

# Geometry, conformal Killing-Yano tensors and conserved “currents”

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**ABSTRACT:** In this paper we discuss the construction of conserved tensors (currents) involving conformal Killing-Yano tensors (CKYT), generalising the corresponding constructions for Killing-Yano tensors (KYTs). As a useful preparation for this, but also of intrinsic interest, we derive identities relating CKYT and geometric quantities. The behaviour of CKYT under conformal transformations is also given, correcting formulae in the literature. We then use the identities derived to construct covariantly conserved “currents”. We find several new CKYT currents and also include a known one by Penrose which shows that “trivial” currents are also useful. We further find that rank- $n$  currents based on rank- $n$  CKYT  $k$  must have a simple form in terms of  $dk$ . By construction, these currents are covariant under a general conformal rescaling of the metric. How currents lead to conserved charges is then illustrated using the Kerr-Newman and the C-metric in four dimensions. Separately, we study a rank-1 current, construct its charge and discuss its relation to the recently constructed Cotton current for the Kerr-Newman black hole.

**KEYWORDS:** Classical Theories of Gravity, Differential and Algebraic Geometry

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## 1 Introduction

In recent papers [1–3], we have investigated Killing tensors (KTs) and Killing-Yano tensors (KYTs), identified new geometric identities constraining the geometries that carry such tensors and shown the existence of covariantly conserved antisymmetric tensors (that we call “currents”) as well as worked out their related charges and, separately, asymptotic charges. So far, we have only considered conformal Killing-Yano tensors (CKYTs) in connection with the Cotton currents in [1], but a natural question to ask is which of the KYT currents can be generalised to include CKYTs. In the present paper, we first derive useful identities and relations for CKYTs, including the transformation properties of a CKYT under a rescaling of the metric, and then discuss a number of currents based on CKYTs. The existence of such currents and charges can be quite rewarding; e.g. one can then look for applications that lead to finding additional supersymmetries or construct asymptotic charges as already done for KYTs.

For the special case of rank- $n$  currents based on rank- $n$  CKYTs we find the unexpected result that the divergences of all currents can be expressed in a trivial form reminiscent of a topological current. In the language of differential forms, they only involve  $dk$  of the

$n$ -form CKYT  $k$ . However this is by no means obvious. We reach this conclusion from a general ansatz making use of a number of identities derived in the preparatory section.

The outline of the paper is as follows: section 2 contains a derivation of identities of general interest, in particular (2.4), (2.9) and (2.10), some of which are needed for the derivation of the rank- $n$  conserved currents in section 4. In section 3 we present our CKYT currents relegating the discussion of the rank- $n$  problems to section 4 with the simpler  $n = 1$  and  $n = 2$  cases in appendix A. Section 5 consists of two subsections in which we illustrate how to find charges from two of the currents discussed; the nontrivial “Einstein current” utilizing the Kerr-Newman metric and the “trivial current” deploying the C-metric. Our conclusions and a discussion are contained in section 6. In appendix B, we review the Cotton current and prove the important property of covariance under conformal transformations for it as well.

Here we try to focus our attention on the discussion of currents, but a list of publications containing useful general background information about (C)KYTs and their applications can be found in [4–12]. Finally it should be noted that throughout, the geometric setting is always curved (pseudo-)Riemannian geometry in  $D$  dimensions with the Levi-Civita connection.

## 2 Rank- $n$ identities and transformation properties

In this section we define a CKYT and derive identities involving CKYT and geometric tensors. These identities are nontrivial and allow us to identify possible constituent terms of conserved currents and to arrive at a general ansatz for such a current. This ansatz applied to a rank- $n$  current involving a rank- $n$  CKYT is shown in section 4 to reduce to a trivial current, in the sense mentioned in the introduction. We also examine the behaviour of CKYT under conformal transformations and correct the literature on this.

### 2.1 Identities

A CKYT  $k$  generalises a KYT in the way a conformal Killing vector generalises a Killing vector. A rank- $n$  CKYT  $k$  can be defined [13] as an  $n$ -form that satisfies

$$\nabla_{a_1} k_{a_2 \dots a_{n+1}} = \nabla_{[a_1} k_{a_2 \dots a_{n+1}]} + n g_{a_1 [a_2} K_{a_3 \dots a_{n+1}]}, \quad (2.1)$$

$$K_{a_1 \dots a_{n-1}} := \frac{1}{(D - n + 1)} \nabla_c k^c_{a_1 \dots a_{n-1}}. \quad (2.2)$$

It follows that

$$\nabla_{a_1} K^{a_1 \dots a_{n-1}} = 0. \quad (2.3)$$

The definition reduces to that of a rank- $n$  KYT  $f$  by setting  $K^{a_1 \dots a_{n-1}} = 0$ . An important identity for KYTs [14] may be generalized to CKYT as follows<sup>1</sup>

$$\begin{aligned} \nabla_a \nabla_b k_{c_1 \dots c_n} &= (-1)^{n+1} \frac{(n+1)}{2} R^d_{a[b c_1} k_{c_2 \dots c_n] d} - (n+1) g_{a[b} \nabla_{c_1} K_{c_2 \dots c_n]} \\ &\quad + n \nabla_a \left( g_{b[c_1} K_{c_2 \dots c_n]} \right). \end{aligned} \quad (2.4)$$

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<sup>1</sup>It has been pointed out that (2.4) is derivable from (1.4) of [13]. We find (2.4) more accessible.

