot \vec{v}_2 - v_2^2 \\
& \left. + 90 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 95 (\vec{v}_1 \cdot \vec{n})^2 + 5 (\vec{v}_2 \cdot \vec{n})^2) - 2 \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (46 \vec{v}_1 \cdot \vec{n} - 85 \vec{v}_2 \cdot \vec{n}) \right] \\
& + \frac{2 G^3 m_2^3}{5 r^4} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5 v_1^2 - 17 \vec{v}_1 \cdot \vec{v}_2 + 12 v_2^2 + 85 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right. \\
& - 25 (\vec{v}_1 \cdot \vec{n})^2 - 60 (\vec{v}_2 \cdot \vec{n})^2) + 8 \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 + 6 \vec{v}_1 \cdot \vec{v}_2 - 7 v_2^2 - 30 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& \left. - 5 (\vec{v}_1 \cdot \vec{n})^2 + 35 (\vec{v}_2 \cdot \vec{n})^2) + 3 \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (11 \vec{v}_1 \cdot \vec{n} + 8 \vec{v}_2 \cdot \vec{n}) \right] \\
& + \frac{G^3 m_1 m_2^2}{192 r^4} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 ((10208 - 1341 \pi^2) v_1^2 - (28544 - 1368 \pi^2) \vec{v}_1 \cdot \vec{v}_2 \right. \\
& + (18336 - 27 \pi^2) v_2^2 + (115360 - 6840 \pi^2) \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (41440 - 6705 \pi^2) (\vec{v}_1 \cdot \vec{n})^2 \\
& \left. - (77088 - 135 \pi^2) (\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 ((5952 - 747 \pi^2) v_1^2 \right. \\
& - (17056 - 2772 \pi^2) \vec{v}_1 \cdot \vec{v}_2 + (11104 - 2025 \pi^2) v_2^2 + (71072 - 13860 \pi^2) \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& \left. - (28320 - 3735 \pi^2) (\vec{v}_1 \cdot \vec{n})^2 - (45920 - 10125 \pi^2) (\vec{v}_2 \cdot \vec{n})^2) \right. \\
& \left. + 2 \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 ((9472 + 63 \pi^2) \vec{v}_1 \cdot \vec{n} - (11824 + 1359 \pi^2) \vec{v}_2 \cdot \vec{n}) \right] \\
& - \frac{31 G^4 m_1 m_2^3}{3 r^5} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right] - \frac{3 G^4 m_2^4}{8 r^5} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right] \\
& - \frac{5 G^4 m_1^2 m_2^2}{3 r^5} \left(\frac{1}{\epsilon} - 4 \log \frac{r}{R_0} \right) \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right] \\
& + \frac{G^4 m_1^2 m_2^2}{6 r^5} \left[(40 - 13 \pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - (40 - 13 \pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right] + (1 \leftrightarrow 2), \quad (\text{A.2})
\end{aligned}$$

where the piece at order G^4 , namely on the last 3 lines, was computed in [30].

Then, we separate the piece with one higher-order time derivative into:

$$V_{3,1}^{(1)} = (V_a)_{3,1} + (V_{\dot{S}})_{3,1}, \quad (\text{A.3})$$

where

$$\begin{aligned}
(V_a)_{3,1} &= \frac{G m_2}{8 r} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (12 v_1^2 \vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 6 \vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 5 v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \right. \\
&\quad \left. - 4 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{v}_2 + v_1^2 \vec{a}_1 \cdot \vec{n} v_2^2 + 4 \vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 v_2^2 + 7 \vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} v_2^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -3\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 v_2^2 - 4\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 v_2^2 - v_1^2 v_2^2 \vec{a}_2 \cdot \vec{n} + 5\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_1 \cdot \vec{a}_2 \\
& + 3v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_1 \cdot \vec{a}_2 - 7\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{a}_2 \\
& - 4v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 - 12\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_2 \cdot \vec{a}_2 - \vec{a}_2 \cdot \vec{n}v_1^4 \\
& + \vec{a}_1 \cdot \vec{n}v_2^4 - 9\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 9\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 3v_1^2 \vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
& + 3v_2^2 \vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 9\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^3 \\
& + 12\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2 \vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 12\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 3\vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^3 + 9\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 3\vec{a}_1 \cdot \vec{n}v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 15\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 + 15\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 \\
& - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{n}v_1^2 v_2^2 - 5v_1^2 \vec{v}_2 \cdot \vec{n}v_2^2 + 2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 v_2^2 \\
& + 2\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 - 3\vec{v}_2 \cdot \vec{n}v_1^4 - 2\vec{v}_1 \cdot \vec{n}v_2^4 - 3\vec{v}_2 \cdot \vec{n}v_2^4 + 9\vec{v}_2 \cdot \vec{n}v_2^2 (\vec{v}_1 \cdot \vec{n})^2 \\
& + 6\vec{v}_1 \cdot \vec{n}v_1^2 (\vec{v}_2 \cdot \vec{n})^2 + 3v_2^2 (\vec{v}_2 \cdot \vec{n})^3 + 6\vec{v}_1 \cdot \vec{n}v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 15(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^3) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (36v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 15v_1^2 v_2^2 + 12\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 30v_1^4 - 7v_2^4 + 12\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} \\
& - 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 7\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 2v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 3v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 6(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 - 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{v}_2 - 3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n}v_2^2 - \vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 v_2^2 \\
& + \vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 v_2^2 + 2\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 v_2^2 + v_1^2 \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + v_1^2 v_2^2 \vec{a}_2 \cdot \vec{n} - v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& - 6\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_1 \cdot \vec{a}_2 + 3\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{a}_2 + 4v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \\
& + 12\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_2 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n}v_2^4 + 9\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - 9\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} \\
& - 3\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 3v_2^2 \vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - 3\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^3 - 12\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^3 \\
& + 3\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 3\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 3\vec{a}_1 \cdot \vec{n}v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 15\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 15\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (36v_1^2 \vec{v}_1 \cdot \vec{a}_1 \\
& - 4\vec{v}_1 \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{v}_2 - 2v_1^2 \vec{a}_1 \cdot \vec{v}_2 - 6\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{v}_2 + 12\vec{v}_1 \cdot \vec{a}_1 v_2^2 - \vec{a}_1 \cdot \vec{v}_2 v_2^2 \\
& - 2v_1^2 \vec{v}_1 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 - 2v_2^2 \vec{v}_1 \cdot \vec{a}_2 + 3v_1^2 \vec{v}_2 \cdot \vec{a}_2 + 12v_2^2 \vec{v}_2 \cdot \vec{a}_2 \\
& + v_1^2 \vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 10\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 2\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 3\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 \\
& - 3v_1^2 \vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n}v_2^2 \vec{a}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& - 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 2\vec{v}_1 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - 3\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 9\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (8v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 9v_1^2 v_2^2 \\
& + 5\vec{v}_1 \cdot \vec{v}_2 v_2^2 + 6(\vec{v}_1 \cdot \vec{v}_2)^2 - 9v_1^4 - 8v_2^4 + 5\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& + 3\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 3v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 3v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 9(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) + \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (2v_1^2 \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& + 3\vec{v}_1 \cdot \vec{n}v_1^2 v_2^2 + 2v_1^2 \vec{v}_2 \cdot \vec{n}v_2^2 - 2\vec{v}_1 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 + 3\vec{v}_1 \cdot \vec{n}v_2^4 - 6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - 3v_2^2 (\vec{v}_1 \cdot \vec{n})^3 - 6\vec{v}_2 \cdot \vec{n}v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 9\vec{v}_1 \cdot \vec{n}v_1^2 (\vec{v}_2 \cdot \vec{n})^2 + 15(\vec{v}_1 \cdot \vec{n})^3 (\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (2v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 2v_1^2 v_2^2 + 2(\vec{v}_1 \cdot \vec{v}_2)^2 - 3v_2^4 - 3\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} \\
& + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 + 2v_2^2 (\vec{v}_1 \cdot \vec{n})^2
\end{aligned}$$

$$\begin{aligned}
& + 3v_1^2(\vec{v}_2 \cdot \vec{n})^2 + 3\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 9(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2(v_1^2\vec{v}_1 \cdot \vec{v}_2 \\
& + v_1^2v_2^2 - 3\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} - \vec{v}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 - v_2^2(\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3) \Big] \\
& - \frac{G^2m_1m_2}{6r^2} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(4v_1^2\vec{a}_1 \cdot \vec{n} + 122\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_1 + 17\vec{v}_1 \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n} \\
& + 14\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 85\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 + 17\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 + 10\vec{a}_1 \cdot \vec{n}v_2^2 + 20v_1^2\vec{a}_2 \cdot \vec{n} \\
& + 3v_2^2\vec{a}_2 \cdot \vec{n} - 74\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 - 6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 120\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \\
& + 6\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 296\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 48\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 128\vec{a}_1 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
& - 44\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 104\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1(90\vec{v}_1 \cdot \vec{n}v_1^2 - 7v_1^2\vec{v}_2 \cdot \vec{n} \\
& - 95\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 17\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 43\vec{v}_1 \cdot \vec{n}v_2^2 + 12\vec{v}_2 \cdot \vec{n}v_2^2 - 36(\vec{v}_1 \cdot \vec{n})^3 \\
& + 16\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 40\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1(104v_1^2 - 43\vec{v}_1 \cdot \vec{v}_2 - 48v_2^2 \\
& + 206\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 244(\vec{v}_1 \cdot \vec{n})^2 + 12(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(16v_1^2\vec{a}_1 \cdot \vec{n} \\
& + 5\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_1 - 19\vec{v}_1 \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n} - 34\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 7\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 + 5\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 \\
& + 13\vec{a}_1 \cdot \vec{n}v_2^2 - 7v_1^2\vec{a}_2 \cdot \vec{n} + 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 - 18\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 112\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 52\vec{a}_1 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 16\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 52\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(61\vec{v}_1 \cdot \vec{a}_1 \\
& + 16\vec{a}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{a}_2 + 48\vec{v}_2 \cdot \vec{a}_2 - 26\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} - 14\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 68\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& - \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2(39v_1^2 - 28\vec{v}_1 \cdot \vec{v}_2 - 64v_2^2 + 230\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 186(\vec{v}_1 \cdot \vec{n})^2 \\
& - 4(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_2(9\vec{v}_1 \cdot \vec{n}v_1^2 - 14v_1^2\vec{v}_2 \cdot \vec{n} - 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& - 9\vec{v}_1 \cdot \vec{n}v_2^2 - 20(\vec{v}_1 \cdot \vec{n})^3 + 32\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2(v_1^2 + 13\vec{v}_1 \cdot \vec{v}_2 \\
& + 12v_2^2 - 56\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 34(\vec{v}_1 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2(7v_1^2 - 8(\vec{v}_1 \cdot \vec{n})^2) \Big] \\
& - \frac{G^2m_2^2}{24r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(6v_1^2\vec{a}_1 \cdot \vec{n} + 21\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_1 - 192\vec{v}_1 \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n} \\
& + 42\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 54\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 + 506\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 - 56\vec{a}_1 \cdot \vec{n}v_2^2 + 156v_1^2\vec{a}_2 \cdot \vec{n} \\
& - 56\vec{v}_1 \cdot \vec{n}_2\vec{a}_2 \cdot \vec{n} - 752v_2^2\vec{a}_2 \cdot \vec{n} + 6\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 304\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 40\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \\
& - 1984\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 96\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 1232\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} \\
& - 624\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 80\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 592\vec{a}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{a}_1(15\vec{v}_1 \cdot \vec{n}v_1^2 - 84v_1^2\vec{v}_2 \cdot \vec{n} + 90\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 454\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& - 112\vec{v}_1 \cdot \vec{n}v_2^2 - 1032\vec{v}_2 \cdot \vec{n}v_2^2 - 96\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 160\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 64(\vec{v}_2 \cdot \vec{n})^3) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1(270v_1^2 - 480\vec{v}_1 \cdot \vec{v}_2 + 208v_2^2 - 192\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 9(\vec{v}_1 \cdot \vec{n})^2 \\
& + 94(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(6\vec{v}_1 \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n} + 3\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 86\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 \\
& + 34\vec{a}_1 \cdot \vec{n}v_2^2 - 47v_1^2\vec{a}_2 \cdot \vec{n} + 195\vec{v}_1 \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} + 52v_2^2\vec{a}_2 \cdot \vec{n} + 29\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& + 63\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 42\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 230\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 24\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 564\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 188\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 112\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 \\
& + 224\vec{a}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(48\vec{v}_1 \cdot \vec{a}_1 + 173\vec{a}_1 \cdot \vec{v}_2 + 140\vec{v}_1 \cdot \vec{a}_2 \\
& - 346\vec{v}_2 \cdot \vec{a}_2 - 24\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} - 55\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 220\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 278\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2(63v_1^2 + 398\vec{v}_1 \cdot \vec{v}_2 - 440v_2^2 - 802\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 54(\vec{v}_1 \cdot \vec{n})^2 \\
& + 724(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_2(3\vec{v}_1 \cdot \vec{n}v_1^2 + 9v_1^2\vec{v}_2 \cdot \vec{n} + 45\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2
\end{aligned}$$

$$\begin{aligned}
& + 7\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 17\vec{v}_1 \cdot \vec{n}v_2^2 + 162\vec{v}_2 \cdot \vec{n}v_2^2 - 4(\vec{v}_1 \cdot \vec{n})^3 - 36\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
& + 22\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 84(\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2(171v_1^2 - 512\vec{v}_1 \cdot \vec{v}_2 \\
& + 410v_2^2 - 272\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 42(\vec{v}_1 \cdot \vec{n})^2 + 266(\vec{v}_2 \cdot \vec{n})^2) - 4\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2(4v_1^2 \\
& - 34\vec{v}_1 \cdot \vec{v}_2 + 7v_2^2 + 152\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 8(\vec{v}_1 \cdot \vec{n})^2 - 143(\vec{v}_2 \cdot \vec{n})^2) \\
& + \frac{G^3 m_1^2 m_2}{450r^3} [12\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2398\vec{a}_1 \cdot \vec{n} + 401\vec{a}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (4358\vec{v}_1 \cdot \vec{n} \\
& - 7829\vec{v}_2 \cdot \vec{n}) - 9\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (3157\vec{a}_1 \cdot \vec{n} + 3\vec{a}_2 \cdot \vec{n}) - 9\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (532\vec{v}_1 \cdot \vec{n} \\
& + 3\vec{v}_2 \cdot \vec{n}) + 12000\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 + 11300\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 - 700\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2] \\
& + \frac{G^3 m_1^2 m_2}{15r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) [3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (161\vec{a}_1 \cdot \vec{n} + 67\vec{a}_2 \cdot \vec{n}) \\
& - 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (39\vec{v}_1 \cdot \vec{n} - 157\vec{v}_2 \cdot \vec{n}) - 6\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (74\vec{a}_1 \cdot \vec{n} + \vec{a}_2 \cdot \vec{n}) \\
& - 6\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (19\vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}) + 200\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 + 305\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \\
& + 105\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2] + \frac{2G^3 m_2^3}{15r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) [3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{a}_1 \cdot \vec{n} - 67\vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (10\vec{v}_1 \cdot \vec{n} - 17\vec{v}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (8\vec{a}_1 \cdot \vec{n} + 111\vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (63\vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}) - 5\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 - 25\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \\
& - 130\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 + 110\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2] + \frac{G^3 m_2^3}{450r^3} [36\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (100\vec{a}_1 \cdot \vec{n} \\
& - 879\vec{a}_2 \cdot \vec{n}) + 9\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (650\vec{v}_1 \cdot \vec{n} - 791\vec{v}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (827\vec{a}_1 \cdot \vec{n} \\
& + 12454\vec{a}_2 \cdot \vec{n}) + 18\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (1642\vec{v}_1 \cdot \vec{n} - 891\vec{v}_2 \cdot \vec{n}) - 2550\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \\
& - 6350\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 - 18600\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 + 14800\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2] \\
& + \frac{G^3 m_1 m_2^2}{288r^3} [3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 ((11680 - 972\pi^2)\vec{a}_1 \cdot \vec{n} - (16704 - 423\pi^2)\vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 ((4544 - 828\pi^2)\vec{v}_1 \cdot \vec{n} - (7792 - 171\pi^2)\vec{v}_2 \cdot \vec{n}) \\
& - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 ((10368 - 675\pi^2)\vec{a}_1 \cdot \vec{n} - (13664 - 1422\pi^2)\vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 ((4016 + 171\pi^2)\vec{v}_1 \cdot \vec{n} - (2016 + 810\pi^2)\vec{v}_2 \cdot \vec{n}) \\
& + 16(374 - 9\pi^2)\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 - 8(374 + 63\pi^2)\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \\
& - 28(908 - 9\pi^2)\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 + 4(4112 - 153\pi^2)\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2] + (1 \leftrightarrow 2), \quad (\text{A.4})
\end{aligned}$$

and

$$\begin{aligned}
(V_{\dot{S}})_{3,1} = & -\frac{Gm_2}{8r} [\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (3v_1^2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{n}v_1^2v_2^2 - 5v_1^2\vec{v}_2 \cdot \vec{n}v_2^2 + 2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 v_2^2 \\
& + 2\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 - 3\vec{v}_2 \cdot \vec{n}v_1^4 - 2\vec{v}_1 \cdot \vec{n}v_2^4 - 3\vec{v}_2 \cdot \vec{n}v_2^4 + 9\vec{v}_2 \cdot \vec{n}v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 6\vec{v}_1 \cdot \vec{n}v_1^2(\vec{v}_2 \cdot \vec{n})^2 + 3v_1^2(\vec{v}_2 \cdot \vec{n})^3 + 6\vec{v}_1 \cdot \vec{n}v_2^2(\vec{v}_2 \cdot \vec{n})^2 - 15(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^3) \\
& + \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (3v_1^2\vec{v}_2 \cdot \vec{n}v_2^2 + 2\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 2\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 + 2\vec{v}_1 \cdot \vec{n}v_2^4 \\
& + 3\vec{v}_2 \cdot \vec{n}v_2^4 - 9\vec{v}_2 \cdot \vec{n}v_2^2(\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2(\vec{v}_2 \cdot \vec{n})^3 - 6\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 \\
& - 6\vec{v}_1 \cdot \vec{n}v_2^2(\vec{v}_2 \cdot \vec{n})^2 + 15(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^3) - \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (2v_1^2\vec{v}_1 \cdot \vec{v}_2 - 9v_1^2v_2^2 \\
& + 2\vec{v}_1 \cdot \vec{v}_2 v_2^2 + 2(\vec{v}_1 \cdot \vec{v}_2)^2 - 9v_1^4 - 7v_2^4 + 5\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} + 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2
\end{aligned}$$

$$\begin{aligned}
& + 3v_2^2(\vec{v}_1 \cdot \vec{n})^2 + 3v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 2\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 \\
& - 2v_2^2(\vec{v}_2 \cdot \vec{n})^2 - 9(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2 + 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 \Big] \\
& - \frac{G^2 m_2^2}{48r^2} \left[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (21\vec{v}_1 \cdot \vec{n}v_1^2 - 176v_1^2\vec{v}_2 \cdot \vec{n} + 152\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \right. \\
& + 924\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 216\vec{v}_1 \cdot \vec{n}v_2^2 - 2064\vec{v}_2 \cdot \vec{n}v_2^2 - 160\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
& + 312\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 128(\vec{v}_2 \cdot \vec{n})^3) + 8\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (4v_1^2\vec{v}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& + 147\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 69\vec{v}_1 \cdot \vec{n}v_2^2 + 18\vec{v}_2 \cdot \vec{n}v_2^2 - 16\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 234\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 \\
& + 154(\vec{v}_2 \cdot \vec{n})^3) + 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (29v_1^2 + 274\vec{v}_1 \cdot \vec{v}_2 - 255v_2^2 - 401\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& \left. - 22(\vec{v}_1 \cdot \vec{n})^2 + 423(\vec{v}_2 \cdot \vec{n})^2) \right] - \frac{G^2 m_1 m_2}{6r^2} \left[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (129\vec{v}_1 \cdot \vec{n}v_1^2 + 44v_1^2\vec{v}_2 \cdot \vec{n} \right. \\
& - 50\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 22\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 70\vec{v}_1 \cdot \vec{n}v_2^2 - 30\vec{v}_2 \cdot \vec{n}v_2^2 + 168(\vec{v}_1 \cdot \vec{n})^3 \\
& - 584\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 272\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) - 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (3\vec{v}_1 \cdot \vec{n}v_1^2 + 19v_1^2\vec{v}_2 \cdot \vec{n} \\
& + 32\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n}v_2^2 - 27\vec{v}_2 \cdot \vec{n}v_2^2 + 52(\vec{v}_1 \cdot \vec{n})^3 \\
& - 196\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 120\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) + \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (34v_1^2 - 18\vec{v}_1 \cdot \vec{v}_2 \\
& \left. + 51v_2^2 - 102\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 4(\vec{v}_1 \cdot \vec{n})^2 + 24(\vec{v}_2 \cdot \vec{n})^2) \right] \\
& + \frac{G^3 m_1^2 m_2}{450r^3} \left[3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (2744\vec{v}_1 \cdot \vec{n} + 1027\vec{v}_2 \cdot \vec{n}) - 9\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (529\vec{v}_1 \cdot \vec{n} \right. \\
& \left. + 56\vec{v}_2 \cdot \vec{n}) - 2306\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] + \frac{G^3 m_2^3}{150r^3} \left[21\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (100\vec{v}_1 \cdot \vec{n} - 113\vec{v}_2 \cdot \vec{n}) \right. \\
& \left. + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (559\vec{v}_1 \cdot \vec{n} + 868\vec{v}_2 \cdot \vec{n}) - 2300\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] \\
& + \frac{2G^3 m_2^3}{5r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (10\vec{v}_1 \cdot \vec{n} - 17\vec{v}_2 \cdot \vec{n}) + \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (13\vec{v}_1 \cdot \vec{n} \right. \\
& \left. + 51\vec{v}_2 \cdot \vec{n}) - 10\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] + \frac{G^3 m_1^2 m_2}{15r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (97\vec{v}_1 \cdot \vec{n} \right. \\
& \left. + 21\vec{v}_2 \cdot \vec{n}) - 12\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (19\vec{v}_1 \cdot \vec{n} - 9\vec{v}_2 \cdot \vec{n}) + 97\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] \\
& + \frac{G^3 m_1 m_2^2}{96r^3} \left[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 ((5520 - 954\pi^2)\vec{v}_1 \cdot \vec{n} - (9392 - 297\pi^2)\vec{v}_2 \cdot \vec{n}) \right. \\
& \left. - \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 ((512 - 378\pi^2)\vec{v}_1 \cdot \vec{n} - (2416 - 1017\pi^2)\vec{v}_2 \cdot \vec{n}) \right. \\
& \left. - (4816 + 27\pi^2)\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] + (1 \leftrightarrow 2). \tag{A.5}
\end{aligned}$$

The pieces with further time derivatives are then:

$$\begin{aligned}
V_{3,1}^{(2)} = & -\frac{1}{24} G m_2 \left[(3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \dot{\vec{a}}_1 v_2^2 - 2\dot{\vec{a}}_1 \cdot \vec{v}_2 v_2^2 - 2v_1^2\vec{v}_1 \cdot \dot{\vec{a}}_2 + 2v_1^2\vec{v}_2 \cdot \dot{\vec{a}}_2 \right. \\
& - \dot{\vec{a}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - \vec{v}_1 \cdot \vec{n}v_1^2\dot{\vec{a}}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 - 2\vec{v}_2 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - 2\vec{v}_1 \cdot \dot{\vec{a}}_1 (\vec{v}_2 \cdot \vec{n})^2 + 2\dot{\vec{a}}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + \dot{\vec{a}}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 + \dot{\vec{a}}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) \\
& + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (v_1^2 v_2^2 + v_2^4 - 3\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - v_1^2(\vec{v}_2 \cdot \vec{n})^2 - v_2^2(\vec{v}_2 \cdot \vec{n})^2 \\
& + 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) + 2\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 (18v_1^2\vec{v}_2 \cdot \vec{n} - 12\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n}v_2^2 \\
& \left. + 9\vec{v}_2 \cdot \vec{n}v_2^2 - 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - (\vec{v}_2 \cdot \vec{n})^3) + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\dot{\vec{a}}_1 \cdot \vec{v}_2 v_2^2 + v_1^2\vec{v}_1 \cdot \dot{\vec{a}}_2 \right. \\
& \left. - \dot{\vec{a}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - \vec{v}_1 \cdot \vec{n}v_1^2\dot{\vec{a}}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 - 2\vec{v}_2 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 \right. \\
& \left. - 2\vec{v}_1 \cdot \dot{\vec{a}}_1 (\vec{v}_2 \cdot \vec{n})^2 + 2\dot{\vec{a}}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + \dot{\vec{a}}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 + \dot{\vec{a}}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3) \right]
\end{aligned}$$

$$\begin{aligned}
& -2v_1^2 \vec{v}_2 \cdot \dot{\vec{a}}_2 + \dot{\vec{a}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 + \vec{v}_1 \cdot \vec{n} v_1^2 \dot{\vec{a}}_2 \cdot \vec{n} - \vec{v}_1 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 + 2\vec{v}_2 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - \dot{\vec{a}}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 - \dot{\vec{a}}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (12\vec{v}_1 \cdot \dot{\vec{a}}_1 \vec{v}_2 \cdot \vec{n} \\
& - 2\vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_1 \cdot \vec{v}_2 - v_1^2 \dot{\vec{a}}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 + \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 (18v_1^2 \vec{v}_2 \cdot \vec{n} + 6\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 18\vec{v}_1 \cdot \vec{n} v_2^2 \\
& + 15\vec{v}_2 \cdot \vec{n} v_2^2 - 18\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + (\vec{v}_2 \cdot \vec{n})^3) - 3\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (v_1^2 \vec{v}_1 \cdot \vec{v}_2 \\
& + v_1^2 v_2^2 - 3\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} - \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 - v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (6\vec{v}_1 \cdot \vec{n} v_1^2 - 9v_1^2 \vec{v}_2 \cdot \vec{n} - 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 6\vec{v}_1 \cdot \vec{n} v_2^2 - 2(\vec{v}_1 \cdot \vec{n})^3 \\
& + 9\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 (3\vec{v}_1 \cdot \vec{n} v_2^2 - (\vec{v}_1 \cdot \vec{n})^3) + (3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 v_2^2 \\
& + v_2^4 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - v_2^2 (\vec{v}_2 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) \\
& - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 v_2^2 + v_2^4 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (6v_1^2 \vec{v}_2 \cdot \vec{n} + 6\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n} v_2^2 \\
& + 3\vec{v}_2 \cdot \vec{n} v_2^2 - 6\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + (\vec{v}_2 \cdot \vec{n})^3) + (3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2a_1^2 v_2^2 \\
& + 10\vec{v}_1 \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 6v_1^2 \vec{a}_1 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{a}_2 + 5v_2^2 \vec{a}_1 \cdot \vec{a}_2 \\
& - 11\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{a}_2 + 8\vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 + 2v_1^2 a_2^2 + v_1^2 \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 4\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} \\
& + 3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + \vec{a}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& - 9\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 4\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 2a_2^2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - 2a_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 9\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (4\vec{v}_1 \cdot \vec{a}_1 v_2^2 - 3\vec{a}_1 \cdot \vec{v}_2 v_2^2 \\
& + 5v_1^2 \vec{v}_1 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 4v_2^2 \vec{v}_1 \cdot \vec{a}_2 - 7v_1^2 \vec{v}_2 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 \\
& - 12v_2^2 \vec{v}_2 \cdot \vec{a}_2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 + 2\vec{v}_1 \cdot \vec{n} v_1^2 \vec{a}_2 \cdot \vec{n} + 3v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} \\
& - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 3\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - 4\vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 3\vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 9\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) \\
& + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (24\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 12\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 2\vec{a}_1 \cdot \vec{n} v_2^2 - 3v_1^2 \vec{a}_2 \cdot \vec{n} - v_2^2 \vec{a}_2 \cdot \vec{n} \\
& + 12\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 11\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 2\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 2\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (5\vec{a}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 7\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{a}_2 + 3v_2^2 \vec{a}_1 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{a}_2 \\
& + 4\vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 + 2v_1^2 a_2^2 + 3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + \vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + \vec{a}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} - \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 5\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 \\
& + 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 4\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& - 2a_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 9\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (12a_1^2 \vec{v}_2 \cdot \vec{n} - \vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - 2\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 5\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 + 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 4\vec{v}_1 \cdot \vec{n} a_2^2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (24\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 6\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 3\vec{a}_1 \cdot \vec{n} v_2^2 - 3v_1^2 \vec{a}_2 \cdot \vec{n} \\
& + \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + v_2^2 \vec{a}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 9\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& + 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& - 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (6\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{v}_2 - 3\vec{v}_1 \cdot \vec{a}_1 v_2^2 + 2\vec{a}_1 \cdot \vec{v}_2 v_2^2 - v_1^2 \vec{v}_1 \cdot \vec{a}_2 + 4v_1^2 \vec{v}_2 \cdot \vec{a}_2 \\
& + 2\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} v_2^2 + 2\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2
\end{aligned}$$

