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Completing the fifth PN precision frontier via the EFT of spinning gravitating objects

Michèle Levi[a] **and Zhewei Yin**[b]

[a] *Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, U.K.*
[b] *Department of Physics and Astronomy, Uppsala University, Box 516, 75120 Uppsala, Sweden*

E-mail: levi@maths.ox.ac.uk, zhewei.yin@physics.uu.se

Abstract: We put forward a broader picture of the effective theory of a spinning particle within the EFT of spinning gravitating objects, through which we derive and establish the new precision frontier at the fifth PN (5PN) order. This frontier includes higher-spin sectors, quadratic and quartic in the spin, which both display novel physical features, due to the extension of the effective theory beyond linear order in the curvature. The quadratic-in-spin sectors give rise to a new tidal effect, and the quartic-in-spin sectors exhibit a new multipolar deformation. We then generalize the concept of tidal operators and of spin-induced multipolar operators, and make conjectures on the numerical values of their Wilson coefficients, and on the effective point-particle action of Kerr black holes. We confirm the generalized actions for generic compact binaries of the NLO quartic-in-spin sectors which were derived via the extension of the EFT of gravitating spinning objects. We first present the corresponding interaction potentials and general Hamiltonians, which consist of 12 distinct sectors, with a new one due to the new multipolar deformation. These Hamiltonians give the full physical information on the binary system, which mostly gets lost in higher-spin sectors, when going to the aligned-spins configuration. Moreover these general Hamiltonians uniquely allow us to find the complete Poincaré algebra at the 5PN order with spins, including the third subleading quadratic-in-spin sectors. We derive consequent observables for GW applications. Finally, to make contact with the scattering problem, we also derive the extrapolated scattering angles for aligned spins. Our completion of the Poincaré algebra provides the strongest validation of our most comprehensive new results, and thus that the 5PN order has now been established as the new precision frontier.

Keywords: Black Holes, Effective Field Theories, Space-Time Symmetries, Scattering Amplitudes

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Contents

1 Introduction

Second-generation gravitational-wave (GW) experiments, currently including Advanced LIGO [1], Advanced Virgo [2], and KAGRA [3], have brought about one of the greatest breakthroughs of the century in physics: measurements of GWs emitted from the inspirals and mergers of two black holes (BHs) [4], neutron stars (NSs) [5], or even of mixed BH-NS binaries [6]. In just a few years the influx of GW data has been rapidly growing [7–9], ushering in an exciting era of accelerated theoretical progress. The modelling of such GW

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l \ n	(N^{0})LO	$\mathrm{N}^{(1)}$LO	N^{2}LO	N^{3}LO	N^{4}LO	N^{5}LO	N^{6}LO
S^{0}	0	1	2	3	4	5	6
S^{1}	1.5	2.5	3.5	**4.5**	5.5		
S^{2}	2	3	4	**5**	6		
S^{3}	3.5	**4.5**	5.5				
S^{4}	4	**5**	6				
S^{5}	5.5						
S^{6}	6						

Table 1. The complete state of the art of PN gravity for the conservative dynamics of generic compact binaries, with an eye towards the next precision frontier at the 6PN order. Using the PN formula we introduced in [28], each correction enters at the order $n + l + \mathrm{Parity}(l)/2$, where n is highest n-loop and l highest spin-induced multipole contained in each of the sectors, and the parity is 0 or 1 for even or odd l, respectively. The new precision frontier at the 4.5 and 5PN orders with spins (in boldface) has been uniquely completed via the EFT of spinning gravitating objects, and has not been fully derived nor verified via independent methods. Most of the sectors at the 5.5 and 6PN orders (in gray) are still unknown. The green area of the table is explained in the text below.

signals is rooted in the analytic post-Newtonian (PN) approximation of General Relativity (GR) [10], as the motion in sources of such signals is characterized by non-relativistic (NR) velocities. The effective-one-body (EOB) approach [11] builds on PN theory to generate theoretical waveforms across the entirety of GW signals. In particular, the progress in recent years on the precision frontier of the conservative dynamics of generic compact binaries has been spectacular.

The state of the art is presented in table 1, with an eye towards the next precision frontier. In the point-mass sector, which occupies the first row of table 1, the fifth PN (5PN) order has been tackled via traditional GR methods [12–14], and the effective field theory (EFT) approach [15, 16], and work on the 6PN order is underway in both methodologies [17–19]. The 5PN state of the art in the point-mass sector constitutes a unique milestone, due to the first appearance of finite-size effects in this sector, and thus of UV physics taking place at the short scales of the individual compact objects. In the spin-orbit sector, which occupies the second row of table 1, finite-size effects are also postponed by 5 PN orders to the high 6.5PN accuracy, similar to the point-mass sector. The spin-orbit sector has been fully completed and verified at the 4.5PN order [20–22], followed by a similar derivation in [23], via the EFT of spinning gravitating objects and the `EFTofPNG` public code [24, 25], and was also approached via traditional GR methods following [12, 13] in [26, 27].

This picture for finite-size effects dramatically changes once higher-spin sectors are considered. To begin with, the leading finite-size effect shows up already at the 2PN order [29], due to the spin-induced quadrupole. Such spin-induced multipolar deformations then show up at all subsequent PN orders, as captured uniquely via the EFT of spinning gravitating objects, that introduced the relevant theory to all orders in spin [24, 30]. In particular the work exclusively carried out via this framework [30–33], which pioneered the

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general treatment of higher-spin sectors, was key to the completion of the state of art at the 4PN order [34, 35]. More recently, we completed the precision frontier at the 4.5PN order, which consists only of sectors with spins, including the complete general Hamiltonians of the third subleading (N^3LO) spin-orbit sector [20, 36], and the next-to-leading order (NLO) cubic-in-spin sectors [22, 28], which were all verified via the full Poincaré algebra in [22]. Moreover, at the 5PN order, which we already noted is a milestone in the point-mass sector, we recently completed the N^3LO quadratic-in-spin sectors in [36–38], followed by [39] that used the same methodology and framework. To complete the 5PN accuracy we also have had to obtain the NLO quartic-in-spin sectors [40].

These different sectors with spins at the 5PN order required an extension of the effective action of a spinning particle from our [24], beyond linear order in the curvature. Similar to the milestone in the point-mass sector at this order, the extension of theory for spin uncovers yet another novel level of intricacy in the finite-size effects as of this order. In the quadratic-in-spin sectors we discovered a new tidal effect [38] that is unique to the presence of spin. Yet, another type of new effect was featured first in [40]. In section 2 of this paper after we review the EFT of higher spin in gravity, we discuss these new finite-size effects in a broader perspective, and generalize their different notions. In section 3 we derive the reduced interaction potentials and general Hamiltonians of the NLO quartic-in-spin sectors, resulting from the generalized actions computed via EFT in [40], which consist of no less than 12 subsectors. In section 4 we proceed to validate our general Hamiltonians across all sectors with spins at the 5PN order via a derivation of the full Poincaré algebra. In that we find the global Poincaré invariants of the binary system, and at the same time carry out a highly non-trivial consistency check for the general Hamiltonians in arbitrary reference frames. Notably, the N^3LO sectors, particularly those that are quadratic in the spin, required a significant scaling of the approach to solve for the Poincaré algebra.

After having thus established the new precision frontier at the 5PN order, we derive in section 5 the consequent simplified Hamiltonians in restricted kinematic configurations, and then the GW observables in the form of gauge-invariant relations among the binding energy, the angular momentum, and the emitted frequency. Furthermore, to make contact with the scattering problem, we also derive the extension to scattering angles, which is relevant only in the restricted aligned-spins configuration. Yet, while studying higher-spin sectors generally provides invaluable input on QCD and gravity theories, the restriction to the aligned-spins configuration in these sectors entails a growing loss of information on the physics of precessing binaries with spins of arbitrary orientations, which have a clear observational signature on the gravitational waveforms. For this reason it is especially crucial in higher-spin sectors to obtain general Hamiltonians, rather than limited scattering angles.

In the scattering problem in a weak-field approximation or so-called post-Minkowskian (PM) expansion, the scattering angle for BHs with aligned spins was first approached in [41] for the NLO quartic-in-spin sectors. NLO PM quartic-in-spin Hamiltonians in the center-of-mass (COM) frame for BHs, and for generic compact binaries, were then presented in [42], and [43], respectively. However, some basic clarifications on such scattering-amplitudes approaches are in place. As far as application to GW measurements from binary mergers is concerned, velocities remain NR in most if not all of the GW signal. As long as the binary is bound, it is only meaningful to obtain precision corrections in the PN expansion:

$$n\text{PN} \sim \sum_{i=0}^{n+1} v^{2i} G^{n+1-i}, \tag{1.1}$$

with the speed of light $c \equiv 1$. As can be easily seen, this requires for each nPN order a computation that includes a piece that is of order G^{n+1}. Then at the brief time that the binary is no longer bound, the weak-field approximation also breaks down, and does not hold any more.

For these reasons, an mPM weak-field correction, which is an expansion to the order of G^m (even if to all orders in the velocity) is useful phenomenologically only for an $(m-1)$PN correction, and only when it can be meaningfully transformed from the scattering to the bound problem. Such a transformation to the bound problem faces an obstacle at and beyond the N^{2}LO, or the 2-loop level, essentially due to radiation-reaction effects that kick in at these orders in gravity, and for which a solution is currently unknown. At the same time, for higher-spin sectors of S^l, which map to scattering of massive particles with higher spin $s = l/2$, the 4-particle amplitude of massive higher-spin particles and gravitons is imperative beyond tree level, namely as of the NLO, and this so-called "Compton amplitude" is as yet murky for $l \geq 5$.

As can be illustrated from table 1 by the green area, these 2 limitations on high loop and higher spin sectors, combine to restrict the applicability of current scattering-amplitudes approaches to the GW precision frontier. In fact already as of the 3PN order, sectors that are required to complete a certain PN accuracy, are no longer within the reach of these approaches. Moreover, it is timely now to tackle sectors at the 5.5 and 6PN orders, whereas resolving the Compton amplitude that is only needed as of the NLO quintic-in-spin sectors at the 6.5PN order, is not experimentally meaningful, while none of the difficult sectors at the 5.5 and 6PN orders have been tackled yet.

In contrast, our free-standing QFT-based approach efficiently uses QFT methods without the irrelevant quantum degrees of freedom (DOFs), has been directly suited to the bound problem, and thus readily gets at the necessary results for GW measurements. Our self-contained EFT approach has no limited reach to any sector, and provides the most general comprehensive results for GW measurements of generic compact binaries. For these very reasons our EFT approach is also key as a guide to such diverse particle-amplitudes methods in tackling the related gravitational scattering problem.

2 Effective Field Theory of higher spin in gravity

At the 5PN order we encounter a leap in the intricate array of finite-size effects for rotating compact objects. Such finite-size effects which provide unique input on unknown physics in small scales, further multiply and diversify at this order. In order to capture all that, we need to go step by step, from the minimal coupling of spin to gravity, to leading spin-induced multipolar deformations, which are the leading finite-size effects of rotating objects, and then further to include tidal and subleading spin-induced multipolar deformations.

The minimal coupling of an extended relativistic spin already presents the first challenge which we resolved in the EFT of spinning gravitating objects [24, 44] via what we coined as "spin-gauge invariance". Second, for the leading spin-induced multipolar deformations of generic compact objects we introduced an infinite tower of operators to all orders in spin [24, 30]. Implementing these two formal developments at higher loop and spin orders already requires a thoughtful treatment [24, 28, 30, 32]. Yet, for the sectors at the 5PN order, one needs to further extend the theory to include couplings beyond linear in the curvature, namely tidal and subleading spin-induced multipolar deformations [36–38, 40]. There are also additional subtleties due to the implementation of spin-gauge invariance in the leading spin-induced multipolar deformations [28, 40]. Here we will review these formal developments, and further elaborate on the more advanced effective theory beyond linear in the curvature.

We recall that our computations for the compact binary inspiral start at the orbital scale from a two-particle system in weak gravity. To arrive at that, we have invoked new DOFs localized on two worldlines for each object that capture all of the small-scale physics which we are suppressing at this stage [15]:

$$S_{\text{eff}} = S_{\text{gr}}[g_{\mu\nu}] + \sum_{a=1}^{2} S_{\text{pp}}(\lambda_a). \tag{2.1}$$

In addition to the purely gravitational action of the weak-field modes S_{gr}, we should then prescribe the infinite point-particle action for a spinning particle, S_{pp}, interacting with weak-field gravity. λ_a parametrizes the worldline of each object.

2.1 EFT of spinning gravitating objects

We start with the initial generic point-particle action we can write for a spinning object as [24, 44–47]:

$$S_{\text{pp}}\left[g_{\mu\nu}, y^\mu, e_A^\mu\right] = \int d\lambda \left[\underbrace{-m\sqrt{u^2}}_{\text{0PN}} \underbrace{-\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu}}_{\text{1.5PN}} \underbrace{+L_{\text{SI}}\left[u^\mu, S_{\mu\nu}, g_{\mu\nu}\left(y^\mu\right)\right]}_{\text{2PN}}\right], \tag{2.2}$$

with y^μ and e_A^μ, the coordinate and tetrad, respectively, as the new worldline DOFs. $u^\mu \equiv dy^\mu/d\lambda$ is then the 4-velocity, and $\Omega^{\mu\nu}(\lambda) \equiv e_A^\mu \frac{De^{A\nu}}{D\lambda}$ the generalized angular velocity. The conjugate of the latter is spin, $S_{\mu\nu}(\lambda)$, which is featured explicitly in the effective action, and in fact accounts for all the non-minimal coupling of the rotating object to gravity, L_{SI}. We noted for each term in eq. (2.2) the PN order, in which it starts to play a role according to PN power-counting.

2.1.1 Spin-gauge invariance

The minimal coupling as it appears in eq. (2.2) actually takes into account only the basic symmetries that play a role when we consider only position DOFs, as in the point-mass sector [24]. Once we also consider rotation, SO(3) invariance plays a key role in the construction of all parts of the spinning-particle action [24]. Our spin-gauge invariance then arises naturally as its consequent gauge redundancy, which is concealed in the form of eq. (2.2) [24]. In our formulation of the effective action of a spinning particle [24], we made this hidden gauge invariance manifest, by deriving from the minimal coupling of spin to gravity, the more general form:

$$\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} = \frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} + \frac{\hat{S}^{\mu\nu}p_\nu}{p^2}\frac{Dp_\mu}{D\sigma}, \tag{2.3}$$

with generic rotational variables at this time, related to a general gauge of the spin (or "SSC"). Notably there is an extra term that shows up, which is kinematic. This term affects all orders in spin in all sectors with spin, but is always already fixed from (the dynamics of) lower perturbative orders. Therefore we have kept it expressed in terms of only worldline DOFs, since at each new order it always contains no new dynamics involving the gravitational field, which therefore does not have to be integrated out of it again. Our formulation then differed from Yee and Bander [48], and later [49] that followed suit, who added an ad-hoc term with curvature/field dependence, which could not capture correctly this significant general kinematic effect for a relativistic rotating object.

In the EFT of spinning gravitating objects we exploit the spin-gauge invariance, and switch to a generalized canonical gauge that we formulated in [24], which enables to land directly on spin variables that satisfy the canonical SO(3) Poisson brackets (or commutation relations) [24, 50]. In order to switch to a generic gauge for the spin also in the non-minimal couplings, one then needs to use the relation:

$$S_{\mu\nu} = \hat{S}_{\mu\nu} - \frac{\hat{S}_{\mu\rho}p^\rho p_\nu}{p^2} + \frac{\hat{S}_{\nu\rho}p^\rho p_\mu}{p^2}. \tag{2.4}$$

2.1.2 Tower of higher-spin couplings

To begin with we construct the non-minimal coupling of the action with the analogue of the Pauli-Lubanski vector, S^μ, which is orthogonal to the linear momentum [24, 30]. This naturally leads to definite-parity SO(3) tensors for the spin-induced multipoles, in a construction that is guided mainly by the symmetries of parity and SO(3) invariance [24, 30]. Using the complete set of symmetries spelled out in [24], and with the definite-parity curvature components, $E_{\mu\nu}$ and $B_{\mu\nu}$, our rigorous analysis gave rise to an infinite tower of leading non-minimal couplings to all orders in spin [24, 30]:

$$\begin{aligned} L_{\text{NMC}} = &\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\frac{C_{ES^{2n}}}{m^{2n-1}} D_{\mu_{2n}}\cdots D_{\mu_3}\frac{E_{\mu_1\mu_2}}{\sqrt{u^2}}S^{\mu_1}S^{\mu_2}\cdots S^{\mu_{2n-1}}S^{\mu_{2n}} \\ &+\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{C_{BS^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}}\cdots D_{\mu_3}\frac{B_{\mu_1\mu_2}}{\sqrt{u^2}}S^{\mu_1}S^{\mu_2}\cdots S^{\mu_{2n-1}}S^{\mu_{2n}}S^{\mu_{2n+1}}. \end{aligned} \tag{2.5}$$

This new set of operators represents spin-induced multipolar deformations, and are preceded by new Wilson coefficients, that match what would be called in traditional GR "multipolar deformation parameters". From this infinite series we need the 3 leading terms for the quartic-in-spin sectors [24, 30]:

$$L^{\leq 5\text{PN}}_{\text{NMC(R)}} = \underbrace{-\frac{C_{ES^2}}{2m}\frac{E_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu}_{\text{2PN}}\underbrace{-\frac{C_{BS^3}}{6m^2}\frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu S^\lambda}_{\text{3.5PN}}\underbrace{+\frac{C_{ES^4}}{24m^3}\frac{D_\kappa D_\lambda E_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu S^\lambda S^\kappa}_{\text{4PN}}, \tag{2.6}$$

with the spin-induced quadrupole [29, 47], octupole, and hexadecapole [30], which enter at the PN orders noted, inferred from power-counting as detailed in [24, 30, 35]. Notably the leading relativistic corrections start only from the octupole [24, 30, 35].

The various approaches which have implemented scattering-amplitudes methods to study scattering of massive higher-spin particles, such as [41–43], have relied on the above theory, introduced in [24, 30]. The S^l couplings to gravity in eq. (2.5) provided the 3-point amplitudes for the scattering of massive particles of spin $s = l/2$, where 3-point amplitudes are the critical building blocks used to derive any amplitude in such methods. Moreover, these approaches all used further input from implementation of the above theory for the case of BHs in traditional GR, e.g. [41], to critically guide their derivations. The dependence of Guevara et al. [41] on our worldline theory for higher-spin from [24, 30] should be noted here in particular, since it was omitted in [41].

At this point we can notice that the iterative substitution of the linear momentum in eqs. (2.3), (2.4), and (2.6), becomes subtle at cubic- and quartic-in-spin orders [28, 40]. In the latter it can give rise to contributions that are already quadratic in the curvature (though still without further new Wilson coefficients) [40]. In any case the 5PN order necessitates an extension of the effective action of a spinning particle, to include operators that are beyond linear in the curvature [36–38, 40].

2.2 Going beyond linear in curvature

It was already noted in [15], which treated the point-mass sector (without spin), that the effective action at the 5PN order should be extended to quadratic order in the curvature. We consider here such an extension with eyes towards the next precision frontier — covering up to the 6PN order, where we restrict the discussion to strictly conservative operators. Indeed, in [36–38, 40] for the present 5PN precision frontier, we approached such an extension of the non-minimal coupling to gravity for all sectors up to quartic order in the spin. Using the complete symmetries, and similar considerations spelled out in [24], we can thus write schematically such an extension:

$$L^{\leq 6\text{PN}}_{\text{NMC(R}^2)} = \underbrace{C_{E^2}\frac{E_{\alpha\beta}E^{\alpha\beta}}{\sqrt{u^2}^3} + C_{E^2S^2}S^\mu S^\nu \frac{E_{\mu\alpha}E_\nu^\alpha}{\sqrt{u^2}^3} + C_{E^2S^4}S^\mu S^\nu S^\kappa S^\rho \frac{E_{\mu\nu}E_{\kappa\rho}}{\sqrt{u^2}^3}}_{5\text{PN}}$$
$$\underbrace{+C_{B^2}\frac{B_{\alpha\beta}B^{\alpha\beta}}{\sqrt{u^2}^3} + C_{B^2S^2}S^\mu S^\nu \frac{B_{\mu\alpha}B_\nu^\alpha}{\sqrt{u^2}^3} + C_{B^2S^4}S^\mu S^\nu S^\kappa S^\rho \frac{B_{\mu\nu}B_{\kappa\rho}}{\sqrt{u^2}^3}}_{6\text{PN}}, \tag{2.7}$$

and beyond the 6PN order:

$$L^{\leq 7\text{PN};\leq S^4}_{\text{NMC(R}^2)} = \underbrace{C_{\nabla EBS}S^\mu \frac{D_\mu E_{\alpha\beta}B^{\alpha\beta}}{\sqrt{u^2}^3} + C_{E\nabla BS}S^\mu \frac{E_{\alpha\beta}D_\mu B^{\alpha\beta}}{\sqrt{u^2}^3}}_{6.5\text{PN}} \tag{2.8}$$
$$\underbrace{+C_{\nabla EBS^3}S^\mu S^\nu S^\kappa \frac{D_\kappa E_{\mu\alpha}B_\nu^\alpha}{\sqrt{u^2}^3} + C_{E\nabla BS^3}S^\mu S^\nu S^\kappa \frac{E_{\mu\alpha}D_\kappa B_\nu^\alpha}{\sqrt{u^2}^3}}_{6.5\text{PN}}$$

$$
\underbrace{+C_{(\nabla E)^2}\frac{D_\mu E_{\alpha\beta}D^\mu E^{\alpha\beta}}{\sqrt{u^2}^3}+C_{(\nabla B)^2}\frac{D_\mu B_{\alpha\beta}D^\mu B^{\alpha\beta}}{\sqrt{u^2}^3}}_{\text{7PN}}
$$

$$
\underbrace{+C_{(\nabla E)^2S^2}S^\mu S^\nu\frac{D_\mu E_{\alpha\beta}D_\nu E^{\alpha\beta}}{\sqrt{u^2}^3}+C_{(\nabla B)^2S^2}S^\mu S^\nu\frac{D_\mu B_{\alpha\beta}D_\nu B^{\alpha\beta}}{\sqrt{u^2}^3}}_{\text{7PN}}
$$

$$
\underbrace{+C_{(\nabla E)^2S^4}S^\mu S^\nu S^\kappa S^\rho\frac{D_\kappa E_{\mu\alpha}D_\rho E^\alpha_\nu}{\sqrt{u^2}^3}+C_{(\nabla B)^2S^4}S^\mu S^\nu S^\kappa S^\rho\frac{D_\kappa B_{\mu\alpha}D_\rho B^\alpha_\nu}{\sqrt{u^2}^3}}_{\text{7PN}},
$$

where we noted the PN order in which the various new operators enter. Notice that since the indices of covariant derivatives and the curvature components are symmetric [24], one can always write the contractions among two curvature components and their covariant derivatives, starting with the indices of curvature components. Notice that only operators quadratic- and quartic-in-spin enter up to the 6PN order, whereas the linear and cubic-in-spin sectors both get such corrections only as of the 6.5PN order. Moreover, from eq. (2.8) it is also easy to realize that an extension to higher orders in the curvature (beyond quadratic) is not relevant even up to the 7PN order.

Thus at the present 5PN order we only need to consider the first line of eq. (2.7), with 3 leading operators. Note however that the 2 first operators do not contribute to the NLO quartic-in-spin sectors, which are further studied in the present paper, due to their lower order in the spin: at the 5PN order they can only contribute to the sector without spin, or to the quadratic-in-spin sectors, as detailed in [36]. Yet even within these leading quadratic-in-curvature operators, we can already notice a clear distinction between the first two, where indices of the curvature components are contracted among themselves, and the operator that is quartic in the spin, where none of the curvature indices are contacted among themselves, similar to the operators in eq. (2.5). Furthermore, as we elaborated in [36, 38], the first two operators in eq. (2.7) both capture tidal effects. These observations then motivate us to make the following useful definitions:

Definition 1. *We call all operators whose indices of curvature components and their covariant derivatives have contractions among themselves — "tidal operators". We call the Wilson coefficients of such operators "tidal coefficients", or "generalized Love numbers".*

Definition 2. *We call all operators, of which no indices of curvature components and their covariant derivatives are contracted among themselves — "spin-induced multipolar operators". We call the Wilson coefficients of such operators "spin-induced multipolar coefficients".*

It is clear that the infinite set in the form of direct products of the leading linear-in-curvature basis of spin-induced multipolar-deformation operators in eq. (2.5), will appear as subleading spin-induced multipolar operators of higher orders in spin and curvature. The first of these subleading operators is for S^4, and is of the form of a direct product of two spin-induced quadrupole operators [40]. It is easy to see that the next similar operator appears at the NLO S^5 sectors, and so on and so forth, as in:

$$
L^{\rm SI}_{\rm NMC(R^2)} \supset \underbrace{C_{E^2S^4}S^\mu S^\nu S^\kappa S^\rho E_{\mu\nu}E_{\kappa\rho},}_{\text{NLO S}^4}\quad \underbrace{C_{EBS^5}S^\mu S^\nu S^\kappa S^\rho S^\lambda E_{\mu\nu}D_\lambda B_{\kappa\rho},}_{\text{NLO S}^5}\ \dots \tag{2.9}
$$

Notice that though no ladder graphs are allowed in our EFT diagrammatic expansion, here we have an infinite tower of ladder-like operators (albeit, localized on the worldline also in time), as subleading corrections to the leading spin-induced multipolar ones in eq. (2.5). Incidentally, it would be inconsistent to include from eq. (2.9) the operator for S^5, but not that for S^4, which is the case in [43], where the latter was omitted.

We can now notice the following interesting patterns for operators at the sectors S^l. For $l = 0, 1$, namely for the sectors dominated by minimal coupling to gravity, corrections to the leading effects enter at a shift of 5 PN orders, and they are of the tidal type. For $l = 2, 3$, corrections to the leading effects, which include the spin-induced multipolar type, enter at a shift of 3 PN orders, and they are also of the tidal type. As of $l \geq 4$, we can also tell from power counting, that corrections to the leading effects, which include the spin-induced multipolar type, will always enter at a shift of 1 PN order, but will also include the new spin-induced multipolar type, from subleading spin-induced multipolar operators as in eq. (2.9). However, we can actually also see from power-counting that in fact, for $l \geq 2$, tidal corrections to the leading effects, always enter at a shift of 3 PN orders.

With the two definitions above, we proceed to make two corresponding conjectures:

Conjecture 1. *All tidal coefficients (or generalized Love numbers) vanish for rotating black holes in Einstein's theory of gravity (in 4 spacetime dimensions).*

Conjecture 2. *All spin-induced multipolar coefficients, when rendered dimensionless, and normalized according to the symmetry of their operator, are of order unity for rotating black holes in Einstein's theory of gravity (in 4 spacetime dimensions).*

Such conjectures should of course be rigorously proved (or disproved) using matching computations of the EFT to full GR. Conjecture 1 would follow from some extended "Love symmetry", see for example some seminal and recent surge in studies of basic Love numbers, namely the traditional ones — from the point-mass sector (i.e. simply of the Wilson coefficient of the first operator in eq. (2.7)) in e.g. [51–54], and [55–61], respectively. As to the spin-induced multipolar coefficients in conjecture 2, they can in general be fixed from studying leading observables, using only the linearized Kerr BH metric (namely of GR at 4 spacetime dimensions), since the leading contributions from these operators would always arise from the linearized curvature components, for which only the linearized Kerr field is needed.

Conjecture 2 then stipulates that the new spin-induced multipolar Wilson coefficient in eq. (2.7) in the NLO quartic-in-spin sectors at the 5PN order, after the proper normalization, satisfies that $\mathrm{C}_{E^2S^4}$ is of order 1 for rotating BHs in standard GR, as in:

$$L_{\mathrm{S}^4(\mathrm{R}^2)} = \frac{C_{E^2S^4}}{4!m^3}\Bigg[\frac{S^{\mu_1}S^{\mu_2}S^{\mu_3}S^{\mu_4}}{\sqrt{u^2}^3}\left(E_{\mu_1\mu_2}E_{\mu_3\mu_4}+E_{\mu_1\mu_3}E_{\mu_2\mu_4}+E_{\mu_1\mu_4}E_{\mu_2\mu_3}\right) \\ -\frac{12}{7}\frac{S^2S^{\mu_1}S^{\mu_2}}{\sqrt{u^2}^3}E_{\mu_1\nu}E_{\mu_2\nu}+\frac{6}{35}\frac{S^4}{\sqrt{u^2}^3}E_{\mu\nu}E_{\mu\nu}\Bigg], \tag{2.10}$$

where here the Wilson coefficient was indeed rendered dimensionless, and normalized according to the symmetry of the operator, and the operator is written in terms of its fully symmteric and tracless rank-4 SO(3) tensor components. Notice that the traces that are removed from the spin-induced multipolar operator constitute tidal deformations, and are accounted for in the respective point-mass and quadratic-in-spin operators/sectors, similar to what happened in the quadratic-in-spin sectors as detailed in [36] and [39]. The Feynman rule due to this operator, recalling that all indices in our Feynman rules are Euclidean, reads:

$$\int dt\, \frac{C_{E^2S^4}}{8m^3}\left[S^iS^jS^kS^l\phi_{,ij}\,\phi_{,kl}-\frac{4}{7}S^2S^iS^j\phi_{,ik}\,\phi_{,jk}+\frac{2}{35}S^4\phi_{,ij}\,\phi_{,ij}\right], \tag{2.11}$$

that is a two-graviton coupling of the KK scalar field as in [40].

