UPPSALA DISSERTATIONS IN MATHEMATICS

133

Structure and representations of bimodule categories

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UPPSALA 2023

Dissertation presented at Uppsala University to be publicly examined in Polhemsalen, Ångströmlaboratoriet, Lägerhyddsvägen 1, Uppsala, Monday, 20 November 2023 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English. Faculty examiner: Professor Pedro Vaz (Université catholique de Louvain).

Abstract
Jonsson, H. 2023. Structure and representations of bimodule categories. *Uppsala Dissertations in Mathematics* 133. Uppsala: Department of Mathematics. ISBN 978-91-506-3019-0.

This thesis consists of four research papers in the field of representation theory. Three of the papers are concerned with bicategories of finite-dimensional bimodules over a family of radical square zero Nakayama algebras. In the first, we study the tensor combinatorics of these bimodules. This amounts to an explicit description of the tensor combinatorics in terms of so called left, right, and two-sided cells, which are inspired by Green's relations for semigroups. In the second and third paper we study the problem of classifying simple transitive birepresentations of these bicategories. This results in a complete classification for all but one possible value of an invariant called apex. The fourth paper is concerned with the cell structure within the bicategory of finite-dimensional bimodules over all algebras over a fixed field. We specifically study two-sided relations between the regular bimodules over different algebras, that is, the question of when the regular bimodule can appear as a direct summand in a tensor product of bimodules.

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ISSN 1401-2049
ISBN 978-91-506-3019-0
URN urn:nbn:se:uu:diva-512433 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-512433)

For Viktor and Majken

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I H. Jonsson. Cell structure of bimodules over radical square zero Nakayama algebras. Comm. Alg. **48** (2020) no. 2, 573–588 .

II H. Jonsson. On simple transitive 2-representations of bimodules over the dual numbers. Alg. Rep. Theory. (2022) https://doi.org/10.1007/s10468-022-10167-w

III H. Jonsson, M. Stroiński. Simple transitive 2-representations of bimodules over radical square zero Nakayama algebras via localization. J. Alg. **613** (2023), 87–114.

IV H. Jonsson. J-equivalence for associative algebras. Submitted. (2023)

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Contents

1. Introduction

The origin and raison d'être of mathematics is describing the natural world, but what makes mathematics so incredibly powerful – and beautiful – is in its tools for generalization, for going beyond the already observed. This thesis has very little to do with describing the natural world. Instead, it deals with theories that may or may not be useful in future applications, within mathematics or other sciences.

The subject of it is abstract algebra, and more precisely representation theory. In abstract algebra we study algebraic structures and structure-preserving functions, known as homomorphisms. This study is done in an abstract, axiomatic way. An algebraic structure is a set of elements with some operations. An everyday example is the integers: 0, 1, -1, 2, -2, and so on. On this set there are the usual operations of addition and multiplication. These operations have various properties. For example, the order of multiplication does not matter for integers: $a \cdot b = b \cdot a$, and $(ab)c = a(bc)$. These properties are called commutativity and associativity, respectively. There is an element 1 which is neutral with respect to the multiplication: $a \cdot 1 = a$. However, we cannot "undo" multiplication, i.e. divide integers, without leaving our chosen set of element: for example $\frac{1}{2}$ is not an integer, so we cannot divide 1 by 2.

The integers is an example of a ring, which we define as a set with two operations satisfying certain properties. A ring homomorphism is then a function between two rings such that all structure is preserved, i.e. $f(1) = 1$, $f(a+b) = f(a) + f(b)$ and so on. Depending on the operations and properties axiomatised, there are many algebraic structures: semigroups, groups, rings, fields, vector spaces, algebras... Such structure might arise in the natural world, for example in a system of particles interacting with each other. They may also arise within mathematics: the continuous functions on the real line form an algebra together with the operations addition and composition. The abstract approach allows us to study *all* structures of a certain kind at the same time.

Linear algebra is devoted to a certain kind of algebraic structure and homomorphisms: vector spaces and linear maps. These structures are not only immensely useful throughout all areas of mathematics and its applications, but also fairly well-understood. The idea of classical representation theory is to use vector spaces and linear maps to represent another structure, for example an algebra. Studying the representations of an algebra can still give a great deal of information about the algebra itself, while it also allows us into the safety of linear algebra. A representation of an algebra is, after changing point of view slightly, also known as a module over the algebra. Central to this thesis are *bimodules over algebras*. These are modules with two simultaneous and compatible module structures, possibly over two different algebras.

One of the main goals when studying the representations of a fixed object – for example, a certain algebra – is to classify the ”smallest” representations, or *indecomposable* modules. As the name suggests, these cannot be divided into several smaller modules. A special class of indecomposable modules are the *simple* modules. These are the building blocks, or atoms, of the modules. The classification is only up to *isomorphism*, meaning that we identify modules which are essentially the same.

Categories take the abstraction one step further. They consist of objects, and morphisms (or arrows) between the objects, together with identity morphisms for each object, and a unitary associative composition of morphisms. Any type of algebraic structure of our choice, together with its structure-preserving homomorphism, form a category. For example, vector spaces (over a fixed field) and linear maps form a category. Moreover, if A is an algebra, all A-modules, together with A-module homomorphisms, form a category. The structure-preserving functions between categories are known as functors. Categories and functors generalize many other concepts, such as algebras and representations. As a language, categories can encode many different types of mathematical objects, similar to how mathematics can encode many different phenomena from other sciences. Another use of category theory is *categorification*, meaning finding categorical equivalents to objects we wish to study. This categorical equivalent will in general be more ”difficult”, but on the other hand have more structure to work with. It is from examples of categorification, such as [4] and [11], that the interest of birepresentations of bicategories stems.

2-categories generalizes categories by allowing two layers of structure: there are objects, 1-morphisms between the objects, and 2-morphisms between the 1-morphisms. Illustrated by diagrams, the typical diagram in (1-)categories respectively 2-categories are as below.

$$\begin{tikzcd} \heartsuit \arrow[r] & \diamondsuit \end{tikzcd} \qquad \begin{tikzcd} \heartsuit \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \diamondsuit \end{tikzcd}$$

Most interesting cases are not strict 2-categories but bicateogries, a slightly looser notion. For the most part of this thesis, the distinction between 2- and bicategories is not important. In particular, when studying their representations, the theories are essentially the same by [14]. The theory of birepresentations is fairly young – the systematic study was initiated by Mazorchuk and Miemietz in [15]. Many of the first examples of bicategories whose birepresentatiosn are classified are so called fiab, satisfying a number of technical conditions inherent in many of the ”easiest” examples.

This thesis consists of four papers. All of them are in some way concerned with bicategories of bimodules, i.e. bicategories whose 1-morphisms are given by bimodules over algebras. Two of them are devoted to the problem of classifying birepresentations of a family of bicategories of bimodules. These bicategories are not fiab – for example, unlike fiab bicategories, they have infinitely many indecomposable 1-morphisms (up to isomorphism). The other two papers investigate the structure of different bicategories of bimodules. More specifically, we study the tensor combinatorics of bimodules. The (balanced) tensor product is a form of ”multiplication” of bimodules, inputting two bimodules and outputting a new one. The questions we ask can be phrased as follows. Starting with a fixed bimodule M, what other bimodules can we get if we tensor M by another bimodule? Since the tensor product is not commutative, we have to ask this question for when we tensor M from the left, right, and both sides. Starting in the other end, we fix a certain bimodule – the regular bimodule, which is the identity with respect to the tensor product. The question is then: when can we tensor two bimodules and get the regular bimodule?

2. Preliminaries

2.1 Algebras and modules

2.1.1 Basics about algebras

Let $\Bbbk$ be a field. An associative unital $\Bbbk$-algebra is a $\Bbbk$-vector space A with a multiplication $A \times A \to A$ which is

- associative: $(ab)c = a(bc)$ for $a, b, c \in A$;
- bilinear: $(a+b)c = ac + bc$, $(\lambda a)b = \lambda(ab)$ and similar in the second component, for $a, b, c \in A$ and $\lambda \in \Bbbk$.

Moreover there should exist a multiplicative identity 1, i.e. an element $1 \in A$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in A$.

If the vector space A is finite-dimensional, then we say that the algebra is finite-dimensional. We will only consider associative, unital, finite-dimensional algebras, and simply refer to them as algebras. The following are examples of algebras.

