

Angular Momentum Loss due to Tidal Effects in the Post-Minkowskian Expansion

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We calculate the tidal corrections to the loss of angular momentum in a two-body collision at leading post-Minkowskian order from an amplitude-based approach. The eikonal operator allows us to efficiently combine elastic and inelastic amplitudes, and captures both the contributions due to genuine gravitational-wave emissions and those due to the static gravitational field. We calculate the former by harnessing powerful collider-physics techniques such as reverse unitarity, thereby reducing them to cut two-loop integrals, and cross check the result by performing an independent calculation in the post-Newtonian limit. For the latter, we can employ the results of P. Di Vecchia *et al.* [Angular momentum of zero-frequency gravitons, *J. High Energy Phys.* **08** (2022) 172.], where static-field effects were calculated for generic gravitational scattering events using the leading soft graviton theorem.

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Introduction.—The steadily increasing sensitivity of gravitational-wave measurements challenges the state of the art of precision calculations for gravitational collisions [1]. In this context, scattering amplitudes have found fertile ground and contributed to advance the precision frontier in the post-Minkowskian (PM) expansion, based on successive approximations labeled by powers of the Newton constant G (see Ref. [2] for our conventions on the units of G , c , and \hbar) and valid for generic velocities [3–7]. This synergy between general relativity and amplitude methods, and its recent successes highlight the importance of pushing these calculations to higher orders and of including all relevant physical effects, such as spin [8–33] and tidal corrections [34–38] that will be vital, in combination with numerical simulations, to provide accurate waveform models [1]. The measurement of effects due to tidal deformations [39–49], in particular, may provide clues on the internal structure of neutron stars [50], on the nature of black holes [51], and on the possible existence of exotic astrophysical objects [52–54].

Amplitudes provide a natural way to organize the G expansion, based on the standard perturbative series where the double copy [55–60], generalized unitarity [61–63], and gauge invariance offer powerful techniques for integrand construction. Resummation methods like the eikonal exponentiation [64–77], effective-field-theory matching [78–81], or the KMOC framework [82–88] then provide the needed bridge between the quantum formulation and the classical

PM regime of scattering at large impact parameter $b \gg Gm$, with m the typical mass (or energy) scale of the colliding objects. Moreover, techniques borrowed from collider physics like integration via differential equations [71,89] and reverse unitarity [83,84,90–93] have recently proven very valuable when applied to the calculation of classical observables as well. Such techniques and ideas have also been exploited in the context of quantum-field-theory-inspired worldline setups that efficiently encode the PM expansion [94–112].

In this Letter we focus on dissipative effects induced by linear tidal deformations corresponding to mass or “electric” and current or “magnetic” quadrupole corrections. Combining eikonal operator [75,77,85,113–116] and reverse unitarity, we first confirm the results of Refs. [108,109] for the radiated energy-momentum and then obtain a totally new prediction: the angular momentum lost due to tidal effects, thus completing the analysis of the Poincaré charges of the gravitational field to leading PM order, performed in Refs. [83,117] for point particles, and initiated in Refs. [108,109] for tidal effects. Two types of contributions are relevant for this calculation. The first is due to the emission of gravitational waves, described by superpositions of dynamically propagating gravitons. The second is due to static-field effects that are localized at the zero-frequency end of the graviton spectrum. Both fit naturally within our approach.

Radiative contributions are calculated by recasting them as Fourier transforms of three-particle cuts, which can be in turn evaluated as cut two-loop integrals [71,83,105]. Static contributions follow from the results of Ref. [113], where they were evaluated for generic processes exploiting the universality of the leading soft graviton theorem, supplemented by

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the tidal corrections to the impulse [34,35,37]. The fluxes of energy and angular momentum serve, in combination with the binding energies, as ingredients for building accurate waveform models [1,118–120] that are crucial for gravitational-wave detection and analysis. For this reason, we also provide the analytic continuation of the result to bound orbits in the high-eccentricity limit and the associated flux, which contains the exact dependence on the velocity and can be used in the future to improve the waveform at large velocities [1]. Appendix A summarizes our kinematics conventions. Appendix B details the notation for integration, Fourier transforms, and index contractions. In Appendix C we quote the tidal effects in the impulse.

Eikonal operator.—The eikonal operator determines the final state of a gravitational collision in terms of the initial one in the classical limit. It combines the eikonal phase $\text{Re}2\delta$, which determines the deflection (see Ref. [71] and references therein) and is sensitive to tidal effects starting at one loop [35], with the gravitational waveform $\tilde{\mathcal{A}}^{\mu\nu}$ [99,100,121,122], obtained to leading order from the five-point amplitude $\mathcal{A}^{\mu\nu}$ via Fourier transform [Eq. (B7)]. Introducing the graviton creation-annihilation operators $\hat{a}_k^\dagger, \hat{a}_k$, up to 3PM order it describes gravitational waves as coherent graviton emissions, [123]

$$\hat{S} = e^{i\text{Re}2\delta} e^{i \int_k (\tilde{\mathcal{A}}(k) \hat{a}_k^\dagger + \tilde{\mathcal{A}}^*(k) \hat{a}_k)} \quad (1)$$

so that $|\Psi_{\text{out}}\rangle = \hat{S}|\Psi_{\text{in}}\rangle$, where $|\Psi_{\text{in}}\rangle$ models two incoming particles with mass m_1, m_2 and impact parameter $b^\alpha = b_1^\alpha - b_2^\alpha$, while $|\Psi_{\text{out}}\rangle$ captures the final configuration. In the following, we will calculate the expectation values of the linear and angular momentum operators of the gravitational field in the final state

$$P^\alpha = \langle \Psi_{\text{in}} | \hat{S}^\dagger \hat{P}^\alpha \hat{S} | \Psi_{\text{in}} \rangle, \quad J^{\alpha\beta} = \langle \Psi_{\text{in}} | \hat{S}^\dagger \hat{J}^{\alpha\beta} \hat{S} | \Psi_{\text{in}} \rangle \quad (2)$$

taking into account tidal corrections. Since the (connected) amplitude $\mathcal{A}^{\mu\nu}$ only includes the standard Weinberg limit of *soft but nonzero* momenta, the quantities in Eq. (2) only include effects due to dynamically propagating gravitons, and involve no contributions localized at zero frequency, i.e., no static terms.