The two conjectures above can be jointly stated as one conjecture for the general effective point-particle theory of Kerr BHs:

Conjecture. *Rotating black holes in Einstein's theory of gravity (in 4 spacetime dimensions) are captured by an effective point-particle theory, which contains an infinite set of only spin-induced multipolar-deformation operators, whose Wilson coefficients are of order unity, once their dimensions and symmetry factors are accounted for.*

At this stage an additional point should be made regarding the construction of our EFT of a spinning particle, and of EFTs in general. EFTs are devised to be an efficient and economic tool for precision computations. Therefore, when operators can be shifted to higher perturbative orders, using EOMs of the lower-order effective theory, namely when such operators are redundant, then they should indeed be removed at the level of the effective theory already (albeit, if left in the EFT, their onset in the observables will anyway be postponed to higher perturbative orders). According to this general EFT philosophy, we argued e.g. in [24] for possible linear-in-curvature operators, that could be recast as quadratic in the curvature, and higher order in the spin, using lower-order EOMs, and thus are omitted from the effective theory.

3 Effective actions and general Hamiltonians

Equipped with the extended effective theory of a spinning particle as discussed above, the NLO quartic-in-spin interactions were evaluated through an EFT computation in [40]. The evaluation involved 28 unique Feynman graphs [40]. The NLO quartic-in-spin generalized actions can then be summarized as follows [40]:

$$L^{\rm NLO}_{\rm S^4}=L^{\rm NLO}_{\rm S_1^2S_2^2}+L^{\rm NLO}_{\rm S_1^3S_2}+L^{\rm NLO}_{\rm S_1^4}+(1\leftrightarrow 2), \tag{3.1}$$

with the simplest part being:

$$\begin{aligned}L^{\rm NLO}_{\rm S_1^2S_2^2}=&\,\frac{1}{2}C_{1(\rm ES^2)}C_{2(\rm ES^2)}\frac{G}{m_1m_2}\left(\frac{3L_{(1)}}{8r^5}+\frac{3L_{(2)}}{4r^4}+\frac{L_{(3)}}{2r^3}\right)\\&-\frac{9}{2}C_{1(\rm ES^2)}C_{2(\rm ES^2)}\frac{G^2}{m_1r^6}L_{(4)}+\frac{1}{2}C_{1(\rm ES^2)}\frac{G^2}{m_1r^6}L_{(5)},\end{aligned} \tag{3.2}$$

l \ n	$(\mathrm{N}^{\mathbf{0}})$LO	$\mathrm{N}^{(\mathbf{1})}$LO
$\mathrm{S}^{\mathbf{0}}$		
$\mathrm{S}^{\mathbf{1}}$	$+$	$++$
$\mathrm{S}^{\mathbf{2}}$		$++$
$\mathrm{S}^{\mathbf{3}}$	$+$	$++$
$\mathrm{S}^{\mathbf{4}}$		$++$

Table 2. 10 sectors contribute through redefinitions to the present NLO quartic-in-spin sectors, $(1,4)$, where in [28] we introduced the notation, (n,l), and the corresponding PN counting. "+" indicates that only position shifts or redefinitions of rotational variables need to be fixed, whereas "++" indicates that redefinitions for both position and rotational variables need to be fixed.

then:

$$L^{\mathrm{NLO}}_{\mathrm{S}_1^3 S_2} = C_{1(\mathrm{BS}^3)} \frac{G}{m_1^2} \left(\frac{L_{[1]}}{4r^5} + \frac{L_{[2]}}{4r^4} + \frac{L_{[3]}}{12r^3} \right) + C_{1(\mathrm{BS}^3)} \frac{G^2}{m_1 r^6} L_{[4]} + C_{1(\mathrm{BS}^3)} \frac{m_2 G^2}{m_1^2 r^6} L_{[5]}$$
$$+ C_{1(\mathrm{ES}^2)} \frac{G^2}{m_1 r^6} L_{[6]} + \frac{3}{2} C_{1(\mathrm{ES}^2)} \frac{G}{m_1^2 r^4} L_{[7]} + 3 C_{1(\mathrm{ES}^2)} \frac{m_2 G^2}{m_1^2 r^6} L_{[8]}, \tag{3.3}$$

and finally:

$$L^{\mathrm{NLO}}_{\mathrm{S}_1^4} = C_{1(\mathrm{ES}^4)} \frac{G m_2}{m_1^3} \left(\frac{L_{\{1\}}}{16r^5} + \frac{L_{\{2\}}}{8r^4} + \frac{L_{\{3\}}}{12r^3} \right) - \frac{1}{8} C_{1(\mathrm{ES}^4)} \frac{G^2 m_2}{r^6 m_1^2} L_{\{4\}}$$
$$- \frac{3}{8} C_{1(\mathrm{ES}^4)} \frac{G^2 m_2^2}{r^6 m_1^3} L_{\{5\}} + C_{1(\mathrm{BS}^3)} \frac{G^2 m_2}{r^6 m_1^2} L_{\{6\}} - \frac{1}{8} C^2_{1(\mathrm{ES}^2)} \frac{G^2 m_2}{r^6 m_1^2} L_{\{7\}}$$
$$+ \frac{1}{2} C_{1(\mathrm{BS}^3)} \frac{G m_2}{r^4 m_1^3} L_{\{8\}} + C_{1(\mathrm{BS}^3)} \frac{G^2 m_2^2}{r^6 m_1^3} L_{\{9\}} + \frac{C_{1(\mathrm{E}^2\mathrm{S}^4)}}{8} \frac{G^2 m_2^2}{r^6 m_1^3} L_{\{10\}}, \tag{3.4}$$

where we confirm the various distinct pieces that make up the above as in [40]. We also present these generalized actions in the supplementary material attached to this publication.

3.1 Reduction of generalized actions

Though the generalized actions provided in [40] already allow to directly derive the EOMs for both the position and spin variables, as we discussed in [62], it is more useful to have at hand reduced actions, that contain only velocity and spin as the variables of highest time-derivative. The reduction involves formal redefinitions of both the position and rotational variables, that was introduced in [62], and recently further extended in [21]. Table 2 shows the build-up of redefinitions, that need to be applied in increasing PN order, up to the present NLO quartic-in-spin sectors, and here we will follow up on the derivations presented in [21, 22, 36].

First, we should note that as in all other higher-spin sectors starting from the NLO quadratic-in-spin [24], position shifts need to be applied beyond linear order. As to the rotational variables in the present sectors no redefinitions are needed to be applied beyond linear order [21]. As table 2 shows, we need to consider the redefinitions fixed from 6 sectors,

to / from	(0P)N	LO S^1	LO S^2	LO S^3
LO S^1			$(\Delta\vec{x})^2$	$\Delta\vec{x}$
NLO S^2			$\Delta\vec{x}$	
LO S^3			$\Delta\vec{S}$	
NLO S^3		$\Delta\vec{x}$		
NLO S^4	$\Delta\vec{x}$, $\Delta\vec{S}$			

Table 3. Contributions to the NLO quartic-in-spin sectors from position shifts and spin redefinitions in lower-order sectors.

where they were provided for the 3 sectors below cubic-in-spin in [21, 36], and for the 2 cubic-in-spin sectors in [22]. Thus here we are left to fix the redefinitions at the present NLO quartic-in-spin sectors, as captured in table 3, following our conventions and notations from [21].

Thus, we recall that the unreduced actions and redefinitions at the LO and NLO spin-orbit, NLO quadratic-in-spin, as well as LO and NLO cubic-in-spin sectors, are detailed in [21, 36], and [22], respectively. We can now proceed to the new position shift that is fixed in the present sectors:

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\rm NLO}_{\rm S^4} =& -\frac{GC_{1\rm ES^4}m_2}{8{m_1}^4r^4}\Big[-5S_1^2\vec{n}(\vec{S}_1\cdot\vec{n})^2+\vec{n}S_1^4+2\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_1\Big]\\
&-\frac{GC_{1\rm BS^3}m_2}{{m_1}^4r^4}\Big[5S_1^2\vec{n}(\vec{S}_1\cdot\vec{n})^2-\vec{n}S_1^4+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_1-5(\vec{S}_1\cdot\vec{n})^3\vec{S}_1\Big]\\
&+\frac{3GC_{1\rm ES^2}m_2}{16{m_1}^4r^4}\Big[5S_1^2\vec{n}(\vec{S}_1\cdot\vec{n})^2-\vec{n}S_1^4+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_1-5(\vec{S}_1\cdot\vec{n})^3\vec{S}_1\Big]\\
&+\frac{3GC_{1\rm ES^2}C_{2\rm ES^2}}{2{m_1}^2m_2r^4}\Big[2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1+\vec{S}_1\cdot\vec{n}S_2^2\vec{S}_1-5\vec{S}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2\vec{S}_1\Big]\\
&-\frac{3G}{8{m_1}^3r^4}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\vec{n}-5\vec{S}_1\cdot\vec{n}S_1^2\vec{n}\vec{S}_2\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1\\
&+5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\vec{S}_1+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\Big]+\frac{3GC_{1\rm ES^2}}{{m_1}^3r^4}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\vec{n}\\
&-5\vec{S}_1\cdot\vec{n}S_1^2\vec{n}\vec{S}_2\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1+5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\vec{S}_1+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\Big]\\
&-\frac{3GC_{2\rm ES^2}}{16{m_1}^2m_2r^4}\Big[S_1^2S_2^2\vec{n}-5S_1^2\vec{n}(\vec{S}_2\cdot\vec{n})^2-2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1-\vec{S}_1\cdot\vec{n}S_2^2\vec{S}_1\\
&+5\vec{S}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2\vec{S}_1+2S_1^2\vec{S}_2\cdot\vec{n}\vec{S}_2\Big]+\frac{GC_{1\rm BS^3}}{2{m_1}^3r^4}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\vec{n}\\
&+5\vec{S}_1\cdot\vec{n}S_1^2\vec{n}\vec{S}_2\cdot\vec{n}-10\vec{S}_1\cdot\vec{S}_2\vec{n}(\vec{S}_1\cdot\vec{n})^2-S_1^2\vec{S}_2\cdot\vec{n}\vec{S}_1+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1\\
&-7\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2+10(\vec{S}_1\cdot\vec{n})^3\vec{S}_2\Big]+\frac{GC_{2\rm BS^3}}{m_1{m_2}^2r^4}\Big[\vec{S}_1\cdot\vec{S}_2S_2^2\vec{n}-5\vec{S}_1\cdot\vec{S}_2\vec{n}(\vec{S}_2\cdot\vec{n})^2\\
&-\vec{S}_1\cdot\vec{n}S_2^2\vec{S}_2+5\vec{S}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2\vec{S}_2\Big].
\end{aligned}
\tag{3.5}
$$

The new redefinitions of the spin in the present sectors are then fixed as:

$$
\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S^4}=\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^4}+\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^3S_2}+\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^2S_2^2}+\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1S_2^3}-(i\leftrightarrow j),
\tag{3.6}
$$

where we present the explicit expressions in appendix A due to their large volume, and in machine-readable format in the supplementary material attached to this publication.

After the above reduction is done, we obtain the following NLO quartic-in-spin potentials, made up of no less than 12 unique sectors:

$$\begin{aligned} V_{\mathrm{S}^4}^{\mathrm{NLO}} =& C_{1\mathrm{ES}^2} V_{(\mathrm{ES}_1^2)\mathrm{S}_1^2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2}^2 V_{(\mathrm{ES}_1^2)^2}^{\mathrm{NLO}} + C_{1\mathrm{BS}^3} V_{(\mathrm{BS}_1^3)\mathrm{S}_1}^{\mathrm{NLO}} + C_{1\mathrm{ES}^4} V_{\mathrm{ES}_1^4}^{\mathrm{NLO}} + C_{1\mathrm{E}^2\mathrm{S}^4} V_{\mathrm{E}^2\mathrm{S}_1^4}^{\mathrm{NLO}} \\ &+ V_{\mathrm{S}_1^3\mathrm{S}_2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2} V_{(\mathrm{ES}_1^2)\mathrm{S}_1\mathrm{S}_2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2}^2 V_{C_{\mathrm{ES}_1^2}^2\mathrm{S}_1^3\mathrm{S}_2}^{\mathrm{NLO}} + C_{1\mathrm{BS}^3} V_{(\mathrm{BS}_1^3)\mathrm{S}_2}^{\mathrm{NLO}} \\ &+ V_{\mathrm{S}_1^2\mathrm{S}_2^2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2} V_{(\mathrm{ES}_1^2)\mathrm{S}_2^2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2} C_{2\mathrm{ES}^2} V_{(\mathrm{ES}_1^2)(\mathrm{ES}_2^2)}^{\mathrm{NLO}} + (1 \leftrightarrow 2), \end{aligned} \tag{3.7}$$

where we present the explicit expressions in appendix B, and in machine-readable format in the supplementary material attached to this publication.

Notice that for generic compact binaries we have 2 distinct sectors that are both proportional to the square of the quadrupolar deformation parameter, one in eq. (B.3), and another unique contribution that is cubic in the individual spins in eq. (B.9), which first emerged in the NLO cubic-in-spin sectors [22]. Note also the new sector in eq. (B.6) due to the new subleading hexadecapole operator at this order from eq. (2.10).

3.2 General Hamiltonians

The final actions that we obtain via the EFT of gravitating spinning objects contain position and spin variables that correspond to the generalized canonical gauge we formulated therein [24]. This enables to directly derive the general Hamiltonians via a Legendre transform only with respect to the position variables, similar to sectors without spin. For this derivation of the Hamiltonian all the sectors in table 2 need to be taken into account consistently. It should be highlighted that we obtain the most general Hamiltonians in an arbitrary reference frame, and these in turn uniquely allow to study the global Poincaré invariants in phase space.

Due to the existence and uniqueness of the Poincaré algebra, the ability to find a closed solution for it, with some given derived Hamiltonian, provides a significant validation of the latter, as will be further discussed in section 4 below. For phenomenological applications these general Hamiltonians can be gradually simplified, starting from their restriction to the COM frame, and then to further specialized configurations, in which one can obtain compact observables, see section 5 below. The Hamiltonians can also be simplified into EOB models, which are crucial to generate GW templates.

Our general Hamiltonians for the NLO quartic-in-spin sectors, similar to the action potentials in eq. (3.7), consist of 12 unique sectors:

$$\begin{aligned} H_{\mathrm{S}^4}^{\mathrm{NLO}} =& C_{1\mathrm{ES}^2} H_{(\mathrm{ES}_1^2)\mathrm{S}_1^2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2}^2 H_{(\mathrm{ES}_1^2)^2}^{\mathrm{NLO}} + C_{1\mathrm{BS}^3} H_{(\mathrm{BS}_1^3)\mathrm{S}_1}^{\mathrm{NLO}} + C_{1\mathrm{ES}^4} H_{\mathrm{ES}_1^4}^{\mathrm{NLO}} + C_{1\mathrm{E}^2\mathrm{S}^4} H_{\mathrm{E}^2\mathrm{S}_1^4}^{\mathrm{NLO}} \\ &+ H_{\mathrm{S}_1^3\mathrm{S}_2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2} H_{(\mathrm{ES}_1^2)\mathrm{S}_1\mathrm{S}_2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2}^2 H_{C_{\mathrm{ES}_1^2}^2\mathrm{S}_1^3\mathrm{S}_2}^{\mathrm{NLO}} + C_{1\mathrm{BS}^3} H_{(\mathrm{BS}_1^3)\mathrm{S}_2}^{\mathrm{NLO}} \\ &+ H_{\mathrm{S}_1^2\mathrm{S}_2^2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2} H_{(\mathrm{ES}_1^2)\mathrm{S}_2^2}^{\mathrm{NLO}} + C_{1\mathrm{ES}^2} C_{2\mathrm{ES}^2} H_{(\mathrm{ES}_1^2)(\mathrm{ES}_2^2)}^{\mathrm{NLO}} + (1 \leftrightarrow 2), \end{aligned} \tag{3.8}$$

where we present the explicit expressions in appendix C, and in machine-readable format in the supplementary material attached to this publication.

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4 Poincaré algebra with spins at the 5PN order

In this section we study the global Poincaré symmetry of the binary system for all the sectors with spins at the 5PN order. This symmetry is realized in phase space, where in sectors with spin variables, we use the generalized Poisson brackets [45, 63]:

$$\{f, g\} \equiv \{f, g\}_x + \{f, g\}_S, \tag{4.1}$$

with

$$\{f, g\}_x = \sum_{I=1}^{2} \left(\frac{\partial f}{\partial x_I} \cdot \frac{\partial g}{\partial p_I} - \frac{\partial f}{\partial p_I} \cdot \frac{\partial g}{\partial x_I} \right), \tag{4.2}$$

$$\{f, g\}_S = \sum_{I=1}^{2} S_I \times \frac{\partial f}{\partial S_I} \cdot \frac{\partial g}{\partial S_I}. \tag{4.3}$$

For our binary system we have:

$$\vec{P} = \vec{p}_1 + \vec{p}_2, \quad \vec{J} = \sum_{I=1}^{2} \left(\vec{x}_I \times \vec{p}_I + \vec{S}_I \right), \tag{4.4}$$

for $\vec{P}$, the total linear momentum, and $\vec{J}$, the total angular momentum. If our PN Hamiltonians for arbitrary reference frames are valid, then they should satisfy the following Poincaré algebra:

$$\{P_i, P_j\} = \{P_i, H\} = \{J_i, H\} = 0, \qquad \{J_i, J_j\} = \epsilon_{ijk} J_k, \qquad \{J_i, P_j\} = \epsilon_{ijk} P_k, \tag{4.5}$$

$$\{G_i, P_j\} = \delta_{ij} H, \qquad \{G_i, H\} = P_i, \qquad \{G_i, G_j\} = -\epsilon_{ijk} J_k, \qquad \{J_i, G_j\} = \epsilon_{ijk} G_k, \tag{4.6}$$

with H for the Hamiltonian, and $\vec{G}$, the center-of-mass generator, related to the boost via $\vec{K} \equiv \vec{G} - t\vec{P}$. While the Poisson brackets in eq. (4.5) are trivially satisfied, those in eq. (4.6), which involve $\vec{G}$, require careful consideration. Our task here is thus to construct the unique solution for $\vec{G}$ that will satisfy the latter equation.

First, we construct $\vec{G}$ from the vectors $\vec{x}_I$, $\vec{p}_I$ and $\vec{S}_I$, such that the last Poisson brackets in eq. (4.6) are automatically satisfied. Then, if we take the following form for $\vec{G}$:

$$\vec{G} = h_1 \vec{x}_1 + h_2 \vec{x}_2 + \vec{Y}, \qquad h_1 + h_2 = H, \tag{4.7}$$

where h_I and $\vec{Y}$ satisfy:

$$\{h_I, P_i\} = \{Y_i, P_j\} = 0, \tag{4.8}$$

then

$$\{G_i, P_j\} = \delta_{ij} H. \tag{4.9}$$

Various considerations [22] then lead to the more specific form:

$$\vec{G} = H(\vec{x}_1 + \vec{x}_2)/2 + \vec{Y}, \tag{4.10}$$

where $\vec{Y}$ is symmetric under the exchange of worldline labels $1 \leftrightarrow 2$, and depends on x_I only through $\vec{n}$ and r. Thus we are left with the task of constructing and constraining $\vec{Y}$, such that $\vec{G}$ uniquely solves the non-trivial Poisson brackets:

$$\{G_i, H\} = P_i. \tag{4.11}$$

This is since it turns out that if the closed form for $\vec{G}$ in flat spacetime [63]:

$$\vec{G}_{\text{flat}} = \sum_{I=1}^{2} \left(\gamma_I m_I \vec{x}_I - \frac{\vec{S}_I \times \vec{p}_I}{m_I (1+\gamma_I)} \right), \tag{4.12}$$

with $\gamma_I = \sqrt{1 + p_I^2/m_I^2}$, is also used to constrain the $\mathcal{O}(G_N^0)$ in $\vec{G}$, then the third Poisson brackets in eq. (4.6) is also automatically satisfied by the solution for $\vec{G}$.

Thus, in order to solve for $\vec{G}$ we first decompose it according to spin orders in each individual spin, PN orders, and dependence in Wilson coefficients. We then construct an ansatz for $\vec{Y}$ using the vectors $\vec{n}$, $\vec{p}_I$, $\vec{S}_I$. We use dimensional analysis and Euclidean covariance to constrain the ansatz, but note that due to the fact that in sectors with spin we have 4 or even 5 vectors to consider in 3-dimensional space, each 4 of them are necessarily linearly dependent, and thus our general ansatz will contain a certain redundancy.

Following these considerations we proceed to solve for $\vec{G}$ at the sectors with spin that make up the new precision frontier at the 5PN order, with the general Hamiltonians for the NLO quartic-in-spin sectors first presented in section 3.2 above, and the general Hamiltonians for the N^{3}LO quadratic-in-spin sectors first presented in [36]. For the latter, the general ansatz to solve for, contains an order of $\sim 10^3$ free dimensionless coefficients, which means that the solution for the problem needed to be scaled significantly, even with respect to the most advanced Poincaré algebra at the 4PN order that we presented in [33].

4.1 NLO quartic-in-spin sectors

We start by decomposing H and $\vec{G}$ to all the sectors relevant to the solution for $\vec{G}$ at the NLO quartic-in-spin sectors. The Hamiltonian consists of:

$$H = H_{\text{N}} + H_{\text{1PN}} + H^{\text{LO}}_{\text{SO}} + H^{\text{LO}}_{\text{S}^2} + H^{\text{NLO}}_{\text{SO}} + H^{\text{NLO}}_{\text{S}^2} + H^{\text{LO}}_{\text{S}^3} + H^{\text{LO}}_{\text{S}^4} + H^{\text{NLO}}_{\text{S}^3} + H^{\text{NLO}}_{\text{S}^4}, \tag{4.13}$$

with:

$$\begin{aligned}
H^{\text{LO}}_{\text{SO}} &= H^{\text{LO}}_{\text{S}_1} + (1 \leftrightarrow 2), \qquad H^{\text{NLO}}_{\text{SO}} = H^{\text{NLO}}_{\text{S}_1} + (1 \leftrightarrow 2), & (4.14)\\
H^{\text{LO}}_{\text{S}^2} &= C_{1\text{ES}^2} H^{\text{LO}}_{\text{ES}_1^2} + H^{\text{LO}}_{\text{S}_1\text{S}_2} + (1 \leftrightarrow 2), & (4.15)\\
H^{\text{NLO}}_{\text{S}^2} &= H^{\text{NLO}}_{\text{S}_1^2} + C_{1\text{ES}^2} H^{\text{NLO}}_{\text{ES}_1^2} + H^{\text{NLO}}_{\text{S}_1\text{S}_2} + (1 \leftrightarrow 2), & (4.16)\\
H^{\text{LO}}_{\text{S}^3} &= C_{1\text{ES}^2} H^{\text{LO}}_{(\text{ES}_1^2)\text{S}_1} + C_{1\text{BS}^3} H^{\text{LO}}_{\text{BS}_1^3} + H^{\text{LO}}_{\text{S}_1^2\text{S}_2} + C_{1\text{ES}^2} H^{\text{LO}}_{(\text{ES}_1^2)\text{S}_2} + (1 \leftrightarrow 2), & (4.17)\\
H^{\text{NLO}}_{\text{S}^3} &= H^{\text{NLO}}_{\text{S}_1^3} + C_{1\text{ES}^2} H^{\text{NLO}}_{(\text{ES}_1^2)\text{S}_1} + C^2_{1\text{ES}^2} H^{\text{NLO}}_{C^2_{\text{ES}_1^2}\text{S}_1^3} + C_{1\text{BS}^3} H^{\text{NLO}}_{\text{BS}_1^3} & \\
&\quad + H^{\text{NLO}}_{\text{S}_1^2 S_2} + C_{1\text{ES}^2} H^{\text{NLO}}_{(\text{ES}_1^2)\text{S}_2} + (1 \leftrightarrow 2), & (4.18)\\
H^{\text{LO}}_{\text{S}^4} &= C_{1\text{ES}^4} H^{\text{NLO}}_{\text{ES}_1^4} + C_{1\text{BS}^3} H^{\text{LO}}_{(\text{BS}_1^3)\text{S}_2} + C_{1\text{ES}^2} C_{2\text{ES}^2} H^{\text{LO}}_{(\text{ES}_1^2)(\text{ES}_2^2)} + (1 \leftrightarrow 2), & (4.19)
\end{aligned}$$

and $H^{\text{NLO}}_{\text{S}^4}$ is given in eq. (3.8). For the COM generator we have:

$$\vec{G} = \vec{G}_{\text{N}} + \vec{G}_{\text{1PN}} + \vec{G}^{\text{LO}}_{\text{SO}} + \vec{G}^{\text{NLO}}_{\text{SO}} + \vec{G}^{\text{NLO}}_{\text{S}^2} + \vec{G}^{\text{NLO}}_{\text{S}^3} + \vec{G}^{\text{NLO}}_{\text{S}^4}, \tag{4.20}$$

where we used that all generators of LO sectors with spin vanish beyond spin-orbit, and with:

$$\vec{G}^{\rm LO}_{\rm SO} = \vec{G}^{\rm LO}_{\rm S_1} + (1 \leftrightarrow 2), \qquad \vec{G}^{\rm NLO}_{\rm SO} = \vec{G}^{\rm NLO}_{\rm S_1} + (1 \leftrightarrow 2), \tag{4.21}$$

$$\vec{G}^{\rm NLO}_{\rm S^2} = \vec{G}^{\rm NLO}_{\rm S_1^2} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm ES_1^2} + \vec{G}^{\rm NLO}_{\rm S_1S_2} + (1 \leftrightarrow 2), \tag{4.22}$$

$$\begin{aligned}\vec{G}^{\rm NLO}_{\rm S^3} &= \vec{G}^{\rm NLO}_{\rm S_1^3} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1} + C^2_{\rm 1ES^2}\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}S_1^3} + C_{\rm 1BS^3}\vec{G}^{\rm NLO}_{\rm BS_1^3} \\ &\quad + \vec{G}^{\rm NLO}_{\rm S_1^2 S_2} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_2} + (1 \leftrightarrow 2),\end{aligned} \tag{4.23}$$

where we solved for the latter in [22], and now we should solve for:

$$\begin{aligned}\vec{G}^{\rm NLO}_{\rm S^4} &= C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1^2} + C^2_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)^2} + C_{\rm 1BS^3}\vec{G}^{\rm NLO}_{\rm (BS_1^3)S_1} + C_{\rm 1ES^4}\vec{G}^{\rm NLO}_{\rm ES_1^4} + C_{\rm 1E^2S^4}\vec{G}^{\rm NLO}_{\rm E^2S_1^4} \\ &\quad + \vec{G}^{\rm NLO}_{\rm S_1^3S_2} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1S_2} + C^2_{\rm 1ES^2}\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}S_1^3S_2} + C_{\rm 1BS^3}\vec{G}^{\rm NLO}_{\rm (BS_1^3)S_2} \\ &\quad + \vec{G}^{\rm NLO}_{\rm S_1^2S_2^2} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_2^2} + C_{\rm 1ES^2}C_{\rm 2ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)(ES_2^2)} + (1 \leftrightarrow 2).\end{aligned} \tag{4.24}$$

The Poisson brackets in eq. (4.11) then decouple to a set of independent equations to solve for the generators of each subsector according to the above decomposition. Let us then list this set of equations.

