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RESEARCH ARTICLE

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Broadcasting-induced colorings of preferential attachment trees

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Abstract

We consider random two-colorings of random linear preferential attachment trees, which includes recursive trees, plane-oriented recursive trees, binary search trees, and a class of d-ary trees. The random coloring is defined by assigning the root the color red or blue with equal probability, and all other vertices are assigned the color of their parent with probability p and the other color otherwise. These colorings have been previously studied in other contexts, including Ising models and broadcasting, and can be considered as generalizations of bond percolation. With the help of Pólya urns, we prove limiting distributions, after proper rescalings, for the number of vertices, monochromatic subtrees, and leaves of each color, as well as the number of fringe subtrees with two-colorings. Using methods from analytic combinatorics, we also provide precise descriptions of the limiting distribution after proper rescaling of the size of the root cluster; the largest monochromatic subtree containing the root.

KEYWORDS

broadcasting, generating functions, Pólya urns, percolation, preferential attachment, random trees

1 | INTRODUCTION

For a rooted tree $T = (V, E)$ with root ρ and $p \in [0, 1]$, we define a *broadcasting-induced coloring* $\sigma_{T,p}$ of T to be a random two-coloring $\sigma_{T,p} : V(T) \to \{red, blue\}$ of the vertices of T such that

(i) $\mathbb{P}(\sigma_{T,p}(\rho) = \text{red}) = \mathbb{P}(\sigma_{T,p}(\rho) = \text{blue}) = \frac{1}{2}$ and
(ii) for all vertices v with parent u, $\mathbb{P}(\sigma_{T,p}(v) = \sigma_{T,p}(u)) = p$ and $\mathbb{P}(\sigma_{T,p}(v) \neq \sigma_{T,p}(u)) = 1 - p$.

The coloring $\sigma_{T,p}$ is induced from a broadcast process, in which the root of T is assigned a bit 0 or 1 uniformly at random, and this bit is propagated along the tree in the following way: any vertex in the tree takes the same bit as its parent with probability p and the other bit with probability $1 - p$. By assigning a vertex the color red if its bit value is 0 and blue if its bit value is 1, we recover the coloring $\sigma_{T,p}$. This broadcast process was described by Evans, Kenyon, Peres, and Schulman [12], where they outline a correspondence of this process to the Ising model (see [12], Section 2.2). The reconstruction problem is then to reconstruct the bit value of the root ρ from the bit values of some subset of vertices in T after broadcasting. This problem has long been studied, see for example the survey [27] of early works. Applications of the reconstruction problem in trees include its connection to stochastic block models [14, 28], a random graph model with applications in machine learning. Of particular interest to this work, Addario-Berry, Devroye, Lugosi, and Velona studied the reconstruction problem in random recursive trees and preferential attachment trees [1].

For a real number α, a random (linear) preferential attachment tree $\mathcal{T}_{\alpha,n}$ is grown recursively in the following manner. The tree $\mathcal{T}_{\alpha,1}$ consists of a single vertex ρ, the root of all trees that follow. The tree $\mathcal{T}_{\alpha,n}$ is grown from $\mathcal{T}_{\alpha,n-1}$ by choosing a vertex v at random and adding a child to v, where v is chosen with probability

$$\frac{\alpha \deg^+(v) + 1}{\sum_{u \in V(\mathcal{T}_{\alpha,n-1})}(\alpha \deg^+(u) + 1)} = \frac{\alpha \deg^+(v) + 1}{\alpha(n-1) + n}, \tag{1}$$

where $\deg^+(u)$ (called the *outdegree* of u) is the number of children of u. To avoid degenerate cases, we only allow $\alpha \in \{\dots, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}\} \bigcup [0, \infty)$. If $\alpha = -1$, then only leaves can be chosen as the parent of a new vertex, resulting in $\mathcal{T}_{-1,n}$ simply being a path of length n. If α is a different negative number outside of $\{\dots, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}\}$, then there may be vertices v in $\mathcal{T}_{\alpha,n}$ for which (1) is negative. This problem is avoided when $\alpha = -\frac{1}{d}$, since (1) is positive when $\deg^+(u) < d$ and is zero when $\deg^+(u) = d$, resulting in a tree $\mathcal{T}_{-1/d,n}$ whose vertices all have outdegree less than or equal to d.

The random tree $\mathcal{T}_{\alpha,n}$ has several names in the literature. When $\alpha = 0$, the vertex v is chosen uniformly at random amongst all the vertices in the tree. This random tree is called a *random recursive tree*, and has been extensively studied for many years; since at least 1967 [33]. When $\alpha = 1$, the tree $\mathcal{T}_n$ is called a *random plane-oriented recursive tree* which was introduced by Szymański [32]. The more general *linear preferential attachment tree* $\mathcal{T}_{\alpha,n}$ coincides with a special case of the preferential attachment model studied by Barabási and Albert [2], but has also been studied in several other contexts (see e.g., [7, 17, 31]). When $\alpha = -\frac{1}{d}$ for a positive integer d, the tree $\mathcal{T}_{-1/d,n}$ is a model of random d-ary trees, and corresponds to a random binary search tree when $d = 2$. The random trees $\mathcal{T}_{\alpha,n}$ also fall into the class of *increasing trees* (see [6, 10]), so named since if we label the vertices $1, \dots, n$ by the time they appear in the tree, then the labels increase along all paths from the root.

For ease of notation, we may sometimes fix α and p, and let $\mathcal{T}_n$ denote the tree $\mathcal{T}_{\alpha,n}$ and let σ_n denote the random broadcasting induced coloring $\sigma_{\mathcal{T}_n,p}$. For fixed α and p we can consider a random sequence $((\mathcal{T}_n, \sigma_n))_{n=1}^{\infty}$ of preferential attachment trees with broadcasting-induced colorings where $\mathcal{T}_n$ is grown from $\mathcal{T}_{n-1}$ in the manner outlined above, and where σ_n restricted to the $n - 1$ vertices of $\mathcal{T}_{n-1}$ is equal

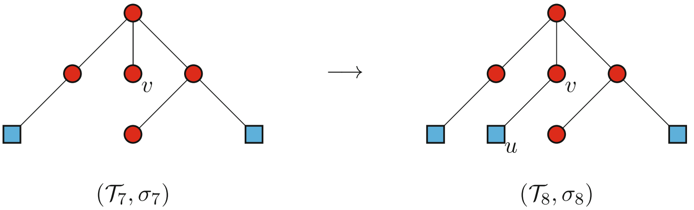

FIGURE 1 A tree $\mathcal{T}_8$ with broadcasting-induced coloring σ_8 grown from $(\mathcal{T}_7, \sigma_7)$.

to σ_{n-1} (and the color of the newest vertex v in $\mathcal{T}_n$ is randomly chosen such that with probability p the color of v is the same as its parent).

As an example of the growth process we describe, consider the trees with broadcasting induced colorings in Figure 1. The tree $\mathcal{T}_8$ is grown from $\mathcal{T}_7$ by choosing v according to the probability (1) and adding a child u (notice that $\deg^+(v) = 0$). The probability that u takes a different color from v is $1-p$, and so

$$\mathbb{P}((\mathcal{T}_8, \sigma_8)|(\mathcal{T}_7, \sigma_7)) = (1-p)\left(\frac{1}{6\alpha+7}\right).$$

Our contribution in this work is to study asymptotic properties of the trees in the sequence $((\mathcal{T}_n, \sigma_n))_{n=1}^{\infty}$. These results are gathered in Section 2, and come in two categories. In Section 2.1 we list global properties of the trees $((\mathcal{T}_n, \sigma_n))_{n=1}^{\infty}$. These include limit laws (after appropriate rescaling) for the number of vertices of each color, the number of clusters of each color (maximal monochromatic subtrees of $\mathcal{T}_n$), the number of leaves of each color, and the number of trees $T_1, \ldots, T_m$ with respective two-colorings $\varsigma_1, \ldots, \varsigma_m$ appearing in the fringe; a fringe subtree consists of a vertex and all its descendants. These results are proved using results on Pólya urns from [18]. The limiting distributions experience different phases. When $p < (3-\alpha)/4$, we observe normal limit laws after rescaling by $\sqrt{n}$. Normal limit laws are also observed when $p = (3-\alpha)/4$ but with a rescaling factor of $\sqrt{n \ln n}$, while convergence to a nonnormal distribution is observed when $p > (3-\alpha)/4$.

In Section 2.2 we study the size of the cluster C_n containing the root ρ, which we denote $|C_n|$. If we consider $\mathcal{T}_n$ with a random broadcasting-induced coloring σ_n and remove edges between two vertices if they do not have the same color, we are left with a forest of trees corresponding to clusters after performing Bernoulli bond percolation with parameter p on $\mathcal{T}_n$; Bernoulli bond percolation with parameter p is a process by which each edge in a graph is kept with probability p and removed with probability $1-p$, independently of every other edge. The size of C_n in the context of percolation, with a connection to memory-reinforced random walks, has previously been studied (see [3, 4, 8, 21]). In particular, Businger [8] has shown that for random recursive trees, $|C_n|/n^p$ converges in distribution to a Mittag–Leffler distribution (see also [4, Theorem 3.1] and [26]). Baur [3] studied $|C_n|$ in linear preferential attachment trees with $\alpha \geq 0$. He showed that $|C_n|/n^{(p+\alpha)/(1+\alpha)}$ converges in distribution to some random variable C, and provided the first two moments of this random variable. In this paper, we reprove these earlier results, and give a more precise description of this random variable C by providing a recursion to calculate the integer moments of C. These results are proved using methods of analytic combinatorics. When $\alpha = 1$ we give a closed form for the integer moments of C. We further extend these results by studying the size of C_n when $\alpha = -\frac{1}{d}$, where we observe different phases. When $p > \frac{1}{d}$, we observe a similar limiting distribution as when $\alpha > 0$, and find closed forms for the integer moments of this limiting distribution when $d = 2$. When $p \leq \frac{1}{d}$, the size of the root cluster C_n is bounded almost surely as $n \to \infty$, and we describe the limiting distribution of $|C_n|$ as the size of a Galton–Watson tree with binomial $\mathrm{Bin}(d, p)$ offspring distribution.

2 | MAIN RESULTS

In this section, we gather our main results. They are separated in two categories: global properties and the size of the root cluster.

2.1 | Global properties

Throughout this section we define a random variable

$$B = \begin{cases} 1 & \text{if the root is red,} \\ -1 & \text{if the root is blue.} \end{cases} \tag{2}$$

Since the root is either red or blue with equal probability, B is a Rademacher random variable.

We start with the number of vertices of each color in a random preferential attachment tree $\mathcal{T}_n = \mathcal{T}_{\alpha,n}$ with a broadcasting induced coloring $\sigma_n = \sigma_{\mathcal{T}_n,p}$.

Theorem 2.1. *Let R_n and B_n denote the number of red and blue vertices, respectively, in a preferential attachment tree $\mathcal{T}_n$ with broadcasting-induced coloring σ_n.*

(i) The following strong law of large numbers holds

$$\frac{1}{n}(R_n, B_n) \xrightarrow{a.s.} \left(\frac{1}{2}, \frac{1}{2}\right).$$

(ii) If $p < (3-\alpha)/4$ or if $p = 1/2$, then the following multivariate normal limit law holds

$$\frac{(R_n, B_n) - n\left(\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_I),$$

where

$$\Sigma_I = c_{\alpha,p}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad c_{\alpha,p} = \begin{cases} \frac{4\alpha p-\alpha-1}{4(4p+\alpha-3)} & p \neq \frac{1}{2}, \\ \frac{1}{4} & p = \frac{1}{2}. \end{cases}$$

(iii) If $p = (3-\alpha)/4$ and $p \neq \frac{1}{2}$ (i.e. $\alpha \neq 1$), then the following multivariate normal limit law holds

$$\frac{(R_n, B_n) - n\left(\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{II}),$$

where

$$\Sigma_{II} = \frac{(\alpha-1)^2}{4(1+\alpha)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

(iv) If $p > (3-\alpha)/4$ and $p \neq \frac{1}{2}$, then the following almost sure convergence holds

$$\frac{(R_n, B_n) - n\left(\frac{1}{2}, \frac{1}{2}\right)}{n^{(2p+\alpha-1)/(1+\alpha)}} \xrightarrow{a.s.} \frac{BZ}{2(2p+\alpha-1)}(2p-1, 1-2p), \tag{3}$$

where Z is a random variable with

$$\mathbb{E}[Z] = \frac{\Gamma(1/(1+\alpha))}{\Gamma((2p+\alpha)/(1+\alpha))},$$

and

$$\mathbb{E}[Z^2] = \frac{\Gamma(1/(1+\alpha))(1+\alpha)(4p+\alpha-2)}{\Gamma((4p+2\alpha-1)/(1+\alpha))(4p+\alpha-3)}.$$

Remark 2.2. When $p = 1/2$, every new vertex added to the tree is either red or blue with probability $1/2$, independent of everything that happened before. Therefore, R_n is simply a sum of independent Bernoulli Be$(1/2)$ random variables, as is $B_n = n - R_n$. We see that in this case, the matrix Σ_I in the convergence (ii) simplifies to

$$\Sigma_I = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and we see that the random variables on the right-hand side of the convergences in (iii) and (iv) degenerate to (0,0).

There is a close connection between Theorem 2.1 and results on a class of reinforced random walks. In [3], Baur defines the *strongly reinforced elephant random walk* as a sum of ± 1 random variables ζ_n, where the value of ζ_n has a positive probability of taking the value of a previous step ζ_i chosen proportional to a weight $k_{n-1}(i)$. The first step ζ_1 is 1 or -1 with equal probability, and is assigned a weight $k_1(1) = 1$. For values $p^* \in (0,1)$ and $\alpha \geq 0$, the nth step is decided by first choosing a previous step ζ_{I_n} with probability

$$\mathbb{P}(I_n = i) = \frac{k_{n-1}(i)}{\sum_{j=1}^{n-1} k_{n-1}(j)}.$$

With probability p^*, the nth step is $\zeta_n = \zeta_{I_n}$, and with probability $1-p^*$, ζ_n is either assigned 1 or -1 with equal probability. Equivalently, $\zeta_n = \zeta_{I_n}$ with probability $p = p^* + \frac{1}{2}(1-p^*)$ and $\zeta_n = -\zeta_{I_n}$ with probability $1-p$. The nth step is then assigned the weight $k_n(n) = 1$, while $k_n(I_n) = k_{n-1}(I_n) + \alpha$ and $k_n(j) = k_{n-1}(j)$ for all other $j \neq I_n$. Letting $S_n = \sum_{i=1}^{n} \zeta_n$, we can quickly see that S_n is distributed as $R_n - B_n$. Indeed, many of the cases of Theorem 2.1 (specifically when $p > 1/2$) are implied by [3, Theorem 3.2] on S_n, proved using Pólya urns, where the parameter p in Theorem 2.1 corresponds to $2p-1$ in [3, Theorem 3.2] and the parameter α corresponds to $\beta > 0$. See also [22] for proofs using a martingale approach.

We now turn to the number of clusters (maximal monochromatic subtrees). If we want to know the total number of clusters, we can first notice that whenever a child takes a different color from its parent, a new cluster is formed. This is also the only way of forming a new cluster (in addition to the initial cluster containing the root). The probability that a newly added vertex does not take the color of its parent is $1-p$, from which we can conclude that the total number of clusters at time n is simply $1+\mathrm{Bin}(n-1, 1-p)$, where Bin denotes a binomial random variable.

Theorem 2.3. *Let R_n^c and B_n^c denote the number of red and blue clusters respectively in a preferential attachment tree $\mathcal{T}_n$ with broadcasting induced coloring σ_n.*

(i) The following strong law of large numbers holds

$$\frac{1}{n}(R_n^c, B_n^c) \xrightarrow{a.s.} \left(\frac{1-p}{2}, \frac{1-p}{2}\right).$$

(ii) If $p < (3-\alpha)/4$, then the following multivariate normal limit law holds

$$\frac{(R_n^c, B_n^c) - n\left(\frac{1-p}{2}, \frac{1-p}{2}\right)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_I^c),$$

where

$$\Sigma_I^c = \frac{1-p}{4(3-\alpha-4p)} \begin{pmatrix} (1-p)(\alpha+4p+1) & 3p-4p^2-\alpha p-\alpha-1 \\ 3p-4p^2-\alpha p-\alpha-1 & (1-p)(\alpha+4p+1) \end{pmatrix}.$$

(iii) If $p = (3-\alpha)/4$, then the following multivariate normal limit law holds

$$\frac{(R_n^c, B_n^c) - n\left(\frac{1-p}{2}, \frac{1-p}{2}\right)}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{II}^c),$$

where

$$\Sigma_{II}^c = \frac{\alpha+1}{16} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

(iv) If $p > (3-\alpha)/4$, then the following almost sure convergence holds

$$\frac{(R_n^c, B_n^c) - n\left(\frac{1-p}{2}, \frac{1-p}{2}\right)}{n^{(2p+\alpha-1)/(1+\alpha)}} \xrightarrow{a.s.} \frac{BZ}{2(2p+\alpha-1)}(p-1, 1-p),$$

where Z is equal almost surely to the random variable in (3).