- The field $\Bbbk$ itself is a $\Bbbk$-algebra, and any field extension $K \supseteq \Bbbk$ is as well.
- The set $M_n(\Bbbk)$ of $n \times n$-matrices with entries from $\Bbbk$ is a $\Bbbk$-algebra. More generally, for any $\Bbbk$-vector space V, the set of linear maps from V to itself, $\mathrm{End}_{\Bbbk}(V)$, is a $\Bbbk$-algebra. The multiplication is given by composition.
- The set $\Bbbk[x]$ of polynomials in one variable and coefficients in $\Bbbk$ is a $\Bbbk$-algebra (of infinite dimension!).
- The *dual numbers* $\Bbbk[x]/(x^2)$, i.e. (linear) polynomials with coefficients in $\Bbbk$, with the rule that $x^2 = 0$, is a $\Bbbk$-algebra.

A *subalgebra* of an algebra A is a linear subspace $B \subseteq A$ such that B is itself an algebra under the restricted operations, and with the same unit as in A. For example, the space $T_n(\Bbbk)$ of upper-triangular $n \times n$-matrices with entries from $\Bbbk$ is a subalgebra of $M_n(\Bbbk)$.

An *ideal* of an algebra A is a linear subspace which is closed under multiplication with A from both the left and right. A subspace closed only under left (right) multiplication by A is called a left (right) ideal. An proper ideal I of an algebra A is called *maximal* if whenever J is an ideal such that $I \subseteq J$, then either $J = I$ or $J = A$. The *radical* of A, $\mathrm{rad}(A)$, is the intersection of all maximal ideals. A is called *local* if it has a unique maximal left ideal.

Let A and B be $\Bbbk$-algebras. A *homomorphism* of $\Bbbk$-algebras is a $\Bbbk$-linear map $\varphi : A \to B$ such that

$$\varphi(1_A) = 1_B \quad \text{and} \quad \varphi(ab) = \varphi(a)\varphi(b)$$

for all $a, b \in A$. An *isomorphism* of $\Bbbk$-algebras is a bijective algebra homomorphism, and two algebras are called *isomorphic* if there is an algebra isomorphism between them. Given a morphism of algebras $\varphi : A \to B$, the kernel of φ is an ideal of A, and the image of φ is a subalgebra of B.

2.1.2 Representations and modules

Let A be a $\Bbbk$-algebra and V a $\Bbbk$-vector space. Then a *representation* of A on V is an algebra homomorphism $\varphi : A \to \mathrm{End}_{\Bbbk}(V)$. Hence each element $a \in A$ is represented by a linear map $\varphi(a) : V \to V$.

A *left A-module* is a $\Bbbk$-vector space M together with a $\Bbbk$-bilinear action

$$\bullet : A \times M \to M$$

such that, for all $m \in M$ and $a, b \in A$,

$$1 \bullet m = m;$$
$$(ab) \bullet m = a \bullet (b \bullet m).$$

If M is a finite-dimensional vector space, then we say that M is a finite-dimensional module. In this thesis we only consider finite-dimensional modules, unless otherwise stated.

The notions of a representation of A and a left A-module are two sides of the same coin. If $\varphi : A \to \mathrm{End}_{\Bbbk}(V)$ is a representation of V, then V is a left A-module with the action $a \bullet v = \varphi(a)(v)$. On the other hand, if M is a left A-module, then the action of each $a \in A$ defines a linear map $a \bullet - : M \to M$. Then $\varphi : A \to \mathrm{End}_{\Bbbk}(M)$ defined by $\varphi(a) = a \bullet -$ is a representation of A.

Example.
- If A is an algebra, then A is also an A-module with the action given by multiplication from the left, i.e. $a \bullet b = ab$ for $a, b \in A$.
- Modules over the field $\Bbbk$ are exactly $\Bbbk$-vector spaces.
- The algebra $M_n(\Bbbk)$ has a natural module $\Bbbk^n$, where the action is given by multiplication.
- The field $\Bbbk$ is a module over the dual numbers, with the action determined by $x \bullet v = 0$ for all $v \in \Bbbk$.

Let A be an algebra. A *homomorphism* of A-modules is a linear map $\varphi : M \to N$ between A-modules which intertwines the A-action, i.e. such that $\varphi(a \bullet m) = a \bullet \varphi(m)$ for all $m \in M$ and $a \in A$. Homomorphisms $M \to M$ are called *endomorphisms*, and the set $\mathrm{End}_A(M)$ of endomorphism of an A-module M forms an algebra – the endomorphism algebra.

A *submodule* of a left A-module M is a linear subspace $N \subseteq M$ which is closed under the action of A. We call a nonzero A-module M *indecomposable* if whenever $M \simeq M_1 \oplus M_2$ for some submodules M_1, M_2, then either $M_1 = 0$

or $M_2 = 0$. A module is indecomposable if and only if its endomorphism algebra is local. Moreover, $M \neq 0$ is called *simple* if it has no proper nontrivial submodules. Any simple module is indecomposable, but the converse is not true in general. A module which is a direct sum of simple modules is called *semisimple*.

Let M be an A-module. Then its *socle*, $\mathrm{soc}(M)$, is the sum of all simple submodules of M. Its *radical*, $\mathrm{rad}(M)$, is $\mathrm{rad}(A)M$, and its top, $\mathrm{top}(M)$, is the quotient $M/\mathrm{rad}(M)$. The top of a module is semisimple.

For an A-module M, let $S_0 = 0$, and for $j \geq 1$, let S_j be the submodule of M such that $S_j/S_{j-1} = \mathrm{soc}(M/S_{j-1})$. The *Loewy length* of M is the minimal j such that $S_j = M$, and the Loewy length of the algebra A, denoted $\ell\ell(A)$, is the Loewy length of the regular A-module A. Further, a *composition series* for M is a sequence of submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M$ such that each subquotient M_j/M_{j-1} is simple. These simple subquotients are called the *composition factors*, and are unique up to permutation.

Theorem (Jordan-Hölder)**.** *Let A be an algebra and M an A-module. Assume that M has two composition series*

$$\begin{gathered} 0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M \\ 0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{k-1} \subseteq N_k = M. \end{gathered}$$

Then $n = k$, and there is some $\sigma \in S_n$ such that $M_j/M_{j-1} \simeq N_{\sigma(j)}/N_{\sigma(j)-1}$ for all j.

When studying representation theory of finite-dimensional algebras, one is in general interested in classifying the indecomposable modules up to isomorphism. According to the following result, knowing the indecomposable modules means that we know *all* modules.

Theorem (Krull-Schmidt)**.** *Let A be a finite-dimensional $\Bbbk$-algebra. Then any nonzero finite-dimensional A-module can be written as a direct sum of indecomposable A-modules, and the summands are unique up to permutation and isomorphism.*

A natural question is now: how many indecomposable A-modules are there? A measure of this is the notions of finite, tame and wild – translating to the classification problem being ”not too difficult”, ”difficult but conceivable”, and ”probably impossible”. More rigorously, an algebra A is of *finite representation type* if there are finitely many indecomposable A-modules. If A is not of finite representation type, then is of *infinite representation type*. In this case, A is of *tame representation type* if for each positive integer d, all but finitely many indecomposable A-modules of dimension d belong to one of finitely many 1-parameter families, and A is of *wild representation type* if for any finite-dimensional algebra B, the representation theory of B can be embedded into that of A.

Theorem (Drozd's trichotomy theorem [5]). *Let $\Bbbk$ be an algebraically closed field. Then any $\Bbbk$-algebra is either of finite, tame, or wild representation type.*

A very important class of modules over any algebra A are the projective modules. An A-module P is *projective* if whenever the is a surjective A-module morphism $f : M \to N$ and an A-module morphism $g : P \to N$, then there is some A-module morphism $h : P \to M$ such that $g = f \circ h$.

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ M \arrow[r, two heads, "f"] & N \end{tikzcd}$$

Direct sums and direct summands of projective modules are again projective. The canonical example of a projective A-module is the regular A-module. Indeed, any indecomposable projective A-module is isomorphic to an indecomposable direct summand of the regular A-module. This means in particular that there are, up to isomorphism, finitely many indecomposable projective A-modules. The indecomposable projective modules also have simple tops. Assume that $P_1, \dots, P_n$ are, up to isomorphism, all indecomposable projective A-modules. Set $S_i = \mathrm{top}(P_i)$. Then $S_1, \dots, S_n$ are all simple A-modules up to isomorphism. Assume that the regular A-module ${}_AA \simeq P_1 \oplus \dots \oplus P_m$, where the P_i are all indecomposable. Then A is called *basic* if the summands $P_1, \dots, P_m$ are pairwise non-isomorphic. An algebra A is called a *Nakayama algebra* if any indecomposable projective A-module has a unique composition series.

2.1.3 Bimodules and tensor algebras

We have defined left A-modules, but may as well define right A-modules using an action $\bullet : M \times A \to M$. Then there is nothing stopping us from having two different actions from left and right on the same vector space. This leads us to the concept of a bimodule.