We solve for $\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1^2}$ from:

$$0 = \{\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1^2}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm (ES_1^2)S_1^2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm (ES_1^2)S_1}\}_x + \{\vec{G}^{\rm NLO}_{\rm S_1^2}, H^{\rm LO}_{\rm ES_1^2}\}_x. \tag{4.25}$$

We solve for $\vec{G}^{\rm NLO}_{\rm (ES_1^2)^2}$ from:

$$0 = \{\vec{G}^{\rm NLO}_{\rm (ES_1^2)^2}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm (ES_1^2)^2}\}_x + \{\vec{G}^{\rm NLO}_{\rm ES_1^2}, H^{\rm LO}_{\rm ES_1^2}\}_x. \tag{4.26}$$

We solve for $\vec{G}^{\rm NLO}_{\rm (BS_1^3)S_1}$ from:

$$0 = \{\vec{G}^{\rm NLO}_{\rm (BS_1^3)S_1}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm (BS_1^3)S_1}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm BS_1^3}\}_x. \tag{4.27}$$

We solve for $\vec{G}^{\rm NLO}_{\rm ES_1^4}$ from:

$$0 = \{\vec{G}^{\rm NLO}_{\rm ES_1^4}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm ES_1^4}\}_x + \{\vec{G}_{\rm 1PN}, H^{\rm LO}_{\rm ES_1^4}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm ES_1^4}\}_S. \tag{4.28}$$

We solve for $\vec{G}^{\rm NLO}_{\rm E^2S_1^4}$ from:

$$0 = \{\vec{G}^{\rm NLO}_{\rm E^2S_1^4}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm E^2S_1^4}\}_x. \tag{4.29}$$

We solve for $\vec{G}^{\rm NLO}_{\rm S_1^3S_2}$ from:

$$0 = \{\vec{G}^{\rm NLO}_{\rm S_1^3S_2}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm S_1^3S_2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm S_1^2S_2}\}_x + \{\vec{G}^{\rm NLO}_{\rm S_1^2}, 2H^{\rm LO}_{\rm S_1S_2}\}_x. \tag{4.30}$$

We solve for $\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1S_2}$ from:

$$\begin{aligned}0 &= \{\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1S_2}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm (ES_1^2)S_1S_2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_2}, H^{\rm LO}_{\rm (ES_1^2)S_1}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm (ES_1^2)S_2}\}_x \\ &\quad + \{2\vec{G}^{\rm NLO}_{\rm S_1S_2}, H^{\rm LO}_{\rm ES_1^2}\}_x + \{\vec{G}^{\rm NLO}_{\rm ES_1^2}, 2H^{\rm LO}_{\rm S_1S_2}\}_x.\end{aligned} \tag{4.31}$$

We solve for $\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}{\rm S}_1^3{\rm S}_2}$ from:

$$0=\{\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}{\rm S}_1^3{\rm S}_2},H_{\rm N}\}_x+\{\vec{G}_{\rm N},H^{\rm NLO}_{C^2_{\rm ES_1^2}{\rm S}_1^3{\rm S}_2}\}_x. \tag{4.32}$$

We solve for $\vec{G}^{\rm NLO}_{({\rm BS}_1^3){\rm S}_2}$ from:

$$\begin{aligned}0=\;&\{\vec{G}^{\rm NLO}_{({\rm BS}_1^3){\rm S}_2},H_{\rm N}\}_x+\{\vec{G}_{\rm N},H^{\rm NLO}_{({\rm BS}_1^3){\rm S}_2}\}_x+\{\vec{G}^{\rm LO}_{{\rm S}_2},H^{\rm LO}_{{\rm BS}_1^3}\}_x+\{\vec{G}_{\rm 1PN},H^{\rm LO}_{({\rm BS}_1^3){\rm S}_2}\}_x\\&+\{\vec{G}^{\rm LO}_{{\rm S}_2},H^{\rm LO}_{{\rm BS}_1^3{\rm S}_2}\}_S+\{\vec{G}^{\rm LO}_{{\rm S}_1},H^{\rm LO}_{{\rm BS}_1^3{\rm S}_2}\}_S.\end{aligned} \tag{4.33}$$

We solve for $\vec{G}^{\rm NLO}_{{\rm S}_1^2{\rm S}_2^2}$ from:

$$\begin{aligned}0=\;&\{2\vec{G}^{\rm NLO}_{{\rm S}_1^2{\rm S}_2^2},H_{\rm N}\}_x+\{\vec{G}_{\rm N},2H^{\rm NLO}_{{\rm S}_1^2{\rm S}_2^2}\}_x+\{\vec{G}^{\rm LO}_{{\rm S}_2},H^{\rm LO}_{{\rm S}_1^2{\rm S}_2}\}_x+\{\vec{G}^{\rm LO}_{{\rm S}_1},H^{\rm LO}_{{\rm S}_1{\rm S}_2^2}\}_x\\&+\{2\vec{G}^{\rm NLO}_{{\rm S}_1{\rm S}_2},2H^{\rm LO}_{{\rm S}_1{\rm S}_2}\}_x.\end{aligned} \tag{4.34}$$

We solve for $\vec{G}^{\rm NLO}_{({\rm ES}_1^2){\rm S}_2^2}$ from:

$$0=\{\vec{G}^{\rm NLO}_{({\rm ES}_1^2){\rm S}_2^2},H_{\rm N}\}_x+\{\vec{G}_{\rm N},H^{\rm NLO}_{({\rm ES}_1^2){\rm S}_2^2}\}_x+\{\vec{G}^{\rm LO}_{{\rm S}_2},H^{\rm LO}_{({\rm ES}_1^2){\rm S}_2}\}_x+\{\vec{G}^{\rm NLO}_{{\rm S}_2^2},H^{\rm LO}_{{\rm ES}_1^2}\}_x. \tag{4.35}$$

We solve for $\vec{G}^{\rm NLO}_{({\rm ES}_1^2)({\rm ES}_2^2)}$ from:

$$\begin{aligned}0=\;&\{2\vec{G}^{\rm NLO}_{({\rm ES}_1^2)({\rm ES}_2^2)},H_{\rm N}\}_x+\{\vec{G}_{\rm N},2H^{\rm NLO}_{({\rm ES}_1^2)({\rm ES}_2^2)}\}_x+\{\vec{G}_{\rm 1PN},2H^{\rm LO}_{({\rm ES}_1^2)({\rm ES}_2^2)}\}_x\\&+\{\vec{G}^{\rm NLO}_{{\rm ES}_2^2},H^{\rm LO}_{{\rm ES}_1^2}\}_x+\{\vec{G}^{\rm NLO}_{{\rm ES}_1^2},H^{\rm LO}_{{\rm ES}_2^2}\}_x+\{\vec{G}^{\rm LO}_{{\rm S}_2},2H^{\rm LO}_{({\rm ES}_1^2)({\rm ES}_2^2)}\}_S+\{\vec{G}^{\rm LO}_{{\rm S}_1},2H^{\rm LO}_{({\rm ES}_1^2)({\rm ES}_2^2)}\}_S.\end{aligned} \tag{4.36}$$

We then write the solution of $\vec{G}^{\rm NLO}_{{\rm S}^4}$ as:

$$\vec{G}^{\rm NLO}_{{\rm S}^4}=H^{\rm LO}_{{\rm S}^4}\frac{\vec{x}_1+\vec{x}_2}{2}+\vec{Y}^{\rm NLO}_{{\rm S}^4}, \tag{4.37}$$

with:

$$\begin{aligned}\vec{Y}^{\rm NLO}_{{\rm S}^4}=&-\frac{GC_{1{\rm ES}^4}m_2}{4{m_1}^3r^4}\Big[3\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_1-5(\vec{S}_1\cdot\vec{n})^3\vec{S}_1\Big]-\frac{GC_{1{\rm BS}^3}m_2}{2{m_1}^3r^4}\Big[5S_1^2\vec{n}(\vec{S}_1\cdot\vec{n})^2-\vec{n}S_1^4\\&+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_1-5(\vec{S}_1\cdot\vec{n})^3\vec{S}_1\Big]+\frac{3GC_{1{\rm ES}^2}m_2}{4{m_1}^3r^4}\Big[5S_1^2\vec{n}(\vec{S}_1\cdot\vec{n})^2-\vec{n}S_1^4+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_1\\&-5(\vec{S}_1\cdot\vec{n})^3\vec{S}_1\Big]\\&-\frac{3G}{2{m_1}^2r^4}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\vec{n}-5\vec{S}_1\cdot\vec{n}S_1^2\vec{n}\vec{S}_2\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1+5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\\&+\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\Big]-\frac{GC_{1{\rm BS}^3}}{4{m_1}^2r^4}\Big[2S_1^2\vec{S}_1\cdot\vec{S}_2\vec{n}-10\vec{S}_1\cdot\vec{S}_2\vec{n}(\vec{S}_1\cdot\vec{n})^2-S_1^2\vec{S}_2\cdot\vec{n}\vec{S}_1\\&+2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1+5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\vec{S}_1+3\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2-5(\vec{S}_1\cdot\vec{n})^3\vec{S}_2\Big]\\&+\frac{3GC_{1{\rm ES}^2}}{2{m_1}^2r^4}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\vec{n}-5\vec{S}_1\cdot\vec{S}_2\vec{n}(\vec{S}_1\cdot\vec{n})^2+2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1-3\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\\&+5(\vec{S}_1\cdot\vec{n})^3\vec{S}_2\Big]\end{aligned}$$

$$-\frac{3GC_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}}{4m_1m_2r^4}\Big[5S_1^2\vec{n}(\vec{S}_2\cdot\vec{n})^2+2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_1+3\vec{S}_1\cdot\vec{n}S_2^2\vec{S}_1$$
$$-5\vec{S}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2\vec{S}_1\Big]-\frac{3GC_{1\mathrm{ES}^2}}{4m_1m_2r^4}\Big[S_1^2S_2^2\vec{n}-5S_2^2\vec{n}(\vec{S}_1\cdot\vec{n})^2+2\vec{S}_1\cdot\vec{n}S_2^2\vec{S}_1$$
$$-S_1^2\vec{S}_2\cdot\vec{n}\vec{S}_2-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{S}_2+5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\vec{S}_2\Big]+(1\leftrightarrow2)\,. \tag{4.38}$$

Thus we found a solution for the Poincaré algebra of the NLO quartic-in-spin sectors, with our new general Hamiltonian from eq. (3.8), and this provides significant confidence in the validity of these new results.

4.2 N^{3}LO quadratic-in-spin sectors

To confirm the new precision frontier at the 5PN order across all sectors, we proceed to solve for the Poincaré algebra of the N^{3}LO quadratic-in-spin sectors. Again, we start by decomposing H and $\vec{G}$ to all the sectors relevant to the solution for $\vec{G}$ at the N^{3}LO quadratic-in-spin sectors in question. The Hamiltonian consists of:

$$H = H_{\mathrm{N}}+H_{\mathrm{1PN}}+H_{\mathrm{SO}}^{\mathrm{LO}}+H_{\mathrm{2PN}}+H_{\mathrm{S}^2}^{\mathrm{LO}}+H_{\mathrm{SO}}^{\mathrm{NLO}}+H_{\mathrm{3PN}}+H_{\mathrm{S}^2}^{\mathrm{NLO}}$$
$$+H_{\mathrm{SO}}^{\mathrm{N^2LO}}+H_{\mathrm{S}^2}^{\mathrm{N^2LO}}+H_{\mathrm{SO}}^{\mathrm{N^3LO}}+H_{\mathrm{S}^2}^{\mathrm{N^3LO}}, \tag{4.39}$$

with $H_{\mathrm{SO}}^{\mathrm{N^3LO}}$ taken from our [21], and $H_{\mathrm{S}^2}^{\mathrm{N^3LO}}$ from our [36], and we have:

$$H_{\mathrm{SO}}^{\mathrm{LO}} = H_{\mathrm{S}_1}^{\mathrm{LO}}+(1\leftrightarrow2),\qquad H_{\mathrm{SO}}^{\mathrm{NLO}}=H_{\mathrm{S}_1}^{\mathrm{NLO}}+(1\leftrightarrow2), \tag{4.40}$$
$$H_{\mathrm{SO}}^{\mathrm{N^2LO}} = H_{\mathrm{S}_1}^{\mathrm{N^2LO}}+(1\leftrightarrow2),\qquad H_{\mathrm{SO}}^{\mathrm{N^3LO}}=H_{\mathrm{S}_1}^{\mathrm{N^3LO}}+(1\leftrightarrow2), \tag{4.41}$$
$$H_{\mathrm{S}^2}^{\mathrm{LO}} = C_{1\mathrm{ES}^2}H_{\mathrm{ES}_1^2}^{\mathrm{LO}}+H_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{LO}}+(1\leftrightarrow2), \tag{4.42}$$
$$H_{\mathrm{S}^2}^{\mathrm{NLO}} = H_{\mathrm{S}_1^2}^{\mathrm{NLO}}+C_{1\mathrm{ES}^2}H_{\mathrm{ES}_1^2}^{\mathrm{NLO}}+H_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{NLO}}+(1\leftrightarrow2), \tag{4.43}$$
$$H_{\mathrm{S}^2}^{\mathrm{N^2LO}} = H_{\mathrm{S}_1^2}^{\mathrm{N^2LO}}+C_{1\mathrm{ES}^2}H_{\mathrm{ES}_1^2}^{\mathrm{N^2LO}}+H_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{N^2LO}}+(1\leftrightarrow2), \tag{4.44}$$
$$H_{\mathrm{S}^2}^{\mathrm{N^3LO}} = H_{\mathrm{S}_1^2}^{\mathrm{N^3LO}}+C_{1\mathrm{ES}^2}H_{\mathrm{ES}_1^2}^{\mathrm{N^3LO}}+C_{1\mathrm{E}^2\mathrm{S}^2}H_{\mathrm{E}^2\mathrm{S}_1^2}^{\mathrm{N^3LO}}+H_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{N^3LO}}+(1\leftrightarrow2). \tag{4.45}$$

For the COM generator we have:

$$\vec{G} = \vec{G}_{\mathrm{N}}+\vec{G}_{\mathrm{1PN}}+\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}+\vec{G}_{\mathrm{2PN}}+\vec{G}_{\mathrm{SO}}^{\mathrm{NLO}}+\vec{G}_{\mathrm{3PN}}+\vec{G}_{\mathrm{S}^2}^{\mathrm{NLO}}+\vec{G}_{\mathrm{SO}}^{\mathrm{N^2LO}}+\vec{G}_{\mathrm{S}^2}^{\mathrm{N^2LO}}$$
$$+\vec{G}_{\mathrm{SO}}^{\mathrm{N^3LO}}+\vec{G}_{\mathrm{S}^2}^{\mathrm{N^3LO}}, \tag{4.46}$$

where we used again that all generators of LO sectors with spin vanish beyond spin-orbit, and with:

$$\vec{G}_{\mathrm{SO}}^{\mathrm{LO}} = \vec{G}_{\mathrm{S}_1}^{\mathrm{LO}}+(1\leftrightarrow2),\qquad \vec{G}_{\mathrm{SO}}^{\mathrm{NLO}}=\vec{G}_{\mathrm{S}_1}^{\mathrm{NLO}}+(1\leftrightarrow2), \tag{4.47}$$
$$\vec{G}_{\mathrm{SO}}^{\mathrm{N^2LO}} = \vec{G}_{\mathrm{S}_1}^{\mathrm{N^2LO}}+(1\leftrightarrow2),\qquad \vec{G}_{\mathrm{SO}}^{\mathrm{N^3LO}}=\vec{G}_{\mathrm{S}_1}^{\mathrm{N^3LO}}+(1\leftrightarrow2), \tag{4.48}$$
$$\vec{G}_{\mathrm{S}^2}^{\mathrm{NLO}} = \vec{G}_{\mathrm{S}_1^2}^{\mathrm{NLO}}+C_{1\mathrm{ES}^2}\vec{G}_{\mathrm{ES}_1^2}^{\mathrm{NLO}}+\vec{G}_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{NLO}}+(1\leftrightarrow2), \tag{4.49}$$
$$\vec{G}_{\mathrm{S}^2}^{\mathrm{N^2LO}} = \vec{G}_{\mathrm{S}_1^2}^{\mathrm{N^2LO}}+C_{1\mathrm{ES}^2}\vec{G}_{\mathrm{ES}_1^2}^{\mathrm{N^2LO}}+\vec{G}_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{N^2LO}}+(1\leftrightarrow2), \tag{4.50}$$

where we solved for $\vec{G}_{\mathrm{SO}}^{\mathrm{N^3LO}}$ in [22], and now we should solve for:

$$\vec{G}_{\mathrm{S}^2}^{\mathrm{N^3LO}} = \vec{G}_{\mathrm{S}_1^2}^{\mathrm{N^3LO}}+C_{1\mathrm{ES}^2}\vec{G}_{\mathrm{ES}_1^2}^{\mathrm{N^3LO}}+C_{1\mathrm{E}^2\mathrm{S}^2}\vec{G}_{\mathrm{E}^2\mathrm{S}_1^2}^{\mathrm{N^3LO}}+\vec{G}_{\mathrm{S}_1\mathrm{S}_2}^{\mathrm{N^3LO}}+(1\leftrightarrow2). \tag{4.51}$$

Let us then list the set of decoupled equations to solve for the generators of each sector.

We solve for $\vec{G}^{\text{N}^3\text{LO}}_{\text{S}_1^2}$ from:

$$0 = \{\vec{G}^{\text{N}^3\text{LO}}_{\text{S}_1^2}, H_{\text{N}}\}_x + \{\vec{G}_{\text{N}}, H^{\text{N}^3\text{LO}}_{\text{S}_1^2}\}_x + \{\vec{G}^{\text{LO}}_{\text{S}_1}, H^{\text{N}^2\text{LO}}_{\text{S}_1}\}_x + \{\vec{G}_{1\text{PN}}, H^{\text{N}^2\text{LO}}_{\text{S}_1^2}\}_x$$
$$+\{\vec{G}^{\text{NLO}}_{\text{S}_1}, H^{\text{NLO}}_{\text{S}_1}\}_x + \{\vec{G}^{\text{NLO}}_{\text{S}_1^2}, H^{\text{N}^2\text{LO}}_{2\text{PN}}\}_x + \{\vec{G}^{\text{N}^2\text{LO}}_{2\text{PN}}, H^{\text{NLO}}_{\text{S}_1^2}\}_x + \{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_1}, H^{\text{LO}}_{\text{S}_1}\}_x$$
$$+\{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_1^2}, H_{1\text{PN}}\}_x + \{\vec{G}^{\text{LO}}_{\text{S}_1}, H^{\text{N}^2\text{LO}}_{\text{S}_1^2}\}_S + \{\vec{G}^{\text{NLO}}_{\text{S}_1}, H^{\text{NLO}}_{\text{S}_1^2}\}_S + \{\vec{G}^{\text{NLO}}_{\text{S}_1^2}, H^{\text{NLO}}_{\text{S}_1}\}_S$$
$$+\{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_1}, H^{\text{LO}}_{\text{S}_1^2}\}_S + \{\vec{G}^{\text{LO}}_{\text{S}_1^2}, H^{\text{N}^2\text{LO}}_{\text{S}_1}\}_S. \tag{4.52}$$

We solve for $\vec{G}^{\text{N}^3\text{LO}}_{\text{ES}_1^2}$ from:

$$0 = \{\vec{G}^{\text{N}^3\text{LO}}_{\text{ES}_1^2}, H_{\text{N}}\}_x + \{\vec{G}_{\text{N}}, H^{\text{N}^3\text{LO}}_{\text{ES}_1^2}\}_x + \{\vec{G}_{1\text{PN}}, H^{\text{N}^2\text{LO}}_{\text{ES}_1^2}\}_x + \{\vec{G}^{\text{NLO}}_{\text{ES}_1^2}, H^{\text{N}^2\text{LO}}_{2\text{PN}}\}_x$$
$$+\{\vec{G}^{\text{N}^2\text{LO}}_{2\text{PN}}, H^{\text{NLO}}_{\text{ES}_1^2}\}_x + \{\vec{G}^{\text{N}^2\text{LO}}_{\text{ES}_1^2}, H_{1\text{PN}}\}_x + \{\vec{G}^{\text{N}^3\text{LO}}_{3\text{PN}}, H^{\text{LO}}_{\text{ES}_1^2}\}_x + \{\vec{G}^{\text{LO}}_{\text{S}_1}, H^{\text{N}^2\text{LO}}_{\text{ES}_1^2}\}_S$$
$$+\{\vec{G}^{\text{NLO}}_{\text{S}_1}, H^{\text{NLO}}_{\text{ES}_1^2}\}_S + \{\vec{G}^{\text{NLO}}_{\text{ES}_1^2}, H^{\text{NLO}}_{\text{S}_1}\}_S + \{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_1}, H^{\text{LO}}_{\text{ES}_1^2}\}_S + \{\vec{G}^{\text{LO}}_{\text{ES}_1^2}, H^{\text{N}^2\text{LO}}_{\text{S}_1}\}_S. \tag{4.53}$$

We solve for $\vec{G}^{\text{N}^3\text{LO}}_{\text{E}^2\text{S}_1^2}$ from:

$$0 = \{\vec{G}^{\text{N}^3\text{LO}}_{\text{E}^2\text{S}_1^2}, H_{\text{N}}\}_x + \{\vec{G}_{\text{N}}, H^{\text{N}^3\text{LO}}_{\text{E}^2\text{S}_1^2}\}_x. \tag{4.54}$$

We solve for $\vec{G}^{\text{N}^3\text{LO}}_{\text{S}_1\text{S}_2}$ from:

$$0 = \{\vec{G}^{\text{N}^3\text{LO}}_{\text{S}_1\text{S}_2}, H_{\text{N}}\}_x + \{\vec{G}_{\text{N}}, H^{\text{N}^3\text{LO}}_{\text{S}_1\text{S}_2}\}_x + \{\vec{G}^{\text{LO}}_{\text{S}_2}, H^{\text{N}^2\text{LO}}_{\text{S}_1}\}_x + \{\vec{G}_{1\text{PN}}, H^{\text{N}^2\text{LO}}_{\text{S}_1\text{S}_2}\}_x$$
$$+\{\vec{G}^{\text{NLO}}_{\text{S}_2}, H^{\text{NLO}}_{\text{S}_1}\}_x + \{\vec{G}^{\text{NLO}}_{\text{S}_1\text{S}_2}, H^{\text{N}^2\text{LO}}_{2\text{PN}}\}_x + \{\vec{G}^{\text{N}^2\text{LO}}_{2\text{PN}}, H^{\text{NLO}}_{\text{S}_1\text{S}_2}\}_x + \{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_2}, H^{\text{LO}}_{\text{S}_1}\}_x$$
$$+\{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_1\text{S}_2}, H_{1\text{PN}}\}_x + \{\vec{G}^{\text{N}^3\text{LO}}_{\text{PN}}, H^{\text{LO}}_{\text{S}_1\text{S}_2}\}_x + \{\vec{G}^{\text{LO}}_{\text{S}_2}, 2H^{\text{N}^2\text{LO}}_{\text{S}_1\text{S}_2}\}_S + \{\vec{G}^{\text{NLO}}_{\text{S}_2}, 2H^{\text{NLO}}_{\text{S}_1\text{S}_2}\}_S$$
$$+\{2\vec{G}^{\text{NLO}}_{\text{S}_1\text{S}_2}, H^{\text{NLO}}_{\text{S}_2}\}_S + \{\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_2}, 2H^{\text{LO}}_{\text{S}_1\text{S}_2}\}_S + \{2\vec{G}^{\text{N}^2\text{LO}}_{\text{S}_1\text{S}_2}, H^{\text{LO}}_{\text{S}_2}\}_S + (1 \leftrightarrow 2)\,. \tag{4.55}$$

We then write the solution of $\vec{G}^{\text{N}^3\text{LO}}_{\text{S}^2}$ as:

$$\vec{G}^{\text{N}^3\text{LO}}_{\text{S}^2} = H^{\text{N}^2\text{LO}}_{\text{S}^2}\frac{\vec{x}_1 + \vec{x}_2}{2} + \left(\vec{Y}^{\text{N}^3\text{LO}}_{\text{S}_1^2} + C_{1\text{ES}^2}\vec{Y}^{\text{N}^3\text{LO}}_{\text{ES}_1^2} + \vec{Y}^{\text{N}^3\text{LO}}_{\text{S}_1\text{S}_2} + (1 \leftrightarrow 2)\right), \tag{4.56}$$

where we present the explicit expressions in appendix D.

Thus, we solved for the Poincaré algebra of the N^{3}LO quadratic-in-spin sectors, with our full general Hamiltonians first presented in [36], and this provides significant confidence in the validity of these results. It should be highlighted that though an agreement with the consequent observables in specific restricted configurations, which we first provided in [38], was found later in [39], the Poincaré algebra which we solved for here, provides the most stringent check on the most comprehensive Hamiltonian results. This then completes the Poincaré algebra across all sectors with spins at the new precision frontier at the 5PN order.

5 Specialized Hamiltonians and observables

Obtaining Hamiltonians is essential to derive EOMs, EOB models, or as we have shown, to study the global Poincaré algebra of the binary system. However, Hamiltonians are also

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critical to get sensible binding energies, which are key to GW applications. To that end, we gradually simplify our general Hamiltonians from section 3.2, by considering specific kinematic configurations, and eventually removing all gauge-dependence to get physical observables. First, we define some binary-mass conventions:

$$m \equiv m_1 + m_2, \qquad q \equiv m_1/m_2, \tag{5.1}$$
$$\mu \equiv m_1 m_2/m, \qquad \nu \equiv m_1 m_2/m^2 = \mu/m = q/(1+q)^2, \tag{5.2}$$

where q and ν are the dimensionless mass ratio and symmetric mass-ratio, respectively, that are used to express the final observables. All the variables are also converted to dimensionless ones, denoted with a tilde, where Gm and μ are the length and mass units, respectively. The general Hamiltonians are then specified to the COM frame, reducing to a single momentum vector, $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$, with which the orbital angular momentum is defined, $\vec{L} \equiv r\vec{n} \times \vec{p}$.

The NLO quartic-in-spin Hamiltonians in the COM frame are then:

$$\begin{aligned}\tilde{H}^{\text{NLO}}_{\text{S}^4} = {} & C_{1\text{ES}^2}\tilde{H}^{\text{NLO}}_{(\text{ES}_1^2)\text{S}_1^2} + C_{1\text{ES}^2}^2\tilde{H}^{\text{NLO}}_{(\text{ES}_1^2)^2} + C_{1\text{BS}^3}\tilde{H}^{\text{NLO}}_{(\text{BS}_1^3)\text{S}_1} + C_{1\text{ES}^4}\tilde{H}^{\text{NLO}}_{\text{ES}_1^4} + C_{1\text{E}^2\text{S}^4}\tilde{H}^{\text{NLO}}_{\text{E}^2\text{S}_1^4} \\ & +\tilde{H}^{\text{NLO}}_{\text{S}_1^3\text{S}_2} + C_{1\text{ES}^2}\tilde{H}^{\text{NLO}}_{(\text{ES}_1^2)\text{S}_1\text{S}_2} + C_{1\text{ES}^2}^2\tilde{H}^{\text{NLO}}_{C^2_{\text{ES}_1^2}\text{S}_1^3\text{S}_2} + C_{1\text{BS}^3}\tilde{H}^{\text{NLO}}_{(\text{BS}_1^3)\text{S}_2} \\ & +\tilde{H}^{\text{NLO}}_{\text{S}_1^2\text{S}_2^2} + C_{1\text{ES}^2}\tilde{H}^{\text{NLO}}_{(\text{ES}_1^2)\text{S}_2^2} + C_{1\text{ES}^2}C_{2\text{ES}^2}\tilde{H}^{\text{NLO}}_{(\text{ES}_1^2)(\text{ES}_2^2)} + (1 \leftrightarrow 2),\end{aligned} \tag{5.3}$$

where we present the explicit expressions in appendix E. Very few scattering-amplitudes approaches to the scattering problem also present analogous Hamiltonians for the bound problem, though only for low loop orders, as explained in section 1 and illustrated in table 1, due to radiation-reaction effects that kick in as of the N^{2}LO. In any case, those analogous Hamiltonians are already restricted to the COM frame [42, 43].

As noted in [22, 42] and [43] differ between them in several ways already at the level of their physical scattering amplitudes, which underlie their subsequent COM Hamiltonians for BHs, and for generic compact objects, respectively. The results in [42] have been known to be discrepant with ours as of our verified cubic-in-spin orders, which was already clarified in [22], and at the present quartic-in-spin sectors.

As was also clarified in [22], the COM Hamiltonians presented in [43] for generic compact binaries, have been known to be discrepant even for BHs, with our results as of the LO cubic-in-spin sectors in [30], which have been well-verified along the years. This discrepancy is due to singularities that show up as of cubic order in spin on [43]. At the NLO the COM Hamiltonians in [43] for the quartic-in-spin sectors exhibit a growing discrepancy, including similar singularities even for BHs. Moreover, at the NLO quartic-in-spin sectors the work in [43] contained contributions with 3 claimed new Wilson coefficients, H_2, H_3, and H_4, which they stipulated in their formulation. Such extra free parameters violate spin-gauge invariance [24], and are also absent and discrepant with other corresponding physical scattering amplitudes, e.g. in [42]. Moreover, there was an inconsistent omission in [43] of the new subleading spin-induced hexadecapular deformation operator, as noted in section 2, and thus the consequent contribution is also missing in their results.

A further simplification is obtained by constraining the spins to be aligned with the orbital angular momentum, thus also requiring $\vec{S}_a \cdot \vec{n} = \vec{S}_a \cdot \vec{p} = 0$. However, this simplification is notably inappropriate at higher-spin sectors, as it entails a growing loss of information on the physics of the system — the higher in spin the sectors are — as of quadratic order in the spin. For the present quartic-in-spin sectors, the aligned-spins simplification leads to the following total Hamiltonian:

$$
\begin{aligned}
\tilde{H}^{\text{NLO}}_{\text{S}^4} =\ & \frac{\nu^2 \tilde{S}_1^4}{\tilde{r}^6}\Biggl[C_{1(\text{ES}^2)}\biggl(\frac{3}{2} - \frac{3\nu}{2} - \frac{3\nu}{4}\frac{\tilde{L}^2}{\tilde{r}} + \tilde{p}_r^2\tilde{r}\left(\frac{45}{16} - \frac{21\nu}{8}\right) \\
& + \frac{1}{\nu q}\biggl(-\frac{3\nu^2}{2} + \frac{9\nu}{2} - \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{4} - \frac{3\nu^2}{2}\right) + \tilde{p}_r^2\tilde{r}\left(-\frac{39\nu^2}{16} + \frac{33\nu}{4} - \frac{45}{16}\right)\biggr)\biggr) \\
& + C^2_{1(\text{ES}^2)}\left(-\frac{\nu}{8} + \frac{1}{q}\left(\frac{1}{8} - \frac{\nu}{8}\right)\right) + C_{1(\text{BS}^3)}\biggl(-\frac{\nu}{2} - \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{2} - \frac{3}{2}\right) \\
& + \tilde{p}_r^2\tilde{r}\left(\frac{3\nu}{2} - \frac{3}{2}\right) + \frac{1}{\nu q}\biggl(-\frac{\nu^2}{2} - \frac{5\nu}{2} + \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu^2}{2} - \frac{9\nu}{2} + \frac{3}{2}\right) \\
& + \tilde{p}_r^2\tilde{r}\left(\frac{3\nu^2}{2} - \frac{9\nu}{2} + \frac{3}{2}\right)\biggr)\biggr) + C_{1(\text{ES}^4)}\biggl(\frac{5\nu}{2} - \frac{23}{8} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{16} + \frac{9}{16}\right) \\
& + \tilde{p}_r^2\tilde{r}\left(\frac{9\nu}{8} + \frac{9}{16}\right) + \frac{1}{\nu q}\biggl(\frac{5\nu^2}{2} - \frac{33\nu}{4} + \frac{23}{8} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu^2}{8} + \frac{15\nu}{16} - \frac{9}{16}\right) \\
& + \tilde{p}_r^2\tilde{r}\left(\frac{9\nu^2}{4} - \frac{9}{16}\right)\biggr)\biggr) + \frac{27}{35}C_{1(\text{E}^2\text{S}^4)}\left(\frac{1}{8} - \frac{\nu}{8} + \frac{1}{\nu q}\left(-\frac{\nu^2}{8} + \frac{3\nu}{8} - \frac{1}{8}\right)\right)\Biggr] \\
& + \frac{\nu^3\tilde{S}_1^3\tilde{S}_2}{\tilde{r}^6}\Biggl[3 + \frac{45}{8}\tilde{p}_r^2\tilde{r} + \frac{1}{\nu q}\left(3\nu - 3 + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{2}\right) + \tilde{p}_r^2\tilde{r}\left(\frac{21\nu}{4} - \frac{45}{8}\right)\right) \\
& + C_{1(\text{ES}^2)}\biggl(\frac{5}{2} - \frac{9}{2}\frac{\tilde{L}^2}{\tilde{r}} - \frac{9}{2}\tilde{p}_r^2\tilde{r} + \frac{1}{\nu q}\biggl(\frac{5\nu}{2} + 6 + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9}{2} - 6\nu\right) \\
& + \tilde{p}_r^2\tilde{r}\left(\frac{9}{2} - \frac{33\nu}{8}\right)\biggr)\biggr) + C_{1(\text{BS}^3)}\biggl(-6 + \frac{3}{2}\frac{\tilde{L}^2}{\tilde{r}} + \frac{3}{2}\tilde{p}_r^2\tilde{r} \\
& + \frac{1}{\nu q}\left(\frac{23}{2} - 6\nu + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{4} - \frac{13}{4}\right) + \tilde{p}_r^2\tilde{r}\left(\frac{7}{4} - 3\nu\right)\right)\biggr)\Biggr] \\
& + \frac{\nu^2\tilde{S}_1^2\tilde{S}_2^2}{2\tilde{r}^6}\Biggl[3 - 3\nu\frac{\tilde{L}^2}{\tilde{r}} + \frac{3\nu}{4}\tilde{p}_r^2\tilde{r} + 2C_{1(\text{ES}^2)}\biggl(4\nu + 3 + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9}{2} - \frac{15\nu}{4}\right) \\
& + \tilde{p}_r^2\tilde{r}\left(\frac{27}{16} - \frac{15\nu}{8}\right) + \frac{\nu}{q}\left(4 - \frac{9}{2}\frac{\tilde{L}^2}{\tilde{r}} - \frac{27}{16}\tilde{p}_r^2\tilde{r}\right)\biggr) \\
& + C_{1(\text{ES}^2)}C_{2(\text{ES}^2)}\left(9 + \frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{9\nu}{8} - \frac{33}{8}\right) + \tilde{p}_r^2\tilde{r}\left(-\frac{27\nu}{4} - \frac{3}{8}\right)\right)\Biggr] + (1 \leftrightarrow 2).
\end{aligned}
\tag{5.4}
$$

Due to the aforementioned loss of information in the aligned-spins configuration, we see that similar to what happens at the NLO cubic-in-sectors, the unique sector in the potentials and in eq. (E.9), that is proportional to both $C^2_{1\text{ES}^2}$ and S_2, vanishes here, and thus it is absent in all the common observables that assume this simplified restriction.