We finish our summary of global properties with fringe subtrees. In a rooted tree $\mathcal{T}$ a fringe subtree T consists of a vertex and all its descendants. The simplest example of a fringe subtree in $\mathcal{T}$ is a vertex with no descendants (a leaf of $\mathcal{T}$). Normal limit laws for the number of leaves in preferential attachment trees are already well known (see [17, 19, 24, 29]).

In this simplest case, we offer covariance matrices for the limiting normal limit laws for the number of leaves of each color in $\mathcal{T}_n$, though the distributions are already markedly more complicated than what we have described above.

Theorem 2.4. *Let R_n^l and B_n^l denote the number of red and blue leaves respectively in a preferential attachment tree $\mathcal{T}_n$ with broadcasting induced coloring σ_n.*

(i) The following strong law of large numbers holds

$$\frac{1}{n}(R_n^l, B_n^l) \xrightarrow{a.s.} \left(\frac{1+\alpha}{4+2\alpha}, \frac{1+\alpha}{4+2\alpha}\right).$$

(ii) If $p < (3-\alpha)/4$ or if $p = 1/2$, then the following multivariate normal limit law holds

$$\frac{(R_n^l, B_n^l) - n\left(\frac{1+\alpha}{4+2\alpha}, \frac{1+\alpha}{4+2\alpha}\right)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_I^l),$$

where

$$\Sigma_I^l = \frac{\alpha+1}{4(2+\alpha)^2(3+\alpha)(2p-3)(4p+\alpha-3)} \begin{pmatrix} \sigma_{1,1}^l & \sigma_{1,2}^l \\ \sigma_{2,1}^l & \sigma_{2,2}^l \end{pmatrix},$$

with

$$\begin{aligned} \sigma_{1,1}^l = \sigma_{2,2}^l &= \left(8p^2 - 6p - 1\right)\alpha^3 + \left(48p^2 - 46p + 1\right)\alpha^2 \\ &\quad + \left(112p^2 - 158p + 49\right)\alpha + 88p^2 - 158p + 71 \\ \sigma_{1,2}^l = \sigma_{2,1}^l &= \left(1 + 6p - 8p^2\right)\alpha^3 - \left(48p^2 - 50p + 7\right)\alpha^2 \\ &\quad - \left(96p^2 - 126p + 37\right)\alpha - 72p^2 + 122p - 53, \end{aligned}$$

when $p \neq 1/2$, and

$$\Sigma_I^l = \frac{\alpha+1}{4(2+\alpha)^2(3+\alpha)} \begin{pmatrix} 7 + 6\alpha + \alpha^2 & -5 - 4\alpha - \alpha^2 \\ -5 - 4\alpha - \alpha^2 & 7 + 6\alpha + \alpha^2 \end{pmatrix},$$

when $p = 1/2$.

(iii) If $p = (3-\alpha)/4$ and $p \neq \frac{1}{2}$ (i.e. $\alpha \neq 1$), then the following multivariate normal limit law holds

$$\frac{(R_n^l, B_n^l) - n\left(\frac{1+\alpha}{4+2\alpha}, \frac{1+\alpha}{4+2\alpha}\right)}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{II}^l),$$

where

$$\Sigma_{II}^l = \frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

(iv) If $p > (3-\alpha)/4$ and $p \neq \frac{1}{2}$, then the following almost sure convergence holds

$$\frac{(R_n^l, B_n^l) - n\left(\frac{1+\alpha}{4+2\alpha}, \frac{1+\alpha}{4+2\alpha}\right)}{n^{(2p+\alpha-1)/(1+\alpha)}} \xrightarrow{a.s.} \frac{BZ}{2\alpha + 4p}(2p-1, 1-2p),$$

where Z is equal almost surely to the random variable in (3).

Remark 2.5. Once more, we witness a special case $p = 1/2$. Though the explanation is not as simple as in Remark 2.2, the color of a newly added leaf is independent of the color of its parent. We see again that the random variables on the right hand side of the convergences in (iii) and (iv) degenerate to (0,0). When $p = 1/2 \neq (3-\alpha)/4$, both matrices in (*ii*) are equal.

Limiting joint distributions for the number of fringe subtrees (without colors) have already been studied [17]. Let $T_1, \dots, T_m$ be a sequence of finite trees of sizes $k_1, \dots, k_m$ with colorings $\varsigma_1, \dots, \varsigma_m$. Let $\sigma_n|_T$ denote the coloring σ_n restricted to the subtree T. We say that two colored rooted trees are isomorphic (and use the symbol $\simeq$) if there is an isomorphism between them that preserves roots and colors.

Theorem 2.6. *Let X_n^i be the number of fringe subtrees T in $\mathcal{T}_n$ isomorphic to T_i with coloring ς_i, and let*

$$\boldsymbol{\mu} = \left(\frac{\mathbb{P}((\mathcal{T}_{k_1}, \sigma_{k_1}) \simeq (T_1, \varsigma_1))\frac{1}{\alpha+1}}{(k_1 + \frac{1}{\alpha+1} - 1)(k_1 + \frac{1}{\alpha+1})}, \dots, \frac{\mathbb{P}((\mathcal{T}_{k_m}, \sigma_{k_m}) \simeq (T_m, \varsigma_m))\frac{1}{\alpha+1}}{(k_m + \frac{1}{\alpha+1} - 1)(k_m + \frac{1}{\alpha+1})} \right). \tag{4}$$

(i) The following strong law of large numbers holds

$$\frac{1}{n}\left(X_n^1, \dots, X_n^m\right) \xrightarrow{a.s} \boldsymbol{\mu}.$$

(ii) If $p < (3-\alpha)/4$ or if $p = 1/2$, then the following multivariate normal limit law holds

$$\frac{\left(X_n^1, \dots, X_n^m\right) - n\boldsymbol{\mu}}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_I^f),$$

for some covariance matrix Σ_I^f.

(iii) If $p = (3-\alpha)/4$ and $p \neq \frac{1}{2}$ (i.e. $\alpha \neq 1$), then the following multivariate normal limit law holds

$$\frac{\left(X_n^1, \dots, X_n^m\right) - n\boldsymbol{\mu}}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{II}^f),$$

for some covariance matrix Σ_{II}^f.

(iv) If $p > (3-\alpha)/4$ and $p \neq \frac{1}{2}$, then the following almost sure convergence holds

$$\frac{\left(X_n^1, \dots, X_n^m\right) - n\boldsymbol{\mu}}{n^{(2p+\alpha-1)/(1+\alpha)}} \xrightarrow{a.s} BZ\boldsymbol{v},$$

for some vector $\boldsymbol{v}$, where Z is equal almost surely to the random variable in (3).

Remark 2.7. The matrices Σ_I^f, Σ_{II}^f, as well as the vector $\boldsymbol{v}$ can be calculated explicitly from the sequence $T_1, \dots, T_m$ of trees and the colorings $\varsigma_1, \dots, \varsigma_m$ (see Theorem 3.1 below).

Remark 2.8. The statements of Theorem 2.1 (iv), Theorem 2.3 (iv), Theorem 2.4 (iv), and Theorem 2.6 (iv), all contain the same random variable BZ in the limit. By studying a Pólya urn process (see Proposition 3.2) which is a linear transformation of each of the Pólya urn processes used in the proofs of Theorems 2.1,2.3,2.4, and 2.6, we show in the proofs of Section 3 that indeed the random variable BZ is equal almost surely in all of these statements.

2.2 Root cluster

We let $\mathcal{C}_n$ denote the *root cluster* in the random preferential attachment tree $\mathcal{T}_n$ with broadcasting induced coloring σ_n, that is, the maximal monochromatic subtree containing the root, and let $|\mathcal{C}_n|$ denote its size. As noted in the introduction, $\mathcal{C}_n$ is identically distributed as the root cluster in $\mathcal{T}_n$ after applying Bernoulli bond percolation with parameter p.

In previous works of Möhle [26], Baur and Bertoin [4], and Businger [8], convergence in distribution of $|\mathcal{C}_n|$ when $\alpha = 0$, once scaled by n^p, to a Mittag–Leffler distribution was proved. That is,

$$\frac{|\mathcal{C}_n|}{n^p} \xrightarrow{d} \mathcal{C},$$

where $\mathcal{C}$ is characterized by its integer moments

$$\mathbb{E}[\mathcal{C}^k] = \frac{k!}{\Gamma(pk+1)},$$

the integer moments of a Mittag–Leffler distribution with parameter p.

Baur [3, Proposition 4.1] proved that $|\mathcal{C}_n|/n^{(p+\alpha)/(1+\alpha)}$ converges in L^2 to a random variable $\mathcal{C}$ when $\alpha > 0$ (though we believe the method may also apply for applicable $\alpha > -1/p$), and provides the first two moments of $\mathcal{C}$. Baur also provides a description of the sizes of the remaining clusters [3, Corollary 4.3]. If the ith added vertex is the root of a cluster then the size of this cluster, scaled by $n^{(p+\alpha)/(1+\alpha)}$, converges in distribution to

$$\beta_i^{(p+\alpha)/(1+\alpha)}\mathcal{C},$$

where $\mathcal{C}$ is the random variable given in Theorem 2.9 and β_i is a Beta$(1/(1+\alpha), i)$ distributed random variable.

We extend the description of the limiting distribution in [3, Proposition 4.1] by finding a recursion for the integer moments. This recursion uses the partial Bell polynomials (see [9, Chapter 3.3])

$$B_{k,j}(x_1, \dots, x_{k-j+1}) = \sum_{\substack{m_1+\cdots+(k-j+1)m_{k-j+1}=k \\ m_1+\cdots+m_{k-j+1}=j}} k! \prod_{i=1}^{k-j+1} \frac{x_i^{m_i}}{m_i!\, i!^{m_i}}.$$

Theorem 2.9. *Let $\alpha > 0$. Then*

$$\frac{|\mathcal{C}_n|}{n^{(p+\alpha)/(1+\alpha)}} \xrightarrow{d} \mathcal{C},$$

where $\mathcal{C}$ has integer moments

$$\mathbb{E}[\mathcal{C}^k] = \frac{C_k(1+\alpha)\Gamma(1/(1+\alpha))}{\alpha\Gamma((kp+\alpha(k-1))/(\alpha+1))},$$

where C_k satisfies the recursion $C_1 = \alpha/(p+\alpha)$ and

$$(k-1)(p/\alpha+1)C_k = \sum_{j=2}^{k} \frac{p^j\Gamma(1/\alpha+j)}{\Gamma(1/\alpha)} B_{k,j}(C_1, \dots, C_{k-j+1}).$$

By using the recursion in Theorem 2.9, we calculate the first two moments of C to be

$$\mathbb{E}[C] = \frac{(1+\alpha)\Gamma\left(\frac{1}{1+\alpha}\right)}{(p+\alpha)\Gamma\left(\frac{p}{1+\alpha}\right)},$$

$$\mathbb{E}[C^2] = \frac{p^2(1+\alpha)^2\Gamma\left(\frac{1}{1+\alpha}\right)}{(p+\alpha)^3\Gamma\left(\frac{2p+\alpha}{1+\alpha}\right)},$$

which agrees with the calculations given in [3].

In the special case $\alpha = 1$, we are able to find a closed form for the recursion given in Theorem 2.9, and with it, a more precise description of the limiting distribution of $|C_n|$ after proper rescaling.

Proposition 2.10. *Let the underlying tree be a random plane-oriented recursive tree, so $\alpha = 1$. Then the integer moments of the limiting random variable C in Theorem 2.9 can be written as*

$$\mathbb{E}[C^k] = \frac{2p^{k-1}\Gamma(kp+k-1)\sqrt{\pi}}{(p+1)^{2k-1}\Gamma(kp)\Gamma((kp+k-1)/2)}.$$

We now turn to the case when $\alpha = -1/d$ for an integer $d \geq 2$, that is, when the underlying tree $\mathcal{T}_n$ is a random increasing d-ary tree. If we consider $\mathcal{T}_n$ as a subtree of an infinite d-ary tree T_d, then the coloring σ_n can be recovered from bond percolation on T_d: start by assigning the root either red or blue, and assign to a vertex v the color of its parent u if the edge joining u and v is still present after performing bond percolation, and the other color otherwise. In this way, the root cluster C_n of $\mathcal{T}_n$ is a subtree of the cluster $\mathcal{K}_d$ of T_d containing the root after performing bond percolation. Using this fact, we can prove the following result on the sizes of $|C_n|$ and $|\mathcal{K}_d|$ (which may be infinite).

Proposition 2.11. *Let $|C_n|$ be the size of the root cluster of $\mathcal{T}_n$ with broadcasting-induced coloring σ_n. Then $|C_n| \xrightarrow{a.s.} |\mathcal{K}_d|$.*

Using well known results on the size of $|\mathcal{K}_d|$, the following corollary is immediate:

Corollary 2.12. *Let $\alpha = -1/d$, where $d \geq 2$ is a positive integer. Then for every positive integer k,*

$$\mathbb{P}\left(\lim_{n\to\infty} |C_n| = k\right) = \frac{1}{k}\binom{kd}{k-1} p^{k-1}(1-p)^{kd-k+1}. \tag{5}$$

Remark 2.13. When $p \leq 1/d$, the probabilities in (5) sum to 1, and so the root cluster is almost surely finite.

When $\alpha = -p$, the root cluster is almost surely finite, though its expected size grows to infinity. In fact, we can describe the asymptotic behaviour of all the moments of $|C_n|$.

Proposition 2.14. *Let $\alpha = -1/d$, where $d \geq 2$ is a positive integer, and let $p = -\alpha = 1/d$. Then*

$$\mathbb{E}[|C_n|^k] \sim E_k \ln^{2k-1} n,$$

where E_k satisfies the recursion $E_1 = 1/(d-1)$ and

$$(2k-1)E_k = \frac{1}{2d}\sum_{j=1}^{k-1}\binom{k}{j}E_jE_{k-j}.$$

In the case $\alpha > -p$, a similar limiting distribution C to that found in Theorem 2.9 exists.

Theorem 2.15. *Let $\alpha = -1/d$, where $d \geq 2$ is a positive integer, and let $p > -\alpha$. Then*

$$\frac{|C_n|}{n^{(pd-1)/(d-1)}} \xrightarrow{d} C,$$

where C has integer moments

$$\mathbb{E}[C^k] = \frac{D_k\Gamma(1/(d-1))}{\Gamma((kpd-k+1)/(d-1))},$$

where D_k satisfies the recursion $D_1 = 1/(pd-1)$ and

$$(k-1)(pd-1)D_k = \sum_{j=2}^{\min\{k,d\}} \frac{p^j d!}{(d-j)!}B_{k,j}(D_1, \dots, D_{k-j+1}).$$

By using the recursion in Theorem 2.15, we calculate the first two moments of C to be

$$\mathbb{E}[C] = \frac{\Gamma\left(\frac{1}{d-1}\right)}{(pd-1)\Gamma\left(\frac{pd}{d-1}\right)},$$

$$\mathbb{E}[C^2] = \frac{p^2d(d-1)\Gamma\left(\frac{1}{d-1}\right)}{(pd-1)^3\Gamma\left(\frac{2pd-1}{d-1}\right)}.$$

When the underlying tree is a binary search tree (when $d = 2$), we can once again find a complete description of the limiting distribution.

Proposition 2.16. *Let the underlying tree be a random binary search tree, so $d = 2$, and let $p > -\alpha = 1/2$. Then the integer moments of the limiting random variable C in Theorem 2.15 can be written as*

$$\mathbb{E}[C^k] = \frac{k!p^{2(k-1)}}{(2p-1)^{2k-1}\Gamma(k(2p-1)+1)}.$$

Remark 2.17. From Theorem 2.12, there is a positive probability that the size of the root cluster is finite, even in the case $p > 1/d$. For any p and d, let p_∞ be the smallest positive solution to

$$1 - x = (1 - px)^d.$$

It is known that p_∞ is the probability that the cluster $\mathcal{K}_d$ containing the root after performing Bernoulli bond percolation on an infinite d-ary tree is infinite (see e.g., [23, Exercise 5.41]).