Let A and B be $\Bbbk$-algebras. An *A-B-bimodule* is a vector space M which is simultaneously a left A-module and right B-module, such that for all $m \in M$, $A \in A$ and $b \in B$, $(a \bullet m) \bullet b = a \bullet (m \bullet b)$.

For example, A itself is an A-A-bimodule with left and right A-action given by multiplication. This bimodule is called the *regular* bimodule.

Let A, B and C be algebras and ${}_AM_B$, ${}_AN_C$ and ${}_CL_B$ bimodules. Then the set of left A-module morphisms $\mathrm{Hom}_{A\text{-}}(M, N)$ is a B-C-bimodule, with actions

$$(b \bullet \varphi)(m) = \varphi(m \bullet b) \quad \text{and} \quad (\varphi \bullet c)(m) = \varphi(m) \bullet c.$$

Similarly, the right B-module morphisms $\mathrm{Hom}_{\text{-}B}(M,L)$ form a C-A-bimodule with

$$(c \bullet \varphi)(m) = c \bullet \varphi(m) \quad \text{and} \quad (\varphi \bullet a)(m) = \varphi(a \bullet m).$$

If A and B are two algebras, then their tensor product $A \otimes_{\Bbbk} B$ is again an algebra, with multiplication defined on simple tensors by

$$(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb').$$

If M is a left A-module, and N is a left B-module, then $M \otimes N$ is a left $A \otimes B$-module via

$$(a \otimes b) \bullet (m \otimes n) = (a \bullet m) \otimes (b \bullet n)$$

We denote by A^{op} the opposite algebra of A, that is the algebra such that $A^{\mathrm{op}} = A$ as vector spaces, but with reversed order of multiplication. If M is a right A-module, it is a left A^{op}-module, and vice versa. Moreover, if M is an A-B-bimodule, it is a left $A \otimes_{\Bbbk} B^{\mathrm{op}}$-module. If A is commutative, then $A \simeq A^{\mathrm{op}}$. Since a left A-module is a vector space, it is a left $\Bbbk$-module. The field is commutative, so we may as well say it is a right $\Bbbk$-module. Therefore a left A-module can be viewed as an A-$\Bbbk$-bimodule. Similarly a right A-module is a $\Bbbk$-A-bimodule.

An A-B-bimodule M is called *projective* if it is projective as a left $A \otimes B^{\mathrm{op}}$-module. There are also weaker notions of projectivity for bimodules: M is *left projective* if it is projective as a left A-module, *right projective* if it is projective as a right B-module, and *left-right projective* if it is both.

For a left A-module M and a right A-module N we define the (balanced) tensor product $N \otimes_A M$ as the quotient of $N \otimes_{\Bbbk} M$ by the vector space spanned by all elements of the form

$$na \otimes m - n \otimes am.$$

If M is an A-B-bimodule, and N is an B-C-bimodule, then $M \otimes_B N$ is an A-C-bimodule. In particular, if M is a left A-module, and N is a right B-module, then $M \otimes_{\Bbbk} N$ is an A-B-bimodule. If additionally M and N are indecomposable, then so is $M \otimes_{\Bbbk} N$. This type of indecomposable A-B-bimodule is called $\Bbbk$-*split*.

If $P_1, \ldots, P_m$ are all indecomposable projective left A-modules, and $Q_1, \ldots, Q_m$ are the indecomposable projective right B-modules, then

$$P_i \otimes_{\Bbbk} Q_j, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

are all indecomposable projective A-B-bimodules. The simple top of $P_i \otimes Q_j$ is $S_i \otimes_{\Bbbk} \tilde{S}_j$, where $S_i = \mathrm{top}(P_i)$ and $\tilde{S}_j = \mathrm{top}(Q_j)$. Note that all indecomposable projective A-B-bimodules are $\Bbbk$-split, as are the simple bimodules.

2.1.4 Quiver algebras

A special type of algebras, which are immensely useful and relatively easy to work with, are quiver algebras. A *quiver* Q is a directed graph (allowing multiple edges and loops). Hence it consists of a set Q_0 of nodes (vertices), a set Q_1 of arrows, and two functions $s,t : Q_1 \to Q_0$ called source and target. An element $\alpha \in Q_1$ is an arrow from $s(\alpha)$ to $t(\alpha)$.

If α and β are arrows such that $t(\alpha) = s(\beta)$, then we can concatenate and form a path $\beta\alpha$. If γ is such that $t(\beta) = s(\gamma)$, then we can concatenate again and form the path $\gamma\beta\alpha$, and so on. The *length* of a path is the number of arrows in it.

Example. In the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

$\gamma\beta\alpha$ is a path of length 3 from 1 to 2, understood as first α, then β, then γ.

Given a quiver Q we can form the *quiver algebra* $\Bbbk Q$. As vector space it is spanned by all paths in Q, including a path ε_i of length zero at each node i. The multiplication of paths is given by concatenation, so for paths $\alpha_m \dots \alpha_1$ and $\beta_n \dots \beta_1$,

$$\beta_n \dots \beta_1 \cdot \alpha_m \dots \alpha_1 = \begin{cases} \beta_n \dots \beta_1 \alpha_m \dots \alpha_1 & \text{if } t(\alpha_m) = s(\beta_1) \\ 0 & \text{else.} \end{cases}$$

The identity is $\sum_{i \in Q_0} \varepsilon_i$.

Denote by R the ideal generated by all arrows of Q. Then an ideal $I \subseteq \Bbbk Q$ is *admissible* if $R^m \subseteq I \subseteq R^2$ for some $m \geq 2$.

A representation of a quiver algebra $\Bbbk Q$ can be constructed as follows: to each node i, assign a finite-dimensional vector space V_i, and to each arrow $\alpha : i \to j$, assign a linear map $\varphi_\alpha : V_i \to V_j$. For all nodes i, $\varphi_{\varepsilon_i} = \mathrm{id}_{V_i}$. For a quiver with relations, or an algebra $\Bbbk Q/I$ for some admissible ideal I, the linear maps φ_α must satisfy the relations imposed by I.

Much of the structure of a quiver algebra can be immediately read off from the quiver (with relations). Let $A = \Bbbk Q/I$ be a path algebra of a quiver Q modulo some admissible ideal I. Assume that the nodes of Q are $Q_0 = \{1, \dots, n\}$. Then the projective A-modules are, up to isomorphism, $A\varepsilon_1, \dots, A\varepsilon_n$, where $A\varepsilon_i$ is the submodule of the regular A-module spanned by all paths starting at the node i. The projective $A\varepsilon_i$ and its simple top S_i is referred to as the projective and simple at the node i, respectively. A quiver algebra $\Bbbk Q/I$ is a

Nakayama algebra if and only if Q is one of the following quivers.

$$1 \longrightarrow 2 \longrightarrow \dots \longrightarrow n \qquad \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd} \qquad \begin{tikzcd} & n \arrow[dl] & \\ 1 \arrow[r] & \dots \arrow[r] & n-1 \arrow[ul] \end{tikzcd}$$

The representation theory of any algebra can be recovered using that of a basic algebra – we shall state this more formally soon. The following result therefore reinforces the usefulness of quiver algebras.

Theorem. *The algebra $\Bbbk Q/I$ is basic. If the field $\Bbbk$ is algebraically closed, any basic $\Bbbk$-algebra is isomorphic to $\Bbbk Q/I$ for some quiver Q and some admissible ideal $I \subseteq \Bbbk Q$.*

Forming tensor algebras of quiver algebras is also straightforward. Given two algebras $A = \Bbbk Q/I$ and $B = \Bbbk\tilde{Q}/\tilde{I}$, the tensor algebra $A \otimes B$ is isomorphic to $\Bbbk(Q \otimes \tilde{Q})/(I \boxtimes \tilde{I})$, where the quiver $Q \otimes \tilde{Q}$ is determined by

$$\begin{aligned} (Q \otimes \tilde{Q})_0 &= Q_0 \times \tilde{Q}_0 \\ (Q \otimes \tilde{Q})_1 &= Q_1 \times \tilde{Q}_0 \cup Q_0 \times \tilde{Q}_1. \end{aligned}$$

The ideal $I \boxtimes \tilde{I}$ is generated by $I \times \tilde{Q}_0$, $Q_0 \times \tilde{I}$, and, for all $\alpha : i \to j$ in Q and $\sigma : k \to l$ in $\tilde{Q}$

$$(\varepsilon_j \otimes \sigma) \circ (\alpha \otimes \tilde{\varepsilon}_k) - (\alpha \otimes \tilde{\varepsilon}_l) \circ (\varepsilon_i \otimes \sigma).$$

This means that the tensor algebra inherits the relations from the tensor factors, and that there are additional commutativity relations. For example, set

$$Q : \ 1 \xrightarrow{\alpha} 2, \quad \tilde{Q} : \ 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3,$$

$I = 0$ and $\tilde{I} = (\gamma \circ \beta)$. Then $Q \otimes \tilde{Q}$ is the quiver

$$\begin{tikzcd} 1|1 \arrow[r, "\beta_1"] \arrow[d, "\alpha_1"'] & 1|2 \arrow[r, "\gamma_1"] \arrow[d, "\alpha_2"] & 1|3 \arrow[d, "\alpha_3"] \\ 2|1 \arrow[r, "\beta_2"] & 2|2 \arrow[r, "\gamma_2"] & 2|3 \end{tikzcd}$$

where $\alpha_i = \alpha \otimes \tilde{\varepsilon}_i$, $\beta_j = \varepsilon_j \otimes \beta$ and similarly for γ_j. The ideal $I \boxtimes \tilde{I}$ is generated by

$$\gamma_1 \circ \beta_1, \ \gamma_2 \circ \beta_2, \ \alpha_2 \circ \beta_1 - \beta_2 \circ \alpha_1, \ \alpha_3 \circ \gamma_1 - \gamma_2 \circ \alpha_2.$$

In other words, both horizontal paths of length 2 are zero, as in $\Bbbk\tilde{Q}/\tilde{I}$, and the squares commute. The latter is a general rule for tensor products of quiver algebras.