Finally, in the long quasi-circular inspiral phase, it is reasonable to assume the circular-orbit condition, $p_r \equiv \vec{p}\cdot\vec{n} = 0 \Rightarrow p^2 = p_r^2 + L^2/r^2 \to L^2/r^2$. Subjecting our aligned-spins

Hamiltonian to this condition then gives rise to:

$$\begin{aligned}
\tilde{H}^{\rm NLO}_{\rm S^4} =\ & \frac{\nu^2\tilde{S}_1^4}{\tilde{r}^6}\left[C_{1({\rm ES}^2)}\left(\frac{3}{2}-\frac{3\nu}{2}-\frac{3\nu}{4}\frac{\tilde{L}^2}{\tilde{r}}+\frac{1}{\nu q}\left(-\frac{3\nu^2}{2}+\frac{9\nu}{2}-\frac{3}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{4}-\frac{3\nu^2}{2}\right)\right)\right)\right.\\
&+C^2_{1({\rm ES}^2)}\left(-\frac{\nu}{8}+\frac{1}{q}\left(\frac{1}{8}-\frac{\nu}{8}\right)\right)+C_{1({\rm BS}^3)}\left(-\frac{\nu}{2}-\frac{3}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{2}-\frac{3}{2}\right)\right.\\
&\left.+\frac{1}{\nu q}\left(-\frac{\nu^2}{2}-\frac{5\nu}{2}+\frac{3}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu^2}{2}-\frac{9\nu}{2}+\frac{3}{2}\right)\right)\right)\\
&+C_{1({\rm ES}^4)}\left(\frac{5\nu}{2}-\frac{23}{8}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{16}+\frac{9}{16}\right)+\frac{1}{\nu q}\left(\frac{5\nu^2}{2}-\frac{33\nu}{4}+\frac{23}{8}\right.\right.\\
&\left.\left.\left.+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu^2}{8}+\frac{15\nu}{16}-\frac{9}{16}\right)\right)\right)+\frac{27}{35}C_{1({\rm E}^2{\rm S}^4)}\left(\frac{1}{8}-\frac{\nu}{8}+\frac{1}{\nu q}\left(-\frac{\nu^2}{8}+\frac{3\nu}{8}-\frac{1}{8}\right)\right)\right]\\
&+\frac{\nu^3\tilde{S}_1^3\tilde{S}_2}{\tilde{r}^6}\left[3+\frac{1}{\nu q}\left(3\nu-3+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{2}\right)\right)+C_{1({\rm ES}^2)}\left(\frac{5}{2}-\frac{9}{2}\frac{\tilde{L}^2}{\tilde{r}}+\frac{1}{\nu q}\left(\frac{5\nu}{2}+6\right.\right.\right.\\
&\left.\left.\left.+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9}{2}-6\nu\right)\right)\right)+C_{1({\rm BS}^3)}\left(-6+\frac{3}{2}\frac{\tilde{L}^2}{\tilde{r}}+\frac{1}{\nu q}\left(\frac{23}{2}-6\nu+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3\nu}{4}-\frac{13}{4}\right)\right)\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2\tilde{S}_2^2}{2\tilde{r}^6}\left[3-3\nu\frac{\tilde{L}^2}{\tilde{r}}+2C_{1({\rm ES}^2)}\left(4\nu+3+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9}{2}-\frac{15\nu}{4}\right)+\frac{\nu}{q}\left(4-\frac{9}{2}\frac{\tilde{L}^2}{\tilde{r}}\right)\right)\right.\\
&\left.+C_{1({\rm ES}^2)}C_{2({\rm ES}^2)}\left(9+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{9\nu}{8}-\frac{33}{8}\right)\right)\right]+(1\leftrightarrow 2). \qquad (5.5)
\end{aligned}$$

5.1 Gravitational-wave observables

Now we can define the binding energies associated with the simplified Hamiltonians above as $e\equiv\tilde{H}$, and relate them to the emitted frequencies of GWs measured in the LIGO, Virgo, or KAGRA experiments. To that end, we need to supplement the circular-orbit condition of constant coordinate-separation with Hamilton's equation, $\dot{p}_r=-\partial\tilde{H}(\tilde{r},\tilde{L})/\partial\tilde{r}=0$, which then removes completely the coordinate dependence from the simplified Hamiltonian in eq. (5.5), and relates, to begin with, the binding energy with the angular momentum:

$$\begin{aligned}
(e)^{\rm NLO}_{\rm S^4}(\tilde{L}) =\ & \frac{\nu^2\tilde{S}_1^4}{\tilde{L}^{12}}\left[\left(\left(\frac{285\nu}{4}+\frac{483}{8}\right)C_{1{\rm ES}^2}+\left(\frac{85\nu}{16}+\frac{417}{16}\right)C^2_{1{\rm ES}^2}\right.\right.\\
&+\left(\frac{17\nu}{2}+\frac{39}{2}\right)C_{1{\rm BS}^3}+\left(\frac{43\nu}{16}+\frac{83}{16}\right)C_{1{\rm ES}^4}+\left(\frac{1}{8}-\frac{\nu}{8}\right)\frac{27}{35}C_{1{\rm E}^2{\rm S}^4}\\
&+\frac{1}{\nu q}\left(\left(\frac{423\nu^2}{8}+\frac{99\nu}{2}-\frac{483}{8}\right)C_{1{\rm ES}^2}+\left(\frac{11\nu^2}{2}+\frac{749\nu}{16}-\frac{417}{16}\right)C^2_{1{\rm ES}^2}\right.\\
&+\left(\frac{17\nu^2}{2}+\frac{61\nu}{2}-\frac{39}{2}\right)C_{1{\rm BS}^3}+\left(\frac{23\nu^2}{8}+\frac{123\nu}{16}-\frac{83}{16}\right)C_{1{\rm ES}^4}\\
&\left.\left.+\left(-\frac{\nu^2}{8}+\frac{3\nu}{8}-\frac{1}{8}\right)\frac{27}{35}C_{1{\rm E}^2{\rm S}^4}\right)\right]+\frac{\nu^3\tilde{S}_1^3\tilde{S}_2}{\tilde{L}^{12}}\left[-\frac{717}{4}-35C_{1{\rm ES}^2}+3C_{1{\rm BS}^3}\right.\\
&\left.+\frac{1}{\nu q}\left(-\frac{285\nu}{2}-\frac{483}{4}+\left(-\frac{145\nu}{2}-\frac{1455}{4}\right)C_{1{\rm ES}^2}+\left(\frac{9\nu}{4}-\frac{207}{4}\right)C_{1{\rm BS}^3}\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2\tilde{S}_2^2}{2\tilde{L}^{12}}\left[-\frac{297\nu}{4}-\frac{2025}{4}+\left(-\frac{383\nu}{2}-\frac{951}{4}\right)C_{1{\rm ES}^2}\right.
\end{aligned}$$

$$+\left(-\frac{3\nu}{2}-\frac{195}{2}\right)C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}-\frac{913}{4}\frac{\nu}{q}C_{1\mathrm{ES}^2}\Big]+(1\leftrightarrow 2). \tag{5.6}$$

To relate to the gauge-invariant frequency, we invoke Hamilton's equation for the orbital phase, $d\phi/d\tilde{t}\equiv\tilde{\omega}=\partial\tilde{H}(\tilde{r},\tilde{L})/\partial\tilde{L}=0$, and define the PN parameter, $x\equiv\tilde{\omega}^{2/3}$. From this we get the angular momentum as a function of GW frequency:

$$\begin{aligned}\frac{1}{\tilde{L}^2}\supset\ &\nu^2x^6\tilde{S}_1^4\Bigg[\left(30\nu+\frac{381}{2}\right)C_{1\mathrm{ES}^2}+\left(\frac{99}{4}-\frac{125\nu}{4}\right)C^2_{1\mathrm{ES}^2}+(-7\nu-69)\,C_{1\mathrm{BS}^3}\\
&+\left(\frac{61\nu}{4}-\frac{59}{4}\right)C_{1\mathrm{ES}^4}+(1-\nu)\,\frac{27}{35}C_{1\mathrm{E}^2\mathrm{S}^4}+\frac{1}{\nu q}\Bigg(\left(\frac{61\nu^2}{2}+351\nu-\frac{381}{2}\right)C_{1\mathrm{ES}^2}\\
&+\left(-\frac{47\nu^2}{2}+\frac{323\nu}{4}-\frac{99}{4}\right)C^2_{1\mathrm{ES}^2}+\left(-7\nu^2-131\nu+69\right)C_{1\mathrm{BS}^3}\\
&+\left(\frac{21\nu^2}{2}-\frac{179\nu}{4}+\frac{59}{4}\right)C_{1\mathrm{ES}^4}+\left(-\nu^2+3\nu-1\right)\frac{27}{35}C_{1\mathrm{E}^2\mathrm{S}^4}\Bigg)\Bigg]\\
&+\nu^3x^6\tilde{S}_1^3\tilde{S}_2\Big[-59+105C_{1\mathrm{ES}^2}-51C_{1\mathrm{BS}^3}\\
&+\frac{1}{\nu q}\left(-60\nu-381+(75\nu-255)\,C_{1\mathrm{ES}^2}+(111-32\nu)\,C_{1\mathrm{BS}^3}\right)\Big]\\
&+\frac{1}{2}\nu^2x^6\tilde{S}_1^2\tilde{S}_2^2\Big[-29\nu-825+(33-2\nu)\,C_{1\mathrm{ES}^2}+(13\nu+5)\,C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}\\
&-\frac{\nu}{q}C_{1\mathrm{ES}^2}\Big]+(1\leftrightarrow 2).\end{aligned} \tag{5.7}$$

Finally, we can express the binding energy as a function of the GW frequency by combining the former two relations:

$$\begin{aligned}(e)^{\mathrm{NLO}}_{\mathrm{S}^4}(x)=\ &\nu^2x^6\tilde{S}_1^4\Bigg[\left(\frac{9\nu}{2}-9\right)C_{1\mathrm{ES}^2}+\left(\frac{135\nu}{16}-\frac{45}{16}\right)C^2_{1\mathrm{ES}^2}+(9-3\nu)\,C_{1\mathrm{BS}^3}\\
&+\left(\frac{21}{16}-\frac{99\nu}{16}\right)C_{1\mathrm{ES}^4}+\left(\frac{3\nu}{8}-\frac{3}{8}\right)\frac{27}{35}C_{1\mathrm{E}^2\mathrm{S}^4}+\frac{1}{\nu q}\Bigg(\left(6\nu^2-\frac{45\nu}{2}+9\right)C_{1\mathrm{ES}^2}\\
&+\left(\frac{33\nu^2}{4}-\frac{225\nu}{16}+\frac{45}{16}\right)C^2_{1\mathrm{ES}^2}+\left(-3\nu^2+21\nu-9\right)C_{1\mathrm{BS}^3}\\
&+\left(-\frac{39\nu^2}{8}+\frac{141\nu}{16}-\frac{21}{16}\right)C_{1\mathrm{ES}^4}+\left(\frac{3\nu^2}{8}-\frac{9\nu}{8}+\frac{3}{8}\right)\frac{27}{35}C_{1\mathrm{E}^2\mathrm{S}^4}\Bigg)\Bigg]\\
&+\nu^3x^6\tilde{S}_1^3\tilde{S}_2\Big[-6-\frac{21}{2}C_{1\mathrm{ES}^2}+\frac{27}{2}C_{1\mathrm{BS}^3}\\
&+\frac{1}{\nu q}\left(18-9\nu+\left(-\frac{27\nu}{4}-\frac{45}{4}\right)C_{1\mathrm{ES}^2}+\left(\frac{33\nu}{4}-\frac{9}{4}\right)C_{1\mathrm{BS}^3}\right)\Big]\\
&+\frac{1}{2}\nu^2x^6\tilde{S}_1^2\tilde{S}_2^2\Big[\frac{27\nu}{4}+\frac{81}{4}+(3\nu-36)\,C_{1\mathrm{ES}^2}+\left(\frac{33}{2}-\frac{15\nu}{2}\right)C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}\\
&+6\frac{\nu}{q}C_{1\mathrm{ES}^2}\Big]+(1\leftrightarrow 2).\end{aligned} \tag{5.8}$$

5.2 Extension to scattering angles

In a weak gravitational field 2-to-2 scattering events can be studied in a perturbative expansion in G, within the so-called post-Minkowskian (PM) approximation. In this setup the scattering angle for aligned spins in the COM frame can be derived, and only in low loop orders, as we explained in section 1, it can then be linked to an extrapolated quantity associated with the NR binary inspiral in some overlap with the PN approximation. At this point it should be highlighted that recently a unique novel approach was put forward in [64], which bypasses the need to link between scattering and inspiral, and is applicable to any loop order. This approach is formulated directly in the bound problem, and uses amplitudes methods to reach unprecedented state-of-the-art results in PN theory [64].

In the present sectors, we can extend our PN aligned-spins Hamiltonian in eq. (5.4) to a scattering energy function, and using general integration considerations [65], we can then compute here such extrapolated scattering angles as we detailed in [36]. We then only need to truncate the expansion in G according to the loop order in the PM approximation, which in this case is only one loop. Using identical notation to [36]:

$$p_\infty = \frac{m_1 m_2}{E}\sqrt{\gamma^2-1}, \qquad E = \sqrt{m_1^2+m_2^2+2m_1m_2\gamma}, \qquad \gamma = \frac{1}{\sqrt{1-v_\infty^2}}, \tag{5.9}$$

and:

$$\tilde{b} = \frac{v_\infty^2}{Gm}b, \qquad \tilde{v} = \frac{v_\infty}{c}, \qquad \tilde{a}_i = \frac{S_i}{bm_ic}, \qquad \Gamma = \frac{E}{mc^2} = \sqrt{1+2\nu(\gamma-1)}, \tag{5.10}$$

we then write the scattering angles in the NLO quartic-in-spin sectors as:

$$\theta_{\mathrm{S}^4} = \theta_{\mathrm{S}_1^4} + \theta_{\mathrm{S}_1^3\mathrm{S}_2} + \theta_{\mathrm{S}_1^2\mathrm{S}_2^2} + (1\leftrightarrow 2), \tag{5.11}$$

with

$$\begin{aligned}\frac{\theta_{\mathrm{S}_1^4}}{\Gamma} = \tilde{a}_1^4\Bigg[&\frac{1}{\tilde{b}}C_{1\mathrm{ES}^4}(2+2\tilde{v}^2) + \frac{\pi}{\tilde{b}^2}\Bigg(-\frac{75\nu}{16}\tilde{v}^2C_{1\mathrm{ES}^2} + \left(\frac{15}{16}+\left(\frac{105\nu}{64}+\frac{105}{32}\right)\tilde{v}^2\right)C_{1\mathrm{ES}^2}^2\\ &+\left(15-\frac{45\nu}{8}\right)\tilde{v}^2C_{1\mathrm{BS}^3} + \left(\frac{45}{16}+\left(\frac{315}{32}-\frac{75\nu}{16}\right)\tilde{v}^2\right)C_{1\mathrm{ES}^4} + \frac{81}{448}\nu\tilde{v}^2C_{1\mathrm{E}^2\mathrm{S}^4}\\ &+\frac{\nu}{q}\tilde{v}^2\left(-\frac{75}{16}C_{1\mathrm{ES}^2}+\frac{105}{64}C_{1\mathrm{ES}^2}^2-\frac{45}{8}C_{1\mathrm{BS}^3}-\frac{75}{16}C_{1\mathrm{ES}^4}+\frac{81}{448}C_{1\mathrm{E}^2\mathrm{S}^4}\right)\Bigg)\Bigg],\end{aligned} \tag{5.12}$$

$$\begin{aligned}\frac{\theta_{\mathrm{S}_1^3\mathrm{S}_2}}{\Gamma} = \tilde{a}_1^3\tilde{a}_2\Bigg[&\frac{1}{\tilde{b}}C_{1\mathrm{BS}^3}\left(8+8\tilde{v}^2\right) + \frac{\pi}{\tilde{b}^2}\Bigg(-\frac{75\nu}{8}\tilde{v}^2 + \left(\frac{15}{4}+\left(\frac{885}{16}-\frac{195\nu}{16}\right)\tilde{v}^2\right)C_{1\mathrm{ES}^2}\\ &+\left(\frac{45}{4}+\left(\frac{735}{16}-\frac{75\nu}{16}\right)\tilde{v}^2\right)C_{1\mathrm{BS}^3}\\ &+\frac{\nu}{q}\tilde{v}^2\left(-\frac{75}{8}-\frac{195}{16}C_{1\mathrm{ES}^2}-\frac{75}{16}C_{1\mathrm{BS}^3}\right)\Bigg)\Bigg],\end{aligned} \tag{5.13}$$

$$\begin{aligned}\frac{\theta_{\mathrm{S}_1^2\mathrm{S}_2^2}}{\Gamma} = \frac{1}{2}\tilde{a}_1^2\tilde{a}_2^2\Bigg[&\frac{1}{\tilde{b}}C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}\left(12+12\tilde{v}^2\right) + \frac{\pi}{\tilde{b}^2}\Bigg(\frac{15}{4}+\frac{165}{16}\tilde{v}^2\\ &+2\left(\frac{315\nu}{16}+\frac{375}{16}\right)\tilde{v}^2C_{1\mathrm{ES}^2} + \left(\frac{75}{4}+\frac{225}{4}\tilde{v}^2\right)C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}\\ &+\frac{315}{8}\frac{\nu}{q}\tilde{v}^2C_{1\mathrm{ES}^2}\Bigg)\Bigg].\end{aligned} \tag{5.14}$$

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When all leading spin-induced multipolar Wilson coefficients are set to unity, as stipulated in [24] for BHs, namely $C_{\mathrm{ES}^2} = C_{\mathrm{BS}^3} = C_{\mathrm{ES}^4} = 1$, and depending on our conjecture 2 in section 2, and the related matching of the new Wilson coefficient $C_{\mathrm{E}^2\mathrm{S}^4}$ for BHs from full GR or real-world data, then our scattering angles may agree or not with those presented for BHs in [41]. In any case, as noted in [22] and in section 2, the derivations and results in Guevara et al. [41] are inherently dependent on our free-standing framework, as they relied on our worldline spin theory and results introduced in [24, 30], which was omitted in [41].

At this point it is important to reiterate that such limited observables provide but a very partial physical picture of the system, and even more so for higher-spin sectors, where the full information is found at the most general Hamiltonians first presented in section 3.2 above. This further highlights the indispensable comprehensive confirmation of the general Hamiltonians provided through the Poincaré algebra that we found in section 4 above.

6 Conclusions

We put forward a broader picture of the effective theory of a spinning particle, and in particular of Kerr BHs, within the EFT of spinning gravitating objects [24]. We also fully derived and established the new precision frontier at the 5PN order for GW measurements from inspirals and mergers of generic compact binaries. The 5PN precision frontier includes higher-spin sectors, quadratic and quartic in the spin, which both display novel physical effects, originating from the extension of the effective theory beyond linear order in the curvature. In the third subleading quadratic-in-spin sectors there is a new tidal effect, and in the quartic-in-spin sectors there is a new spin-induced multipolar effect. Using these observations, and with eyes towards the next precision frontier at the 6PN order, we generalized the concept of tidal and of spin-induced multipolar operators, and made conjectures on the numerical values of their Wilson coefficients for rotating BHs in GR in 4 spacetime dimensions.

We then confirmed the generalized actions for generic compact binaries of the complete NLO quartic-in-spin sectors, which were computed via the extension of the EFT of gravitating spinning objects in [40]. We derived for the first time the NLO quartic-in spin interaction potentials, that consist of no less than 12 distinct sectors, with a new one due to the new spin-induced multipolar operator that is quadratic in the curvature. We also derived for the first time the corresponding general Hamiltonians in an arbitrary reference frame. These Hamiltonians give the full physical information on the binary system, which mostly gets lost in higher-spin sectors, when going to common observables which assume an aligned-spins configuration. This is since generic spin orientations of the rotating components of the binary have an observational signature in the gravitational waveforms.

Moreover, the general Hamiltonians obtained exclusively via our framework uniquely enable to find the global Poincaré algebra, which we carried out successfully in all the sectors with spins that contribute at the present 5PN order. It should be noted that in order to accomplish that, we needed to significantly scale our approach to the solution of the Poincaré algebra. Thus we solved for the full Poincaré algebra of the third subleading quadratic-in-spin Hamiltonians first presented in [36], and of the NLO quartic-in-spin sectors computed in [40]. This solution of the Poincaré algebra also provides the most stringent consistency check to the validity of our new comprehensive state-of-the-art results.

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We proceeded to derive simplified Hamiltonians in restricted kinematic configurations, starting with the COM Hamiltonians. We then derived observables for GW applications, namely gauge-invariant relations among binding energies, angular momentum, and GW frequency. Furthermore, to make contact with the scattering problem, we also derived the extrapolated scattering angles, relevant only in the aligned-spins configuration. We could then specify our consequent scattering angles to the scattering of BHs, by fixing all leading spin-induced multipolar Wilson coefficients to 1, as we prescribed in [24]; and depending on our conjecture 2 in section 2, and the related matching of the new Wilson coefficient in the present NLO quartic-in-spin sectors, $C_{\mathrm{E}^2\mathrm{S}^4}$, for BHs from full GR or real-world data. In any case, as noted in [22] and in section 2, the derivations and results in the few overlapping scattering-amplitudes studies are inherently dependent on our free-standing framework, as they relied on our worldline spin theory and prior results introduced in [24, 30]. We reiterate however, that such limited scattering observables can provide but very partial physical input on the system, especially at higher-spin sectors. The solution of the Poincaré algebra on the other hand provides the strongest confirmation that the 5PN order has now been established as the new precision frontier.

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A Redefinition of rotational variables

As noted in section 3.1, the new redefinitions of the spin in the present sectors are fixed as:

$$\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S^4} = \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^4} + \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^3S_2} + \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^2S_2^2} + \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1S_2^3} - (i \leftrightarrow j), \tag{A.1}$$

where

$$\begin{aligned}
\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^4} = & -\frac{GC_{1\rm BS^3}m_2}{4{m_1}^3r^4}\Big[20S_1^2S_1^{ki}n^k\vec{v}_1\cdot\vec{n}n^j + 2S_1^2S_1^{ki}v_1^kn^j - 2S_1^2S_1^{kj}n^kv_1^i \\
& +8S_1^2S_1^{ij}\vec{v}_1\cdot\vec{n} - 10S_1^{ki}v_1^kn^j(\vec{S}_1\cdot\vec{n})^2 + 5S_1^{ij}\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 \\
& -30\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1n^jS_1^{ki}n^k - 13\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_1^{ij}\Big] \\
& +\frac{3GC_{1\rm ES^2}m_2}{8{m_1}^3r^4}\Big[20S_1^2S_1^{ki}n^k\vec{v}_1\cdot\vec{n}n^j + 5S_1^2S_1^{ki}v_1^kn^j - S_1^2S_1^{kj}n^kv_1^i \\
& +9S_1^2S_1^{ij}\vec{v}_1\cdot\vec{n} - 15S_1^{ki}v_1^kn^j(\vec{S}_1\cdot\vec{n})^2 + 5S_1^{kj}n^kv_1^i(\vec{S}_1\cdot\vec{n})^2 \\
& -20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1n^jS_1^{ki}n^k - 9\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_1^{ij}\Big] \\
& +\frac{GC_{1\rm ES^4}m_2}{8{m_1}^3r^4}\Big[10S_1^2S_1^{ki}n^k(\vec{v}_1\cdot\vec{n}n^j - 4\vec{v}_2\cdot\vec{n}n^j) - 2S_1^2S_1^{ki}v_1^kn^j \\
& -4S_1^2S_1^{ki}v_2^kn^j + 2S_1^2S_1^{kj}n^k(v_1^i + 2v_2^i) + S_1^2S_1^{ij}(\vec{v}_1\cdot\vec{n} - 19\vec{v}_2\cdot\vec{n}) \\
& +70S_1^{ki}n^k\vec{v}_2\cdot\vec{n}n^j(\vec{S}_1\cdot\vec{n})^2 + 10S_1^{ki}v_2^kn^j(\vec{S}_1\cdot\vec{n})^2 + 5S_1^{ij}(\vec{v}_1\cdot\vec{n}
\end{aligned}$$

$$+4\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+30\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2n^jS_1^{ki}n^k-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_1^{ij}$$
$$+11\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2S_1^{ij}\Big], \tag{A.2}$$