From Proposition 2.11, $|C_n|$ converges almost surely to $|\mathcal{K}_d|$. The limiting random variable C in Theorem 2.15 can be broken down in the following way:

$$C = \begin{cases} 0 & \text{with probability } \ 1 - p_\infty, \\ C_\infty & \text{with probability } \ p_\infty, \end{cases}$$

where C_∞ has moments

$$\mathbb{E}[C_\infty^k] = p_\infty^{-1}\mathbb{E}[C^k].$$

When $d = 2$, then $p_\infty = (2p-1)/p^2$, and C_∞ has moments

$$\mathbb{E}[C_\infty^k] = \frac{k!p^{2k}}{(2p-1)^{2k}\Gamma(k(2p-1)+1)},$$

and so C_∞ is distributed as $p^2/(2p-1)^2$ times a random variable with Mittag–Leffler distribution with parameter $(2p-1)$.

3 | PROOFS OF GLOBAL PROPERTIES

We start by summarizing some results on generalized Pólya urns that will be used throughout this section. A generalized Pólya urn process $(\boldsymbol{X}_n)_{n=0}^\infty$ is defined as follows. There are q types (or colors) $1, 2, \dots, q$ of balls, and for each vector $\boldsymbol{X}_n = (X_{n,1}, X_{n,2}, \dots, X_{n,q})$, the entry $X_{n,i} \geq 0$ is the number of balls of type i in the urn at time n. For each type i, an activity $a_i \geq 0$ is assigned, as well as a random vector $\boldsymbol{\xi}_i = (\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,q})$ such that $\xi_{i,j} \geq 0$ for $i \neq j$ and $\xi_{i,i} \geq -1$. The urn process begins with a given vector $\boldsymbol{X}_0$. At time $n \geq 1$, a ball is drawn uniformly at random from the urn, so that the probability that a ball of color i is chosen is

$$\frac{a_i X_{n-1,i}}{\sum_{j=1}^q a_j X_{n-1,j}}.$$

If the drawn ball is of type i, then we set $\boldsymbol{X}_n = \boldsymbol{X}_{n-1} + \Delta\boldsymbol{X}_n$, where $\Delta\boldsymbol{X}_n \sim \boldsymbol{\xi}_i$ and $\Delta\boldsymbol{X}_n$ is independent of everything that has happened so far. The *intensity matrix* of the Pólya urn is the $q \times q$ matrix

$$A := \left(a_j \mathbb{E}[\xi_{j,i}]\right)_{i,j=1}^q.$$

Note that while several authors place $\mathbb{E}[\boldsymbol{\xi}_i]$ for row i of A, we follow the notation of [18] by placing $\mathbb{E}[\boldsymbol{\xi}_j]$ for column j of A. As noted in [18], since all off-diagonal entries of A are nonnegative, A has a largest real eigenvalue λ_1 such that $\lambda_1 > \mathrm{Re}\lambda$ for all other eigenvalues λ of A. A type i is called *dominating* if for all other types j, it is possible to find a ball of type j in an urn beginning with a single ball of type i. By ordering the types such that every dominating type i is smaller than every nondominating type j, the matrix A will be a block diagonal matrix. We say that an eigenvalue λ of A belongs to the dominating class if it is also an eigenvalue of the submatrix of A restricted to the dominating types.

The following six assumptions appear in [18] (the assumption (A1) is a generalization from [18, Remark 4.2], note the indices of the variables in (A1)):

(A1) For each $i = 1, \dots, q$, either

(a) there is a real number $d_i > 0$ such that $X_{0,i}$ and $\xi_{1,i}, \xi_{2,i}, \dots, \xi_{q,i}$ are multiples of d_i and $\xi_{i,i} \geq -d_i$, or

(b) $\xi_{i,i} \geq 0$.

(A2) $\mathbb{E}[\xi_{i,j}^2] < \infty$ for all $i, j = 1, \dots, q$.

(A3) The largest real eigenvalue λ_1 of A is positive.

(A4) The largest real eigenvalue λ_1 is simple.

(A5) There exists a dominating type i with $X_{0,i} > 0$.

(A6) λ_1 belongs to the dominating class.

We add the following simplifying assumption.

(A7) For each $n \geq 1$ there exists a ball of dominating type in the urn.

We further add the following assumption which will make the covariance matrix calculations simpler.

(A8) There exists $c > 0$ such that $\sum_{i=1}^{q} a_i \mathbb{E}[\xi_{j,i}] = c$ for every $j = 1, \dots, q$ where $a_j > 0$.

All of these assumptions are satisfied in the particular Pólya urns we use in the proofs that follow.

All vectors $\boldsymbol{v}$ for the remainder of this discussion are assumed to be column vectors. Let $\boldsymbol{a} = (a_1, \dots, a_q)^T$ be the vector of activities. Let $\boldsymbol{v}_1$ and $\boldsymbol{u}_1$ be the right and left eigenvectors associated with λ_1 normalized such that $\boldsymbol{a}^T \boldsymbol{v}_1 = 1$ and $\boldsymbol{u}_1^T \boldsymbol{v}_1 = 1$. Order the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_q$ such that $\lambda_1 \geq \mathrm{Re}\lambda_2 \geq \mathrm{Re}\lambda_3 \geq \cdots \geq \mathrm{Re}\lambda_q$. If A is diagonalizable, then there are q linearly independent right eigenvectors of A and q linearly independent left eigenvectors of A. Let $\boldsymbol{v}_i$ and $\boldsymbol{u}_i^T$ be dual bases for the eigenspaces of A, that is, right and left eigenvectors of A associated with λ_i such that $\boldsymbol{u}_i^T \boldsymbol{v}_j = \delta_{i,j}$ for all $i, j = 1, \dots, q$, where

$$\delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Denote $\boldsymbol{v}_1 := (v_{1,1}, v_{1,2}, \dots, v_{1,q})^T$ and define the matrices

$$B := \sum_{i=1}^{q} a_i v_{1,i} \mathbb{E}\left[\boldsymbol{\xi}_i \boldsymbol{\xi}_i^T\right],$$

and

$$\Sigma_I = \sum_{j,k=2}^{q} \frac{\boldsymbol{u}_j^T B \boldsymbol{u}_k}{\lambda_1 - \lambda_j - \lambda_k} \boldsymbol{v}_j \boldsymbol{v}_k^T,$$

whenever none of the denominators is equal to zero (which holds in the cases relevant to us). Let $P = I - \boldsymbol{v}_1 \boldsymbol{u}_1$ and

$$\Sigma_I^{\dagger} = \int_0^{\infty} P e^{sA} B e^{sA^T} P^T e^{-\lambda_1 s} ds. \tag{6}$$

If λ_2 is real and $\lambda_2 > \mathrm{Re}\lambda_3$, then define the matrix

$$\Sigma_{II} := (\boldsymbol{u}_2^T B \boldsymbol{u}_2) \boldsymbol{v}_2 \boldsymbol{v}_2^T. \tag{7}$$

We are now ready to gather results from [18].

Theorem 3.1 (Janson [18]). *Suppose an urn process $(\boldsymbol{X}_n)_{n=0}^{\infty}$ satisfies (A1)–(A7). The following hold*:

(i) a strong law of large numbers,

$$\frac{\boldsymbol{X}_n}{n} \xrightarrow{a.s.} \lambda_1 \boldsymbol{v}_1,$$

(ii) if (A8) is satisfied and $\lambda_1 > 2\mathrm{Re}\lambda_2$, then

$$\frac{\boldsymbol{X}_n - n\lambda_1\boldsymbol{v}_1}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\boldsymbol{0}, \lambda_1 \Sigma_I^{\dagger}),$$

where $\mathcal{N}$ denotes a multivariate normal distribution, and $\Sigma_I^{\dagger}$ is defined as in (6). If A is diagonalizable, then $\Sigma_I^{\dagger}$ can be replaced with Σ_I,

(iii) if (A8) is satisfied, $\lambda_1 = 2\lambda_2$, $\lambda_2 > \mathrm{Re}\lambda_3$, and A is diagonalizable, then

$$\frac{\boldsymbol{X}_n - n\lambda_1\boldsymbol{v}_1}{\sqrt{n\ln n}} \xrightarrow{d} \mathcal{N}(\boldsymbol{0}, \Sigma_{II}),$$

where $\mathcal{N}$ denotes a multivariate normal distribution, and Σ_{II} is defined as in (7),

(iv) if λ_2 is real, $\lambda_1 < 2\lambda_2$, and $\lambda_1 > 2\mathrm{Re}\lambda_i$ for all $i = 3, \dots, q$, then

$$\frac{\boldsymbol{X}_n - n\lambda_1\boldsymbol{v}_1}{n^{\lambda_2/\lambda_1}} \xrightarrow{a.s} \widehat{Z}\boldsymbol{v}_2,$$

where $\widehat{Z}$ is a real random variable.

Proof. The convergence in (i) follows from [18, Theorem 3.21] (essential nonexistence is always guaranteed if (A7) holds), the convergence in (ii) follows from [18, Theorem 3.22], and the convergence in (iii) follows from [18, Theorem 3.23], while the covariance matrix calculations in (ii) and (iii) follow from [18, Lemma 5.3(i), Lemma 5.4], where we note that the proof of [18, Lemma 5.4] follows exactly the same with the slightly more general assumption (A8). The convergence in (iv) follows from [18, Theorem 3.24] by letting $\widehat{Z} = \boldsymbol{u}_2^T \widehat{W}$ (the random vector $\widehat{W}$ is an element of the eigenspace of λ_2). ■

We are now ready to prove our results of global properties for $\mathcal{T}_n := \mathcal{T}_{\alpha,n}$ with broadcasting-induced coloring $\sigma_n := \sigma_{\mathcal{T}_n,p}$. Let $\alpha\,\deg^+(v) + 1$ be the *weight* of the vertex v in $\mathcal{T}_n$. We consider an urn with two colors of balls: red r and blue b, both with activity 1. In this urn, the total activity of red and blue balls at time n will correspond to the sum of the total weights of red and blue vertices in the tree $\mathcal{T}_n$ with coloring σ_n, respectively. When a ball is picked, with probability p it is replaced with an additional $1 + \alpha$ balls of the same color; 1 corresponding to the addition of a new vertex, while the extra α corresponds to the increase in weight of the selected vertex. With probability $1 - p$, the chosen ball is replaced along with α balls of the same color (corresponding to the increase in weight), while an additional 1 ball of the other color is added (corresponding to the new vertex added). Let R_n^w and B_n^w be the total activity of red and blue balls, respectively, at time n, which is also the total weight of the vertices of each color in $\mathcal{T}_n$. We therefore have the following activity matrix for our urn:

$$\begin{array}{ccc} & \begin{array}{cc} r & \quad b \end{array} & \\ A = & \begin{pmatrix} \alpha + p & 1 - p \\ 1 - p & \alpha + p \end{pmatrix} & \begin{array}{c} r \\ b \end{array} \end{array}.$$

This particular Pólya urn process was previously studied in the context of preferential attachment trees by Baur and Bertoin to study elephant random walks [5]. The eigenvalues of A are $\lambda_1 = 1+\alpha$ and $\lambda_2 = 2p+\alpha-1$, while A satisfies (A1)–(A8). Therefore, Theorem 3.1 applies with $\boldsymbol{v}_1 = (1/2, 1/2)$, $\boldsymbol{u}_1 = (1, 1)$, $\boldsymbol{v}_2 = (1/2, -1/2)$, and $\boldsymbol{u}_2 = (1, -1)$.

We can say something more about the limiting distribution in this case when $2\lambda_2 > \lambda_1$ (so when $p > (3-\alpha)/4$). Recall the random variable B defined in (2) ($B = 1$ if the root is red and $B = -1$ if the root is blue, so B is a Rademacher random variable).

Proposition 3.2. *Let R_n^w and B_n^w be the total weight of red and blue balls, respectively, and suppose that $p > (3-\alpha)/4$. Then*

$$\frac{(R_n^w, B_n^w) - n\left(\frac{1}{2}, \frac{1}{2}\right)}{n^{(2p+\alpha-1)/(1+\alpha)}} \xrightarrow{a.s.} BZ\left(\frac{1}{2}, -\frac{1}{2}\right), \tag{8}$$

where Z is a real random variable with

$$\mathbb{E}[Z] = \frac{\Gamma(1/(1+\alpha))}{\Gamma((2p+\alpha)/(1+\alpha))},$$

and

$$\mathbb{E}[Z^2] = \frac{\Gamma(1/(1+\alpha))(1+\alpha)(4p+\alpha-2)}{\Gamma((4p+2\alpha-1)/(1+\alpha))(4p+\alpha-3)}.$$

Proof. First, suppose we always start with a red root (so start the urn with a red ball). Then the convergence in (8) with $B = 1$ follows from 3.1 (iv). For the calculation of the expected value and second moment (again assuming we start with a red ball), we appeal to [18, Theorem 3.10, Theorem 3.26]. The random variable $Z_1 = \boldsymbol{u}_2^T W_{\lambda_2,1}$ corresponds in this case to starting with a single ball of color r, and $Z_2 = \boldsymbol{u}_2^T W_{\lambda_2,2}$ corresponds to starting with a single ball of color b. The expected value of Z_1 is the first component of $\boldsymbol{u}_2$. By symmetry, $\sigma_1^2 = \mathrm{Var}[Z_1] = \mathrm{Var}[Z_2] = \sigma_2^2$, and so

$$\begin{aligned}(2\lambda_2 - (\alpha+p) - (1-p))\sigma_1^2 &= \lambda_2^2 + \mathbb{E}[(\boldsymbol{u}_2^T\xi_1)^2] - (\boldsymbol{u}_2^T\mathbb{E}[\xi_1])^2 \\ &= p(1+\alpha)^2 + (1-p)(\alpha-1)^2.\end{aligned}$$

Rearranging for σ_1^2 and adding $(\mathbb{E}[Z_1])^2 = 1$ gives

$$\mathbb{E}[Z_1^2] = \frac{(1+\alpha)(4p+\alpha-2)}{4p+\alpha-3}.$$

Then by applying [18, Eq. 3.21], we get

$$\mathbb{E}[Z] = \frac{\Gamma(1/(1+\alpha))}{\Gamma((2p+\alpha)/(1+\alpha))}\mathbb{E}[Z_1] = \frac{\Gamma(1/(1+\alpha))}{\Gamma((2p+\alpha)/(1+\alpha))},$$

and

$$\mathbb{E}[Z^2] = \frac{\Gamma(1/\lambda_1)}{\Gamma((1+2\lambda_2)/\lambda_1)}\mathbb{E}[Z_1^2] = \frac{\Gamma(1/(1+\alpha))(1+\alpha)(4p+\alpha-2)}{\Gamma((4p+2\alpha-1)/(1+\alpha))(4p+\alpha-3)}.$$

Next we multiply by B since the urn starts with a single red ball with probability $1/2$, and a single blue ball with probability $1/2$. ■

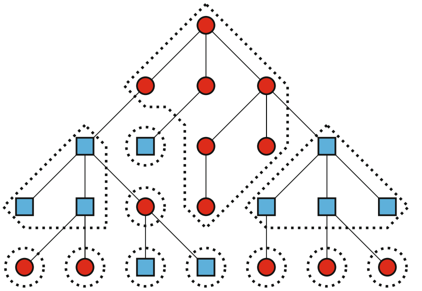

FIGURE 2 A tree $\mathcal{T}_{23}$ with broadcasting-induced coloring σ_{23} with the clusters identified.

While a Pólya urn can be used to study the number of vertices of each color, a simpler proof follows from limit laws for the number of clusters (maximal monochromatic subtrees) of each color. We therefore start with studying the clusters in $\mathcal{T}_n$.

Proof of Theorem 2.3. Consider an urn with four colors of balls: r, b with activity 1, and r^c, b^c, with activity 0. Let R_n^w and B_n^w be the total number of balls (and so the total activity of the balls) of colors r and b respectively, and let R_n^c and B_n^c be the total number of balls of colors r^c and r^b respectively. As in the urn above, the balls r and b represent the weights of the red and blue vertices in $\mathcal{T}_n$ with coloring $\sigma_{\mathcal{T}_n}$. The balls of colors r^c and b^c represent clusters of color red and blue respectively. We start the urn with a ball of color r and a ball of color r^c if the root is red, and a ball of color b and a ball of color b^c if the root is blue. Therefore the number of red and blue clusters at time n is exactly R_n^c and B_n^c, respectively.

For example in Figure 2, there are seven red clusters and five blue clusters, so $R_n^c = 7$ and $B_n^c = 5$. Each vertex v contributes $\alpha\deg^+(v) + 1$ to the total weight of its color. Summing over all red vertices yields $R_n^w = 13 + 11\alpha$ and summing over all blue vertices yields $B_n^w = 10 + 11\alpha$.