Using (2.1) and (2.4) in the commutator  $[\nabla_a, \nabla_b] \nabla_c k_{c_1 \dots c_n}$ , and contracting the index pairs  $(a, c_n)$  and  $(b, c)$  on both sides of the resultant expression, we get

$$\begin{aligned} (n-1) \left( \nabla^b R^a_{[c_1]} k_{c_2 \dots c_{n-1}]ab} + \frac{1}{2} (\nabla^a R) k_{ac_1 \dots c_{n-1}} + R K_{c_1 \dots c_{n-1}} + (D-n) \square K_{c_1 \dots c_{n-1}} \right. \\ \left. - \frac{1}{3} (n-1)(n-2)(D-n-1) K^{ab}_{[c_1 \dots c_{n-3}} \left( R_{c_{n-2}|a|c_{n-1}]b} + R_{c_{n-2}c_{n-1}]ab} \right) \right. \\ \left. + (n-1)(D-n-2) K_{[c_1 \dots c_{n-2}}{}^a R_{c_{n-1}]a} - (n-1)(D-n) \nabla_a \nabla_{[c_1} K^a_{c_2 \dots c_{n-1}]} \right) = 0. \end{aligned} \quad (2.5)$$

Since

$$2K^{ab}_{[c_1 \dots c_{n-3}} R_{c_{n-2}|a|c_{n-1}]b} = K^{ab}_{[c_1 \dots c_{n-3}} R_{c_{n-2}c_{n-1}]ab}, \quad (2.6)$$

we have

$$-\frac{1}{3} K^{ab}_{[c_1 \dots c_{n-3}} \left( R_{c_{n-2}|a|c_{n-1}]b} + R_{c_{n-2}c_{n-1}]ab} \right) = -\frac{1}{2} K^{ab}_{[c_1 \dots c_{n-3}} R_{c_{n-2}c_{n-1}]ab}. \quad (2.7)$$

Moreover

$$\begin{aligned} \nabla_a \nabla_{[c_1} K^a_{c_2 \dots c_{n-1}]} &= [\nabla_a, \nabla_{[c_1}] K^a_{c_2 \dots c_{n-1}]} \\ &= R_{a[c_1} K^a_{c_2 \dots c_{n-1}]} - (n-2) R_{b[c_1 c_2|a]} K^{ab}_{c_3 \dots c_{n-1}]} \\ \Rightarrow \nabla_a \nabla_{[c_1} K^a_{c_2 \dots c_{n-1}]} &= K_{[c_1 \dots c_{n-2}}{}^a R_{c_{n-1}]a} - (n-2) R_{b[c_{n-2}c_{n-1}|a]} K^{ab}_{c_1 \dots c_{n-3}]} \end{aligned} \quad (2.8)$$

These can be used for simplifying the second and third lines of (2.5) to arrive at

$$\begin{aligned} (n-1) \left( \nabla^b R^a_{[c_1]} k_{c_2 \dots c_{n-1}]ab} + \frac{1}{2} (\nabla^a R) k_{ac_1 \dots c_{n-1}} + (D-n) \square K_{c_1 \dots c_{n-1}} + R K_{c_1 \dots c_{n-1}} \right. \\ \left. - 2(n-1) K_{[c_1 \dots c_{n-2}}{}^a R_{c_{n-1}]a} + \frac{1}{2} (n-1)(n-2) K^{ab}_{[c_1 \dots c_{n-3}} R_{c_{n-2}c_{n-1}]ab} \right) = 0. \end{aligned} \quad (2.9)$$

On the other hand, by contracting the  $(a, b)$  indices in (2.4), we also have

$$\square k^{c_1 \dots c_n} = \frac{(n-1)}{2} k^{ab[c_1 \dots c_{n-2}} R^{c_{n-1}c_n]_{ab}} - R^{a[c_1} k_a^{c_2 \dots c_n]} + (2n-D) \nabla^{[c_1} K^{c_2 \dots c_n]}. \quad (2.10)$$

These may be combined to define the following current

$$F^{c_1 \dots c_n} = \square k^{c_1 \dots c_n} - n \nabla^{[c_1} K^{c_2 \dots c_n]} = \nabla_a \left( \nabla^{[a} k^{c_1 \dots c_n]} \right). \quad (2.11)$$

It is easy to show that this current is indeed covariantly conserved, i.e.

$$\nabla_{c_1} F^{c_1 \dots c_n} = 0. \quad (2.12)$$

This kind of current will be called “trivial”. It is conserved because it is the covariant divergence of an antisymmetric tensor, not due to any other property of the CKYT. The combination  $\nabla_a T^{ac_2 \dots c_n}$  will be divergence-free for any completely antisymmetric tensor  $T$ .

## 2.2 CKYTs under conformal transformations

In this subsection we give the transformation properties of CKYTs under conformal transformations and show that the covariant-conservation of the trivial current (2.11) is left-invariant under such transformations.

Under conformal transformations  $\tilde{g}_{ab} = \Omega^2 g_{ab}$ , a generic rank- $n$  CKYT  $k$  (2.1) and its rank- $(n-1)$  companion  $K$  (2.2) transform as

$$\tilde{k}^{a_1 \dots a_n} = \Omega^{1-n} k^{a_1 \dots a_n} \quad \text{and} \quad \tilde{K}_{a_1 \dots a_{n-1}} = \Omega^{n-1} K_{a_1 \dots a_{n-1}} + \Omega^{n-2} (\nabla_c \Omega) k^c_{a_1 \dots a_{n-1}}. \quad (2.13)$$

Note that for all  $n \neq 2$ , the second part in the transformations of  $\tilde{K}$  differs significantly from the formulae given in appendix A of [15].