To include such terms, it is sufficient to perform the following dressing [113,116]:

$$|\text{out/in}\rangle = e^{\int_k (F_{\text{out/in}}(k) \hat{a}_k^\dagger - F_{\text{out/in}}^*(k) \hat{a}_k)} |\Psi_{\text{out/in}}\rangle, \quad (3)$$

where, introducing a soft scale ω^* (to be later sent to 0),

$$F_{\text{out/in}}^{\mu\nu}(k) = \Theta(\omega^* - k^0) \sum_{n \in \text{out/in}} \frac{\eta_n \sqrt{8\pi G} p_n^\mu p_n^\nu}{p_n \cdot k - i0}, \quad (4)$$

and $\eta_n = +1$ if n is outgoing, and $\eta_n = -1$ if n is incoming, which recovers the static effects via the $-i0$

prescription [100,105,113,117]. In this way, $|\text{out}\rangle = \hat{S}|\text{in}\rangle$ provided

$$\hat{S} = e^{i2\tilde{\delta}} e^{\int_k (F(k) \hat{a}_k^\dagger - F^*(k) \hat{a}_k)} e^{i \int_k (\tilde{\mathcal{A}}(k) \hat{a}_k^\dagger + \tilde{\mathcal{A}}^*(k) \hat{a}_k)} \quad (5)$$

where $F^{\mu\nu} = F_{\text{out}}^{\mu\nu} - F_{\text{in}}^{\mu\nu}$ is the total soft factor and $2\tilde{\delta} = \text{Re}2\delta - 2\delta^{\text{RR}}$ is the *conservative* eikonal phase [116]. The dressed expectation values [8,124–126]

$$P^\alpha = \langle \Psi_{\text{in}} | \hat{S}^\dagger \hat{P}^\alpha \hat{S} | \Psi_{\text{in}} \rangle, \quad J^{\alpha\beta} = \langle \Psi_{\text{in}} | \hat{S}^\dagger \hat{J}^{\alpha\beta} \hat{S} | \Psi_{\text{in}} \rangle \quad (6)$$

then also capture the effects of the static gravitational field. The distinction between Eqs. (2) and (6) is irrelevant for the linear momentum, $P^\alpha = \mathbf{P}^\alpha$, but the angular momentum is sensitive to it [113,116,127–131]: $J^{\alpha\beta} = \mathbf{J}^{\alpha\beta} + \mathcal{J}^{\alpha\beta}$, with the former term due to radiative modes and the latter due to static modes [100,105,113,116,117].

Tidal effects in the five-point amplitude.—The $2 \rightarrow 3$ amplitude in the classical limit $\mathcal{A}^{\mu\nu}$ for graviton emissions up to linear order in the tidal couplings [34,35,37] can be obtained, at tree level, from the stress-energy tensors $t^{\mu\nu}$ calculated in Refs. [100,105,108] via $\mathcal{A}^{\mu\nu} = 4(8\pi G)^{3/2} m_1^2 m_2^2 t^{\mu\nu} / (q_1^2 q_2^2)$ [132]. We shall follow the notation of Ref. [108] and denote by $c_{E_i^2}, c_{B_i^2}$, where $i = 1, 2$ labels the two colliding objects, the couplings associated to mass-electric-type and current-magnetic-type effects, $X = E, B$ for short [137]. These are related to the Love numbers $k_i^{(2)}, j_i^{(2)}$ by $c_{E_i^2} = \frac{1}{6} k_i^{(2)} R_i^5 / G$ and $c_{B_i^2} = (1/32) j_i^{(2)} R_i^5 / G$ with R_i the radius of object i . Note that $R_i = Gm_i / K_i$, with K_i an additional perturbative parameter characterizing the star's “compactness,” roughly of order 0.1–0.2 for typical neutron stars [138,139]. In this way, tidal effects can compete with point-particle effects, controlled by Gm_i/b [140].

We first restrict for simplicity to the case $c_{X_2^2} = 0$; the general case will be obtained by symmetrizing over particle labels. Accordingly, $\mathcal{A}^{\mu\nu} = \mathcal{A}_{\text{pp}}^{\mu\nu} + \mathcal{A}_{E_1^2}^{\mu\nu} + \mathcal{A}_{B_1^2}^{\mu\nu}$, where $\mathcal{A}_{\text{pp}}^{\mu\nu}$ is the point-particle contribution [94,98,133] and the $\mathcal{A}_{X_1^2}^{\mu\nu}$ capture the linear tidal effects [108]. We provide their expressions in an ancillary file [141]. Note that $\mathcal{A}^{\mu\nu}$ obeys the conservation condition only up to contact terms, $k_\mu \mathcal{A}^{\mu\nu} = C^\nu$, where C^ν is analytic in q_1^2 and q_2^2 and thus vanishes upon Fourier transform [Eq. (B7)] for large b . We checked that our results are unchanged if we add contact terms to $\mathcal{A}^{\mu\nu}$.