If a red vertex is chosen at step n, this corresponds to choosing a ball of color r. Then with probability p it is replaced with additional $1 + \alpha$ balls of color r, just as above. With probability $1 - p$ however, the ball is replaced along with α balls of color r along with 1 ball of color b (just as above), with an additional ball of color b^c added representing the new blue cluster that is formed. Therefore, we have

$$\mathbb{E}[\xi_r] = (\alpha + p, 1 - p, 0, 1 - p)^T,$$

the first column in the intensity matrix A^c for this urn. The symmetric argument for balls of color b holds, contributing to the second column of A^c. Balls of color r^c and b^c have activity 0, and so the intensity matrix for this urn is

$$A^c = \begin{array}{cc} \begin{array}{cccc} r & b & r^c & b^c \end{array} & \\ \begin{pmatrix} \alpha + p & 1 - p & 0 & 0 \\ 1 - p & \alpha + p & 0 & 0 \\ 0 & 1 - p & 0 & 0 \\ 1 - p & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} r \\ b \\ r^c \\ b^c \end{array} \end{array}.$$

The eigenvalues of A^c are $\lambda_1 = 1 + \alpha, \lambda_2 = 2p + \alpha - 1, \lambda_3 = \lambda_4 = 0$. We see that the assumptions (A1)–(A8) hold. The matrix A^c is diagonalizable when $\alpha \neq 1 - 2p$, and a dual basis for the eigenspaces of A in this case is given by

$$
\begin{aligned}
\boldsymbol{v}_1 &= \frac{1}{2}\left(1, 1, \frac{1-p}{1+\alpha}, \frac{1-p}{1+\alpha}\right),\\
\boldsymbol{v}_2 &= \frac{1}{2}\left(1, -1, \frac{p-1}{2p+\alpha-1}, \frac{1-p}{2p+\alpha-2}\right),\\
\boldsymbol{v}_3 &= \frac{1}{(1+\alpha)(2p+\alpha-1)}(0,0,1,0),\\
\boldsymbol{v}_4 &= \frac{1}{(1+\alpha)(2p+\alpha-1)}(0,0,0,1),\\
\boldsymbol{u}_1 &= (1,1,0,0),\\
\boldsymbol{u}_2 &= (1,-1,0,0),\\
\boldsymbol{u}_3 &= \left((1-p)^2, (p-1)(\alpha+p), (1+\alpha)(2p+\alpha-1), 0\right),\\
\boldsymbol{u}_4 &= \left((p-1)(\alpha+p), (1-p)^2, 0, (1+\alpha)(2p+\alpha-1)\right).
\end{aligned}
$$

We can therefore apply Theorem 3.1. Using MATHEMATICA, the covariance matrices for the limiting distribution for this urn when $1+\alpha = \lambda_1 > 2\lambda_2 = 4p-2+2\alpha$ and $1+\alpha = \lambda_1 = 2\lambda_2 = 4p-2+2\alpha$ are calculated (see Appendix A.1). If $\alpha = 1-2p$ (and so A^c is not diagonalizable), then $1+\alpha = \lambda_1 > 2\lambda_2 = 4p-2+2\alpha$ (since $\alpha > -1$), and the calculation for $\Sigma_1^\dagger$ from (6) yields the same result as the calculation for Σ_I. When $1+\alpha = \lambda_1 < 2\lambda_2 = 4p-2+2\alpha$, we conclude from Theorem 3.1 (iv) that $n^{-(2p+\alpha-1)/(1+\alpha)}(R_n^w, B_n^w, R_n^c, B_n^c)$ converges almost surely to $\widehat{Z}\boldsymbol{v}_2$, for some random variable $\widehat{Z}$. If we restrict to R_n^w and B_n^w, we see that $\widehat{Z}$ is the same random variable BZ as in (8). Restricted to R_n^c and B_n^c, the results of Theorem 2.3 follow. ∎

A similar urn process to the one above (with balls of activity 0 representing the vertices) can be used to find limit laws for the number of vertices of each color. But we can instead use the following observation: if a vertex of one color contributes to the weight of another vertex of a different color, then it must be the root of a cluster. Therefore, from the previous proof, we can now derive convergence for the number of vertices of each color.

Proof of Theorem 2.1. If we consider again the urn in the previous proof, we can recover the number of vertices R_n and B_n of each color in our tree. Each red vertex contributes $(1+\alpha)$ to the value R_n^w, except those that are roots of red clusters; these contribute 1 to R_n^w. The root of a blue cluster contributes α to the weight of its parent, and so α to R_n^w. The only root of a blue cluster that does not contribute to R_n^w is the root of $\mathcal{T}_n$ if this root is blue. Using B defined in (2), we see that $R_n^w = (1+\alpha)R_n - \alpha R_n^c + \alpha\left(B_n^c + \frac{B-1}{2}\right)$. Performing the symmetric analysis for B_n^w and rearranging gives

$$
R_n = \frac{R_n^w + \alpha R_n^c - \alpha\left(\frac{B-1}{2}\right)}{1+\alpha},
$$
$$
B_n = \frac{B_n^w + \alpha B_n^c + \alpha\left(\frac{B+1}{2}\right)}{1+\alpha}.
$$

When scaled by $\sqrt{n}$, $\sqrt{\ln n}$, or $n^{(2p+\alpha-1)(1+\alpha)}$, the last term of each of the equations above vanishes. For $\lambda_1 \geq 2\lambda_2$, since $R_n^c, B_n^c, R_n^w, B_n^w$ converge jointly in distribution, so do linear combinations of these random variables by the continuous mapping theorem. The limiting distributions are normal in these cases, and the covariance matrices can be calculated from the covariance matrices in Appendix A.1.

As discussed in Remark 2.2, we can treat the special case when $p = 1/2$ directly since the number of red vertices is simply given by $R_n = \sum_{i=1}^{n} X_i$ where $X_i \sim \mathrm{Be}(1/2)$ are independent Bernoulli random variables. Then we can apply the central limit theorem to get

$$\frac{R_n - \frac{n}{2}}{\sqrt{n}} \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4}\right).$$

A multivariate normal limit law for the number of red and blue vertices follows since $B_n = n - R_n$ and

$$a\left(\frac{R_n - \frac{n}{2}}{\sqrt{n}}\right) + b\left(\frac{B_n - \frac{n}{2}}{\sqrt{n}}\right) = (a-b)\left(\frac{R_n - \frac{n}{2}}{\sqrt{n}}\right),$$

converges in distribution to a normal distribution for all $a, b \in \mathbb{R}$, so the Cramér–Wold theorem applies [15, ch. 5, Theorem 10.5]. Finally, a quick calculation shows that $\mathrm{Cov}(R_n, B_n) = -\mathrm{Var}(R_n)$, implying the convergence in Theorem 2.1(ii) when $p = 1/2$. ∎

We turn now to the number of leaves of each color.

Proof of Theorem 2.4. Consider an urn with four colors of balls: r^l, b^l, r^u, b^u, each with activity 1. Let R_n^l, B_n^l, R_n^u, and B_n^u be the total number balls of colors r^l, b^l, r^u, and b^u, respectively, at time n. The balls of color r^l and b^l represent red and blue leaves, respectively. The other balls represent the remaining weights of the red and blue vertices respectively.

If a red leaf is chosen at step n, this corresponds to choosing a ball of color r^l. Then with probability $1-p$ it is removed and replaced with one ball of color b^l for the new blue leaf that is added, and $1+\alpha$ balls of color r^u, representing the weight of the now nonleaf vertex that was chosen. With probability p, the ball is placed back in the urn for the new red leaf that was added, along with $1+\alpha$ balls of color r^u, representing the weight of the now nonleaf vertex that was chosen. Therefore, we have

$$\mathbb{E}[\xi_{r^l}] = (p-1, 1-p, \alpha+1, 0)^T,$$

the first column of the intensity matrix A^l for this urn. If a red vertex that is not a leaf is chosen, then additional α balls of color r^u are added (for the increase in weight of that vertex), along with either one ball of color r^l with probability p, or one ball of color b^l with probability $1-p$. Therefore, we have

$$\mathbb{E}[\xi_{r^u}] = (p, 1-p, \alpha, 0)^T,$$

the third column of A^l. The symmetric arguments hold when balls of color b^l or b^u are chosen. Therefore, the intensity matrix for this urn is

$$A^l = \begin{array}{cc} \begin{array}{cccc} r^l & b^l & r^u & b^u \end{array} & \\ \begin{pmatrix} p-1 & 1-p & p & 1-p \\ 1-p & p-1 & 1-p & p \\ \alpha+1 & 0 & \alpha & 0 \\ 0 & \alpha+1 & 0 & \alpha \end{pmatrix} & \begin{array}{c} r^l \\ b^l \\ r^u \\ b^u \end{array} \end{array}. \tag{9}$$

We see immediately that assumptions (A1)–(A8) hold. The eigenvalues of A are $\lambda_1 = 1+\alpha$, $\lambda_2 = 2p-1+\alpha$, $\lambda_3 = \lambda_4 = -1$. The matrix A^l is diagonalizable when $\alpha \neq -2p$, and a dual basis for the eigenspaces of A in this case is given by

$$
\begin{aligned}
\boldsymbol{v}_1 &= \frac{1}{4+2\alpha}(1,1,1+\alpha,1+\alpha),\\
\boldsymbol{v}_2 &= \frac{1}{2\alpha+4p}(2p-1,1-2p,1+\alpha,-(1+\alpha)),\\
\boldsymbol{v}_3 &= \frac{1}{(2+\alpha)(\alpha+2p)}(1,0,-1,0),\\
\boldsymbol{v}_4 &= \frac{1}{(2+\alpha)(\alpha+2p)}(0,1,0,-1),\\
\boldsymbol{u}_1 &= (1,1,1,1),\\
\boldsymbol{u}_2 &= (1,-1,1,-1),\\
\boldsymbol{u}_3 &= ((1+\alpha)(1+\alpha+p),(1+\alpha)(p-1),1-(3+\alpha)p,(1+\alpha)(p-1)),\\
\boldsymbol{u}_4 &= ((1+\alpha)(p-1),(1+\alpha)(1+\alpha+p),(1+\alpha)(p-1),1-(3+\alpha)p).
\end{aligned}
$$

We can therefore apply Theorem 3.1. Using MATHEMATICA, the covariance matrix for the limiting distribution for this urn when $1+\alpha = \lambda_1 > 2\lambda_2 = 4p-2+2\alpha$ and $1+\alpha = \lambda_1 = 2\lambda_2 = 4p-2+2\alpha$ are calculated (see Appendix A.2). If $\alpha = -2p$, then $1+\alpha = \lambda_1 > 2\lambda_2 = 4p-2+2\alpha$ (since $\alpha > -1$), and the calculation for $\Sigma_1^{\dagger}$ from (6) yields the same result as the calculation for Σ_I. When $1+\alpha = \lambda_1 < 2\lambda_2 = 4p-2+2\alpha$, we can conclude from Theorem 3.1 (iv) that $n^{-(2p+\alpha-1)/(1+\alpha)}(R_n^l, B_n^l, R_n^u, B_n^u)$ converges almost surely to $\widehat{Z}\boldsymbol{v}_2$ for some random variable $\widehat{Z}$. Notice also $R_n^l + R_n^u = R_n^w$ and $B_n^l + B_n^u = B_n^w$, and by the uniqueness of convergence almost surely, we see that $\widehat{Z} = BZ$ from (8) almost surely.

When $p = 1/2$, the color of a newly added vertex does not depend on the color of its parent. In this case, consider an urn with three colors of balls: r^l, b^l, v^u, each with activity 1. The balls of color r^l and b^l represent red and blue leaves, respectively, while v^u represents the remaining weights of all nonleaf vertices. Performing a similar analysis as above, we get the following intensity matrix for this urn:

$$
A^l = \begin{array}{c@{}c}
\begin{array}{ccc} r^l & b^l & v^u \end{array} & \\
\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \alpha+1 & \alpha+1 & \alpha \end{pmatrix} & \begin{array}{c} r^l \\ b^l \\ v^u \end{array}
\end{array}. \tag{10}
$$

The eigenvalues of A are $\lambda_1 = 1+\alpha$, $\lambda_2 = \lambda_3 = -1$, and the matrix is diagonalizable for all valid values of α. A dual basis for the eigenspaces of A is given by

$$
\begin{aligned}
\boldsymbol{v}_1 &= \frac{1}{4+2\alpha}(1,1,2+2\alpha),\\
\boldsymbol{v}_2 &= \frac{1}{4+2\alpha}(1,-3-2\alpha,2+2\alpha),\\
\boldsymbol{v}_3 &= \frac{1}{4+2\alpha}(1,1,-2),\\
\boldsymbol{u}_1 &= (1,1,1),\\
\boldsymbol{u}_2 &= (1,-1,0),\\
\boldsymbol{u}_3 &= (2(1+\alpha),0,-1).
\end{aligned}
$$

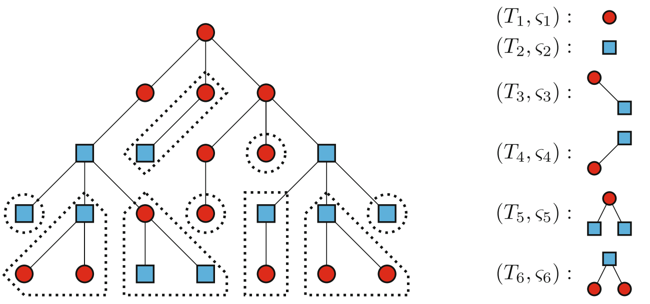

FIGURE 3 A tree $\mathcal{T}_{23}$ with broadcasting induced coloring σ_{23} with fringe subtrees identified.

Once more, assumptions (A1)–(A8) hold. By looking at the eigenvalues of A, we see immediately that Theorem 3.1 (ii) applies. The covariance matrix for this case is included in Appendix A.2.

Restricted to R_n^l and B_n^l, the results of Theorem 2.4 follow. ■

The proof of Theorem 2.6 follows much the same way as the proof of [17, Theorem 3.9]. Consider a partial ordering $\preccurlyeq$ on the set of all pairs (T, ς), where T is a rooted tree and ς is a two-coloring of the vertices, such that $(T_1, \varsigma_1) \preccurlyeq (T_2, \varsigma_2)$ if T_1 is a subtree of T_2 (preserving the root) and $\varsigma_2|_{T_1} = \varsigma_1$. Let $S = \{(T_1, \varsigma_1), \dots, (T_q, \varsigma_q)\}$ such that if $(T, \varsigma) \in S$ and $(T', \varsigma') \preccurlyeq (T, \varsigma)$, then $(T', \varsigma') \in S$. Assume that the pairs $(T_1, \varsigma_1), \dots, (T_q, \varsigma_q)$ are indexed so that if $(T_i, \varsigma_i) \preccurlyeq (T_j, \varsigma_j)$ then $i < j$, and assume that (T_1, ς_1) corresponds to a single red vertex, and (T_2, ς_2) corresponds to a single blue vertex. We define an urn such that for the tree $\mathcal{T}_n$ with coloring σ_n, if a vertex v is the root of a fringe subtree T isomorphic to T_i with $\sigma_n|_T = \varsigma_i$ for which $(T_i, \varsigma_i) \in S$ and if v does not belong to another fringe subtree T' isomorphic to T_j with $\sigma_n|_{T'} = \varsigma_j$ such that $(T_i, \varsigma_i) \preccurlyeq (T_j, \varsigma_j) \in S$, then v is represented in the urn by the ball of type i. If v is not the root of a fringe subtree isomorphic to a tree with coloring in S, then v is represented by $\alpha \deg^+(v) + 1$ balls of special type $*_r$ if v is red, and $*_b$ if v is blue. Let Y_n^i be the number of balls of type i at time n, and let $Y_n^{*_r}$ and $Y_n^{*_b}$ be the number of balls of special type $*_r$ and $*_b$ respectively at time n, and let $\boldsymbol{Y}_n = (Y_n^1, \dots, Y_n^q, Y_n^{*_r}, Y_n^{*_b})$.

For example, consider $S = \{(T_1, \varsigma_1), \dots, (T_6, \varsigma_6)\}$, where (T_i, ς_i) are identified on the right side of Figure 3. A tree $\mathcal{T}_{23}$ with coloring σ_{23} is given in Figure 3. Then the urn we consider will contain two balls of type 1, two balls of type 2, one ball of type 3, one ball of type 4, one ball of type 5, and two balls of type 6. There are a further $7\alpha + 4$ balls of type $*_r$ for the remaining red vertices, and $6\alpha + 2$ balls of type $*_b$ for the remaining blue vertices. Note that only two red leaves contribute balls of type 1, since the remaining red leaves are subtrees of fringe subtrees isomorphic to (T_4, σ_4) or (T_6, σ_6).