$$\begin{aligned}
& -3\vec{v}_1 \cdot \vec{n} v_1^2 \vec{a}_2 \cdot \vec{n} + \vec{v}_1 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - 4\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 - 9\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (14\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 \\
& + 3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + 2\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 2\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - 2\vec{a}_1 \cdot \vec{n} v_2^2 \\
& + 3v_1^2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \\
& + 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - 3\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 (7\vec{v}_1 \cdot \vec{n} v_1^2 + 9v_1^2 \vec{v}_2 \cdot \vec{n} + 4\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 5\vec{v}_1 \cdot \vec{n} v_2^2 - 2\vec{v}_2 \cdot \vec{n} v_2^2 - 9\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& + 3\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 (3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + \vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \\
& + \vec{a}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n})) + (3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (4\vec{v}_1 \cdot \vec{a}_1 v_2^2 - 3\vec{a}_1 \cdot \vec{v}_2 v_2^2 \\
& + 5v_1^2 \vec{v}_1 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 4v_2^2 \vec{v}_1 \cdot \vec{a}_2 - 7v_1^2 \vec{v}_2 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 \\
& - 12v_2^2 \vec{v}_2 \cdot \vec{a}_2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 + 2\vec{v}_1 \cdot \vec{n} v_1^2 \vec{a}_2 \cdot \vec{n} + 3v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} \\
& - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 3\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - 4\vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 3\vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 9\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) + 6\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (v_1^2 v_2^2 \\
& + v_2^4 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - v_2^2 (\vec{v}_2 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) \\
& + 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 (12v_1^2 \vec{v}_2 \cdot \vec{n} - 8\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n} v_2^2 \\
& + 7\vec{v}_2 \cdot \vec{n} v_2^2 - 4\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - (\vec{v}_2 \cdot \vec{n})^3) + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{v}_2 v_2^2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& - 2v_2^2 \vec{v}_1 \cdot \vec{a}_2 + 3v_1^2 \vec{v}_2 \cdot \vec{a}_2 + 12v_2^2 \vec{v}_2 \cdot \vec{a}_2 + 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - 3v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& - 3\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 9\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) \\
& + 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (24\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 3\vec{a}_1 \cdot \vec{n} v_2^2 - 3v_1^2 \vec{a}_2 \cdot \vec{n} \\
& + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 2v_2^2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& + 8\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& + 3\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 (12v_1^2 \vec{v}_2 \cdot \vec{n} + 8\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 12\vec{v}_1 \cdot \vec{n} v_2^2 \\
& + 9\vec{v}_2 \cdot \vec{n} v_2^2 - 12\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + (\vec{v}_2 \cdot \vec{n})^3) + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (3v_1^2 v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^2 \\
& + 3v_2^4 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - 3v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 9(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2) - 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 (7\vec{v}_1 \cdot \vec{n} v_1^2 + 9v_1^2 \vec{v}_2 \cdot \vec{n} + 8\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n} v_2^2 - 4\vec{v}_2 \cdot \vec{n} v_2^2 - 9\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 6\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& + 3\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 (3v_1^2 \vec{v}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2)) \Big] \\
& + \frac{G^2 m_1 m_2}{6r} \Big[(2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (18\vec{v}_1 \cdot \dot{\vec{a}}_1 - 10\dot{\vec{a}}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \dot{\vec{a}}_2 - 4\dot{\vec{a}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& + 4\vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 4\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (6v_1^2 + 2\vec{v}_1 \cdot \vec{v}_2 - 7v_2^2 - 28\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 9(\vec{v}_1 \cdot \vec{n})^2 \\
& + 14(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 (24\vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}) - 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \dot{\vec{a}}_1 \cdot \vec{n} \\
& + 4\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 (16\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) + 2\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (7v_1^2 - 8(\vec{v}_1 \cdot \vec{n})^2) \\
& + 26\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n}) + (\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (15v_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 - 10v_2^2 - 68\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& + 24(\vec{v}_1 \cdot \vec{n})^2 + 20(\vec{v}_2 \cdot \vec{n})^2) + 2\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (6v_1^2 - 4\vec{v}_1 \cdot \vec{v}_2 + 5v_2^2 + 20\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}
\end{aligned}$$

$$\begin{aligned}
& -6(\vec{v}_1 \cdot \vec{n})^2 - 10(\vec{v}_2 \cdot \vec{n})^2 + 2\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (25\vec{v}_1 \cdot \vec{n} - 18\vec{v}_2 \cdot \vec{n}) \\
& + (13\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (a_1^2 - 4\vec{a}_1 \cdot \vec{a}_2 - 4\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (143\vec{v}_1 \cdot \vec{a}_1 \\
& - 68\vec{a}_1 \cdot \vec{v}_2 - 14\vec{v}_1 \cdot \vec{a}_2 - 14\vec{v}_2 \cdot \vec{a}_2 + 80\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} - 128\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& + 16\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 4\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (21\vec{a}_1 \cdot \vec{n} + 8\vec{a}_2 \cdot \vec{n}) \\
& + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (5a_1^2 + 4\vec{a}_1 \cdot \vec{a}_2 + 13\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 12\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{n} \\
& + 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (9\vec{v}_1 \cdot \vec{a}_1 - 2\vec{a}_1 \cdot \vec{v}_2 - 27\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} + 25\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) \\
& + 28\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{a}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 (61\vec{v}_1 \cdot \vec{n} - 9\vec{v}_2 \cdot \vec{n}) - 24\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{a}_1 \cdot \vec{n} \\
& + (2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (43\vec{v}_1 \cdot \vec{a}_1 - 34\vec{a}_1 \cdot \vec{v}_2 - 16\vec{v}_1 \cdot \vec{a}_2 + 21\vec{v}_2 \cdot \vec{a}_2 + 28\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} \\
& - 40\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 34\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (16v_1^2 + 38\vec{v}_1 \cdot \vec{v}_2 \\
& - 53v_2^2 - 148\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 46(\vec{v}_1 \cdot \vec{n})^2 + 64(\vec{v}_2 \cdot \vec{n})^2) + 2\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 (69\vec{v}_1 \cdot \vec{n} \\
& - 7\vec{v}_2 \cdot \vec{n}) + 4\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (4\vec{a}_1 \cdot \vec{v}_2 + 9\vec{v}_1 \cdot \vec{a}_2 - 15\vec{v}_2 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} \\
& + 10\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 15\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{n} + 3\vec{a}_2 \cdot \vec{n}) \\
& + 26\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 (5\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}) + 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (10v_1^2 + \vec{v}_1 \cdot \vec{v}_2 - 15v_2^2 \\
& + 64\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 50(\vec{v}_1 \cdot \vec{n})^2) + 2\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 (48\vec{v}_1 \cdot \vec{n} - 17\vec{v}_2 \cdot \vec{n}) \\
& - 34\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n}) \Big] + \frac{G^2 m_2^2}{24r} \Big[(2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5\dot{\vec{a}}_1 \cdot \vec{v}_2 - 4\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) \\
& + 2\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (10\vec{v}_1 \cdot \vec{v}_2 - 12v_2^2 - 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 9(\vec{v}_2 \cdot \vec{n})^2) - 92\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 \vec{v}_2 \cdot \vec{n} \\
& - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\dot{\vec{a}}_1 \cdot \vec{v}_2 - 36\vec{v}_1 \cdot \dot{\vec{a}}_2 + 8\vec{v}_2 \cdot \dot{\vec{a}}_2 + 4\dot{\vec{a}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 24\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n} \\
& - 4\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (5\dot{\vec{a}}_1 \cdot \vec{n} + 12\dot{\vec{a}}_2 \cdot \vec{n}) - 10\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 (2\vec{v}_1 \cdot \vec{n} \\
& + 35\vec{v}_2 \cdot \vec{n}) + 4\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (16v_1^2 + v_2^2 + 48\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 32(\vec{v}_1 \cdot \vec{n})^2 + (\vec{v}_2 \cdot \vec{n})^2) \\
& + 32\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (4\vec{v}_1 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n}) - 72\vec{S}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n}) + (3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{v}_1 \cdot \vec{v}_2 \\
& - 8v_2^2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 6(\vec{v}_2 \cdot \vec{n})^2) - 2\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - 35v_2^2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& + 61(\vec{v}_2 \cdot \vec{n})^2) - 5\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (3\vec{v}_1 \cdot \vec{n} + 70\vec{v}_2 \cdot \vec{n})) - (\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (123\vec{a}_1 \cdot \vec{a}_2 \\
& + 268a_2^2 + 156\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (30\vec{a}_1 \cdot \vec{v}_2 + 51\vec{v}_1 \cdot \vec{a}_2 \\
& + 20\vec{v}_2 \cdot \vec{a}_2 - 24\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 312\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 308\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& - 78\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} - 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (38\vec{a}_1 \cdot \vec{a}_2 + 63a_2^2 + 47\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& + 6\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (5\vec{a}_1 \cdot \vec{n} + 42\vec{a}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (3\vec{v}_1 \cdot \vec{a}_1 + 36\vec{a}_1 \cdot \vec{v}_2 \\
& + 32\vec{v}_1 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} - 18\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 140\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 12\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (23\vec{a}_1 \cdot \vec{n} + 40\vec{a}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 (153\vec{v}_1 \cdot \vec{n} - 188\vec{v}_2 \cdot \vec{n}) \\
& - 16\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{a}_1 \cdot \vec{n}) + (\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (25\vec{a}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{a}_2 \\
& + 20\vec{v}_2 \cdot \vec{a}_2 - 20\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 312\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 308\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& + \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (35\vec{v}_1 \cdot \vec{v}_2 - 48v_2^2 - 28\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 36(\vec{v}_2 \cdot \vec{n})^2) \\
& - 92\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 4\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{v}_2 - 14\vec{v}_1 \cdot \vec{a}_2 + 3\vec{v}_2 \cdot \vec{a}_2 + 4\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 95\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 141\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (25\vec{a}_1 \cdot \vec{n} + 348\vec{a}_2 \cdot \vec{n})
\end{aligned}$$

$$\begin{aligned}
& -35\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} + 20\vec{v}_2 \cdot \vec{n}) + 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (2v_1^2 + 16\vec{v}_1 \cdot \vec{v}_2 - v_2^2 \\
& + 22\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 - (\vec{v}_2 \cdot \vec{n})^2) + \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 (203\vec{v}_1 \cdot \vec{n} - 228\vec{v}_2 \cdot \vec{n}) \\
& - 8\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 (3\vec{v}_1 \cdot \vec{n} + 35\vec{v}_2 \cdot \vec{n})) \Big] + \frac{2G^2 m_1 m_2}{3r} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{a}_1 \cdot \vec{n})^2 \\
& - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{n})^2 \Big] + \frac{G^2 m_2^2}{6r} \Big[5\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{a}_2 \cdot \vec{n})^2 - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_2 \cdot \vec{n})^2 \Big] \\
& - \frac{G^3 m_1^2 m_2}{18r^2} \Big[210\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 + 142\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 - (178\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 - 87\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2) \Big] \\
& - \frac{G^3 m_1^2 m_2}{3r^2} \Big(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \Big) \Big[14\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 + 10\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 - (5\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 \\
& + 8\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2) \Big] - \frac{2G^3 m_2^3}{3r^2} \Big(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \Big) \Big[\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 + (\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \\
& + \ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2) + (2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 + 9\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2) \Big] - \frac{G^3 m_2^3}{18r^2} \Big[30\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \\
& + (30\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 + 22\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2) + (60\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 + 291\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2) \Big] \\
& - \frac{G^3 m_1 m_2^2}{288r^2} \Big[576(16 - \pi^2)\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 + (2(3296 - 225\pi^2)\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \\
& - 7(160 - 9\pi^2)\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2) + (18(416 - 29\pi^2)\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 \\
& + 3(1456 + 33\pi^2)\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2) \Big] + (1 \leftrightarrow 2), \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
V_{3,1}^{(3)} = & -\frac{1}{24} G m_2 r \Big[(\vec{S}_1 \times \vec{n} \cdot \ddot{\vec{a}}_1 (3\vec{v}_2 \cdot \vec{n} v_2^2 - (\vec{v}_2 \cdot \vec{n})^3) + 6\vec{S}_1 \times \vec{v}_1 \cdot \ddot{\vec{a}}_1 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) \\
& + 9\vec{S}_1 \times \ddot{\vec{a}}_1 \cdot \vec{v}_2 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{n} \cdot \ddot{\vec{a}}_2 (3\vec{v}_1 \cdot \vec{n} v_1^2 - (\vec{v}_1 \cdot \vec{n})^3) \\
& + 3\vec{S}_1 \times \vec{v}_1 \cdot \ddot{\vec{a}}_2 (v_1^2 + (\vec{v}_1 \cdot \vec{n})^2)) + (\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_2 \cdot \vec{n} v_2^2 - (\vec{v}_2 \cdot \vec{n})^3) \\
& - \ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (3\vec{v}_2 \cdot \vec{n} v_2^2 - (\vec{v}_2 \cdot \vec{n})^3) + 9\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2)) \\
& - (3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} - 2\vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 5\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& - \vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} + 2\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 5\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 - 2\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \\
& + \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_1^2 \vec{a}_2 \cdot \vec{n} - 4\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& - \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (2\vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_2 \cdot \vec{a}_2 - \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_1^2 \vec{a}_2 \cdot \vec{n} + v_2^2 \vec{a}_2 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& + 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 6\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (6\vec{v}_1 \cdot \vec{a}_2 \\
& - 5\vec{v}_2 \cdot \vec{a}_2 - 2\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 6\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_1 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 3\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + \vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} \\
& - \vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{a}_2 + 2\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \\
& - \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (2\vec{a}_1 \cdot \vec{a}_2 + 3\vec{a}_1 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& - 3\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_2 \cdot \vec{a}_2 - \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (4\vec{v}_1 \cdot \vec{a}_2 \\
& - 11\vec{v}_2 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (2\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - \vec{a}_1 \cdot \vec{n} v_2^2 \\
& + \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (14\vec{v}_1 \cdot \vec{a}_1 - 4\vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 (7v_1^2 + 2\vec{v}_1 \cdot \vec{v}_2 + 6v_2^2 + 15\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (\vec{v}_2 \cdot \vec{n})^2)
\end{aligned}$$

$$\begin{aligned}
& + 3\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 (\dot{\vec{a}}_1 \cdot \vec{v}_2 - \dot{\vec{a}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (3\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + \vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (2\vec{v}_1 \cdot \vec{a}_1 + \vec{a}_1 \cdot \vec{v}_2 \\
& + 2\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2 (5v_1^2 - \vec{v}_1 \cdot \vec{v}_2 - v_2^2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& + 5(\vec{v}_1 \cdot \vec{n})^2) + 6\vec{S}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{a}_1 + \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n}) + (3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 4\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (3\vec{v}_2 \cdot \vec{n} v_2^2 \\
& - (\vec{v}_2 \cdot \vec{n})^3) + 12\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) - 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) + 6\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \dot{\vec{a}}_2 - 2\vec{v}_2 \cdot \dot{\vec{a}}_2 \\
& - \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 27\dot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) - 3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (3v_1^2 \vec{v}_2 \cdot \vec{n} \\
& + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) + 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (5v_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 \\
& - 2v_2^2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 5(\vec{v}_1 \cdot \vec{n})^2) - 6\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 (v_1^2 + (\vec{v}_1 \cdot \vec{n})^2)) \\
& - (3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 \dot{\vec{a}}_2 \cdot \vec{n} + v_2^2 \dot{\vec{a}}_2 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 \\
& + 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (3\vec{v}_2 \cdot \vec{n} v_2^2 - (\vec{v}_2 \cdot \vec{n})^3) \\
& - 6\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + v_2^2 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 + 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 9\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (3\vec{v}_2 \cdot \dot{\vec{a}}_2 + 2\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 27\ddot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 (v_2^2 \\
& + (\vec{v}_2 \cdot \vec{n})^2) - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n} v_2^2 \\
& + 2\vec{v}_2 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) + 3\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (7v_1^2 + 6\vec{v}_1 \cdot \vec{v}_2 + 4v_2^2 \\
& + 15\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 3(\vec{v}_2 \cdot \vec{n})^2) - 3\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{v}_2 + v_2^2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \\
& - (6\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (a_1^2 \vec{a}_2 \cdot \vec{n} - \vec{a}_1 \cdot \vec{n} a_2^2) + 3\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (4\vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \\
& - 9\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \dot{\vec{a}}_2 + 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - 4\vec{v}_1 \cdot \vec{n} a_2^2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n})) \\
& - 6\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 (7\vec{a}_1 \cdot \vec{a}_2 + a_2^2 - \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 6\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{n} a_2^2 \\
& - 3\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (9\vec{a}_1 \cdot \vec{a}_2 - 2a_2^2 - 3\vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (3\vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} \\
& - \vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 - 5\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \dot{\vec{a}}_2 + 4\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 3\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (14a_1^2 - 5\vec{a}_1 \cdot \vec{a}_2 + 3\vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 (28\vec{v}_1 \cdot \vec{a}_1 \\
& - 9\vec{a}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{a}_2 + 4\vec{v}_2 \cdot \vec{a}_2 + 9\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 3\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n})) \\
& - (3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (4\vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 9\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \dot{\vec{a}}_2 + 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& - 4\vec{v}_1 \cdot \vec{n} a_2^2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 6\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (v_1^2 \vec{a}_2 \cdot \vec{n} + v_2^2 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 3\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \dot{\vec{a}}_2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 + 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& - 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 (12\vec{v}_1 \cdot \vec{a}_2 - 11\vec{v}_2 \cdot \vec{a}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 5\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \dot{\vec{a}}_2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n} a_2^2 \\
& + 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (5\vec{a}_1 \cdot \vec{a}_2 - 4a_2^2 - 3\vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& - 3\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 (4\vec{v}_1 \cdot \vec{a}_2 - 21\vec{v}_2 \cdot \vec{a}_2 - 12\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 3\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n})
\end{aligned}$$

$$\begin{aligned}
& + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (2\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 - 3\vec{a}_1 \cdot \vec{n}v_2^2 + 3v_1^2\vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& + 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 3\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 3\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) + 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 (28\vec{v}_1 \cdot \vec{a}_1 \\
& - 5\vec{a}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{a}_2 + 8\vec{v}_2 \cdot \vec{a}_2 + 9\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 6\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& + 3\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2 (14v_1^2 + 8\vec{v}_1 \cdot \vec{v}_2 + 11v_2^2 + 30\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 3(\vec{v}_2 \cdot \vec{n})^2) \\
& + 3\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 (\vec{a}_1 \cdot \vec{v}_2 - 3\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n})) \\
& + \frac{1}{144}G^2m_2^2 \left[(468\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1\vec{a}_2 \cdot \vec{n} + 12\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\dot{\vec{a}}_1 \cdot \vec{n} + 384\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2\vec{a}_1 \cdot \vec{n} \right. \\
& - 347\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 - 253\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2 + 648\vec{S}_1 \times \vec{a}_2 \cdot \dot{\vec{a}}_2) - (144\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2\dot{\vec{a}}_2 \cdot \vec{n} \\
& - 384\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (2\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) + 410\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 - 340\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2) \\
& + (468\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1\vec{a}_2 \cdot \vec{n} - 576\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} + 12\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n} - 20\vec{v}_2 \cdot \vec{n}) \\
& - 347\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 - 23\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2) + (936\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1\vec{a}_2 \cdot \vec{n} + 24\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{a}_1 \cdot \vec{n} \\
& - 26\vec{a}_2 \cdot \vec{n}) - 694\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2) \left. \right] + \frac{1}{6}G^2m_1m_2 \left(\frac{1}{\epsilon} - 2\log \frac{r}{R_0} \right) \left[(32\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_1 \right. \\
& - 40\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 - 8\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2) - (16\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 + 5\dot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 \\
& + 11\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2) - (32\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 - 3\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 + 43\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 \\
& - 8\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2) - 80\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2 \left. \right] - \frac{1}{6}G^2m_2^2 \left(\frac{1}{\epsilon} - 2\log \frac{r}{R_0} \right) \left[(40\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 \right. \\
& + 8\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2) + (16\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 - 5\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2) + (40\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 \\
& - 5\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2) + 80\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2 \left. \right] + \frac{1}{36}G^2m_1m_2 \left[(12\vec{S}_1 \times \vec{n} \cdot \vec{a}_1\vec{a}_2 \cdot \vec{n} \right. \\
& - 24\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (4\vec{a}_1 \cdot \vec{n} - 7\vec{a}_2 \cdot \vec{n}) + 144\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2\vec{a}_1 \cdot \vec{n} + 340\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_1 \\
& - 152\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 - 40\vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2) - (12\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (8\vec{v}_1 \cdot \vec{n} - 7\vec{v}_2 \cdot \vec{n}) \\
& - 204\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2\vec{v}_1 \cdot \vec{n} + 164\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 + 13\dot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 + 25\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2) \\
& + (60\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1\vec{a}_2 \cdot \vec{n} + 12\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (8\vec{v}_1 \cdot \vec{n} - 9\vec{v}_2 \cdot \vec{n}) - 60\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} \\
& + 12\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (12\vec{v}_1 \cdot \vec{n} - 13\vec{v}_2 \cdot \vec{n}) - 406\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 + 21\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 \\
& - 155\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 + 28\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2) - (24\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (4\vec{a}_1 \cdot \vec{n} - 9\vec{a}_2 \cdot \vec{n}) \\
& \left. \left. - 96\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2\vec{a}_1 \cdot \vec{n} + 358\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2 \right) \right] + (1 \leftrightarrow 2), \tag{A.7}
\end{aligned}$$

$$\begin{aligned} \overset{(4)}{V}_{3,1} = & \frac{1}{24} G m_2 r^2 \left[(\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (8\dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2 - \dot{\vec{a}}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (2\vec{v}_1 \cdot \dot{\vec{a}}_2 - 2\vec{v}_2 \cdot \dot{\vec{a}}_2 \right. \\ & - \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 \dot{\vec{a}}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (5\dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2 - \dot{\vec{a}}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) \\ & + \vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} - 3\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (\dot{\vec{a}}_1 \cdot \vec{v}_2 - \dot{\vec{a}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \dot{\vec{a}}_1 \cdot \vec{n} \\ & - 3\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2 (8\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) + 2\vec{S}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \dot{\vec{a}}_1 \cdot \vec{n}) + (3\vec{S}_1 \times \vec{n} \cdot \ddot{\vec{a}}_1 (\vec{v}_2 \cdot \vec{a}_2 \\ & + \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 6\vec{S}_1 \times \vec{v}_1 \cdot \ddot{\vec{a}}_1 \vec{a}_2 \cdot \vec{n} + 9\vec{S}_1 \times \ddot{\vec{a}}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 21\vec{S}_1 \times \ddot{\vec{a}}_1 \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n} \\ & - 3\vec{S}_1 \times \vec{n} \cdot \ddot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{a}_1 + \vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{v}_1 \cdot \ddot{\vec{a}}_2 \vec{a}_1 \cdot \vec{n} - 3\vec{S}_1 \times \vec{a}_1 \cdot \ddot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n}) \\ & - (3\dot{\vec{S}}_1 \times \vec{n} \cdot \ddot{\vec{a}}_2 (v_1^2 + (\vec{v}_1 \cdot \vec{n})^2) + 6\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \ddot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n}) + (3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \dot{\vec{a}}_2 \right. \end{aligned}$$

$$\begin{aligned}
& -2\vec{v}_2 \cdot \dot{\vec{a}}_2 - \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \dot{\vec{a}}_2 - 2\vec{v}_2 \cdot \dot{\vec{a}}_2 - \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) \\
& + 3\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} + 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{v}_2 + v_2^2 + 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \\
& - 3\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 (8\vec{v}_1 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n}) + 6\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n}) + (3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_2 \cdot \dot{\vec{a}}_2 \\
& + \vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_2 \cdot \dot{\vec{a}}_2 + \vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 9\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) + 21\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n} - 3\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n}) \\
& + (3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (5\vec{a}_1 \cdot \dot{\vec{a}}_2 - \vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 6\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 a_2^2 + 6\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_1 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (3\dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2 - \dot{\vec{a}}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 3\ddot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 15\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2 \dot{\vec{a}}_1 \cdot \vec{n}) + (3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{a}_1 \cdot \dot{\vec{a}}_2 - \vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) \\
& + 6\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (2\vec{v}_1 \cdot \dot{\vec{a}}_2 - 2\vec{v}_2 \cdot \dot{\vec{a}}_2 - \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 3\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 \dot{\vec{a}}_2 \cdot \vec{n} \\
& + 9\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (\vec{v}_2 \cdot \dot{\vec{a}}_2 + \vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 12\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 \vec{a}_2 \cdot \vec{n} - 3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (3\vec{a}_1 \cdot \dot{\vec{a}}_2 \\
& - \vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 3\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} + 27\dot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 63\dot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n} \\
& - 3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (\vec{a}_1 \cdot \vec{v}_2 - 3\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - 15\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{a}_1 \cdot \vec{n} \\
& - 3\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2 (16\vec{v}_1 \cdot \vec{n} - 3\vec{v}_2 \cdot \vec{n}) + 6\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{a}_1 \cdot \vec{n}) - (6\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 a_2^2 \\
& - 9\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (\vec{v}_2 \cdot \dot{\vec{a}}_2 + \vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 6\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} - 6\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 a_2^2 \\
& - 27\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{a}_2 - 4\vec{v}_2 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) \\
& - 9\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{a}_2 \cdot \vec{n} - 63\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n}) - (12\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 a_2^2 \\
& + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (5\vec{a}_1 \cdot \vec{a}_2 - 3\vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 9\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2 \vec{a}_2 \cdot \vec{n}) \Big] \\
& - \frac{1}{18} G^2 m_1 m_2 r \left[9\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 + 14\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \right] - \frac{157}{144} G^2 m_2^2 r \ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \\
& - \frac{4}{3} G^2 m_1 m_2 r \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 - \frac{4}{3} G^2 m_2^2 r \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \\
& + (1 \leftrightarrow 2), \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
{}^{(5)}V_{3,1} = & -\frac{1}{72} G m_2 r^3 \left[(3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 \dot{\vec{a}}_2 \cdot \vec{n} + 3\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \dot{\vec{a}}_1 \cdot \vec{n} + 11\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2 \right. \\
& + \vec{S}_1 \times \vec{a}_1 \cdot \ddot{\vec{a}}_2) + (9\ddot{\vec{S}}_1 \times \vec{n} \cdot \ddot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n} + 3\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \ddot{\vec{a}}_2) + (3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 3\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} - 9\ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{v}_2 \cdot \vec{n} + 11\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 - 2\ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2) \\
& + (9\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 \dot{\vec{a}}_2 \cdot \vec{n} + 33\dot{\vec{S}}_1 \times \dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2) + (9\dot{\vec{S}}_1 \times \vec{n} \cdot \ddot{\vec{a}}_2 \vec{a}_1 \cdot \vec{n} + 3\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \ddot{\vec{a}}_2) \\
& \left. + (9\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 \dot{\vec{a}}_2 \cdot \vec{n} + 33\ddot{\vec{S}}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2) - 9\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 \dot{\vec{a}}_2 \cdot \vec{n} \right] + (1 \leftrightarrow 2), \tag{A.9}
\end{aligned}$$

$${}^{(6)}V_{3,1} = \frac{1}{72} G m_2 r^4 \ddot{\vec{S}}_1 \times \vec{n} \cdot \ddot{\vec{a}}_2 + (1 \leftrightarrow 2). \tag{A.10}$$

B Redefinition of position and spin

In this appendix we provide the explicit unreduced potentials, and redefinitions that are fixed in order to reduce them, throughout the reduction process that is described in sec-

tion 4.2. First we consider the 3PN sector. Our unreduced potential can be expressed as:

$$V_{3\text{PN}} = \sum_{i=0}^4 V_{3,0}^{(i)}, \quad (\text{B.1})$$

with

$$\begin{aligned} V_{3,0}^{(0)} = & -\frac{5}{128}m_1v_1^8 - \frac{5}{128}m_2v_2^8 \\ & + \frac{Gm_1m_2}{16r} \left[19v_1^2\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 6v_1^2(\vec{v}_1 \cdot \vec{v}_2)^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^3 - 6v_2^2(\vec{v}_1 \cdot \vec{v}_2)^2 - 11v_1^6 \right. \\ & + 17\vec{v}_1 \cdot \vec{v}_2 v_1^4 - 8v_2^2 v_1^4 - 8v_1^2 v_2^4 + 17\vec{v}_1 \cdot \vec{v}_2 v_2^4 - 11v_2^6 - 4\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\ & + 5\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 + 3v_1^2 v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 5\vec{v}_1 \cdot \vec{v}_2 v_2^2 (\vec{v}_1 \cdot \vec{n})^2 \\ & - 5v_1^2 \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 10\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{v}_2)^2 + 3v_1^2 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 + 7\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_1^4 \\ & + 3(\vec{v}_2 \cdot \vec{n})^2 v_1^4 + 3(\vec{v}_1 \cdot \vec{n})^2 v_2^4 + 7\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^4 - 3\vec{v}_2 \cdot \vec{n} v_2^2 (\vec{v}_1 \cdot \vec{n})^3 \\ & - 9v_1^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 - 3\vec{v}_1 \cdot \vec{n} v_1^2 (\vec{v}_2 \cdot \vec{n})^3 + 15\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \\ & \left. - 9v_2^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 5(\vec{v}_1 \cdot \vec{n})^3 (\vec{v}_2 \cdot \vec{n})^3 \right] \\ & - \frac{G^2 m_1^2 m_2}{48r^2} \left[408v_1^2 \vec{v}_1 \cdot \vec{v}_2 + 168v_1^2 v_2^2 - 396\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 48(\vec{v}_1 \cdot \vec{v}_2)^2 - 264v_1^4 \right. \\ & + 135v_2^4 - 816\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} + 1008\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 192\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 \\ & + 816v_1^2 (\vec{v}_1 \cdot \vec{n})^2 - 936\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 + 120v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 96v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \\ & + 96\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 12v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 8(\vec{v}_1 \cdot \vec{n})^4 + 48(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \\ & - 64\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 \left. \right] + \frac{G^2 m_1 m_2^2}{48r^2} \left[396v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 168v_1^2 v_2^2 - 408\vec{v}_1 \cdot \vec{v}_2 v_2^2 \right. \\ & + 48(\vec{v}_1 \cdot \vec{v}_2)^2 - 135v_1^4 + 264v_2^4 + 192\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} - 1008\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\ & + 816\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - 12v_1^2 (\vec{v}_1 \cdot \vec{n})^2 - 96\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 + 96v_2^2 (\vec{v}_1 \cdot \vec{n})^2 \\ & - 120v_1^2 (\vec{v}_2 \cdot \vec{n})^2 + 936\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 816v_2^2 (\vec{v}_2 \cdot \vec{n})^2 + 64\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 \\ & \left. - 48(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 8(\vec{v}_2 \cdot \vec{n})^4 \right] \\ & - \frac{G^3 m_1^3 m_2}{3r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[3v_1^2 - 5\vec{v}_1 \cdot \vec{v}_2 + 2v_2^2 + 15\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 9(\vec{v}_1 \cdot \vec{n})^2 \right. \\ & \left. - 6(\vec{v}_2 \cdot \vec{n})^2 \right] - \frac{G^3 m_1 m_2^3}{3r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[2v_1^2 - 5\vec{v}_1 \cdot \vec{v}_2 + 3v_2^2 + 15\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right. \\ & \left. - 6(\vec{v}_1 \cdot \vec{n})^2 - 9(\vec{v}_2 \cdot \vec{n})^2 \right] - \frac{G^3 m_1^3 m_2}{36r^3} \left[127v_1^2 - 241\vec{v}_1 \cdot \vec{v}_2 + 105v_2^2 \right. \\ & + 471\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 300(\vec{v}_1 \cdot \vec{n})^2 - 198(\vec{v}_2 \cdot \vec{n})^2 \left. \right] - \frac{G^3 m_1 m_2^3}{36r^3} \left[105v_1^2 - 241\vec{v}_1 \cdot \vec{v}_2 \right. \\ & + 127v_2^2 + 471\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 198(\vec{v}_1 \cdot \vec{n})^2 - 300(\vec{v}_2 \cdot \vec{n})^2 \left. \right] \\ & - \frac{G^3 m_1^2 m_2^2}{576r^3} \left[(11200 - 783\pi^2)v_1^2 - (23264 - 1566\pi^2)\vec{v}_1 \cdot \vec{v}_2 \right. \\ & \left. + (11200 - 783\pi^2)v_2^2 + (69216 - 4698\pi^2)\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (35904 - 2349\pi^2)(\vec{v}_1 \cdot \vec{n})^2 \right. \end{aligned}$$

$$\begin{aligned}
& - (35904 - 2349\pi^2)(\vec{v}_2 \cdot \vec{n})^2 \Big] \\
& + \frac{3G^4 m_1^4 m_2}{8r^4} + \frac{6G^4 m_1^3 m_2^2}{r^4} + \frac{6G^4 m_1^2 m_2^3}{r^4} + \frac{3G^4 m_1 m_2^4}{8r^4}, \tag{B.2}
\end{aligned}$$

where the kinetic term is included and:

$$\begin{aligned}
{}^{(1)}V_{3,0} = & -\frac{1}{16}Gm_1m_2 \left[28v_1^2\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 8\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 10v_1^2\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \right. \\
& - 10\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{v}_2 + 3v_1^2\vec{a}_1 \cdot \vec{n} v_2^2 + 12\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 v_2^2 + 7\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} v_2^2 \\
& - 5\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 - 13\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 v_2^2 - 10\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 v_2^2 + 5v_1^2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - 3v_1^2 v_2^2 \vec{a}_2 \cdot \vec{n} + 10\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_1 \cdot \vec{a}_2 + 13v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 10\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 \\
& + 10\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_1 \cdot \vec{a}_2 - 7\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{a}_2 - 12v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 8\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 \\
& - 28\vec{v}_1 \cdot \vec{n} v_2^2 \vec{v}_2 \cdot \vec{a}_2 - 3\vec{a}_2 \cdot \vec{n} v_1^4 + 3\vec{a}_1 \cdot \vec{n} v_1^4 - 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 \\
& + 3\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 3v_1^2 \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 5\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \\
& + 3v_2^2 \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 13\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 + \vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^3 \\
& + 12\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2 \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - 12\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 \\
& - \vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^3 + 5\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 13\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 3\vec{a}_1 \cdot \vec{n} v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 3\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 + 3\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 \Big] \\
& + \frac{G^2 m_1^2 m_2}{6r} \left[32v_1^2 \vec{a}_1 \cdot \vec{n} + 124\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 - 24\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 21\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right. \\
& - 81\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 29\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - 11\vec{a}_1 \cdot \vec{n} v_2^2 + 5v_1^2 \vec{a}_2 \cdot \vec{n} - 5\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - 68\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 5\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 25\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 42\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 20\vec{a}_1 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 22\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \Big] \\
& - \frac{G^2 m_1 m_2^2}{6r} \left[25\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 5\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 5\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - 68\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \right. \\
& + 5\vec{a}_1 \cdot \vec{n} v_2^2 - 11v_1^2 \vec{a}_2 \cdot \vec{n} - 21\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 32v_2^2 \vec{a}_2 \cdot \vec{n} + 29\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& - 81\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 24\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 124\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& - 42\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 22\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + 20\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \Big] \\
& - \frac{5G^3 m_1 m_2^3}{3r^2} \left[\vec{a}_1 \cdot \vec{n} - 7\vec{a}_2 \cdot \vec{n} \right] - \frac{2G^3 m_1 m_2^3}{3r^2} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[\vec{a}_1 \cdot \vec{n} - 7\vec{a}_2 \cdot \vec{n} \right] \\
& - \frac{5G^3 m_1^3 m_2}{3r^2} \left[7\vec{a}_1 \cdot \vec{n} - \vec{a}_2 \cdot \vec{n} \right] - \frac{2G^3 m_1^3 m_2}{3r^2} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[7\vec{a}_1 \cdot \vec{n} - \vec{a}_2 \cdot \vec{n} \right] \\
& - \frac{2G^3 m_1^2 m_2^2}{r^2} \left[16\vec{a}_1 \cdot \vec{n} - \pi^2 \vec{a}_1 \cdot \vec{n} - (16 - \pi^2) \vec{a}_2 \cdot \vec{n} \right], \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
{}^{(2)}V_{3,0} = & -\frac{1}{48}Gm_1m_2r \left[(18\vec{v}_1 \cdot \dot{\vec{a}}_1 v_2^2 - 21\dot{\vec{a}}_1 \cdot \vec{v}_2 v_2^2 - 21v_1^2 \vec{v}_1 \cdot \dot{\vec{a}}_2 + 18v_1^2 \vec{v}_2 \cdot \dot{\vec{a}}_2 \right. \\
& + 3\dot{\vec{a}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 + 3\vec{v}_1 \cdot \vec{n} v_1^2 \dot{\vec{a}}_2 \cdot \vec{n} - 21\vec{v}_1 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 + 18\vec{v}_2 \cdot \dot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& + 18\vec{v}_1 \cdot \dot{\vec{a}}_1 (\vec{v}_2 \cdot \vec{n})^2 - 21\dot{\vec{a}}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 - \dot{\vec{a}}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) \\
& + 3(6a_1^2 v_2^2 + 20\vec{v}_1 \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{a}_2 - 3\vec{a}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 13v_1^2 \vec{a}_1 \cdot \vec{a}_2 + 5\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{a}_2 \\
& + 13v_2^2 \vec{a}_1 \cdot \vec{a}_2 - 11\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{a}_2 + 20\vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 + 6v_1^2 a_2^2 - 3v_1^2 \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \right]
\end{aligned}$$

$$\begin{aligned}
& -12\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_1\vec{a}_2 \cdot \vec{n} - 3\vec{v}_1 \cdot \vec{a}_1\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 5\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} \\
& + 13\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} - 3\vec{a}_1 \cdot \vec{n}v_2^2\vec{a}_2 \cdot \vec{n} + 13\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& + 29\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 12\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 6a_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 6a_1^2(\vec{v}_2 \cdot \vec{n})^2 + 3\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& + \frac{1}{72}G^2m_1^2m_2[484a_1^2 + 347\vec{a}_1 \cdot \vec{a}_2 + 132\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}] + \frac{1}{72}G^2m_1m_2^2[347\vec{a}_1 \cdot \vec{a}_2 \\
& + 484a_2^2 + 132\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}] + \frac{11}{3}G^2m_1^2m_2\left(\frac{1}{\epsilon} - 2\log\frac{r}{R_0}\right)[a_1^2 + 2\vec{a}_1 \cdot \vec{a}_2] \\
& + \frac{11}{3}G^2m_1m_2^2\left(\frac{1}{\epsilon} - 2\log\frac{r}{R_0}\right)[2\vec{a}_1 \cdot \vec{a}_2 + a_2^2] - \frac{1}{3}G^2m_1^2m_2(\vec{a}_1 \cdot \vec{n})^2 \\
& - \frac{1}{3}G^2m_1m_2^2(\vec{a}_2 \cdot \vec{n})^2,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
V_{3,0}^{(3)} = & \frac{1}{16}Gm_1m_2r^2[(6\vec{v}_1 \cdot \dot{\vec{a}}_1\vec{a}_2 \cdot \vec{n} - 7\dot{\vec{a}}_1 \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} - 15\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_1 \cdot \vec{a}_2 + \dot{\vec{a}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \\
& - \vec{v}_1 \cdot \vec{a}_1\dot{\vec{a}}_2 \cdot \vec{n} + 7\vec{a}_1 \cdot \vec{n}\vec{v}_1 \cdot \dot{\vec{a}}_2 + 15\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \dot{\vec{a}}_2 - 6\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \dot{\vec{a}}_2 + \dot{\vec{a}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} \\
& - \vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}) + 6(a_1^2\vec{a}_2 \cdot \vec{n} - \vec{a}_1 \cdot \vec{n}a_2^2)],
\end{aligned} \tag{B.5}$$

$$V_{3,0}^{(4)} = \frac{1}{144}Gm_1m_2r^3[23\dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2 - 3\dot{\vec{a}}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}]. \tag{B.6}$$

The redefinition of position that we fix for the 3PN sector can be written as:

$$(\Delta\vec{x}_1)_{\text{3PN}} = \sum_{i=0}^3 \Delta\vec{x}_{1(3,0)}^{(i)}, \tag{B.7}$$

with:

$$\begin{aligned}
\Delta\vec{x}_{1(3,0)}^{(0)} = & \frac{1}{48}Gm_2[6v_1^2v_2^2\vec{n} - 18\vec{v}_1 \cdot \vec{v}_2v_2^2\vec{n} + 12v_2^4\vec{n} - 24v_1^2\vec{v}_2 \cdot \vec{n}\vec{v}_1 + 24\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_1 \\
& + 12\vec{v}_1 \cdot \vec{n}v_2^2\vec{v}_1 + 72\vec{v}_2 \cdot \vec{n}v_2^2\vec{v}_1 + 12v_1^2\vec{v}_2 \cdot \vec{n}\vec{v}_2 + 12\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_2 - 18\vec{v}_1 \cdot \vec{n}v_2^2\vec{v}_2 \\
& - 90\vec{v}_2 \cdot \vec{n}v_2^2\vec{v}_2 - 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2\vec{n} - 6v_1^2(\vec{v}_2 \cdot \vec{n})^2\vec{n} + 18\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2\vec{n} \\
& - 15v_2^2(\vec{v}_2 \cdot \vec{n})^2\vec{n} - 12\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{v}_1 - 20(\vec{v}_2 \cdot \vec{n})^3\vec{v}_1 + 18\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{v}_2 \\
& + 20(\vec{v}_2 \cdot \vec{n})^3\vec{v}_2 + 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3\vec{n} + 3(\vec{v}_2 \cdot \vec{n})^4\vec{n}] \\
& - \frac{G^2m_1m_2}{96r}[488v_1^2\vec{n} - 150\vec{v}_1 \cdot \vec{v}_2\vec{n} - 254v_2^2\vec{n} + 1891\vec{v}_1 \cdot \vec{n}\vec{v}_1 - 735\vec{v}_2 \cdot \vec{n}\vec{v}_1 \\
& - 1203\vec{v}_1 \cdot \vec{n}\vec{v}_2 + 740\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 108\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - 130(\vec{v}_1 \cdot \vec{n})^2\vec{n} + 217(\vec{v}_2 \cdot \vec{n})^2\vec{n}] \\
& - \frac{G^2m_2^2}{12r}[10\vec{v}_1 \cdot \vec{v}_2\vec{n} - v_2^2\vec{n} + 4\vec{v}_2 \cdot \vec{n}\vec{v}_1 + 10\vec{v}_1 \cdot \vec{n}\vec{v}_2 + 73\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} \\
& + 2(\vec{v}_2 \cdot \vec{n})^2\vec{n}] \\
& + \frac{7G^3m_1^2m_2}{9r^2}\vec{n} + \frac{5G^3m_2^3}{3r^2}\vec{n} + \frac{G^3m_1m_2^2}{2r^2}(47 - 4\pi^2)\vec{n},
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
\Delta\vec{x}_{1(3,0)}^{(1)} = & \frac{1}{64}Gm_2r[34\vec{v}_1 \cdot \vec{a}_2\vec{v}_1 - 90\vec{v}_2 \cdot \vec{a}_2\vec{v}_1 - 34\vec{v}_1 \cdot \vec{a}_2\vec{v}_2 + 90\vec{v}_2 \cdot \vec{a}_2\vec{v}_2 + 11v_1^2\vec{a}_2 \\
& - 50\vec{v}_1 \cdot \vec{v}_2\vec{a}_2 + 99v_2^2\vec{a}_2 - 5v_1^2\vec{a}_2 \cdot \vec{n}\vec{n} + 14\vec{v}_1 \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n}\vec{n} - 13v_2^2\vec{a}_2 \cdot \vec{n}\vec{n}]
\end{aligned}$$

$$\begin{aligned}
& +30\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2\vec{n} - 2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2\vec{n} - 38\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2\vec{n} + 2\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{v}_1 \\
& - 74\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{v}_1 - 2\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{v}_2 + 80\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{v}_2 - 2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \\
& + 88(\vec{v}_2 \cdot \vec{n})^2\vec{a}_2 + 2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n} + 8\vec{a}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{n} \\
& - \frac{1}{288}G^2m_2^2[1099\vec{a}_2 + 201\vec{a}_2 \cdot \vec{n}\vec{n}] - \frac{1}{144}G^2m_1m_2[502\vec{a}_1 + 561\vec{a}_2 - 48\vec{a}_1 \cdot \vec{n}\vec{n} \\
& + 69\vec{a}_2 \cdot \vec{n}\vec{n}] - \frac{11}{3}G^2m_2^2\left(\frac{1}{\epsilon} - 2\log\frac{r}{R_0}\right)\vec{a}_2 - \frac{11}{3}G^2m_1m_2\left(\frac{1}{\epsilon} - 2\log\frac{r}{R_0}\right)[\vec{a}_1 \\
& + \vec{a}_2], \tag{B.9}
\end{aligned}$$

$$\begin{aligned}
\Delta\vec{x}_{1(3,0)}^{(2)} = & -\frac{1}{96}Gm_2r^2[(\vec{v}_1 \cdot \vec{a}_2\vec{n} - 7\vec{v}_2 \cdot \vec{a}_2\vec{n} - 35\dot{\vec{a}}_2 \cdot \vec{n}\vec{v}_1 + 41\dot{\vec{a}}_2 \cdot \vec{n}\vec{v}_2 - 23\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \\
& + 113\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 + \vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}\vec{n} - 7\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}\vec{n}) - 6(3\vec{a}_2^2\vec{n} + 2\vec{a}_2 \cdot \vec{n}\vec{a}_1 \\
& - 22\vec{a}_2 \cdot \vec{n}\vec{a}_2)] + \frac{1}{16}Gm_2r^2\vec{n}(\vec{a}_2 \cdot \vec{n})^2, \tag{B.10}
\end{aligned}$$

$$\Delta\vec{x}_{1(3,0)}^{(3)} = \frac{1}{288}Gm_2r^3[23\ddot{\vec{a}}_2 - 3\ddot{\vec{a}}_2 \cdot \vec{n}\vec{n}]. \tag{B.11}$$

We proceed to consider the N²LO spin-orbit sector at the 3.5PN order. Our unreduced potential can be expressed as:

$$V_{\text{N}^2\text{LO}}^{\text{SO}} = \sum_{i=0}^4 V_{2,1}^{(i)} + (1 \leftrightarrow 2), \tag{B.12}$$

with:

$$\begin{aligned}
V_{2,1}^{(0)} = & -\frac{Gm_2}{4r^2}[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(5v_1^2\vec{v}_1 \cdot \vec{v}_2 - 3v_1^2v_2^2 + 4\vec{v}_1 \cdot \vec{v}_2v_2^2 - 3v_1^4 - 3v_2^4 + 6\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} \\
& - 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 3v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 3v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 15(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2(v_1^2v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^2 \\
& + 3v_2^4 - 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - 3v_2^2(\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2(\vec{v}_2 \cdot \vec{n})^2 + 15(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n}v_1^2 - 2\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n}v_2^2 + 2\vec{v}_2 \cdot \vec{n}v_2^2 - 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2)] \\
& + \frac{G^2m_2^2}{4r^3}[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(13v_1^2 - 41\vec{v}_1 \cdot \vec{v}_2 + 28v_2^2 - 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 8(\vec{v}_1 \cdot \vec{n})^2 \\
& + 12(\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(v_1^2 + 7\vec{v}_1 \cdot \vec{v}_2 - 8v_2^2 - 56\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 \\
& + 62(\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(7\vec{v}_1 \cdot \vec{n} - 38\vec{v}_2 \cdot \vec{n})] - \frac{2G^2m_1m_2}{r^3}[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(2v_1^2 \\
& + \vec{v}_1 \cdot \vec{v}_2 - 3v_2^2 + 12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 16(\vec{v}_1 \cdot \vec{n})^2 + 4(\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{v}_2 \\
& - v_2^2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(2\vec{v}_1 \cdot \vec{n} - 5\vec{v}_2 \cdot \vec{n})] \\
& + \frac{G^3m_1^2m_2}{r^4}[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2] + \frac{5G^3m_1m_2^2}{r^4}[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2] \\
& + \frac{G^3m_2^3}{r^4}[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2], \tag{B.13}
\end{aligned}$$

and we separate terms with one higher-order time derivative into:

$$\overset{(1)}{V}_{2,1} = (V_a)_{2,1} + (V_{\dot{S}})_{2,1}, \tag{B.14}$$

with:

$$\begin{aligned}
(V_a)_{2,1} = & -\frac{5}{16}\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 v_1^4 \\
& \frac{Gm_2}{4r} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (4\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n} v_2^2 - v_1^2 \vec{a}_2 \cdot \vec{n} \right. \\
& + 4\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_1^2 \vec{v}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n} v_2^2 + \vec{v}_2 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \\
& - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (12v_1^2 - 12\vec{v}_1 \cdot \vec{v}_2 + 5v_2^2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (2\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - \vec{a}_1 \cdot \vec{n} v_2^2 + v_1^2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (6\vec{v}_1 \cdot \vec{a}_1 \\
& - \vec{a}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{a}_2 + 2\vec{v}_2 \cdot \vec{a}_2 - \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (6v_1^2 - 4\vec{v}_1 \cdot \vec{v}_2 \\
& + 5v_2^2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + (\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (v_1^2 \vec{v}_2 \cdot \vec{n} \\
& + \vec{v}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (v_1^2 - 2v_2^2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - (\vec{v}_1 \cdot \vec{n})^2) \\
& \left. - \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 (v_1^2 - (\vec{v}_1 \cdot \vec{n})^2) \right] \\
& - \frac{2G^2 m_1 m_2}{r^2} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{a}_1 \cdot \vec{n} + \vec{a}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3\vec{v}_1 \cdot \vec{n} + 2\vec{v}_2 \cdot \vec{n}) \right. \\
& - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \left. \right] - \frac{G^2 m_2^2}{4r^2} \left[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n} \right. \\
& + 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{n} + 14\vec{a}_2 \cdot \vec{n}) \\
& - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n} - 18\vec{v}_2 \cdot \vec{n}) + 14\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 + 27\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \\
& \left. + 12\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \right], \tag{B.15}
\end{aligned}$$

and

$$\begin{aligned}
(V_{\dot{S}})_{2,1} = & \frac{Gm_2}{2r} \left[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 \vec{v}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n} v_2^2 + \vec{v}_2 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) \right. \\
& - \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} v_2^2 + \vec{v}_2 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (3v_1^2 \\
& \left. + 2v_2^2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + (\vec{v}_2 \cdot \vec{n})^2) \right] \\
& - \frac{G^2 m_1 m_2}{r^2} \left[6\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (7\vec{v}_1 \cdot \vec{n} - 5\vec{v}_2 \cdot \vec{n}) \right. \\
& + 7\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \left. \right] - \frac{G^2 m_2^2}{4r^2} \left[\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}) + 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} \right. \\
& \left. + 15\vec{v}_2 \cdot \vec{n}) + 27\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right], \tag{B.16}
\end{aligned}$$

and the pieces with further time derivatives:

$$\begin{aligned}
\overset{(2)}{V}_{2,1} = & -\frac{1}{4}Gm_2 \left[(\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_1 \vec{v}_2 \cdot \vec{n} + 6\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right. \\
& - \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (v_1^2 - (\vec{v}_1 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n}) + (\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) \\
& - \ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) + 6\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n}) + (\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{a}_1 \cdot \vec{a}_2 \\
& \left. + \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (2\vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_2 \cdot \vec{a}_2 + \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \right]
\end{aligned}$$

$$\begin{aligned}
& -\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (3\vec{a}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{a}_1 \cdot \vec{n} - 2\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 (4\vec{v}_1 \cdot \vec{n} \\
& - \vec{v}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{a}_1 \cdot \vec{n}) + (2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_2 \cdot \vec{a}_2 + \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) + 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{a}_2 \\
& - 2\vec{v}_2 \cdot \vec{a}_2 + \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 2\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 12\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \\
& + 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (v_2^2 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 (2\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) \\
& + 2\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n}) \\
& - \frac{G^2 m_1 m_2}{r} [\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 + 6\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2] - \frac{17G^2 m_2^2}{4r} \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2,
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
V_{2,1}^{(3)} = & \frac{1}{4} G m_2 r \left[(\vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 \vec{a}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{a}_1 \cdot \vec{n} + 7\vec{S}_1 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 + \vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2) \right. \\
& + (2\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{v}_1 \cdot \vec{n} + 2\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2) + (\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_2 \cdot \vec{n} - \ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - 2\ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n} + 7\ddot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 - \ddot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2) + (2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} \\
& \left. + 14\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2) \right], \tag{B.18}
\end{aligned}$$

$$V_{2,1}^{(4)} = \frac{1}{4} G m_2 r^2 \ddot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2. \tag{B.19}$$

The new redefinitions in the N²LO spin-orbit sector of both the position and spin variables can be expressed as:

$$(\Delta \vec{x}_1)_{\text{N}^2\text{LO}}^{\text{SO}} = \sum_{i=0}^3 \Delta \vec{x}_{1(2,1)}^{(i)}, \tag{B.20}$$

$$(\omega^{ij})_{\text{N}^2\text{LO}}^{\text{SO}} = \sum_{k=0}^1 \omega_{1(2,1)}^{ij(k)} - (i \leftrightarrow j), \tag{B.21}$$

with:

$$\begin{aligned}
\Delta \vec{x}_{1(2,1)}^{(0)} = & \frac{1}{16m_1} \vec{S}_1 \times \vec{v}_1 v_1^4 \\
& - \frac{G m_2}{8m_1 r} \left[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_2^2 \vec{n} - 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 3(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_2^2 \vec{n} \right. \\
& - 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 3(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (4\vec{v}_1 - 5\vec{v}_2) - 2\vec{S}_1 \times \vec{n} (4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - \vec{v}_1 \cdot \vec{n} v_2^2 - 5\vec{v}_2 \cdot \vec{n} v_2^2 + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + 3(\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_1 \times \vec{v}_1 (5v_1^2 - 16\vec{v}_1 \cdot \vec{v}_2 \\
& \left. + 8v_2^2 - 2(\vec{v}_2 \cdot \vec{n})^2) - 4\vec{S}_1 \times \vec{v}_2 (5\vec{v}_1 \cdot \vec{v}_2 - 5v_2^2 + 2(\vec{v}_2 \cdot \vec{n})^2) \right] \\
& - \frac{G}{16r} \left[4\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (v_2^2 \vec{n} - 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 3(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (7v_1^2 \vec{n} \right. \\
& - 14\vec{v}_1 \cdot \vec{v}_2 \vec{n} + 4v_2^2 \vec{n} + 14\vec{v}_1 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 14\vec{v}_1 \cdot \vec{n} \vec{v}_2 \\
& + 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} - 9(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 9(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 2\vec{S}_2 \times \vec{v}_1 \cdot \vec{v}_2 (9\vec{v}_1 \\
& - 13\vec{v}_2 - 13\vec{v}_1 \cdot \vec{n} \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{n}) - 4\vec{S}_2 \times \vec{n} (2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_2 \cdot \vec{n} v_2^2 \\
& \left. + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + 3(\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_2 \times \vec{v}_2 (9v_1^2 - 26\vec{v}_1 \cdot \vec{v}_2 + 24v_2^2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right]
\end{aligned}$$

$$\begin{aligned}
& -13(\vec{v}_1 \cdot \vec{n})^2 - 7(\vec{v}_2 \cdot \vec{n})^2) \Big] \\
& + \frac{G^2 m_2}{32r^2} \left[25\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} + 164\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 4\vec{S}_1 \times \vec{n} (73\vec{v}_1 \cdot \vec{n} - 13\vec{v}_2 \cdot \vec{n}) \right. \\
& \left. + 185\vec{S}_1 \times \vec{v}_1 - 20\vec{S}_1 \times \vec{v}_2 \right] + \frac{G^2 m_2^2}{4m_1 r^2} \left[4\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{n} \right. \\
& \left. + 4\vec{S}_1 \times \vec{n} (\vec{v}_1 \cdot \vec{n} + 2\vec{v}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{v}_1 + 9\vec{S}_1 \times \vec{v}_2 \right] + \frac{G^2 m_2}{8r^2} \left[5\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} \right. \\
& \left. - 8\vec{S}_2 \times \vec{n} \vec{v}_2 \cdot \vec{n} - 38\vec{S}_2 \times \vec{v}_2 \right] + \frac{G^2 m_1}{32r^2} \left[101\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} + 116\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} \right. \\
& \left. + \vec{S}_2 \times \vec{n} (487\vec{v}_1 \cdot \vec{n} - 287\vec{v}_2 \cdot \vec{n}) - 405\vec{S}_2 \times \vec{v}_1 + 270\vec{S}_2 \times \vec{v}_2 \right], \tag{B.22}
\end{aligned}$$

$$\begin{aligned}
\Delta \vec{x}_{1(2,1)}^{(1)} = & \frac{G m_2}{8m_1} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{a}_2 + \vec{a}_2 \cdot \vec{n} \vec{n}) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (3\vec{a}_2 + \vec{a}_2 \cdot \vec{n} \vec{n}) - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{v}_2 \right. \\
& \left. + \vec{v}_2 \cdot \vec{n} \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{n} + \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{n} + 2\vec{S}_1 \times \vec{n} (3\vec{v}_1 \cdot \vec{a}_2 - 5\vec{v}_2 \cdot \vec{a}_2 \right. \\
& \left. + 3\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) - 4\vec{S}_1 \times \vec{v}_1 \vec{a}_2 \cdot \vec{n} - 8\vec{S}_1 \times \vec{a}_1 \vec{v}_2 \cdot \vec{n} + 9\vec{S}_1 \times \vec{v}_2 \vec{a}_2 \cdot \vec{n} \right. \\
& \left. - 6\vec{S}_1 \times \vec{a}_2 (\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}) \right] + \frac{1}{16} G \left[2\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (3\vec{a}_2 + \vec{a}_2 \cdot \vec{n} \vec{n}) \right. \\
& \left. + \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (5\vec{a}_2 - \vec{a}_2 \cdot \vec{n} \vec{n}) - 8\vec{S}_2 \times \vec{a}_1 \cdot \vec{v}_2 \vec{n} - 2\vec{S}_2 \times \vec{n} \cdot \vec{a}_2 (3\vec{v}_1 - 3\vec{v}_2 - \vec{v}_1 \cdot \vec{n} \vec{n} \right. \\
& \left. + \vec{v}_2 \cdot \vec{n} \vec{n}) + 16\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_2 \vec{n} - 5\vec{S}_2 \times \vec{v}_2 \cdot \vec{a}_2 \vec{n} + 4\vec{S}_2 \times \vec{n} (2\vec{v}_1 \cdot \vec{a}_2 - 3\vec{v}_2 \cdot \vec{a}_2 \right. \\
& \left. + 3\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 2\vec{S}_2 \times \vec{v}_1 \vec{a}_2 \cdot \vec{n} - \vec{S}_2 \times \vec{v}_2 (8\vec{a}_1 \cdot \vec{n} + 9\vec{a}_2 \cdot \vec{n}) - 2\vec{S}_2 \times \vec{a}_2 (7\vec{v}_1 \cdot \vec{n} \right. \\
& \left. + 5\vec{v}_2 \cdot \vec{n}) \right] + \frac{G m_2}{4m_1} \left[\dot{\vec{S}}_1 \times \vec{n} (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) - 6\dot{\vec{S}}_1 \times \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] \\
& + \frac{1}{8} G \left[4\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_2 + \vec{v}_2 \cdot \vec{n} \vec{n}) + \dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 - \vec{v}_2 + \vec{v}_1 \cdot \vec{n} \vec{n} - 3\vec{v}_2 \cdot \vec{n} \vec{n}) \right. \\
& \left. + 11\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2 \vec{n} + 2\dot{\vec{S}}_2 \times \vec{n} (2\vec{v}_1 \cdot \vec{v}_2 - 3v_2^2 + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 3(\vec{v}_2 \cdot \vec{n})^2) \right. \\
& \left. - \dot{\vec{S}}_2 \times \vec{v}_2 (11\vec{v}_1 \cdot \vec{n} + 3\vec{v}_2 \cdot \vec{n}) \right] \\
& + \frac{33G^2 m_2}{8r} \dot{\vec{S}}_1 \times \vec{n} - \frac{237G^2 m_1}{32r} \dot{\vec{S}}_2 \times \vec{n} - \frac{3G^2 m_2}{r} \ddot{\vec{S}}_2 \times \vec{n}, \tag{B.23}
\end{aligned}$$

$$\begin{aligned}
\Delta \vec{x}_{1(2,1)}^{(2)} = & \frac{1}{16} Gr \left[(\vec{S}_2 \times \vec{n} \cdot \vec{a}_2 \vec{n} + 4\vec{S}_2 \times \vec{n} \vec{a}_2 \cdot \vec{n} + 19\vec{S}_2 \times \vec{a}_2) + (4\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} \right. \\
& \left. - 3\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 4\ddot{\vec{S}}_2 \times \vec{n} (\vec{v}_1 \cdot \vec{n} + 3\vec{v}_2 \cdot \vec{n}) - \ddot{\vec{S}}_2 \times \vec{v}_2) - (2\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 \vec{n} \right. \\
& \left. - 12\dot{\vec{S}}_2 \times \vec{n} \vec{a}_2 \cdot \vec{n} - 14\dot{\vec{S}}_2 \times \vec{a}_2) \right] + \frac{G m_2 r}{4m_1} \left[\vec{S}_1 \times \vec{n} \vec{a}_2 \cdot \vec{n} - 7\vec{S}_1 \times \vec{a}_2 \right], \tag{B.24}
\end{aligned}$$

$$\Delta \vec{x}_{1(2,1)}^{(3)} = -\frac{1}{4} Gr^2 \ddot{\vec{S}}_2 \times \vec{n}, \tag{B.25}$$

and:

$$\begin{aligned}
\omega_{1(2,1)}^{(0)} = & -\frac{G m_2}{4r} \left[6v_1^2 v_2^i v_1^j - 6\vec{v}_1 \cdot \vec{v}_2 v_2^i v_1^j + 10v_2^2 v_2^i v_1^j - 6v_1^2 v_1^i v_2^j + 6\vec{v}_1 \cdot \vec{v}_2 v_1^i v_2^j \right. \\
& \left. - 10v_2^2 v_1^i v_2^j - 2v_1^2 \vec{v}_2 \cdot \vec{n} v_1^i n^j + 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i n^j - \vec{v}_1 \cdot \vec{n} v_2^2 v_1^i n^j - 5\vec{v}_2 \cdot \vec{n} v_2^2 v_1^i n^j \right. \\
& \left. - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^i n^j + \vec{v}_1 \cdot \vec{n} v_2^2 v_2^i n^j + 5\vec{v}_2 \cdot \vec{n} v_2^2 v_2^i n^j + 2v_1^2 \vec{v}_2 \cdot \vec{n} n^i v_1^j \right. \\
& \left. - 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 n^i v_1^j + \vec{v}_1 \cdot \vec{n} v_2^2 n^i v_1^j + 5\vec{v}_2 \cdot \vec{n} v_2^2 n^i v_1^j - 4(\vec{v}_2 \cdot \vec{n})^2 v_2^i v_1^j \right]
\end{aligned}$$

$$\begin{aligned}
& + 2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 n^i v_2^j - \vec{v}_1 \cdot \vec{n}v_2^2 n^i v_2^j - 5\vec{v}_2 \cdot \vec{n}v_2^2 n^i v_2^j + 4(\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j \\
& + 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j + 3(\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j - 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j - 3(\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j \\
& - 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 n^i v_1^j - 3(\vec{v}_2 \cdot \vec{n})^3 n^i v_1^j + 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 n^i v_2^j + 3(\vec{v}_2 \cdot \vec{n})^3 n^i v_2^j \\
& - \frac{G^2 m_2^2}{8r^2} \left[54v_2^i v_1^j - 54v_1^i v_2^j + 6\vec{v}_1 \cdot \vec{n}v_1^i n^j + 18\vec{v}_2 \cdot \vec{n}v_1^i n^j + 4\vec{v}_1 \cdot \vec{n}v_2^i n^j \right. \\
& \left. + 15\vec{v}_2 \cdot \vec{n}v_2^i n^j - 6\vec{v}_1 \cdot \vec{n}n^i v_1^j - 18\vec{v}_2 \cdot \vec{n}n^i v_1^j - 4\vec{v}_1 \cdot \vec{n}n^i v_2^j - 15\vec{v}_2 \cdot \vec{n}n^i v_2^j \right] \\
& - \frac{G^2 m_1 m_2}{2r^2} \left[13v_2^i v_1^j - 13v_1^i v_2^j + 31\vec{v}_1 \cdot \vec{n}v_1^i n^j - 10\vec{v}_2 \cdot \vec{n}v_1^i n^j - 17\vec{v}_1 \cdot \vec{n}v_2^i n^j \right. \\
& \left. + 13\vec{v}_2 \cdot \vec{n}v_2^i n^j - 31\vec{v}_1 \cdot \vec{n}n^i v_1^j + 10\vec{v}_2 \cdot \vec{n}n^i v_1^j + 17\vec{v}_1 \cdot \vec{n}n^i v_2^j - 13\vec{v}_2 \cdot \vec{n}n^i v_2^j \right], \\
\end{aligned} \tag{B.26}$$

$$\begin{aligned}
\omega_1^{ij}_{(2,1)} = & \frac{1}{4} G m_2 \left[2\vec{v}_2 \cdot \vec{a}_2 v_1^i n^j + v_2^2 a_1^i n^j - 2\vec{v}_2 \cdot \vec{a}_2 v_2^i n^j - v_2^2 a_2^i n^j - 2\vec{v}_2 \cdot \vec{a}_2 n^i v_1^j \right. \\
& + 6\vec{a}_2 \cdot \vec{n}v_2^i v_1^j + 6\vec{v}_2 \cdot \vec{n}a_2^i v_1^j - v_2^2 n^i a_1^j + 6\vec{v}_2 \cdot \vec{n}v_2^i a_1^j + 2\vec{v}_2 \cdot \vec{a}_2 n^i v_2^j - 6\vec{a}_2 \cdot \vec{n}v_1^i v_2^j \\
& - 6\vec{v}_2 \cdot \vec{n}a_1^i v_2^j + v_2^2 n^i a_2^j - 6\vec{v}_2 \cdot \vec{n}v_1^i a_2^j - 2\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}v_1^i n^j - (\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j \\
& + 2\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}v_2^i n^j + (\vec{v}_2 \cdot \vec{n})^2 a_2^i n^j + 2\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}n^i v_1^j + (\vec{v}_2 \cdot \vec{n})^2 n^i a_1^j \\
& \left. - 2\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}n^i v_2^j - (\vec{v}_2 \cdot \vec{n})^2 n^i a_2^j \right] \\
& + \frac{G^2 m_1 m_2}{4r} \left[5a_1^i n^j + a_2^i n^j - 5n^i a_1^j - n^i a_2^j \right]. \\
\end{aligned} \tag{B.27}$$