Radiative modes.—In view of Eq. (1), the formula expressing the total radiated energy momentum [Eq. (2)] in terms of $\tilde{\mathcal{A}}^{\mu\nu}$ is given by [71,83,84,117]

$$P^\alpha = \int_k \tilde{\mathcal{A}}^{\alpha} \tilde{\mathcal{A}}^*. \quad (7)$$

Since $\tilde{\mathcal{A}}^{\mu\nu}$ involves a complicated dependence on Bessel functions while $\mathcal{A}^{\mu\nu}$ is a rational function, it pays off to recast this integral as the Fourier transform of a convolution, i.e., an integral over the three-particle phase space,

$$P^\alpha = \text{FT} \int d(\text{LIPS}) k^\alpha \quad (8)$$

Here FT is the Fourier transform defined in Eq. (B5), each five-point amplitude represents $\mathcal{A}^{\mu\nu}$ as in Eq. (B6), and $d(\text{LIPS})$ stands for the Lorentz-invariant phase space measure in the soft region [71,84,89],

$$\frac{d^D k}{(2\pi)^D} 2\pi\theta(k^0)\delta(k^2) \frac{d^D q_1}{(2\pi)^D} 2\pi\delta(2p_1 \cdot q_1) 2\pi\delta[2p_2 \cdot (q_1 + k)].$$

To evaluate these integrals, we use reverse unitarity [83,84,90–92]. Starting from Eq. (8), we first rewrite the phase-space delta functions as “cut” propagators, via the identity $2i\pi\delta(x) = [1/(x - i0)] - [1/(x + i0)]$, and then apply integration by parts (IBP) identities to the resulting integrals of rational functions to recast them as linear combinations of the master integrals (MIs) calculated in Refs. [68,71,84]. We employ the *Mathematica* package LiteRed [142,143] for the IBP reduction. We refer to Ref. [71], Secs. 3 and 6.1 for more details about the integration and the MIs. Focusing on the terms linear in the tidal effects, we thus confirm the results of Refs. [108,109],

$$P_{\text{tid}}^\alpha = R_f \sum_X \frac{c_{X_1^2}}{m_1} (\mathcal{E}^X \check{u}_1^\alpha + \mathcal{F}^X \check{u}_2^\alpha), \quad (9)$$

where $R_f = 15\pi G^3 m_1^2 m_2^2 / (64b^7)$, while

$$\mathcal{E}^X = f_1^X + f_2^X \log \frac{\sigma + 1}{2} + f_3^X \frac{\sigma \text{arccosh} \sigma}{2\sqrt{\sigma^2 - 1}}, \quad (10)$$

with $f_3^X = -(\sigma^2 - \frac{3}{2})f_2^X/(\sigma^2 - 1)$, and $f_1^X, f_2^X, \mathcal{F}^X$ are given in Table I as functions of $\sigma = -u_1 \cdot u_2$.

The expression (9) for the radiated energy momentum holds for $c_{X_2^2} = 0$, and the generic case is obtained by symmetrizing over $1 \leftrightarrow 2$. The relation between f_3^X and f_2^X is due to the fact that the tidal interactions under consideration are linear (no “H topology”) [108]. Verifying that the coefficient of b^μ in Eq. (9) vanishes provides an internal cross-check for the calculation. Indeed, since the integrand of Eq. (8) is real, a component along b^μ would originate from a term of the type $f(\sigma, q^2)q^\mu$ with real f , whose Fourier transform [Eq. (B5)] is purely imaginary.

From the eikonal operator [Eq. (1)], one can derive the following formulas expressing the radiated angular momentum [Eq. (2)] in terms of $\tilde{\mathcal{A}}^{\mu\nu}$ [113,116,117]: $J_{\alpha\beta} = J_{\alpha\beta}^{(o)} + J_{\alpha\beta}^{(s)}$,

TABLE I. Functions entering the radiated energy-momentum due to linear tidal effects [108].

$f_1^E = \frac{(\sigma^2-1)^{\frac{1}{2}}}{2(\sigma+1)^3} [937\sigma^9 + 1551\sigma^8 - 2463\sigma^7 - 5645\sigma^6 + 20415\sigma^5 + 65965\sigma^4 - 349541\sigma^3 + 535057\sigma^2 - 360356\sigma + 92160]$
$f_2^E = 30\sqrt{\sigma^2-1}(21\sigma^4 - 14\sigma^2 + 9)$
$\mathcal{F}^E = \frac{3(\sigma^2-1)^{\frac{3}{2}}}{(\sigma+1)^5} [42\sigma^8 + 210\sigma^7 + 315\sigma^6 - 105\sigma^5 - 944\sigma^4 - 1528\sigma^3 + 22011\sigma^2 - 33201\sigma + 16272]$
$f_1^B = \frac{\sqrt{\sigma^2-1}}{4(\sigma+1)^4} [1559\sigma^8 + 3716\sigma^7 - 1630\sigma^6 - 11660\sigma^5 + 28288\sigma^4 + 155292\sigma^3 - 543442\sigma^2 + 535212\sigma - 180775]$
$f_2^B = 210(\sigma^2 - 1)^{\frac{3}{2}}(3\sigma^2 + 1)$
$\mathcal{F}^B = \frac{-3(105\sigma^5 + 1630\sigma^4 + 1840\sigma^3 + 3690\sigma^2 - 17769\sigma + 15984)}{(\sigma+1)^6(\sigma^2-1)^{\frac{3}{2}}}$

$$iJ_{\alpha\beta}^{(o)} = \int_k k_{[\alpha} \frac{\partial \tilde{\mathcal{A}}}{\partial k^{\beta]}} \tilde{\mathcal{A}}^*, \quad J_{\alpha\beta}^{(s)} = i \int_k 2\tilde{\mathcal{A}}_{[\alpha}^\mu \tilde{\mathcal{A}}_{\beta]\mu}^*. \quad (11)$$

Under a translation [Eq. (B8)] [117],

$$J^{\alpha\beta} \rightarrow J^{\alpha\beta} + a^{[\alpha} P^{\beta]}. \quad (12)$$