With these, we find

$$\tilde{\nabla}^c [\tilde{k}^{a_1 \dots a_n}] = \Omega^{-(n+1)} \nabla^c [k^{a_1 \dots a_n}] + (n+1) \Omega^{-(n+2)} (\nabla^c \Omega) k^{a_1 \dots a_n} \quad (2.14)$$

leading to the transformation rule

$$\begin{aligned} \tilde{F}^{a_1 \dots a_n} &= \Omega^{-(n+1)} F^{a_1 \dots a_n} + (D-n-1) \Omega^{-(n+2)} (\nabla_c \Omega) \nabla^c k^{a_1 \dots a_n} \\ &\quad + (n+1) \Omega^{-(n+2)} \left( \nabla_c ((\nabla^c \Omega) k^{a_1 \dots a_n}) + (D-n-2) \Omega^{-1} (\nabla_c \Omega) (\nabla^c \Omega) k^{a_1 \dots a_n} \right) \end{aligned} \quad (2.15)$$

for a generic rank- $n$  current  $F$ . Since for any skew-symmetric rank- $(n+1)$  tensor  $\tilde{T}$ , one has

$$[\tilde{\nabla}_{a_1}, \tilde{\nabla}_c] \tilde{T}^{ca_1 \dots a_n} = 2 \tilde{\nabla}_{a_1} \tilde{\nabla}_c \tilde{T}^{ca_1 \dots a_n} = 0, \quad (2.16)$$

which can be shown by the expansion of the commutator, the symmetries of the curvature tensors and the first Bianchi identity, it follows straightforwardly that  $\tilde{\nabla}_{a_1} \tilde{F}^{a_1 \dots a_n} = 0$  by the identifications  $\tilde{T}^{ca_1 \dots a_n} = \tilde{\nabla}^c [\tilde{k}^{a_1 \dots a_n}]$  and  $\tilde{F}^{a_1 \dots a_n} = \tilde{\nabla}_c \tilde{T}^{ca_1 \dots a_n}$ . So the covariant-conservation of the trivial current  $F$  is left invariant under conformal transformations.

## 3 CKYT currents

In this section we present cases when CKYTs give rise to conserved currents. The conserved currents are characterised by their rank, the rank of the CKYT involved and the number of derivatives. One of the currents presented is related to the Kastor-Traschen current (KT-current) [14] to which we return below in the special case of a rank-2 CKYT and rank-2 current.

From the definitions (2.3) we see that a conserved current involving a rank-2 CKYT  $k$  in  $D = 4$  dimensions is [16]

$$j^a := \epsilon^{abcd} \nabla_b k_{cd}. \quad (3.1)$$

Here  $\nabla_a j^a = 0$  follows easily from (2.1). In this case the current is rank-1. In fact, since the Hodge dual  $\star k$  of a CKYT  $k$  is again a CKYT [17]  $\hat{k} := \star k$ , one has

$$j^a = \nabla_b \hat{k}^{ab}, \quad (3.2)$$

so that this is a trivial current as described at the end of subsection 2.1. It is used in [16] as the starting point for a discussion of KT vs KYTs, and quasilocal charges.

In [1], we showed that there are also other rank-1 currents constructed out of the Cotton tensor  $C$  and a rank-2 KYT  $f$  or a rank-2 CKYT  $k$ , the Killing-Yano Cotton currents

$$J^a := C^{abc} f_{bc}, \quad \tilde{J}^a := C^{abc} k_{bc}. \quad (3.3)$$

These are covariantly conserved in arbitrary  $D \geq 3$  dimensions. In showing that the Cotton currents are divergence free, we use the fact that the Cotton tensor is traceless on all index pairs and satisfies

$$C_{abc} = C_{a[bc]}, \quad C_{[abc]} = 0, \quad \nabla^a C_{abc} = 0. \quad (3.4)$$

The conservation of the currents follow from an interplay between the geometry and (C)KYT properties. It is therefore useful to explore such relations in some detail, which we have done in subsection 2.1 and shall elaborate further in section 4.

The Hodge dual of a KYT  $f$  is a closed conformal Killing-Yano tensor<sup>2</sup> (CCKYT)  $h$  and vice versa [17], i.e.

$$f = \star h, \quad dh = 0. \quad (3.5)$$

Specifically when the rank of the CCKYT  $h$  is  $n = D - 1$ , one has

$$\nabla_a h_{b_1 \dots b_{D-1}} = (D-1) g_{a[b_1} H_{b_2 \dots b_{D-1}]}, \quad (3.6)$$

so that e.g. the Hodge dual of the rank-1 current (3.1) in  $D = 4$  naturally gives rise to another conserved current of rank-2, as mentioned in the context of (3.2). Likewise the Hodge dualisation of the rank-1 Cotton currents (3.3) give rise to two more conserved currents of rank-2 and rank- $(D-2)$ , respectively. Explicitly, the rank- $(D-2)$  dual of a rank-2 CKYT  $k$  brings about the current

$$\mathcal{J}^a = C^{abc} \epsilon_{bcd_1 \dots d_{D-2}} k^{d_1 \dots d_{D-2}}. \quad (3.7)$$

In three dimensions this relates the Cotton current  $\tilde{J}^a$  for a rank-2 CKYT to a current for its rank-1 dual defined with the dual of the Cotton tensor, the York tensor.