Finally, we consider the N³LO spin-orbit sector at the 4.5PN order. Following the discussion in section 4.2, and in particular table 8, we implemented the following new redefinition of the position and spin variables in the present sector:

$$(\Delta \vec{x}_1)_{\text{N}^3\text{LO}}^{\text{SO}} = \sum_{i=0}^5 \Delta \vec{x}_{1(3,1)}^{(i)}, \tag{B.28}$$

$$(\omega^{ij})_{\text{N}^3\text{LO}}^{\text{SO}} = \sum_{k=0}^2 \omega_{1(3,1)}^{ij} - (i \leftrightarrow j), \tag{B.29}$$

where

$$\begin{aligned}
\Delta \vec{x}_{1(3,1)}^{(0)} = & \frac{5}{128m_1} \vec{S}_1 \times \vec{v}_1 v_1^6 \\
& - \frac{G m_2}{32m_1 r} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 v_2^2 \vec{n} - 4\vec{v}_1 \cdot \vec{v}_2 v_2^2 \vec{n} + 8v_2^4 \vec{n} + 2v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \right. \\
& + 26\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 - 7\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_1 + 5\vec{v}_2 \cdot \vec{n}v_2^2 \vec{v}_1 - 6v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_2 \\
& - 36\vec{v}_2 \cdot \vec{n}v_2^2 \vec{v}_2 - 24\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}v_2^2 \vec{n} - 3v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \vec{n} + 12\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \vec{n} \\
& - 36v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \vec{n} + 21\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 - 5(\vec{v}_2 \cdot \vec{n})^3 \vec{v}_1 + 12\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_2 \\
& + 20(\vec{v}_2 \cdot \vec{n})^3 \vec{v}_2 + 40\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 \vec{n} + 20(\vec{v}_2 \cdot \vec{n})^4 \vec{n}) + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 v_2^2 \vec{n} - 2v_2^4 \vec{n} \\
& - 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 + 2\vec{v}_1 \cdot \vec{n}v_2^2 \vec{v}_1 - 2\vec{v}_2 \cdot \vec{n}v_2^2 \vec{v}_1 - 2v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_2 + 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \\
& \left. + 4\vec{v}_2 \cdot \vec{n}v_2^2 \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}v_2^2 \vec{n} - 3v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \vec{n} + 9v_2^2 (\vec{v}_2 \cdot \vec{n})^2 \vec{n} \right]
\end{aligned}$$

$$\begin{aligned}
& -6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{v}_1 + 2(\vec{v}_2 \cdot \vec{n})^3\vec{v}_1 - 2(\vec{v}_2 \cdot \vec{n})^3\vec{v}_2 - 10\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3\vec{n} \\
& -5(\vec{v}_2 \cdot \vec{n})^4\vec{n}) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (2v_1^2\vec{v}_1 - 46\vec{v}_1 \cdot \vec{v}_2\vec{v}_1 - 59v_2^2\vec{v}_1 + 2v_1^2\vec{v}_2 + 16\vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \\
& + 72v_2^2\vec{v}_2 + 6v_1^2\vec{v}_2 \cdot \vec{n}\vec{n} + 8\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{n} - 12\vec{v}_2 \cdot \vec{n}v_2^2\vec{n} + 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \\
& + 27(\vec{v}_2 \cdot \vec{n})^2\vec{v}_1 + 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 32(\vec{v}_2 \cdot \vec{n})^2\vec{v}_2 + 12(\vec{v}_2 \cdot \vec{n})^3\vec{n}) \\
& - \vec{S}_1 \times \vec{n}(6v_1^2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n}v_1^2v_2^2 - v_1^2\vec{v}_2 \cdot \vec{n}v_2^2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2v_2^2 \\
& + 36\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2v_2^2 - 4\vec{v}_2 \cdot \vec{n}v_1^4 - 8\vec{v}_1 \cdot \vec{n}v_2^4 - 44\vec{v}_2 \cdot \vec{n}v_2^4 + 12\vec{v}_2 \cdot \vec{n}v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 3\vec{v}_1 \cdot \vec{n}v_1^2(\vec{v}_2 \cdot \vec{n})^2 + v_1^2(\vec{v}_2 \cdot \vec{n})^3 - 12\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 - 20\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^3 \\
& + 36\vec{v}_1 \cdot \vec{n}v_2^2(\vec{v}_2 \cdot \vec{n})^2 + 52v_2^2(\vec{v}_2 \cdot \vec{n})^3 - 20(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^3 - 20\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^4 \\
& - 20(\vec{v}_2 \cdot \vec{n})^5) + \vec{S}_1 \times \vec{v}_1 (18v_1^2\vec{v}_1 \cdot \vec{v}_2 - 7v_1^2v_2^2 + 51\vec{v}_1 \cdot \vec{v}_2v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^2 - 18v_1^4 \\
& - 32v_2^4 + 14\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} - 12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 3\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& - 7v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 19\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 + 24v_2^2(\vec{v}_2 \cdot \vec{n})^2 - 3(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2 \\
& + 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 - 6(\vec{v}_2 \cdot \vec{n})^4) - \vec{S}_1 \times \vec{v}_2 (2v_1^2\vec{v}_1 \cdot \vec{v}_2 + 9v_1^2v_2^2 + 72\vec{v}_1 \cdot \vec{v}_2v_2^2 \\
& + 8(\vec{v}_1 \cdot \vec{v}_2)^2 - 12v_1^4 - 76v_2^4 + 6\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} + 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& - 12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 - 9v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 32\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 + 64v_2^2(\vec{v}_2 \cdot \vec{n})^2 \\
& + 12\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 - 24(\vec{v}_2 \cdot \vec{n})^4) \Big] - \frac{G}{32r} \Big[4\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (v_2^4\vec{n} + 4\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_1 \\
& - 2\vec{v}_1 \cdot \vec{n}v_2^2\vec{v}_1 + 2\vec{v}_2 \cdot \vec{n}v_2^2\vec{v}_1 - 4\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_2 - 2\vec{v}_2 \cdot \vec{n}v_2^2\vec{v}_2 - 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2\vec{n} \\
& - 6v_2^2(\vec{v}_2 \cdot \vec{n})^2\vec{n} + 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{v}_1 - 2(\vec{v}_2 \cdot \vec{n})^3\vec{v}_1 + 2(\vec{v}_2 \cdot \vec{n})^3\vec{v}_2 \\
& + 10\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3\vec{n} + 5(\vec{v}_2 \cdot \vec{n})^4\vec{n}) - \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (16v_1^2\vec{v}_1 \cdot \vec{v}_2\vec{n} - 10v_1^2v_2^2\vec{n} \\
& + 18\vec{v}_1 \cdot \vec{v}_2v_2^2\vec{n} - 2(\vec{v}_1 \cdot \vec{v}_2)^2\vec{n} - 11v_1^4\vec{n} - 7v_2^4\vec{n} - 16\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_1 - 2v_2^2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \\
& + 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_1 + 4\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_1 - 20\vec{v}_1 \cdot \vec{n}v_2^2\vec{v}_1 + 16\vec{v}_2 \cdot \vec{n}v_2^2\vec{v}_1 + 16\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \\
& + 2v_1^2\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_2 - 4\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{v}_2 + 18\vec{v}_1 \cdot \vec{n}v_2^2\vec{v}_2 - 10\vec{v}_2 \cdot \vec{n}v_2^2\vec{v}_2 \\
& + 24\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n}\vec{n} - 36\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{n} + 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2\vec{n} + 48v_1^2(\vec{v}_1 \cdot \vec{n})^2\vec{n} \\
& - 60\vec{v}_1 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2\vec{n} + 30v_2^2(\vec{v}_1 \cdot \vec{n})^2\vec{n} + 9v_1^2(\vec{v}_2 \cdot \vec{n})^2\vec{n} - 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{n} \\
& - 21v_2^2(\vec{v}_2 \cdot \vec{n})^2\vec{n} + 20(\vec{v}_1 \cdot \vec{n})^3\vec{v}_1 + 18\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2\vec{v}_1 + 18\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{v}_1 \\
& - 12(\vec{v}_2 \cdot \vec{n})^3\vec{v}_1 - 20(\vec{v}_1 \cdot \vec{n})^3\vec{v}_2 - 18\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2\vec{v}_2 \\
& - 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{v}_2 - 25(\vec{v}_1 \cdot \vec{n})^4\vec{n} - 20\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3\vec{n} - 15(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2\vec{n} \\
& + 30\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3\vec{n} + 15(\vec{v}_2 \cdot \vec{n})^4\vec{n}) + 2\vec{S}_2 \times \vec{v}_1 \cdot \vec{v}_2 (8v_1^2\vec{v}_1 + 10\vec{v}_1 \cdot \vec{v}_2\vec{v}_1 + 10v_2^2\vec{v}_1 \\
& - 8v_1^2\vec{v}_2 - 10\vec{v}_1 \cdot \vec{v}_2\vec{v}_2 - 13v_2^2\vec{v}_2 - 32\vec{v}_1 \cdot \vec{n}v_1^2\vec{n} + 4v_1^2\vec{v}_2 \cdot \vec{n}\vec{n} + 12\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{n} \\
& - 6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2\vec{n} - 10\vec{v}_1 \cdot \vec{n}v_2^2\vec{n} - 5\vec{v}_2 \cdot \vec{n}v_2^2\vec{n} + 14\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 - 6(\vec{v}_1 \cdot \vec{n})^2\vec{v}_1 \\
& - (\vec{v}_2 \cdot \vec{n})^2\vec{v}_1 - 6\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_2 + 6(\vec{v}_1 \cdot \vec{n})^2\vec{v}_2 + (\vec{v}_2 \cdot \vec{n})^2\vec{v}_2 + 38(\vec{v}_1 \cdot \vec{n})^3\vec{n} \\
& - 6\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2\vec{n} + 9\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2\vec{n} + 5(\vec{v}_2 \cdot \vec{n})^3\vec{n}) + 4\vec{S}_2 \times \vec{n} (v_1^2\vec{v}_2 \cdot \vec{n}v_2^2 \\
& - 2\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2v_2^2 - 2\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 + \vec{v}_1 \cdot \vec{n}v_2^4 + 5\vec{v}_2 \cdot \vec{n}v_2^4 - 3\vec{v}_2 \cdot \vec{n}v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& - v_1^2(\vec{v}_2 \cdot \vec{n})^3 + 2\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^3 - 6\vec{v}_1 \cdot \vec{n}v_2^2(\vec{v}_2 \cdot \vec{n})^2 - 10v_2^2(\vec{v}_2 \cdot \vec{n})^3 \\
& + 5(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^3 + 5\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^4 + 5(\vec{v}_2 \cdot \vec{n})^5) + \vec{S}_2 \times \vec{v}_2 (16v_1^2\vec{v}_1 \cdot \vec{v}_2 \\
& - 10v_1^2v_2^2 + 26\vec{v}_1 \cdot \vec{v}_2v_2^2 + 10(\vec{v}_1 \cdot \vec{v}_2)^2 - 13v_1^4 - 37v_2^4 - 8\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n}
\end{aligned}$$

$$\begin{aligned}
& +12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 10\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 32v_1^2(\vec{v}_1 \cdot \vec{n})^2 - 12\vec{v}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 10v_2^2(\vec{v}_1 \cdot \vec{n})^2 + v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 2\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 + 19v_2^2(\vec{v}_2 \cdot \vec{n})^2 - 19(\vec{v}_1 \cdot \vec{n})^4 \\
& + 4\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 9(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2 - 10\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 - 11(\vec{v}_2 \cdot \vec{n})^4 \Big] \\
& + \frac{G^2 m_2}{384r^2} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3047v_1^2 \vec{n} - 4702\vec{v}_1 \cdot \vec{v}_2 \vec{n} + 4159v_2^2 \vec{n} + 11830\vec{v}_1 \cdot \vec{n}\vec{v}_1 \right. \\
& + 3194\vec{v}_2 \cdot \vec{n}\vec{v}_1 - 13618\vec{v}_1 \cdot \vec{n}\vec{v}_2 + 2726\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 9086\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} \\
& + 2614(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 2342(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (643v_1^2 \vec{n} - 1756\vec{v}_1 \cdot \vec{v}_2 \vec{n} \\
& + 835v_2^2 \vec{n} - 238\vec{v}_1 \cdot \vec{n}\vec{v}_1 - 532\vec{v}_2 \cdot \vec{n}\vec{v}_1 + 98\vec{v}_1 \cdot \vec{n}\vec{v}_2 - 298\vec{v}_2 \cdot \vec{n}\vec{v}_2 \\
& + 2086\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - 496(\vec{v}_1 \cdot \vec{n})^2 \vec{n} - 2158(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 6\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (1110\vec{v}_1 \\
& - 154\vec{v}_2 - 1307\vec{v}_1 \cdot \vec{n}\vec{n} + 205\vec{v}_2 \cdot \vec{n}\vec{n}) + 4\vec{S}_1 \times \vec{n} (2160\vec{v}_1 \cdot \vec{n}v_1^2 - 891v_1^2 \vec{v}_2 \cdot \vec{n} \\
& - 3912\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 3028\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 2855\vec{v}_1 \cdot \vec{n}v_2^2 \\
& - 1935\vec{v}_2 \cdot \vec{n}v_2^2 - 437(\vec{v}_1 \cdot \vec{n})^3 + 2118\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 5744\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 \\
& + 2528(\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_1 \times \vec{v}_1 (155v_1^2 + 1922\vec{v}_1 \cdot \vec{v}_2 - 1261v_2^2 + 7814\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 4042(\vec{v}_1 \cdot \vec{n})^2 - 2140(\vec{v}_2 \cdot \vec{n})^2) - 4\vec{S}_1 \times \vec{v}_2 (526v_1^2 - 584\vec{v}_1 \cdot \vec{v}_2 \\
& + 565v_2^2 - 776\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 230(\vec{v}_1 \cdot \vec{n})^2 + 418(\vec{v}_2 \cdot \vec{n})^2) \Big] \\
& + \frac{G^2 m_2^2}{192m_1 r^2} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (24v_1^2 \vec{n} + 272\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 141v_2^2 \vec{n} - 120\vec{v}_1 \cdot \vec{n}\vec{v}_1 + 32\vec{v}_2 \cdot \vec{n}\vec{v}_1 \right. \\
& + 272\vec{v}_1 \cdot \vec{n}\vec{v}_2 + 1314\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 512\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - 963(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (24v_1^2 \vec{n} - 16\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 495v_2^2 \vec{n} + 48\vec{v}_1 \cdot \vec{n}\vec{v}_1 - 19\vec{v}_2 \cdot \vec{n}\vec{v}_1 \\
& - 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 - 2076\vec{v}_2 \cdot \vec{n}\vec{v}_2 + 256\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} + 1485(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& - 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (572\vec{v}_1 - 1196\vec{v}_2 + 56\vec{v}_1 \cdot \vec{n}\vec{n} + 73\vec{v}_2 \cdot \vec{n}\vec{n}) + \vec{S}_1 \times \vec{n} (24\vec{v}_1 \cdot \vec{n}v_1^2 \\
& - 272v_1^2 \vec{v}_2 \cdot \vec{n} + 272\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 3192\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 303\vec{v}_1 \cdot \vec{n}v_2^2 \\
& - 7896\vec{v}_2 \cdot \vec{n}v_2^2 - 256\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 609\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 456(\vec{v}_2 \cdot \vec{n})^3) \\
& + 3\vec{S}_1 \times \vec{v}_1 (92v_1^2 - 208\vec{v}_1 \cdot \vec{v}_2 - 147v_2^2 + 160\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 8(\vec{v}_1 \cdot \vec{n})^2 \\
& - 483(\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{v}_2 (584v_1^2 - 1528\vec{v}_1 \cdot \vec{v}_2 + 1554v_2^2 + 245\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& + 56(\vec{v}_1 \cdot \vec{n})^2 - 162(\vec{v}_2 \cdot \vec{n})^2) \Big] - \frac{G^2 m_2}{96r^2} \left[8\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (35v_2^2 \vec{n} - 24\vec{v}_2 \cdot \vec{n}\vec{v}_1 \right. \\
& + 94\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 71(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (531v_1^2 \vec{n} - 1024\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 3v_2^2 \vec{n} \\
& + 1350\vec{v}_1 \cdot \vec{n}\vec{v}_1 - 4414\vec{v}_2 \cdot \vec{n}\vec{v}_1 - 922\vec{v}_1 \cdot \vec{n}\vec{v}_2 + 3126\vec{v}_2 \cdot \vec{n}\vec{v}_2 + 3682\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} \\
& - 1362(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 351(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 3\vec{S}_2 \times \vec{v}_1 \cdot \vec{v}_2 (107\vec{v}_1 - 156\vec{v}_2 + 17\vec{v}_1 \cdot \vec{n}\vec{n} \\
& - 194\vec{v}_2 \cdot \vec{n}\vec{n}) - 4\vec{S}_2 \times \vec{n} (24v_1^2 \vec{v}_2 \cdot \vec{n} - 188\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 70\vec{v}_1 \cdot \vec{n}v_2^2 + 33\vec{v}_2 \cdot \vec{n}v_2^2 \\
& + 142\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 115(\vec{v}_2 \cdot \vec{n})^3) + 3\vec{S}_2 \times \vec{v}_2 (193v_1^2 - 358\vec{v}_1 \cdot \vec{v}_2 - 77v_2^2 \\
& + 344\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 388(\vec{v}_1 \cdot \vec{n})^2 + 67(\vec{v}_2 \cdot \vec{n})^2) \Big] + \frac{G^2 m_1}{384r^2} \left[\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (3281v_1^2 \vec{n} \right. \\
& + 8942\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 917v_2^2 \vec{n} + 6340\vec{v}_1 \cdot \vec{n}\vec{v}_1 + 3644\vec{v}_2 \cdot \vec{n}\vec{v}_1 + 4004\vec{v}_1 \cdot \vec{n}\vec{v}_2 \\
& + 2750\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 14408\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} + 3886(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 4454(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& \left. - 2\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (7670v_1^2 \vec{n} - 2293\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 182v_2^2 \vec{n} + 14077\vec{v}_1 \cdot \vec{n}\vec{v}_1 + 1367\vec{v}_2 \cdot \vec{n}\vec{v}_1 \right]
\end{aligned}$$

$$\begin{aligned}
& -2194\vec{v}_1 \cdot \vec{n}\vec{v}_2 - 631\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 224\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - 137(\vec{v}_1 \cdot \vec{n})^2\vec{n} + 2408(\vec{v}_2 \cdot \vec{n})^2\vec{n}) \\
& + 2\vec{S}_2 \times \vec{v}_1 \cdot \vec{v}_2 (6790\vec{v}_1 - 2805\vec{v}_2 - 2237\vec{v}_1 \cdot \vec{n}\vec{n} + 2100\vec{v}_2 \cdot \vec{n}\vec{n}) \\
& + \vec{S}_2 \times \vec{n} (7983\vec{v}_1 \cdot \vec{n}\vec{v}_1^2 - 2619\vec{v}_1^2\vec{v}_2 \cdot \vec{n} + 1986\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 10090\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& + 2045\vec{v}_1 \cdot \vec{n}\vec{v}_2^2 - 6009\vec{v}_2 \cdot \vec{n}\vec{v}_2^2 + 6330(\vec{v}_1 \cdot \vec{n})^3 - 19692\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
& + 3958\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 + 3960(\vec{v}_2 \cdot \vec{n})^3) - \vec{S}_2 \times \vec{v}_1 (2939\vec{v}_1^2 - 3358\vec{v}_1 \cdot \vec{v}_2 \\
& + 2597\vec{v}_2^2 - 14206\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 6356(\vec{v}_1 \cdot \vec{n})^2 + 2570(\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_2 \times \vec{v}_2 (1999\vec{v}_1^2 \\
& - 2203\vec{v}_1 \cdot \vec{v}_2 - 403\vec{v}_2^2 + 7361\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4913(\vec{v}_1 \cdot \vec{n})^2 - 856(\vec{v}_2 \cdot \vec{n})^2) \\
& - \frac{G^3 m_1 m_2}{7200 r^3} [227766\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} - 41658\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{n} - 6\vec{S}_1 \times \vec{n} (49539\vec{v}_1 \cdot \vec{n} \\
& - 76382\vec{v}_2 \cdot \vec{n}) + 214175\vec{S}_1 \times \vec{v}_1 - 78450\vec{S}_1 \times \vec{v}_2] \\
& + \frac{13G^3 m_1 m_2}{2r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) [3\vec{S}_1 \times \vec{n} (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 + \vec{S}_1 \times \vec{v}_2] \\
& + \frac{29G^3 m_2^2}{6r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) [3\vec{S}_1 \times \vec{n} (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 + \vec{S}_1 \times \vec{v}_2] \\
& - \frac{G^3 m_2^3}{1800 m_1 r^3} [16200\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} + 19149\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 9\vec{S}_1 \times \vec{n} (1800\vec{v}_1 \cdot \vec{n} \\
& - 1289\vec{v}_2 \cdot \vec{n}) - 450\vec{S}_1 \times \vec{v}_1 + 14525\vec{S}_1 \times \vec{v}_2] \\
& - \frac{G^3 m_2^2}{1152 r^3} [27\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3163 - 350\pi^2)\vec{n} - 12\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (8213 - 675\pi^2)\vec{n} \\
& - 6\vec{S}_1 \times \vec{n} ((11267 + 504\pi^2)\vec{v}_1 \cdot \vec{n} - (5635 - 810\pi^2)\vec{v}_2 \cdot \vec{n}) \\
& + (31805 - 2142\pi^2)\vec{S}_1 \times \vec{v}_1 + 2(13 + 2160\pi^2)\vec{S}_1 \times \vec{v}_2] \\
& - \frac{29G^3 m_1^2}{6r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) [3\vec{S}_2 \times \vec{n} (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - \vec{S}_2 \times \vec{v}_1 + \vec{S}_2 \times \vec{v}_2] \\
& - \frac{29G^3 m_1 m_2}{6r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) [3\vec{S}_2 \times \vec{n} (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - \vec{S}_2 \times \vec{v}_1 + \vec{S}_2 \times \vec{v}_2] \\
& - \frac{G^3 m_1^2}{14400 r^3} [324234\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} - 181158\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 3\vec{S}_2 \times \vec{n} (339703\vec{v}_1 \cdot \vec{n} \\
& - 227636\vec{v}_2 \cdot \vec{n}) - 104425\vec{S}_2 \times \vec{v}_1 + 13350\vec{S}_2 \times \vec{v}_2] + \frac{G^3 m_2^2}{14400 r^3} [864\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} \\
& + 230691\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 288\vec{S}_2 \times \vec{n} (3\vec{v}_1 \cdot \vec{n} + 857\vec{v}_2 \cdot \vec{n}) + 155875\vec{S}_2 \times \vec{v}_2] \\
& - \frac{G^3 m_1 m_2}{1152 r^3} [9\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (15415 - 1896\pi^2)\vec{n} - 54\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (2769 - 53\pi^2)\vec{n} \\
& + 3\vec{S}_2 \times \vec{n} ((29371 - 3240\pi^2)\vec{v}_1 \cdot \vec{n} - (26609 - 684\pi^2)\vec{v}_2 \cdot \vec{n}) \\
& + (22607 - 2448\pi^2)\vec{S}_2 \times \vec{v}_1 - 30(1483 - 9\pi^2)\vec{S}_2 \times \vec{v}_2], \tag{B.30}
\end{aligned}$$

$$\begin{aligned}
\Delta \vec{x}_{1(3,1)}^{(1)} = & \frac{G m_2}{192 m_1} [3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1 \cdot \vec{a}_2 \vec{v}_1 - 15\vec{v}_2 \cdot \vec{a}_2 \vec{v}_1 - 12\vec{v}_1 \cdot \vec{a}_2 \vec{v}_2 + 52\vec{v}_2 \cdot \vec{a}_2 \vec{v}_2 + 14\vec{v}_1^2 \vec{a}_2 \\
& - 24\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 + 66\vec{v}_2^2 \vec{a}_2 + 2\vec{v}_1^2 \vec{a}_2 \cdot \vec{n}\vec{n} - 8\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n}\vec{n} + 18\vec{v}_2^2 \vec{a}_2 \cdot \vec{n}\vec{n} \\
& - 20\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \vec{n} + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 44\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 5\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{v}_1]
\end{aligned}$$

$$\begin{aligned}
& -\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 + 4\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 - 40\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \\
& - 56(\vec{v}_2 \cdot \vec{n})^2 \vec{a}_2 - 12\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{n} - 48\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_2^2 \vec{v}_1 \\
& - 4v_2^2 \vec{v}_2 + 4\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} - (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 + 4(\vec{v}_2 \cdot \vec{n})^2 \vec{v}_2 - 4(\vec{v}_2 \cdot \vec{n})^3 \vec{n}) \\
& + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (7v_2^2 \vec{n} - 16\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 34\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 7(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& + 6\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (6\vec{v}_1 \cdot \vec{a}_2 \vec{v}_1 + 2\vec{v}_2 \cdot \vec{a}_2 \vec{v}_1 - 6\vec{v}_1 \cdot \vec{a}_2 \vec{v}_2 - 6\vec{v}_2 \cdot \vec{a}_2 \vec{v}_2 + 3v_1^2 \vec{a}_2 \\
& - 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 - 19v_2^2 \vec{a}_2 + v_1^2 \vec{a}_2 \cdot \vec{n} \vec{n} + 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \vec{n} - 9v_2^2 \vec{a}_2 \cdot \vec{n} \vec{n} + 6\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \vec{n} \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \vec{n} - 18\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 + 8\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 + 16(\vec{v}_2 \cdot \vec{n})^2 \vec{a}_2 \\
& + 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{n} + 24\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (8\vec{v}_1 \cdot \vec{a}_2 \vec{n} - 16\vec{v}_2 \cdot \vec{a}_2 \vec{n} \\
& + 9\vec{a}_2 \cdot \vec{n} \vec{v}_1 - 36\vec{a}_2 \cdot \vec{n} \vec{v}_2 + 16\vec{v}_1 \cdot \vec{n} \vec{a}_2 - 60\vec{v}_2 \cdot \vec{n} \vec{a}_2 + 12\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{n}) \\
& + 6\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (6v_2^2 \vec{n} - 7\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 6(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& + 12\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 + v_2^2 \vec{v}_1 + v_1^2 \vec{v}_2 - 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 - 5v_2^2 \vec{v}_2 + v_1^2 \vec{v}_2 \cdot \vec{n} \vec{n} \\
& + 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{n} - \vec{v}_1 \cdot \vec{n} v_2^2 \vec{n} - 9\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 - (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 + 4(\vec{v}_2 \cdot \vec{n})^2 \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n} + 8(\vec{v}_2 \cdot \vec{n})^3 \vec{n}) \\
& + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (2v_1^2 \vec{n} - 6v_2^2 \vec{n} - 35\vec{v}_1 \cdot \vec{n} \vec{v}_1 - 33\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 20\vec{v}_1 \cdot \vec{n} \vec{v}_2 + 112\vec{v}_2 \cdot \vec{n} \vec{v}_2 \\
& + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} + 4(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 6\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 (v_1^2 \vec{n} + 2\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 9v_2^2 \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \\
& - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 + 8\vec{v}_2 \cdot \vec{n} \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} + 8(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& + 3\vec{S}_1 \times \vec{n} (\vec{v}_1 \cdot \vec{a}_1 v_2^2 - 4\vec{a}_1 \cdot \vec{v}_2 v_2^2 + 11v_1^2 \vec{v}_1 \cdot \vec{a}_2 - 24\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 + 72v_2^2 \vec{v}_1 \cdot \vec{a}_2 \\
& + v_1^2 \vec{v}_2 \cdot \vec{a}_2 + 72\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 - 176v_2^2 \vec{v}_2 \cdot \vec{a}_2 + 4\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 + 5\vec{v}_1 \cdot \vec{n} v_1^2 \vec{a}_2 \cdot \vec{n} \\
& - 9v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} - 8\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 40\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 12\vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} \\
& + 104\vec{v}_2 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} - 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 24\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 4\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - \vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 + 4\vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 56\vec{v}_1 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 + 128\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 12\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 4\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 - 48\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \\
& - 80\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) - 3\vec{S}_1 \times \vec{v}_1 (48\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 34\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 7\vec{a}_1 \cdot \vec{n} v_2^2 \\
& - 10v_1^2 \vec{a}_2 \cdot \vec{n} - 69\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 88v_2^2 \vec{a}_2 \cdot \vec{n} + 34\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 137\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& - 49\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 224\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 17\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} + 7\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \\
& - 7\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - 60\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{a}_1 (24v_1^2 \vec{v}_2 \cdot \vec{n} - 102\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& + 21\vec{v}_1 \cdot \vec{n} v_2^2 + 132\vec{v}_2 \cdot \vec{n} v_2^2 - 21\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - 28(\vec{v}_2 \cdot \vec{n})^3) \\
& - 3\vec{S}_1 \times \vec{v}_2 (14\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + 8\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - 12\vec{a}_1 \cdot \vec{n} v_2^2 + 37v_1^2 \vec{a}_2 \cdot \vec{n} \\
& + 60\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 134v_2^2 \vec{a}_2 \cdot \vec{n} - 24\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 108\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 92\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \\
& - 348\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 12\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} - 12\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 12\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \\
& + 96\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{a}_2 (81\vec{v}_1 \cdot \vec{n} v_1^2 - 201v_1^2 \vec{v}_2 \cdot \vec{n} - 24\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - 480\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 108\vec{v}_1 \cdot \vec{n} v_2^2 + 1116\vec{v}_2 \cdot \vec{n} v_2^2 + 84\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \\
& - 12\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - 272(\vec{v}_2 \cdot \vec{n})^3) \Big] - \frac{1}{96}G \Big[3\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \vec{a}_2 \vec{v}_1 + 2\vec{v}_2 \cdot \vec{a}_2 \vec{v}_1 \\
& - 6\vec{v}_1 \cdot \vec{a}_2 \vec{v}_2 - 2\vec{v}_2 \cdot \vec{a}_2 \vec{v}_2 - 3v_1^2 \vec{a}_2 - 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 - 13v_2^2 \vec{a}_2 - v_1^2 \vec{a}_2 \cdot \vec{n} \vec{n} \Big]
\end{aligned}$$