$$\begin{aligned}\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^3S_2} = &-\frac{GC_{\rm 1BS^3}}{8{m_1}^2r^4}\Big[20\vec{S}_1\cdot\vec{S}_2S_1^{ki}n^k(4\vec{v}_1\cdot\vec{n}n^j-3\vec{v}_2\cdot\vec{n}n^j)-10S_1^2S_2^{ki}n^k(2\vec{v}_1\cdot\vec{n}n^j\\
&-5\vec{v}_2\cdot\vec{n}n^j)-4\vec{S}_1\cdot\vec{v}_2S_1^{ki}S_2^kn^j-16\vec{S}_1\cdot\vec{S}_2S_1^{ki}v_1^kn^j+4S_1^2S_2^{ki}v_1^kn^j\\
&+24\vec{S}_1\cdot\vec{S}_2S_1^{ki}v_2^kn^j-10S_1^2S_2^{ki}v_2^kn^j+4\vec{S}_1\cdot\vec{S}_2S_1^{kj}n^k(4v_1^i-5v_2^i)\\
&-2S_1^2S_2^{kj}n^k(14v_1^i-11v_2^i)+4S_1^iS_1^{kj}S_2^k\vec{v}_2\cdot\vec{n}+4\vec{S}_1\cdot\vec{S}_2S_1^{ij}(5\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n})\\
&+4S_1^2S_2^{ij}(\vec{v}_1\cdot\vec{n}+3\vec{v}_2\cdot\vec{n})-210S_2^{ki}n^k\vec{v}_2\cdot\vec{n}n^j(\vec{S}_1\cdot\vec{n})^2+20S_2^{ki}v_2^kn^j(\vec{S}_1\cdot\vec{n})^2\\
&+10S_2^{ij}(2\vec{v}_1\cdot\vec{n}-11\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2-120\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1n^jS_1^{ki}n^k\\
&+40\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2n^jS_1^{ki}n^k+90\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2n^jS_1^{ki}n^k-20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2n^jS_2^{ki}n^k\\
&-10\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}n^jS_1^{ki}v_2^k-4\vec{S}_2\cdot\vec{n}S_1^iS_1^{kj}v_2^k+4\vec{S}_1\cdot\vec{n}S_1^iS_2^{kj}v_2^k\\
&-5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(4\vec{v}_1\cdot\vec{n}-9\vec{v}_2\cdot\vec{n})S_1^{ij}+4\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_1^{ij}-32\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1S_1^{ij}\\
&+14\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2S_1^{ij}+23\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2S_1^{ij}-8\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_2^{ij}\Big]\\
&-\frac{3GC_{\rm 1ES^2}}{4{m_1}^2r^4}\Big[3\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1S_1^in^j+20\vec{S}_1\cdot\vec{S}_2S_1^{ki}n^k\vec{v}_1\cdot\vec{n}n^j-3\vec{S}_1\cdot\vec{v}_1S_1^{ki}S_2^kn^j\\
&-3\vec{S}_1\cdot\vec{S}_2S_1^{ki}v_1^kn^j-3S_1^2S_2^{ki}v_1^kn^j+2\vec{S}_1\cdot\vec{S}_2S_1^{kj}n^kv_1^i-4S_1^2S_2^{kj}n^kv_1^i\\
&-4\vec{S}_1\cdot\vec{n}S_1^{kj}S_2^kv_1^i+4S_1^iS_1^{kj}S_2^k\vec{v}_1\cdot\vec{n}+2\vec{S}_1\cdot\vec{S}_2S_1^{ij}\vec{v}_1\cdot\vec{n}+3S_1^2S_2^{ij}\vec{v}_1\cdot\vec{n}\\
&-10\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1n^jS_1^i+20\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1n^jS_2^i\\
&-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1n^jS_1^{ki}n^k-10\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1n^jS_2^{ki}n^k-10\vec{S}_1\times\vec{n}\cdot\vec{v}_1S_2^iS_1^j\\
&-4\vec{S}_2\cdot\vec{v}_1S_1^iS_1^{kj}n^k+4\vec{S}_1\cdot\vec{v}_1S_1^iS_2^{kj}n^k+5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}S_1^{ij}\\
&-3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1S_1^{ij}-7\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_2^{ij}\Big]-\frac{3G}{8{m_1}^2r^4}\Big[2\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1S_1^in^j\\
&-20S_1^2S_2^{ki}n^k\vec{v}_1\cdot\vec{n}n^j-\vec{S}_1\cdot\vec{v}_1S_1^{ki}S_2^kn^j-13\vec{S}_1\cdot\vec{S}_2S_1^{ki}v_1^kn^j-S_1^2S_2^{ki}v_1^kn^j\\
&-\vec{S}_1\times\vec{n}\cdot\vec{S}_2S_1^jv_1^i+5\vec{S}_1\cdot\vec{S}_2S_1^{kj}n^kv_1^i-3S_1^2S_2^{kj}n^kv_1^i-\vec{S}_1\cdot\vec{n}S_1^{kj}S_2^kv_1^i\\
&-S_1^iS_1^{kj}S_2^k\vec{v}_1\cdot\vec{n}-5\vec{S}_1\cdot\vec{S}_2S_1^{ij}\vec{v}_1\cdot\vec{n}-7S_1^2S_2^{ij}\vec{v}_1\cdot\vec{n}+20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1n^jS_2^{ki}n^k\\
&+30\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}n^jS_1^{ki}v_1^k-10\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}v_1^iS_1^{kj}n^k-2\vec{S}_2\cdot\vec{v}_1S_1^iS_1^{kj}n^k\\
&+2\vec{S}_1\cdot\vec{v}_1S_1^iS_2^{kj}n^k+3\vec{S}_2\cdot\vec{n}S_1^iS_1^{kj}v_1^k-3\vec{S}_1\cdot\vec{n}S_1^iS_2^{kj}v_1^k+5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1S_1^{ij}\\
&+7\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_2^{ij}\Big]\\
&-\frac{GC_{\rm 1BS^3}}{24{m_1}^2r^3}\Big[6\vec{S}_1\cdot\dot{\vec{S}}_2S_1^{ki}n^kn^j+9S_1^2\dot{S}_2^{ki}n^kn^j-45\dot{S}_2^{ki}n^kn^j(\vec{S}_1\cdot\vec{n})^2\\
&+9\vec{S}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}S_1^{ij}+2\vec{S}_1\cdot\dot{\vec{S}}_2S_1^{ij}+4S_1^2\dot{S}_2^{ij}-21(\vec{S}_1\cdot\vec{n})^2\dot{S}_2^{ij}\Big],\end{aligned} \tag{A.3}$$

$$\begin{aligned}\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^2S_2^2} = &-\frac{3GC_{\rm 2ES^2}}{8m_1m_2r^4}\Big[\vec{S}_2\cdot\vec{v}_1S_1^{ki}S_2^kn^j+2S_2^2S_1^{ki}v_1^kn^j+\vec{S}_1\cdot\vec{S}_2S_2^{ki}v_1^kn^j\\
&-\vec{S}_1\times\vec{n}\cdot\vec{S}_2S_2^jv_1^i+2S_2^2S_1^{kj}n^kv_1^i-\vec{S}_1\cdot\vec{S}_2S_2^{kj}n^kv_1^i+\vec{S}_2\cdot\vec{n}S_1^{kj}S_2^kv_1^i\\
&+S_2^iS_1^{kj}S_2^k\vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2S_2^{ij}\vec{v}_1\cdot\vec{n}-5S_1^{ki}v_1^kn^j(\vec{S}_2\cdot\vec{n})^2-5S_1^{kj}n^kv_1^i(\vec{S}_2\cdot\vec{n})^2\end{aligned}$$

$$
\begin{aligned}
&-\vec{S}_2\cdot\vec{n}S_2^iS_1^{kj}v_1^k+\vec{S}_1\cdot\vec{n}S_2^iS_2^{kj}v_1^k-\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1S_2^{ij}\Big]\\
&+\frac{3GC_{1\mathrm{ES}^2}}{4m_1m_2r^4}\Big[2\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2S_2^in^j+10S_2^2S_1^{ki}n^k\vec{v}_2\cdot\vec{n}n^j+3\vec{S}_2\cdot\vec{v}_2S_1^{ki}S_2^kn^j\\
&-3S_2^2S_1^{ki}v_2^kn^j-\vec{S}_1\cdot\vec{S}_2S_2^{ki}v_2^kn^j+3S_2^2S_1^{kj}n^kv_2^i-3\vec{S}_1\cdot\vec{S}_2S_2^{kj}n^kv_2^i\\
&-3\vec{S}_2\cdot\vec{n}S_1^{kj}S_2^kv_2^i+4S_2^iS_1^{kj}S_2^k\vec{v}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{S}_2S_2^{ij}\vec{v}_2\cdot\vec{n}\\
&-10\vec{S}_2\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2n^jS_1^i+20\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2n^jS_2^i\\
&-10\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2n^jS_1^{ki}n^k-10\vec{S}_2\times\vec{n}\cdot\vec{v}_2S_2^iS_1^j-3\vec{S}_2\cdot\vec{v}_2S_2^iS_1^{kj}n^k\\
&+3\vec{S}_1\cdot\vec{v}_2S_2^iS_2^{kj}n^k-\vec{S}_2\cdot\vec{n}S_2^iS_1^{kj}v_2^k+\vec{S}_1\cdot\vec{n}S_2^iS_2^{kj}v_2^k-2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2S_2^{ij}\Big]\\
&+\frac{3GC_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}}{16m_1m_2r^4}\Big[20S_2^2S_1^{ki}n^k\vec{v}_2\cdot\vec{n}n^j-40\vec{S}_1\cdot\vec{S}_2S_2^{ki}n^k\vec{v}_2\cdot\vec{n}n^j\\
&+8S_2^2S_1^{ki}v_1^kn^j+16\vec{S}_1\cdot\vec{S}_2S_2^{ki}v_1^kn^j-20S_2^2S_1^{ki}v_2^kn^j-24\vec{S}_1\cdot\vec{S}_2S_2^{ki}v_2^kn^j\\
&-4S_2^2S_1^{kj}n^k(2v_1^i-5v_2^i)-8\vec{S}_1\cdot\vec{S}_2S_2^{kj}n^kv_2^i+8S_2^2S_1^{ij}(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})\\
&+8\vec{S}_1\cdot\vec{S}_2S_2^{ij}(\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n})+140S_1^{ki}n^k\vec{v}_2\cdot\vec{n}n^j(\vec{S}_2\cdot\vec{n})^2\\
&-20S_1^{ki}v_1^kn^j(\vec{S}_2\cdot\vec{n})^2+30S_1^{ki}v_2^kn^j(\vec{S}_2\cdot\vec{n})^2+10S_1^{kj}n^k(2v_1^i-3v_2^i)(\vec{S}_2\cdot\vec{n})^2\\
&-10S_1^{ij}(2\vec{v}_1\cdot\vec{n}-7\vec{v}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})^2-20\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2n^jS_1^{ki}n^k\\
&-20\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1n^jS_2^{ki}n^k+30\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2n^jS_2^{ki}n^k-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2n^jS_2^{ki}n^k\\
&-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}n^jS_2^{ki}v_1^k+30\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}n^jS_2^{ki}v_2^k+2\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2S_1^{ij}\\
&-5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(2\vec{v}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n})S_2^{ij}-2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1S_2^{ij}+3\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2S_2^{ij}\\
&-6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2S_2^{ij}\Big]\\
&-\frac{GC_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}}{8m_1m_2r^3}\Big[42\vec{S}_2\cdot\dot{\vec{S}}_2S_1^{ki}n^kn^j-6\vec{S}_1\cdot\dot{\vec{S}}_2S_2^{ki}n^kn^j-6\vec{S}_1\cdot\vec{S}_2\dot{S}_2^{ki}n^kn^j\\
&-30\vec{S}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}n^jS_1^{ki}n^k-18\vec{S}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}S_1^{ij}+16\vec{S}_2\cdot\dot{\vec{S}}_2S_1^{ij}\\
&-3\vec{S}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}S_2^{ij}-3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\dot{S}_2^{ij}\Big],
\end{aligned}\tag{A.4}
$$

$$
\begin{aligned}
\left(\omega_1^{ij}\right)^{\mathrm{NLO}}_{\mathrm{S_1S_2^3}}=&-\frac{3G}{4{m_2}^2r^4}\Big[10S_2^2S_2^{ki}n^k\vec{v}_2\cdot\vec{n}n^j-2S_2^2S_2^{ki}v_2^kn^j+3S_2^2S_2^{ij}\vec{v}_2\cdot\vec{n}\\
&-10\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2n^jS_2^{ki}n^k-3\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2S_2^{ij}\Big]+\frac{3GC_{2\mathrm{ES}^2}}{2{m_2}^2r^4}\Big[3S_2^2S_2^{ki}v_2^kn^j\\
&+S_2^2S_2^{ij}\vec{v}_2\cdot\vec{n}-5S_2^{ki}v_2^kn^j(\vec{S}_2\cdot\vec{n})^2-\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2S_2^{ij}\Big]\\
&-\frac{GC_{2\mathrm{BS}^3}}{8{m_2}^2r^4}\Big[30S_2^2S_2^{ki}n^k\vec{v}_2\cdot\vec{n}n^j-6S_2^2S_2^{ki}v_2^kn^j-2S_2^2S_2^{kj}n^k(4v_1^i-3v_2^i)\\
&+8S_2^2S_2^{ij}\vec{v}_2\cdot\vec{n}-70S_2^{ki}n^k\vec{v}_2\cdot\vec{n}n^j(\vec{S}_2\cdot\vec{n})^2+10S_2^{ki}v_2^kn^j(\vec{S}_2\cdot\vec{n})^2\\
&+5S_2^{ij}(4\vec{v}_1\cdot\vec{n}-5\vec{v}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})^2-40\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1n^jS_2^{ki}n^k\\
&+30\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2n^jS_2^{ki}n^k-20\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1S_2^{ij}+13\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2S_2^{ij}\Big]\\
&-\frac{GC_{2\mathrm{BS}^3}}{24{m_2}^2r^3}\Big[18\vec{S}_2\cdot\dot{\vec{S}}_2S_2^{ki}n^kn^j+9S_2^2\dot{S}_2^{ki}n^kn^j-45\dot{S}_2^{ki}n^kn^j(\vec{S}_2\cdot\vec{n})^2\\
&+3\vec{S}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}S_2^{ij}+8\vec{S}_2\cdot\dot{\vec{S}}_2S_2^{ij}+4S_2^2\dot{S}_2^{ij}-21(\vec{S}_2\cdot\vec{n})^2\dot{S}_2^{ij}\Big].
\end{aligned}\tag{A.5}
$$

B Effective actions

After the reduction in section 3.1 is done, we obtain the NLO quartic-in-spin potentials, made up of no less than 12 unique sectors:

$$
\begin{aligned}
V^{\rm NLO}_{\rm S^4} = {} & C_{\rm 1ES^2} V^{\rm NLO}_{\rm (ES_1^2)S_1^2} + C^2_{\rm 1ES^2} V^{\rm NLO}_{\rm (ES_1^2)^2} + C_{\rm 1BS^3} V^{\rm NLO}_{\rm (BS_1^3)S_1} + C_{\rm 1ES^4} V^{\rm NLO}_{\rm ES_1^4} + C_{\rm 1E^2S^4} V^{\rm NLO}_{\rm E^2S_1^4} \\
& + V^{\rm NLO}_{\rm S_1^3S_2} + C_{\rm 1ES^2} V^{\rm NLO}_{\rm (ES_1^2)S_1S_2} + C^2_{\rm 1ES^2} V^{\rm NLO}_{C^2_{\rm ES_1^2} \rm S_1^3S_2} + C_{\rm 1BS^3} V^{\rm NLO}_{\rm (BS_1^3)S_2} \\
& + V^{\rm NLO}_{\rm S_1^2S_2^2} + C_{\rm 1ES^2} V^{\rm NLO}_{\rm (ES_1^2)S_2^2} + C_{\rm 1ES^2} C_{\rm 2ES^2} V^{\rm NLO}_{\rm (ES_1^2)(ES_2^2)} + (1 \leftrightarrow 2),
\end{aligned}
\tag{B.1}
$$

where

$$
\begin{aligned}
V^{\rm NLO}_{\rm (ES_1^2)S_1^2} = {} & -\frac{3Gm_2}{16{m_1}^3r^5}\Big[10\vec{S}_1\cdot\vec{n}S_1^2(3\vec{v}_1\cdot\vec{n}+2\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1-40\vec{S}_1\cdot\vec{n}S_1^2\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2 \\
& -4S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-5S_1^2(3v_1^2+4\vec{v}_1\cdot\vec{v}_2-28\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+21(\vec{v}_1\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2 \\
& +70\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^3+35v_1^2(\vec{S}_1\cdot\vec{n})^4+60\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2 \\
& -30(\vec{S}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{v}_1)^2+(4\vec{v}_1\cdot\vec{v}_2-20\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+15(\vec{v}_1\cdot\vec{n})^2)S_1^4\Big] \\
& -\frac{3G^2{m_2}^2}{8{m_1}^3r^6}\Big[5(\vec{S}_1\cdot\vec{n})^4-6S_1^2(\vec{S}_1\cdot\vec{n})^2+S_1^4\Big],
\end{aligned}
\tag{B.2}
$$

$$
V^{\rm NLO}_{\rm (ES_1^2)^2} = -\frac{3G^2{m_2}^2}{2{m_1}^3r^6}\Big[5(\vec{S}_1\cdot\vec{n})^4-3S_1^2(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^2m_2}{8{m_1}^2r^6}\Big[9(\vec{S}_1\cdot\vec{n})^4-6S_1^2(\vec{S}_1\cdot\vec{n})^2+S_1^4\Big],
\tag{B.3}
$$

$$
\begin{aligned}
V^{\rm NLO}_{\rm (BS_1^3)S_1} = {} & \frac{Gm_2}{2{m_1}^3r^5}\Big[15\vec{S}_1\cdot\vec{n}S_1^2\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1-15\vec{S}_1\cdot\vec{n}S_1^2\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2+3S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2 \\
& -30S_1^2(v_1^2-\vec{v}_1\cdot\vec{v}_2)(\vec{S}_1\cdot\vec{n})^2-35\vec{S}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+35\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3 \\
& +35(v_1^2-\vec{v}_1\cdot\vec{v}_2)(\vec{S}_1\cdot\vec{n})^4-15\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2+15(\vec{S}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{v}_1)^2 \\
& -3S_1^2(\vec{S}_1\cdot\vec{v}_1)^2+3(v_1^2-\vec{v}_1\cdot\vec{v}_2)S_1^4\Big] \\
& +\frac{G^2m_2}{{m_1}^2r^6}\Big[5(\vec{S}_1\cdot\vec{n})^4-3S_1^2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
V^{\rm NLO}_{\rm ES_1^4} = {} & -\frac{Gm_2}{16{m_1}^3r^5}\Big[60\vec{S}_1\cdot\vec{n}S_1^2(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1-60\vec{S}_1\cdot\vec{n}S_1^2\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2 \\
& +12S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-30S_1^2(3v_1^2-7\vec{v}_1\cdot\vec{v}_2+3v_2^2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2 \\
& -140\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^3+140\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+35(3v_1^2 \\
& -7\vec{v}_1\cdot\vec{v}_2+3v_2^2-9\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^4-60\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2 \\
& +60(\vec{S}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{v}_1)^2-12S_1^2(\vec{S}_1\cdot\vec{v}_1)^2+3(3v_1^2-7\vec{v}_1\cdot\vec{v}_2 \\
& +3v_2^2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})S_1^4\Big] \\
& +\frac{G^2m_2}{8{m_1}^2r^6}\Big[35(\vec{S}_1\cdot\vec{n})^4-30S_1^2(\vec{S}_1\cdot\vec{n})^2+3S_1^4\Big]+\frac{G^2{m_2}^2}{8{m_1}^3r^6}\Big[285(\vec{S}_1\cdot\vec{n})^4 \\
& -240S_1^2(\vec{S}_1\cdot\vec{n})^2+23S_1^4\Big],
\end{aligned}
\tag{B.5}
$$

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$$
\begin{aligned}
V^{\mathrm{NLO}}_{\mathrm{E^2S_1^4}} = & -\frac{G^2{m_2}^2}{8{m_1}^3r^6}\Big[9(\vec{S}_1\cdot\vec{n})^4-\frac{54}{7}S_1^2(\vec{S}_1\cdot\vec{n})^2+\frac{27}{35}S_1^4\Big],
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
V^{\mathrm{NLO}}_{\mathrm{S_1^3S_2}} = & -\frac{3G}{8{m_1}^2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\big(4\vec{v}_1\cdot\vec{v}_2-20\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+15(\vec{v}_1\cdot\vec{n})^2\big)\\
& -5\vec{S}_1\cdot\vec{n}S_1^2\big(4\vec{v}_1\cdot\vec{v}_2-28\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+21(\vec{v}_1\cdot\vec{n})^2\big)\vec{S}_2\cdot\vec{n}+20S_1^2\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\\
& -20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2(\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1+10\vec{S}_1\cdot\vec{n}S_1^2(3\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n})\vec{S}_2\cdot\vec{v}_1\\
& -4S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1-20S_1^2\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
& -8\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+4S_1^2\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-20\vec{S}_1\cdot\vec{n}S_1^2\vec{v}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\\
& -15\vec{S}_1\cdot\vec{S}_2v_1^2(\vec{S}_1\cdot\vec{n})^2+70\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2\\
& +35\vec{S}_2\cdot\vec{n}v_1^2(\vec{S}_1\cdot\vec{n})^3-10\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2+20\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\\
& -20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{v}_1)^2+4\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{v}_1)^2\Big]\\
& +\frac{3G^2m_2}{4{m_1}^2r^6}\Big[4\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-S_1^2\vec{S}_1\cdot\vec{S}_2-5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+2\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
V^{\mathrm{NLO}}_{\mathrm{(ES_1^2)S_1S_2}} = & \frac{3G}{8{m_1}^2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\big(12v_1^2-8\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-15\vec{S}_1\cdot\vec{n}S_1^2\big(4v_1^2\\
& -3\vec{v}_1\cdot\vec{v}_2\big)\vec{S}_2\cdot\vec{n}+20S_1^2\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1+10\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\big(4\vec{v}_1\cdot\vec{n}\\
& -\vec{v}_2\cdot\vec{n}\big)\vec{S}_1\cdot\vec{v}_1+15\vec{S}_1\cdot\vec{n}S_1^2\vec{v}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1-4S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1\\
& -15S_1^2\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2-40\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\\
& -30\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+8\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{S}_2\big(12v_1^2\\
& -8\vec{v}_1\cdot\vec{v}_2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2-140\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
& +105\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_2\cdot\vec{n}\big(4v_1^2-3\vec{v}_1\cdot\vec{v}_2\big)(\vec{S}_1\cdot\vec{n})^3\\
& -35\vec{S}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+20\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{v}_1)^2\\
& -8\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{v}_1)^2\Big]\\
& +\frac{3G^2m_2}{{m_1}^2r^6}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2-3\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^2}{m_1r^6}\Big[15\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-4S_1^2\vec{S}_1\cdot\vec{S}_2\\
& -39\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+18\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{B.8}
$$

$$
V^{\mathrm{NLO}}_{C^2_{\mathrm{ES_1^2}}\mathrm{S_1^3S_2}} = \frac{9G^2m_2}{{m_1}^2r^6}\Big[\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3-\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\tag{B.9}
$$

$$
\begin{aligned}
V^{\mathrm{NLO}}_{\mathrm{(BS_1^3)S_2}} = & -\frac{G}{4{m_1}^2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\big(13v_1^2-23\vec{v}_1\cdot\vec{v}_2+7v_2^2+25\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-20(\vec{v}_1\cdot\vec{n})^2\\
& -20(\vec{v}_2\cdot\vec{n})^2\big)-15\vec{S}_1\cdot\vec{n}S_1^2\big(3v_1^2-5\vec{v}_1\cdot\vec{v}_2+v_2^2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)\vec{S}_2\cdot\vec{n}\\
& +25S_1^2\vec{S}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1-30\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1\\
& +15\vec{S}_1\cdot\vec{n}S_1^2(4\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n})\vec{S}_2\cdot\vec{v}_1-17S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1-5S_1^2\vec{S}_2\cdot\vec{n}\big(7\vec{v}_1\cdot\vec{n}\\
& -4\vec{v}_2\cdot\vec{n}\big)\vec{S}_1\cdot\vec{v}_2+10\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2(\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_2
\end{aligned}
$$

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$$
\begin{aligned}
&-90\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+2\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+13S_1^2\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&-15\vec{S}_1\cdot\vec{n}S_1^2(3\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})\vec{S}_2\cdot\vec{v}_2+11S_1^2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2-7S_1^2\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2\\
&-5\vec{S}_1\cdot\vec{S}_2(13v_1^2-23\vec{v}_1\cdot\vec{v}_2+7v_2^2+35\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-28(\vec{v}_1\cdot\vec{n})^2\\
&-28(\vec{v}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2-175\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2\\
&+35\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2(7\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_2\cdot\vec{n}(3v_1^2-5\vec{v}_1\cdot\vec{v}_2\\
&+v_2^2-9\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^3-35\vec{S}_2\cdot\vec{v}_1(4\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^3\\
&+35\vec{S}_2\cdot\vec{v}_2(3\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^3+85\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2\\
&-65\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2-55\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\\
&+50\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{v}_1)^2-2\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{v}_1)^2+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^2}{m_1r^6}\Big[19\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-4S_1^2\vec{S}_1\cdot\vec{S}_2-45\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+20\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{G^2m_2}{2{m_1}^2r^6}\Big[105\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-19S_1^2\vec{S}_1\cdot\vec{S}_2-240\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3\\
&+96\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
V^{\rm NLO}_{\rm S_1^2S_2^2}=&-\frac{3G}{8m_1m_2r^5}\Big[S_1^2S_2^2(9\vec{v}_1\cdot\vec{v}_2-10\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})\\
&+5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(9\vec{v}_1\cdot\vec{v}_2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{S}_2-10\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{v}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\\
&+20\vec{S}_1\cdot\vec{n}S_2^2\vec{v}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1+10S_1^2\vec{S}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1-30\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{v}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\\
&-18S_2^2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-30\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+13\vec{S}_1\cdot\vec{S}_2\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&-10\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2+9\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\\
&+70\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-10S_1^2\vec{v}_1\cdot\vec{v}_2(\vec{S}_2\cdot\vec{n})^2-35\vec{v}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2\\
&-(13\vec{v}_1\cdot\vec{v}_2-15\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{S}_2)^2+10\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_2\cdot\vec{n})^2\Big]\\
&-\frac{6G^2}{m_1r^6}\Big[5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-5(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2-(\vec{S}_1\cdot\vec{S}_2)^2\Big]\\
&-\frac{6G^2}{m_2r^6}S_1^2(\vec{S}_2\cdot\vec{n})^2,
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
V^{\rm NLO}_{\rm (ES_1^2)S_2^2}=&-\frac{3G}{16m_1m_2r^5}\Big[S_1^2S_2^2(92\vec{v}_1\cdot\vec{v}_2-80v_2^2-100\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+85(\vec{v}_2\cdot\vec{n})^2)\\
&-80\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{v}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1+40\vec{S}_1\cdot\vec{n}S_2^2\vec{v}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1+120S_1^2\vec{S}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\\
&-20\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2(4\vec{v}_1\cdot\vec{n}-7\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_2+40\vec{S}_1\cdot\vec{n}S_2^2(\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_2\\
&-72S_2^2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+64\vec{S}_1\cdot\vec{S}_2\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+10S_1^2\vec{S}_2\cdot\vec{n}(10\vec{v}_1\cdot\vec{n}\\
&-19\vec{v}_2\cdot\vec{n})\vec{S}_2\cdot\vec{v}_2-40\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})\vec{S}_2\cdot\vec{v}_2\\
&-40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2+72\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2-92S_1^2\vec{S}_2\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\\
&+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2-120\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2-5S_2^2(28\vec{v}_1\cdot\vec{v}_2\\
&-24v_2^2-28\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+21(\vec{v}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2-280\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2
\end{aligned}
$$

$$
\begin{aligned}
&-70\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2(2\vec{v}_1\cdot\vec{n}-5\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+140\vec{S}_2\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\\
&-15S_1^2(8\vec{v}_1\cdot\vec{v}_2-7v_2^2)(\vec{S}_2\cdot\vec{n})^2+35(8\vec{v}_1\cdot\vec{v}_2-7v_2^2)(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2\\
&-2(32\vec{v}_1\cdot\vec{v}_2-28v_2^2-40\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+35(\vec{v}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{S}_2)^2\\
&+80\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_2\cdot\vec{n})^2-70(\vec{S}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{v}_2)^2+64S_2^2(\vec{S}_1\cdot\vec{v}_2)^2\\
&-120(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{v}_2)^2+80S_1^2(\vec{S}_2\cdot\vec{v}_2)^2\Big]\\
&+\frac{3G^2}{8m_2r^6}\Big[2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-S_1^2S_2^2+3S_2^2(\vec{S}_1\cdot\vec{n})^2-5(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2\\
&+S_1^2(\vec{S}_2\cdot\vec{n})^2\Big]-\frac{G^2}{2m_1r^6}\Big[30\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-5S_1^2S_2^2+12S_2^2(\vec{S}_1\cdot\vec{n})^2\\
&-75(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+15S_1^2(\vec{S}_2\cdot\vec{n})^2+3(\vec{S}_1\cdot\vec{S}_2)^2\Big],
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
V^{\rm NLO}_{(\mathrm{ES}_1^2)(\mathrm{ES}_2^2)}=&-\frac{3G}{16m_1m_2r^5}\Big[S_1^2S_2^2(10v_1^2-11\vec{v}_1\cdot\vec{v}_2+15\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-20(\vec{v}_1\cdot\vec{n})^2)\\
&-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(6v_1^2-7\vec{v}_1\cdot\vec{v}_2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{S}_2+40\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2(\vec{v}_1\cdot\vec{n}\\
&-\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1+20\vec{S}_1\cdot\vec{n}S_2^2(\vec{v}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n})\vec{S}_1\cdot\vec{v}_1-20S_1^2\vec{S}_2\cdot\vec{n}(4\vec{v}_1\cdot\vec{n}\\
&-\vec{v}_2\cdot\vec{n})\vec{S}_2\cdot\vec{v}_1-40\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{v}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1\\
&-8\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1-4S_2^2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&+4\vec{S}_1\cdot\vec{S}_2\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2+4\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\\
&-30S_2^2v_1^2(\vec{S}_1\cdot\vec{n})^2+140\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-10S_1^2(5v_1^2-9\vec{v}_1\cdot\vec{v}_2\\
&+7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-14(\vec{v}_1\cdot\vec{n})^2)(\vec{S}_2\cdot\vec{n})^2-140\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})^2\\
&+35(6v_1^2-7\vec{v}_1\cdot\vec{v}_2-9\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+2(6v_1^2\\
&-7\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{S}_2)^2-20\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(\vec{S}_2\cdot\vec{n})^2\\
&+20(\vec{S}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{v}_1)^2-4S_2^2(\vec{S}_1\cdot\vec{v}_1)^2+8S_1^2(\vec{S}_2\cdot\vec{v}_1)^2\Big]\\
&-\frac{3G^2}{2m_1r^6}\Big[32\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-3S_1^2S_2^2-62(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+12S_1^2(\vec{S}_2\cdot\vec{n})^2\\
&-3(\vec{S}_1\cdot\vec{S}_2)^2\Big]-\frac{15G^2}{m_2r^6}S_1^2(\vec{S}_2\cdot\vec{n})^2.
\end{aligned}
\tag{B.13}
$$

C General Hamiltonians

As noted in section 3.2, our general Hamiltonians for the NLO quartic-in-spin sectors, similar to the action potentials in eq. (3.7), consist of 12 unique sectors:

$$
\begin{aligned}
H^{\rm NLO}_{\rm S^4}=&\;C_{1\mathrm{ES}^2}H^{\rm NLO}_{(\mathrm{ES}_1^2)\mathrm{S}_1^2}+C^2_{1\mathrm{ES}^2}H^{\rm NLO}_{(\mathrm{ES}_1^2)^2}+C_{1\mathrm{BS}^3}H^{\rm NLO}_{(\mathrm{BS}_1^3)\mathrm{S}_1}+C_{1\mathrm{ES}^4}H^{\rm NLO}_{\mathrm{ES}_1^4}+C_{1\mathrm{E}^2\mathrm{S}^4}H^{\rm NLO}_{\mathrm{E}^2\mathrm{S}_1^4}\\
&+H^{\rm NLO}_{\mathrm{S}_1^3\mathrm{S}_2}+C_{1\mathrm{ES}^2}H^{\rm NLO}_{(\mathrm{ES}_1^2)\mathrm{S}_1\mathrm{S}_2}+C^2_{1\mathrm{ES}^2}H^{\rm NLO}_{C^2_{\mathrm{ES}_1^2}\mathrm{S}_1^3\mathrm{S}_2}+C_{1\mathrm{BS}^3}H^{\rm NLO}_{(\mathrm{BS}_1^3)\mathrm{S}_2}\\
&+H^{\rm NLO}_{\mathrm{S}_1^2\mathrm{S}_2^2}+C_{1\mathrm{ES}^2}H^{\rm NLO}_{(\mathrm{ES}_1^2)\mathrm{S}_2^2}+C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}H^{\rm NLO}_{(\mathrm{ES}_1^2)(\mathrm{ES}_2^2)}+(1\leftrightarrow 2),
\end{aligned}
\tag{C.1}
$$

with:

$$
H^{\rm NLO}_{(\mathrm{ES}_1^2)\mathrm{S}_1^2}=-\frac{3G}{4{m_1}^4r^5}\Big[5\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}S_1^2-10\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}S_1^2-\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1S_1^2
$$