In a slight detour from the main thrust of this section, we also note that given a CKYT  $k$  (2.1), one may also construct a current from the associated trace  $K$  (2.2). For a rank-2 CKYT  $k$ , the associated rank-1 trace  $K^a$  is trivially conserved  $\nabla_a K^a = 0$  (2.3). Moreover, the “Einstein current”

$$\mathcal{J}^a := G^{ac} K_c, \quad (3.8)$$

is also covariantly conserved (with a manifest potential, cf. (A.8)), thanks to

$$R^{ac} \nabla_a K_c = R^{ac} \nabla_{(a} K_{c)} = 0, \quad (3.9)$$

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<sup>2</sup>So a CCKYT  $h$  satisfies (2.1) with the first term on the right hand side vanishing identically.

derived in [1]. When  $K^a$  is itself a Killing vector,<sup>3</sup> we again find a dual  $(D - 1)$ -form CCKYT and an associated rank-1 current.

The duality relation (3.5) also makes it possible to find a version of the KT-current involving CCKYTs. The KT-current [14] is defined for a rank- $n$  KYT  $f$  as

$$j^{a_1 \dots a_n} = N_n \delta_{b_1 \dots b_n c_1 c_2}^{a_1 \dots a_n d_1 d_2} f^{b_1 \dots b_n} R_{d_1 d_2}{}^{c_1 c_2}, \quad (3.10)$$

where  $\delta_{b_1 \dots b_m}^{a_1 \dots a_m} = \delta_{b_1}^{[a_1} \dots \delta_{b_m}^{a_m]}$  is the generalised Kronecker delta, which is totally antisymmetric in all up and down indices,  $N_n := -(n+1)(n+2)/4n$  and  $R$  is the Riemann curvature tensor. The covariant divergence of the KT-current vanishes thanks to the Bianchi identities for  $R$ . Thus, dualising  $f$  to  $h$  via (3.5) in (3.10), we find a new covariantly conserved current based on  $h$

$$j^{a_1 \dots a_n} \sim \epsilon^{a_1 \dots a_n b_1 b_2} R_{b_1 b_2}{}^{c_1 c_2} h_{c_1 c_2} \sim \star R^{a_1 \dots a_n, c_1 c_2} h_{c_1 c_2}, \quad (3.11)$$

where star denotes the left dual.

The preceding construction involving CCKYTs in the KT-current prompts the question of whether one can replace the KYTs in the KT-current by CKYTs, modulo some modifications. This turns out not to give a conserved current, as we discuss in the next section.

## 4 Rank- $n$ currents from rank- $n$ CKYTs

In this section, we use some of the formulae derived in section 2 to investigate a rank- $n$  current constructed from a rank- $n$  CKYT. More specifically, we look for a covariantly conserved rank- $n$  tensor  $J$  linear in the CKYT and its contraction, formed from geometric tensors and covariant derivatives, such that  $J$  is second-order in covariant derivatives. This specification is in line with previous currents constructed using KYTs [2, 14].

It must be pointed out that there are many options for the terms in such a current and our final result, that a large class of them lead to trivial currents, is by no means obvious. It is only with the help of a number of identities, some of which are derived in section 2 that we are able to show this (somewhat disappointing) result.

We start by listing a number of relations between terms of the right kind:

$$R^{[a_1 | cd | a_2 k^{a_3 \dots a_n}]_{cd}} = -\frac{1}{2} R_{cd}^{[a_1 a_2 k^{a_3 \dots a_n}]^{cd}}, \quad (4.1)$$

$$\nabla_c \nabla^{[a_1 k^{a_2 \dots a_n}]^c} = \nabla_c \nabla^{[a_1 k^{a_2 \dots a_n} c]} + (-1)^{n+1} \nabla^{[a_1 K^{a_2 \dots a_n}]}, \quad (4.2)$$

$$\nabla^{[a_1} \nabla_c k^{a_2 \dots a_n]c} = (-1)^{n+1} (D + 1 - n) \nabla^{[a_1 K^{a_2 \dots a_n]}}, \quad (4.3)$$

$$[\nabla_c, \nabla^{[a_1} k^{a_2 \dots a_n]c}] = (-1)^n \frac{(n-1)}{2} R_{cd}^{[a_1 a_2 k^{a_3 \dots a_n}]^{cd}} + R_c^{[a_1} k^{a_2 \dots a_n]c}, \quad (4.4)$$

$$\square k^{a_1 \dots a_n} = \nabla_c \nabla^{[c} k^{a_1 \dots a_n]} + n \nabla^{[a_1 K^{a_2 \dots a_n]}. \quad (4.5)$$

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<sup>3</sup>As is the case for the trace part of a CKYT in  $D = 4$ .

Notice that the difference of (4.2) and (4.3) gives (4.4), and thus

$$F^{a_1 \dots a_n} + (D - n) \nabla^{[a_1} K^{a_2 \dots a_n]} = \frac{(n-1)}{2} R_{cd}^{[a_1 a_2} k^{a_3 \dots a_n] cd} + (-1)^n R_c^{[a_1} k^{a_2 \dots a_n] c}. \quad (4.6)$$

These relations let us choose the following rank- $n$ , skew-symmetric, independent combinations

$$R_{cd}^{[a_1 a_2} k^{a_3 \dots a_n] cd}, R_c^{[a_1} k^{a_2 \dots a_n] c}, R k^{a_1 \dots a_n}, \nabla^{[a_1} K^{a_2 \dots a_n]}, \quad (4.7)$$

as the building blocks of a possible current. For computational ease, without loss of generality, we replace the first two terms in the list (4.7) by

$$\check{K}^{a_1 \dots a_n} := -\frac{(n-1)}{4} R^{[a_1 a_2}{}_{bc} k^{a_3 \dots a_n] bc} + \frac{1}{2n} R k^{a_1 \dots a_n}, \quad (4.8)$$

and

$$\hat{K}^{a_1 \dots a_n} := R_c^{[a_1} k^{a_2 \dots a_n] c} + \frac{(-1)^n}{n} R k^{a_1 \dots a_n}, \quad (4.9)$$

modeled after two conserved currents in [2].