$$\begin{aligned}
& +2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \vec{n} - 7v_2^2 \vec{a}_2 \cdot \vec{n} \vec{n} + 6\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \vec{n} - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \vec{n} - 14\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \vec{n} \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 + 8\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 \\
& - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 + 16(\vec{v}_2 \cdot \vec{n})^2 \vec{a}_2 + 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{n} + 24\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& - 6\vec{S}_2 \times \vec{n} \cdot \vec{a}_1 (v_2^2 \vec{v}_2 - \vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} - (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_2 + (\vec{v}_2 \cdot \vec{n})^3 \vec{n}) \\
& + 3\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (24\vec{v}_1 \cdot \vec{a}_1 \vec{v}_1 - 8\vec{a}_1 \cdot \vec{v}_2 \vec{v}_1 + 4\vec{v}_1 \cdot \vec{a}_2 \vec{v}_1 + 6\vec{v}_2 \cdot \vec{a}_2 \vec{v}_1 + 21v_1^2 \vec{a}_1 \\
& - 14\vec{v}_1 \cdot \vec{v}_2 \vec{a}_1 - 7v_2^2 \vec{a}_1 - 38\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 + 22\vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 + 4\vec{v}_1 \cdot \vec{a}_2 \vec{v}_2 - 18\vec{v}_2 \cdot \vec{a}_2 \vec{v}_2 \\
& - 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 - 5v_1^2 \vec{a}_1 \cdot \vec{n} \vec{n} - 11\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 \vec{n} + 14\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{n} + 4\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{n} \\
& + 4\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \vec{n} - 14\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 \vec{n} + \vec{a}_1 \cdot \vec{n} v_2^2 \vec{n} - 2v_1^2 \vec{a}_2 \cdot \vec{n} \vec{n} + 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \vec{n} \\
& + 4v_2^2 \vec{a}_2 \cdot \vec{n} \vec{n} + 14\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \vec{n} + 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \vec{n} - 12\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \vec{n} \\
& + 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 \vec{n} - 23\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_1 + 8\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 - 8\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 \\
& + 6\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_1 + 14\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_1 - 21(\vec{v}_1 \cdot \vec{n})^2 \vec{a}_1 + 7(\vec{v}_2 \cdot \vec{n})^2 \vec{a}_1 \\
& + 28\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 - 6\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 + 8\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{v}_2 + 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \\
& + 7(\vec{v}_1 \cdot \vec{n})^2 \vec{a}_2 - 6(\vec{v}_2 \cdot \vec{n})^2 \vec{a}_2 - 12\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} + 15\vec{a}_1 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \vec{n} \\
& + 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \vec{n} - 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n} - 18\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& - 3\vec{S}_2 \times \vec{v}_1 \cdot \vec{v}_2 (21\vec{v}_1 \cdot \vec{a}_1 \vec{n} - 4\vec{a}_1 \cdot \vec{v}_2 \vec{n} - 8\vec{v}_1 \cdot \vec{a}_2 \vec{n} + 12\vec{v}_2 \cdot \vec{a}_2 \vec{n} - 23\vec{a}_1 \cdot \vec{n} \vec{v}_1 \\
& - 10\vec{a}_2 \cdot \vec{n} \vec{v}_1 - 18\vec{v}_1 \cdot \vec{n} \vec{a}_1 - 10\vec{v}_2 \cdot \vec{n} \vec{a}_1 - 12\vec{a}_1 \cdot \vec{n} \vec{v}_2 + 2\vec{a}_2 \cdot \vec{n} \vec{v}_2 - 16\vec{v}_1 \cdot \vec{n} \vec{a}_2 \\
& + 2\vec{v}_2 \cdot \vec{n} \vec{a}_2 - 14\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{n} + 20\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} - 10\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \vec{n}) \\
& - 3\vec{S}_2 \times \vec{a}_1 \cdot \vec{v}_2 (19v_1^2 \vec{n} - 28\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 9v_2^2 \vec{n} + 17\vec{v}_1 \cdot \vec{n} \vec{v}_1 - 40\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 52\vec{v}_1 \cdot \vec{n} \vec{v}_2 \\
& + 22\vec{v}_2 \cdot \vec{n} \vec{v}_2 + 44\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} - 19(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 6\vec{S}_2 \times \vec{n} \cdot \vec{a}_2 (5v_1^2 \vec{v}_1 \\
& - 6\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 - v_1^2 \vec{v}_2 + 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 + 2v_2^2 \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n} v_1^2 \vec{n} + 2v_1^2 \vec{v}_2 \cdot \vec{n} \vec{n} \\
& - 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{n} - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{n} + 3\vec{v}_1 \cdot \vec{n} v_2^2 \vec{n} - 4\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} + 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \\
& + 3(\vec{v}_1 \cdot \vec{n})^2 \vec{v}_1 - 3(\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 - 8\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 - 3(\vec{v}_1 \cdot \vec{n})^2 \vec{v}_2 - 2(\vec{v}_1 \cdot \vec{n})^3 \vec{n} \\
& - 3\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 6(\vec{v}_2 \cdot \vec{n})^3 \vec{n}) - 3\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_2 (17v_1^2 \vec{n} - 6\vec{v}_1 \cdot \vec{v}_2 \vec{n} + 13v_2^2 \vec{n} \\
& + 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 - 18\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 + 8\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 14\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} - 20(\vec{v}_1 \cdot \vec{n})^2 \vec{n} \\
& - 4(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 3\vec{S}_2 \times \vec{v}_2 \cdot \vec{a}_2 (v_1^2 \vec{n} - 6\vec{v}_1 \cdot \vec{v}_2 \vec{n} + 9v_2^2 \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \\
& + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 + 4\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 12\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} - (\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 4(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& + 6\vec{S}_2 \times \vec{n} (\vec{a}_1 \cdot \vec{v}_2 v_2^2 - v_1^2 \vec{v}_1 \cdot \vec{a}_2 - 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 - 7v_2^2 \vec{v}_1 \cdot \vec{a}_2 - v_1^2 \vec{v}_2 \cdot \vec{a}_2 \\
& - 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 + 20v_2^2 \vec{v}_2 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - \vec{v}_1 \cdot \vec{n} v_1^2 \vec{a}_2 \cdot \vec{n} + v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \\
& + 2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 3\vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} - 20\vec{v}_2 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} \\
& + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + \vec{v}_2 \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 8\vec{v}_1 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 - 20\vec{v}_2 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 - 3\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 \\
& + 12\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + 20\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3) + 3\vec{S}_2 \times \vec{v}_1 (v_1^2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} \\
& - v_2^2 \vec{a}_2 \cdot \vec{n} + 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 - 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \\
& + 4\vec{S}_2 \times \vec{a}_1 (3\vec{v}_2 \cdot \vec{n} v_2^2 - (\vec{v}_2 \cdot \vec{n})^3) + 3\vec{S}_2 \times \vec{v}_2 (67v_1^2 \vec{a}_1 \cdot \vec{n} + 157\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 \\
& - 34\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 60\vec{a}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 76\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 22\vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 11\vec{a}_1 \cdot \vec{n} v_2^2
\end{aligned}$$

$$\begin{aligned}
& -6v_1^2 \vec{a}_2 \cdot \vec{n} + 2\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 20v_2^2 \vec{a}_2 \cdot \vec{n} - 22\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 - 4\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \\
& + 32\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 22\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + 44\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \\
& - 67\vec{a}_1 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 3\vec{a}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 - 22\vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \\
& + 2\vec{S}_2 \times \vec{a}_2 (27\vec{v}_1 \cdot \vec{n} v_1^2 - 12v_1^2 \vec{v}_2 \cdot \vec{n} - 12\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 30\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 36\vec{v}_1 \cdot \vec{n} v_2^2 \\
& - 6\vec{v}_2 \cdot \vec{n} v_2^2 - 10(\vec{v}_1 \cdot \vec{n})^3 - 6\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 24\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + 4(\vec{v}_2 \cdot \vec{n})^3) \\
& - \frac{G m_2}{64m_1} \left[2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (v_2^2 \vec{v}_1 + 4v_2^2 \vec{v}_2 + 8\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} - (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 \right. \\
& \left. - 4(\vec{v}_2 \cdot \vec{n})^2 \vec{v}_2 - 8(\vec{v}_2 \cdot \vec{n})^3 \vec{n}) - 16\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} - (\vec{v}_2 \cdot \vec{n})^3 \vec{n}) \right. \\
& \left. - 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (6v_2^2 \vec{n} - 7\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 6(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + \dot{\vec{S}}_1 \times \vec{n} (v_1^2 v_2^2 + 8\vec{v}_1 \cdot \vec{v}_2 v_2^2 \right. \\
& \left. - 16v_2^4 + 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 8\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \right. \\
& \left. + 24v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 16\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 - 8(\vec{v}_2 \cdot \vec{n})^4) + 12\dot{\vec{S}}_1 \times \vec{v}_1 v_1^2 \vec{v}_2 \cdot \vec{n} \right. \\
& \left. - 2\dot{\vec{S}}_1 \times \vec{v}_2 (7v_1^2 \vec{v}_2 \cdot \vec{n} - 12\vec{v}_1 \cdot \vec{n} v_2^2 - 48\vec{v}_2 \cdot \vec{n} v_2^2 + 12\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 + 8(\vec{v}_2 \cdot \vec{n})^3) \right] \\
& - \frac{1}{48} G \left[12\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 + v_2^2 \vec{v}_1 - 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 - v_2^2 \vec{v}_2 - \vec{v}_1 \cdot \vec{n} v_2^2 \vec{n} \right. \\
& \left. - 2\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 - (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 + (\vec{v}_2 \cdot \vec{n})^2 \vec{v}_2 + 3\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n} \right. \\
& \left. + 2(\vec{v}_2 \cdot \vec{n})^3 \vec{n}) + 3\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (v_1^2 \vec{v}_1 - 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 - 9v_2^2 \vec{v}_1 - v_2^2 \vec{v}_2 + 2\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \right. \\
& \left. + 5v_2^2 \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} v_1^2 \vec{n} - 3v_1^2 \vec{v}_2 \cdot \vec{n} \vec{n} + 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{n} + 2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{n} + 7\vec{v}_2 \cdot \vec{n} v_2^2 \vec{n} \right. \\
& \left. - 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 - 3(\vec{v}_1 \cdot \vec{n})^2 \vec{v}_1 + 6(\vec{v}_2 \cdot \vec{n})^2 \vec{v}_1 + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 + 3(\vec{v}_1 \cdot \vec{n})^2 \vec{v}_2 \right. \\
& \left. + 2(\vec{v}_1 \cdot \vec{n})^3 \vec{n} + 3\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 \vec{n} - 9\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \vec{n} - 6(\vec{v}_2 \cdot \vec{n})^3 \vec{n}) \right. \\
& \left. - 3\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2 (8v_1^2 \vec{n} - 2\vec{v}_1 \cdot \vec{v}_2 \vec{n} + 8v_2^2 \vec{n} - 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 + 10\vec{v}_2 \cdot \vec{n} \vec{v}_1 - 2\vec{v}_1 \cdot \vec{n} \vec{v}_2 \right. \\
& \left. - 6\vec{v}_2 \cdot \vec{n} \vec{v}_2 - 14\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{n} - 10(\vec{v}_1 \cdot \vec{n})^2 \vec{n} - 3(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 6\dot{\vec{S}}_2 \times \vec{n} (v_1^2 v_2^2 \right. \\
& \left. - 2\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^2 + 5v_2^4 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 - v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \right. \\
& \left. + 2\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 - 10v_2^2 (\vec{v}_2 \cdot \vec{n})^2 + 3(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 4\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 \right. \\
& \left. + 5(\vec{v}_2 \cdot \vec{n})^4) + \dot{\vec{S}}_2 \times \vec{v}_2 (24\vec{v}_1 \cdot \vec{n} v_1^2 + 15v_1^2 \vec{v}_2 \cdot \vec{n} - 6\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 18\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right. \\
& \left. + 24\vec{v}_1 \cdot \vec{n} v_2^2 + 21\vec{v}_2 \cdot \vec{n} v_2^2 - 10(\vec{v}_1 \cdot \vec{n})^3 - 21\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 - 9\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 \right. \\
& \left. - 16(\vec{v}_2 \cdot \vec{n})^3) \right] \\
& + \frac{G^2 m_2^2}{384m_1 r} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (283\vec{a}_2 + 1063\vec{a}_2 \cdot \vec{n} \vec{n}) - 16\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 (5\vec{v}_2 - 4\vec{v}_2 \cdot \vec{n} \vec{n}) \right. \\
& \left. - 6\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (184\vec{a}_2 + 273\vec{a}_2 \cdot \vec{n} \vec{n}) + 80\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{n} + \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (1037\vec{v}_1 \right. \\
& \left. - 2092\vec{v}_2 + 32\vec{v}_1 \cdot \vec{n} \vec{n} - 874\vec{v}_2 \cdot \vec{n} \vec{n}) + 56\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{n} + 1292\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \vec{n} \right. \\
& \left. + \vec{S}_1 \times \vec{n} (80\vec{a}_1 \cdot \vec{v}_2 + 650\vec{v}_1 \cdot \vec{a}_2 - 1054\vec{v}_2 \cdot \vec{a}_2 - 64\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 55\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} \right. \\
& \left. + 344\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) + 730\vec{S}_1 \times \vec{v}_1 \vec{a}_2 \cdot \vec{n} + 256\vec{S}_1 \times \vec{a}_1 \vec{v}_2 \cdot \vec{n} + 4\vec{S}_1 \times \vec{v}_2 (20\vec{a}_1 \cdot \vec{n} \right. \\
& \left. + 107\vec{a}_2 \cdot \vec{n}) - 3\vec{S}_1 \times \vec{a}_2 (607\vec{v}_1 \cdot \vec{n} - 1322\vec{v}_2 \cdot \vec{n}) \right] - \frac{G^2 m_2}{384r} \left[2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (296\vec{a}_1 \right. \\
& \left. + 394\vec{a}_2 + 54\vec{a}_1 \cdot \vec{n} \vec{n} + 305\vec{a}_2 \cdot \vec{n} \vec{n}) + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (217\vec{v}_1 + 537\vec{v}_2 - 176\vec{v}_1 \cdot \vec{n} \vec{n} \right]
\end{aligned}$$

$$\begin{aligned}
& -375\vec{v}_2 \cdot \vec{n}\vec{n}) + 2677\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1\vec{n} + 8\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(172\vec{a}_1 + 97\vec{a}_2 - 26\vec{a}_1 \cdot \vec{n}\vec{n} \\
& + 94\vec{a}_2 \cdot \vec{n}\vec{n}) + 2294\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2\vec{n} + 8\vec{S}_1 \times \vec{n} \cdot \vec{a}_2(60\vec{v}_1 + 34\vec{v}_2 - 15\vec{v}_1 \cdot \vec{n}\vec{n} \\
& + 175\vec{v}_2 \cdot \vec{n}\vec{n}) - 866\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\vec{n} - 408\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2\vec{n} + 2\vec{S}_1 \times \vec{n}(538\vec{v}_1 \cdot \vec{a}_1 \\
& - 1502\vec{a}_1 \cdot \vec{v}_2 - 1012\vec{v}_1 \cdot \vec{a}_2 + 2128\vec{v}_2 \cdot \vec{a}_2 - 458\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} + 1771\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& + 3488\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 4088\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1(3665\vec{a}_1 \cdot \vec{n} + 3508\vec{a}_2 \cdot \vec{n}) \\
& - 2\vec{S}_1 \times \vec{a}_1(2931\vec{v}_1 \cdot \vec{n} + 1823\vec{v}_2 \cdot \vec{n}) + 6\vec{S}_1 \times \vec{v}_2(607\vec{a}_1 \cdot \vec{n} + 152\vec{a}_2 \cdot \vec{n}) \\
& + 16\vec{S}_1 \times \vec{a}_2(427\vec{v}_1 \cdot \vec{n} - 324\vec{v}_2 \cdot \vec{n})] + \frac{G^2 m_2}{192r} [8\vec{S}_2 \times \vec{n} \cdot \vec{v}_1(41\vec{a}_2 + 98\vec{a}_2 \cdot \vec{n}\vec{n}) \\
& + \vec{S}_2 \times \vec{n} \cdot \vec{v}_2(60\vec{a}_1 - 503\vec{a}_2 - 84\vec{a}_1 \cdot \vec{n}\vec{n} - 1069\vec{a}_2 \cdot \vec{n}\vec{n}) - 543\vec{S}_2 \times \vec{a}_1 \cdot \vec{v}_2\vec{n} \\
& - 4\vec{S}_2 \times \vec{n} \cdot \vec{a}_2(174\vec{v}_1 - 95\vec{v}_2 - 126\vec{v}_1 \cdot \vec{n}\vec{n} + 133\vec{v}_2 \cdot \vec{n}\vec{n}) + 632\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_2\vec{n} \\
& + 79\vec{S}_2 \times \vec{v}_2 \cdot \vec{a}_2\vec{n} + 4\vec{S}_2 \times \vec{n}(151\vec{v}_1 \cdot \vec{a}_2 - 206\vec{v}_2 \cdot \vec{a}_2 + 13\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} \\
& + 256\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + 324\vec{S}_2 \times \vec{v}_1\vec{a}_2 \cdot \vec{n} - 3\vec{S}_2 \times \vec{v}_2(469\vec{a}_1 \cdot \vec{n} - 229\vec{a}_2 \cdot \vec{n}) \\
& - 8\vec{S}_2 \times \vec{a}_2(103\vec{v}_1 \cdot \vec{n} - 280\vec{v}_2 \cdot \vec{n})] + \frac{G^2 m_1}{384r} [2\vec{S}_2 \times \vec{n} \cdot \vec{v}_1(2180\vec{a}_1 + 584\vec{a}_2 \\
& + 17\vec{a}_1 \cdot \vec{n}\vec{n} - 67\vec{a}_2 \cdot \vec{n}\vec{n}) - \vec{S}_2 \times \vec{n} \cdot \vec{a}_1(1005\vec{v}_1 - 1771\vec{v}_2 - 3504\vec{v}_1 \cdot \vec{n}\vec{n} \\
& + 452\vec{v}_2 \cdot \vec{n}\vec{n}) + 845\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_1\vec{n} - \vec{S}_2 \times \vec{n} \cdot \vec{v}_2(4628\vec{a}_1 - 1291\vec{a}_2 + 82\vec{a}_1 \cdot \vec{n}\vec{n} \\
& + 95\vec{a}_2 \cdot \vec{n}\vec{n}) + 978\vec{S}_2 \times \vec{a}_1 \cdot \vec{v}_2\vec{n} - \vec{S}_2 \times \vec{n} \cdot \vec{a}_2(2044\vec{v}_1 - 1520\vec{v}_2 + 1696\vec{v}_1 \cdot \vec{n}\vec{n} \\
& - 23\vec{v}_2 \cdot \vec{n}\vec{n}) + 3040\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_2\vec{n} - 38\vec{S}_2 \times \vec{v}_2 \cdot \vec{a}_2\vec{n} + \vec{S}_2 \times \vec{n}(79\vec{v}_1 \cdot \vec{a}_1 \\
& + 1119\vec{a}_1 \cdot \vec{v}_2 + 2086\vec{v}_1 \cdot \vec{a}_2 - 3135\vec{v}_2 \cdot \vec{a}_2 - 4328\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} + 1404\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 206\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 3432\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 3\vec{S}_2 \times \vec{v}_1(77\vec{a}_1 \cdot \vec{n} - 346\vec{a}_2 \cdot \vec{n}) \\
& - 28\vec{S}_2 \times \vec{a}_1(35\vec{v}_1 \cdot \vec{n} - 59\vec{v}_2 \cdot \vec{n}) - 8\vec{S}_2 \times \vec{v}_2(233\vec{a}_1 \cdot \vec{n} + 76\vec{a}_2 \cdot \vec{n}) \\
& - 25\vec{S}_2 \times \vec{a}_2(88\vec{v}_1 \cdot \vec{n} + 23\vec{v}_2 \cdot \vec{n})] - \frac{G^2 m_2}{384r} [4\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1(31\vec{v}_1 + 97\vec{v}_2 \\
& + 742\vec{v}_1 \cdot \vec{n}\vec{n} - 629\vec{v}_2 \cdot \vec{n}\vec{n}) + 8\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2(64\vec{v}_1 + 175\vec{v}_2 - 80\vec{v}_1 \cdot \vec{n}\vec{n} \\
& + 133\vec{v}_2 \cdot \vec{n}\vec{n}) + 1606\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{n} - \dot{\vec{S}}_1 \times \vec{n}(435v_1^2 - 2104\vec{v}_1 \cdot \vec{v}_2 \\
& + 1852v_2^2 - 2368\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 504(\vec{v}_1 \cdot \vec{n})^2 + 760(\vec{v}_2 \cdot \vec{n})^2) + 2\dot{\vec{S}}_1 \times \vec{v}_1(532\vec{v}_1 \cdot \vec{n} \\
& - 771\vec{v}_2 \cdot \vec{n}) - 8\dot{\vec{S}}_1 \times \vec{v}_2(782\vec{v}_1 \cdot \vec{n} - 565\vec{v}_2 \cdot \vec{n})] \\
& - \frac{G^2 m_2^2}{24m_1 r} [3\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1(5\vec{v}_2 - 4\vec{v}_2 \cdot \vec{n}\vec{n}) - 2\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_2 + 4\vec{v}_2 \cdot \vec{n}\vec{n}) \\
& - 15\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{n} + 3\dot{\vec{S}}_1 \times \vec{n}(5\vec{v}_1 \cdot \vec{v}_2 - 6v_2^2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 5(\vec{v}_2 \cdot \vec{n})^2) \\
& + 18\dot{\vec{S}}_1 \times \vec{v}_1\vec{v}_2 \cdot \vec{n} + \dot{\vec{S}}_1 \times \vec{v}_2(15\vec{v}_1 \cdot \vec{n} + 221\vec{v}_2 \cdot \vec{n})] + \frac{G^2 m_2}{48r} [24\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1(4\vec{v}_2 \\
& + 11\vec{v}_2 \cdot \vec{n}\vec{n}) - \dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2(125\vec{v}_1 - 165\vec{v}_2 - 142\vec{v}_1 \cdot \vec{n}\vec{n} + 420\vec{v}_2 \cdot \vec{n}\vec{n}) \\
& + 229\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2\vec{n} + \dot{\vec{S}}_2 \times \vec{n}(96\vec{v}_1 \cdot \vec{v}_2 - 29v_2^2 + 264\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 149(\vec{v}_2 \cdot \vec{n})^2) \\
& - \dot{\vec{S}}_2 \times \vec{v}_2(340\vec{v}_1 \cdot \vec{n} - 593\vec{v}_2 \cdot \vec{n})] + \frac{G^2 m_1}{768r} [4\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1(719\vec{v}_1 \\
& + 865\vec{v}_2 - 3148\vec{v}_1 \cdot \vec{n}\vec{n} + 2137\vec{v}_2 \cdot \vec{n}\vec{n}) - 2\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2(1126\vec{v}_1
\end{aligned}$$

$$\begin{aligned}
& -435\vec{v}_2 - 1898\vec{v}_1 \cdot \vec{n}\vec{n} + 3519\vec{v}_2 \cdot \vec{n}\vec{n}) + 12180\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2 \vec{n} - \dot{\vec{S}}_2 \times \vec{n}(1966v_1^2 \\
& - 9236\vec{v}_1 \cdot \vec{v}_2 + 6903v_2^2 - 4292\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 11728(\vec{v}_1 \cdot \vec{n})^2 - 7884(\vec{v}_2 \cdot \vec{n})^2 \\
& + 8\dot{\vec{S}}_2 \times \vec{v}_1(835\vec{v}_1 \cdot \vec{n} + 92\vec{v}_2 \cdot \vec{n}) - 4\dot{\vec{S}}_2 \times \vec{v}_2(3575\vec{v}_1 \cdot \vec{n} + 354\vec{v}_2 \cdot \vec{n})] \\
& - \frac{225G^3m_1m_2}{16r^2}\dot{\vec{S}}_1 \times \vec{n} + \frac{5G^3m_2^3}{3m_1r^2}\dot{\vec{S}}_1 \times \vec{n} - \frac{17G^3m_1m_2}{3r^2}\left(\frac{1}{\epsilon} - 3\log\frac{r}{R_0}\right)\dot{\vec{S}}_1 \times \vec{n} \\
& - \frac{29G^3m_2^2}{6r^2}\left(\frac{1}{\epsilon} - 3\log\frac{r}{R_0}\right)\dot{\vec{S}}_1 \times \vec{n} + \frac{2G^3m_2^3}{3m_1r^2}\left(\frac{1}{\epsilon} - 3\log\frac{r}{R_0}\right)\dot{\vec{S}}_1 \times \vec{n} \\
& - \frac{G^3m_2^2}{576r^2}(2767 - 108\pi^2)\dot{\vec{S}}_1 \times \vec{n} + \frac{3325G^3m_1^2}{288r^2}\dot{\vec{S}}_2 \times \vec{n} + \frac{23G^3m_2^2}{3r^2}\dot{\vec{S}}_2 \times \vec{n} \\
& + \frac{37G^3m_1^2}{6r^2}\left(\frac{1}{\epsilon} - 3\log\frac{r}{R_0}\right)\dot{\vec{S}}_2 \times \vec{n} + \frac{29G^3m_1m_2}{6r^2}\left(\frac{1}{\epsilon} - 3\log\frac{r}{R_0}\right)\dot{\vec{S}}_2 \times \vec{n} \\
& + \frac{8G^3m_2^2}{3r^2}\left(\frac{1}{\epsilon} - 3\log\frac{r}{R_0}\right)\dot{\vec{S}}_2 \times \vec{n} + \frac{G^3m_1m_2}{288r^2}(2441 - 99\pi^2)\dot{\vec{S}}_2 \times \vec{n}, \tag{B.31}
\end{aligned}$$