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$$-5S_1^2\big(\vec p_1\cdot\vec p_2-7\vec p_1\cdot\vec n\vec p_2\cdot\vec n\big)(\vec S_1\cdot\vec n)^2-35\vec p_1\cdot\vec S_1\vec p_2\cdot\vec n(\vec S_1\cdot\vec n)^3$$
$$+15\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1(\vec S_1\cdot\vec n)^2+\big(\vec p_1\cdot\vec p_2-5\vec p_1\cdot\vec n\vec p_2\cdot\vec n\big)S_1^4\Big]$$
$$-\frac{15Gm_2}{16{m_1}^5r^5}\Big[6\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\vec p_1\cdot\vec nS_1^2-3S_1^2\big(p_1^2+7(\vec p_1\cdot\vec n)^2\big)(\vec S_1\cdot\vec n)^2$$
$$+14\vec p_1\cdot\vec S_1\vec p_1\cdot\vec n(\vec S_1\cdot\vec n)^3+7p_1^2(\vec S_1\cdot\vec n)^4-6(\vec S_1\cdot\vec n)^2(\vec p_1\cdot\vec S_1)^2+3(\vec p_1\cdot\vec n)^2S_1^4\Big]$$
$$-\frac{3G^2{m_2}^2}{2{m_1}^3r^6}\Big[5(\vec S_1\cdot\vec n)^4-6S_1^2(\vec S_1\cdot\vec n)^2+S_1^4\Big], \tag{C.2}$$

$$H^{\rm NLO}_{({\rm ES}_1^2)^2}=-\frac{3G^2{m_2}^2}{2{m_1}^3r^6}\Big[5(\vec S_1\cdot\vec n)^4-3S_1^2(\vec S_1\cdot\vec n)^2\Big]+\frac{G^2m_2}{8{m_1}^2r^6}\Big[9(\vec S_1\cdot\vec n)^4-6S_1^2(\vec S_1\cdot\vec n)^2+S_1^4\Big], \tag{C.3}$$

$$H^{\rm NLO}_{({\rm BS}_1^3){\rm S}_1}=\frac{Gm_2}{2{m_1}^5r^5}\Big[15\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\vec p_1\cdot\vec nS_1^2-30S_1^2p_1^2(\vec S_1\cdot\vec n)^2-35\vec p_1\cdot\vec S_1\vec p_1\cdot\vec n(\vec S_1\cdot\vec n)^3$$
$$+35p_1^2(\vec S_1\cdot\vec n)^4+15(\vec S_1\cdot\vec n)^2(\vec p_1\cdot\vec S_1)^2-3S_1^2(\vec p_1\cdot\vec S_1)^2+3p_1^2S_1^4\Big]$$
$$-\frac{G}{2{m_1}^4r^5}\Big[15\vec S_1\cdot\vec n\vec p_2\cdot\vec S_1\vec p_1\cdot\vec nS_1^2-3\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1S_1^2-30S_1^2\vec p_1\cdot\vec p_2(\vec S_1\cdot\vec n)^2$$
$$-35\vec p_2\cdot\vec S_1\vec p_1\cdot\vec n(\vec S_1\cdot\vec n)^3+35\vec p_1\cdot\vec p_2(\vec S_1\cdot\vec n)^4+15\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1(\vec S_1\cdot\vec n)^2$$
$$+3\vec p_1\cdot\vec p_2S_1^4\Big]$$
$$+\frac{3G^2{m_2}^2}{2{m_1}^3r^6}\Big[5(\vec S_1\cdot\vec n)^4-6S_1^2(\vec S_1\cdot\vec n)^2+S_1^4\Big]+\frac{G^2m_2}{{m_1}^2r^6}\Big[15(\vec S_1\cdot\vec n)^4-15S_1^2(\vec S_1\cdot\vec n)^2$$
$$+2S_1^4\Big], \tag{C.4}$$

$$H^{\rm NLO}_{{\rm ES}_1^4}=\frac{G}{16{m_1}^4r^5}\Big[60\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\vec p_2\cdot\vec nS_1^2+60\vec S_1\cdot\vec n\vec p_2\cdot\vec S_1\vec p_1\cdot\vec nS_1^2-12\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1S_1^2$$
$$-210S_1^2\big(\vec p_1\cdot\vec p_2+\vec p_1\cdot\vec n\vec p_2\cdot\vec n\big)(\vec S_1\cdot\vec n)^2-140\vec p_1\cdot\vec S_1\vec p_2\cdot\vec n(\vec S_1\cdot\vec n)^3$$
$$-140\vec p_2\cdot\vec S_1\vec p_1\cdot\vec n(\vec S_1\cdot\vec n)^3+35\big(7\vec p_1\cdot\vec p_2+9\vec p_1\cdot\vec n\vec p_2\cdot\vec n\big)(\vec S_1\cdot\vec n)^4$$
$$+60\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1(\vec S_1\cdot\vec n)^2+3\big(7\vec p_1\cdot\vec p_2+5\vec p_1\cdot\vec n\vec p_2\cdot\vec n\big)S_1^4\Big]$$
$$-\frac{Gm_2}{16{m_1}^5r^5}\Big[60\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\vec p_1\cdot\vec nS_1^2-90S_1^2p_1^2(\vec S_1\cdot\vec n)^2-140\vec p_1\cdot\vec S_1\vec p_1\cdot\vec n(\vec S_1\cdot\vec n)^3$$
$$+105p_1^2(\vec S_1\cdot\vec n)^4+60(\vec S_1\cdot\vec n)^2(\vec p_1\cdot\vec S_1)^2-12S_1^2(\vec p_1\cdot\vec S_1)^2+9p_1^2S_1^4\Big]$$
$$+\frac{3G}{16{m_1}^3m_2r^5}\Big[30S_1^2p_2^2(\vec S_1\cdot\vec n)^2-35p_2^2(\vec S_1\cdot\vec n)^4-3p_2^2S_1^4\Big]$$
$$+\frac{G^2m_2}{8{m_1}^2r^6}\Big[35(\vec S_1\cdot\vec n)^4-30S_1^2(\vec S_1\cdot\vec n)^2+3S_1^4\Big]+\frac{G^2{m_2}^2}{8{m_1}^3r^6}\Big[285(\vec S_1\cdot\vec n)^4$$
$$-240S_1^2(\vec S_1\cdot\vec n)^2+23S_1^4\Big], \tag{C.5}$$

$$H^{\rm NLO}_{{\rm E}^2{\rm S}_1^4}=-\frac{G^2{m_2}^2}{8{m_1}^3r^6}\Big[9(\vec S_1\cdot\vec n)^4-\frac{54}{7}S_1^2(\vec S_1\cdot\vec n)^2+\frac{27}{35}S_1^4\Big], \tag{C.6}$$

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$$
\begin{aligned}
H^{\rm NLO}_{\rm S_1^3S_2} =&-\frac{3G}{8{m_1}^4r^5}\Big[15S_1^2\vec{S}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{n})^2-105\vec{S}_1\cdot\vec{n}S_1^2(\vec{p}_1\cdot\vec{n})^2\vec{S}_2\cdot\vec{n}\\
&+20\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}+30\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2-4\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{S}_2\\
&-20\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2+70\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-15\vec{S}_1\cdot\vec{S}_2p_1^2(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_2\cdot\vec{n}p_1^2(\vec{S}_1\cdot\vec{n})^3-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2\\
&-10\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2+4\vec{S}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&-\frac{3G}{2{m_1}^3m_2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\\
&-5\vec{S}_1\cdot\vec{n}S_1^2(\vec{p}_1\cdot\vec{p}_2-7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\vec{S}_2\cdot\vec{n}-5\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\\
&+10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}-5\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2+\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{S}_2\\
&-5\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2+10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{S}_1\cdot\vec{S}_2\\
&-35\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2+5\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{3G^2m_2}{{m_1}^2r^6}\Big[4\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-S_1^2\vec{S}_1\cdot\vec{S}_2-5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+2\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
H^{\rm NLO}_{\rm (ES_1^2)S_1S_2} =&\frac{3G}{2{m_1}^4r^5}\Big[3S_1^2\vec{S}_1\cdot\vec{S}_2p_1^2-15\vec{S}_1\cdot\vec{n}S_1^2p_1^2\vec{S}_2\cdot\vec{n}+5\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\\
&-\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{S}_2+10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-35\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-15\vec{S}_1\cdot\vec{S}_2p_1^2(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_2\cdot\vec{n}p_1^2(\vec{S}_1\cdot\vec{n})^3+10\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2\\
&+5\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2-2\vec{S}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{S}_1)^2\Big]-\frac{3G}{8{m_1}^3m_2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2(8\vec{p}_1\cdot\vec{p}_2\\
&+5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})-45\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_1\cdot\vec{p}_2\vec{S}_2\cdot\vec{n}+15\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\\
&+30\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}-15\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\\
&+10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2+40\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-8\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{S}_1\cdot\vec{S}_2\\
&-105\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-5\vec{S}_1\cdot\vec{S}_2(8\vec{p}_1\cdot\vec{p}_2+7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2\\
&+105\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^3+35\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3\Big]\\
&-\frac{3G^2m_2}{{m_1}^2r^6}\Big[4\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-2S_1^2\vec{S}_1\cdot\vec{S}_2-5\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+5\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{G^2}{2m_1r^6}\Big[57\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-17S_1^2\vec{S}_1\cdot\vec{S}_2-123\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3\\
&+63\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{C.8}
$$

$$
H^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3S_2} =\frac{9G^2m_2}{{m_1}^2r^6}\Big[\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3-\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\tag{C.9}
$$

$$
\begin{aligned}
H^{\rm NLO}_{\rm (BS_1^3)S_2} =&-\frac{G}{4{m_1}^2{m_2}^2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2(7p_2^2-20(\vec{p}_2\cdot\vec{n})^2)-15\vec{S}_1\cdot\vec{n}S_1^2p_2^2\vec{S}_2\cdot\vec{n}\\
&+20\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_2\cdot\vec{n}\vec{S}_2\cdot\vec{n}+15\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-7\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_2\cdot\vec{S}_2
\end{aligned}
$$

$$
\begin{aligned}
&-40\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-140\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-35\vec{S}_1\cdot\vec{S}_2\big(p_2^2-4(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_2\cdot\vec{n}p_2^2(\vec{S}_1\cdot\vec{n})^3\\
&-35\vec{p}_2\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{p}_2\cdot\vec{S}_1)^2+35\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{G}{4{m_1}^4r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2\big(13p_1^2-20(\vec{p}_1\cdot\vec{n})^2\big)-45\vec{S}_1\cdot\vec{n}S_1^2p_1^2\vec{S}_2\cdot\vec{n}\\
&+25\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}+60\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2-17\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{S}_2\\
&-30\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-175\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-5\vec{S}_1\cdot\vec{S}_2\big(13p_1^2-28(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2+105\vec{S}_2\cdot\vec{n}p_1^2(\vec{S}_1\cdot\vec{n})^3\\
&-140\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+50\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2+85\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\\
&-2\vec{S}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{S}_1)^2\Big]+\frac{G}{4{m_1}^3m_2r^5}\Big[S_1^2\vec{S}_1\cdot\vec{S}_2(23\vec{p}_1\cdot\vec{p}_2-25\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\\
&-15\vec{S}_1\cdot\vec{n}S_1^2(5\vec{p}_1\cdot\vec{p}_2+7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\vec{S}_2\cdot\vec{n}+25\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_2\cdot\vec{n}\vec{S}_2\cdot\vec{n}\\
&+35\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}+90\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}+45\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\\
&-13\vec{p}_2\cdot\vec{S}_1S_1^2\vec{p}_1\cdot\vec{S}_2+45\vec{S}_1\cdot\vec{n}S_1^2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-11\vec{p}_1\cdot\vec{S}_1S_1^2\vec{p}_2\cdot\vec{S}_2\\
&-30\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-10\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{S}_1\cdot\vec{S}_2\\
&-175\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-245\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-5\vec{S}_1\cdot\vec{S}_2(23\vec{p}_1\cdot\vec{p}_2-35\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+35\vec{S}_2\cdot\vec{n}(5\vec{p}_1\cdot\vec{p}_2\\
&+9\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^3-105\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3-105\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3\\
&+65\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2+55\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{G^2}{2m_1r^6}\Big[41\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-11S_1^2\vec{S}_1\cdot\vec{S}_2-105\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3\\
&+55\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^2m_2}{2{m_1}^2r^6}\Big[109\vec{S}_1\cdot\vec{n}S_1^2\vec{S}_2\cdot\vec{n}-23S_1^2\vec{S}_1\cdot\vec{S}_2\\
&-260\vec{S}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^3+116\vec{S}_1\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
H^{\rm NLO}_{\rm S_1^2S_2^2}=&-\frac{3G}{8{m_1}^2{m_2}^2r^5}\Big[S_1^2S_2^2(9\vec{p}_1\cdot\vec{p}_2-10\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})+10S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\\
&-30\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2+20S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\\
&-10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-18S_1^2\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{S}_2\\
&+5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(9\vec{p}_1\cdot\vec{p}_2-7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\vec{S}_1\cdot\vec{S}_2-10\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\\
&-30\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2+13\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2+9\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2\\
&+70\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2-10S_1^2\vec{p}_1\cdot\vec{p}_2(\vec{S}_2\cdot\vec{n})^2-35\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2\\
&+10\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(\vec{S}_2\cdot\vec{n})^2-(13\vec{p}_1\cdot\vec{p}_2-15\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{S}_2)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3G^2}{m_1 r^6}\Big[5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-5(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2-(\vec{S}_1\cdot\vec{S}_2)^2\Big]\\
&-\frac{3G^2}{m_2 r^6}S_1^2(\vec{S}_2\cdot\vec{n})^2, \qquad\qquad (\mathrm{C.11})
\end{aligned}
$$

$$
\begin{aligned}
H^{\rm NLO}_{(\mathrm{ES}_1^2)\mathrm{S}_2^2}=&-\frac{3G}{4{m_1}^2{m_2}^2r^5}\Big[S_1^2S_2^2(23\vec{p}_1\cdot\vec{p}_2-25\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})+30S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\\
&+25S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-23S_1^2\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{S}_2\\
&-20\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-20\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\\
&-10\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2+16\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2+18\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2\\
&+10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}S_2^2+10\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}S_2^2-18\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1S_2^2\\
&-70\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-35\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-35S_2^2(\vec{p}_1\cdot\vec{p}_2\\
&-\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+35\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{S}_2(\vec{S}_1\cdot\vec{n})^2-30S_1^2\vec{p}_1\cdot\vec{p}_2(\vec{S}_2\cdot\vec{n})^2\\
&+70\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+20\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(\vec{S}_2\cdot\vec{n})^2\\
&-4(4\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{S}_2)^2\Big]+\frac{3G}{16m_1{m_2}^3r^5}\Big[5S_1^2S_2^2(16p_2^2-17(\vec{p}_2\cdot\vec{n})^2)\\
&+190S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-40\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\\
&-140\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-40\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\\
&+120\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2+80\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}S_2^2\\
&-350\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-15S_2^2(8p_2^2-7(\vec{p}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2\\
&-64S_2^2(\vec{p}_2\cdot\vec{S}_1)^2-105S_1^2p_2^2(\vec{S}_2\cdot\vec{n})^2+245p_2^2(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2\\
&+70(\vec{p}_2\cdot\vec{S}_1)^2(\vec{S}_2\cdot\vec{n})^2+120(\vec{S}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{S}_2)^2-80S_1^2(\vec{p}_2\cdot\vec{S}_2)^2\\
&-14(4p_2^2-5(\vec{p}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{S}_2)^2\Big]\\
&-\frac{3G^2}{m_2r^6}\Big[2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-S_1^2S_2^2+3S_2^2(\vec{S}_1\cdot\vec{n})^2-5(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2\\
&+S_1^2(\vec{S}_2\cdot\vec{n})^2\Big]-\frac{G^2}{2m_1r^6}\Big[54\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-17S_1^2S_2^2+48S_2^2(\vec{S}_1\cdot\vec{n})^2\\
&-135(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+27S_1^2(\vec{S}_2\cdot\vec{n})^2+3(\vec{S}_1\cdot\vec{S}_2)^2\Big], \qquad\qquad (\mathrm{C.12})
\end{aligned}
$$

$$
\begin{aligned}
H^{\rm NLO}_{(\mathrm{ES}_1^2)(\mathrm{ES}_2^2)}=&-\frac{3G}{8m_1{m_2}^3r^5}\Big[10S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2-15S_1^2p_2^2(\vec{S}_2\cdot\vec{n})^2-2S_1^2(\vec{p}_2\cdot\vec{S}_2)^2\Big]\\
&+\frac{3G}{16{m_1}^2{m_2}^2r^5}\Big[S_1^2S_2^2(11\vec{p}_1\cdot\vec{p}_2-15\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})-20S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\\
&+20\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2-20S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\\
&+20\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2+4S_1^2\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{S}_2-140\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2\\
&+\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\vec{S}_1\cdot\vec{S}_2+40\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2+40\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\\
&-4\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2-4\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2-140\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2\\
&-140\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2-10S_1^2(9\vec{p}_1\cdot\vec{p}_2-7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})^2
\end{aligned}
$$

$$
\begin{aligned}
&+35\big(7\vec{p}_1\cdot\vec{p}_2+9\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+20\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(\vec{S}_2\cdot\vec{n})^2\\
&+2\big(7\vec{p}_1\cdot\vec{p}_2+5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{S}_2)^2\Big]-\frac{3G}{8{m_1}^3m_2r^5}\Big[5S_1^2S_2^2\big(p_1^2-2(\vec{p}_1\cdot\vec{n})^2\big)\\
&-40S_1^2\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2+20\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2-60\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}p_1^2\vec{S}_1\cdot\vec{S}_2\\
&+20\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-4\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{S}_2\\
&-70\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2-5S_1^2\big(5p_1^2-14(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_2\cdot\vec{n})^2\\
&+105p_1^2(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+10(\vec{p}_1\cdot\vec{S}_1)^2(\vec{S}_2\cdot\vec{n})^2+4S_1^2(\vec{p}_1\cdot\vec{S}_2)^2+6p_1^2(\vec{S}_1\cdot\vec{S}_2)^2\Big]\\
&-\frac{3G^2}{2m_1r^6}\Big[32\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2-3S_1^2S_2^2-62(\vec{S}_1\cdot\vec{n})^2(\vec{S}_2\cdot\vec{n})^2+12S_1^2(\vec{S}_2\cdot\vec{n})^2\\
&-3(\vec{S}_1\cdot\vec{S}_2)^2\Big]-\frac{15G^2}{m_2r^6}S_1^2(\vec{S}_2\cdot\vec{n})^2. \qquad\qquad (C.13)
\end{aligned}
$$

D COM generator of the N^{3}LO quadratic-in-spin sectors

As noted in section 4.2, we write the solution of the COM generator of the N^{3}LO quadratic-in-spin sectors, $\vec{G}^{\rm N^3LO}_{\rm S^2}$, as:

$$
\vec{G}^{\rm N^3LO}_{\rm S^2}=H^{\rm N^2LO}_{\rm S^2}\frac{\vec{x}_1+\vec{x}_2}{2}+\Big(\vec{Y}^{\rm N^3LO}_{\rm S_1^2}+C_{\rm 1ES^2}\vec{Y}^{\rm N^3LO}_{\rm ES_1^2}+\vec{Y}^{\rm N^3LO}_{\rm S_1S_2}+(1\leftrightarrow 2)\Big)\,, \qquad (D.1)
$$

with:

$$
\begin{aligned}
\vec{Y}^{\rm N^3LO}_{\rm S_1^2}=&-\frac{Gm_2}{32{m_1}^5r^2}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(p_1^2\vec{p}_1-9\vec{p}_1\cdot\vec{n}p_1^2\vec{n}\big)-5S_1^2\big(3p_1^4\vec{n}\\
&+4\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1-12p_1^2(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)-2\vec{S}_1\cdot\vec{n}\vec{S}_1p_1^4-33\vec{p}_1\cdot\vec{S}_1\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^2\\
&-15p_1^4\vec{n}(\vec{S}_1\cdot\vec{n})^2+3\big(5p_1^2\vec{n}+18\vec{p}_1\cdot\vec{n}\vec{p}_1-12(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&-\frac{G}{32{m_1}^4r^2}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(18\vec{p}_1\cdot\vec{p}_2\vec{p}_1+14p_1^2\vec{p}_2+9p_1^2\vec{p}_2\cdot\vec{n}\vec{n}+69\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}\\
&-57\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1-12(\vec{p}_1\cdot\vec{n})^2\vec{p}_2+60\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)\\
&-\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(38p_1^2\vec{p}_1-63\vec{p}_1\cdot\vec{n}p_1^2\vec{n}+3(\vec{p}_1\cdot\vec{n})^2\vec{p}_1\big)-\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(11p_1^2\vec{n}\\
&+99\vec{p}_1\cdot\vec{n}\vec{p}_1+15(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)+S_1^2\big(99p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n}+46p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1+11\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\\
&-9\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2-168\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{n}-57\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\vec{n}+102\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_1\\
&+12(\vec{p}_1\cdot\vec{n})^3\vec{p}_2-60\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\vec{n}\big)+\vec{S}_1\cdot\vec{n}\vec{S}_1\big(10p_1^2\vec{p}_1\cdot\vec{p}_2+9\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-3\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\big)+\vec{p}_1\cdot\vec{S}_1\vec{S}_1\big(22p_1^2\vec{p}_2\cdot\vec{n}+67\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-63\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)\\
&+3\vec{p}_2\cdot\vec{S}_1\vec{S}_1\big(5\vec{p}_1\cdot\vec{n}p_1^2+(\vec{p}_1\cdot\vec{n})^3\big)-3\big(20p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n}+5p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\\
&-\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\big)(\vec{S}_1\cdot\vec{n})^2-\big(92\vec{p}_1\cdot\vec{p}_2\vec{n}+68\vec{p}_2\cdot\vec{n}\vec{p}_1\\
&-15\vec{p}_1\cdot\vec{n}\vec{p}_2-183\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]-\frac{G}{16m_1{m_2}^3r^2}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(p_2^2\vec{p}_2\\
&+3\vec{p}_2\cdot\vec{n}p_2^2\vec{n}+3(\vec{p}_2\cdot\vec{n})^2\vec{p}_2-15(\vec{p}_2\cdot\vec{n})^3\vec{n}\big)+S_1^2\big(-p_2^4\vec{n}+38\vec{p}_2\cdot\vec{n}p_2^2\vec{p}_2\\
&+18p_2^2(\vec{p}_2\cdot\vec{n})^2\vec{n}-54(\vec{p}_2\cdot\vec{n})^3\vec{p}_2\big)+\vec{S}_1\cdot\vec{n}\vec{S}_1\big(p_2^4-18p_2^2(\vec{p}_2\cdot\vec{n})^2\big)\\
&-2\vec{p}_2\cdot\vec{S}_1\vec{S}_1\big(19\vec{p}_2\cdot\vec{n}p_2^2-27(\vec{p}_2\cdot\vec{n})^3\big)-6\big(\vec{p}_2\cdot\vec{n}p_2^2\vec{p}_2-5(\vec{p}_2\cdot\vec{n})^3\vec{p}_2\big)(\vec{S}_1\cdot\vec{n})^2
\end{aligned}
$$

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$$
\begin{aligned}
&- 2\big(p_2^2\vec n + 3(\vec p_2\cdot\vec n)^2\vec n\big)(\vec p_2\cdot\vec S_1)^2\Big] + \frac{G}{32{m_1}^3 m_2 r^2}\Big[\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\big(23p_2^2\vec p_1\\
&- 22\vec p_1\cdot\vec p_2\vec p_2 - 174\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\vec n + 135\vec p_1\cdot\vec n p_2^2\vec n - 45(\vec p_2\cdot\vec n)^2\vec p_1\\
&- 114\vec p_1\cdot\vec n\vec p_2\cdot\vec n\vec p_2 - 135\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\vec n\big) + 8\vec S_1\cdot\vec n\vec p_2\cdot\vec S_1\big(\vec p_1\cdot\vec p_2\vec p_1 + 12p_1^2\vec p_2\cdot\vec n\vec n\\
&- 3\vec p_1\cdot\vec n\vec p_2\cdot\vec n\vec p_1\big) + 2\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1\big(23\vec p_1\cdot\vec p_2\vec n - 53\vec p_2\cdot\vec n\vec p_1 - 15\vec p_1\cdot\vec n\vec p_2\\
&+ 69\vec p_1\cdot\vec n\vec p_2\cdot\vec n\vec n\big) + S_1^2\big(105p_1^2p_2^2\vec n + 2(\vec p_1\cdot\vec p_2)^2\vec n - 78\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\vec p_1\\
&- 27\vec p_1\cdot\vec n p_2^2\vec p_1 - 16p_1^2\vec p_2\cdot\vec n\vec p_2 - 48\vec p_1\cdot\vec n\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\vec n - 45p_2^2(\vec p_1\cdot\vec n)^2\vec n\\
&- 174p_1^2(\vec p_2\cdot\vec n)^2\vec n + 93\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\vec p_1 + 162\vec p_2\cdot\vec n(\vec p_1\cdot\vec n)^2\vec p_2\\
&+ 45(\vec p_1\cdot\vec n)^2(\vec p_2\cdot\vec n)^2\vec n\big) - 3\vec S_1\cdot\vec n\vec S_1\big(3p_1^2p_2^2 - 10(\vec p_1\cdot\vec p_2)^2 + 8\vec p_1\cdot\vec n\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\\
&- 7p_2^2(\vec p_1\cdot\vec n)^2 + 2p_1^2(\vec p_2\cdot\vec n)^2 + 35(\vec p_1\cdot\vec n)^2(\vec p_2\cdot\vec n)^2\big) + \vec p_1\cdot\vec S_1\vec S_1\big(162\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\\
&+ 19\vec p_1\cdot\vec n p_2^2 - 51\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\big) + 6\vec p_2\cdot\vec S_1\vec S_1\big(4p_1^2\vec p_2\cdot\vec n\\
&+ 4\vec p_1\cdot\vec n\vec p_1\cdot\vec p_2 - 15\vec p_2\cdot\vec n(\vec p_1\cdot\vec n)^2\big) - 3\big(32p_1^2p_2^2\vec n - 26\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\vec p_1\\
&+ 5\vec p_1\cdot\vec n p_2^2\vec p_1 - 40p_1^2(\vec p_2\cdot\vec n)^2\vec n - 25\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\vec p_1\big)(\vec S_1\cdot\vec n)^2 - \big(111p_2^2\vec n\\
&- 14\vec p_2\cdot\vec n\vec p_2 - 183(\vec p_2\cdot\vec n)^2\vec n\big)(\vec p_1\cdot\vec S_1)^2 - 2\big(24p_1^2\vec n + 5\vec p_1\cdot\vec n\vec p_1\big)(\vec p_2\cdot\vec S_1)^2\Big]\\
&- \frac{G}{32{m_1}^2{m_2}^2r^2}\Big[3\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\big(5p_2^2\vec p_2 - 35\vec p_2\cdot\vec n p_2^2\vec n - 3(\vec p_2\cdot\vec n)^2\vec p_2 + 25(\vec p_2\cdot\vec n)^3\vec n\big)\\
&+ 2\vec S_1\cdot\vec n\vec p_2\cdot\vec S_1\big(3p_2^2\vec p_1 - 4\vec p_1\cdot\vec p_2\vec p_2 + 18\vec p_1\cdot\vec n p_2^2\vec n\\
&- 45(\vec p_2\cdot\vec n)^2\vec p_1 - 90\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\vec n\big) - \vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1\big(41p_2^2\vec n\\
&+ 90\vec p_2\cdot\vec n\vec p_2 - 183(\vec p_2\cdot\vec n)^2\vec n\big) + S_1^2\big(37\vec p_1\cdot\vec p_2 p_2^2\vec n - 134\vec p_2\cdot\vec n p_2^2\vec p_1\\
&- 22\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\vec p_2 - 51\vec p_1\cdot\vec n p_2^2\vec p_2 + 33\vec p_1\cdot\vec n\vec p_2\cdot\vec n p_2^2\vec n - 171\vec p_1\cdot\vec p_2(\vec p_2\cdot\vec n)^2\vec n\\
&+ 150(\vec p_2\cdot\vec n)^3\vec p_1 + 117\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\vec p_2 + 45\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^3\vec n\big)\\
&+ 3\vec S_1\cdot\vec n\vec S_1\big(\vec p_1\cdot\vec p_2 p_2^2 - 3\vec p_1\cdot\vec n\vec p_2\cdot\vec n p_2^2 + 9\vec p_1\cdot\vec p_2(\vec p_2\cdot\vec n)^2 - 15\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^3\big)\\
&+ 2\vec p_1\cdot\vec S_1\vec S_1\big(67\vec p_2\cdot\vec n p_2^2 - 75(\vec p_2\cdot\vec n)^3\big) + \vec p_2\cdot\vec S_1\vec S_1\big(62\vec p_2\cdot\vec n\vec p_1\cdot\vec p_2\\
&+ 31\vec p_1\cdot\vec n p_2^2 - 57\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\big) - 3\big(8\vec p_1\cdot\vec p_2 p_2^2\vec n - 27\vec p_2\cdot\vec n p_2^2\vec p_1\\
&+ 4\vec p_1\cdot\vec n p_2^2\vec p_2 - 40\vec p_1\cdot\vec p_2(\vec p_2\cdot\vec n)^2\vec n + 25(\vec p_2\cdot\vec n)^3\vec p_1 - 20\vec p_1\cdot\vec n(\vec p_2\cdot\vec n)^2\vec p_2\big)(\vec S_1\cdot\vec n)^2\\
&+ 2\big(4\vec p_1\cdot\vec p_2\vec n + 25\vec p_2\cdot\vec n\vec p_1\big)(\vec p_2\cdot\vec S_1)^2\Big] - \frac{G^2{m_2}^2}{32{m_1}^3r^3}\Big[2\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\big(26\vec p_1\\
&+ 167\vec p_1\cdot\vec n\vec n\big) + S_1^2\big(205p_1^2\vec n - 60\vec p_1\cdot\vec n\vec p_1 - 122(\vec p_1\cdot\vec n)^2\vec n\big)\\
&- 2\vec S_1\cdot\vec n\vec S_1\big(27p_1^2 - 32(\vec p_1\cdot\vec n)^2\big) + 60\vec p_1\cdot\vec S_1\vec S_1\vec p_1\cdot\vec n - \big(217p_1^2\vec n\\
&+ 64\vec p_1\cdot\vec n\vec p_1\big)(\vec S_1\cdot\vec n)^2 - 198\vec n(\vec p_1\cdot\vec S_1)^2\Big] - \frac{G^2m_2}{48{m_1}^2r^3}\Big[\vec S_1\cdot\vec n\vec p_1\cdot\vec S_1\big(491\vec p_1 + 138\vec p_2\\
&+ 3604\vec p_1\cdot\vec n\vec n + 432\vec p_2\cdot\vec n\vec n\big) - 3\vec S_1\cdot\vec n\vec p_2\cdot\vec S_1\big(46\vec p_1 + 151\vec p_1\cdot\vec n\vec n\big)\\
&+ 462\vec p_1\cdot\vec S_1\vec p_2\cdot\vec S_1\vec n + S_1^2\big(833p_1^2\vec n - 999\vec p_1\cdot\vec p_2\vec n + 433\vec p_1\cdot\vec n\vec p_1 + 627\vec p_2\cdot\vec n\vec p_1\\
&+ 72\vec p_1\cdot\vec n\vec p_2 - 477\vec p_1\cdot\vec n\vec p_2\cdot\vec n\vec n - 1187(\vec p_1\cdot\vec n)^2\vec n\big) - \vec S_1\cdot\vec n\vec S_1\big(272p_1^2\\
&- 705\vec p_1\cdot\vec p_2 - 792\vec p_1\cdot\vec n\vec p_2\cdot\vec n - 920(\vec p_1\cdot\vec n)^2\big) - 5\vec p_1\cdot\vec S_1\vec S_1\big(113\vec p_1\cdot\vec n + 195\vec p_2\cdot\vec n\big)\\
&+ 3\vec p_2\cdot\vec S_1\vec S_1\vec p_1\cdot\vec n - \big(1157p_1^2\vec n - 546\vec p_1\cdot\vec p_2\vec n + 1480\vec p_1\cdot\vec n\vec p_1 + 555\vec p_2\cdot\vec n\vec p_1
\end{aligned}
$$