The activity of each ball of type i is given by the sum of the weights of the vertices in the tree T_i, which is $a_i := |T_i|(\alpha + 1) - \alpha$. The activities of the balls of special type are 1. When a ball of type i is picked, this corresponds to adding a child u to a vertex v that lies in a fringe subtree isomorphic to T_i. Let (T_j, ς_j) denote the fringe subtree with u attached and colored. If $(T_j, \varsigma_j) \in S$, then the ball of type i is removed and replaced with a ball of type j. If $(T_j, \varsigma_j) \notin S$, then the ball of type i is removed, the root ρ_j of T_j is now represented by $\alpha \deg^+(\rho_j) + 1$ balls of special type (with the appropriate colors) that are newly added, and the children of ρ_j are roots to $\deg^+(\rho_j)$ newly considered fringe subtrees. If these subtrees (along with their coloring) appear in S, then balls representing them are added. Otherwise, balls of special type are added for the root, and the subtrees of that vertex are considered, continuing this process until all vertices are represented by balls in the urn. If a new vertex u added to $\mathcal{T}_n$ is the child of a vertex v that is represented by balls of special type in the urn, then α balls of special type with the appropriate color are added to the urn, representing the increase in the weight of v, while either a ball of type 1 or 2 is added as well, representing the new leaf u added to $\mathcal{T}_n$.

For $4 \leq k \leq q+2$, let $S_k = \{(T_1, \varsigma_1), \dots, (T_{k-2}, \varsigma_{k-2})\}$, and let A_k be the intensity matrix for the urn with balls of type $1, \dots, k-2$ along with $*_r$ and $*_b$. Let $a_i := |T_i|(\alpha+1) - \alpha$.

Proof of Theorem 2.6. We start with convergence of the random vector $\boldsymbol{Y}_n$. We would like to know the eigenvalues of the matrix A_{q+2}. We proceed by induction on k. Let $4 \leq k \leq q+2$ and consider A_k. Let $(T_{i(r)}, \varsigma_{i(r)})$ and $(T_{i(b)}, \varsigma_{i(b)})$ be the longest red and blue path respectively in S_k. Then $(A_k)_{ii} = -a_i$ for all $i \neq i(r)$ and $i \neq i(b)$, $(A_k)_{i(r),i(r)} = p - a_{i(r)}$, $(A_k)_{i(b),i(b)} = p - a_{i(b)}$, and $(A_k)_{k-1,k-1} = (A_k)_{kk} = \alpha$. Therefore, we see that

$$\operatorname{tr}(A_k) = \alpha + 1 - \sum_{j=1}^{k-2} a_j.$$

The base case is A_4, which is precisely the intensity matrix A^l in (9), and has eigenvalues $\lambda_1 = 1 + \alpha, \lambda_2 = 2p - 1 + \alpha, \lambda_3 = \lambda_4 = -1$. The induction step is identical to the one in the proofs of [17, Theorem 6.2, Theorem 8.2], and the eigenvalues of A_{k+1} are inherited from A_k. The last eigenvalue is then given by

$$\lambda_{k+1} = \operatorname{tr}(A_{k+1}) - \operatorname{tr}(A_k) = -a_k.$$

Therefore, the eigenvalues of A_{q+2} are

$$\lambda_1 = 1 + \alpha, \lambda_2 = 2p - 1 + \alpha, -a_1, -a_2, \dots, -a_q.$$

All of the types of balls in the urn are dominating types. This follows since there will always eventually be balls of special type. When a ball of special type is chosen, then either a ball of type 1 or 2 is added (corresponding to the new leaf added to the tree). Since for every $(T_i, \varsigma_i) \in S$, we have either $(T_1, \varsigma_1) \preccurlyeq (T_i, \varsigma_i)$ or $(T_2, \varsigma_2) \preccurlyeq (T_i, \varsigma_i)$, there is a positive probability of a ball of type i appearing. Finally, for any $(T_i, \varsigma_i) \in S$, there is a positive probability that vertices in a fringe subtree isomorphic to T_i are chosen often enough so that the tree eventually decomposes to balls of special type (and other balls of other types). So if we start the urn with a single ball of any type, then there is a positive probability that any other type of ball will eventually appear.

All the conditions are met for convergence of the urn process, and we can apply Theorem 3.1. For appropriate right eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we get

$$\frac{\boldsymbol{Y}_n}{n} \xrightarrow{a.s.} \boldsymbol{v}_1, \tag{11}$$

$$\frac{\boldsymbol{Y}_n - n\boldsymbol{v}_1}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\boldsymbol{0}, \Sigma_I^g) \qquad \text{if } p < \frac{3-\alpha}{4}, \tag{12}$$

$$\frac{\boldsymbol{Y}_n - n\boldsymbol{v}_1}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(\boldsymbol{0}, \Sigma_{II}^g) \qquad \text{if } p = \frac{3-\alpha}{4}, \tag{13}$$

$$\frac{\boldsymbol{Y}_n - n\boldsymbol{v}_1}{n^{(2p+\alpha-1)/(1+\alpha)}} \xrightarrow{a.s.} BZ\boldsymbol{v}_2 \qquad \text{if } p > \frac{3-\alpha}{4}. \tag{14}$$

Similar to the proofs above, R_n^w and B_n^w are linear combinations of $Y_n^1, \dots, Y_n^q, Y_n^{*_r}, Y_n^{*_b}$, and so the random variable Z is the same almost surely as in (8) (with the appropriate choice of $\boldsymbol{v}_2$).

The case $p = 1/2$ is treated similarly by looking at $\boldsymbol{Y}_n' = (Y_n^1, \dots, Y_n^q, Y_n^*)$, where $Y_n^* = Y_n^{*_r} + Y_n^{*_b}$ counts all balls $*$ of special type. Since the color of a new vertex is independent of the color of its parent, when a ball of type $*$ is chosen, α balls of special type $*$ are added, while either a ball of type 1 or 2 is added with equal probability. For $3 \le k \le q+1$, let A_k' be the intensity matrix for the urn with balls of type $1, \dots, k-1$ along with balls of type $*$. Similar arguments as above hold, but in this case, the base case A_3' is the matrix A^l from (10). Thus, the eigenvalues of A_{q+1}' are

$$\lambda_1 = 1 + \alpha, -a_1, -a_2, \dots, -a_q.$$

The conditions are once again met for convergence of the urn process, and by applying Theorem 3.1 with the appropriate right eigenvector $\boldsymbol{v}_1'$ of A_{q+1}', we get

$$\frac{\boldsymbol{Y}_n' - n\boldsymbol{v}_1'}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(\boldsymbol{0}, \Sigma_I^g). \tag{15}$$

The random variables $X_n^1, \dots, X_n^q$ are linear combinations of $Y_n^1, \dots, Y_n^q$, and so the convergences of Theorem 2.6 hold by (11)–(15) above and the continuous mapping theorem, though we need to replace $\boldsymbol{\mu}$ from (4) with some vector $\boldsymbol{\mu}'$ for now. We can show that $\boldsymbol{\mu}' = \boldsymbol{\mu}$ by looking at $\mathbb{E}[(X_n^1, \dots, X_n^q)]/n$. We have just argued that $(X_n^1, \dots, X_n^q)/n$ converges almost surely to $\boldsymbol{\mu}'$, and since no number of fringe trees exceeds the number of vertices, $(X_n^1, \dots, X_n^q)/n$ is uniformly bounded. Therefore, $(X_n^1, \dots, X_n^q)/n$ converges in mean to $\boldsymbol{\mu}'$, and so $\mathbb{E}[(X_n^1, \dots, X_n^q)]/n$ converges to $\boldsymbol{\mu}'$. From [17, Remark 3.10] we know that the expected number of fringe subtrees X^{T_i} isomorphic to T_i is given by

$$\mathbb{E}[X_n^{T_i}] = \frac{\mathbb{P}(\mathcal{T}_{\alpha,k_i} \simeq T_i)\frac{1}{1+\alpha}}{(k_i - 1 + \frac{1}{1+\alpha})(k_i + \frac{1}{1+\alpha})} n + O(1),$$

where k_i is the number of vertices in T_i. Since the root of $\mathcal{T}_n$ is red or blue with equal probability, then by symmetry, the root of a fringe subtree T isomorphic to T_i is red or blue with equal probability. Then by definition of σ_n, the coloring ς of T follows the same distribution as σ_T. From this, we conclude that

$$\begin{aligned}
\mathbb{E}[X_n^i] &= \mathbb{P}(\sigma_{T_i} = \varsigma_i)\mathbb{E}[X_n^{T_i}] \\
&= \mathbb{P}(\sigma_{T_i} = \varsigma_i)\frac{\mathbb{P}(\mathcal{T}_{\alpha,k_i} \simeq T_i)\frac{1}{1+\alpha}}{(k_i - 1 + \frac{1}{1+\alpha})(k_i + \frac{1}{1+\alpha})} n + O(1) \\
&= \frac{\mathbb{P}((\mathcal{T}_{\alpha,k_i}, \sigma_{k_i}) \simeq (T_i, \varsigma_i))\frac{1}{\alpha+1}}{(k_i + \frac{1}{\alpha+1} - 1)(k_i + \frac{1}{\alpha+1})} n + O(1).
\end{aligned}$$

Therefore, $\mathbb{E}[(X_n^1, \dots, X_n^q)]/n$ converges to $\boldsymbol{\mu}$, and so $\boldsymbol{\mu}' = \boldsymbol{\mu}$. ∎

4 | PROOFS OF PROPERTIES OF THE ROOT CLUSTER

As mentioned in the introduction, convergence in distribution for the size of the root cluster has previously been proven [3, 4, 8, 26] using random walks and branching processes. Here we use results from analytic combinatorics to get recursions for the moments of the limiting distributions.

We start by a useful description of the trees studied. Since we are only interested in the size of the root cluster and not the color of this cluster, we can assume without loss of generality that the root is red. In the following, we define $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\phi(\delta) = \begin{cases} 1 & \alpha = 0, \\ \frac{\Gamma(\delta+1/\alpha)}{\Gamma(1/\alpha)} & \alpha > 0, \\ \frac{d!}{(d-\delta)!} & \alpha = -\frac{1}{d}, d \in \mathbb{Z}^+. \end{cases} \tag{16}$$

For a particular tree T on n vertices, define the *weight* of T to be

$$w(T) = \prod_{v \in V(T)} \phi(\deg^+(v)). \tag{17}$$

Then the probability of producing the tree T is given by

$$\mathbb{P}(\mathcal{T}_n = T) = \frac{w(T)}{\sum_{T'} w(T')},$$

see for example [10, Section 1.3.3]. The probability of producing T with a broadcasting induced coloring σ_T being the two-coloring ς is then given by

$$\mathbb{P}((\mathcal{T}_n, \sigma_n) = (T, \varsigma) | \sigma_n(\rho) = \text{red}) = \frac{2\mathbb{P}(\sigma_T = \varsigma) w(T)}{\sum_{T'} w(T')},$$

the factor of 2 appearing since we condition on the root being colored red. If we define the weight $\omega(T, \varsigma) = 2\mathbb{P}(\sigma_T = \varsigma) w(T)$, then

$$\mathbb{P}((\mathcal{T}_n, \sigma_n) = (T, \varsigma) | \sigma_n(\rho) = \text{red}) = \frac{\omega(T, \varsigma)}{\sum_{(T', \varsigma')} \omega(T', \varsigma')},$$

where (T', ς') ranges over all rooted trees on n vertices and over all two-colorings of the vertices such that the root is red. Symmetrically, define $\omega'(T, \varsigma)$ to be the weight of T and ς where ς is conditioned such that the color of the root is blue.

Let $r_{n,k}$ be the sum of the weights $\omega(T, \varsigma)$ over all trees with n vertices whose root vertex is red and whose root cluster has size k, and let b_n be the sum of the weights $\omega'(T, \varsigma)$ over all trees on n vertices with a blue root. Equivalently, b_n is the sum of the weights $w(T)$ over all trees T on n vertices (and so $b_n = \sum_k r_{n,k}$). Then notice that

$$\mathbb{P}((\mathcal{T}_n, \sigma_n) = (T, \varsigma)) = \frac{\omega(T, \varsigma)}{\sum_k r_{n,k}} = \frac{\omega(T, \varsigma)}{b_n},$$

and the probability that $\mathcal{T}_n$ with coloring σ_n has a root cluster of size k' is given by $r_{n,k'} / \sum_k r_{n,k}$. We develop a recursion formula for $r_{n,k}$. Take any tree T on n vertices, with coloring ς conditioned on the root ρ being red, whose root cluster is of size k, and with δ subtrees rooted at the children of the root ρ.

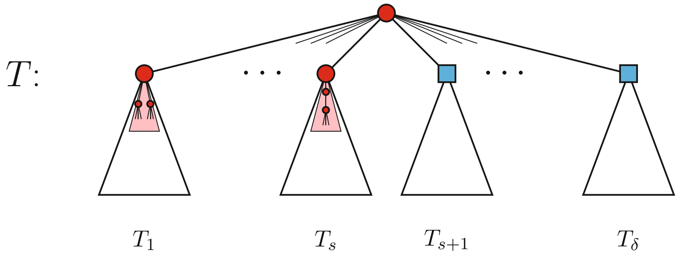

FIGURE 4 A recursion for the weight of T is established by examining the subtrees rooted at children of the root of T.

Suppose we order the subtrees such that the first s subtrees $T_1, \dots, T_s$ have red roots, and the remaining subtrees $T_{s+1}, \dots, T_\delta$ have blue roots (Figure 4). Then the weight of T and ς can be written as

$$\omega(T,\varsigma) = \phi(\delta)\prod_{i=1}^{s} p\omega(T_i,\varsigma_i)\prod_{j=s+1}^{\delta}(1-p)\omega'(T_j,\varsigma_j),$$

where the ς_i's and ς_j's are the coloring ς restricted to the subtrees T_i and T_j, respectively. If the trees $T_1, \dots, T_\delta$ are of size $n_1, \dots, n_\delta$, then $n_1 + \cdots + n_\delta = n-1$. If the trees $T_1, \dots, T_s$ have root clusters of size $k_1, \dots, k_s$, then $k_1 + \cdots + k_s = k-1$. Now sum over all such trees T on n vertices with root clusters of size k. The degree δ of the root can range from 0 to $n-1$. The number s of children with the color red ranges from 0 to δ. There are $\binom{\delta}{s}$ ways of choosing these s children. There are $\binom{n-1}{n_1,\dots,n_\delta}$ ways of distributing the remaining $n-1$ vertices to the δ subtrees $T_1, \dots, T_\delta$. Finally, to unorder the subtrees we divide by $\delta!$ to get the recursion

$$\begin{aligned} r_{n,k} &= \sum_{\delta=0}^{n-1}\sum_{s=0}^{\delta}\binom{\delta}{s}\frac{\phi(\delta)}{\delta!}\sum_{n_1,\dots,n_\delta}\binom{n-1}{n_1,\dots,n_\delta}\sum_{k_1,\dots,k_s}\prod_{i=1}^{s} p r_{n_i,k_i}\prod_{j=s+1}^{\delta}(1-p)b_{n_j} \\ &= \sum_{\delta=0}^{\infty}\sum_{s=0}^{\delta}\binom{\delta}{s}\frac{\phi(\delta)}{\delta!}\sum_{n_1,\dots,n_\delta}\binom{n-1}{n_1,\dots,n_\delta}\sum_{k_1,\dots,k_s} p^s\prod_{i=1}^{s} r_{n_i,k_i}(1-p)^{\delta-s}\prod_{j=s+1}^{\delta} b_{n_j}, \end{aligned} \tag{18}$$

where $n_1, \dots, n_\delta$ range over all nonnegative integers with $n_1 + \cdots + n_\delta = n-1$, and $k_1, \dots, k_s$ range over all nonnegative integers with $k_1 + \cdots + k_s = k-1$. Finally we can let δ range to infinity since $r_{0,k} = b_0 = 0$.

We start with the case $\alpha = 0$. It is already known that the size of the root cluster converges to a Mittag–Leffler distribution after proper rescaling in this case. To help outline our more general methods, we show how to prove convergence in distribution by using the recursion in (18) and the method of moments.

Let $R(x,u)$ be the bivariate (exponential) generating function for $r_{n,k}$, so

$$R(x,u) = \sum_{n,k\geq 1}\frac{r_{n,k}}{n!}x^n u^k,$$

and let $B(x)$ be the exponential generating function for b_n. The first thing to notice is that b_n and $\sum_{k=1}^{n} r_{n,k}$ are simply the number of recursive trees of size n, which is $(n-1)!$. Therefore,

$$B(x) = R(x,1) = -\ln(1-x).$$

Using (18), we establish a partial differential equation for $R(x,u)$. The resulting differential equation is then solved to get the following closed form for $R(x,u)$.