$$\begin{aligned}
\Delta\vec{x}_{1(3,1)}^{(2)} = & -\frac{Gm_2r}{768m_1}\left[(4\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1(4\vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{n} - 28\vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{n} + 7\dot{\vec{a}}_2 \cdot \vec{n}\vec{v}_1 + 44\dot{\vec{a}}_2 \cdot \vec{n}\vec{v}_2 \right. \\
& \left. - 32\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 + 152\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 - 4\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}\vec{n} + 28\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}\vec{n}) \right. \\
& \left. - 16\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{n} - 7\vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{n} + \dot{\vec{a}}_2 \cdot \vec{n}\vec{v}_1 + 5\dot{\vec{a}}_2 \cdot \vec{n}\vec{v}_2 - 5\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \right. \\
& \left. + 23\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 - \vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}\vec{n} + 7\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}\vec{n}) + 144\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 \dot{\vec{a}}_2 \right. \\
& \left. - 48\dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2(\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 3v_2^2\vec{n} + \vec{v}_2 \cdot \vec{n}\vec{v}_1 - \vec{v}_1 \cdot \vec{n}\vec{v}_2 + 4\vec{v}_2 \cdot \vec{n}\vec{v}_2 - \vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} \right. \\
& \left. + 3(\vec{v}_2 \cdot \vec{n})^2\vec{n}) + 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2(53\vec{v}_1 + 100\vec{v}_2 + 8\vec{v}_1 \cdot \vec{n}\vec{n} - 44\vec{v}_2 \cdot \vec{n}\vec{n}) \right. \\
& \left. - 16\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2(2\vec{v}_1 + 7\vec{v}_2 + 2\vec{v}_1 \cdot \vec{n}\vec{n} - 11\vec{v}_2 \cdot \vec{n}\vec{n}) + 4\dot{\vec{S}}_1 \times \vec{n}(7v_1^2\dot{\vec{a}}_2 \cdot \vec{n} \right. \\
& \left. + 52\vec{v}_1 \cdot \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} - 104v_2^2\dot{\vec{a}}_2 \cdot \vec{n} - 28\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \dot{\vec{a}}_2 + 160\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \dot{\vec{a}}_2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \dot{\vec{a}}_2 \right. \\
& \left. - 280\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \dot{\vec{a}}_2 + 20\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n} - 4\dot{\vec{a}}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 80\dot{\vec{a}}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 \right. \\
& \left. + 4\dot{\vec{S}}_1 \times \vec{v}_1(307\vec{v}_1 \cdot \dot{\vec{a}}_2 - 352\vec{v}_2 \cdot \dot{\vec{a}}_2 + 7\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n} - 184\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}) \right. \\
& \left. - 16\dot{\vec{S}}_1 \times \vec{v}_2(49\vec{v}_1 \cdot \dot{\vec{a}}_2 - 115\vec{v}_2 \cdot \dot{\vec{a}}_2 - 2\vec{v}_1 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n} - 49\vec{v}_2 \cdot \vec{n}\dot{\vec{a}}_2 \cdot \vec{n}) \right. \\
& \left. - 4\dot{\vec{S}}_1 \times \dot{\vec{a}}_2(53v_1^2 + 356\vec{v}_1 \cdot \vec{v}_2 - 628v_2^2 + 44\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 8(\vec{v}_1 \cdot \vec{n})^2 \right. \\
& \left. - 460(\vec{v}_2 \cdot \vec{n})^2) - (32\ddot{\vec{S}}_1 \times \vec{n}(3\vec{v}_2 \cdot \vec{n}v_2^2 - (\vec{v}_2 \cdot \vec{n})^3) - 288\ddot{\vec{S}}_1 \times \vec{v}_2(v_2^2 + (\vec{v}_2 \cdot \vec{n})^2)) \right. \\
& \left. - (4\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1(40a_2^2\vec{n} - \vec{a}_2 \cdot \vec{n}\vec{a}_1 - 168\vec{a}_2 \cdot \vec{n}\vec{a}_2) - 96\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1(\vec{a}_2 \cdot \vec{n}\vec{v}_2 \right. \\
& \left. + 2\vec{v}_2 \cdot \vec{n}\vec{a}_2) + 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_1(97\vec{a}_2 - 15\vec{a}_2 \cdot \vec{n}\vec{n}) - 96\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2(a_2^2\vec{n} - 4\vec{a}_2 \cdot \vec{n}\vec{a}_2) \right. \\
& \left. + 96\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{v}_2(\vec{a}_2 - \vec{a}_2 \cdot \vec{n}\vec{n}) + 64\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2(2\vec{v}_1 \cdot \vec{a}_2 \vec{n} - 6\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 3\vec{a}_2 \cdot \vec{n}\vec{v}_2 \right. \\
& \left. - 2\vec{v}_1 \cdot \vec{n}\vec{a}_2 + 12\vec{v}_2 \cdot \vec{n}\vec{a}_2 + 6\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) - 4\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2(15\vec{a}_1 \right. \\
& \left. + 304\vec{a}_2 - 24\vec{a}_2 \cdot \vec{n}\vec{n}) - 96\dot{\vec{S}}_1 \times \vec{a}_1 \cdot \vec{a}_2(\vec{v}_2 + 2\vec{v}_2 \cdot \vec{n}\vec{n}) + 192\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2(2\vec{a}_2 \right. \\
& \left. - \vec{a}_2 \cdot \vec{n}\vec{n}) - 4\dot{\vec{S}}_1 \times \vec{n}(3\vec{v}_1 \cdot \vec{a}_1 \vec{a}_2 \cdot \vec{n} + 168\vec{a}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 24\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{a}_2 \right. \\
& \left. - 24\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 384\vec{a}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 8\vec{v}_1 \cdot \vec{n}a_2^2 - 264\vec{v}_2 \cdot \vec{n}a_2^2 \right. \\
& \left. + 24\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 8\dot{\vec{S}}_1 \times \vec{v}_1(97\vec{a}_1 \cdot \vec{a}_2 - 126a_2^2 - 15\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \right. \\
& \left. - 12\dot{\vec{S}}_1 \times \vec{a}_1(97\vec{v}_1 \cdot \vec{a}_2 - 120\vec{v}_2 \cdot \vec{a}_2 - 15\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 56\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \right. \\
& \left. + 192\dot{\vec{S}}_1 \times \vec{v}_2(2\vec{a}_1 \cdot \vec{a}_2 - 8a_2^2 - \vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + 4\dot{\vec{S}}_1 \times \vec{a}_2(45\vec{v}_1 \cdot \vec{a}_1 + 48\vec{a}_1 \cdot \vec{v}_2 \right)
\end{aligned}$$

$$\begin{aligned}
& +496\vec{v}_1 \cdot \vec{a}_2 - 1440\vec{v}_2 \cdot \vec{a}_2 - 120\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 24\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 816\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& +(48\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 2\vec{a}_2 \cdot \vec{n}\vec{v}_1 + \vec{a}_2 \cdot \vec{n}\vec{v}_2 + \vec{v}_2 \cdot \vec{n}\vec{a}_2 - \vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) \\
& -48\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{a}_2 \vec{n} + \vec{a}_2 \cdot \vec{n}\vec{v}_2 + \vec{v}_2 \cdot \vec{n}\vec{a}_2 - \vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) \\
& -48\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{a}_2 - 3\vec{a}_2 \cdot \vec{n}\vec{n}) - 48\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 (v_2^2 \vec{n} + 2\vec{v}_2 \cdot \vec{n}\vec{v}_2 - (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \\
& +48\dot{\vec{S}}_1 \times \vec{v}_1 \cdot \vec{a}_2 (14\vec{v}_1 + 3\vec{v}_2 + 7\vec{v}_2 \cdot \vec{n}\vec{n}) - 48\dot{\vec{S}}_1 \times \vec{v}_2 \cdot \vec{a}_2 (\vec{v}_2 + \vec{v}_2 \cdot \vec{n}\vec{n}) \\
& +3\dot{\vec{S}}_1 \times \vec{n} (17v_1^2 \vec{a}_2 \cdot \vec{n} - 32\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 48v_2^2 \vec{a}_2 \cdot \vec{n} - 32\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& -32\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 96\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 32\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 64\vec{a}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) \\
& +6\dot{\vec{S}}_1 \times \vec{v}_1 (81\vec{v}_1 \cdot \vec{a}_2 - 88\vec{v}_2 \cdot \vec{a}_2 - 31\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 24\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& +192\dot{\vec{S}}_1 \times \vec{a}_1 (v_2^2 + (\vec{v}_2 \cdot \vec{n})^2) + 48\dot{\vec{S}}_1 \times \vec{v}_2 (2\vec{v}_1 \cdot \vec{a}_2 - \vec{v}_2 \cdot \vec{a}_2 + 6\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} \\
& -15\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 3\dot{\vec{S}}_1 \times \vec{a}_2 (127v_1^2 - 160\vec{v}_1 \cdot \vec{v}_2 + 192v_2^2 - 224\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& +304(\vec{v}_2 \cdot \vec{n})^2)) - \frac{1}{768}Gr\left[(16\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{a}_2 \vec{n} - 7\vec{v}_2 \cdot \vec{a}_2 \vec{n} + \vec{a}_2 \cdot \vec{n}\vec{v}_1 \right. \\
& \left.+ 5\vec{a}_2 \cdot \vec{n}\vec{v}_2 - 5\vec{v}_1 \cdot \vec{n}\vec{a}_2 + 23\vec{v}_2 \cdot \vec{n}\vec{a}_2 - \vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n} + 7\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) \right. \\
& \left.+ 16\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (36\vec{v}_1 \cdot \vec{a}_1 \vec{n} - 33\vec{a}_1 \cdot \vec{v}_2 \vec{n} + 10\vec{v}_1 \cdot \vec{a}_2 \vec{n} - 4\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 36\vec{a}_1 \cdot \vec{n}\vec{v}_1 \right. \\
& \left.+ \vec{a}_2 \cdot \vec{n}\vec{v}_1 + 63\vec{v}_1 \cdot \vec{n}\vec{a}_1 + 21\vec{v}_2 \cdot \vec{n}\vec{a}_1 - 33\vec{a}_1 \cdot \vec{n}\vec{v}_2 - \vec{a}_2 \cdot \vec{n}\vec{v}_2 + 19\vec{v}_1 \cdot \vec{n}\vec{a}_2 \right. \\
& \left.- 4\vec{v}_2 \cdot \vec{n}\vec{a}_2 - 15\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{n} - 3\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} + 2\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n} - 5\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) \right. \\
& \left.- 16\vec{S}_2 \times \vec{v}_1 \cdot \vec{v}_2 (9\vec{a}_1 + 4\vec{a}_2 + 57\vec{a}_1 \cdot \vec{n}\vec{n} - 5\vec{a}_2 \cdot \vec{n}\vec{n}) - 48\vec{S}_2 \times \vec{a}_1 \cdot \vec{v}_2 (40\vec{v}_1 - 25\vec{v}_2 \right. \\
& \left.+ 47\vec{v}_1 \cdot \vec{n}\vec{n} - 5\vec{v}_2 \cdot \vec{n}\vec{n}) - 24\vec{S}_2 \times \vec{n} \cdot \vec{a}_2 (3v_1^2 \vec{n} - 4\vec{v}_1 \cdot \vec{v}_2 \vec{n} - v_2^2 \vec{n} + 10\vec{v}_1 \cdot \vec{n}\vec{v}_1 \right. \\
& \left.+ 2\vec{v}_2 \cdot \vec{n}\vec{v}_1 - 10\vec{v}_1 \cdot \vec{n}\vec{v}_2 - 2\vec{v}_2 \cdot \vec{n}\vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - (\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 4(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) \right. \\
& \left.+ 16\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_2 (8\vec{v}_1 - 14\vec{v}_2 + 20\vec{v}_1 \cdot \vec{n}\vec{n} + 25\vec{v}_2 \cdot \vec{n}\vec{n}) - 16\vec{S}_2 \times \vec{v}_2 \cdot \vec{a}_2 (34\vec{v}_1 \right. \\
& \left.- 34\vec{v}_2 + 22\vec{v}_1 \cdot \vec{n}\vec{n} + 11\vec{v}_2 \cdot \vec{n}\vec{n}) + 16\vec{S}_2 \times \vec{n} (v_1^2 \vec{a}_2 \cdot \vec{n} + 7\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 20v_2^2 \vec{a}_2 \cdot \vec{n} \right. \\
& \left.- 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 25\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 - 5\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 40\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \right. \\
& \left.+ 5\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - \vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 20\vec{a}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) - 16\vec{S}_2 \times \vec{v}_1 (5\vec{v}_1 \cdot \vec{a}_2 \right. \\
& \left.- 2\vec{v}_2 \cdot \vec{a}_2 - \vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + 48\vec{S}_2 \times \vec{v}_2 (32\vec{v}_1 \cdot \vec{a}_1 - 17\vec{a}_1 \cdot \vec{v}_2 \right. \\
& \left.+ 4\vec{v}_1 \cdot \vec{a}_2 - 5\vec{v}_2 \cdot \vec{a}_2 + 39\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} + 3\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 8\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \right. \\
& \left.- 8\vec{S}_2 \times \vec{a}_2 (13v_1^2 + 4\vec{v}_1 \cdot \vec{v}_2 + 37v_2^2 + 100\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 19(\vec{v}_1 \cdot \vec{n})^2 - 2(\vec{v}_2 \cdot \vec{n})^2) \right. \\
& \left.- (192\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (v_2^2 \vec{n} - \vec{v}_1 \cdot \vec{n}\vec{v}_1 + \vec{v}_2 \cdot \vec{n}\vec{v}_1 - \vec{v}_2 \cdot \vec{n}\vec{v}_2 - \vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} \right. \\
& \left.- (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 6\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (12v_1^2 \vec{n} - 8\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 27v_2^2 \vec{n} + 24\vec{v}_1 \cdot \vec{n}\vec{v}_1 - 48\vec{v}_2 \cdot \vec{n}\vec{v}_1 \right. \\
& \left.- 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 + 24\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 24(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 48\ddot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2 (9\vec{v}_1 \right. \\
& \left.- 5\vec{v}_2 + 11\vec{v}_1 \cdot \vec{n}\vec{n} + \vec{v}_2 \cdot \vec{n}\vec{n}) + 32\ddot{\vec{S}}_2 \times \vec{n} (3v_1^2 \vec{v}_2 \cdot \vec{n} - 6\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n}\vec{v}_2^2 \right. \\
& \left.+ 30\vec{v}_2 \cdot \vec{n}\vec{v}_2^2 - 3\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 10(\vec{v}_2 \cdot \vec{n})^3) + 6\ddot{\vec{S}}_2 \times \vec{v}_2 (36v_1^2 \right. \\
& \left.- 40\vec{v}_1 \cdot \vec{v}_2 - 5v_2^2 + 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 44(\vec{v}_1 \cdot \vec{n})^2 + 40(\vec{v}_2 \cdot \vec{n})^2) \right. \\
& \left.- (96\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (a_2^2 \vec{n} - 4\vec{a}_2 \cdot \vec{n}\vec{a}_2) - 64\vec{S}_2 \times \vec{n} \cdot \vec{a}_1 (\vec{a}_2 \cdot \vec{n}\vec{v}_2 + 2\vec{v}_2 \cdot \vec{n}\vec{a}_2) \right. \\
& \left.- 4\vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (54a_1^2 \vec{n} - 90\vec{a}_1 \cdot \vec{a}_2 \vec{n} - 19a_2^2 \vec{n} + 216\vec{a}_1 \cdot \vec{n}\vec{a}_1 + 58\vec{a}_2 \cdot \vec{n}\vec{a}_1 \right)
\end{aligned}$$

$$\begin{aligned}
& -42\vec{a}_1 \cdot \vec{n}\vec{a}_2 - 36\vec{a}_2 \cdot \vec{n}\vec{a}_2 - 6\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) + 8\vec{S}_2 \times \vec{a}_1 \cdot \vec{v}_2 (84\vec{a}_1 - 91\vec{a}_2 \\
& + 162\vec{a}_1 \cdot \vec{n}\vec{n} - 3\vec{a}_2 \cdot \vec{n}\vec{n}) - 4\vec{S}_2 \times \vec{n} \cdot \vec{a}_2 (84\vec{v}_1 \cdot \vec{a}_1 \vec{n} - 84\vec{a}_1 \cdot \vec{v}_2 \vec{n} + 36\vec{v}_1 \cdot \vec{a}_2 \vec{n} \\
& - 3\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 48\vec{a}_1 \cdot \vec{n}\vec{v}_1 + 24\vec{a}_2 \cdot \vec{n}\vec{v}_1 + 84\vec{v}_1 \cdot \vec{n}\vec{a}_1 + 116\vec{v}_2 \cdot \vec{n}\vec{a}_1 - 36\vec{a}_1 \cdot \vec{n}\vec{v}_2 \\
& + 132\vec{v}_1 \cdot \vec{n}\vec{a}_2 - 72\vec{v}_2 \cdot \vec{n}\vec{a}_2 - 24\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{n} - 12\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} + 12\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n} \\
& - 72\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) + 48\vec{S}_2 \times \vec{v}_1 \cdot \vec{a}_2 (5\vec{a}_1 - 3\vec{a}_2 + 10\vec{a}_1 \cdot \vec{n}\vec{n} - 3\vec{a}_2 \cdot \vec{n}\vec{n}) \\
& + 16\vec{S}_2 \times \vec{a}_1 \cdot \vec{a}_2 (60\vec{v}_1 - 11\vec{v}_2 + 66\vec{v}_1 \cdot \vec{n}\vec{n} + 21\vec{v}_2 \cdot \vec{n}\vec{n}) - 4\vec{S}_2 \times \vec{v}_2 \cdot \vec{a}_2 (6\vec{a}_1 \\
& + 23\vec{a}_2 - 30\vec{a}_1 \cdot \vec{n}\vec{n} + 15\vec{a}_2 \cdot \vec{n}\vec{n}) + 32\vec{S}_2 \times \vec{n} (\vec{a}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 12\vec{a}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \\
& - \vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{a}_2 + 3\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 30\vec{a}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 3\vec{v}_1 \cdot \vec{n}\vec{a}_2^2 \\
& + 15\vec{v}_2 \cdot \vec{n}\vec{a}_2^2 - 3\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + 96\vec{S}_2 \times \vec{a}_1 (\vec{v}_2 \cdot \vec{a}_2 + \vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) \\
& - 4\vec{S}_2 \times \vec{v}_2 (282a_1^2 - 158\vec{a}_1 \cdot \vec{a}_2 - 45a_2^2 - 30\vec{a}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 4\vec{S}_2 \times \vec{a}_2 (60\vec{v}_1 \cdot \vec{a}_1 \\
& - 68\vec{a}_1 \cdot \vec{v}_2 - 36\vec{v}_1 \cdot \vec{a}_2 - 117\vec{v}_2 \cdot \vec{a}_2 + 120\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} + 12\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 132\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + 48\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n})) + (48\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1 \cdot \vec{a}_2 \vec{n} - 8\vec{v}_2 \cdot \vec{a}_2 \vec{n} \\
& - \vec{a}_2 \cdot \vec{n}\vec{v}_1 + 4\vec{a}_2 \cdot \vec{n}\vec{v}_2 - \vec{v}_1 \cdot \vec{n}\vec{a}_2 + 16\vec{v}_2 \cdot \vec{n}\vec{a}_2 + \vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n} + 8\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) \\
& + 48\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_1 (v_2^2 \vec{n} + 2\vec{v}_2 \cdot \vec{n}\vec{v}_2 - (\vec{v}_2 \cdot \vec{n})^2 \vec{n}) + 12\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (28\vec{v}_1 \cdot \vec{a}_1 \vec{n} \\
& - 28\vec{a}_1 \cdot \vec{v}_2 \vec{n} + 8\vec{v}_1 \cdot \vec{a}_2 \vec{n} + 7\vec{v}_2 \cdot \vec{a}_2 \vec{n} + 16\vec{a}_1 \cdot \vec{n}\vec{v}_1 + 12\vec{a}_2 \cdot \vec{n}\vec{v}_1 + 28\vec{v}_1 \cdot \vec{n}\vec{a}_1 \\
& + 28\vec{v}_2 \cdot \vec{n}\vec{a}_1 - 12\vec{a}_1 \cdot \vec{n}\vec{v}_2 + 32\vec{v}_1 \cdot \vec{n}\vec{a}_2 - 24\vec{v}_2 \cdot \vec{n}\vec{a}_2 - 8\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n}\vec{n} \\
& - 4\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{n} - 24\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}\vec{n}) - 48\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2 (5\vec{a}_1 + 6\vec{a}_2 + 10\vec{a}_1 \cdot \vec{n}\vec{n} \\
& - 6\vec{a}_2 \cdot \vec{n}\vec{n}) - 48\dot{\vec{S}}_2 \times \vec{a}_1 \cdot \vec{v}_2 (20\vec{v}_1 - 3\vec{v}_2 + 22\vec{v}_1 \cdot \vec{n}\vec{n} + 9\vec{v}_2 \cdot \vec{n}\vec{n}) \\
& - 3\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 (32v_1^2 \vec{n} - 32\vec{v}_1 \cdot \vec{v}_2 \vec{n} - 61v_2^2 \vec{n} + 128\vec{v}_1 \cdot \vec{n}\vec{v}_1 - 96\vec{v}_2 \cdot \vec{n}\vec{v}_1 \\
& - 128\vec{v}_1 \cdot \vec{n}\vec{v}_2 - 16(\vec{v}_1 \cdot \vec{n})^2 \vec{n} + 96(\vec{v}_2 \cdot \vec{n})^2 \vec{n}) - 48\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{a}_2 (7\vec{v}_1 \\
& + 2\vec{v}_2 - 15\vec{v}_1 \cdot \vec{n}\vec{n} - 6\vec{v}_2 \cdot \vec{n}\vec{n}) + 6\dot{\vec{S}}_2 \times \vec{v}_2 \cdot \vec{a}_2 (8\vec{v}_1 - \vec{v}_2 - 48\vec{v}_1 \cdot \vec{n}\vec{n} + 31\vec{v}_2 \cdot \vec{n}\vec{n}) \\
& - 48\dot{\vec{S}}_2 \times \vec{n} (2\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n}v_2^2 - v_1^2 \vec{a}_2 \cdot \vec{n} - 4\vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} + 20v_2^2 \vec{a}_2 \cdot \vec{n} \\
& - 2\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 - 16\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 + 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 \\
& + 40\vec{v}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 - 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} + \vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - \vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 \\
& - 20\vec{a}_2 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) - 48\dot{\vec{S}}_2 \times \vec{v}_1 (\vec{v}_1 \cdot \vec{a}_2 - 3\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - 96\dot{\vec{S}}_2 \times \vec{a}_1 (v_2^2 \\
& + (\vec{v}_2 \cdot \vec{n})^2) + 12\dot{\vec{S}}_2 \times \vec{v}_2 (20\vec{v}_1 \cdot \vec{a}_1 - 28\vec{a}_1 \cdot \vec{v}_2 + 24\vec{v}_1 \cdot \vec{a}_2 - 63\vec{v}_2 \cdot \vec{a}_2 \\
& + 40\vec{v}_1 \cdot \vec{n}\vec{a}_1 \cdot \vec{n} + 4\vec{a}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 48\vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n} - 64\vec{v}_2 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) + 3\dot{\vec{S}}_2 \times \vec{a}_2 (48v_1^2 \\
& - 160\vec{v}_1 \cdot \vec{v}_2 + 19v_2^2 - 384\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 96(\vec{v}_1 \cdot \vec{n})^2 + 160(\vec{v}_2 \cdot \vec{n})^2)) \Big] \\
& - \frac{Gm_2 r}{16m_1} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{a}_2 \cdot \vec{n})^2 \vec{n} - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_2 \cdot \vec{n})^2 \vec{n} + 2\vec{S}_1 \times \vec{n} (\vec{v}_1 \cdot \vec{n}(\vec{a}_2 \cdot \vec{n})^2 \\
& + 5\vec{v}_2 \cdot \vec{n}(\vec{a}_2 \cdot \vec{n})^2) - 9\vec{S}_1 \times \vec{v}_1 (\vec{a}_2 \cdot \vec{n})^2 + 12\vec{S}_1 \times \vec{v}_2 (\vec{a}_2 \cdot \vec{n})^2 \Big] \\
& - \frac{1}{32} Gr \Big[4\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (\vec{a}_2 \cdot \vec{n})^2 \vec{n} - \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (5(\vec{a}_1 \cdot \vec{n})^2 \vec{n} + 3(\vec{a}_2 \cdot \vec{n})^2 \vec{n}) \\
& + 4\vec{S}_2 \times \vec{n} (\vec{v}_1 \cdot \vec{n}(\vec{a}_2 \cdot \vec{n})^2 + 5\vec{v}_2 \cdot \vec{n}(\vec{a}_2 \cdot \vec{n})^2) + \vec{S}_2 \times \vec{v}_2 (51(\vec{a}_1 \cdot \vec{n})^2 - 11(\vec{a}_2 \cdot \vec{n})^2) \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{G^2 m_2^2}{576 m_1} \left[(48 \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{n} + 1404 \vec{S}_1 \times \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} + 4952 \vec{S}_1 \times \dot{\vec{a}}_2) - (333 \dot{\vec{S}}_1 \times \vec{n} \vec{a}_2 \cdot \vec{n} \right. \\
& \left. - 1647 \vec{S}_1 \times \vec{a}_2) \right] + \frac{1}{6} G^2 m_2 \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \left[(10 \vec{S}_1 \times \dot{\vec{a}}_1 + 40 \vec{S}_1 \times \vec{a}_2) - (6 \ddot{\vec{S}}_1 \times \vec{v}_1 \right. \\
& \left. - 8 \ddot{\vec{S}}_1 \times \vec{v}_2) - (28 \dot{\vec{S}}_1 \times \vec{a}_1 - 5 \dot{\vec{S}}_1 \times \vec{a}_2) \right] + \frac{1}{576} G^2 m_2 \left[(51 \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_1 \vec{n} \right. \\
& \left. + 288 \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{n} - 3 \vec{S}_1 \times \vec{n} (383 \dot{\vec{a}}_1 \cdot \vec{n} - 812 \dot{\vec{a}}_2 \cdot \vec{n}) + 3720 \vec{S}_1 \times \dot{\vec{a}}_1 \right. \\
& \left. + 2608 \vec{S}_1 \times \vec{a}_2) + (756 \ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} - 588 \ddot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 \vec{n} - 36 \ddot{\vec{S}}_1 \times \vec{n} (20 \vec{v}_1 \cdot \vec{n} \right. \\
& \left. - 89 \vec{v}_2 \cdot \vec{n}) + 52 \vec{S}_1 \times \vec{v}_1 + 576 \vec{S}_1 \times \vec{v}_2) - (117 \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_1 \vec{n} + 360 \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 \vec{n} \right. \\
& \left. + 1068 \dot{\vec{S}}_1 \times \vec{n} (\vec{a}_1 \cdot \vec{n} - 2 \vec{a}_2 \cdot \vec{n}) - 47 \dot{\vec{S}}_1 \times \vec{a}_1 + 1368 \dot{\vec{S}}_1 \times \vec{a}_2) \right] \\
& + \frac{1}{576} G^2 m_1 \left[(624 \vec{S}_2 \times \vec{n} \cdot \dot{\vec{a}}_1 \vec{n} + 750 \vec{S}_2 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{n} + 15 \vec{S}_2 \times \vec{n} (32 \dot{\vec{a}}_1 \cdot \vec{n} + 71 \dot{\vec{a}}_2 \cdot \vec{n}) \right. \\
& \left. - 1320 \vec{S}_2 \times \dot{\vec{a}}_1 - 2174 \vec{S}_2 \times \dot{\vec{a}}_2) + (2124 \ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} - 1479 \ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} \right. \\
& \left. + 6 \ddot{\vec{S}}_2 \times \vec{n} (44 \vec{v}_1 \cdot \vec{n} + 625 \vec{v}_2 \cdot \vec{n}) + 1649 \ddot{\vec{S}}_2 \times \vec{v}_1 - 1590 \ddot{\vec{S}}_2 \times \vec{v}_2) \right. \\
& \left. + (273 \dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_1 \vec{n} - 216 \dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 \vec{n} + 3 \dot{\vec{S}}_2 \times \vec{n} (95 \vec{a}_1 \cdot \vec{n} + 1093 \vec{a}_2 \cdot \vec{n}) \right. \\
& \left. - 322 \dot{\vec{S}}_2 \times \vec{a}_1 - 1558 \dot{\vec{S}}_2 \times \vec{a}_2) \right] - \frac{1}{6} G^2 m_1 \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \left[30 \vec{S}_2 \times \dot{\vec{a}}_2 - (8 \ddot{\vec{S}}_2 \times \vec{v}_1 \right. \\
& \left. - 6 \ddot{\vec{S}}_2 \times \vec{v}_2) - (5 \dot{\vec{S}}_2 \times \vec{a}_1 - 28 \dot{\vec{S}}_2 \times \vec{a}_2) \right] - \frac{1}{6} G^2 m_2 \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \left[30 \vec{S}_2 \times \dot{\vec{a}}_2 \right. \\
& \left. - 2 \ddot{\vec{S}}_2 \times \vec{v}_2 + 23 \dot{\vec{S}}_2 \times \vec{a}_2 \right] + \frac{1}{288} G^2 m_2 \left[(243 \vec{S}_2 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{n} + 936 \vec{S}_2 \times \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} \right. \\
& \left. - 2063 \vec{S}_2 \times \dot{\vec{a}}_2) + (480 \vec{S}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} - 315 \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 24 \vec{S}_2 \times \vec{n} (20 \vec{v}_1 \cdot \vec{n} \right. \\
& \left. + 13 \vec{v}_2 \cdot \vec{n}) + 87 \vec{S}_2 \times \vec{v}_2) - (318 \dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 \vec{n} - 1674 \dot{\vec{S}}_2 \times \vec{n} \vec{a}_2 \cdot \vec{n} + 2058 \dot{\vec{S}}_2 \times \vec{a}_2) \right] \\
& + \frac{20 G^2 m_2^2}{3 m_1} \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \vec{S}_1 \times \dot{\vec{a}}_2, \tag{B.32}
\end{aligned}$$

$$\begin{aligned}
\Delta \vec{x}_{1(3,1)}^{(3)} = & \frac{G m_2 r^2}{48 m_1} \left[(\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (8 \ddot{\vec{a}}_2 - \ddot{\vec{a}}_2 \cdot \vec{n} \vec{n}) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (5 \ddot{\vec{a}}_2 - \ddot{\vec{a}}_2 \cdot \vec{n} \vec{n}) \right. \\
& - 3 \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (\vec{v}_2 - \vec{v}_2 \cdot \vec{n} \vec{n}) - 2 \vec{S}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{n} + 2 \vec{S}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{n} + \vec{S}_1 \times \vec{n} (10 \vec{v}_1 \cdot \dot{\vec{a}}_2 \right. \\
& \left. - 16 \vec{v}_2 \cdot \dot{\vec{a}}_2 + \vec{v}_1 \cdot \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n} - 10 \vec{v}_2 \cdot \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n}) - 10 \vec{S}_1 \times \vec{v}_1 \ddot{\vec{a}}_2 \cdot \vec{n} + 13 \vec{S}_1 \times \vec{v}_2 \ddot{\vec{a}}_2 \cdot \vec{n} \right. \\
& \left. - \vec{S}_1 \times \dot{\vec{a}}_2 (20 \vec{v}_1 \cdot \vec{n} - 83 \vec{v}_2 \cdot \vec{n}) \right) + (\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (8 \dot{\vec{a}}_2 - \dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) \right. \\
& \left. - \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (23 \vec{a}_2 - 7 \dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) - 3 \vec{S}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (7 \vec{a}_2 - 3 \vec{a}_2 \cdot \vec{n} \vec{n}) - 2 \vec{S}_1 \times \vec{a}_1 \cdot \dot{\vec{a}}_2 \vec{n} \right. \\
& \left. + 2 \vec{S}_1 \times \vec{a}_2 \cdot \dot{\vec{a}}_2 \vec{n} + \vec{S}_1 \times \vec{n} (8 \vec{a}_1 \cdot \dot{\vec{a}}_2 - 50 \vec{a}_2 \cdot \dot{\vec{a}}_2 - \vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} - 20 \vec{a}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) \right. \\
& \left. - 12 \vec{S}_1 \times \vec{a}_1 \dot{\vec{a}}_2 \cdot \vec{n} + 69 \vec{S}_1 \times \vec{a}_2 \dot{\vec{a}}_2 \cdot \vec{n} + \vec{S}_1 \times \dot{\vec{a}}_2 (2 \vec{a}_1 \cdot \vec{n} + 115 \vec{a}_2 \cdot \vec{n}) \right) \\
& + 6 \dot{\vec{S}}_1 \times \vec{n} (\vec{a}_2 \cdot \vec{n})^2 + (\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_1 (8 \dot{\vec{a}}_2 - \dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) - \dot{\vec{S}}_1 \times \vec{n} \cdot \vec{v}_2 (5 \dot{\vec{a}}_2 - \dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) \right. \\
& \left. - 3 \dot{\vec{S}}_1 \times \vec{n} \cdot \dot{\vec{a}}_2 (\vec{v}_2 - \vec{v}_2 \cdot \vec{n} \vec{n}) - 2 \dot{\vec{S}}_1 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{n} + 2 \dot{\vec{S}}_1 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{n} + \dot{\vec{S}}_1 \times \vec{n} (7 \vec{v}_1 \cdot \dot{\vec{a}}_2 \right. \\
& \left. - \vec{v}_2 \cdot \dot{\vec{a}}_2 - 2 \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} + 5 \vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - \dot{\vec{S}}_1 \times \vec{v}_1 \dot{\vec{a}}_2 \cdot \vec{n} - 20 \dot{\vec{S}}_1 \times \vec{v}_2 \dot{\vec{a}}_2 \cdot \vec{n} \right. \\
& \left. + \dot{\vec{S}}_1 \times \dot{\vec{a}}_2 (13 \vec{v}_1 \cdot \vec{n} - 34 \vec{v}_2 \cdot \vec{n}) \right) + (3 \ddot{\vec{S}}_1 \times \vec{n} (\vec{v}_2 \cdot \vec{a}_2 + \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n}) \right]
\end{aligned}$$