2.2 Categories and functors

A category $\mathscr{C}$ consists of a class of objects $\{X,Y,\dots\}$, and for each pair of objects X,Y, a class of morphisms $\mathscr{C}(X,Y)$. For each object X there must exist an identity morphism 1_X, and there should be an associative, unitary composition of morphisms (for compositions that makes sense). The following are examples of categories.

- **Set** consisting of sets and functions.
- $\Bbbk$-vec consisting of finite-dimensional $\Bbbk$-vector spaces and linear maps.
- Given algebras A and B, A-mod consisting of finite-dimensional left A-modules and module homomorphisms, and A-mod-B consisting of finite dimensional A-B-bimodules and bimodule homomorphisms.

A category $\mathscr{C}$ is *additive* if all $\mathscr{C}(X,Y)$ form abelian groups and composition is additive. We also require the existence of a zero object and of direct sums. $\mathscr{C}$ is $\Bbbk$-linear if $\mathscr{C}(X,Y)$ is a finite-dimensional vector space for all X,Y, and composition is $\Bbbk$-bilinear. Examples of $\Bbbk$-linear categories are $\Bbbk$-vec and A-mod. Another important example is the category $\mathscr{C}_A$, for some $\Bbbk$-algebra A. It has one object $\heartsuit$, and $\mathscr{C}(\heartsuit,\heartsuit) = A$, with composition of morphisms given by multiplication within A. Indeed, for any object X in any $\Bbbk$-linear category $\mathscr{C}$, the morphism space $\mathscr{C}(X,X)$ is an algebra. So, in a sense, an algebra is a special case of a $\Bbbk$-linear category.

If $\mathscr{C}$ is a category such that for all objects X,Y, the homomorphisms $\mathscr{C}(X,Y)$ form a set (and not a proper class), then $\mathscr{C}$ is called locally small. If also the objects of $\mathscr{C}$ form a set, then $\mathscr{C}$ is called *small*. A $\Bbbk$-linear category $\mathscr{C}$ is *finitary* if it has finitely many objects, and each morphism space is a finite-dimensional vector space.

A very important property of the category A-mod is the Krull-Schmidt theorem, i.e. that any nonzero object can be uniquely written as a direct sum of indecomposable objects. Recall that a module is indecomposable if and only if its endomorphism algebra is local. The following definition therefore captures the property of the theorem: A $\Bbbk$-linear category $\mathscr{C}$ is *Krull-Schmidt* if any nonzero object is isomorphic to a unique direct sum of finitely many objects, such that each summand X and satisfies that the algebra $\mathscr{C}(X,X)$ is local.

A morphism $f : X \to X$ such that $f \circ f = f$ is called *idempotent*. An idempotent $f : X \to X$ is *split* if there is an object Y, and morphisms $g : X \to Y$ and $h : Y \to X$, such that $f = h \circ g$ and $g \circ h = \mathrm{id}_Y$. A category is *idempotent split* if all idempotents are split.

A category of the form A-mod-B, for algebras A and B, is a small, $\Bbbk$-linear, Krull-Schmidt, idempotent split category.

Given a category $\mathscr{C}$, a *subcategory* $\mathscr{D}$ consists of a collection of objects from $\mathscr{C}$, and for every pair of objects X,Y in $\mathscr{D}$, a collection $\mathscr{D}(X,Y)$ of morphisms from $\mathscr{C}(X,Y)$ which is closed under composition, and such that $\mathrm{id}_X \in \mathscr{D}(X,X)$ for any object X in $\mathscr{D}$. Omitting the latter condition yields a subsemicategory. A subcategory is *full* if $\mathscr{D}(X,Y) = \mathscr{C}(X,Y)$ for all X,Y in $\mathscr{D}$.

If $\mathscr{C}$ is $\Bbbk$-linear, an ideal in $\mathscr{C}$ is a subsemicategory closed under composition with 1-morphisms from $\mathscr{C}$.

Given categories $\mathscr{C}$ and $\mathscr{D}$, a *(covariant) functor* $F : \mathscr{C} \to \mathscr{D}$ maps every diagram $X \xrightarrow{f} Y$ in $\mathscr{C}$ to a diagram $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathscr{D}$. Moreover, it should respect composition and identity morphisms. If $F : \mathscr{C} \to \mathscr{D}$ is as above except that it reverses morphism, i.e. maps any diagram $X \xrightarrow{f} Y$ in $\mathscr{C}$ to a diagram $F(Y) \xrightarrow{F(f)} F(X)$ in $\mathscr{D}$, then F is a *contravariant functor*.

A functor between $\Bbbk$-linear categories is called $\Bbbk$-linear if it also respects the $\Bbbk$-linear structure.

Example.

- Given a category $\mathscr{C}$, there is the identity functor $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, which is the identity on objects and morphisms.
- Let A, B and C be algebras and M an A-B-bimodule. Then there are the so called tensor functors

$$\begin{aligned} M \otimes_B - &: B\text{-mod-}C \to A\text{-mod-}C \\ - \otimes_A M &: C\text{-mod-}A \to C\text{-mod-}B. \end{aligned}$$

  Moreover, to M we can associate four Hom-functors, two of which are covariant:

$$\begin{aligned} \mathrm{Hom}_{A\text{-}}(M,-) &: A\text{-mod-}C \to B\text{-mod-}C \\ \mathrm{Hom}_{\text{-}B}(M,-) &: C\text{-mod-}B \to C\text{-mod-}A \end{aligned}$$

  and two of which are contravariant:

$$\begin{aligned} \mathrm{Hom}_{A\text{-}}(-,M) &: A\text{-mod-}C \to C\text{-mod-}B \\ \mathrm{Hom}_{\text{-}B}(-,M) &: C\text{-mod-}B \to A\text{-mod-}C. \end{aligned}$$

Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and F and G functors $\mathscr{C} \to \mathscr{D}$. Then a *natural transformation* $\eta : F \to G$ is a family of morphisms in $\mathscr{D}$ indexed by elements in $\mathscr{C}$, such that for each diagram $X \xrightarrow{f} Y$ we have the following commutative diagram in $\mathscr{D}$.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

If each η_X is an isomorphism in $\mathscr{D}$, then η is called a *natural isomorphism*.

For any categories $\mathscr{C}, \mathscr{D}$, there is the so called *functor category* $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$, whose objects are functors $\mathscr{C} \to \mathscr{D}$, and morphisms are natural transformations. Given a functor $F : \mathscr{C} \to \mathscr{D}$, the collection of identity morphisms $1_{F(X)}$

in $\mathscr{D}$ gives rise to a natural transformation id_F, which is the identity morphism on F in $\mathrm{Fun}(\mathscr{C},\mathscr{D})$.

Two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there are functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] & \mathscr{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that there are natural isomorphisms $G\circ F \simeq \mathrm{id}_{\mathscr{C}}$ and $F\circ G \simeq \mathrm{id}_{\mathscr{D}}$.

If A and B are algebras such that A-mod is equivalent to B-mod, then A and B are *Morita equivalent*. We are now able to state the aforementioned result about the representation theory of basic algebras covering that of any algebra.