It is straightforward to express $J_{\alpha\beta}^{(s)}$ as the Fourier transform of a three-particle cut, as we did for P^α in Eq. (8), with appropriate index contractions. This step is more delicate for $J_{\alpha\beta}^{(o)}$, which involves derivatives with respect to k^μ that can act on the mass-shell delta functions. Nevertheless, in a frame where $b_1^\mu = b^\mu$ and $b_2^\mu = 0$ where Eq. (B9) applies, one can recast it in the form of [116]

$$iJ_{\alpha\beta}^{(o)} = \text{FT} \int k_{[\alpha} \frac{\partial}{\partial k^{\beta]}} \left[d(\text{LIPS}) \right] \quad (13)$$

where the derivative in the first line can act both on $\mathcal{A}^{\mu\nu}$ and on $d(\text{LIPS})$, and $q_{||2} = -u_2 \cdot q^\mu$. The integrals to be performed then belong to the same family as for Eq. (8), so we can evaluate them in the same way.

Translating the result to a frame where $b_1^\mu = 0$ and $b_2^\mu = -b^\mu$, using the simple transformation law [Eq. (12)] and the explicit result [Eq. (9)] for P_{tid}^α , we obtain the

TABLE II. Functions entering the radiated angular momentum due to E_1^2 tidal coupling.

$$\begin{aligned}
 g_1^E &= \frac{(\sigma^2-1)^{-\frac{3}{2}}}{10(\sigma+1)^3} [2573\sigma^9 + 9819\sigma^8 + 13143\sigma^7 + 1845\sigma^6 \\
 &\quad - 897603\sigma^5 + 3221239\sigma^4 - 5046195\sigma^3 + 4203751\sigma^2 \\
 &\quad - 1862318\sigma + 351826] \\
 g_2^E &= -6(35\sigma^4 - 50\sigma^2 - 1)/\sqrt{\sigma^2 - 1} \\
 h_1^E &= \frac{4(\sigma^2-1)^{-\frac{3}{2}}}{5(\sigma+1)^2} [492\sigma^7 + 564\sigma^6 - 609\sigma^5 - 722\sigma^4 \\
 &\quad - 4636\sigma^3 + 13478\sigma^2 - 14143\sigma + 5096] \\
 h_2^E &= 48\sigma(7\sigma^2 + 1)/\sqrt{\sigma^2 - 1}
 \end{aligned}$$

following new result for the radiated angular momentum due to linear tidal effects,

$$\mathbf{J}_{\text{tid}}^{\alpha\beta} = R_f \sum_X \frac{c_{X1}^2}{m_1} (\mathcal{C}^X b^{[\alpha} u_1^{\beta]} + \mathcal{D}^X u_2^{[\alpha} b^{\beta]}) \quad (14)$$

where R_f is given below Eq. (9),

$$\mathcal{C}^X = g_1^X + g_2^X \log \frac{\sigma+1}{2} + g_3^X \frac{\sigma \text{arccosh} \sigma}{2\sqrt{\sigma^2-1}}, \quad (15)$$

$$\mathcal{D}^X = h_1^X + h_2^X \log \frac{\sigma+1}{2} + h_3^X \frac{\sigma \text{arccosh} \sigma}{2\sqrt{\sigma^2-1}}, \quad (16)$$

with $g_3^X = -(\sigma^2 - \frac{3}{2})g_2^X/(\sigma^2 - 1)$ and similarly $h_3^X = -(\sigma^2 - \frac{3}{2})h_2^X/(\sigma^2 - 1)$, and the functions g_j^X , h_j^X for $j = 1, 2$ are detailed in Table II for $X = E$ and in Table III for $X = B$. Verifying that the coefficient of $u_1^{[\alpha} u_2^{\beta]}$ in Eq. (14) vanishes serves an internal consistency check analogous to one discussed for $\mathbf{P}_{\text{tid}}^a$. Each integrand on the right-hand side of Eq. (13) is real and terms $f(\sigma, q^2) u_1^{[\alpha} u_2^{\beta]}$ with real f would be real in b space, and hence contribute imaginary terms to the radiated angular momentum. In fact, such terms do appear separately in each line of Eq. (13), but they crucially cancel out in the sum.

Since Eq. (14) holds for $c_{X2}^2 = 0$ in a frame where $b_1^a = 0$, $b_2^a = -b^a$, interchanging all particle labels in it

 TABLE III. Functions entering the radiated angular momentum due to B_1^2 tidal coupling.

$$\begin{aligned}
 g_1^B &= \frac{20(\sigma^2-1)^{-\frac{1}{2}}}{(\sigma+1)^4} [4495\sigma^8 + 22180\sigma^7 + 46630\sigma^6 + 50020\sigma^5 \\
 &\quad - 1748636\sigma^4 + 4687932\sigma^3 - 5397990\sigma^2 + 3026428\sigma - 681459] \\
 g_2^B &= -30\sqrt{\sigma^2-1}(7\sigma^2-3) \\
 h_1^B &= \frac{2(\sigma^2-1)^{-\frac{1}{2}}}{5(\sigma+1)^3} [879\sigma^6 + 1797\sigma^5 - 492\sigma^4 - 2908\sigma^3 \\
 &\quad - 10491\sigma^2 + 18815\sigma - 9280] \\
 h_2^B &= 336\sigma\sqrt{\sigma^2-1}
 \end{aligned}$$

yields the radiated angular momentum for $c_{X1}^2 = 0$ in a frame where $b_1^a = b^a$, $b_2^a = 0$ instead, but one can obtain $\mathbf{J}_{\text{tid}}^{\alpha\beta}$ in any desired translation frame with the help of the simple transformation law [Eq. (12)] and the explicit form [Eq. (9)] for $\mathbf{P}_{\text{tid}}^a$.