$$\begin{aligned}
& +6\ddot{\vec{S}}_1 \times \vec{v}_1 \vec{a}_2 \cdot \vec{n} - 9\ddot{\vec{S}}_1 \times \vec{v}_2 \vec{a}_2 \cdot \vec{n} - 21\ddot{\vec{S}}_1 \times \vec{a}_2 \vec{v}_2 \cdot \vec{n}) - (8\dot{\vec{S}}_1 \times \vec{n} \cdot \vec{a}_2 \vec{a}_2 \\
& - 2\dot{\vec{S}}_1 \times \vec{n} \vec{a}_2^2 - 12\dot{\vec{S}}_1 \times \vec{a}_1 \vec{a}_2 \cdot \vec{n} + 60\dot{\vec{S}}_1 \times \vec{a}_2 \vec{a}_2 \cdot \vec{n})] + \frac{1}{96} Gr^2 [(2\vec{S}_2 \times \vec{n} \cdot \vec{v}_1 (5\ddot{\vec{a}}_2 \\
& - \ddot{\vec{a}}_2 \cdot \vec{n} \vec{n}) + \vec{S}_2 \times \vec{n} \cdot \vec{v}_2 (7\ddot{\vec{a}}_2 + \ddot{\vec{a}}_2 \cdot \vec{n} \vec{n}) + 24\vec{S}_2 \times \ddot{\vec{a}}_1 \cdot \vec{v}_2 \vec{n} - 2\vec{S}_2 \times \vec{n} \cdot \ddot{\vec{a}}_2 (4\vec{v}_1 \\
& - 4\vec{v}_2 + \vec{v}_1 \cdot \vec{n} \vec{n} - \vec{v}_2 \cdot \vec{n} \vec{n}) + 24\vec{S}_2 \times \vec{v}_1 \cdot \ddot{\vec{a}}_2 \vec{n} - 17\vec{S}_2 \times \vec{v}_2 \cdot \ddot{\vec{a}}_2 \vec{n} + 2\vec{S}_2 \times \vec{n} (7\vec{v}_1 \cdot \ddot{\vec{a}}_2 \\
& - 10\vec{v}_2 \cdot \ddot{\vec{a}}_2 + \vec{v}_1 \cdot \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n} - 10\vec{v}_2 \cdot \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n}) + 4\vec{S}_2 \times \vec{v}_1 \ddot{\vec{a}}_2 \cdot \vec{n} + 3\vec{S}_2 \times \vec{v}_2 (8\ddot{\vec{a}}_1 \cdot \vec{n} \\
& - 5\ddot{\vec{a}}_2 \cdot \vec{n}) - 20\vec{S}_2 \times \ddot{\vec{a}}_2 (\vec{v}_1 \cdot \vec{n} + 2\vec{v}_2 \cdot \vec{n})) - (8\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 - \vec{v}_2 + \vec{v}_1 \cdot \vec{n} \vec{n} \\
& + 2\vec{v}_2 \cdot \vec{n} \vec{n}) - 6\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (2\vec{v}_1 + \vec{v}_1 \cdot \vec{n} \vec{n} + 2\vec{v}_2 \cdot \vec{n} \vec{n}) + 2\ddot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{v}_2 \vec{n} \\
& + 4\ddot{\vec{S}}_2 \times \vec{n} (v_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 + 10v_2^2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + (\vec{v}_1 \cdot \vec{n})^2 + 10(\vec{v}_2 \cdot \vec{n})^2) \\
& - 2\ddot{\vec{S}}_2 \times \vec{v}_2 (\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}) + (2\vec{S}_2 \times \vec{n} \cdot \vec{a}_1 (5\dot{\vec{a}}_2 - \dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) + 2\vec{S}_2 \times \vec{n} \cdot \vec{a}_2 (21\dot{\vec{a}}_1 \\
& - 4\dot{\vec{a}}_2 + 3\dot{\vec{a}}_1 \cdot \vec{n} \vec{n} + 5\dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) - 42\vec{S}_2 \times \dot{\vec{a}}_1 \cdot \vec{a}_2 \vec{n} + 3\vec{S}_2 \times \vec{n} \cdot \dot{\vec{a}}_2 (11\vec{a}_1 - 2\vec{a}_2 \\
& + \vec{a}_1 \cdot \vec{n} \vec{n} + 4\vec{a}_2 \cdot \vec{n} \vec{n}) - 43\vec{S}_2 \times \dot{\vec{a}}_1 \cdot \dot{\vec{a}}_2 \vec{n} - 16\vec{S}_2 \times \vec{a}_2 \cdot \dot{\vec{a}}_2 \vec{n} + 2\vec{S}_2 \times \vec{n} (5\vec{a}_1 \cdot \dot{\vec{a}}_2 \\
& - 20\vec{a}_2 \cdot \dot{\vec{a}}_2 - \vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} - 20\vec{a}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 2\vec{S}_2 \times \vec{a}_2 (45\dot{\vec{a}}_1 \cdot \vec{n} + 2\dot{\vec{a}}_2 \cdot \vec{n}) \\
& + \vec{S}_2 \times \dot{\vec{a}}_2 (19\vec{a}_1 \cdot \vec{n} + 2\vec{a}_2 \cdot \vec{n})) - 60\dot{\vec{S}}_2 \times \vec{n} (\vec{a}_2 \cdot \vec{n})^2 \\
& + (2\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (23\dot{\vec{a}}_2 - 7\dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) + 2\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (21\dot{\vec{a}}_1 - 4\dot{\vec{a}}_2 + 3\dot{\vec{a}}_1 \cdot \vec{n} \vec{n} \\
& + 5\dot{\vec{a}}_2 \cdot \vec{n} \vec{n}) - 42\dot{\vec{S}}_2 \times \dot{\vec{a}}_1 \cdot \vec{v}_2 \vec{n} - 6\dot{\vec{S}}_2 \times \vec{n} \cdot \dot{\vec{a}}_2 (\vec{v}_1 - \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n} \vec{n} - 4\vec{v}_2 \cdot \vec{n} \vec{n}) \\
& + 74\dot{\vec{S}}_2 \times \vec{v}_1 \cdot \dot{\vec{a}}_2 \vec{n} - 22\dot{\vec{S}}_2 \times \vec{v}_2 \cdot \dot{\vec{a}}_2 \vec{n} + 10\dot{\vec{S}}_2 \times \vec{n} (5\vec{v}_1 \cdot \dot{\vec{a}}_2 - 8\vec{v}_2 \cdot \dot{\vec{a}}_2 - \vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} \\
& - 8\vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) + 4\dot{\vec{S}}_2 \times \vec{v}_1 \dot{\vec{a}}_2 \cdot \vec{n} + 18\dot{\vec{S}}_2 \times \vec{v}_2 (5\dot{\vec{a}}_1 \cdot \vec{n} - 2\dot{\vec{a}}_2 \cdot \vec{n}) \\
& - 2\dot{\vec{S}}_2 \times \dot{\vec{a}}_2 (71\vec{v}_1 \cdot \vec{n} - 11\vec{v}_2 \cdot \vec{n})) + (24\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 (2\vec{a}_2 - \vec{a}_2 \cdot \vec{n} \vec{n}) \\
& + 6\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_1 (\vec{v}_2 + \vec{v}_2 \cdot \vec{n} \vec{n}) + 3\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 (7\vec{a}_1 - 6\vec{a}_2 + \vec{a}_1 \cdot \vec{n} \vec{n} + 6\vec{a}_2 \cdot \vec{n} \vec{n}) \\
& - 51\ddot{\vec{S}}_2 \times \vec{a}_1 \cdot \vec{v}_2 \vec{n} + 18\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 (\vec{v}_1 + 2\vec{v}_2 \cdot \vec{n} \vec{n}) + 24\ddot{\vec{S}}_2 \times \vec{v}_1 \cdot \vec{a}_2 \vec{n} \\
& + 12\ddot{\vec{S}}_2 \times \vec{v}_2 \cdot \vec{a}_2 \vec{n} - 6\ddot{\vec{S}}_2 \times \vec{n} (\vec{a}_1 \cdot \vec{v}_2 - 8\vec{v}_1 \cdot \vec{a}_2 + 20\vec{v}_2 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \\
& + 4\vec{v}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} + 20\vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 12\ddot{\vec{S}}_2 \times \vec{a}_1 \vec{v}_2 \cdot \vec{n} + 15\ddot{\vec{S}}_2 \times \vec{v}_2 (\vec{a}_1 \cdot \vec{n} - 2\vec{a}_2 \cdot \vec{n}) \\
& - 96\ddot{\vec{S}}_2 \times \vec{a}_2 (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n})) + (16\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_1 \vec{a}_2 + 2\dot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 (29\vec{a}_1 - 18\vec{a}_2 \\
& + 3\vec{a}_1 \cdot \vec{n} \vec{n} + 18\vec{a}_2 \cdot \vec{n} \vec{n}) - 90\dot{\vec{S}}_2 \times \vec{a}_1 \cdot \vec{a}_2 \vec{n} + 4\dot{\vec{S}}_2 \times \vec{n} (\vec{a}_1 \cdot \vec{a}_2 \\
& - 15a_2^2 - 3\vec{a}_1 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n}) - 12\dot{\vec{S}}_2 \times \vec{a}_1 \vec{a}_2 \cdot \vec{n} + 30\dot{\vec{S}}_2 \times \vec{a}_2 (\vec{a}_1 \cdot \vec{n} + 2\vec{a}_2 \cdot \vec{n})) \\
& + \frac{559}{576} G^2 m_1 r \ddot{\vec{S}}_2 \times \vec{n} + \frac{7}{6} G^2 m_2 r \ddot{\vec{S}}_1 \times \vec{n} + \frac{77}{48} G^2 m_2 r \ddot{\vec{S}}_2 \times \vec{n} \\
& + \frac{4}{3} G^2 m_1 r \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \ddot{\vec{S}}_2 \times \vec{n} + \frac{4}{3} G^2 m_2 r \left(\frac{1}{\epsilon} - 2 \log \frac{r}{R_0} \right) \ddot{\vec{S}}_2 \times \vec{n}, \quad (B.33)
\end{aligned}$$

$$\begin{aligned}
\Delta \overset{(4)}{\vec{x}}_{1(3,1)} &= \frac{G m_2 r^3}{144 m_1} [(6\vec{S}_1 \times \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n} - 22\vec{S}_1 \times \ddot{\vec{a}}_2) + (3\dot{\vec{S}}_1 \times \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} - 11\dot{\vec{S}}_1 \times \ddot{\vec{a}}_2) \\
&\quad - (3\ddot{\vec{S}}_1 \times \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} - 11\ddot{\vec{S}}_1 \times \ddot{\vec{a}}_2)] + \frac{1}{288} Gr^3 [(3\vec{S}_2 \times \vec{n} \cdot \ddot{\vec{a}}_2 \vec{n} + 12\vec{S}_2 \times \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n} \\
&\quad + 27\vec{S}_2 \times \ddot{\vec{a}}_2) + (12\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_1 \vec{n} - 9\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{v}_2 \vec{n} + 12\ddot{\vec{S}}_2 \times \vec{n} (\vec{v}_1 \cdot \vec{n} + 5\vec{v}_2 \cdot \vec{n})]
\end{aligned}$$

$$\begin{aligned}
& - \ddot{\vec{S}}_2 \times \vec{v}_2) - (6\dot{\vec{S}}_2 \times \vec{n} \cdot \ddot{\vec{a}}_2 \vec{n} - 60\dot{\vec{S}}_2 \times \vec{n} \ddot{\vec{a}}_2 \cdot \vec{n} - 46\dot{\vec{S}}_2 \times \ddot{\vec{a}}_2) - (36\ddot{\vec{S}}_2 \times \vec{n} \cdot \dot{\vec{a}}_2 \vec{n} \\
& - 120\ddot{\vec{S}}_2 \times \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} + 20\ddot{\vec{S}}_2 \times \dot{\vec{a}}_2) - (6\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_1 \vec{n} + 36\ddot{\vec{S}}_2 \times \vec{n} \cdot \vec{a}_2 \vec{n} \\
& - 6\ddot{\vec{S}}_2 \times \vec{n} (\vec{a}_1 \cdot \vec{n} + 20\vec{a}_2 \cdot \vec{n}) - 4\ddot{\vec{S}}_2 \times \vec{a}_1 + 44\ddot{\vec{S}}_2 \times \vec{a}_2), \tag{B.34}
\end{aligned}$$

$$\Delta \overset{(5)}{\vec{x}}_{1(3,1)} = -\frac{1}{72} Gr^4 \overset{(5)}{\vec{S}}_2 \times \vec{n}, \tag{B.35}$$

and

$$\begin{aligned}
\overset{(0)}{\omega}_{1(3,1)}^{ij} = & -\frac{Gm_2}{64r} \left[50v_1^2 \vec{v}_1 \cdot \vec{v}_2 v_1^i v_2^j - 75v_1^2 v_2^2 v_1^i v_2^j + 80\vec{v}_1 \cdot \vec{v}_2 v_2^2 v_1^i v_2^j + 16(\vec{v}_1 \cdot \vec{v}_2)^2 v_1^i v_2^j \right. \\
& - 72v_1^4 v_1^i v_2^j - 152v_2^4 v_1^i v_2^j + 38v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i n^j - 9\vec{v}_1 \cdot \vec{n} v_1^2 v_2^2 v_1^i n^j \\
& - 45v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 v_1^i n^j + 8\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 v_1^i n^j + 72\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 v_1^i n^j \\
& - 24\vec{v}_2 \cdot \vec{n} v_1^4 v_1^i n^j - 16\vec{v}_1 \cdot \vec{n} v_2^4 v_1^i n^j - 88\vec{v}_2 \cdot \vec{n} v_2^4 v_1^i n^j + 8v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 v_2^i n^j \\
& - 32\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 v_1^i n^j - 16\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{v}_2)^2 v_2^i n^j + 16\vec{v}_1 \cdot \vec{n} v_2^4 v_2^i n^j \\
& + 88\vec{v}_2 \cdot \vec{n} v_2^4 v_2^i n^j + 6\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} v_1^i v_2^j + 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i v_2^j \\
& - 24\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 v_1^i v_2^j + 27v_1^2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j - 32\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j \\
& + 128v_2^2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j + 24\vec{v}_2 \cdot \vec{n} v_2^2 (\vec{v}_1 \cdot \vec{n})^2 v_1^i n^j + 27\vec{v}_1 \cdot \vec{n} v_1^2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j \\
& + 29v_1^2 (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j - 24\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j - 40\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j \\
& + 72\vec{v}_1 \cdot \vec{n} v_2^2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j + 104v_2^2 (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j - 24\vec{v}_2 \cdot \vec{n} v_2^2 (\vec{v}_1 \cdot \vec{n})^2 v_2^i n^j \\
& - 8v_1^2 (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j + 16\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j - 72\vec{v}_1 \cdot \vec{n} v_2^2 (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j \\
& - 104v_2^2 (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j + 24\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 v_1^i v_2^j \\
& - 48(\vec{v}_2 \cdot \vec{n})^4 v_1^i v_2^j - 40(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j - 40\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^4 v_1^i n^j \\
& - 40(\vec{v}_2 \cdot \vec{n})^5 v_1^i n^j + 40(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j + 40\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^4 v_2^i n^j \\
& \left. + 40(\vec{v}_2 \cdot \vec{n})^5 v_2^i n^j \right] \\
& + \frac{G^2 m_2^2}{96r^2} \left[172v_1^2 v_1^i v_2^j + 988\vec{v}_1 \cdot \vec{v}_2 v_1^i v_2^j - 570v_2^2 v_1^i v_2^j - 42\vec{v}_1 \cdot \vec{n} v_1^2 v_1^i n^j \right. \\
& + 214v_1^2 \vec{v}_2 \cdot \vec{n} v_1^i n^j - 136\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i n^j - 1146\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i n^j + 51\vec{v}_1 \cdot \vec{n} v_2^2 v_1^i n^j \\
& + 3366\vec{v}_2 \cdot \vec{n} v_2^2 v_1^i n^j + 127v_1^2 \vec{v}_2 \cdot \vec{n} v_2^i n^j - 16\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^i n^j \\
& - 2868\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^i n^j - 267\vec{v}_1 \cdot \vec{n} v_2^2 v_2^i n^j - 252\vec{v}_2 \cdot \vec{n} v_2^2 v_2^i n^j - 548\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_1^i v_2^j \\
& - 56(\vec{v}_1 \cdot \vec{n})^2 v_1^i v_2^j + 222(\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j + 128\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 v_1^i n^j \\
& + 129\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j + 394(\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j + 128\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 v_2^i n^j \\
& + 1665\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j - 1408(\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j \left. \right] + \frac{G^2 m_1 m_2}{96r^2} \left[2043v_1^2 v_1^i v_2^j \right. \\
& - 2364\vec{v}_1 \cdot \vec{v}_2 v_1^i v_2^j + 1620v_2^2 v_1^i v_2^j - 2295\vec{v}_1 \cdot \vec{n} v_1^2 v_1^i n^j - 318v_1^2 \vec{v}_2 \cdot \vec{n} v_1^i n^j \\
& + 3060\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i n^j - 1300\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i n^j - 2780\vec{v}_1 \cdot \vec{n} v_2^2 v_1^i n^j \\
& + 1536\vec{v}_2 \cdot \vec{n} v_2^2 v_1^i n^j + 192\vec{v}_1 \cdot \vec{n} v_1^2 v_2^i n^j + 64v_1^2 \vec{v}_2 \cdot \vec{n} v_2^i n^j + 488\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^i n^j \\
& + 936\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^i n^j + 672\vec{v}_1 \cdot \vec{n} v_2^2 v_2^i n^j - 1416\vec{v}_2 \cdot \vec{n} v_2^2 v_2^i n^j \\
& \left. + 2772\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_1^i v_2^j - 2508(\vec{v}_1 \cdot \vec{n})^2 v_1^i v_2^j - 1668(\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j - 729(\vec{v}_1 \cdot \vec{n})^3 v_1^i n^j \right]
\end{aligned}$$

$$\begin{aligned}
& + 2364 \vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 v_1^i n^j + 1808 \vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j - 1296 (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j \\
& + 328 (\vec{v}_1 \cdot \vec{n})^3 v_2^i n^j - 1552 \vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 v_2^i n^j - 384 \vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j \\
& + 1056 (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j \Big] \\
& - \frac{19 G^3 m_1^2 m_2}{30 r^3} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[v_1^i v_2^j - 9 \vec{v}_1 \cdot \vec{n} v_1^i n^j + 9 \vec{v}_2 \cdot \vec{n} v_1^i n^j + 12 \vec{v}_1 \cdot \vec{n} v_2^i n^j \right. \\
& \left. - 12 \vec{v}_2 \cdot \vec{n} v_2^i n^j \right] + \frac{G^3 m_1^2 m_2}{1800 r^3} \left[20299 v_1^i v_2^j + 107253 \vec{v}_1 \cdot \vec{n} v_1^i n^j - 55251 \vec{v}_2 \cdot \vec{n} v_1^i n^j \right. \\
& \left. - 42594 \vec{v}_1 \cdot \vec{n} v_2^i n^j + 34434 \vec{v}_2 \cdot \vec{n} v_2^i n^j \right] + \frac{G^3 m_2^3}{1800 r^3} \left[20525 v_1^i v_2^j + 12600 \vec{v}_1 \cdot \vec{n} v_1^i n^j \right. \\
& \left. - 11001 \vec{v}_2 \cdot \vec{n} v_1^i n^j + 14799 \vec{v}_1 \cdot \vec{n} v_2^i n^j + 85548 \vec{v}_2 \cdot \vec{n} v_2^i n^j \right] \\
& + \frac{G^3 m_1 m_2^2}{288 r^3} \left[(11024 + 468\pi^2) v_1^i v_2^j + (12063 - 1512\pi^2) \vec{v}_1 \cdot \vec{n} v_1^i n^j \right. \\
& \left. - (11112 + 459\pi^2) \vec{v}_2 \cdot \vec{n} v_1^i n^j - (13689 - 945\pi^2) \vec{v}_1 \cdot \vec{n} v_2^i n^j \right. \\
& \left. + (16194 - 2862\pi^2) \vec{v}_2 \cdot \vec{n} v_2^i n^j \right], \tag{B.36}
\end{aligned}$$

$$\begin{aligned}
\omega_{1(3,1)}^{ij} = & \frac{1}{192} G m_2 \left[42 \vec{v}_1 \cdot \vec{a}_1 v_2^2 v_1^i n^j - 24 \vec{a}_1 \cdot \vec{v}_2 v_2^2 v_1^i n^j - 48 v_2^2 \vec{v}_1 \cdot \vec{a}_2 v_1^i n^j + 42 v_1^2 \vec{v}_2 \cdot \vec{a}_2 v_1^i n^j \right. \\
& - 96 \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 v_1^i n^j + 264 v_2^2 \vec{v}_2 \cdot \vec{a}_2 v_1^i n^j + 21 v_1^2 v_2^2 a_1^i n^j - 48 \vec{v}_1 \cdot \vec{v}_2 v_2^2 a_1^i n^j \\
& + 72 v_2^4 a_1^i n^j + 24 v_2^2 \vec{v}_1 \cdot \vec{a}_2 v_2^i n^j + 48 \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 v_2^i n^j - 264 v_2^2 \vec{v}_2 \cdot \vec{a}_2 v_2^i n^j \\
& + 24 \vec{v}_1 \cdot \vec{v}_2 v_2^2 a_2^i n^j - 72 v_2^4 a_2^i n^j + 24 \vec{v}_2 \cdot \vec{n} v_2^2 v_1^i a_1^j - 204 \vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} v_1^i v_2^j \\
& - 72 \vec{a}_1 \cdot \vec{n} v_2^2 v_1^i v_2^j - 102 v_1^2 \vec{a}_2 \cdot \vec{n} v_1^i v_2^j + 144 \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} v_1^i v_2^j - 432 v_2^2 \vec{a}_2 \cdot \vec{n} v_1^i v_2^j \\
& + 144 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 v_1^i v_2^j - 864 \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 v_1^i v_2^j - 102 v_1^2 \vec{v}_2 \cdot \vec{n} a_1^i v_2^j \\
& + 144 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 a_1^i v_2^j - 480 \vec{v}_2 \cdot \vec{n} v_2^2 a_1^i v_2^j - 102 v_1^2 \vec{v}_2 \cdot \vec{n} v_1^i a_2^j + 144 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_1^i a_2^j \\
& - 528 \vec{v}_2 \cdot \vec{n} v_2^2 v_1^i a_2^j + 24 \vec{v}_2 \cdot \vec{n} v_2^2 v_2^i a_2^j - 48 \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 v_1^i n^j - 42 v_1^2 \vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_1^i n^j \\
& + 96 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} v_1^i n^j - 24 \vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} v_1^i n^j - 216 \vec{v}_2 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} v_1^i n^j \\
& - 48 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 v_1^i n^j - 42 \vec{v}_1 \cdot \vec{a}_1 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j + 24 \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j \\
& + 48 \vec{v}_1 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j - 216 \vec{v}_2 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j - 24 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 a_1^i n^j \\
& - 21 v_1^2 (\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j + 48 \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j - 120 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j \\
& + 48 \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 v_2^i n^j - 48 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \vec{a}_2 \cdot \vec{n} v_2^i n^j + 24 \vec{v}_1 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} v_2^i n^j \\
& + 216 \vec{v}_2 \cdot \vec{n} v_2^2 \vec{a}_2 \cdot \vec{n} v_2^i n^j + 48 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 v_2^i n^j - 24 \vec{v}_1 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j \\
& + 216 \vec{v}_2 \cdot \vec{a}_2 (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j + 24 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 a_2^i n^j - 24 \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 a_2^i n^j \\
& + 120 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 a_2^i n^j - 8 (\vec{v}_2 \cdot \vec{n})^3 v_1^i a_1^j + 72 \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j \\
& + 288 \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_1^i v_2^j + 112 (\vec{v}_2 \cdot \vec{n})^3 a_1^i v_2^j + 128 (\vec{v}_2 \cdot \vec{n})^3 v_1^i a_2^j - 8 (\vec{v}_2 \cdot \vec{n})^3 v_2^i a_2^j \\
& + 48 \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j + 72 \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j + 168 \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 v_1^i n^j \\
& + 24 \vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 a_1^i n^j + 48 (\vec{v}_2 \cdot \vec{n})^4 a_1^i n^j - 48 \vec{a}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j \\
& - 72 \vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j - 168 \vec{a}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 v_2^i n^j - 24 \vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 a_2^i n^j \\
& \left. - 48 (\vec{v}_2 \cdot \vec{n})^4 a_2^i n^j \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{G^2 m_1 m_2}{96r} \left[174\vec{v}_1 \cdot \vec{a}_1 v_1^i n^j + 160\vec{a}_1 \cdot \vec{v}_2 v_1^i n^j + 376\vec{v}_1 \cdot \vec{a}_2 v_1^i n^j - 836\vec{v}_2 \cdot \vec{a}_2 v_1^i n^j \right. \\
& + 185v_1^2 a_1^i n^j + 180\vec{v}_1 \cdot \vec{v}_2 a_1^i n^j - 428v_2^2 a_1^i n^j + 168\vec{v}_1 \cdot \vec{a}_1 v_2^i n^j + 112\vec{a}_1 \cdot \vec{v}_2 v_2^i n^j \\
& - 8\vec{v}_1 \cdot \vec{a}_2 v_2^i n^j + 344\vec{v}_2 \cdot \vec{a}_2 v_2^i n^j + 16v_1^2 a_2^i n^j + 128\vec{v}_1 \cdot \vec{v}_2 a_2^i n^j + 164v_2^2 a_2^i n^j \\
& - 196\vec{v}_1 \cdot \vec{n} v_1^i a_1^j - 20\vec{v}_2 \cdot \vec{n} v_1^i a_1^j - 1364\vec{a}_1 \cdot \vec{n} v_1^i v_2^j + 996\vec{a}_2 \cdot \vec{n} v_1^i v_2^j - 1500\vec{v}_1 \cdot \vec{n} a_1^i v_2^j \\
& + 1012\vec{v}_2 \cdot \vec{n} a_1^i v_2^j - 1168\vec{v}_1 \cdot \vec{n} v_1^i a_2^j + 1016\vec{v}_2 \cdot \vec{n} v_1^i a_2^j - 136\vec{v}_1 \cdot \vec{n} v_2^i a_2^j \\
& + 16\vec{v}_2 \cdot \vec{n} v_2^i a_2^j + 444\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} v_1^i n^j - 452\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_1^i n^j - 884\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_1^i n^j \\
& + 688\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_1^i n^j - 492\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} a_1^i n^j + 26(\vec{v}_1 \cdot \vec{n})^2 a_1^i n^j + 364(\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j \\
& - 168\vec{v}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{n} v_2^i n^j + 184\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^i n^j + 424\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_2^i n^j \\
& - 496\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_2^i n^j + 152\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} a_2^i n^j + 52(\vec{v}_1 \cdot \vec{n})^2 a_2^i n^j - 232(\vec{v}_2 \cdot \vec{n})^2 a_2^i n^j \Big] \\
& - \frac{G^2 m_2^2}{96r} \left[60\vec{a}_1 \cdot \vec{v}_2 v_1^i n^j + 60\vec{v}_1 \cdot \vec{a}_2 v_1^i n^j - 336\vec{v}_2 \cdot \vec{a}_2 v_1^i n^j + 60\vec{v}_1 \cdot \vec{v}_2 a_1^i n^j \right. \\
& - 168v_2^2 a_1^i n^j - 8\vec{a}_1 \cdot \vec{v}_2 v_2^i n^j - 8\vec{v}_1 \cdot \vec{a}_2 v_2^i n^j + 902\vec{v}_2 \cdot \vec{a}_2 v_2^i n^j - 8\vec{v}_1 \cdot \vec{v}_2 a_2^i n^j \\
& + 451v_2^2 a_2^i n^j + 60\vec{a}_1 \cdot \vec{n} v_1^i v_2^j + 1316\vec{a}_2 \cdot \vec{n} v_1^i v_2^j + 60\vec{v}_1 \cdot \vec{n} a_1^i v_2^j + 1316\vec{v}_2 \cdot \vec{n} a_1^i v_2^j \\
& + 60\vec{v}_1 \cdot \vec{n} v_1^i a_2^j + 1316\vec{v}_2 \cdot \vec{n} v_1^i a_2^j - 48\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_1^i n^j - 48\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_1^i n^j \\
& - 24\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_1^i n^j - 48\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} a_1^i n^j - 12(\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j - 32\vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^i n^j \\
& - 32\vec{v}_1 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_2^i n^j - 634\vec{v}_2 \cdot \vec{n} \vec{a}_2 \cdot \vec{n} v_2^i n^j - 32\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} a_2^i n^j \\
& \left. - 317(\vec{v}_2 \cdot \vec{n})^2 a_2^i n^j \right] \\
& - \frac{G^3 m_1^2 m_2}{72r^2} \left[829a_1^i n^j - 102a_2^i n^j \right] + \frac{2G^3 m_2^3}{3r^2} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) \left[a_1^i n^j + a_2^i n^j \right] \\
& + \frac{G^3 m_2^3}{9r^2} \left[15a_1^i n^j + 11a_2^i n^j \right] + \frac{G^3 m_1 m_2^2}{288r^2} \left[(5813 - 450\pi^2)a_1^i n^j \right. \\
& \left. + (358 + 63\pi^2)a_2^i n^j \right] - \frac{5G^3 m_1^2 m_2}{2r^2} \left(\frac{1}{\epsilon} - 3 \log \frac{r}{R_0} \right) a_1^i n^j, \tag{B.37}
\end{aligned}$$

$$\begin{aligned}
\overset{(2)}{\omega_1^{ij}}_{(3,1)} &= \frac{1}{24} G m_2 r \left[(18\vec{v}_2 \cdot \dot{\vec{a}}_2 v_1^i v_2^j + 9v_2^2 \dot{a}_1^i v_2^j + 9v_2^2 v_1^i \dot{a}_2^j - 3v_2^2 \vec{a}_2 \cdot \vec{n} v_1^i n^j - 6\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 v_1^i n^j \right. \\
& - 3\vec{v}_2 \cdot \vec{n} v_2^2 \dot{a}_1^i n^j + 3v_2^2 \dot{\vec{a}}_2 \cdot \vec{n} v_2^i n^j + 6\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \dot{\vec{a}}_2 v_2^i n^j + 3\vec{v}_2 \cdot \vec{n} v_2^2 \dot{a}_2^i n^j \\
& + 18\vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} v_2^i v_2^j + 9(\vec{v}_2 \cdot \vec{n})^2 \dot{a}_1^i v_2^j + 9(\vec{v}_2 \cdot \vec{n})^2 v_1^i \dot{a}_2^j + 3\dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_1^i n^j \\
& + (\vec{v}_2 \cdot \vec{n})^3 \dot{a}_1^i n^j - 3\dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 v_2^i n^j - (\vec{v}_2 \cdot \vec{n})^3 \dot{a}_2^i n^j) + 6(3a_2^2 v_1^i v_2^j \right. \\
& + 6\vec{v}_2 \cdot \vec{a}_2 a_1^i v_2^j + 6\vec{v}_2 \cdot \vec{a}_2 v_1^i a_2^j + 3v_2^2 a_1^i a_2^j - 2\vec{a}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 v_1^i n^j - \vec{v}_2 \cdot \vec{n} a_2^2 v_1^i n^j \\
& - v_2^2 \vec{a}_2 \cdot \vec{n} a_1^i n^j - 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 a_1^i n^j + 2\vec{a}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 v_2^i n^j + \vec{v}_2 \cdot \vec{n} a_2^2 v_2^i n^j \\
& + v_2^2 \vec{a}_2 \cdot \vec{n} a_2^i n^j + 2\vec{v}_2 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 a_2^i n^j + 6\vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} a_1^i v_2^j + 6\vec{v}_2 \cdot \vec{n} \dot{\vec{a}}_2 \cdot \vec{n} v_1^i a_2^j \\
& \left. + 3(\vec{v}_2 \cdot \vec{n})^2 a_1^i a_2^j + \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 a_1^i n^j - \dot{\vec{a}}_2 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^2 a_2^i n^j \right] \\
& + \frac{1}{4} G m_2 r \left[3(\vec{a}_2 \cdot \vec{n})^2 v_1^i v_2^j + \vec{v}_2 \cdot \vec{n} (\vec{a}_2 \cdot \vec{n})^2 v_1^i n^j - \vec{v}_2 \cdot \vec{n} (\vec{a}_2 \cdot \vec{n})^2 v_2^i n^j \right] \\
& - \frac{1}{24} G^2 m_1 m_2 \left[(12\dot{a}_1^i v_2^j + 12v_1^i \dot{a}_2^j + 49\dot{\vec{a}}_1 \cdot \vec{n} v_1^i n^j + 5\dot{\vec{a}}_2 \cdot \vec{n} v_1^i n^j + 49\vec{v}_1 \cdot \vec{n} \dot{a}_1^i n^j \right.
\end{aligned}$$

$$\begin{aligned}
& + 5\vec{v}_2 \cdot \vec{n} \dot{a}_1^i n^j - 34\dot{a}_1 \cdot \vec{n} v_2^i n^j + 4\dot{a}_2 \cdot \vec{n} v_2^i n^j - 34\vec{v}_1 \cdot \vec{n} \dot{a}_2^i n^j + 4\vec{v}_2 \cdot \vec{n} \dot{a}_2^i n^j \\
& + 2(12a_1^i a_2^j + 49\vec{a}_1 \cdot \vec{n} a_1^i n^j + 5\vec{a}_2 \cdot \vec{n} a_1^i n^j - 34\vec{a}_1 \cdot \vec{n} a_2^i n^j + 4\vec{a}_2 \cdot \vec{n} a_2^i n^j) \Big].
\end{aligned} \quad (\text{B.38})$$