Theorem. *Every algebra is Morita equivalent to a basic algebra.*

Given categories and functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] & \mathscr{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

(F,G) is an *adjoint pair* if there are isomorphism $f_{X,Y} : \mathscr{D}\big(F(X),Y\big) \simeq \mathscr{C}\big(X,G(Y)\big)$ for all objects X in $\mathscr{C}$ and Y in $\mathscr{D}$, natural in X and Y. Equivalently, (F,G) is an adjoint pair if there exist natural transformations $\varepsilon : FG \to \mathrm{id}_{\mathscr{D}}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ such that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon_F} F \quad G \xrightarrow{\eta_G} GFG \xrightarrow{G\varepsilon} G$$

are the identity natural transformations on F and G, respectively. Then ε and η are called counit and unit of adjunction, respectively.

For example, given algebras A and B and an A-B-bimodule M, the functors

$$\begin{aligned} M\otimes_B - :& B\text{-mod} \to A\text{-mod} \\ \mathrm{Hom}_{A\text{-}}(M,-) :& A\text{-mod} \to B\text{-mod} \end{aligned}$$

form an adjoint pair $\big(M\otimes_B -, \mathrm{Hom}_{A\text{-}}(M,-)\big)$.

Given categories $\mathscr{C}$ and $\mathscr{M}$, a *representation* of $\mathscr{C}$ on $\mathscr{M}$ is a functor $\mathscr{C} \longrightarrow \mathscr{M}$. If $\mathscr{C}$ is $\Bbbk$-linear, we often consider $\Bbbk$-linear representations of $\mathscr{C}$, i.e. such that the target category $\mathscr{M}$, and the functor $\mathscr{C} \to \mathscr{M}$, are $\Bbbk$-linear. A representation of an algebra A can be identified with a $\Bbbk$-linear functor $\mathscr{C}_A \to \Bbbk$-vec, or a ($\Bbbk$-linear) representation of $\mathscr{C}_A$ on $\Bbbk$-vec.

2.3 2-categories and 2-representations

2.3.1 2-categories and 2-functors

A 2-category $\mathscr{C}$ has two categorical levels. It consists of a class of objects $\{\mathtt{i},\mathtt{j},\ldots\}$, and for each pair of objects $\mathtt{i},\mathtt{j}$, a small morphism category $\mathscr{C}(\mathtt{i},\mathtt{j})$. The objects $F,G,\ldots$ of the categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ are called *1-morphisms*, and the morphisms $\alpha,\beta,\ldots$ are called *2-morphisms*. In other words, 1-morphisms go between objects, and 2-morphisms between 1-morphisms. We illustate it by diagrams as below.

$$\begin{tikzcd} \mathtt{i} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathtt{j} \end{tikzcd}$$

The identity 1-morphism at $\mathtt{i}$ is denoted $\mathbb{1}_{\mathtt{i}}$, and the identity 2-morphism at F by id_F. Composition of 1-morphisms is required to be bifunctorial. There are two types of composition of 2-morphisms: *vertical composition* $\circ_v$, within the categories $\mathscr{C}(\mathtt{i},\mathtt{j})$, and *horizontal composition* $\circ_h$, along 1-morphisms. In the below diagram, α can be vertically composed with β and horizontally with γ and δ.

$$\begin{tikzcd} \mathtt{i} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & \mathtt{j} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta"] & \mathtt{k} \end{tikzcd}$$

Bifunctoriality of the composition of 1-morphisms implies the following interchange law:

$$(\delta\circ_v\gamma)\circ_h(\beta\circ_v\alpha)=(\delta\circ_h\beta)\circ_v(\gamma\circ_h\alpha).$$

The following are examples of 2-categories.

- **Cat**, whose objects are small categories, 1-morphisms are functors, and 2-morphisms are natural transformations of functors.
- $\mathfrak{A}^f_{\mathbb{k}}$, the 2-category of finitary $\mathbb{k}$-linear 2-categories, whose objects are categories equivalent to A-proj where A is a finite-dimensional $\mathbb{k}$-algebra, 1-morphisms are additive $\mathbb{k}$-linear functors, and 2-morphisms are natural transformations of functors.
- $\mathfrak{R}_{\mathbb{k}}$, the 2-category of finitary $\mathbb{k}$-linear abelian 2-categories, whose objects are categories equivalent to $A-$proj where A is a finite-dimensional $\mathbb{k}$-algebra, 1-morphisms are right exact additive $\mathbb{k}$-linear functors, and 2-morphisms are natural transformations of functors.

A 2-category $\mathscr{C}$ is $\mathbb{k}$-linear if each morphism category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is a $\mathbb{k}$-linear category, and all compositions are $\mathbb{k}$-bilinear.

An *ideal* $\mathscr{I}$ in a 2-category $\mathscr{C}$ is a family of ideals $\mathscr{I}(\mathtt{i},\mathtt{j})$ in each $\mathscr{C}(\mathtt{i},\mathtt{j})$ which are closed (horizontal) composition with anything in $\mathscr{C}$.

Given 2-categories $\mathscr{C}$ and $\mathscr{D}$, a *2-functor* $\Phi : \mathscr{C} \to \mathscr{D}$ sends any diagram

$$\begin{tikzcd} \mathtt{i} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathtt{j} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

to a diagram

$$\begin{tikzcd} \Phi(\mathtt{i}) \arrow[rr, bend left, "\Phi(F)"{name=U}] \arrow[rr, bend right, "\Phi(G)"'{name=D}] & & \Phi(\mathtt{j}) \arrow[from=U, to=D, Rightarrow, "\Phi(\alpha)"] \end{tikzcd}$$

in such a way that is respects all identities and compositions.

Let $\Phi, \Psi : \mathscr{C} \to \mathscr{D}$ be 2-functors. A *2-natural transformation* $\eta : \Phi \to \Psi$ consists of

- for each object $\mathtt{i}$ of $\mathscr{C}$, a 1-morphism $\eta_{\mathtt{i}} : \Phi(\mathtt{i}) \to \Psi(\mathtt{i})$;
- for each 1-morphism F of $\mathscr{C}$, a 2-morphism $\eta_F : \Phi(F) \to \Psi(F)$,

such that the following diagrams commute for all $\mathtt{i}, \mathtt{j}, F, G, \alpha$.

$$\begin{tikzcd} \Phi(\mathtt{i}) \arrow[r, "\Phi(F)"] \arrow[d, "\eta_{\mathtt{i}}"'] & \Phi(\mathtt{j}) \arrow[d, "\eta_{\mathtt{j}}"] \\ \Psi(\mathtt{i}) \arrow[r, "\Psi(F)"] & \Psi(\mathtt{j}) \end{tikzcd} \qquad \begin{tikzcd} \Phi(F) \arrow[r, "\Phi(\alpha)"] \arrow[d, "\eta_F"'] & \Phi(G) \arrow[d, "\eta_G"] \\ \Psi(F) \arrow[r, "\Psi(\alpha)"] & \Psi(G) \end{tikzcd}$$

2.3.2 2-representations

A *2-representation* of a 2-category $\mathscr{C}$ is a 2-functor $\mathbf{M} : \mathscr{C} \to \mathbf{Cat}$. Hence, objects are represented by categories, 1-morphisms by functors, and 2-morphisms by natural transformations. For example, the axioms for a 2-category $\mathscr{C}$ yields for each object $\mathtt{i}$ the *principal 2-representation* $\mathscr{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$.

Assume that $\mathscr{C}$ is additive, $\Bbbk$-linear and Krull-Schmidt, and that for each object $\mathtt{i}$ the identity 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable. Then a *finitary 2-representation* of $\mathscr{C}$ is a $\Bbbk$-linear 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$. A finitary 2-representation $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ is called *simple* if it has no non-trivial $\mathscr{C}$-stable ideals. $\mathbf{M}$ is *transitive* if for any objects $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{j})$, there is a 1-morphisms F in $\mathscr{C}$ such that Y is a direct summand of $\mathbf{M}(F)(X)$. While simplicity implies transitivity, we still use the terminology *simple transitive 2-representation*.

Two 2-representations $\mathbf{M}, \mathbf{N}$ of a 2-category $\mathscr{C}$ are *equivalent* if there is a 2-natural transformation $\eta : \mathbf{M} \to \mathbf{N}$ such that $\eta_{\mathtt{i}}$ is an equivalence for each object $\mathtt{i}$ in $\mathscr{C}$.

2.3.3 2-categories of bimodules

Let A be an algebra. Then A-mod is not small, but it is equivalent to some small category $\mathscr{C}$. Then there is a 2-category with one object $\heartsuit$, which we think of as A-mod (or $\mathscr{C}$). The 1-morphisms are endofunctors of $\mathscr{C}$ isomorphic to tensoring with finite-dimensional A-A-bimodules. Composition of 1-morphisms is composition of functors, which corresponds to the tensor product of bimodules. The 2-morphisms are natural transformations of functors, which correspond to bimodule homomorphisms. We denote this 2-category by $\mathscr{D}_A$. Then $\mathscr{D}_A(\heartsuit,\heartsuit)$ is equivalent to A-mod-A, and we call $\mathscr{D}_A$ the 2-category of A-A-bimodules.