In addition to translations, Eq. (14) is also covariant under Lorentz transformations. The physical meaning of \mathcal{C}^X and \mathcal{D}^X becomes transparent in frames where not only $b_1^a = 0$ but one of the two particles is also initially at rest, where they are proportional to the angular momentum of gravitational waves. For definiteness, we align the impact parameter along the y axis, $b^a = (0, 0, b, 0)$, and the motion of the incoming particle along the x axis. In a frame where particle 1 is at rest, $u_1^a = (1, 0)$, $u_2^a = (\sigma, p_\infty, 0, 0)$ with $p_\infty = \sqrt{\sigma^2 - 1}$, so

$$\mathbf{J}_{\text{tid}}^{xy} = R_f b p_\infty \sum_X \frac{c_{X1}^2}{m_1} \mathcal{D}^X, \quad (17)$$

while in a frame where particle 2 is at rest $u_1^a = (\sigma, -p_\infty, 0, 0)$, $u_2^a = (1, 0)$, the same formula applies with \mathcal{C}^X replaced by \mathcal{D}^X . In the nonrelativistic limit $p_\infty \rightarrow 0$,

$$\begin{aligned}
 \mathcal{C}^E &= \frac{1056}{5} p_\infty - \frac{349}{35} p_\infty^3 + \mathcal{O}(p_\infty^5) \\
 \mathcal{D}^E &= \frac{1056}{5} p_\infty - \frac{324}{7} p_\infty^3 + \mathcal{O}(p_\infty^5) \\
 \mathcal{C}^B &= 40 p_\infty^3 + \frac{3833}{35} p_\infty^5 + \mathcal{O}(p_\infty^7) \\
 \mathcal{D}^B &= -\frac{168}{5} p_\infty^3 + \frac{1471}{10} p_\infty^5 + \mathcal{O}(p_\infty^7). \quad (18)
 \end{aligned}$$

As expected, in this limit, B contributions are suppressed by an extra power of $p_\infty^2 \sim v^2$ compared with E -type ones.

Let us now start again from Eq. (14), which holds in a frame where $b_1^a = 0$, and perform a translation $b_i^a \rightarrow b_i'^a = b_i^a + a^a$ that places the center of mass (or “center of energy”) in the origin of the transverse plane, $(1-w)b_1'^a + w b_2'^a = 0$ with $w = p_2 \cdot (p_1 + p_2)/(p_1 + p_2)^2$. This sets $a^a = w b^a$ and by the transformation law [Eq. (12)] we can find the radiated angular momentum tensor in this new frame, $\mathbf{J}_{\text{tid}}'^{\alpha\beta}$. Its expression is obtained from Eq. (14) by replacing $\mathcal{C}^X \rightarrow \mathcal{C}'^X = \mathcal{C}^X + w \check{\mathcal{E}}^X$ and $\mathcal{D}^X \rightarrow \mathcal{D}'^X = \mathcal{D}^X - w \check{\mathcal{F}}^X$ where $\mathcal{E}^X = \sigma \check{\mathcal{F}}^X + \check{\mathcal{E}}^X$ and $\mathcal{F}^X = \sigma \check{\mathcal{E}}^X + \check{\mathcal{F}}^X$. Moreover, going to a frame where the center of mass is also at rest, say $b = (0, 0, b, 0)$, $-p_1 = (E_1, -p, 0, 0)$, $-p_2 = (E_2, p, 0, 0)$, we find, for the component of the angular momentum orthogonal to the scattering plane,

$$\mathbf{J}_{\text{tid}} \equiv \mathbf{J}_{\text{tid}}'^{xy} = R_f J \sum_X \frac{c_{X1}^2}{m_1} \left(\frac{\mathcal{C}'^X}{m_1} + \frac{\mathcal{D}'^X}{m_2} \right), \quad (19)$$

where $J = pb$ is the initial angular momentum in the center-of-mass frame. At low energies, for small p_∞ , we find, introducing the symmetric mass ratio $\nu = m_1 m_2 / m^2$, with $m = m_1 + m_2$, and $\Delta = (m_1 - m_2)/m$,

$$\frac{J_{\text{tid}}}{R_f b p_\infty} = \frac{c_{E_1^2}}{m_1} \left[\frac{1056}{5} p_\infty + \left(\frac{6}{5} (49 - 328\nu) - \frac{3678\Delta}{35} \right) p_\infty^3 \right] + \frac{c_{B_1^2}}{m_1} \left(\frac{96\Delta}{5} - \frac{264}{5} \right) p_\infty^3 + \mathcal{O}(p_\infty^5). \quad (20)$$

In the formal ultrarelativistic limit $\sigma \rightarrow \infty$ instead

$$\frac{J_{\text{tid}}}{R_f J} = \frac{c_{E_1^2}}{m_1^2} 63\sigma^5 - \frac{c_{E_1^2} + c_{B_1^2}}{2m_1 m_2} 315\sigma^4 \log \sigma + \mathcal{O}(\sigma^4). \quad (21)$$

Taking into account that the leading deflection angle scales as $\Theta_s \sim Gm\sqrt{\sigma}/b$, and that $c_{E_1^2} \sim G^4 m^5$, we see that $J_{\text{tid}}/J \sim \Theta_s^4 (\sqrt{\sigma}\Theta_s)^3$. Therefore, if we were to take σ arbitrarily large for fixed small Θ_s , the system could radiate an arbitrarily large amount of angular momentum. The perturbative PM expansion is however limited to $\sqrt{\sigma}\Theta_s \lesssim 1$ [77,144–146], so that the true ultrarelativistic limit lies beyond the scope of these calculations.