With these preambles, we make an ansatz for the conserved current:

$$J^{a_1 \dots a_n} = 2\check{K}^{a_1 \dots a_n} + \alpha \hat{K}^{a_1 \dots a_n} + \beta R k^{a_1 \dots a_n} + \gamma \nabla^{[a_1} K^{a_2 \dots a_n]}. \quad (4.10)$$

To this, we may add any amount of the conserved current  $F^{a_1 \dots a_n}$  in (2.11), its coefficient will remain arbitrary.

To proceed, we rewrite the identity (2.10) or (4.6) in terms of the tensors (4.8) and (4.9):

$$F^{a_1 \dots a_n} = (-1)^n \hat{K}^{a_1 \dots a_n} - 2\check{K}^{a_1 \dots a_n} + (n - D) \nabla^{[a_1} K^{a_2 \dots a_n]}. \quad (4.11)$$

Using this identity in (4.10) yields for the divergence

$$\begin{aligned} \nabla_{a_1} J^{a_1 \dots a_n} &= (\alpha + (-1)^n) \nabla_{a_1} \hat{K}^{a_1 \dots a_n} + (\gamma + n - D) \nabla_{a_1} \nabla^{[a_1} K^{a_2 \dots a_n]} + \beta \nabla_{a_1} (R k^{a_1 \dots a_n}) \\ &\quad + \nabla_{a_1} F^{a_1 \dots a_n}. \end{aligned} \quad (4.12)$$

The last term is zero, of course, and the coefficients of the others give  $\alpha = (-1)^{n+1}$ ,  $\beta = 0$  and  $\gamma = D - n$ . Plugging these values into (4.10) and using (4.11), the rank- $n$  current is thus shown to be trivial, i.e. proportional to the current  $F^{a_1 \dots a_n}$  in (2.11).

We thus see that we cannot construct a nontrivial rank- $n$  conserved current from the ansatz (4.10). This ansatz is quite general and covers, e.g., possible generalisations of the KT-current, that is local, geometric, linear in  $k$  and second-order in  $\nabla$ s.

As an additional illustration of the problem, we further discuss the cases  $n = 1$  and  $n = 2$  in appendix A.

## 5 Conserved charges

One obvious reason for constructing currents is to use them to find conserved charges. In this section we illustrate the procedure for doing that on two well-known solutions. For the case of the Kerr-Newman metric used in the Einstein current (3.8), we are able to derive a closed-form charge expression which reduces to a complicated integral on which we comment. Unfortunately though, the corresponding expression for the C-metric with the trivial current (2.11) diverges.



## 5.1 The Kerr-Newman metric

The current (3.8) is an example of a non-trivial current and it can be used for defining a charge in the usual way,<sup>4</sup> e.g. as described in section 3 of [1]:

$$\mathcal{Q} := \int_{\Sigma_t} d^3x \sqrt{\gamma} n_a \mathcal{J}^a. \quad (5.1)$$

Let us see how things go on the example of the Kerr-Newman metric

$$ds^2 = -\frac{Q(r)}{\rho^2} \left( dt - a \sin^2 \theta d\varphi \right)^2 + \frac{\rho^2}{Q(r)} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left( a dt - (r^2 + a^2) d\varphi \right)^2, \quad (5.2)$$

where

$$Q(r) = (r - r_+)(r - r_-), \quad \text{with} \quad r_{\pm} := m \pm \sqrt{m^2 - a^2 - e^2 - g^2}, \quad (5.3)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad (5.4)$$

with the rank-2 CKYT [18]

$$k = r dr \wedge \left[ dt - a \sin^2 \theta d\varphi \right] + a \cos \theta \sin \theta d\theta \wedge \left[ a dt - (r^2 + a^2) d\varphi \right], \quad (5.5)$$

that remarkably has  $K^a = -(\partial_t)^a$ , the timelike Killing vector of (5.2) at the same time. These give

$$\mathcal{Q} = 2\pi \left( e^2 + g^2 \right) \int_{r_+}^{\infty} dr \int_0^{\pi} \frac{d\theta \sin \theta \left( a^2 + 2(r - r_-)(r - r_+) - a^2 \cos 2\theta \right)}{(a^2 + 2r^2 + a^2 \cos 2\theta) \left( (r - r_-)(r - r_+) - a^2 \sin^2 \theta \right)}. \quad (5.6)$$

This integral is equivalent to the integral found in section 5.3 of [1], but here in a different context obviously. It is convergent and finite, but as discussed in [1], even though the  $\theta$  integral can be taken exactly, we were unable to evaluate the  $r$ -integral. See section 5.3 of [1] for details.

Finally, going back to the trivial rank-2 current (2.11) for the Kerr-Newman metric, we immediately see that the CKYT (5.5) is a CCKYT, i.e.  $\nabla_{[a} k_{bc]} = 0$  identically. So it is not suitable for defining a conserved charge using the trivial current (2.11) for  $n = 2$ .