Proposition 4.1. *For $\alpha = 0$, the bivariate generating function $R(x,u)$ is given by*

$$R(x,u) = \frac{-1}{p}\ln(1-u+u(1-x)^p).$$

Proof. From (18), where $\phi(\delta) = 1$ for all δ (recall (16)), we get the partial differential equation

$$\begin{aligned}
\frac{\partial}{\partial x}R(x,u) &= \sum_{n,k} n\frac{r_{n,k}}{n!}x^{n-1}u^k \\
&= u\sum_{n,k}\sum_{\delta=0}^{n-1}\sum_{s=0}^{\delta}\binom{\delta}{s}\frac{1}{\delta!}\sum_{n_1,\dots,n_\delta}\sum_{k_1,\dots,k_s}\prod_{i=1}^{s}\frac{pr_{n_i,k_i}x^{n_i}u^{k_i}}{n_i!}\prod_{j=s+1}^{\delta}\frac{(1-p)b_{n_j}x^{n_j}}{n_j!} \\
&= u\sum_{\delta=0}^{\infty}\sum_{s=0}^{\delta}\frac{1}{\delta!}\binom{\delta}{s}(pR(x,u))^s((1-p)B(x))^{\delta-s} \\
&= u\sum_{\delta=0}^{\infty}\frac{1}{\delta!}(pR(x,u)+(1-p)B(x))^{\delta} \\
&= u\exp(pR(x,u)+(1-p)B(x)).
\end{aligned}$$

Replacing $B(x)$ with $-\ln(1-x)$ and with the initial condition $R(0,u) = 0$, this linear differential equation has the solution

$$R(x,u) = \frac{-1}{p}\ln(1-u+u(1-x)^p).$$

■

From Proposition 4.1, we calculate

$$\left.\frac{\partial^k}{\partial u^k}R(x,u)\right|_{u=1} = \frac{1}{p}(k-1)!((1-x)^{-p}-1)^k.$$

We then extract the coefficients,

$$[x^n]\left.\frac{\partial^k}{\partial u^k}R(x,u)\right|_{u=1} \sim [x^n]\frac{1}{p}(k-1)!(1-x)^{-pk} \sim \frac{(k-1)!n^{pk-1}}{p\Gamma(pk)}.$$

Let C_n be the root cluster at time n. The factorial moments of $|C_n|$ are extracted from the bivariate generating function (see e.g., [13, Proposition III.2]) to get

$$\mathbb{E}\left[|C_n|(|C_n|-1)\cdots(|C_n|-k+1)\right] = \frac{[x^n]\left.\frac{\partial^k}{\partial u^k}R(x,u)\right|_{u=1}}{[x^n]R(x,1)} \sim \frac{(k-1)!n^{pk}}{p\Gamma(pk)}.$$

It can be seen (say by induction), that once expanded and scaled by n^{pk}, all but the $\mathbb{E}[|C_n|^k]$ term on the left hand side of the above equation vanish to zero, and thus

$$\mathbb{E}\left[\frac{|C_n|^k}{n^{pk}}\right] \sim \frac{(k-1)!n^{pk}}{n^{pk}p\Gamma(pk)} = \frac{(k-1)!}{p\Gamma(pk)} = \frac{k!}{\Gamma(pk+1)},$$

which are the moments of the Mittag–Leffler distribution with parameter p. The Mittag–Leffler distribution is uniquely determined by its moments (since its moment generating function, the Mittag–Leffler function $E_p(s) = \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(pn+1)}$, converges for all values of s [25]). Therefore,

$$\frac{|C_n|}{n^p} \xrightarrow{d} M_p,$$

where M_p has the Mittag–Leffler distribution with parameter p.

We move on to the case $\alpha > 0$. We again let

$$R(x,u) = \sum_{n,k\geq 1} \frac{r_{n,k}}{n!} x^n u^k \quad \text{and} \quad B(x) = \sum_{n\geq 1} \frac{b_n}{n!} x^n.$$

The functions $B(x)$ and $R(x,1)$ are simply the generating functions of preferential attachment trees, which is already known (see e.g., [10, p. 252]) to be

$$B(x) = R(x,1) = 1 - (1-(1+1/\alpha)x)^{\frac{\alpha}{1+\alpha}}.$$

Unlike the case when $\alpha = 0$, we were unable to derive a closed form for $R(x,u)$. But to apply the method of moments, we only need the kth partial derivatives of $R(x,u)$ with respect to u. Define

$$R_k(x) := \left.\frac{\partial^k}{\partial u^k} R(x,u)\right|_{u=1},$$

and

$$R_0(x) := R(x,1) = 1 - (1-(1+1/\alpha)x)^{\frac{\alpha}{1+\alpha}}.$$

Throughout the remainder of this section, we will make use of the partial Bell polynomials, which are defined to be

$$B_{k,j}(x_1, \dots, x_{k-j+1}) = \sum_{\substack{m_1+\cdots+(k-j+1)m_{k-j+1}=k \\ m_1+\cdots+m_{k-j+1}=j}} k! \prod_{i=1}^{k-j+1} \frac{x_i^{m_i}}{m_i! i!^{m_i}}.$$

Lemma 4.2. *Let $\alpha > 0$. Then $R_k(x)$ is analytic on the cut plane*

$$\mathbb{C} \setminus [1/(1+1/\alpha), \infty),$$

and

$$R_k(x) = C_k(1-(1+1/\alpha)x)^{\frac{-kp-\alpha(k-1)}{1+\alpha}} + O\left((1-(1+1/\alpha)x)^{\frac{-kp-\alpha(k-1)}{1+\alpha}+\varepsilon}\right),$$

for some $\varepsilon > 0$, where C_k satisfies the recursion $C_1 = \alpha/(p+\alpha)$ and

$$(k-1)(p/\alpha+1)C_k = \sum_{j=2}^{k} \frac{p^j \Gamma(j+1/\alpha)}{\Gamma(1/\alpha)} B_{k,j}(C_1, \dots, C_{k-j+1}).$$

Proof. Using the recursion in (18), where $\phi(d) = \Gamma(d+1/\alpha)/\Gamma(1/\alpha)$ (recall (16)), we get the following partial differential equation:

$$\begin{aligned}
\frac{\partial}{\partial x}R(x,u) &= \sum_{n,k} n\frac{r_{n,k}}{n!}x^{n-1}u^k \\
&= u\sum_{n,k}\sum_{\delta=0}^{n-1}\sum_{s=0}^{\delta}\binom{\delta}{s}\frac{\phi(\delta)}{\delta!}\sum_{n_1,\dots,n_\delta}\sum_{k_1,\dots,k_s}\prod_{i=1}^{s}\frac{pr_{n_i,k_i}x^{n_i}u^{k_i}}{n_i!}\prod_{j=s+1}^{\delta}\frac{(1-p)b_{n_j}x^{n_j}}{n_j!} \\
&= u\sum_{\delta=0}^{\infty}\frac{\Gamma(\delta+1/\alpha)}{\Gamma(1/\alpha)\delta!}\sum_{s=0}^{\delta}\binom{\delta}{s}(pR(x,u))^s((1-p)B(x))^{\delta-s} \\
&= u\sum_{\delta=0}^{\infty}\frac{\Gamma(\delta+1/\alpha)}{\Gamma(1/\alpha)\delta!}(pR(x,u)+(1-p)B(x))^{\delta} \\
&= u(1-(pR(x,u)+(1-p)B(x)))^{-1/\alpha} \\
&= u\left(1-\left(pR(x,u)+(1-p)\left(1-(1-(1+1/\alpha)x)^{\alpha/(1+\alpha)}\right)\right)\right)^{-1/\alpha}.
\end{aligned}$$

We proceed by strong induction. Using the above differential equation, we see that

$$R_1'(x) = \left.\frac{\partial^2}{\partial u\partial x}R(x,u)\right|_{u=1} = \frac{p}{\alpha-(1+\alpha)x}R_1(x)+(1-(1+1/\alpha)x)^{\frac{-1}{1+\alpha}}.$$

Solving this differential equation with the initial condition $R_1(0)=0$ yields

$$R_1(x) = \frac{\alpha}{p+\alpha}\left((1-(1+1/\alpha)x)^{\frac{-p}{1+\alpha}}-(1-(1+1/\alpha)x)^{\frac{\alpha}{1+\alpha}}\right),$$

which is analytic on the desired cut plane.

For the inductive step, using the product rule at higher orders of partial differentiation produces

$$\begin{aligned}
R_k'(x) &= \left.\frac{\partial^{k+1}}{\partial u^k\partial x}R(x,u)\right|_{u=1} \\
&= \left.\frac{\partial^k}{\partial u^k}u\left(1-\left(pR(x,u)+(1-p)\left(1-(1-(1+1/\alpha)x)^{\alpha/(1+\alpha)}\right)\right)\right)^{-1/\alpha}\right|_{u=1} \\
&= \left.\left(u\frac{\partial^k}{\partial u^k}f(R(x,u))+k\frac{\partial^{k-1}}{\partial u^{k-1}}f(R(x,u))\right)\right|_{u=1},
\end{aligned}$$

where $f(y)=(1-(py+(1-p)(1-(1-(1+1/\alpha)x)^{\frac{\alpha}{1+\alpha}})))^{-1/\alpha}$. Define

$$f^{(m)}(y) := \frac{d^m}{dy^m}f(y).$$

Then

$$f^{(m)}(R_0(x)) = \frac{\Gamma(m+1/\alpha)}{\Gamma(1/\alpha)}p^m(1-(1+1/\alpha)x)^{\frac{-1-\alpha m}{1+\alpha}}, \tag{19}$$

and by using Faá di Bruno's formula for higher order derivatives (see [9, p. 139, Theorem C]), we see that

$$\begin{aligned}\left.\frac{\partial^k}{\partial u^k} f(R(x,u))\right|_{u=1} &= \sum_{j=1}^{k} f^{(j)}(R_0(x)) B_{k,j}\left(R_1(x), \dots, R_{k-j+1}(x)\right) \\ &= \frac{p}{\alpha}(1-(1+1/\alpha)x)^{-1} R_k(x) + g_k(x), \end{aligned} \tag{20}$$

where

$$g_k(x) = \sum_{j=2}^{k} f^{(j)}(R_0(x)) B_{k,j}\left(R_1(x), \dots, R_{k-j+1}(x)\right).$$

Since analyticity is preserved under arithmetic operations as well as integration, the analyticity of $R_k(x)$ on the desired cut plane follows by the analyticity in the induction hypothesis. Using the forms of $R_j(x)$ in the induction hypothesis and (19), we find that for some $\epsilon > 0$,

$$g_k(x) = G_k(1-(1+1/\alpha)x)^{\frac{-kp-k\alpha-1}{1+\alpha}} + O\left((1-(1+1/\alpha)x)^{\frac{-kp-k\alpha-1}{1+\alpha}+\epsilon}\right),$$

where

$$G_k = \sum_{j=2}^{k} \frac{p^j \Gamma(j+1/\alpha)}{\Gamma(1/\alpha)} B_{k,j}(C_1, \dots, C_{k-j+1}).$$

From (20), the induction hypothesis, and the assumption $\alpha > 0$, we can also conclude that

$$\left. k\frac{\partial^{k-1}}{\partial u^{k-1}} f(R(x,u))\right|_{u=1} = O\left((1-(1+1/\alpha)x)^{\frac{-kp-k\alpha-1+p+\alpha}{1+\alpha}}\right) = O\left((1-(1+1/\alpha)x)^{\frac{-kp-k\alpha-1}{1+\alpha}+\epsilon}\right),$$

for some $\epsilon > 0$. By solving the differential equation

$$\begin{aligned} R_k'(x) &= \left.\left(u\frac{\partial^k}{\partial u^k} f(R(x,u)) + k\frac{\partial^{k-1}}{\partial u^{k-1}} f(R(x,u))\right)\right|_{u=1} \\ &= \frac{p}{\alpha}(1-(1+1/\alpha)x)^{-1} R_k(x) + G_k(1-(1+1/\alpha)x)^{\frac{-kp-k\alpha-1}{1+\alpha}} \\ &\quad + O\left((1-(1+1/\alpha)x)^{\frac{-kp-k\alpha-1}{1+\alpha}+\epsilon}\right), \end{aligned}$$

we get that

$$R_k(x) = C_k(1-(1+1/\alpha)x)^{\frac{-kp-\alpha(k-1)}{1+\alpha}} + O\left((1-(1+1/\alpha)x)^{\frac{-kp-\alpha(k-1)}{1+\alpha}+\epsilon}\right),$$

where

$$C_k = \frac{G_k}{(k-1)(p/\alpha+1)},$$

concluding the proof of the lemma. ∎

When proving Theorem 2.9, we need to show that our limiting distribution is uniquely determined by its moments. This is accomplished by verifying that the moment generating function exists for some positive radius. To prove this fact, we will instead show that the exponential generating function for the coefficients C_k from the previous lemma exists for some positive radius around $x = 0$.

Lemma 4.3. *The differential equation*

$$xc'(x) = \left(\frac{\alpha}{p+\alpha}\right)\left(c(x) - 1 + \frac{1}{(1-pc(x))^{1/\alpha}}\right), \tag{21}$$

has a unique analytic solution for some neighbourhood around $x = 0$. Furthermore, this solution can be written as

$$c(x) := \sum_{k=1}^{\infty} \frac{C_k}{k!} x^k.$$

Proof. By using the Taylor expansion we see that

$$\frac{1}{(1-px)^{1/\alpha}} = 1 + \frac{p}{\alpha}x + O(x^2),$$

which is analytic on $|x| < 1/p$. Therefore, we can rewrite the differential equation in (21) as

$$\begin{aligned} xc'(x) &= \left(\frac{\alpha}{p+\alpha}\right)\left(c(x) - 1 + 1 + \frac{p}{\alpha}c(x) + O\left((c(x))^2\right)\right) \\ &= \left(\frac{\alpha}{p+\alpha}\right)\left(\left(\frac{p+\alpha}{\alpha}\right)c(x) + O\left((c(x))^2\right)\right) \\ &= c(x)f(c(x)) \end{aligned}$$

where $f(x) = 1 + O(x)$, so in particular, $f(0) = 1$. Furthermore, $f(x)$ maintains the same radius of convergence as $\frac{1}{(1-px)^{1/\alpha}}$. We solve the separable differential equation above

$$\int \frac{dx}{x} = \int \frac{dc}{c} + \int \frac{(1-f(c))dc}{cf(c)},$$

to get

$$ce^{F(c)} = Kx,$$

for some constant K, where $F(c) = \int \frac{(1-f(c))dc}{cf(c)}$. The analyticity of $F(x)$ in some neighborhood of $x = 0$ is guaranteed by preservation of analyticity through integration and the analyticity of $\frac{1-f(c)}{cf(c)}$, which is itself analytic due to the analyticity of $f(x)$ and the fact that $f(0) = 1$. Thus, using the implicit value theorem, there exists a unique analytic function $c(x)$ in the neighborhood of $x = 0$ such that $c(0) = 0$.

To prove the last part of the lemma, it suffices to show that the power series

$$c(x) := \sum_{k=1}^{\infty} \frac{C_k}{k!} x^k,$$

satisfies the differential equation (21). Recall the recursion for C_k given in Lemma 4.2, which states that

$$\sum_{j=2}^{k} p^j \frac{\Gamma(j+1/\alpha)}{\Gamma(1/\alpha)} B_{k,j}(C_1, \dots, C_{k-j+1}) = G_k = (k-1)(p/\alpha+1)C_k.$$

Recall that

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+1/\alpha)p^k}{\Gamma(1/\alpha)k!} x^k = \frac{1}{(1-px)^{1/\alpha}} - 1.$$

Then by using known results about composition of functions and Bell polynomials (see e.g. [9, p. 137, Theorem A]),

$$\begin{aligned}
\frac{1}{(1-pc(x))^{1/\alpha}} - 1 &= \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k} p^j \frac{\Gamma(j+1/\alpha)}{\Gamma(1/\alpha)} B_{k,j}(C_1, \dots, C_{k-j+1})}{k!} x^k \\
&= \sum_{k=1}^{\infty} \frac{\alpha G_k + pC_k}{\alpha k!} x^k \\
&= \sum_{k=1}^{\infty} \frac{k(p+\alpha)C_k}{\alpha k!} x^k - \sum_{k=1}^{\infty} \frac{C_k}{k!} x^k \\
&= \left(\frac{p+\alpha}{\alpha}\right) xc'(x) - c(x),
\end{aligned}$$

which can be rearranged to give (21). ∎

We now have all the tools necessary to prove Theorem 2.9.