Consistency check.—To obtain a cross-check of the functions in Tables II and III, we perform an independent calculation of the angular momentum in the post-Newtonian (PN) limit as in Ref. [117]. We start from $\mathcal{A}^{\mu\nu}(k)$ and expand it for small p_∞ in the relevant scaling region $k^\alpha \sim \mathcal{O}(p_\infty)$ [100,145,146]. We then perform the Fourier transform [Eq. (B7)] term by term in the PN expansion in the frame $b_1^\alpha = 0$. Finally, we substitute into the expression for $J^{\alpha\beta}$ [Eq. (11)] and directly perform the integration over k , without using reverse unitarity. This involves integrals of Bessel functions, conveniently evaluated in Mathematica. Contracting Eq. (14) with $(u_{2\alpha} - u_{1\alpha})b_\beta$ and $u_{1\alpha}b_\beta$, we obtain the small- p_∞ expansions $\frac{1}{2}(\mathcal{C}^E + \mathcal{D}^E) = (1056/5)p_\infty + o(p_\infty)$, $\mathcal{C}^E - \mathcal{D}^E = o(p_\infty)$, and $\frac{1}{2}(\mathcal{C}^B + \mathcal{D}^B) = (16/5)p_\infty^3 + o(p_\infty^3)$, $\mathcal{C}^E - \mathcal{D}^E = (368/5)p_\infty^3 + o(p_\infty^3)$, in perfect agreement with Eq. (18). To obtain these results it is enough to retain the leading PN waveform for E contributions $\mathcal{O}(p_\infty)$, while it is necessary to resolve also the first subleading correction for point-particle $\mathcal{O}(p_\infty^{-1}) + \mathcal{O}(p_\infty^0)$ and B contributions $\mathcal{O}(p_\infty^2) + \mathcal{O}(p_\infty^3)$.

Static modes.—We now complete the result for the angular momentum loss by adding the zero-frequency contribution, i.e., the effect of the static gravitational field, which arises when calculating the expectation value on dressed states [Eq. (6)] using the eikonal operator [Eq. (5)],

$$\mathcal{J}_{\alpha\beta} = -i \int_k \left(F^* k_{[\alpha} \frac{\partial F}{\partial k^{\beta]}} + 2F^*_{\mu[\alpha} F^{\mu}_{\beta]} \right). \quad (22)$$

TABLE IV. Functions and coefficients entering the static terms. Here $n, m = 1, 2, 3, 4$ and $\eta_n = +1$ ($\eta_n = -1$) if the n th state is outgoing (incoming).

$\sigma_{nm} = -\eta_n \eta_m \frac{p_n \cdot p_m}{m_n m_m},$
$\Delta_{nm} = \frac{\text{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}},$
$c_{nm} = 2G \left[\left(\sigma_{nm}^2 - \frac{1}{2} \right) \frac{\sigma_{nm} \Delta_{nm} - 1}{\sigma_{nm}^2 - 1} - 2\sigma_{nm} \Delta_{nm} \right],$
$2\mathcal{G} = c_{14} + c_{23} - 2c_{13}$
$\frac{1}{2}\mathcal{I} = \frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{\sigma(2\sigma^2-3)\text{arccosh}\sigma}{(\sigma^2-1)^{3/2}}.$

To this end, we use Eq. (3.30) of Ref. [113], which provides this contribution for generic gravitational processes. Indeed, Eq. (22) relies only on the form of the leading soft factor, which is universal, and thus the resulting expression holds independently of the details of the collision. In terms of the coefficients defined in Table IV, the angular momentum due to static modes then evaluates to [113]

$$\mathcal{J}^{\alpha\beta} = - \sum_{n=1,2} \sum_{m=3,4} c_{nm} p_n^{[\alpha} p_m^{\beta]}. \quad (23)$$

Like the leading soft theorem, this result only depends on the momenta of the hard particles, and to obtain explicit expressions it is sufficient to substitute $p_4^\alpha = Q^\alpha - p_1^\alpha$, $p_3 = -Q^\alpha - p_2^\alpha$ and the PM expansion of the impulse Q .

Let us note that the static contribution [Eq. (23)] is invariant under translations and covariant under Lorentz transformations. In the center-of-mass frame (aligning the axes as above), we find [113] $\mathcal{J}^{xy} = GpQ\mathcal{I}$ up to $\mathcal{O}(G^4)$ corrections, with \mathcal{I} given in Table IV. In view of the overall power of G , since the $\mathcal{O}(G)$ impulse is unaffected by tidal terms, there is no tidal angular momentum loss to $\mathcal{O}(G^2 c_{X_1^2})$, and therefore (via linear response [147,148]) no tidal radiation-reaction in the deflection angle to $\mathcal{O}(G^3 c_{X_1^2})$, as noted in Ref. [108]. The leading tidal effects in \mathcal{J}^{xy} are $\mathcal{O}(G^3 c_{X_1^2})$ and can be obtained by substituting Eq. (C1) in it, finding the new result,

$$\mathcal{J}_{\text{tid}} = R_f J (\hat{Q}_{E_1^2} + \hat{Q}_{B_1^2}) \mathcal{I}, \quad (24)$$

up to $\mathcal{O}(G^4 c_{X_1^2})$ corrections. The leading, i.e., $\mathcal{O}(G^4)$, tidal radiation reaction on the angle or impulse due to static modes Q^α can be obtained by [116] $Q^\alpha = \frac{1}{2}(\partial Q^2 / \partial b_\alpha) \mathcal{G}$. To leading order in the tidal effects [Eq. (C1)], we then have $Q_{\text{tid}}^\alpha = -b^\alpha Q_{\text{tid}} / b$ with $Q_{\text{tid}} = \frac{7}{2} G b^{-1} Q_{\text{1PM}} (Q_{E_1^2} + Q_{B_1^2}) \mathcal{I}$. This result agrees with the one obtained by applying the linear-response formula [148,149] $Q_{\text{tid}} = -\frac{1}{2}(\partial Q / \partial J) \mathcal{J}_{\text{tid}}$.