2.3.4 Bicategories and birepresentations

A bicategory is a more relaxed version of a 2-representation, where we omit the condition that morphism categories are small, and require composition of 1-morphisms to be associative and unital only up to natural isomorphisms. In many cases, a bicategory can be "strictified" to a 2-category. For example, the bicategory with one object and morphism category A-mod-A is not a 2-category, but it is biequivalent to the 2-category of bimodules $\mathscr{D}_A$ as described above. Importantly, for the representation theory it does not matter if we consider bi- or 2-categories. However, categorical constructions may differ.

2.3.5 Cells in 2-categories

Let $\mathscr{C}$ be an additive, Krull-Schmidt, idempotent split 2-category. The combinatorial structure of the 1-morphisms can be described in ways similar to Green's relations for semigroups [8]. Define the *left preorder* on the set of isomorphism classes of indecomposable 1-morphisms of $\mathscr{C}$ by $F \geq_L G$ if there is some 1-morphism H such that F is a direct summand of $H \circ G$. This is indeed a preorder, and the induced equivalence relation is called *left equivalence* and denoted $\sim_L$. The equivalence classes are called left cells. Similarly, define right preorder $\geq_R$, right equivalence $\sim_R$, and right cells by composing from the right, and the two-sided preorder $\geq_J$, two-sided equivalence $\sim_J$, and two-sided cells by composing from both the left and the right. These notions were introduced in [15] and are of great importance for the study of 2-representations.

First of all, if $\mathscr{C}$ is a 2-category and $\mathbf{M}$ a simple transitive 2-representation of $\mathscr{C}$, then there is a unique maximal two-sided cell which is not annihilated by $\mathbf{M}$. This cell is called the *apex* of the 2-representation. Moreover, let $\mathscr{L}$ be a left cell. Then there is an object $\mathtt{i}_{\mathscr{L}}$ in $\mathscr{C}$ such that all 1-morphisms in $\mathscr{L}$ start at $\mathtt{i}_{\mathscr{L}}$. The principal 2-representation $\mathscr{P}_{\mathtt{i}_{\mathscr{L}}}$ has a subrepresentation given by the additive closure of all 1-morphisms $F \geq_L \mathscr{L}$. This subrepresentation has a unique simple transitive quotient $\mathbf{C}_{\mathscr{L}}$ – the *cell 2-representation* associated to $\mathscr{L}$.

One of the first classes of 2-categories whose 2-representations were studied consists of 2-categories of projective bimodules, that is, the full subcategories of the categories $\mathscr{D}_A$ whose indecomposable 1-morphisms are isomorphic to the indecomposable projective bimodules and the regular bimodule. If A is self-inejctive, these 2-categories are fiat in the sense of Mazorchuk and Miemietz – finitary, with a weak involution and certain adjunction 2-morphisms. It was proven in [18] that any simple transitive 2-representation of a 2-category of projective bimodules is equivalent to a cell 2-representation.

2.3.6 Rank and action matrices

In the special case of a 2-category with only one object, tools from linear algebra become accessible when studying the representation theory.

Let $\mathscr{C}$ be a $\Bbbk$-linear additive, Krull-Schmidt 2-category with one object $\heartsuit$, and $\mathbf{M}$ a finitary simple transitive 2-representation of $\mathscr{C}$. Then $\mathbf{M}(\heartsuit) \simeq B$-proj for some algebra B. The *rank* of $\mathbf{M}$ is the number of indecomposable objects in the category $\mathbf{M}(\heartsuit)$, i.e. the number of indecomposable projective B-modules up to isomorphism. Assume that $\mathbf{M}$ has rank r, and denote the indecomposable projective B-modules by $P_1,\ldots,P_r$. Then, for each 1-morphism F and each $j = 1,\ldots,r$,

$$\mathbf{M}(F)(P_j) \simeq \bigoplus_{i=1}^{r} P_i^{\oplus a_{ij}}$$

for some non-negative integers a_{ij}. The $r \times r$-matrix $[\mathbf{M}(F)] = (a_{ij})_{i,j=1}^{r}$ is the *action matrix* of $\mathbf{M}(F)$.

2.3.7 (Co)algebras and (co)modules in 2-categories

Algebras and their modules can be generalized further, and exist on the 1-morphism level in 2-categories. In this setting, the notions can also be dualized, and so we can define coalgebras and their comodules. Let $\mathscr{C}$ be an additive 2-category. An *algebra 1-morphism* in $\mathscr{C}$ is a 1-morphism $A : \mathtt{i} \to \mathtt{i}$ together with 2-morphisms $\eta : \mathbb{1}_{\mathtt{i}} \to A$ and $\mu : AA \to A$, such that the following diagrams commute, identifying A with $A\mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{i}}A$.

$$\begin{tikzcd} AAA \arrow[r, "A\mu"] \arrow[d, "\mu_A"'] & AA \arrow[d, "\mu"] \\ AA \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\mathrm{id}_A"'] & AA \arrow[d, "\mu"] & A \arrow[l, "\eta_A"'] \arrow[dl, "\mathrm{id}_A"] \\ & A & \end{tikzcd}$$

In particular, take $\mathscr{C}$ to be the 2-category with one object $\heartsuit$, and morphism category $\mathscr{C}(\heartsuit,\heartsuit) = \Bbbk$-vec – the category of finite-dimensional $\Bbbk$-vector spaces. Then an algebra 1-morphism in $\mathscr{C}$ is exactly a finite-dimensional, associative, unital $\Bbbk$-algebra. Similarly, the concept of left (and right) modules can be generalized to a categorical setting.

If $\mathscr{C}$ is a 2-category and A an algebra 1-morphism in $\mathscr{C}$, then a left A-module is a 1-morphism M together with a 2-morphism $\lambda : AM \to M$ such that the following diagrams commute.

$$\begin{tikzcd} AAM \arrow[r, "A\lambda"] \arrow[d, "\mu_M"'] & AM \arrow[d, "\lambda"] \\ AM \arrow[r, "\lambda"] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "\mathrm{id}_M"'] & AM \arrow[d, "\lambda"] \\ & M \end{tikzcd}$$

Similarly we can define right A-modules.

If (M,λ) and (M',λ') are left A-modules, then an A-module morphism is a 2-morphism $f : M \to M'$ such that $f \circ \lambda = \lambda' \circ Af$. The A-modules and A-module morphisms in $\mathscr{C}$ form a category $\mathrm{Mod}_{\mathscr{C}}(A)$, and similarly for right modules.

Now, these definitions can be dualized by reversing all arrows. As usual in the categorical language, this results in a co-structure.

A *coalgebra 1-morphism* in a 2-category $\mathscr{C}$ is a 1-morphism $C : \mathtt{i} \to \mathtt{i}$ together with 2-morphisms $\Delta : C \to CC$ and $\varepsilon : C \to \mathbb{1}_{\mathtt{i}}$ such that the following diagrams commute.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & CC \arrow[d, "C\Delta"] \\ CC \arrow[r, "\Delta_C"] & CCC \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}_C"] & \\ C & CC \arrow[l, "\varepsilon_C"'] \arrow[r, "C\varepsilon"] & C \end{tikzcd}$$

Given a coalgebra C, a left C-comodule is a 1-morphism M together with a 2-morphism $\tau : M \to CM$ such that the following diagrams commute.

$$\begin{tikzcd} M \arrow[r, "\tau"] \arrow[d, "\tau"'] & CM \arrow[d, "C\tau"] \\ CM \arrow[r, "\Delta_M"] & CCM \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\tau"'] \arrow[dr, "\mathrm{id}_M"] & \\ CM \arrow[r, "\varepsilon_M"] & M \end{tikzcd}$$

Similarly we define right comodules. Again dually to the above, we can define (left or right) comodule morphisms, and get a category $\mathrm{Comod}_{\mathscr{C}}(C)$ of (left or right) C-comodules.

3. Summary of papers

Throughout, $\Bbbk$ is a field. In papers I, II and III we assume that $\Bbbk$ is algebraically closed and has characteristic 0. For $n \geq 2$, we set Q_n to be the quiver

$$\begin{tikzcd}
 & & n \arrow[dll] & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \dots \arrow[r] & n-1 \arrow[ull]
\end{tikzcd}$$

of extended Dynkin type $\tilde{A}_{n-1}$.
Moreover, Q_1 denotes the quiver below.

$$\begin{tikzcd}
1 \arrow[loop right]
\end{tikzcd}$$

3.1 Paper I

Let Λ_n be the path algebra of Q_n modulo the relations that the concatenation of any two arrows is zero. The algebras Λ_n are examples of radical square zero Nakayama algebras.