Proof of Theorem 2.9. Using a transfer theorem (see [13, Corollary VI.1]) and Lemma 4.2,

$$[x^n]\frac{\partial^k}{\partial u^k} R(x,u)\bigg|_{u=1} \sim \frac{C_k(1+1/\alpha)^n n^{\frac{kp+\alpha(k-1)}{1+\alpha}-1}}{\Gamma((kp+\alpha(k-1))/(1+\alpha))},$$

and

$$[x^n]R(x,1) \sim -\frac{(1+1/\alpha)^n n^{-\frac{\alpha}{1+\alpha}-1}}{\Gamma(-\alpha/(1+\alpha))}.$$

Let C_n be the root cluster at time n. The factorial moments of $|C_n|$ are extracted from the bivariate generating function (see e.g. [13, Proposition III.2]) to get

$$\mathbb{E}[|C_n|(|C_n|-1)\cdots(|C_n|-k+1)] = \frac{[x^n]R_k(x)}{[x^n]R(x,1)} \sim \frac{C_k n^{\frac{k(p+\alpha)}{1+\alpha}}(1+\alpha)\Gamma(1/(1+\alpha))}{\alpha\Gamma((kp+\alpha(k-1))/(\alpha+1))}.$$

It can be seen (say by induction) that once expanded and scaled by $n^{k(p+\alpha)/(1+\alpha)}$, all but the $\mathbb{E}[|C_n|^k]$ term on the left hand side of the above equation vanish to zero, and thus

$$\mathbb{E}\left[\frac{|C_n|^k}{n^{k(p+\alpha)/(1+\alpha)}}\right] \to \frac{C_k(1+\alpha)\Gamma(1/(1+\alpha))}{\alpha\Gamma((kp+\alpha(k-1))/(\alpha+1))} = M_k.$$

For all k large enough, $M_k < C_k$, and so $m(x) = 1 + \sum_{k=1}^{\infty} \frac{M_k}{k!} x^k$ has greater or equal radius of convergence as $c(x) = \sum_{k=1}^{\infty} \frac{C_k}{k!} x^k$, which is guaranteed to be nonzero by Lemma 4.3. Let $\mathcal{C}$ be the distribution uniquely determined by its moments M_k. Then by using the method of moments, we have shown that

$$\frac{|\mathcal{C}_n|}{n^{(p+\alpha)/(1+\alpha)}} \xrightarrow{d} \mathcal{C}.$$

∎

In general, we were unable to derive a closed form for C_k. We were, however, able to derive a closed form when $\alpha = 1$.

Proof of Proposition 2.10. We use Lemma 4.3, and replace α with 1 to get

$$c'(x) = \frac{1}{x(p+1)} \left(\frac{1}{1 - pc(x)} + c(x) - 1 \right) = \frac{c(x)(1 + p - pc(x))}{x(p+1)(1 - pc(x))},$$

which is rewritten as

$$\int \frac{dx}{x} = \int \frac{(p+1)(1-pc)}{c(1+p-pc)} \, dc = \int \left(\frac{1}{c} - \frac{p^2}{1+p-pc} \right) \, dc.$$

So

$$\ln x = \ln c + p \ln(1 + p - pc) + K.$$

Since we know that $c(x) = \frac{x}{p+1} + O(x^2)$, the constant K is $(1-p)\ln(p+1)$. So

$$\ln c(x) = \ln x - p \ln(1 + p - pc(x)) + (p-1)\ln(p+1),$$

or

$$c(x) = \frac{x}{p+1} \left(1 - \frac{p}{p+1} c(x) \right)^{-p}.$$

Applying the Lagrange inversion formula (see e.g. [13, Theorem A.2]) to this functional equation yields

$$[x^k]c(x) = \frac{1}{k}[t^{k-1}] \frac{1}{(p+1)^k} \left(1 - \frac{pt}{p+1} \right)^{-kp} = \frac{1}{k(p+1)^k} \left(\frac{p}{p+1} \right)^{k-1} \binom{kp+k-2}{k-1}.$$

So finally

$$C_k = k![x^k]c(x) = \frac{(k-1)!p^{k-1}}{(p+1)^{2k-1}} \binom{kp+k-2}{k-1} = \frac{p^{k-1}\Gamma(kp+k-1)}{(p+1)^{2k-1}\Gamma(kp)}.$$

Proposition 2.10 now follows from the above derivation and Theorem 2.9. ∎

We turn our attention to the case $\alpha = -1/d$ for some integer $d \geq 2$, and $\alpha > -p$. In this case the functions $B(x)$ and $R(x, 1)$ are equal to the generating function for increasing d-ary trees, which is known (see e.g., [10, Lemma 6.5]) to be

$$B(x) = R(x, 1) = (1 - (d-1)x)^{-\frac{1}{d-1}} - 1.$$

Once more, we were unable to derive a closed form for $R(x, u)$ in this case. Recall the notation

$$R_k(x) := \frac{\partial^k}{\partial u^k} R(x, u)\bigg|_{u=1},$$

and

$$R_0(x) := R(x, 1) = (1 - (d-1)x)^{-\frac{1}{d-1}} - 1.$$

Lemma 4.4. *Let $d \geq 2$ be a positive integer and let $p > 1/d$. Then $R_k(x)$ is analytic on the cut plane*

$$\mathbb{C} \setminus [1/(d-1), \infty),$$

and

$$R_k(x) = D_k(1 - (d-1)x)^{\frac{-kpd+k-1}{d-1}} + O\left((1 - (d-1)x)^{\frac{-kpd+k-1}{d-1}+\varepsilon}\right),$$

for some $\varepsilon > 0$, where D_k satisfies the recursion $D_1 = 1/(pd-1)$ and

$$(k-1)(pd-1)D_k = \sum_{j=2}^{\min\{k,d\}} \frac{p^j d!}{(d-j)!} B_{k,j}(D_1, \dots, D_{k-j+1}).$$

Since the proof of Lemma 4.4 follows much the same way as the proof of Lemma 4.2, the argument is relegated to Appendix B. Much like the case above for $\alpha > 0$, we will prove the existence of the moment generating function of our limiting distribution in a neighborhood of 0, and this is done by studying the exponential generating function of D_k.

Lemma 4.5. *The differential equation*

$$xt'(x) = \frac{(1 + pt(x))^d - t(x) - 1}{pd - 1}, \tag{22}$$

has a unique analytic solution for some neighborhood around $x = 0$. Furthermore, this solution can be written as

$$t(x) := \sum_{k=1}^{\infty} \frac{D_k}{k!} x^k.$$

Proof. By using the Binomial Theorem, we rewrite the differential equation as

$$\begin{aligned} xt'(x) &= \frac{1 + pd\,t(x) + \sum_{k=2}^{d} \binom{d}{k} (pt(x))^k - t(x) - 1}{pd - 1} \\ &= t(x) + \left(\frac{1}{pd-1}\right) \sum_{k=2}^{d} \binom{d}{k} (pt(x))^k \\ &= t(x) g(t(x)), \end{aligned}$$

where

$$g(x) = 1 + \left(\frac{1}{pd-1}\right) \sum_{k=2}^{d} \binom{d}{k} p^k x^{k-1},$$

which is simply a polynomial (and so an entire function), and $g(0) = 1$. The remainder of the existence part of the proof now follows much the same as that of Lemma 4.3.

To prove the last part of the theorem, recall the recursion of D_k given in Lemma 4.4. Then

$$\sum_{j=2}^{\min\{k,d\}} p^j \frac{d!}{(d-j)!} B_{k,j}(D_1, \dots, D_{k-j+1}) = (k-1)(pd-1)D_k.$$

By using known results about composition of functions and Bell polynomials,

$$\begin{aligned}
(1+pt(x))^d - 1 &= \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{\min\{k,d\}} p^j \frac{d!}{(d-j)!} B_{k,j}(D_1, \dots, D_{k-j+1})}{k!} x^k \\
&= \sum_{k=1}^{\infty} \frac{(k-1)(pd-1)D_k + pdD_k}{k!} x^k \\
&= \sum_{k=1}^{\infty} \frac{k(pd-1)D_k}{k!} x^k + \sum_{k=1}^{\infty} \frac{D_k}{k!} x^k \\
&= (pd-1)xt'(x) + t(x),
\end{aligned}$$

which can be rearranged to give (22). ■

The proof of Theorem 2.15 now follows in much the same way as the proof of Theorem 2.9.

Proof of Theorem 2.15. Using a transfer theorem (see again [13, Corollary VI.1]) and Lemma 4.4,

$$[x^n] \frac{\partial^k}{\partial u^k} R(x,u)\bigg|_{u=1} \sim \frac{D_k (d-1)^n n^{\frac{kpd-k+1}{d-1}-1}}{\Gamma((kpd-k+1)/(d-1))}$$

and

$$[x^n] R(x,1) \sim \frac{(d-1)^n n^{\frac{1}{d-1}-1}}{\Gamma(1/(d-1))}.$$

Let C_n be the root cluster at time n. The factorial moments of $|C_n|$ are extracted from the bivariate generating function (see e.g. [13, Proposition III.2]), and once scaled by $n^{k(pd-1)/(d-1)}$, we get

$$\mathbb{E}\left[\frac{|C_n|^k}{n^{k(pd-1)/(d-1)}}\right] \to \frac{D_k \Gamma(1/(d-1))}{\Gamma((kpd-k+1)/(d-1))} = M_k.$$

For all k large enough, $M_k < D_k$, and so $m(x) = 1 + \sum_{k=1}^{\infty} \frac{M_k}{k!} x^k$ has a greater or equal radius of convergence as $t(x) = \sum_{k=1}^{\infty} \frac{D_k}{k!} x^k$, which is guaranteed to be nonzero by Lemma 4.5. Let C be the distribution uniquely determined by its moments M_k. Then by using the method of moments, we have shown that

$$\frac{|C_n|}{n^{(pd-1)/(d-1)}} \xrightarrow{d} C.$$

■

We were unable to find a closed form for D_k in general. However, a closed form can be found in the case of binary search trees, when $d = 2$.

Proof of Proposition 2.16. We use Lemma 4.5 and replace d with 2 to get

$$t'(x) = \frac{t(x)(p^2t(x) + 2p - 1)}{(2p - 1)x},$$

which is rewritten as

$$\int \frac{dx}{x} = \int \frac{2p - 1}{t(x)(p^2t(x) + 2p - 1)} dt(x) = \int \left(\frac{1}{t(x)} - \frac{p^2}{p^2t(x) + 2p - 1} \right) dt(x).$$

So

$$\ln x = \ln t(x) - \ln(p^2t(x) + 2p - 1) + K.$$

Since $t(x) = \frac{x}{2p-1} + O(x^2)$, the constant K is $2\ln(2p - 1)$, so

$$t(x) = x \left(\frac{1}{2p - 1} + \frac{p^2t(x)}{(2p - 1)^2} \right).$$

The Lagrange inversion formula yields

$$[x^k]t(x) = \frac{1}{k}[y^{k-1}] \frac{1}{(2p - 1)^k} \left(1 + \frac{p^2y}{2p - 1} \right)^k = \binom{k}{k-1} \frac{p^{2(k-1)}}{k(2p - 1)^{2k-1}}.$$

So

$$D_k = k![x^k]t(x) = \frac{k!p^{2(k-1)}}{(2p - 1)^{2k-1}}.$$

Proposition 2.16 now follows from the above derivation and Theorem 2.15. ■

We now look at the cases when the root cluster is finite. Our strategy in these cases is to look at bond percolation on the complete infinite d-ary tree T_d. The root cluster $\mathcal{K}_d$ after performing bond percolation on T_d is distributed as a Galton–Watson tree with binomial $\mathrm{Bin}(d, p)$ offspring distribution. The size (total progeny) of such (finite) trees is known to follow

$$\mathbb{P}(|\mathcal{K}_d| = k) = \frac{1}{k}\mathbb{P}(X_1 + \cdots + X_k = k - 1),$$

where $X_1, \ldots, X_k$ are independent binomial random variables $X_i \sim \mathrm{Bin}(d, p)$ (this result was proved by Otter [30]; a more general result was proved by Dwass [11]. See also [16, Exercises 2.2–2.4]). Thus

$$\mathbb{P}(|\mathcal{K}_d| = k) = \frac{1}{k} \binom{kd}{k-1} p^{k-1}(1 - p)^{kd-k+1}. \tag{23}$$

If we now let $\mathcal{T}_n$ be the rooted subtree of T_d corresponding to a random increasing d-ary tree at time n, then $C_n \sim \mathcal{T}_n \cap \mathcal{K}_d$ is distributed as the root cluster of $\mathcal{T}_n$ with a random broadcasting induced coloring σ_n, where the intersection is the subtree of both $\mathcal{T}_n$ and $\mathcal{K}_d$. For example in Figure 5, we see a tree $\mathcal{T}_9$, with thick edges in the figure, grown on a complete infinite 3-ary tree T_3. Bond percolation has been performed on T_3 (dashed edges represent edges that were removed), and the root

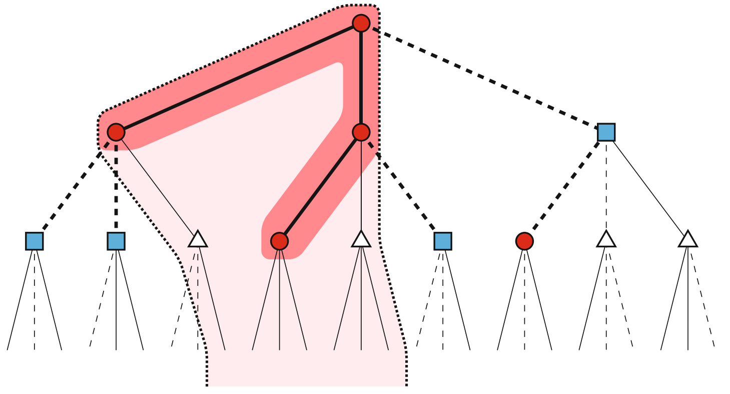

FIGURE 5 A random increasing 3-ary tree is grown on a complete infinite 3-ary tree with bond percolation performed. The root cluster $\mathcal{C}_9$ has four vertices at this stage.

cluster $\mathcal{K}_3$ is shown surrounded by dotted lines. The root cluster $\mathcal{C}_9$ of $\mathcal{T}_9$ is the intersection of $\mathcal{K}_3$ and $\mathcal{T}_9$.

Proof of Proposition 2.11. The result is immediate from the fact that $\mathcal{T}_n$ converges to T_d in a local sense. Suppose a vertex v has at most $d-1$ children in $\mathcal{T}_n$. By (1), the probability that v is not selected in the next step is given by

$$1-(d-\deg^+(v))/(dn-n+1) \leq 1-1/(dn-n+1).$$

Since the product

$$\prod_{k=n}^{\infty}\left(1-\frac{1}{dk-k+1}\right),$$

is 0, the probability for v never to be selected after the nth step is 0. Repeating this argument d times, we see that $\deg^+(v) \xrightarrow{a.s.} d$, and so all of the potential children of v will almost surely appear in $\mathcal{T}_n$ as n grows. Thus any vertex of T_d will almost surely appear in $\mathcal{T}_n$ eventually, including all of $\mathcal{K}_d$ if the latter is finite. If $\mathcal{K}_d$ is infinite, then $\mathcal{C}_n$ will continue to grow as new vertices of $\mathcal{K}_d$ appear in $\mathcal{T}_n$. Therefore, $|\mathcal{C}_n| \xrightarrow{a.s.} |\mathcal{K}_d|$. ∎

Proof of Corollary 2.12. This follows immediately from Proposition 2.11 and (23). ∎

When $p < 1/d$, the distribution described by (23) has finite moments. Since $\{|\mathcal{C}_n|\}_{n=1}^{\infty}$ consists of increasing positive random variables bounded by $\mathcal{K}_d$, their moments are uniformly bounded by the moments of $\mathcal{K}_d$. Thus, along with the almost sure convergence of Proposition 2.11, convergence in all moments holds as well (see [15, ch. 5, Theorem 5.2]).

However, when $p = 1/d$, the distribution described by (23) does not even have finite expectation. We can, however, derive asymptotic results for the moments of $|\mathcal{C}_n|$.

We start by once more approximating the functions $R_k(x)$.

Lemma 4.6. *Let $d \geq 2$ be a positive integer and let $p = 1/d$. Then $R_k(x)$ is analytic on the cut plane*

$$\mathbb{C} \setminus [1/(d-1), \infty),$$

and

$$R_k(x) = -E_k(1-(d-1)x)^{\frac{-1}{d-1}}\ln^{2k-1}(1-(d-1)x) + O\left((1-(d-1)x)^{\frac{-1}{d-1}}\ln^{2k-2}(1-(d-1)x)\right).$$

where E_k satisfies the recursion $E_1 = 1/(d-1)$ and

$$(2k-1)E_k = \frac{1}{2d}\sum_{j=1}^{k-1}\binom{k}{j}E_jE_{k-j}.$$

Since the proof of Lemma 4.6 also follows much the same way as the proof of Lemma 4.2, the argument is relegated to Appendix B.