## 5.2 The C-metric

The C-metric in “spherical-type coordinates” reads [19]

$$ds^2 = \Omega^2(r, \theta) \left( -Q(r) dt^2 + \frac{dr^2}{Q(r)} + \frac{r^2 d\theta^2}{P(\theta)} + P(\theta) r^2 \sin^2 \theta d\phi^2 \right), \quad (5.7)$$

where the conformal factor and the metric functions are

$$\Omega(r, \theta) = \frac{1}{1 + \alpha r \cos \theta}, \quad Q(r) = \left( 1 - \frac{2m}{r} \right) (1 - \alpha^2 r^2), \quad P(\theta) = 1 + 2\alpha m \cos \theta. \quad (5.8)$$

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<sup>4</sup>Recall what is meant by “conservation” of charge in the classical field theory sense. For example, consider Maxwell theory in flat Minkowski space. One defines charge  $Q := \int d^3x J^0$ , so that at  $t = x^0 = \text{const.}$  surfaces one has “conservation”  $dQ/dt = 0$ . So, it is only natural that the charge-integral hypersurface is chosen as spacelike, with an everywhere timelike normal.

The analytic extension of the C-metric is thought to represent a pair of black holes that accelerate from each other due to the presence of a string that is represented by a conical singularity [20, 21]. The C-metric (5.7) reduces to the Schwarzschild black hole when  $\alpha = 0$  and has an obvious curvature singularity at  $r = 0$ . In fact, the parameter  $m > 0$  has to do with the mass of the source, whereas the parameter  $\alpha$ , where  $0 < 2\alpha m < 1$ , can be interpreted as the acceleration of the black hole with an acceleration horizon at  $r = 1/\alpha > 2m$  [19].

In what follows, we will take the  $t$ -coordinate as temporal and the  $r$ -coordinate as spatial, for which one needs  $r \in (2m, 1/\alpha)$ . To stay away from the coordinate poles, we also take  $\theta \in (0, \pi)$ . Finally, we restrict  $\phi \in (-C\pi, C\pi)$ , where the parameter  $C$  determines the balance between the deficit/excess angles on the two halves of the symmetry axis of the C-metric (with  $t, r$  kept constant) [19]. Note also that the choice

$$C = \frac{1}{1 + 2\alpha m} \quad (5.9)$$

removes the conical singularity at  $\theta = 0$  [19].

The C-metric (5.7) admits two rank-2 CKYT

$$k_1 = \Omega^3(r, \theta) r^3 \sin \theta d\theta \wedge d\phi \quad \text{with} \quad K_1 = \alpha \partial_\phi, \quad (5.10)$$

$$k_2 = \Omega^3(r, \theta) r dr \wedge dt \quad \text{with} \quad K_2 = -\partial_t. \quad (5.11)$$

For the trivial current  $F^{ac}$  (2.11), one may, in analogy to the discussion in subsection 5.1 (see also further details given in section 3 of [1]), define the charge

$$\mathcal{Q}^c := \int_{\Sigma_t} d^3x \sqrt{\gamma} n_a F^{ac}. \quad (5.12)$$

Adapted to the C-metric (5.7), a  $t = \text{const.}$  hypersurface  $\Sigma_t$  has the following unit normal vector  $n^a$  and volume element  $\sqrt{\gamma}$

$$n^a = -\frac{1}{\Omega(r, \theta) \sqrt{Q(r)}} (\partial_t)^a, \quad n_a dx^a = \Omega(r, \theta) \sqrt{Q(r)} dt, \quad \sqrt{\gamma} = \frac{\Omega^3(r, \theta) r^2 \sin \theta}{\sqrt{Q(r)}}. \quad (5.13)$$

We first find that the integrand in (5.12) vanishes identically for the CKYT  $k_1$  (5.10). Using the remaining CKYT  $k_2$  (5.11) and identifying the integrand in (5.12) as  $\mathcal{P}^c := \sqrt{\gamma} n_a F^{ac}$  for convenience, we next find that

$$\begin{aligned} \mathcal{P}^r &= \frac{\alpha r^2 \Omega^3 \sin \theta}{2} \left( \cos \theta (7\alpha^2 m r + 4) + \alpha(2m + 4r + 6m \cos 2\theta + \alpha m r \cos 3\theta) \right), \\ \mathcal{P}^\theta &= -2\alpha r \Omega^3 \sin^2 \theta (1 + 2m\alpha \cos \theta). \end{aligned}$$

It easily follows that the nontrivial  $\theta$  integration of  $\mathcal{P}^r$  over the  $(0, \pi)$  interval vanishes and one is just left with  $\mathcal{P}^\theta$ . We find

$$\int_{-C\pi}^{C\pi} d\phi \int_0^\pi d\theta \mathcal{P}^\theta = -\frac{2\pi^2 C}{\alpha r^2} \left( \frac{m \left( 2\alpha^2 r^2 (2\sqrt{1 - \alpha^2 r^2} - 3) - 4\sqrt{1 - \alpha^2 r^2} + 4 \right) + \alpha^2 r^3}{(1 - \alpha^2 r^2)^{3/2}} \right). \quad (5.14)$$

The final step is the  $r$  integration. which formally can be done, as

$$\begin{aligned} & \int \frac{m \left( 2\alpha^2 r^2 \left( 2\sqrt{1 - \alpha^2 r^2} - 3 \right) - 4\sqrt{1 - \alpha^2 r^2} + 4 \right) + \alpha^2 r^3}{r^2 (1 - \alpha^2 r^2)^{3/2}} \\ &= \frac{2m \left( \alpha^2 r^2 + 2\sqrt{1 - \alpha^2 r^2} - 2 \right) + r}{r\sqrt{1 - \alpha^2 r^2}} \end{aligned} \quad (5.15)$$

Unfortunately though, the right hand side clearly diverges as  $r \rightarrow 1/\alpha$ .