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$$
\begin{aligned}
&+180\vec{p}_1\cdot\vec{n}\vec{p}_2+438(\vec{p}_1\cdot\vec{n})^2\vec{n})(\vec{S}_1\cdot\vec{n})^2-1166\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&-\frac{G^2}{8m_2r^3}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(61\vec{p}_2+10\vec{p}_2\cdot\vec{n}\vec{n})+S_1^2(75p_2^2\vec{n}-146\vec{p}_2\cdot\vec{n}\vec{p}_2\\
&+45(\vec{p}_2\cdot\vec{n})^2\vec{n})-2\vec{S}_1\cdot\vec{n}\vec{S}_1(35p_2^2+29(\vec{p}_2\cdot\vec{n})^2)+128\vec{p}_2\cdot\vec{S}_1\vec{S}_1\vec{p}_2\cdot\vec{n}-(29p_2^2\vec{n}\\
&+46\vec{p}_2\cdot\vec{n}\vec{p}_2-3(\vec{p}_2\cdot\vec{n})^2\vec{n})(\vec{S}_1\cdot\vec{n})^2-106\vec{n}(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^2}{48m_1r^3}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(200\vec{p}_2+541\vec{p}_2\cdot\vec{n}\vec{n})+4\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(58\vec{p}_1-84\vec{p}_2\\
&+752\vec{p}_1\cdot\vec{n}\vec{n}-51\vec{p}_2\cdot\vec{n}\vec{n})-1700\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{n}+S_1^2(1394\vec{p}_1\cdot\vec{p}_2\vec{n}-822p_2^2\vec{n}\\
&-310\vec{p}_2\cdot\vec{n}\vec{p}_1+62\vec{p}_1\cdot\vec{n}\vec{p}_2+369\vec{p}_2\cdot\vec{n}\vec{p}_2-902\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}+585(\vec{p}_2\cdot\vec{n})^2\vec{n})\\
&-\vec{S}_1\cdot\vec{n}\vec{S}_1(716\vec{p}_1\cdot\vec{p}_2-840p_2^2-206\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+327(\vec{p}_2\cdot\vec{n})^2)\\
&+538\vec{p}_1\cdot\vec{S}_1\vec{S}_1\vec{p}_2\cdot\vec{n}-\vec{p}_2\cdot\vec{S}_1\vec{S}_1(128\vec{p}_1\cdot\vec{n}+531\vec{p}_2\cdot\vec{n})-(1208\vec{p}_1\cdot\vec{p}_2\vec{n}-147p_2^2\vec{n}\\
&-406\vec{p}_2\cdot\vec{n}\vec{p}_1+1088\vec{p}_1\cdot\vec{n}\vec{p}_2-9\vec{p}_2\cdot\vec{n}\vec{p}_2+780\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\\
&+144(\vec{p}_2\cdot\vec{n})^2\vec{n})(\vec{S}_1\cdot\vec{n})^2+414\vec{n}(\vec{p}_2\cdot\vec{S}_1)^2\Big]+\frac{G^3{m_2}^3}{2m_1r^4}\Big[28S_1^2\vec{n}-15\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-13\vec{S}_1\cdot\vec{n}\vec{S}_1\Big]+\frac{3G^3m_1m_2}{245r^4}\Big[1471S_1^2\vec{n}-1860\vec{n}(\vec{S}_1\cdot\vec{n})^2+72\vec{S}_1\cdot\vec{n}\vec{S}_1\Big]\\
&+\frac{G^3{m_2}^2}{64r^4}\Big[S_1^2(2452-45\pi^2)\vec{n}-25(124-9\pi^2)\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-2(188+45\pi^2)\vec{S}_1\cdot\vec{n}\vec{S}_1\Big],
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\vec{Y}^{\mathrm{N^3LO}}_{\mathrm{ES}_1^2}=&-\frac{5G}{16m_1{m_2}^3r^2}\Big[S_1^2p_2^4\vec{n}-\vec{S}_1\cdot\vec{n}\vec{S}_1p_2^4\Big]+\frac{G}{8{m_1}^2{m_2}^2r^2}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1p_2^2\vec{p}_2\\
&-2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1p_2^2\vec{p}_1+S_1^2(2\vec{p}_1\cdot\vec{p}_2p_2^2\vec{n}+3\vec{p}_2\cdot\vec{n}p_2^2\vec{p}_1\\
&+\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_2-6\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2\vec{n})-2\vec{S}_1\cdot\vec{n}\vec{S}_1(\vec{p}_1\cdot\vec{p}_2p_2^2-3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2)\\
&-2\vec{p}_1\cdot\vec{S}_1\vec{S}_1\vec{p}_2\cdot\vec{n}p_2^2-2\vec{p}_2\cdot\vec{S}_1\vec{S}_1\vec{p}_1\cdot\vec{n}p_2^2+3(\vec{p}_2\cdot\vec{n}p_2^2\vec{p}_1-\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_2)(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{Gm_2}{16{m_1}^5r^2}\Big[3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1p_1^2\vec{p}_1-S_1^2(5p_1^4\vec{n}+4\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1)+5\vec{S}_1\cdot\vec{n}\vec{S}_1p_1^4\\
&+3\vec{p}_1\cdot\vec{S}_1\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^2-2\vec{p}_1\cdot\vec{n}\vec{p}_1(\vec{p}_1\cdot\vec{S}_1)^2\Big]-\frac{G}{8{m_1}^4r^2}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(6\vec{p}_1\cdot\vec{p}_2\vec{p}_1\\
&-p_1^2\vec{p}_2+6\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1-3(\vec{p}_1\cdot\vec{n})^2\vec{p}_2)+2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1p_1^2\vec{p}_1\\
&-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{p}_1-S_1^2(2p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n}+10\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\\
&+4\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2-6\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{n}+9\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_1-3(\vec{p}_1\cdot\vec{n})^3\vec{p}_2)\\
&+2\vec{S}_1\cdot\vec{n}\vec{S}_1(p_1^2\vec{p}_1\cdot\vec{p}_2-3\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n})+\vec{p}_1\cdot\vec{S}_1\vec{S}_1(p_1^2\vec{p}_2\cdot\vec{n}+6\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&+3\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2)+2\vec{p}_2\cdot\vec{S}_1\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^2-3(p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1-\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2)(\vec{S}_1\cdot\vec{n})^2\\
&-2(\vec{p}_2\cdot\vec{n}\vec{p}_1-\vec{p}_1\cdot\vec{n}\vec{p}_2)(\vec{p}_1\cdot\vec{S}_1)^2\Big]+\frac{G}{16{m_1}^3m_2r^2}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(21p_2^2\vec{p}_1\\
&-29\vec{p}_1\cdot\vec{p}_2\vec{p}_2+18\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}-48\vec{p}_1\cdot\vec{n}p_2^2\vec{n}-6(\vec{p}_2\cdot\vec{n})^2\vec{p}_1+30\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2)\\
&-\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(5\vec{p}_1\cdot\vec{p}_2\vec{p}_1-19p_1^2\vec{p}_2+18p_1^2\vec{p}_2\cdot\vec{n}\vec{n}-48\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}
\end{aligned}
$$

$$
\begin{aligned}
&+ 36\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1 - 12(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\big) + \vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(4\vec{p}_1\cdot\vec{p}_2\vec{n} - 9\vec{p}_2\cdot\vec{n}\vec{p}_1 \\
&- 31\vec{p}_1\cdot\vec{n}\vec{p}_2 + 36\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\big) + S_1^2\big(5p_1^2p_2^2\vec{n} - 9\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1 - 42\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_1 \\
&- 6p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_2 + 25\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_2 + 30p_2^2(\vec{p}_1\cdot\vec{n})^2\vec{n} + 15\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_1 \\
&- 27\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_2 + 15(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\vec{n}\big) - 5\vec{S}_1\cdot\vec{n}\vec{S}_1\big(4p_1^2p_2^2 - 3(\vec{p}_1\cdot\vec{p}_2)^2 \\
&+ 6\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2 + 3p_2^2(\vec{p}_1\cdot\vec{n})^2 - 3p_1^2(\vec{p}_2\cdot\vec{n})^2 + 3(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big) \\
&+ \vec{p}_1\cdot\vec{S}_1\vec{S}_1\big(7\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2 + 39\vec{p}_1\cdot\vec{n}p_2^2 - 24\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big) + \vec{p}_2\cdot\vec{S}_1\vec{S}_1\big(3p_1^2\vec{p}_2\cdot\vec{n} \\
&- 23\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2 + 36\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big) + 3\big(5p_1^2p_2^2\vec{n} - 5(\vec{p}_1\cdot\vec{p}_2)^2\vec{n} - 2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1 \\
&+ 7\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_1 - 5p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_2 - 5\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_1 + 5\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\big)(\vec{S}_1\cdot\vec{n})^2 \\
&- \big(2p_2^2\vec{n} - 7\vec{p}_2\cdot\vec{n}\vec{p}_2 + 3(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)(\vec{p}_1\cdot\vec{S}_1)^2 - \big(2p_1^2\vec{n} - 33\vec{p}_1\cdot\vec{n}\vec{p}_1 \\
&+ 33(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big] - \frac{G^2m_2}{48{m_1}^2r^3}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(323\vec{p}_1 - 46\vec{p}_2 - 179\vec{p}_1\cdot\vec{n}\vec{n} \\
&- 380\vec{p}_2\cdot\vec{n}\vec{n}\big) + 2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(497\vec{p}_1 - 677\vec{p}_1\cdot\vec{n}\vec{n}\big) + 136\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{n} \\
&+ S_1^2\big(256p_1^2\vec{n} - 784\vec{p}_1\cdot\vec{p}_2\vec{n} + 140\vec{p}_1\cdot\vec{n}\vec{p}_1 - 856\vec{p}_2\cdot\vec{n}\vec{p}_1 - 796\vec{p}_1\cdot\vec{n}\vec{p}_2 \\
&+ 2992\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n} + 227(\vec{p}_1\cdot\vec{n})^2\vec{n}\big) + \vec{S}_1\cdot\vec{n}\vec{S}_1\big(407p_1^2 \\
&+ 1000\vec{p}_1\cdot\vec{p}_2 - 2104\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n} - 161(\vec{p}_1\cdot\vec{n})^2\big) + 4\vec{p}_1\cdot\vec{S}_1\vec{S}_1\big(19\vec{p}_1\cdot\vec{n} \\
&+ 142\vec{p}_2\cdot\vec{n}\big) + 958\vec{p}_2\cdot\vec{S}_1\vec{S}_1\vec{p}_1\cdot\vec{n} - 2\big(461p_1^2\vec{n} + 376\vec{p}_1\cdot\vec{p}_2\vec{n} + 637\vec{p}_1\cdot\vec{n}\vec{p}_1 \\
&+ 637\vec{p}_2\cdot\vec{n}\vec{p}_1 + 31\vec{p}_1\cdot\vec{n}\vec{p}_2 - 573\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n} - 255(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2 \\
&+ 47\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2\Big] + \frac{G^2{m_2}^2}{12{m_1}^3r^3}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(\vec{p}_1 - 70\vec{p}_1\cdot\vec{n}\vec{n}\big) - 2S_1^2\big(28p_1^2\vec{n} \\
&+ 41\vec{p}_1\cdot\vec{n}\vec{p}_1 - 31(\vec{p}_1\cdot\vec{n})^2\vec{n}\big) + \vec{S}_1\cdot\vec{n}\vec{S}_1\big(23p_1^2 - 8(\vec{p}_1\cdot\vec{n})^2\big) + 67\vec{p}_1\cdot\vec{S}_1\vec{S}_1\vec{p}_1\cdot\vec{n} \\
&+ \big(83p_1^2\vec{n} + 40\vec{p}_1\cdot\vec{n}\vec{p}_1 - 12(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2 - 13\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2\Big] \\
&+ \frac{G^2}{24m_2r^3}\Big[4\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(7\vec{p}_2 - \vec{p}_2\cdot\vec{n}\vec{n}\big) - S_1^2\big(38p_2^2\vec{n} - 2\vec{p}_2\cdot\vec{n}\vec{p}_2 - 17(\vec{p}_2\cdot\vec{n})^2\vec{n}\big) \\
&+ 14\vec{S}_1\cdot\vec{n}\vec{S}_1\big(4p_2^2 - (\vec{p}_2\cdot\vec{n})^2\big) + 4\vec{p}_2\cdot\vec{S}_1\vec{S}_1\vec{p}_2\cdot\vec{n} - \big(76p_2^2\vec{n} \\
&+ 62\vec{p}_2\cdot\vec{n}\vec{p}_2 - 15(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2 + 2\vec{n}(\vec{p}_2\cdot\vec{S}_1)^2\Big] \\
&+ \frac{G^2}{48m_1r^3}\Big[4\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(121\vec{p}_2 - 37\vec{p}_2\cdot\vec{n}\vec{n}\big) + 2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(32\vec{p}_1 \\
&- 163\vec{p}_2 - 86\vec{p}_1\cdot\vec{n}\vec{n} - 842\vec{p}_2\cdot\vec{n}\vec{n}\big) + 40\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{n} + S_1^2\big(332\vec{p}_1\cdot\vec{p}_2\vec{n} - 2p_2^2\vec{n} \\
&+ 20\vec{p}_2\cdot\vec{n}\vec{p}_1 + 128\vec{p}_1\cdot\vec{n}\vec{p}_2 + 236\vec{p}_2\cdot\vec{n}\vec{p}_2 + 172\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n} + 1781(\vec{p}_2\cdot\vec{n})^2\vec{n}\big) \\
&+ 4\vec{S}_1\cdot\vec{n}\vec{S}_1\big(70\vec{p}_1\cdot\vec{p}_2 + 110p_2^2 - 37\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n} - 239(\vec{p}_2\cdot\vec{n})^2\big) + 16\vec{p}_1\cdot\vec{S}_1\vec{S}_1\vec{p}_2\cdot\vec{n} \\
&+ 4\vec{p}_2\cdot\vec{S}_1\vec{S}_1\big(13\vec{p}_1\cdot\vec{n} + 241\vec{p}_2\cdot\vec{n}\big) - \big(740\vec{p}_1\cdot\vec{p}_2\vec{n} + 1177p_2^2\vec{n} + 188\vec{p}_2\cdot\vec{n}\vec{p}_1 \\
&+ 920\vec{p}_1\cdot\vec{n}\vec{p}_2 + 1550\vec{p}_2\cdot\vec{n}\vec{p}_2 - 480\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n} - 897(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2 \\
&+ 179\vec{n}(\vec{p}_2\cdot\vec{S}_1)^2\Big] - \frac{G^3{m_2}^2}{8r^4}\Big[165S_1^2\vec{n} - 181\vec{n}(\vec{S}_1\cdot\vec{n})^2 - 132\vec{S}_1\cdot\vec{n}\vec{S}_1\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^3{m_2}^3}{4m_1r^4}\Big[51S_1^2\vec{n}-46\vec{n}(\vec{S}_1\cdot\vec{n})^2-11\vec{S}_1\cdot\vec{n}\vec{S}_1\Big]+\frac{G^3m_1m_2}{28r^4}\Big[66S_1^2\vec{n}\\
&-197\vec{n}(\vec{S}_1\cdot\vec{n})^2+69\vec{S}_1\cdot\vec{n}\vec{S}_1\Big],
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
\vec{Y}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}}=&-\frac{G}{16{m_1}^4r^2}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(-p_1^4\vec{n}+\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1-5(\vec{p}_1\cdot\vec{n})^3\vec{p}_1\big)\\
&-\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(5p_1^2\vec{p}_1-6\vec{p}_1\cdot\vec{n}p_1^2\vec{n}+18(\vec{p}_1\cdot\vec{n})^2\vec{p}_1\big)\\
&+2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(4p_1^2\vec{p}_1-3\vec{p}_1\cdot\vec{n}p_1^2\vec{n}+15(\vec{p}_1\cdot\vec{n})^3\vec{n}\big)\\
&-2\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(p_1^2\vec{n}-9(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)-2\vec{S}_1\cdot\vec{S}_2\big(-p_1^4\vec{n}+\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\\
&+12p_1^2(\vec{p}_1\cdot\vec{n})^2\vec{n}-15(\vec{p}_1\cdot\vec{n})^3\vec{p}_1\big)+\vec{S}_2\cdot\vec{n}\vec{S}_1\big(7p_1^4+24p_1^2(\vec{p}_1\cdot\vec{n})^2\big)\\
&-6\vec{p}_1\cdot\vec{S}_2\vec{S}_1\big(\vec{p}_1\cdot\vec{n}p_1^2+5(\vec{p}_1\cdot\vec{n})^3\big)-7\vec{S}_1\cdot\vec{n}\vec{S}_2p_1^4+5\vec{p}_1\cdot\vec{S}_1\vec{S}_2\vec{p}_1\cdot\vec{n}p_1^2\Big]\\
&+\frac{G}{32{m_1}^3m_2r^2}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(11p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n}+12p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1+49\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\\
&-43\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2+40\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\vec{n}+20\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_1-15(\vec{p}_1\cdot\vec{n})^3\vec{p}_2\big)\\
&+\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(32\vec{p}_1\cdot\vec{p}_2\vec{p}_1-14p_1^2\vec{p}_2-12p_1^2\vec{p}_2\cdot\vec{n}\vec{n}-63\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}\\
&-84\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1+162(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\big)\\
&+\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(10p_1^2\vec{p}_1-138(\vec{p}_1\cdot\vec{n})^2\vec{p}_1-15(\vec{p}_1\cdot\vec{n})^3\vec{n}\big)-\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(8\vec{p}_1\cdot\vec{p}_2\vec{p}_1\\
&+6p_1^2\vec{p}_2-27p_1^2\vec{p}_2\cdot\vec{n}\vec{n}+60\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}+111\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1-168(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\\
&+180\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)+\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(8\vec{p}_1\cdot\vec{p}_2\vec{n}+16\vec{p}_2\cdot\vec{n}\vec{p}_1-145\vec{p}_1\cdot\vec{n}\vec{p}_2\\
&+27\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\big)-\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(24p_1^2\vec{n}-77\vec{p}_1\cdot\vec{n}\vec{p}_1-96(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)\\
&+4\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(2p_1^2\vec{p}_1-21(\vec{p}_1\cdot\vec{n})^2\vec{p}_1+15(\vec{p}_1\cdot\vec{n})^3\vec{n}\big)+60\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}\vec{p}_1\\
&+\vec{S}_1\cdot\vec{S}_2\big(24p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n}-50p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1-161\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\\
&+165\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2-96\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\vec{n}+222\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_1-159(\vec{p}_1\cdot\vec{n})^3\vec{p}_2\\
&+15\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\vec{n}\big)-12\vec{S}_2\cdot\vec{n}\vec{S}_1\big(2p_1^2\vec{p}_1\cdot\vec{p}_2-4\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\big)\\
&+4\vec{p}_1\cdot\vec{S}_2\vec{S}_1\big(2p_1^2\vec{p}_2\cdot\vec{n}+23\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-18\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)\\
&-12\vec{p}_2\cdot\vec{S}_2\vec{S}_1\big(5\vec{p}_1\cdot\vec{n}p_1^2-2(\vec{p}_1\cdot\vec{n})^3\big)-\vec{S}_1\cdot\vec{n}\vec{S}_2\big(10p_1^2\vec{p}_1\cdot\vec{p}_2-45\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&+36\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+15\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)+\vec{p}_1\cdot\vec{S}_1\vec{S}_2\big(22p_1^2\vec{p}_2\cdot\vec{n}\\
&+45\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-90\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)-\vec{p}_2\cdot\vec{S}_1\vec{S}_2\big(65\vec{p}_1\cdot\vec{n}p_1^2-87(\vec{p}_1\cdot\vec{n})^3\big)\Big]\\
&-\frac{G}{32{m_1}^2{m_2}^2r^2}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(32\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1-40p_1^2(\vec{p}_2\cdot\vec{n})^2\vec{n}\\
&+5\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_1\big)-3\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(12p_2^2\vec{p}_1-18\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}+4\vec{p}_1\cdot\vec{n}p_2^2\vec{n}\\
&+5(\vec{p}_2\cdot\vec{n})^2\vec{p}_1-60\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)+3\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(40\vec{p}_1\cdot\vec{p}_2\vec{p}_1-8p_1^2\vec{p}_2\cdot\vec{n}\vec{n}\\
&-26\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n}-18\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1+5\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)\\
&+2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(17p_2^2\vec{p}_1-27(\vec{p}_2\cdot\vec{n})^2\vec{p}_1\big)+3\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(6p_2^2\vec{n}-53(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)\\
&+16\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}\vec{p}_1+44\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}\vec{p}_1-8\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}\vec{p}_1\\
&-3\vec{S}_1\cdot\vec{S}_2\big(64\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1-2\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_1-51p_1^2(\vec{p}_2\cdot\vec{n})^2\vec{n}-17\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_1\big)
\end{aligned}
$$

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$$
\begin{aligned}
&+\vec{S}_2\cdot\vec{n}\vec{S}_1\big(30p_1^2p_2^2-82(\vec{p}_1\cdot\vec{p}_2)^2-48\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+30p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&-3p_1^2(\vec{p}_2\cdot\vec{n})^2-105(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)+18\vec{p}_2\cdot\vec{S}_2\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&+28\vec{p}_1\cdot\vec{S}_1\vec{S}_2\vec{p}_1\cdot\vec{n}p_2^2+\vec{p}_2\cdot\vec{S}_1\vec{S}_2\big(46p_1^2\vec{p}_2\cdot\vec{n}\\
&-134\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-69\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)\Big]+\frac{G^2m_2}{32{m_1}^2r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(34p_1^2\vec{n}\\
&+77\vec{p}_1\cdot\vec{n}\vec{p}_1\big)+6\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(53\vec{p}_1-40\vec{p}_1\cdot\vec{n}\vec{n}\big)-2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(280\vec{p}_1\\
&+163\vec{p}_1\cdot\vec{n}\vec{n}\big)+354\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{n}-\vec{S}_1\cdot\vec{S}_2\big(378p_1^2\vec{n}\\
&+301\vec{p}_1\cdot\vec{n}\vec{p}_1-465(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)-\vec{S}_2\cdot\vec{n}\vec{S}_1\big(298p_1^2+601(\vec{p}_1\cdot\vec{n})^2\big)\\
&+861\vec{p}_1\cdot\vec{S}_2\vec{S}_1\vec{p}_1\cdot\vec{n}+680\vec{S}_1\cdot\vec{n}\vec{S}_2p_1^2-640\vec{p}_1\cdot\vec{S}_1\vec{S}_2\vec{p}_1\cdot\vec{n}\Big]\\
&-\frac{G^2}{96m_2r^3}\Big[2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(2524\vec{p}_1\cdot\vec{p}_2\vec{n}-425\vec{p}_2\cdot\vec{n}\vec{p}_1+174\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\big)\\
&+2\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(566\vec{p}_1-6767\vec{p}_1\cdot\vec{n}\vec{n}\big)+4333\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{n}\\
&-2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(817\vec{p}_1-910\vec{p}_1\cdot\vec{n}\vec{n}\big)-803\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{n}-\vec{S}_1\cdot\vec{S}_2\big(2690\vec{p}_1\cdot\vec{p}_2\vec{n}\\
&-955\vec{p}_2\cdot\vec{n}\vec{p}_1-6896\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\big)+2884\vec{p}_1\cdot\vec{S}_2\vec{S}_1\vec{p}_2\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{n}\vec{S}_2\big(3949\vec{p}_1\cdot\vec{p}_2-1246\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)-2819\vec{p}_1\cdot\vec{S}_1\vec{S}_2\vec{p}_2\cdot\vec{n}\Big]\\
&+\frac{G^2}{96m_1r^3}\Big[2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(3301p_1^2\vec{n}+3176\vec{p}_1\cdot\vec{n}\vec{p}_1+1699\vec{p}_2\cdot\vec{n}\vec{p}_1-357(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)\\
&+\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(1355\vec{p}_1-13178\vec{p}_1\cdot\vec{n}\vec{n}\big)-\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(347\vec{p}_1+6730\vec{p}_1\cdot\vec{n}\vec{n}\big)\\
&-\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(3163\vec{p}_1+2426\vec{p}_1\cdot\vec{n}\vec{n}\big)+4142\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\vec{n}\\
&+\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(1015\vec{p}_1+1922\vec{p}_1\cdot\vec{n}\vec{n}\big)-\vec{S}_1\cdot\vec{S}_2\big(3632p_1^2\vec{n}+1477\vec{p}_1\cdot\vec{n}\vec{p}_1\\
&+2081\vec{p}_2\cdot\vec{n}\vec{p}_1-6581(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)-\vec{S}_2\cdot\vec{n}\vec{S}_1\big(2603p_1^2-2462(\vec{p}_1\cdot\vec{n})^2\big)\\
&+\vec{p}_1\cdot\vec{S}_2\vec{S}_1\big(929\vec{p}_1\cdot\vec{n}+976\vec{p}_2\cdot\vec{n}\big)+\vec{S}_1\cdot\vec{n}\vec{S}_2\big(2593p_1^2\\
&-4751\vec{p}_1\cdot\vec{p}_2-568\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-4909(\vec{p}_1\cdot\vec{n})^2\big)+\vec{p}_1\cdot\vec{S}_1\vec{S}_2\big(1238\vec{p}_1\cdot\vec{n}\\
&+1633\vec{p}_2\cdot\vec{n}\big)\Big]-\frac{G^3{m_1}^2}{480r^4}\Big[21400\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{n}-7013\vec{S}_1\cdot\vec{S}_2\vec{n}-8168\vec{S}_2\cdot\vec{n}\vec{S}_1\Big]\\
&+\frac{1091G^3m_1m_2}{32r^4}\vec{S}_2\cdot\vec{n}\vec{S}_1+\frac{277G^3{m_2}^2}{480r^4}\vec{S}_2\cdot\vec{n}\vec{S}_1. \qquad \text{(D.4)}
\end{aligned}
$$

E COM Hamiltonians

As noted in section 5, the NLO quartic-in-spin Hamiltonians restricted to the COM frame are written as:

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}^4}=\;&C_{1\mathrm{ES}^2}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)\mathrm{S}_1^2}+C^2_{1\mathrm{ES}^2}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)^2}+C_{1\mathrm{BS}^3}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{BS}_1^3)\mathrm{S}_1}+C_{1\mathrm{ES}^4}\tilde{H}^{\mathrm{NLO}}_{\mathrm{ES}_1^4}+C_{1\mathrm{E}^2\mathrm{S}^4}\tilde{H}^{\mathrm{NLO}}_{\mathrm{E}^2\mathrm{S}_1^4}\\
&+\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}_1^3\mathrm{S}_2}+C_{1\mathrm{ES}^2}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)\mathrm{S}_1\mathrm{S}_2}+C^2_{1\mathrm{ES}^2}\tilde{H}^{\mathrm{NLO}}_{C^2_{\mathrm{ES}_1^2}\mathrm{S}_1^3\mathrm{S}_2}+C_{1\mathrm{BS}^3}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{BS}_1^3)\mathrm{S}_2}\\
&+\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2^2}+C_{1\mathrm{ES}^2}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)\mathrm{S}_2^2}+C_{1\mathrm{ES}^2}C_{2\mathrm{ES}^2}\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)(\mathrm{ES}_2^2)}+(1\leftrightarrow2), \qquad \text{(E.1)}
\end{aligned}
$$