For each fixed positive integer n we study the tensor combinatorics of Λ_n-Λ_n-bimodules in terms of the left-, right-, and two-sided relations in the sense of [15]. The enveloping algebra $\Lambda_n \otimes_{\Bbbk} \Lambda_n^{\mathrm{op}}$ is a special biserial algebra of tame representation type, which allows us to explicitly classify indecomposable bimodules using results from [2, 23]. Up to isomorphism, they are the following.

- Projective-injective bimodules $P_{i|j}$, where $i, j \in \{1, \dots, n\}$.
- String bimodules $W_{i|j}^{(k)}$, $S_{i|j}^{(k)}$, $N_{i|j}^{(k)}$ and $M_{i|j}^{(k)}$, where $i, j \in \{1, \dots, n\}$ and $k \in \{0, 1, 2, \dots\}$. W, S, N or M is the *shape*, $i|j$ the *initial vertex*, and k the number of *valleys*.
- Band bimodules $B(m, j, \lambda)$ where m is a positive integer, $j \in \{1, \dots, n\}$ and $\lambda \in \Bbbk^*$.

The $\Bbbk$-split bimodules are the projective-injectives and the string bimodules of shapes W, S and N with 0 valleys. By [17] the $\Bbbk$-split bimodules will constitute the maximal two-sided cell.

The regular bimodule is the band bimodule $B(1, 1, 1)$, and it will necessarily be contained in the minimal two-sided cell.

Our main result is an explicit description of left- right-, and two-sided cells.

Theorem. *The two-sided cells are the following.*

- $\mathscr{J}_{\Bbbk\text{-split}}$ *consisting of all* $\Bbbk$*-split bimodules;*
- $\mathscr{J}_{M_0}$ *consisting of all* $M^{(0)}_{i|j}$*;*
- *for each* $k \geq 1$, $\mathscr{J}_k$ *consisting of all string bimodules with k valleys.*
- $\mathscr{J}_{band}$ *consisting of all band bimodules.*

The two-sided cells are linearly ordered as follows.

$$\mathscr{J}_{\Bbbk\text{-split}} >_J \mathscr{J}_{M_0} >_J \mathscr{J}_1 >_J \mathscr{J}_2 >_J \cdots >_J \mathscr{J}_{band}.$$

Theorem. *Within the two-sided cells, the one-sided cell structures are as follows.*

- *Within* $\mathscr{J}_{split}$*, left cells are indexed by indecomposable right* Λ_n*-modules, and right cells by indecomposable left* Λ_n*-modules.*
- *Within* $\mathscr{J}_{M_0}$*, left cells consist of all bimodules with the same second coordinate in the initial vertex. Right cells consist of all bimodules with the same first coordinate in the initial vertex.*
- *Within* $\mathscr{J}_k$*,* $k \geq 1$*, left cells consist of all string bimodules of shape W and S, or N and M, with the same second coordinate in the initial vertex. Right cells consist of all string bimodules of shape W and N, or S and M, with the same first coordinate in the initial vertex.*
- *The two-sided cell* $\mathscr{J}_{band}$ *is also a left and right cell.*

In particular, within all two-sided cells except the minimal one, the intersection of any left- and right cell contains exactly one element.

3.2 Paper II

In Paper II we study the problem of classifying simple transitive birepresentations of the bicategory $\mathscr{D}_D$ of bimodules over the dual numbers $D = \Bbbk[x]/(x^2)$. Note that $D \simeq \Lambda_1$ from Paper I. We do a case-by-case analysis depending on the apex of the birepresentation. The main result is the following.

Theorem. *(i) Any simple transitive birepresentation of* $\mathscr{D}_D$ *with apex* $\mathscr{J}_{\Bbbk\text{-split}}$ *is equivalent to a cell birepresentation.*

(ii) Any simple transitive birepresentation of $\mathscr{D}_D$ *with apex* $\mathscr{J}_k$*, for* $k \geq 1$*, has rank 1 or 2.*

(iii) If $\mathbf{M}$ *is a simple transitive birepresentation of* $\mathscr{D}_D$ *with apex* $\mathscr{J}_k$ *which has rank 2, then* $\mathbf{M}$ *is equivalent to the cell birepresentation corresponding to the left cell* $\{N^{(k)}, M^{(k)}\}$ *(or* $\{W^{(k)}, S^{(k)}\}$*).*

(iv) There exists a simple transitive birepresentation of $\mathscr{D}_D$ *with apex* $\mathscr{J}_1$ *which has rank 1.*

The cell $\mathscr{J}_{M_0} = \{M^{(0)}\}$ is not the apex of any simple transitive birepresentation. We conjecture the following.

Conjecture. *For each $k \geq 1$ there is a unique simple transitive birepresentation of $\mathscr{D}_D$ with apex $\mathscr{J}_k$ which has rank 1.*

Part (i) is an application of results about fiab bicategories from [18], and for part (iv), we need only note that the cell birepresentation corresponding to the cell $\mathscr{J}_{M_0}$ is a rank 1-representation with apex J_1.

The proof of part (ii) is obtained by analysing the action matrices. Fix a simple transitive birepresentation $\mathbf{M}$ of $\mathscr{D}_D$ with apex $\mathscr{J}_k$ for some k. An important tool is our explicit knowledge about the multiplication table of the indecomposable 1-morphisms in $\mathscr{J}_k$, as the action matrices must satisfy the same multiplication table. Moreover, setting $F = W_k \oplus S_k \oplus N_k \oplus M_k$ yields, for the action matrix,

$$[F]^2 = 4[F].$$

Since $\mathbf{M}$ is simple transitive, the action matrix $[F]$ has strictly positive integer entries. Using results from [22], we find a finite number of candidates for $[F]$ and do a case-by-case analysis of them. This is the result.

Proposition. *Let $\mathbf{M}$ be a simple transitive birepresentation of $\mathscr{D}_D$ with apex $\mathscr{J}_k$ for some $k \geq 1$. Then the action matrices of the indecomposable 1-morphisms in $\mathscr{J}_k$ are, up to renumbering of the indecomposable objects in $\mathbf{M}(\mathtt{i})$, either all $[1]$ or*

$$[N_k] = [W_k] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad [M_k] = [S_k] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Further analysis of the rank 2-case also yields part (iii) of the main theorem.

Inspired by work on birepresentations of fiab bicateogries, we consider some constructions that illuminate similarities and differences between these and the birepresentations of $\mathscr{D}_D$.

In fiab bicategories, each left cell $\mathscr{L}$ contains a distinguished 1-morphism G, called the *Duflo 1-morphism*. The Duflo morphism is used in the original construction of cell birepresentations in [15]. This concept is generalized beyond the fiab case in [24], and there are examples of a Duflo 1-morphism for a left cell $\mathscr{L}$ which is not an element of the cell. We propose yet another generalization, and prove that the notions coincide in the fiab case. As a categorical contruction, it can be dualized. We prove that each left cell within any of the two-sided cells $\mathscr{J}_k$, $k \geq 1$, contains either a generalized Duflo 1-morphism, or the dual version, a generalized co-Duflo 1-morphism.

The Duflo 1-morphism in a fiab bicategory often has the structure of a coalgebra. It was proven in [13] that simple transitive birepresentations of fiab bicateories can be recovered using categories of right comodules over coalgebra 1-morphisms. In the bicategory $\mathscr{D}_n$, we find that the generalized Duflo 1-morphisms indeed are coalgebra-1-morphisms, and the generalized co-Duflo 1-morphisms are algebra-1-morphisms. However, we prove also that categories of (co)modules, over *any* (co)algebra 1-morphism, cannot give rise to simple transitive birepresentations of $\mathscr{D}_D$ of rank 1 with apex $\mathscr{J}_k$ for $k \geq 2$.

3.3 Paper III

Fix a positive integer n. We let $\mathscr{D}_n = \mathscr{D}_{\Lambda_n}$ be the bicategory of bimodules over the algebra Λ_n from Paper I. In Paper III we study the problem of classifying simple transitive birepresentations of $\mathscr{D}_n$. The main result is the following complete classification of simple transitive birepresentations of $\mathscr{D}_n$ with finite apex.

Theorem. *Fix a positive integer k.*

(i) Any simple transitive birepresentation of $\mathscr{D}_n$ with apex $\mathscr{J}_{split}$ is equivalent to a cell birepresentation.

(ii) Any simple transitive birepresentation of $\mathscr{D}_n$ with apex $\mathscr{J}_k$ has rank between n and 2n.

(iii) For each $0 \leq j \leq 2n$ there are exactly $\binom{n}{j}$ pairwise non-equivalent simple transitive birepresentation of $\mathscr{D}_n$ with apex $\mathscr{J}_k$ which have rank $n+j$.