Finally, one may wonder what the Cotton charge of the C-metric (5.7) is. Since the Ricci tensor identically vanishes for the C-metric (5.7), so does the Cotton tensor. So the Killing-Yano Cotton current and, hence, charge is trivially zero.

## 6 Discussion

After introducing a number of useful identities and relations for CKYTs, including their correct transformations under conformal transformations, we have discussed the construction of conserved “currents”, i.e. covariantly divergence-free tensors, constructed out of geometric tensors and CKYTs. We found a number of currents such as the Einstein current (3.8), the Cotton current (3.3) and their related expressions involving duals of the (C)KYTs. Based on the Einstein current for the Kerr-Newman metric and a rank-2 trivial current for the  $C$ -metric, we illustrated the construction of charges for General Relativity solutions.

Contrary to expectations, however, we found that naive generalisations of rank- $n$  KYT currents, such as the KT-current, to rank- $n$  CKYT currents yield trivial currents (2.11). To show this requires a number of the identities and relations which we introduced in section 4.

In hindsight, it is worth mentioning the following regarding the charge  $\mathcal{Q}$  (5.6) of the Kerr-Newman metric that we found in subsection 5.1: as argued in [1], the Cotton current  $\tilde{J}^a$  (3.3) for the Kerr-Newman metric (see section 5.3 of [1] for details) can be written in terms of a 2-potential  $\ell$  as  $\tilde{J}^a = \nabla_a \ell^{ac}$ , which reduces to  $\ell^{ac} = 8\nabla^{[a} K^{c]}$  in terms of the notation used in the present work. On the other hand, since the curvature scalar  $R = 0$  and  $K^a$  is identical to the timelike Killing vector of the Kerr-Newman metric (5.2), the Einstein current  $\mathcal{J}^a$  (3.8) in fact reduces to  $\tilde{\mathcal{J}}^a = R^{ac} K_c$  in this case. Now it is a well-known fact that  $\nabla_c \left( \nabla^{[c} K^{a]} \right) = -R^{ac} K_c$  for a Killing vector, so the Cotton current  $\tilde{J}^a$  (3.3) and the current  $\mathcal{J}^a$  (3.8) or  $\tilde{\mathcal{J}}^a$  are indeed proportional to each other for the Kerr-Newman metric. In that sense, it is interesting that the Einstein current  $\mathcal{J}^a$  (3.8),  $\tilde{\mathcal{J}}^a$  and the Cotton current  $\tilde{J}^a$  (3.3) [1] all define the same charge (up to some proportionality constant)  $\mathcal{Q}$  (5.6) for the Kerr-Newman metric. In fact, the 3-surface integral of any one of these on a  $t = \text{const.}$  hypersurface, say  $\Sigma_t$ , should be proportional, up to some possible constants, to the Komar “mass”,<sup>5</sup> which is an integral over the “boundary” 2-surface,  $\partial\Sigma_t$  of constant  $t$  and  $r$ , spanned by  $\Sigma_t$ .

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<sup>5</sup>See e.g. section 11.2 of the seminal work [22].

In [1] the Cotton current was lifted to supergravity in three dimensions. An interesting open problem is to find supergravity versions also of the trivial currents (2.11).

As a final comment we note that the physical meaning of currents and charges constructed from KTs, KYTs and CKYT is not always obvious, unlike those from Killing vectors which describe isometries. For this reason the charges based on such tensors are often said to generate hidden symmetries and their study is quite rewarding.

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## A Some special cases of trivial currents

Since we are considering rank- $n$  forms, we are restricted by the dimension  $D$  of the underlying manifold. Interesting cases are therefore  $n = 1, \dots, D/2$  in even and  $n = 1, \dots, (D-1)/2$  in odd dimensions (modulo their Hodge duals). Below we comment on the special cases  $n = 1$  and  $n = 2$ . Separately, we also introduce a rank-1 current, construct its charge and discuss its relation to the recently constructed Cotton current for the Kerr-Newman metric.

### A.1 $n = 1$

This is the special case when the CKYT is a CKV. Thus consider the following contravariant vector

$$J^a = R^{ab}k_b + \alpha Rk^a + \beta \nabla^a K. \quad (\text{A.1})$$

Then

$$\nabla_a J^a = k^a \nabla_a R \left( \frac{1}{2} + \alpha + \frac{\beta}{2(1-D)} \right) + KR \left( 1 + \alpha D + \frac{\beta}{(1-D)} \right). \quad (\text{A.2})$$

Demanding  $\nabla_a J^a = 0$  implies that either “ $D = 2, \beta = 1 + 2\alpha$  with  $\alpha$  left arbitrary” or “ $D \neq 2, \alpha = 0, \beta = D - 1$ ”. However note that when  $D = 2$ , one automatically has  $R_{ab} = (R/2)g_{ab}$ . So one has a conserved vector  $J^a$  for

$$D = 2 : J^a = R^{ab}k_b + \alpha Rk^a + (1 + 2\alpha)\nabla^a K = (1 + 2\alpha)(\nabla^a K + 2Rk^a), \quad (\text{A.3})$$

$$D \neq 2 : J^a = R^{ab}k_b + (D - 1)\nabla^a K = -\square k^a + \nabla^a K = -F^a.$$

This follows the pattern described for  $D \neq 2$  in section 4.