C Generic action

As noted in section 4.3, after our reduction process, we get the following generic action:

$$\begin{aligned}
V_{\text{N}^3\text{LO}}^{\text{SO}} = & \frac{Gm_2}{128r^2} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (100v_1^2 \vec{v}_1 \cdot \vec{v}_2 v_2^2 - 76v_1^2 (\vec{v}_1 \cdot \vec{v}_2)^2 - 128v_2^2 (\vec{v}_1 \cdot \vec{v}_2)^2 + 5v_1^6 \right. \\
& + 32\vec{v}_1 \cdot \vec{v}_2 v_1^4 - 14v_2^2 v_1^4 - 26v_1^2 v_2^4 + 160\vec{v}_1 \cdot \vec{v}_2 v_2^4 - 48v_2^6 + 120\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
& - 204\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 + 96\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 - 54v_1^2 v_2^2 (\vec{v}_1 \cdot \vec{n})^2 \\
& - 252v_1^2 \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 96\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{v}_2)^2 + 192(\vec{v}_2 \cdot \vec{n})^2 (\vec{v}_1 \cdot \vec{v}_2)^2 \\
& + 228v_1^2 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 768\vec{v}_1 \cdot \vec{v}_2 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 144\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_1^4 + 78(\vec{v}_2 \cdot \vec{n})^2 v_1^4 \\
& - 48(\vec{v}_1 \cdot \vec{n})^2 v_2^4 - 144\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^4 + 480(\vec{v}_2 \cdot \vec{n})^2 v_2^4 + 240\vec{v}_2 \cdot \vec{n} v_2^2 (\vec{v}_1 \cdot \vec{n})^3 \\
& + 270v_1^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 180\vec{v}_1 \cdot \vec{n} v_1^2 (\vec{v}_2 \cdot \vec{n})^3 - 180v_1^2 (\vec{v}_2 \cdot \vec{n})^4 \\
& + 480\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^4 + 240v_2^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 480v_2^2 (\vec{v}_2 \cdot \vec{n})^4 - 560(\vec{v}_1 \cdot \vec{n})^3 (\vec{v}_2 \cdot \vec{n})^3 + 140(\vec{v}_2 \cdot \vec{n})^6) \\
& - 16\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 \vec{v}_1 \cdot \vec{v}_2 v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^3 - 2v_2^2 (\vec{v}_1 \cdot \vec{v}_2)^2 + v_1^2 v_2^4 - 3\vec{v}_1 \cdot \vec{v}_2 v_2^4 \\
& + 5v_2^6 - 9\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{v}_2 v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2 \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 6\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{v}_2)^2 - 3v_1^2 v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 3(\vec{v}_1 \cdot \vec{n})^2 v_2^4 - 9\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^4 \\
& + 15\vec{v}_2 \cdot \vec{n} v_2^2 (\vec{v}_1 \cdot \vec{n})^3 + 15\vec{v}_1 \cdot \vec{n} v_1^2 (\vec{v}_2 \cdot \vec{n})^3 + 15\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& + 15v_2^2 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 - 35(\vec{v}_1 \cdot \vec{n})^3 (\vec{v}_2 \cdot \vec{n})^3) - 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (22\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_1 \cdot \vec{v}_2 \\
& - 30v_1^2 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 31\vec{v}_1 \cdot \vec{n} v_1^2 v_2^2 + 38v_1^2 \vec{v}_2 \cdot \vec{n} v_2^2 + 64\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 \\
& - 128\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 v_2^2 + 24\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{v}_2)^2 - 32\vec{v}_1 \cdot \vec{n} v_1^4 + 15\vec{v}_2 \cdot \vec{n} v_1^4 - 56\vec{v}_1 \cdot \vec{n} v_2^4 \\
& + 124\vec{v}_2 \cdot \vec{n} v_2^4 + 9v_1^2 \vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^2 + 24\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 - 36\vec{v}_2 \cdot \vec{n} v_2^2 (\vec{v}_1 \cdot \vec{n})^2 \\
& + 33\vec{v}_1 \cdot \vec{n} v_1^2 (\vec{v}_2 \cdot \vec{n})^2 - 24v_1^2 (\vec{v}_2 \cdot \vec{n})^3 - 120\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 96\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^3 \\
& + 168\vec{v}_1 \cdot \vec{n} v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 192v_2^2 (\vec{v}_2 \cdot \vec{n})^3 + 60(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^3 - 120\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^4 \\
& \left. + 90(\vec{v}_2 \cdot \vec{n})^5 \right] \\
& + \frac{G^2 m_2^2}{96r^3} \left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (66v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 136v_1^2 v_2^2 + 5331\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 1196(\vec{v}_1 \cdot \vec{v}_2)^2 + 6v_1^4 \right. \\
& - 4110v_2^4 - 228\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} + 3154\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 13254\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 \\
& + 168v_1^2 (\vec{v}_1 \cdot \vec{n})^2 + 672\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 - 614v_2^2 (\vec{v}_1 \cdot \vec{n})^2 + 697v_1^2 (\vec{v}_2 \cdot \vec{n})^2 \\
& - 4725\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 + 14760v_2^2 (\vec{v}_2 \cdot \vec{n})^2 - 768\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 \\
& - 258(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 + 186\vec{v}_1 \cdot \vec{n} (\vec{v}_2 \cdot \vec{n})^3 - 42(\vec{v}_2 \cdot \vec{n})^4) \\
& - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (192v_1^2 \vec{v}_1 \cdot \vec{v}_2 + 319v_1^2 v_2^2 - 69\vec{v}_1 \cdot \vec{v}_2 v_2^2 + 1568(\vec{v}_1 \cdot \vec{v}_2)^2 \\
& - 2010v_2^4 - 492\vec{v}_1 \cdot \vec{n} v_1^2 \vec{v}_2 \cdot \vec{n} - 11362\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + 1626\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2 \\
& - 192\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n})^2 - 1894v_2^2 (\vec{v}_1 \cdot \vec{n})^2 - 1105v_1^2 (\vec{v}_2 \cdot \vec{n})^2 + 9927\vec{v}_1 \cdot \vec{v}_2 (\vec{v}_2 \cdot \vec{n})^2 \\
& \left. + 4566v_2^2 (\vec{v}_2 \cdot \vec{n})^2 + 768\vec{v}_2 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n})^3 + 8988(\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n})^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -14166\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 + 2520(\vec{v}_2 \cdot \vec{n})^4) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(522\vec{v}_1 \cdot \vec{n}v_1^2 - 1025v_1^2\vec{v}_2 \cdot \vec{n} \\
& - 3760\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 7756\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 3716\vec{v}_1 \cdot \vec{n}v_2^2 - 5970\vec{v}_2 \cdot \vec{n}v_2^2 \\
& + 224(\vec{v}_1 \cdot \vec{n})^3 + 1994\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 6572\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 + 2590(\vec{v}_2 \cdot \vec{n})^3) \Big] \\
& + \frac{G^2 m_1 m_2}{96 r^3} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3180v_1^2\vec{v}_1 \cdot \vec{v}_2 - 1142v_1^2v_2^2 + 1776\vec{v}_1 \cdot \vec{n}v_2^2 \\
& - 2668(\vec{v}_1 \cdot \vec{v}_2)^2 - 1461v_1^4 + 270v_2^4 - 6510\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} + 17264\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& - 14232\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 8025v_1^2(\vec{v}_1 \cdot \vec{n})^2 - 9633\vec{v}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 + 7256v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 410v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 5472\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 + 2724v_2^2(\vec{v}_2 \cdot \vec{n})^2 + 1110(\vec{v}_1 \cdot \vec{n})^4 \\
& - 7518\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 2802(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2 + 12372\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 \\
& - 2730(\vec{v}_2 \cdot \vec{n})^4) + 8\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (126v_1^2\vec{v}_1 \cdot \vec{v}_2 - 62v_1^2v_2^2 + 120\vec{v}_1 \cdot \vec{n}v_2^2 \\
& - 112(\vec{v}_1 \cdot \vec{v}_2)^2 - 72v_2^4 - 336\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} + 398\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& + 138\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 15v_1^2(\vec{v}_1 \cdot \vec{n})^2 - 426\vec{v}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 + 5v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
& + 158v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 102(\vec{v}_1 \cdot \vec{n})^4 + 798\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 - 588(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (1386\vec{v}_1 \cdot \vec{n}v_1^2 - 1550v_1^2\vec{v}_2 \cdot \vec{n} + 620\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 292\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
& - 890\vec{v}_1 \cdot \vec{n}v_2^2 - 30\vec{v}_2 \cdot \vec{n}v_2^2 - 4489(\vec{v}_1 \cdot \vec{n})^3 + 6686\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
& - 976\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - 380(\vec{v}_2 \cdot \vec{n})^3) \Big] \\
& + \frac{G^3 m_1^2 m_2}{1200 r^4} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (68911v_1^2 - 97654\vec{v}_1 \cdot \vec{v}_2 + 36343v_2^2 + 445520\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 248930(\vec{v}_1 \cdot \vec{n})^2 - 113690(\vec{v}_2 \cdot \vec{n})^2) - 8\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (2296v_1^2 - 3587\vec{v}_1 \cdot \vec{v}_2 \\
& + 1116v_2^2 + 9985\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4355(\vec{v}_1 \cdot \vec{n})^2 + 195(\vec{v}_2 \cdot \vec{n})^2) \\
& + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (41559\vec{v}_1 \cdot \vec{n} - 21695\vec{v}_2 \cdot \vec{n}) \Big] + \frac{G^3 m_2^3}{2400 r^4} \Big[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (10950v_1^2 \\
& - 16134\vec{v}_1 \cdot \vec{v}_2 + 9659v_2^2 + 101520\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 48000(\vec{v}_1 \cdot \vec{n})^2 - 1945(\vec{v}_2 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (1832v_1^2 + 92232\vec{v}_1 \cdot \vec{v}_2 - 94289v_2^2 - 361860\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - 105160(\vec{v}_1 \cdot \vec{n})^2 + 401995(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (15733\vec{v}_1 \cdot \vec{n} + 30874\vec{v}_2 \cdot \vec{n}) \Big] \\
& + \frac{G^3 m_1 m_2^2}{384 r^4} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 ((25844 - 2313\pi^2)v_1^2 - (45404 - 1998\pi^2)\vec{v}_1 \cdot \vec{v}_2 \\
& + (35144 + 315\pi^2)v_2^2 + (172012 - 9990\pi^2)\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} \\
& - (69688 - 11565\pi^2)(\vec{v}_1 \cdot \vec{n})^2 - (44056 + 1575\pi^2)(\vec{v}_2 \cdot \vec{n})^2) \\
& - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 ((8386 - 747\pi^2)v_1^2 - (12010 - 2772\pi^2)\vec{v}_1 \cdot \vec{v}_2 + (9392 - 2025\pi^2)v_2^2 \\
& + (61394 - 13860\pi^2)\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - (19286 - 3735\pi^2)(\vec{v}_1 \cdot \vec{n})^2 \\
& - (25072 - 10125\pi^2)(\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 ((16640 + 495\pi^2)\vec{v}_1 \cdot \vec{n} \\
& - (2140 + 3087\pi^2)\vec{v}_2 \cdot \vec{n}) \Big] \\
& - \frac{G^4 m_2^4}{1800 r^5} \Big[23175\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - 14801\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \Big] + \frac{G^4 m_1^3 m_2}{1800 r^5} \Big[6773\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \\
& + 5892\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \Big] - \frac{G^4 m_1 m_2^3}{7200 r^5} \Big[3(222643 - 22425\pi^2)\vec{S}_1 \times \vec{n} \cdot \vec{v}_1
\end{aligned}$$

$$\begin{aligned}
& - (1489367 - 59625\pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \Big] - \frac{G^4 m_1^2 m_2^2}{3600 r^5} \Big[(297683 - 7500\pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \\
& - 3(179518 - 17425\pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \Big]. \tag{C.1}
\end{aligned}$$

D General Hamiltonian

As noted in section 5, we obtain the following general Hamiltonian for the present sector:

$$\begin{aligned}
H_{N^3LO}^{SO} = & \frac{G}{64m_1^5 m_2^2 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (58p_1^2(\vec{p}_1 \cdot \vec{p}_2)^2 - 7p_2^2 p_1^4 - 84\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& + 45p_1^2 p_2^2 (\vec{p}_1 \cdot \vec{n})^2 + 3(\vec{p}_2 \cdot \vec{n})^2 p_1^4 - 225p_1^2 (\vec{p}_1 \cdot \vec{n})^2 (\vec{p}_2 \cdot \vec{n})^2) \\
& + 24\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (p_1 \cdot \vec{p}_2 p_1^4 + 3\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_1^4) + 2\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (50\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_1 \cdot \vec{p}_2 \\
& - 15\vec{p}_2 \cdot \vec{n} p_1^4 - 9p_1^2 \vec{p}_2 \cdot \vec{n} (\vec{p}_1 \cdot \vec{n})^2) \Big] - \frac{G}{32m_1^4 m_2^2 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (11p_1^2 \vec{p}_1 \cdot \vec{p}_2 p_2^2 \\
& + 27\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} p_2^2 - 33p_1^2 \vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^2 \\
& - 24\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} (\vec{p}_1 \cdot \vec{p}_2)^2 - 60\vec{p}_2 \cdot \vec{n} p_2^2 (\vec{p}_1 \cdot \vec{n})^3 - 45\vec{p}_1 \cdot \vec{n} p_1^2 (\vec{p}_2 \cdot \vec{n})^3 \\
& + 140(\vec{p}_1 \cdot \vec{n})^3 (\vec{p}_2 \cdot \vec{n})^3) + 8\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (2p_1^2 (\vec{p}_1 \cdot \vec{p}_2)^2 - p_2^2 p_1^4 + 3p_1^2 p_2^2 (\vec{p}_1 \cdot \vec{n})^2 \\
& + 3(\vec{p}_2 \cdot \vec{n})^2 p_1^4 - 15p_1^2 (\vec{p}_1 \cdot \vec{n})^2 (\vec{p}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (34p_1^2 \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& + 5\vec{p}_1 \cdot \vec{n} p_1^2 p_2^2 + 24\vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{n})^2 - 15\vec{p}_1 \cdot \vec{n} p_1^2 (\vec{p}_2 \cdot \vec{n})^2) \Big] \\
& - \frac{G}{64m_1^3 m_2^3 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (5p_1^2 p_2^4 + 48\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 p_2^2 - 96(\vec{p}_2 \cdot \vec{n})^2 (\vec{p}_1 \cdot \vec{p}_2)^2 \\
& - 30p_1^2 p_2^2 (\vec{p}_2 \cdot \vec{n})^2 - 24(\vec{p}_1 \cdot \vec{n})^2 p_2^4 + 60p_1^2 (\vec{p}_2 \cdot \vec{n})^4 + 120p_2^2 (\vec{p}_1 \cdot \vec{n})^2 (\vec{p}_2 \cdot \vec{n})^2) \\
& - 8\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (p_1^2 \vec{p}_1 \cdot \vec{p}_2 p_2^2 + 2(\vec{p}_1 \cdot \vec{p}_2)^3 + 15\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} p_2^2 + 3\vec{p}_1 \cdot \vec{p}_2 p_2^2 (\vec{p}_1 \cdot \vec{n})^2 \\
& + 3p_1^2 \vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^2 - 6\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} (\vec{p}_1 \cdot \vec{p}_2)^2 - 15\vec{p}_2 \cdot \vec{n} p_2^2 (\vec{p}_1 \cdot \vec{n})^3 \\
& - 15\vec{p}_1 \cdot \vec{n} p_1^2 (\vec{p}_2 \cdot \vec{n})^3 - 15\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{n})^2 (\vec{p}_2 \cdot \vec{n})^2 + 35(\vec{p}_1 \cdot \vec{n})^3 (\vec{p}_2 \cdot \vec{n})^3) \\
& + 4\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (5p_1^2 \vec{p}_2 \cdot \vec{n} p_2^2 + 8\vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 p_2^2 \\
& + 12\vec{p}_2 \cdot \vec{n} (\vec{p}_1 \cdot \vec{p}_2)^2 - 18\vec{p}_2 \cdot \vec{n} p_2^2 (\vec{p}_1 \cdot \vec{n})^2 + 6p_1^2 (\vec{p}_2 \cdot \vec{n})^3 - 60\vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^2 \\
& + 30(\vec{p}_1 \cdot \vec{n})^2 (\vec{p}_2 \cdot \vec{n})^3) \Big] - \frac{G}{4m_1^2 m_2^4 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (\vec{p}_1 \cdot \vec{p}_2 p_2^4 \\
& + 6\vec{p}_1 \cdot \vec{p}_2 p_2^2 (\vec{p}_2 \cdot \vec{n})^2 - 15\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^4) + \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (2p_2^2 (\vec{p}_1 \cdot \vec{p}_2)^2 - p_1^2 p_2^4 \\
& + 3p_1^2 p_2^2 (\vec{p}_2 \cdot \vec{n})^2 + 3(\vec{p}_1 \cdot \vec{n})^2 p_2^4 - 15p_2^2 (\vec{p}_1 \cdot \vec{n})^2 (\vec{p}_2 \cdot \vec{n})^2) \\
& - \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (4\vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 p_2^2 + \vec{p}_1 \cdot \vec{n} p_2^4 - 12\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^3 - 12\vec{p}_1 \cdot \vec{n} p_2^2 (\vec{p}_2 \cdot \vec{n})^2 \\
& + 15\vec{p}_1 \cdot \vec{n} (\vec{p}_2 \cdot \vec{n})^4) \Big] + \frac{G}{32m_1 m_2^5 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (4p_2^6 - 60p_2^2 (\vec{p}_2 \cdot \vec{n})^4 \\
& + 35(\vec{p}_2 \cdot \vec{n})^6) + 12\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{p}_2 p_2^4 + 3\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_2^4) \\
& - 6\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (2\vec{p}_2 \cdot \vec{n} p_2^4 - 8p_2^2 (\vec{p}_2 \cdot \vec{n})^3 + 15(\vec{p}_2 \cdot \vec{n})^5) \Big] \\
& - \frac{G}{8m_1^6 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (2\vec{p}_1 \cdot \vec{p}_2 p_1^4 + 9\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_1^4) - 4\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{p}_1 \cdot \vec{n} p_1^4 \Big] \\
& - \frac{45Gm_2}{128m_1^7 r^2} \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 p_1^6
\end{aligned}$$

$$\begin{aligned}
& -\frac{G^2 m_2^2}{16m_1^5 r^3} \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (41p_1^4 + 44p_1^2(\vec{p}_1 \cdot \vec{n})^2) - \frac{G^2 m_2}{96m_1^4 r^3} [3\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (34p_1^2 \vec{p}_1 \cdot \vec{p}_2 \\
& + 335p_1^4 + 380\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} - 1027p_1^2(\vec{p}_1 \cdot \vec{n})^2 - 224\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{n})^2 - 370(\vec{p}_1 \cdot \vec{n})^4 \\
& + 256\vec{p}_2 \cdot \vec{n}(\vec{p}_1 \cdot \vec{n})^3) - 3\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (31p_1^4 - 64p_1^2(\vec{p}_1 \cdot \vec{n})^2) \\
& + 2\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (417\vec{p}_1 \cdot \vec{n} p_1^2 + 112(\vec{p}_1 \cdot \vec{n})^3)] \\
& + \frac{G^2}{96m_1^3 r^3} [\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (2964p_1^2 \vec{p}_1 \cdot \vec{p}_2 - 34p_1^2 p_2^2 \\
& - 212(\vec{p}_1 \cdot \vec{p}_2)^2 - 3228\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} + 1738\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& - 8721\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{n})^2 + 58p_2^2(\vec{p}_1 \cdot \vec{n})^2 + 469p_1^2(\vec{p}_2 \cdot \vec{n})^2 - 3918\vec{p}_2 \cdot \vec{n}(\vec{p}_1 \cdot \vec{n})^3 \\
& - 4290(\vec{p}_1 \cdot \vec{n})^2(\vec{p}_2 \cdot \vec{n})^2) + 6\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (99p_1^2 \vec{p}_1 \cdot \vec{p}_2 - 36p_1^4 + 114\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} \\
& + 180p_1^2(\vec{p}_1 \cdot \vec{n})^2 + 32\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{n})^2 - 136(\vec{p}_1 \cdot \vec{n})^4 - 128\vec{p}_2 \cdot \vec{n}(\vec{p}_1 \cdot \vec{n})^3) \\
& + \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (1638\vec{p}_1 \cdot \vec{n} p_1^2 + 1733p_1^2 \vec{p}_2 \cdot \vec{n} + 5512\vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 - 4513(\vec{p}_1 \cdot \vec{n})^3 \\
& - 1490\vec{p}_2 \cdot \vec{n}(\vec{p}_1 \cdot \vec{n})^2)] - \frac{G^2}{96m_2^3 r^3} [6\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (31p_2^4 - 178p_2^2(\vec{p}_2 \cdot \vec{n})^2 \\
& + 275(\vec{p}_2 \cdot \vec{n})^4) - 3\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (144\vec{p}_1 \cdot \vec{p}_2 p_2^2 + 81p_2^4 + 128\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_2^2 \\
& - 77p_2^2(\vec{p}_2 \cdot \vec{n})^2 - 909(\vec{p}_2 \cdot \vec{n})^4) - 2\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (759\vec{p}_2 \cdot \vec{n} p_2^2 - 1486(\vec{p}_2 \cdot \vec{n})^3)] \\
& - \frac{G^2}{96m_1^2 m_2 r^3} [\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (1388p_1^2 p_2^2 - 4245\vec{p}_1 \cdot \vec{p}_2 p_2^2 \\
& + 1228(\vec{p}_1 \cdot \vec{p}_2)^2 - 12680\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 + 13092\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_2^2 - 7304p_2^2(\vec{p}_1 \cdot \vec{n})^2 \\
& + 226p_1^2(\vec{p}_2 \cdot \vec{n})^2 + 1725\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^2 + 7806(\vec{p}_1 \cdot \vec{n})^2(\vec{p}_2 \cdot \vec{n})^2 \\
& - 3714\vec{p}_1 \cdot \vec{n}(\vec{p}_2 \cdot \vec{n})^3) - \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (864p_1^2 \vec{p}_1 \cdot \vec{p}_2 - 88p_1^2 p_2^2 \\
& - 2342(\vec{p}_1 \cdot \vec{p}_2)^2 - 2352\vec{p}_1 \cdot \vec{n} p_1^2 \vec{p}_2 \cdot \vec{n} + 11278\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& - 3408\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{n})^2 + 1117p_2^2(\vec{p}_1 \cdot \vec{n})^2 + 679p_1^2(\vec{p}_2 \cdot \vec{n})^2 + 3072\vec{p}_2 \cdot \vec{n}(\vec{p}_1 \cdot \vec{n})^3 \\
& - 4857(\vec{p}_1 \cdot \vec{n})^2(\vec{p}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (466p_1^2 \vec{p}_2 \cdot \vec{n} - 1774\vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& + 5024\vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 + 2476\vec{p}_1 \cdot \vec{n} p_2^2 - 3193\vec{p}_2 \cdot \vec{n}(\vec{p}_1 \cdot \vec{n})^2 - 3364\vec{p}_1 \cdot \vec{n}(\vec{p}_2 \cdot \vec{n})^2)] \\
& + \frac{G^2}{96m_1 m_2^2 r^3} [6\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (162\vec{p}_1 \cdot \vec{p}_2 p_2^2 - 584p_2^4 - 1736\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_2^2 \\
& - 252\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^2 + 2229p_2^2(\vec{p}_2 \cdot \vec{n})^2 + 1864\vec{p}_1 \cdot \vec{n}(\vec{p}_2 \cdot \vec{n})^3 + 17(\vec{p}_2 \cdot \vec{n})^4) \\
& - \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (40p_1^2 p_2^2 - 1569\vec{p}_1 \cdot \vec{p}_2 p_2^2 + 1040(\vec{p}_1 \cdot \vec{p}_2)^2 - 3952\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& + 2406\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} p_2^2 - 88p_2^2(\vec{p}_1 \cdot \vec{n})^2 - 544p_1^2(\vec{p}_2 \cdot \vec{n})^2 + 10659\vec{p}_1 \cdot \vec{p}_2 (\vec{p}_2 \cdot \vec{n})^2 \\
& + 528(\vec{p}_1 \cdot \vec{n})^2(\vec{p}_2 \cdot \vec{n})^2 - 10710\vec{p}_1 \cdot \vec{n}(\vec{p}_2 \cdot \vec{n})^3) - 2\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (1778\vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \\
& + 1078\vec{p}_1 \cdot \vec{n} p_2^2 - 4251\vec{p}_2 \cdot \vec{n} p_2^2 - 1072\vec{p}_1 \cdot \vec{n}(\vec{p}_2 \cdot \vec{n})^2 + 1889(\vec{p}_2 \cdot \vec{n})^3)] \\
& + \frac{7G^3 m_2^3}{16m_1^3 r^4} \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (9p_1^2 - 152(\vec{p}_1 \cdot \vec{n})^2) + \frac{3G^3 m_1^2}{50m_2^2 r^4} \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (p_2^2 - 5(\vec{p}_2 \cdot \vec{n})^2) \\
& + \frac{G^3 m_2^2}{9600m_1^2 r^4} [\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 ((1292900 - 57825\pi^2)p_1^2 - 165072\vec{p}_1 \cdot \vec{p}_2 \\
& + 279360\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} - (4348600 - 289125\pi^2)(\vec{p}_1 \cdot \vec{n})^2)
\end{aligned}$$

$$\begin{aligned}
& + 16 \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (3383 p_1^2 - 30190 (\vec{p}_1 \cdot \vec{n})^2) + 371728 \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{p}_1 \cdot \vec{n} \Big] \\
& - \frac{G^3 m_1}{4800 m_2 r^4} \Big[4 \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (14957 p_2^2 + 68690 (\vec{p}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (153184 \vec{p}_1 \cdot \vec{p}_2 \\
& - (940700 - 50625 \pi^2) p_2^2 - 319520 \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} + (1920400 - 253125 \pi^2) (\vec{p}_2 \cdot \vec{n})^2) \\
& + 397960 \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{p}_2 \cdot \vec{n} \Big] + \frac{G^3 m_2}{4800 m_1 r^4} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (382144 p_1^2 \\
& - (1143250 - 24975 \pi^2) \vec{p}_1 \cdot \vec{p}_2 + 114686 p_2^2 + (3907550 - 124875 \pi^2) \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \\
& - 1735220 (\vec{p}_1 \cdot \vec{n})^2 + 339170 (\vec{p}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 ((662950 - 18675 \pi^2) p_1^2 \\
& - 455964 \vec{p}_1 \cdot \vec{p}_2 + 595020 \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} - (1479350 - 93375 \pi^2) (\vec{p}_1 \cdot \vec{n})^2) \\
& + \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 ((747200 + 12375 \pi^2) \vec{p}_1 \cdot \vec{n} + 281792 \vec{p}_2 \cdot \vec{n}) \Big] \\
& - \frac{G^3}{9600 r^4} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 (492032 \vec{p}_1 \cdot \vec{p}_2 \\
& - (1377500 + 7875 \pi^2) p_2^2 - 3978760 \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} + (2523700 + 39375 \pi^2) (\vec{p}_2 \cdot \vec{n})^2) \\
& + 4 \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 (85336 p_1^2 - (747575 - 34650 \pi^2) \vec{p}_1 \cdot \vec{p}_2 + 265364 p_2^2 \\
& + (1851325 - 173250 \pi^2) \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} - 149480 (\vec{p}_1 \cdot \vec{n})^2 - 441820 (\vec{p}_2 \cdot \vec{n})^2) \\
& - 2 \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 (677472 \vec{p}_1 \cdot \vec{n} - (294700 + 77175 \pi^2) \vec{p}_2 \cdot \vec{n}) \Big] \\
& - \frac{303 G^4 m_2^4}{8 m_1 r^5} \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 - \frac{G^4 m_2^3}{14400 r^5} \Big[(3373058 - 193275 \pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \\
& - 560008 \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \Big] + \frac{1497 G^4 m_1^3}{50 r^5} \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \\
& - \frac{G^4 m_1 m_2^2}{14400 r^5} \Big[4(927908 - 27075 \pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \\
& - (5053634 - 177975 \pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \Big] - \frac{G^4 m_1^2 m_2}{3600 r^5} \Big[104254 \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \\
& - 3(369443 - 23950 \pi^2) \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \Big]. \tag{D.1}
\end{aligned}$$

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