Proof of Proposition 2.14. From the approximations of the functions $R_k(x)$ in the previous proofs, we conclude by a transfer theorem [13, Corollary VI.1] (or also [20, Théorème A]) that

$$[x^n]\frac{\partial^k}{\partial u^k}R(x,u)\bigg|_{u=1} \sim \frac{E_k(d-1)^n n^{\frac{1}{d-1}-1}}{\Gamma(1/(d-1))}\ln^{2k-1}n,$$

and

$$[x^n]R(x,1) \sim \frac{(d-1)^n n^{\frac{1}{d-1}-1}}{\Gamma(1/(d-1))}.$$

Therefore, we see that

$$\mathbb{E}[|C_n|^k] = \frac{[x^n]R_k(x)}{[x^n]R(x,1)} \sim E_k\ln^{2k-1}n.$$

∎

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APPENDIX A: COVARIANCE MATRICES

A.1 | Number of clusters of each color

For $p < (3-\alpha)/4$:

$$\Sigma_I^c = \frac{1}{4(\alpha-3+4p)}\begin{pmatrix} \sigma_{1,1}^c & \sigma_{1,2}^c & \sigma_{1,3}^c & \sigma_{1,4}^c \\ \sigma_{2,1}^c & \sigma_{2,2}^c & \sigma_{2,3}^c & \sigma_{2,4}^c \\ \sigma_{3,1}^c & \sigma_{3,2}^c & \sigma_{3,3}^c & \sigma_{3,4}^c \\ \sigma_{4,1}^c & \sigma_{4,2}^c & \sigma_{4,3}^c & \sigma_{4,4}^c \end{pmatrix}$$

where

$$\begin{aligned}
\sigma_{1,1}^c &= \sigma_{2,2}^c = -(\alpha+1)\left(\alpha^2+(4p-2)\alpha+1\right) \\
\sigma_{2,1}^c &= \sigma_{1,2}^c = (\alpha+1)\left(\alpha^2+(4p-2)\alpha+1\right) \\
\sigma_{3,3}^c &= \sigma_{4,4}^c = -(p-1)^2(\alpha+4p+1) \\
\sigma_{3,4}^c &= \sigma_{4,3}^c = -(p-1)\left(4p^2+\alpha p-3p+\alpha+1\right) \\
\sigma_{3,1}^c &= \sigma_{4,2}^c = \sigma_{1,3}^c = \sigma_{2,4}^c = (\alpha+1)(\alpha+2p-1) \\
\sigma_{3,2}^c &= \sigma_{4,1}^c = \sigma_{2,3}^c = \sigma_{1,4}^c = -(\alpha+1)(\alpha+2p-1).
\end{aligned}$$

For $p = (3-\alpha)/4$:

$$\Sigma_{II}^c = \frac{1}{4}\begin{pmatrix} \alpha+1 & -\alpha-1 & \frac{-\alpha-1}{2} & \frac{\alpha+1}{2} \\ -\alpha-1 & \alpha+1 & \frac{\alpha+1}{2} & \frac{-\alpha-1}{2} \\ \frac{-\alpha-1}{2} & \frac{\alpha+1}{2} & \frac{\alpha+1}{4} & \frac{-\alpha-1}{4} \\ \frac{\alpha+1}{2} & \frac{-\alpha-1}{2} & \frac{-\alpha-1}{4} & \frac{\alpha+1}{4} \end{pmatrix}$$

A.2 | Number of leaves of each color

For $p < (3-\alpha)/4$:

$$\Sigma_I^l = \frac{\alpha+1}{4(2+\alpha)^2(3+\alpha)(2p-3)(4p+\alpha-3)}\begin{pmatrix} \sigma_{1,1}^l & \sigma_{1,2}^l & \sigma_{1,3}^l & \sigma_{1,4}^l \\ \sigma_{2,1}^l & \sigma_{2,2}^l & \sigma_{2,3}^l & \sigma_{2,4}^l \\ \sigma_{3,1}^l & \sigma_{3,2}^l & \sigma_{3,3}^l & \sigma_{3,4}^l \\ \sigma_{4,1}^l & \sigma_{4,2}^l & \sigma_{4,3}^l & \sigma_{4,4}^l \end{pmatrix}$$

where

$$\begin{aligned}
\sigma_{1,1}^c = \sigma_{2,2}^c &= \left(8p^2-6p-1\right)\alpha^3+\left(48p^2-46p+1\right)\alpha^2 \\
&\quad+\left(112p^2-158p+49\right)\alpha+88p^2-158p+71 \\
\sigma_{2,1}^c = \sigma_{1,2}^c &= -\left(\left(8p^2-6p-1\right)\alpha^3+\left(48p^2-50p+7\right)\alpha^2\right. \\
&\quad\left.+\left(96p^2-126p+37\right)\alpha+72p^2-122p+53\right) \\
\sigma_{3,3}^c = \sigma_{4,4}^c &= -(\alpha+1)\left((2p-3)\alpha^4+2\left(4p^2-7\right)\alpha^3+4\left(10p^2-9p-4\right)\alpha^2\right. \\
&\quad\left.+\left(56p^2-60p-4\right)\alpha+8p^2+14p-23\right)
\end{aligned}$$

$$\sigma^c_{3,1} = \sigma^c_{4,2} = \sigma^c_{1,3} = \sigma^c_{2,4} = (\alpha+1)\left((2p-1)\alpha^3 + \left(-8p^2+22p-9\right)\alpha^2 + \left(-32p^2+62p-21\right)\alpha - 40p^2+74p-29\right)$$

$$\sigma^c_{3,2} = \sigma^c_{4,1} = -(\alpha+1)\left((2p-1)\alpha^3 + \left(-8p^2+22p-9\right)\alpha^2 + \left(-32p^2+66p-27\right)\alpha - 24p^2+38p-11\right)$$

$$\sigma^c_{3,4} = \sigma^c_{4,3} = (\alpha+1)^2\left((2p-3)\alpha^3 + \left(8p^2-2p-11\right)\alpha^2 + \left(32p^2-34p-5\right)\alpha + 24p^2-22p-5\right)$$

For $p = (3-\alpha)/4$:

$$\Sigma^l_{II} = \begin{pmatrix} \frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & -\frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & -\frac{(\alpha+1)^2(\alpha-1)}{2(3+\alpha)^2} & \frac{(\alpha+1)^2(\alpha-1)}{2(3+\alpha)^2} \\ -\frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & \frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & \frac{(\alpha+1)^2(\alpha-1)}{2(3+\alpha)^2} & -\frac{(\alpha+1)^2(\alpha-1)}{2(3+\alpha)^2} \\ -\frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & \frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & \frac{(1+\alpha)^3}{(3+\alpha)^2} & -\frac{(1+\alpha)^3}{(3+\alpha)^2} \\ \frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & -\frac{(\alpha-1)^2(\alpha+1)}{4(3+\alpha)^2} & -\frac{(1+\alpha)^3}{(3+\alpha)^2} & \frac{(1+\alpha)^3}{(3+\alpha)^2} \end{pmatrix}.$$

For $p = 1/2$:

$$\Sigma^l_I = \frac{\alpha+1}{4(2+\alpha^2)(3+\alpha)} \begin{pmatrix} 7+6\alpha+\alpha^2 & -5-4\alpha-\alpha^2 & -2(1+\alpha) \\ -5-4\alpha-\alpha^2 & 7+6\alpha+\alpha^2 & -2(1+\alpha) \\ -2(1+\alpha) & -2(1+\alpha) & 4(1+\alpha) \end{pmatrix}.$$

APPENDIX B: PROOFS OF LEMMAS 4.4 AND 4.6

Proof of Lemma 4.4. Using the recursion in (18), where $\phi(\delta) = d!/(d-\delta)!$ (recall (16)), we get the following partial differential equation:

$$\begin{aligned} \frac{\partial}{\partial x}R(x,u) &= \sum_{n,k} n\frac{r_{n,k}}{n!}x^{n-1}u^k \\ &= u\sum_{n,k}\sum_{\delta=0}^{d}\sum_{s=0}^{\delta}\binom{\delta}{s}\frac{\phi(\delta)}{\delta!}\sum_{n_1,\dots,n_\delta}\sum_{k_1,\dots,k_s}\prod_{i=1}^{s}\frac{pr_{n_i,k_i}x^{n_i}u^{k_i}}{n_i!}\prod_{j=s+1}^{\delta}\frac{(1-p)b_{n_j}x^{n_j}}{n_j!} \\ &= u\sum_{\delta=0}^{d}\binom{d}{\delta}\sum_{s=0}^{\delta}\binom{\delta}{s}(pR(x,u))^s((1-p)B(x))^{\delta-s} \\ &= u\sum_{\delta=0}^{d}\binom{d}{\delta}(pR(x,u)+(1-p)B(x))^{\delta} \\ &= u(1+pR(x,u)+(1-p)B(x))^d \\ &= u\left(1+pR(x,u)+(1-p)\left((1-(d-1)x)^{-\frac{1}{d-1}}-1\right)\right)^d. \end{aligned} \tag{B1}$$

We proceed by strong induction. Using the differential equation above, we see that

$$R_1'(x) = \left.\frac{\partial^2}{\partial u\partial x}R(x,u)\right|_{u=1} = \frac{pd}{(1-(d-1)x)}R_1(x) + (1-(d-1)x)^{\frac{-d}{d-1}}. \tag{B2}$$

Solving this differential equation with the initial condition $R_1(0) = 0$ yields

$$R_1(x) = \frac{1}{pd-1}\left((1-(d-1)x)^{\frac{-pd}{d-1}} - (1-(d-1)x)^{-\frac{1}{d-1}}\right),$$

which is analytic on the desired cut plane.

For the inductive step, using the product rule at higher orders of partial differentiation produces

$$\begin{aligned} R_k'(x) &= \left.\frac{\partial^{k+1}}{\partial u^k \partial x} R(x,u)\right|_{u=1} \\ &= \left.\frac{\partial^k}{\partial u^k} u\left(1 + pR(x,u) + (1-p)\left((1-(d-1)x)^{-\frac{1}{d-1}} - 1\right)\right)^d\right|_{u=1} \\ &= \left.\left(u\frac{\partial^k}{\partial u^k} f(R(x,u)) + k\frac{\partial^{k-1}}{\partial u^{k-1}} f(R(x,u))\right)\right|_{u=1}, \end{aligned} \tag{B3}$$

where $f(y) = \left(1 + py + (1-p)\left((1-(d-1)x)^{-\frac{1}{d-1}} - 1\right)\right)^d$. Define

$$f^{(m)}(y) := \frac{d^m}{dy^m} f(y).$$

Then

$$f^{(m)}(y) = \frac{d!}{(d-m)!} p^m \left(1 + py + (1-p)\left((1-(d-1)x)^{-\frac{1}{d-1}} - 1\right)\right)^{d-m},$$

for $0 \leq m \leq d$, and $f^{(m)} = 0$ for $m > d$. In particular

$$f^{(m)}(R_0(x)) = \frac{d!}{(d-m)!} p^m (1-(d-1)x)^{-\frac{d-m}{d-1}}, \tag{B4}$$

for $0 \leq m \leq d$. By using Faá di Bruno's formula for higher-order derivatives, we see that

$$\begin{aligned} \left.\frac{\partial^k}{\partial u^k} f(R(x,u))\right|_{u=1} &= \sum_{j=1}^{k} f^{(j)}(R_0(x)) B_{k,j}(R_1(x), \ldots, R_{k-j+1}(x)) \\ &= pd(1-(d-1)x))^{-1} R_k(x) + h_k(x), \end{aligned} \tag{B5}$$

where

$$h_k(x) = \sum_{j=2}^{k} f^{(j)}(R_0(x)) B_{k,j}(R_1(x), \ldots, R_{k-j+1}(x)).$$

Since analyticity is preserved under arithmetic operations as well as integration, the analyticity of $R_k(x)$ on the desired cut plane follows by the induction hypothesis. By using the forms of $R_j(x)$ in the induction hypothesis and (B4), then

$$h_k(x) = H_k(1-(d-1)x)^{\frac{-kpd+k-d}{d-1}} + O\left((1-(d-1)x)^{\frac{-kpd+k-d}{d-1}+\epsilon}\right),$$

where

$$H_k = \sum_{j=2}^{\min\{k,d\}} \frac{p^j d!}{(d-j)!} B_{k,j}(D_1, \dots, D_{k-j+1}).$$

From (B5), the induction hypothesis, and the assumption $p > 1/d$, we can also conclude that,

$$\begin{aligned} k\frac{\partial^{k-1}}{\partial u^{k-1}} f(R(x,u))\Big|_{u=1} &= O\left((1-(d-1)x)^{\frac{-kpd+pd+k-1-d}{d-1}}\right) \\ &= O\left((1-(d-1)x)^{\frac{-kpd+k-d}{d-1}+\varepsilon}\right), \end{aligned}$$

for some $\varepsilon > 0$. By solving the differential equation

$$\begin{aligned} R_k'(x) &= \left(u\frac{\partial^k}{\partial u^k} f(R(x,u)) + k\frac{\partial^{k-1}}{\partial u^{k-1}} f(R(x,u))\right)\Bigg|_{u=1} \\ &= pd(1-(d-1)x))^{-1} R_k(x) + H_k(1-(d-1)x)^{\frac{-kpd+k-d}{d-1}} \\ &\quad + O\left((1-(d-1)x)^{\frac{-kpd+k-d}{d-1}+\varepsilon}\right), \end{aligned}$$

we get that

$$R_k(x) = \frac{H_k}{(k-1)(pd-1)}(1-(d-1)x)^{\frac{-kpd+k-1}{d-1}} + O\left((1-(d-1)x)^{\frac{-kpd+k-1}{d-1}+\varepsilon}\right).$$

Setting

$$D_k = \frac{H_k}{(k-1)(pd-1)},$$

concludes the proof of the lemma. ■

Proof of Lemma 4.6. The derivation in (B1) applies here as well. Solving the differential equation in (B2) with $p = 1/d$ yields

$$R_1(x) = \frac{-1}{d-1}(1-(d-1)x)^{\frac{-1}{d-1}} \ln(1-(d-1)x),$$

which is analytic on the desired cut plane.

For the inductive step, the derivation (B3) holds here as well. We get that

$$f^{(m)}(R_0(x)) = \frac{d!}{d^m(d-m)!}(1-(d-1)x)^{-\frac{d-m}{d-1}}.$$

By following the same steps as the proof of Lemma 4.4, we see that

$$\begin{aligned} \frac{\partial^k}{\partial u^k} f(R(x,u))\Big|_{u=1} &= \sum_{j=1}^{k} f^{(j)}(R_0(x)) B_{k,j}(R_1(x), \dots, R_{k-j+1}(x)) \\ &= (1-(d-1)x)^{-1} R_k(x) + l_k(x), \end{aligned}$$

where

$$l_k(x) = \sum_{j=2}^{k} f^{(j)}(R_0(x))B_{k,j}(R_1(x), \dots, R_{k-j+1}(x)).$$

As before, analyticity is preserved. By using the induction hypothesis and the simplification

$$B_{k,2}(x_1, \dots, x_{k-1}) = \frac{1}{2}\sum_{i=1}^{k-1} \binom{k}{i} x_i x_{k-i},$$

then

$$\begin{aligned} l_k(x) &= L_k(1-(d-1)x)^{\frac{-d}{d-1}} \ln^{2k-2}(1-(d-1)x) \\ &\quad + O((1-(d-1)x)^{\frac{-d}{d-1}} \ln^{2k-3}(1-(d-1)x)), \end{aligned}$$

where

$$L_k = \frac{d-1}{2d}\sum_{j=1}^{k-1} \binom{k}{j} E_j E_{k-j}.$$

We can also conclude that

$$k\frac{\partial^{k-1}}{\partial u^{k-1}} f(R(x,u))\bigg|_{u=1} = O\left((1-(d-1)x)^{\frac{-d}{d-1}} \ln^{2k-3}(1-(d-1)x)\right).$$

Solving the differential equation

$$\begin{aligned} R_k'(x) &= (1-(d-1)x)^{-1}R_k(x) + L_k(1-(d-1)x)^{\frac{-d}{d-1}} \ln^{2k-2}(1-(d-1)x) \\ &\quad + O\left((1-(d-1)x)^{\frac{-d}{d-1}} \ln^{2k-3}(1-(d-1)x)\right), \end{aligned}$$

we get that

$$\begin{aligned} R_k(x) &= \frac{-L_k}{(2k-1)(d-1)}(1-(d-1)x)^{\frac{-1}{d-1}} \ln^{2k-1}(1-(d-1)x) \\ &\quad + O\left((1-(d-1)x)^{\frac{-1}{d-1}} \ln^{2k-2}(1-(d-1)x)\right). \end{aligned}$$

Setting

$$E_k = \frac{L_k}{(2k-1)(d-1)},$$

concludes the proof of the lemma. ∎