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<sup>6</sup>However the entire responsibility for the publication is ours. The financial support received from TÜBİTAK does not mean that the content of the publication is approved in a scientific sense by TÜBİTAK.

Since we are dealing with a CKV, a natural question to ask is on the transformation properties of  $J^a$  under Weyl scaling of the metric  $\tilde{g}_{ab} = \Omega^2 g_{ab}$ . A CKV transforms as

$$\tilde{k}^a = k^a \quad \text{and} \quad \tilde{K} = K + \Omega^{-1}(\nabla_a \Omega)k^a. \quad (\text{A.4})$$

The transformations of the Ricci and scalar curvature is given by

$$\begin{aligned} \tilde{R}_{ab} &= R_{ab} - \Omega^{-1}((D-2)\nabla_a \nabla_b \Omega + g_{ab}\square\Omega) \\ &\quad + \Omega^{-2}(2(D-2)(\nabla_a \Omega)\nabla_b \Omega + (3-D)g_{ab}(\nabla_c \Omega)\nabla^c \Omega), \\ \tilde{R} &= \Omega^{-2}(R + 2(1-D)\Omega^{-1}\square\Omega + (D-1)(4-D)\Omega^{-2}(\nabla_c \Omega)\nabla^c \Omega). \end{aligned} \quad (\text{A.5})$$

For  $D = 2$ , these give

$$\begin{aligned} \tilde{J}^a &= \Omega^{-2}J^a - (1+2\alpha)\Omega^{-2}(k^a\square\ln\Omega + \nabla^a(k^c\nabla_c\ln\Omega)), \\ \tilde{\nabla}_a \tilde{J}^a &= -(1+2\alpha)\Omega^{-2}(2K\square\ln\Omega + k^a\nabla_a\square\ln\Omega + \square(k^a\nabla_a\ln\Omega)). \end{aligned} \quad (\text{A.6})$$

So one has  $\tilde{\nabla}_a \tilde{J}^a = 0$  only when  $\alpha = -1/2$ , for which  $J^a = 0$ . The  $D \neq 2$  case can be obtained from the discussion of the Weyl scalings of a generic rank- $n$  current  $F$  given in appendix 2.2, by setting  $n = 1$ , of course.

## A.2 $n = 2$

For  $n = 2$ , following the general case we find the conserved (trivial) current

$$J^{ab} = -\frac{1}{2}\nabla_c \nabla^{[a} k^{bc]}. \quad (\text{A.7})$$

As may be seen from the following relations, proven in [1]

$$(D-3)G^{ac}K_c = \nabla_c \left( G^{b[c} k_b^{a]} + (D-2)\nabla^{[a} K^{c]} \right) \quad (\text{A.8})$$

and

$$\nabla_a \check{K}^{ab} = \nabla_a \hat{K}^{ab} + \frac{1}{2}(D-3)G^b{}_c K^c, \quad (\text{A.9})$$

the case  $D = 3$  requires special attention.

When  $D = 3$ , the Riemann tensor can be expressed in terms of the Ricci tensor and the curvature scalar only, and one finds  $\check{K}^{ab} = \hat{K}^{ab}$  identically. This implies

$$\frac{1}{2}R^{abcd}k_{cd} + R_c{}^{[a}k^{b]c} = -\hat{K}^{ab} = -\nabla^{[a}K^{b]} + \square k^{ab}, \quad (\text{A.10})$$

where the last equality follows from (2.10) evaluated for  $n = 2$ . On the other hand, when  $D = 3$  one has

$$\nabla_c \left( \hat{K}^{ac} + \nabla^{[a} K^{c]} \right) = 0 \quad (\text{A.11})$$

from (A.8). The discrepancy between (A.10) and (A.11) is accounted for by the trivial current (2.11) for  $n = 2$  also in  $D = 3$ .

## B Killing-Yano Cotton current

In [1], we showed that the Killing-Yano Cotton current  $J^a := C^{abc}k_{bc}$ , constructed out of a rank-2 CKYT  $k$  and the Cotton tensor  $C$ , is covariantly conserved. It is known that the Cotton tensor transforms as

$$\tilde{C}_{abc} = C_{abc} - (D-2)(\nabla_d \ln \Omega)W^d_{abc} \quad (\text{B.1})$$

under conformal transformations  $\tilde{g}_{ab} = \Omega^2 g_{ab}$ . Meanwhile, one has<sup>7</sup>

$$\tilde{k}_{ab} = \Omega^3 k_{ab} \quad \text{and} \quad \tilde{K}_c = \Omega K_c + (\nabla_a \Omega)k^a_c. \quad (\text{B.2})$$

These give

$$\tilde{J}^a = \Omega^{-3} \left( J^a - (D-2)(\nabla_d \ln \Omega)W^{da}_{bc}k^{bc} \right). \quad (\text{B.3})$$

Using these and the property  $(D-3)C_{bcd} = (D-2)\nabla_a W^a_{bcd}$ , it is straightforward to show that

$$\tilde{\nabla}_a \tilde{J}^a = \nabla_a \tilde{J}^a + D(\nabla_a \ln \Omega)\tilde{J}^a = 0,$$

independent of the dimension  $D$ .

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<sup>7</sup>Here we correct an error in [1], where a factor of  $(D-1)$  was inadvertently forgotten in the second term on the right-hand-side of the second equation in eq. (2.12) in that paper, with no effect on the ensuing discussion.

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