Expanding for small p_∞ one finds

$$\frac{m}{2m_2 R_f b p_\infty} \mathcal{J}_{\text{tid}} = \frac{c_{E_1^2}}{m_1} \left[\frac{384}{5} p_\infty - \frac{192}{35} (7\nu - 29) p_\infty^3 \right] + \frac{c_{B_1^2}}{m_1} 192 p_\infty^3 + \mathcal{O}(p_\infty^5) \quad (25)$$

while as $\sigma \rightarrow \infty$

$$\frac{\mathcal{J}_{\text{tid}}}{R_f J} \approx \frac{c_{E_1^2} + c_{B_1^2}}{m_1^2} [420\sigma^3 \log \sigma + 70\sigma^3 (6 \log 2 - 5)] \quad (26)$$

up to $\mathcal{O}(\sigma \log \sigma)$ corrections.

Complete result and analytic continuation.—The total angular momentum of the gravitational field due to tidal effects constitutes the main original result of the paper and is given by the sum of the radiative piece [Eq. (14)] and the static piece [Eq. (23)]. In the center-of-mass frame, it reads as $J_{\text{tid}} = \mathbf{J}_{\text{tid}} + \mathcal{J}_{\text{tid}}$ [see Eqs. (19) and (24)], so that

$$J_{\text{tid}} = R_f J A(\sigma), \quad (27)$$

where we isolated a b -independent function $A(\sigma)$,

$$A(\sigma) = \sum_X \frac{c_{X_1^2}}{m_1} \left(\frac{\mathcal{C}^{IX}}{m_1} + \frac{\mathcal{D}^{IX}}{m_2} \right) + (\hat{Q}_{E_1^2} + \hat{Q}_{B_1^2}) \mathcal{I}. \quad (28)$$

Following Refs. [84,108,150–153], one can also use Eq. (27) to obtain the angular momentum that is radiated by a bound system in the high eccentricity limit (large J) during one orbital revolution. The first step is to write the total momentum radiated in the center-of-mass frame as $J_{\text{tid}}(J, \sigma) = J^{-6} f(\sigma)$ where

$$f(\sigma) = \frac{15\pi G^3 m^{11} \nu^9}{64h^7} (\sigma^2 - 1)^{7/2} A(\sigma) \quad (29)$$

with $h = \sqrt{1 + 2\nu(\sigma - 1)}$. The function $f(\sigma)$ is analytic for $\text{Re} \sigma > -1$, as one can easily check since it only involves rational combinations, together with the functions $\log[(\sigma + 1)/2]$ and $(\text{arccosh} \sigma / \sqrt{\sigma^2 - 1}) = (\arccos \sigma / \sqrt{1 - \sigma^2})$ that only have branch cuts for $\sigma < -1$. Using the boundary-to-bound map $J_{\text{tid}}^{\text{bound}}(J, \sigma) = J_{\text{tid}}(J, \sigma) + J_{\text{tid}}(-J, \sigma)$ [152,153], we find

$$J_{\text{tid}}^{\text{bound}}(J, \sigma) = \frac{2}{J^6} \tilde{f}(\sigma), \quad (30)$$

with $\tilde{f}(\sigma)$ the analytic continuation of $f(\sigma)$ to the interval $-1 < \sigma < 1$. In this fashion, $J_{\text{tid}}^{\text{bound}}(J, \sigma)$ gives the leading tidal correction, for large J , to the angular momentum loss for bound orbits with energy $E = mh < m$.

Angular momentum flux.—Let us assume that the relation between the tidal angular momentum loss and the averaged

flux F_{tid} in isotropic gauge reads as [108,153–159]

$$J_{\text{tid}} = \int_{-\infty}^{+\infty} F_{\text{tid}}(r, \sigma) dt = 2 \int_b^\infty F_{\text{tid}}(r, \sigma) \frac{dr}{\dot{r}}, \quad (31)$$

where to leading order we can employ the straight-line trajectory, $r^2(t) \simeq b^2 + v_{\text{rel}}^2 t^2$. Here $v_{\text{rel}} = p/(\xi E)$ with $\xi = E_1 E_2 / (E_1 + E_2)^2$. Dimensional analysis fixes the r dependence of the flux to be $F_{\text{tid}} \sim 1/r^7$. Performing the integral [Eq. (31)] by noting that $2 \int_b^\infty dr / (r^7 \dot{r}) = 16 / (15 v_{\text{rel}} b^6)$ [160] and matching to Eq. (27) then determines the overall r -independent factor and yields

$$F_{\text{tid}} = \frac{225\pi G^3 m^5 \nu^4 (\sigma^2 - 1)}{1024 h^3 \xi r^7} A(\sigma). \quad (32)$$

Conclusions.—In this Letter, we obtained a new result for the total angular momentum that is lost during a two-body scattering due to linear tidal effects, exploiting amplitude-based methods. We also provided the corresponding flux and the analytic continuation to bound orbits. This work opens up several avenues for future work. A natural generalization concerns the dissipation of angular momentum in scattering with spin [161] and in supersymmetric theories [71,84]. For bounded binaries, it would also be interesting to further compare with the PN literature [162–164] by performing a suitable eccentricity resummation needed to access the regime of quasicircular orbits [150,153]. A crucial next step will be to study quantitatively the impact of the present results on waveform models [119,120] and, of course, to extend them by calculating $J^{\alpha\beta}$ to subleading order, three loops on the amplitude side.