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with

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)\mathrm{S}_1^2} = {}& \frac{\nu^2\tilde{S}_1^4}{\tilde{r}^6}\left[\frac{3}{2}-\frac{3\nu}{2}+\tilde{p}_r^2\tilde{r}\left(\frac{45}{16}-\frac{21\nu}{8}\right)+\frac{1}{\nu q}\left(-\frac{3\nu^2}{2}+\frac{9\nu}{2}-\frac{3}{2}\right.\right.\\
&\left.\left.+\tilde{p}_r^2\tilde{r}\left(-\frac{39\nu^2}{16}+\frac{33\nu}{4}-\frac{45}{16}\right)\right)\right]+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[-\frac{3\nu}{4}+\frac{1}{q}\left(\frac{3}{4}-\frac{3\nu}{2}\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[9\nu-9+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{69\nu}{8}-\frac{135}{16}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{63\nu}{4}-\frac{135}{8}\right)\right.\\
&+\frac{1}{\nu q}\left(9\nu^2-27\nu+9+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{141\nu^2}{16}-\frac{51\nu}{2}+\frac{135}{16}\right)\right.\\
&\left.\left.+\tilde{p}_r^2\tilde{r}\left(\frac{117\nu^2}{8}-\frac{99\nu}{2}+\frac{135}{8}\right)\right)\right]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^4}{\tilde{r}^6}\left[\frac{15}{2}-\frac{15\nu}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{195}{16}-\frac{105\nu}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{225}{16}-\frac{105\nu}{8}\right)\right.\\
&+\frac{1}{\nu q}\left(-\frac{15\nu^2}{2}+\frac{45\nu}{2}-\frac{15}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{225\nu^2}{16}+\frac{75\nu}{2}-\frac{195}{16}\right)\right.\\
&\left.\left.+\tilde{p}_r^2\tilde{r}\left(-\frac{195\nu^2}{16}+\frac{165\nu}{4}-\frac{225}{16}\right)\right)\right]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{45}{8}+\frac{1}{\nu q}\left(\frac{45\nu^2}{8}+\frac{45\nu}{4}-\frac{45}{8}\right)\right]\\
&+\frac{\nu^2\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{45}{8}-6\nu+\frac{1}{\nu q}\left(-\frac{51\nu^2}{8}+\frac{69\nu}{4}-\frac{45}{8}\right)\right]\\
&+\frac{\nu^2\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{15}{8}+\frac{1}{\nu q}\left(\frac{15\nu^2}{8}+\frac{15\nu}{4}-\frac{15}{8}\right)\right],
\end{aligned}
\tag{E.2}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)^2} = {}& \frac{\nu^3\tilde{S}_1^4}{\tilde{r}^6}\left[-\frac{1}{8}+\frac{1}{\nu q}\left(\frac{1}{8}-\frac{\nu}{8}\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[\frac{21\nu}{4}-\frac{9}{2}+\frac{1}{\nu q}\left(\frac{21\nu^2}{4}-\frac{57\nu}{4}+\frac{9}{2}\right)\right]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^4}{\tilde{r}^6}\left[\frac{15}{2}-\frac{69\nu}{8}+\frac{1}{\nu q}\left(-\frac{69\nu^2}{8}+\frac{189\nu}{8}-\frac{15}{2}\right)\right],
\end{aligned}
\tag{E.3}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{(\mathrm{BS}_1^3)\mathrm{S}_1} = {}& \frac{\nu^2\tilde{S}_1^4}{\tilde{r}^6}\left[-\frac{\nu}{2}-\frac{3}{2}+\tilde{p}_r^2\tilde{r}\left(\frac{3\nu}{2}-\frac{3}{2}\right)+\frac{1}{\nu q}\left(-\frac{\nu^2}{2}-\frac{5\nu}{2}+\frac{3}{2}\right.\right.\\
&\left.\left.+\tilde{p}_r^2\tilde{r}\left(\frac{3\nu^2}{2}-\frac{9\nu}{2}+\frac{3}{2}\right)\right)\right]+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{3\nu}{2}-\frac{3}{2}+\frac{1}{\nu q}\left(\frac{3\nu^2}{2}-\frac{9\nu}{2}+\frac{3}{2}\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[6\nu+9+\frac{\tilde{L}^2}{\tilde{r}}(6-6\nu)+\tilde{p}_r^2\tilde{r}(9-9\nu)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\nu q}\Bigg(6\nu^2+12\nu-9+\frac{\tilde{L}^2}{\tilde{r}}\left(-6\nu^2+18\nu-6\right)+\tilde{p}_r^2\tilde{r}\left(-9\nu^2+27\nu-9\right)\Bigg)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^4}{\tilde{r}^6}\Bigg[-\frac{15\nu}{2}-\frac{15}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(10\nu-10\right)+\tilde{p}_r^2\tilde{r}\left(\frac{15\nu}{2}-\frac{15}{2}\right)\\
&+\frac{1}{\nu q}\Bigg(-\frac{15\nu^2}{2}-\frac{15\nu}{2}+\frac{15}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(10\nu^2-30\nu+10\right)\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{15\nu^2}{2}-\frac{45\nu}{2}+\frac{15}{2}\right)\Bigg)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{15}{2}-\frac{15\nu}{2}+\frac{1}{\nu q}\left(-\frac{15\nu^2}{2}+\frac{45\nu}{2}-\frac{15}{2}\right)\right]\\
&+\frac{\nu^2\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{9\nu}{2}-\frac{9}{2}+\frac{1}{\nu q}\left(\frac{9\nu^2}{2}-\frac{27\nu}{2}+\frac{9}{2}\right)\right]\\
&+\frac{\nu^2\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{5}{2}-\frac{5\nu}{2}+\frac{1}{\nu q}\left(-\frac{5\nu^2}{2}+\frac{15\nu}{2}-\frac{5}{2}\right)\right],
\end{aligned}
\tag{E.4}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{\mathrm{ES}_1^4}=&\frac{\nu^2\tilde{S}_1^4}{\tilde{r}^6}\Bigg[\frac{5\nu}{2}-\frac{23}{8}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15\nu}{16}-\frac{3}{16}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{9\nu}{8}+\frac{9}{16}\right)\\
&+\frac{1}{\nu q}\Bigg(\frac{5\nu^2}{2}-\frac{33\nu}{4}+\frac{23}{8}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9\nu^2}{8}-\frac{21\nu}{16}+\frac{3}{16}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{9\nu^2}{4}-\frac{9}{16}\right)\Bigg)\Bigg]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{3}{4}-\frac{3\nu}{4}+\frac{1}{\nu q}\left(-\frac{3\nu^2}{4}+\frac{9\nu}{4}-\frac{3}{4}\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\Bigg[30-\frac{105\nu}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{51\nu}{8}-\frac{9}{8}\right)+\tilde{p}_r^2\tilde{r}\left(-\frac{57\nu}{4}-\frac{21}{8}\right)\\
&+\frac{1}{\nu q}\Bigg(-\frac{105\nu^2}{4}+\frac{345\nu}{4}-30+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{33\nu^2}{4}+\frac{33\nu}{8}+\frac{9}{8}\right)\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{51\nu^2}{2}+9\nu+\frac{21}{8}\right)\Bigg)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^4}{\tilde{r}^6}\Bigg[\frac{125\nu}{4}-\frac{285}{8}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{95\nu}{16}+\frac{45}{16}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{145\nu}{8}+\frac{25}{16}\right)\\
&+\frac{1}{\nu q}\Bigg(\frac{125\nu^2}{4}-\frac{205\nu}{2}+\frac{285}{8}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{65\nu^2}{8}-\frac{5\nu}{16}-\frac{45}{16}\right)\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{125\nu^2}{4}-15\nu-\frac{25}{16}\right)\Bigg)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{15\nu}{4}-\frac{15}{4}+\frac{1}{\nu q}\left(\frac{15\nu^2}{4}-\frac{45\nu}{4}+\frac{15}{4}\right)\right]
\end{aligned}
$$

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$$+\frac{\nu^2\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{3\nu}{2}+\frac{9}{4}+\frac{1}{\nu q}\left(\frac{21\nu^2}{4}+3\nu-\frac{9}{4}\right)\right]$$

$$+\frac{\nu^2\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{15\nu}{2}-\frac{5}{4}+\frac{1}{\nu q}\left(-\frac{65\nu^2}{4}+5\nu+\frac{5}{4}\right)\right], \tag{E.5}$$

$$\tilde{H}^{\rm NLO}_{\rm E^2S_1^4}=\frac{27\nu^2\tilde{S}_1^4}{35\tilde{r}^6}\left[\frac{1}{8}-\frac{\nu}{8}+\frac{1}{\nu q}\left(-\frac{\nu^2}{8}+\frac{3\nu}{8}-\frac{1}{8}\right)\right]$$

$$+\frac{9\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{7\tilde{r}^6}\left[\frac{3\nu}{4}-\frac{3}{4}+\frac{1}{\nu q}\left(\frac{3\nu^2}{4}-\frac{9\nu}{4}+\frac{3}{4}\right)\right]$$

$$+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^4}{\tilde{r}^6}\left[\frac{9}{8}-\frac{9\nu}{8}+\frac{1}{\nu q}\left(-\frac{9\nu^2}{8}+\frac{27\nu}{8}-\frac{9}{8}\right)\right], \tag{E.6}$$

$$\tilde{H}^{\rm NLO}_{\rm S_1^3S_2}=\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[3+\frac{45}{8}\tilde{p}_r^2\tilde{r}+\frac{1}{\nu q}\left(3\nu-3+\tilde{p}_r^2\tilde{r}\left(\frac{21\nu}{4}-\frac{45}{8}\right)\right)\right]$$

$$+\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[\frac{3}{2}+\frac{1}{\nu q}\left(\frac{3\nu}{2}-\frac{3}{2}\right)\right]+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^7}\left[-\frac{3}{2}+\frac{3}{2}\frac{1}{\nu q}\right]$$

$$+\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[-12-6\frac{\tilde{L}^2}{\tilde{r}}-\frac{177}{8}\tilde{p}_r^2\tilde{r}\right.$$

$$\left.+\frac{1}{\nu q}\left(12-12\nu+\frac{\tilde{L}^2}{\tilde{r}}(6-6\nu)+\tilde{p}_r^2\tilde{r}\left(\frac{177}{8}-\frac{81\nu}{4}\right)\right)\right]$$

$$+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[-6-\frac{87}{8}\frac{\tilde{L}^2}{\tilde{r}}-\frac{93}{8}\tilde{p}_r^2\tilde{r}\right.$$

$$\left.+\frac{1}{\nu q}\left(6-6\nu+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{87}{8}-\frac{45\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{93}{8}-\frac{45\nu}{4}\right)\right)\right]$$

$$+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[15+\frac{195}{8}\frac{\tilde{L}^2}{\tilde{r}}+\frac{225}{8}\tilde{p}_r^2\tilde{r}\right.$$

$$\left.+\frac{1}{\nu q}\left(15\nu-15+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{105\nu}{4}-\frac{195}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{105\nu}{4}-\frac{225}{8}\right)\right)\right]$$

$$+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[\frac{15}{4}-\frac{15}{4}\frac{1}{\nu q}\right]+\frac{\nu^3\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{15}{2}-\frac{15}{2}\frac{1}{\nu q}\right]$$

$$+\frac{\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{39}{4}+\frac{1}{\nu q}\left(6\nu-\frac{39}{4}\right)\right]$$

$$+\frac{\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{15}{4}+\frac{15}{4}\frac{1}{\nu q}\right]$$

$$+\frac{\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[6+\frac{1}{\nu q}(6\nu-6)\right]$$

$$
\begin{aligned}
&+\frac{\nu^3\tilde{p}_r\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{9}{2}+\frac{9}{2}\frac{1}{\nu q}\right]\\
&+\frac{\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{15}{2}-\frac{15}{2}\frac{1}{\nu q}\right],
\end{aligned}
\tag{E.7}
$$

$$
\begin{aligned}
\tilde{H}^{\rm NLO}_{({\rm ES}_1^2){\rm S}_1{\rm S}_2}=&\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[\frac{5}{2}-\frac{9}{2}\tilde{p}_r^2\tilde{r}+\frac{1}{\nu q}\left(\frac{5\nu}{2}+6+\tilde{p}_r^2\tilde{r}\left(\frac{9}{2}-\frac{33\nu}{8}\right)\right)\right]\\
&+\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[-\frac{3}{2}+\frac{1}{\nu q}\left(\frac{3}{2}-3\nu\right)\right]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^7}\left[-3+\frac{1}{\nu q}(3-3\nu)\right]\\
&+\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{33}{2}+6\frac{\tilde{L}^2}{\tilde{r}}+\frac{33}{2}\tilde{p}_r^2\tilde{r}\right.\\
&\left.+\frac{1}{\nu q}\left(-\frac{33\nu}{2}-12+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{51\nu}{8}-6\right)+\tilde{p}_r^2\tilde{r}\left(\frac{129\nu}{8}-\frac{33}{2}\right)\right)\right]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[-\frac{33}{2}+12\frac{\tilde{L}^2}{\tilde{r}}+\frac{21}{2}\tilde{p}_r^2\tilde{r}\right.\\
&\left.+\frac{1}{\nu q}\left(-\frac{33\nu}{2}-15+\frac{\tilde{L}^2}{\tilde{r}}(12\nu-12)+\tilde{p}_r^2\tilde{r}\left(\frac{69\nu}{8}-\frac{21}{2}\right)\right)\right]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[\frac{93}{2}-30\frac{\tilde{L}^2}{\tilde{r}}-\frac{45}{2}\tilde{p}_r^2\tilde{r}\right.\\
&\left.+\frac{1}{\nu q}\left(\frac{93\nu}{2}+15+\frac{\tilde{L}^2}{\tilde{r}}\left(30-\frac{255\nu}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{45}{2}-\frac{165\nu}{8}\right)\right)\right]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[\frac{15}{2}+\frac{1}{\nu q}\left(15\nu-\frac{15}{2}\right)\right]\\
&+\frac{\nu^3\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[15+\frac{1}{\nu q}\left(\frac{75\nu}{4}-15\right)\right]\\
&+\frac{\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{3}{2}+\frac{1}{\nu q}\left(-\frac{21\nu}{8}-\frac{3}{2}\right)\right]\\
&+\frac{\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{15}{2}+\frac{1}{\nu q}\left(\frac{15}{2}-\frac{15\nu}{8}\right)\right]\\
&+\frac{\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-6+\frac{1}{\nu q}\left(6-\frac{51\nu}{8}\right)\right]\\
&+\frac{\nu^3\tilde{p}_r\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-9+\frac{1}{\nu q}\left(9-\frac{21\nu}{4}\right)\right]\\
&+\frac{\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[15+\frac{1}{\nu q}\left(\frac{105\nu}{8}-15\right)\right],
\end{aligned}
\tag{E.8}
$$

$$\tilde{H}^{\rm NLO}_{C^2_{\rm ES^2_1}\rm S^3_1S_2} = \frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[9+\frac{1}{\nu q}(9\nu-9)\right]+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[-9+\frac{1}{\nu q}(9-9\nu)\right], \tag{E.9}$$

$$\begin{aligned}
\tilde{H}^{\rm NLO}_{(\rm BS^3_1)S_2} = &\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\Bigg[-6-\frac{3}{2}\frac{\tilde{L}^2}{\tilde{r}}+\frac{3}{2}\tilde{p}_r^2\tilde{r}\\
&+\frac{1}{\nu q}\left(\frac{23}{2}-6\nu+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3}{2}-\frac{9\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{7}{4}-3\nu\right)\right)\Bigg]\\
&+\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[\frac{5}{2}+\frac{1}{\nu q}\left(\frac{5\nu}{2}-\frac{17}{4}\right)\right]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^7}\left[\frac{1}{2}+\frac{1}{\nu q}\left(\frac{\nu}{2}-\frac{1}{2}\right)\right]\\
&+\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\Bigg[34-\frac{5}{2}\frac{\tilde{L}^2}{\tilde{r}}+\frac{5}{2}\tilde{p}_r^2\tilde{r}\\
&+\frac{1}{\nu q}\left(34\nu-\frac{109}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{5\nu}{4}-\frac{11}{2}\right)+\tilde{p}_r^2\tilde{r}\left(25\nu-\frac{23}{4}\right)\right)\Bigg]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\Bigg[\frac{61}{2}+\frac{11}{2}\frac{\tilde{L}^2}{\tilde{r}}-\frac{11}{2}\tilde{p}_r^2\tilde{r}\\
&+\frac{1}{\nu q}\left(\frac{61\nu}{2}-58+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{37\nu}{4}-\frac{11}{2}\right)+\tilde{p}_r^2\tilde{r}\left(17\nu-\frac{43}{4}\right)\right)\Bigg]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\Bigg[-\frac{155}{2}+\frac{5}{2}\frac{\tilde{L}^2}{\tilde{r}}-\frac{5}{2}\tilde{p}_r^2\tilde{r}\\
&+\frac{1}{\nu q}\left(130-\frac{155\nu}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15}{2}-\frac{25\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{75}{4}-55\nu\right)\right)\Bigg]\\
&+\frac{\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[-\frac{25}{2}+\frac{1}{\nu q}\left(\frac{85}{4}-\frac{25\nu}{2}\right)\right]\\
&+\frac{\nu^3\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[-\frac{5}{2}+\frac{1}{\nu q}\left(\frac{25}{2}-\frac{5\nu}{2}\right)\right]\\
&+\frac{\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{35}{4}+\frac{1}{\nu q}\left(5\nu-\frac{43}{4}\right)\right]\\
&+\frac{\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^3\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{55}{4}+\frac{1}{\nu q}\left(\frac{55}{4}-5\nu\right)\right]\\
&+\frac{\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{5}{4}+\frac{1}{\nu q}(-5\nu-2)\right]
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{\nu^3\tilde{p}_r\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{3}{2}+\frac{1}{\nu q}\left(\frac{17}{2}-6\nu\right)\right]\\
&+\frac{\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{35}{4}+\frac{1}{\nu q}\left(35\nu-\frac{5}{2}\right)\right],
\end{aligned}
\tag{E.10}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2^2}=&\frac{\nu^3\tilde{S}_1^2\tilde{\vec{S}}_2^2}{2\tilde{r}^6}\left[-\frac{27}{4}\frac{\tilde{L}^2}{\tilde{r}}-\frac{3}{4}\tilde{p}_r^2\tilde{r}\right]+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2)^2}{2\tilde{r}^6}\left[3+\frac{27\nu}{4}\frac{\tilde{L}^2}{\tilde{r}}+\frac{3\nu}{2}\tilde{p}_r^2\tilde{r}\right]\\
&+\frac{27\nu^3\tilde{S}_1^2(\tilde{\vec{S}}_2\cdot\tilde{\vec{L}})^2}{4\tilde{r}^7}-\frac{33\nu^3\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{4\tilde{r}^7}\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_2\cdot\vec{n})^2}{\tilde{r}^6}\left[3\nu-3+\frac{27\nu}{4}\frac{\tilde{L}^2}{\tilde{r}}+\frac{3\nu}{4}\tilde{p}_r^2\tilde{r}+\frac{3\nu}{q}\right]\\
&+\frac{\nu^2\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{2\tilde{r}^6}\left[-15-\frac{51\nu}{4}\frac{\tilde{L}^2}{\tilde{r}}-6\nu\tilde{p}_r^2\tilde{r}\right]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^2(\tilde{\vec{S}}_2\cdot\vec{n})^2}{2\tilde{r}^6}\left[15-\frac{15\nu}{4}\frac{\tilde{L}^2}{\tilde{r}}+\frac{15\nu}{4}\tilde{p}_r^2\tilde{r}\right]-\frac{15\nu^3(\tilde{\vec{S}}_1\cdot\vec{n})^2(\tilde{\vec{S}}_2\cdot\tilde{\vec{L}})^2}{4\tilde{r}^7}\\
&+\frac{15\nu^3\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}-\frac{9\nu^3\tilde{p}_r\tilde{S}_1^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{4\tilde{r}^6}\\
&+\frac{3\nu^3\tilde{p}_r\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{2\tilde{r}^6}+\frac{15\nu^3\tilde{p}_r(\tilde{\vec{S}}_1\cdot\vec{n})^2\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{4\tilde{r}^6},
\end{aligned}
\tag{E.11}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)\mathrm{S}_2^2}=&\frac{\nu^2\tilde{S}_1^2\tilde{\vec{S}}_2^2}{\tilde{r}^6}\left[\frac{11\nu}{2}+3+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{87\nu}{4}-\frac{201}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{3\nu}{8}-\frac{15}{16}\right)\right.\\
&\left.+\frac{\nu}{q}\left(\frac{11}{2}+\frac{201}{8}\frac{\tilde{L}^2}{\tilde{r}}+\frac{15}{16}\tilde{p}_r^2\tilde{r}\right)\right]+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2)^2}{\tilde{r}^6}\left[-\frac{3\nu}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{201}{8}-\frac{87\nu}{4}\right)\right.\\
&\left.+\tilde{p}_r^2\tilde{r}\left(\frac{21}{8}-\frac{9\nu}{4}\right)+\frac{\nu}{q}\left(-\frac{3}{2}-\frac{201}{8}\frac{\tilde{L}^2}{\tilde{r}}-\frac{21}{8}\tilde{p}_r^2\tilde{r}\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_2\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{225}{8}-24\nu-\frac{225}{8}\frac{\nu}{q}\right]+\frac{\nu^2\tilde{S}_2^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\left[\frac{201}{8}-\frac{87\nu}{4}-\frac{201}{8}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^7}\left[42\nu-\frac{195}{4}+\frac{195}{4}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde{S}_1^2(\tilde{\vec{S}}_2\cdot\vec{n})^2}{\tilde{r}^6}\left[-\frac{21\nu}{2}-3+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{345}{16}-\frac{147\nu}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{15}{16}-\frac{3\nu}{8}\right)\right.\\
&\left.+\frac{\nu}{q}\left(-\frac{21}{2}-\frac{345}{16}\frac{\tilde{L}^2}{\tilde{r}}-\frac{15}{16}\tilde{p}_r^2\tilde{r}\right)\right]\\
&+\frac{\nu^2\tilde{S}_2^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[-15\nu-9+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{201}{8}-\frac{87\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{9\nu}{8}+\frac{3}{16}\right)\right.\\
&\left.+\frac{\nu}{q}\left(-15-\frac{201}{8}\frac{\tilde{L}^2}{\tilde{r}}-\frac{3}{16}\tilde{p}_r^2\tilde{r}\right)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{\nu^2\tilde{\vec S}_1\cdot\tilde{\vec S}_2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_2\cdot\vec n}{\tilde r^6}\left[-21\nu-6+\frac{\tilde L^2}{\tilde r}\left(\frac{99\nu}{2}-\frac{225}{4}\right)+\tilde p_r^2\tilde r\left(\frac{21\nu}{2}-\frac{45}{4}\right)\right.\\
&\left.+\frac{\nu}{q}\left(-21+\frac{225}{4}\frac{\tilde L^2}{\tilde r}+\frac{45}{4}\tilde p_r^2\right)\right]\\
&+\frac{\nu^2(\tilde{\vec S}_1\cdot\vec n)^2(\tilde{\vec S}_2\cdot\vec n)^2}{\tilde r^6}\left[\frac{105\nu}{2}+15+\frac{\tilde L^2}{\tilde r}\left(\frac{285}{16}-\frac{135\nu}{8}\right)+\tilde p_r^2\tilde r\left(\frac{135}{16}-\frac{75\nu}{8}\right)\right.\\
&\left.+\frac{\nu}{q}\left(\frac{105}{2}-\frac{285}{16}\frac{\tilde L^2}{\tilde r}-\frac{135}{16}\tilde p_r^2\tilde r\right)\right]+\frac{\nu^2(\tilde{\vec S}_1\cdot\vec n)^2(\tilde{\vec S}_2\cdot\tilde{\vec L})^2}{\tilde r^7}\left[30\nu-\frac{285}{8}+\frac{285}{8}\frac{\nu}{q}\right]\\
&+\frac{\nu^2(\tilde{\vec S}_2\cdot\vec n)^2(\tilde{\vec S}_1\cdot\tilde{\vec L})^2}{\tilde r^7}\left[\frac{45\nu}{2}-\frac{105}{4}+\frac{105}{4}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_1\cdot\tilde{\vec L}\tilde{\vec S}_2\cdot\tilde{\vec L}}{\tilde r^7}\left[\frac{135}{4}-30\nu-\frac{135}{4}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r\tilde S_1^2\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_2\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[\frac{45}{8}-\frac{9\nu}{2}-\frac{45}{8}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r\tilde{\vec S}_1\cdot\tilde{\vec S}_2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_2\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[15-12\nu-15\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r(\tilde{\vec S}_1\cdot\vec n)^2\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_2\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[\frac{45\nu}{2}-\frac{225}{8}+\frac{225}{8}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r\tilde S_2^2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_1\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[6\nu-9+9\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r\tilde{\vec S}_1\cdot\tilde{\vec S}_2\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_1\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[3\nu-\frac{15}{4}+\frac{15}{4}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r(\tilde{\vec S}_2\cdot\vec n)^2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_1\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[\frac{75}{4}-15\nu-\frac{75}{4}\frac{\nu}{q}\right],
\end{aligned}
\tag{E.12}
$$

$$
\begin{aligned}
\tilde H^{\rm NLO}_{({\rm ES}_1^2)({\rm ES}_2^2)}=&\frac{\nu^3\tilde S_1^2\tilde{\vec S}_2^2}{2\tilde r^6}\left[\frac{9}{2}+\frac{\tilde L^2}{\tilde r}\left(-\frac{3\nu}{8}-\frac{21}{8}\right)+\tilde p_r^2\tilde r\left(\frac{15}{8}-\frac{9\nu}{4}\right)\right]\\
&+\frac{\nu^2(\tilde{\vec S}_1\cdot\tilde{\vec S}_2)^2}{2\tilde r^6}\left[\frac{9}{2}+\frac{\tilde L^2}{\tilde r}\left(-\frac{3\nu}{4}-\frac{3}{4}\right)+\tilde p_r^2\tilde r\left(-\frac{9\nu}{2}-\frac{9}{4}\right)\right]\\
&+\frac{\nu^2\tilde S_1^2(\tilde{\vec S}_2\cdot\tilde{\vec L})^2}{\tilde r^7}\left[\frac{9\nu}{4}-\frac{3}{4}+\frac{9}{4}\frac{\nu}{q}\right]-\frac{3\nu^2\tilde{\vec S}_1\cdot\tilde{\vec L}\tilde{\vec S}_2\cdot\tilde{\vec L}\tilde{\vec S}_1\cdot\tilde{\vec S}_2}{4\tilde r^7}\\
&+\frac{\nu^2\tilde S_1^2(\tilde{\vec S}_2\cdot\vec n)^2}{\tilde r^6}\left[-3\nu-15+\frac{\tilde L^2}{\tilde r}\left(\frac{33\nu}{8}+\frac{39}{8}\right)+\tilde p_r^2\tilde r\left(\frac{21\nu}{4}+\frac{21}{8}\right)\right.\\
&\left.+\frac{3\nu}{q}\left(-3+\frac{9}{4}\frac{\tilde L^2}{\tilde r}-6\tilde p_r^2\tilde r\right)\right]\\
&+\frac{\nu^2\tilde{\vec S}_1\cdot\tilde{\vec S}_2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_2\cdot\vec n}{2\tilde r^6}\left[-48+\frac{\tilde L^2}{\tilde r}\left(\frac{15\nu}{2}+\frac{27}{2}\right)+\tilde p_r^2\tilde r\left(45\nu+\frac{33}{2}\right)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{\nu^2(\tilde{\vec S}_1\cdot\vec n)^2(\tilde{\vec S}_2\cdot\vec n)^2}{2\tilde r^6}\left[93+\frac{\tilde L^2}{\tilde r}\left(-\frac{105\nu}{8}-\frac{225}{8}\right)+\tilde p_r^2\tilde r\left(-\frac{315\nu}{4}-\frac{195}{8}\right)\right]\\
&+\frac{\nu^2(\tilde{\vec S}_1\cdot\vec n)^2(\tilde{\vec S}_2\cdot\tilde{\vec L})^2}{\tilde r^7}\left[\frac{15}{4}-\frac{15\nu}{4}-\frac{15}{4}\frac{\nu}{q}\right]\\
&+\frac{15\nu^2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_1\cdot\tilde{\vec L}\tilde{\vec S}_2\cdot\tilde{\vec L}}{4\tilde r^7}+\frac{\nu^2\tilde p_r\tilde S_1^2\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_2\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[\frac{21\nu}{2}-\frac{9}{4}+\frac{57}{4}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r\tilde{\vec S}_1\cdot\tilde{\vec S}_2\tilde{\vec S}_1\cdot\vec n\tilde{\vec S}_2\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[-6+\frac{15}{2}\frac{\nu}{q}\right]\\
&+\frac{\nu^2\tilde p_r(\tilde{\vec S}_1\cdot\vec n)^2\tilde{\vec S}_2\cdot\vec n\tilde{\vec S}_2\times\tilde{\vec L}\cdot\vec n}{\tilde r^6}\left[\frac{15\nu}{2}+\frac{45}{4}-\frac{75}{4}\frac{\nu}{q}\right]. \qquad \text{(E.13)}
\end{aligned}
$$

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