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Appendix A: Kinematics.—All momenta are regarded as outgoing, so $-p_i$ for $i = 1, 2$ are the physical momenta of the incoming states. The Minkowski metric is $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$, so that $p_i^2 + m_i^2 = 0$. Four velocities are defined by $u_i^\mu = -p_i^\mu / m_i$, with $u_i^2 = -1$. We denote the relative Lorentz factor by $\sigma = -u_1 \cdot u_2 = 1/\sqrt{1 - v^2}$, and v is the speed of body 1 as seen from the rest frame of body 2 (or vice versa). A useful variable in the PN limit is $p_\infty = \sqrt{\sigma^2 - 1}$. The spatial momentum in the center-of-mass frame is instead denoted by p . It is also convenient to define variables \check{u}_i^μ which obey $\check{u}_i \cdot u_j = -\delta_{ij}$ by letting $u_1^\mu = \sigma \check{u}_2^\mu + \check{u}_1^\mu$ and $u_2^\mu = \sigma \check{u}_1^\mu + \check{u}_2^\mu$. The relative impact parameter is defined by $b^\mu = b_1^\mu - b_2^\mu$, where b_1^μ and b_2^μ are the impact

parameters of each particle, transverse to the incoming directions $b_i \cdot u_j = 0$. Finally, the symmetric mass ratio is defined by $\nu = m_1 m_2 / m^2$, with $m = m_1 + m_2$, while we let $\Delta = (m_1 - m_2) / m$.

Appendix B: Integration and index contraction.—We employ the shorthand notation

$$\int_k = \int \frac{d^D k}{(2\pi)^D} 2\pi \theta(k^0) \delta(k^2). \quad (\text{B1})$$

Moreover, we define

$$\tilde{\mathcal{A}}(k)\hat{a}_k^\dagger = \sum_i \epsilon_{\mu\nu}^{(i)}(k)^* \tilde{\mathcal{A}}^{\mu\nu}(k)\hat{a}_i^\dagger(k) \quad (\text{B2})$$

where $i = 1, 2$ labels the two physical graviton polarizations, with polarization “tensors” $\epsilon_{\mu\nu}^{(i)}(k)$, and similarly for the Hermitian conjugate of (B2) and for analogous expressions involving $F^{\mu\nu}$. The creation and annihilation operators obey canonical commutation relations,

$$2\pi\theta(k^0)\delta(k^2)[\hat{a}_i(k), \hat{a}_j^\dagger(k')] = (2\pi)^D\delta^{(D)}(k - k')\delta_{ij}. \quad (\text{B3})$$

For convenience, we suppress contractions between five-point amplitudes (unless written otherwise), letting

$$\mathcal{A}\mathcal{A}' = \mathcal{A}_{\mu\nu}\mathcal{A}'^{\mu\nu} - \frac{1}{D-2}\mathcal{A}_{\mu}^{\mu}\mathcal{A}'_{\nu}{}^{\nu}, \quad (\text{B4})$$

and similarly for $F^{\mu\nu}$.

We define the Fourier transform $\text{FT}\mathcal{M}$ by

$$\text{FT}\mathcal{M} = \int \frac{d^D q}{(2\pi)^D} 2\pi\delta(2p_1 \cdot q) 2\pi\delta(2p_2 \cdot q) e^{ib \cdot q} \mathcal{M}(q). \quad (\text{B5})$$

The relation between the momentum space $2 \rightarrow 3$ amplitude in the classical limit (the drawing inside the dashed bubble only serves as a visual help to recall the definition of q_1 , q_2 and does not represent an actual Feynman diagram)

$$\mathcal{A}^{\mu\nu}(q_1, q_2, k) = \text{Diagram (B6)} \quad (\text{B6})$$

and its b -space counterpart $\tilde{\mathcal{A}}^{\mu\nu}(k)$ is given by

$$\begin{aligned} \tilde{\mathcal{A}}^{\mu\nu}(k) = & \int \frac{d^D q_1}{(2\pi)^D} 2\pi\delta(2p_1 \cdot q_1) 2\pi\delta(2p_2 \cdot q_2) \\ & \times e^{ib_1 \cdot q_1 + ib_2 \cdot q_2} \mathcal{A}^{\mu\nu}(q_1, q_2, k), \end{aligned} \quad (\text{B7})$$

with $q_1 + q_2 + k = 0$. Under a translation,

$$b_{1,2}^\mu \rightarrow b_{1,2}^\mu + a^\mu, \quad \tilde{\mathcal{A}}^{\mu\nu}(k) \rightarrow e^{-ia \cdot k} \tilde{\mathcal{A}}^{\mu\nu}(k). \quad (\text{B8})$$

In a frame where $b_2 = 0$, we find

$$\begin{aligned} \tilde{\mathcal{A}}^{\mu\nu}(k) = & \int \frac{d^D q_1}{(2\pi)^D} 2\pi\delta(2p_1 \cdot q_1) e^{ib \cdot q_1} \\ & \times 2\pi\delta[2p_2 \cdot (q_1 + k)] \mathcal{A}^{\mu\nu}(q_1, q_2, k)|_{q_2=-q_1-k}, \end{aligned} \quad (\text{B9})$$

its advantage being that k only enters the second line.

Appendix C: PM impulse.—We collect here for completeness the $\mathcal{O}(G)$ and $\mathcal{O}(G^2 c_{X_1^2})$ terms of the impulse [34,35,95],

$$\begin{aligned} Q_{1\text{PM}} &= \frac{4Gm_1m_2}{b} \frac{\sigma^2 - \frac{1}{2}}{\sqrt{\sigma^2 - 1}}, \\ Q_{E_1^2} &= \frac{R_f b}{G} \frac{3c_{E_1^2}}{m_1^2} \frac{35\sigma^4 - 30\sigma^2 + 11}{\sqrt{\sigma^2 - 1}} \equiv \frac{R_f b}{G} \hat{Q}_{E_1^2}, \\ Q_{B_1^2} &= \frac{R_f b}{G} \frac{15c_{B_1^2}}{m_1^2} \sqrt{\sigma^2 - 1} (7\sigma^2 + 1) \equiv \frac{R_f b}{G} \hat{Q}_{B_1^2}, \quad (\text{C1}) \end{aligned}$$

where R_f is given below Eq. (9).

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