As in Paper II, the first part of the main theorem is an application of results for fiab bicategories. The proof of the statements about birepresentations with apex $\mathscr{J}_k$ consist of two components - action matrices and localization.

For the action matrices, set $F = \bigoplus_{U \in \mathscr{J}_k} U$. Similar to Paper II, the action matrix $[F]$ is a positive integer matrix satisfying $[F]^2 = 4n[F]$. We do a block decomposition

$$[F] = \begin{bmatrix} F_{11} & \cdots & F_{1n} \\ \vdots & \ddots & \cdots \\ F_{n1} & \cdots & F_{nn} \end{bmatrix}$$

and prove that each diagonal block must be as in the case of the dual numbers from Paper II, i.e. for each $i = 1, \dots, n$ we have either

$$F_{ii} = [4] \quad \text{or} \quad F_{ii} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

The n diagonal blocks are pairwise independent, and choosing all diagonal blocks uniquely determines all action matrices of elements from $\mathscr{J}_k$. This proves that the rank of the representation is between n and $2n$.

Further analysis of the case when $\operatorname{rank}(\mathbf{M}) = 2n$ yields that $\mathbf{M}(\heartsuit) \simeq B$-proj, where B is the direct sum of n copies of the path algebra of the quiver $1 \longrightarrow 2$. Each such algebra has two indecomposable projectives, and there is one nonzero morphism between them. The smaller action matrices, corresponding to birepresentations of rank smaller than $2n$, would necessarily have that some of these summands were replaced by a copy of $\Bbbk$. On a categorical level, this would require that the two projectives over each of those summands became identified - i.e. that the morphism between them would be made invertible. We realize that this would, within B-proj, be a case of localization. Indeed, the general idea of localization is, loosely, to make some chosen elements invertible, and to do this in a universal way. We use higher categorical constructions to define the notion of localization of birepresentations, and prove the following.

Theorem. *If* $\mathbf{M}$ *is a simple transitive birepresentation of a bicategory* $\mathscr{C}$*, and* $\mathscr{S}$ *is a* $\mathscr{C}$*-stable collection in* $\mathbf{M}$*, then the localization* $\mathbf{M}[\mathscr{S}^{-1}]$ *is also a simple transitive birepresentation of* $\mathscr{C}$*.*

This allows us to construct the lower-rank simple transitive birepresentations as localizations of the cell birepresentation of rank $2n$. The universal property of the localization assures the uniqueness of the constructed representations.

3.4 Paper IV

In the classification results of Paper II and III, knowing the explicit cell structure from Paper I was a key. However, for most algebras, the category of bimodules is of wild type, making results like those in Paper I unlikely to be found. In Paper IV we take some first steps trying to compare the cell structures of the bimodules over different algebras. Denote by $\mathscr{BM}_\Bbbk$ the bicategory of bimodules over finite-dimensional associative $\Bbbk$-algebras. The objects of this bicategory are finite-dimensional associative $\Bbbk$-algebras. For algebras A and B, the 1-morphisms from A to B are given by finite-dimensional B-A-bimodules, and composition of 1-morphisms is tensor product of bimodules. The 2-morphisms are bimodules morphisms.

In paper IV, we study the two-sided relations between the identity 1-morphisms in the bicategory $\mathscr{BM}_\Bbbk$. Explicitly, the identity 1-morphism on A is the regular A-A-bimodule ${}_AA \otimes_A -$, so $A \geq_J B$ if there is an isomorphism of A-A-bimodules

$$M \otimes_B N \simeq A \oplus X$$

for some A-B-bimodule M, B-A-bimodule N, and A-A-bimodule X.

J-equivalence in $\mathscr{BM}_\Bbbk$ generalizes Morita equivalence, and several other equivalences of "Morita type". Most general of these is so called separable equivalence, which is a J-equivalence induced by left-right projective bimodules.

We study the structure of bimodules inducing J-relation, proving for example that if ${}_AA_A$ is a direct summand of ${}_AM \otimes_B N_A$, then the projective covers of ${}_AM$ and N_A are generators. Moreover, we note that surjective algebra homomorphisms and separable extensions of algebras induce J-relation.

We also use certain constructions involving groups actions. Given a group G acting on an algebra A via automorphisms, we define as in [20] the *skew group algebra* $A * G$ as the vector space, $A * G = A \otimes_{\Bbbk} \Bbbk G$ with multiplication

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh.$$

Further, denote by A^G the subalgebra of A consisting of invariants under the action of G.

Theorem. *Let A be a $\Bbbk$-algebra and G a finite group acting on A via automorphisms. Assume that* $\operatorname{char}(\Bbbk)$ *does not divide the order of G. Then the following holds.*
(i) $A \sim_J A * G$.
(ii) If G is abelian, then $A \sim A^G$.

In [19], Peacock proves results about separable equivalence of algebras, some of which can be directly generalized to J-equivalence. Most important are the following two results.

Theorem. *If A and B are algebras such that $A \geq_J B$, then for any algebra C it holds that $A \otimes_{\Bbbk} C \geq_J B \otimes_{\Bbbk} C$ and $C \otimes_{\Bbbk} A \geq_J C \otimes_{\Bbbk} B$.*

Theorem. *Let $\Bbbk$ be an algebraically closed field, and A and B algebras such that $A \geq_J B$.*
(i) If B is of finite representation type, then so is A.
(ii) If B is of tame representation type, then A is of tame or finite representation type.

We provide a number of examples of J-related algebras.

For positive integers n, k, set $\Lambda_n^{(k)}$ to be the path algebra of the quiver Q_n modulo the relations that any path of length k is zero. In particular, $\Lambda_1^{(n)} \simeq \Bbbk[x]/(x^n)$, and $\Lambda_n^{(2)}$ is the algebra denoted by Λ_n in papers I-III. Denote by A_n the path algebra of $1 \to 2 \to \ldots \to n$ modulo the square of the radical. Moreover, denote by Θ the path algebra of the Kronecker quiver $1 \rightrightarrows 2$ and by A_3' the path algebra of $1 \to 2 \leftarrow 3$. The following result summarizes out main examples of J-related algebras.

Theorem. *Let $\Bbbk$ be an algebraically closed field, and n a positive integer such that* $\operatorname{char}(\Bbbk)$ *does not divide n. Then the following holds.*

(i) $A_2 >_J A_3 >_J \ldots >_J \Bbbk[x]/(x^2) >_J \Bbbk[x]/(x^3) >_J \Bbbk[x]/(x^4) >_J \Bbbk[x]/(x^5)$.

(ii) For all $k \geq 2$, $\Bbbk[x]/(x^k) \sim_J \Lambda_n^{(k)}$.

(iii) $\Bbbk[x]/(x^2) >_J \Theta >_J \Bbbk[x,y]/(x^2, xy, y^2)$.

(iv) $A_3' >_J \Theta$.

Many of the J-relations we prove are induced by left-right projective bimodules, and therefore examples of separable division in the sense of [19]. Left-right projective bimodules are not very well-studied studied in general. We make a small contribution to the topic. This gives information about the possibility of separable division between certain algebras.

Theorem. *If A is a directed algebra and B is a self-injective algebra, then any left-right projective A-B-bimodule is projective.*

Finally, we formulate a conjecture regarding the relation of the Loewy lengths of J-related algebras.

Conjecture. *Let A and B be algebras such that $A \geq_J B$. Then $\ell\ell(A) \leq \ell\ell(B)$.*

4. Summary in Swedish – Sammanfattning på svenska

4.1 Bakgrund

Inom abstrakt algebra studeras *algebraiska strukturer* och *homomorfier* mellan dem. En algebraisk struktur består av en mängd element, och operationer på elementen med vissa bestämda egenskaper. Exempel på algebraiska strukturer är (halv)grupper, ringar, kroppar, vektorrum och algebror. En algebra är till exempel ett vektorrum som har en bilinjär associativ multiplikation, och dessutom ett element 1 som är neutralt med avseende på multiplikationen. En homomorfi mellan två strukturer av en viss typ är en funktion som respekterar operationerna, det vill säga uppfyller $f(a \cdot b) = f(a) \cdot f(b)$, $f(1) = 1$ och dylikt. Algebraiska strukturer kan användas för att beskriva fenomen som observeras i naturen, till exempel partiklar som interagerar med varandra. Genom att abstrakt beskriva partiklarnas interaktioner – som vi kallar operationer – och reglerna de uppfyller kan vi förstå en viss typ av partikelsystem som en viss typ av algebraisk struktur. Vi kan då axiomatiskt studera *alla* strukturer med dessa typer av operationer som uppfyller samma regler.

Representationsteori är studier av objekt via *representationer*, eller *moduler*. Tanken är att använda mindre komplicerade objekt för att förstå mer komplicerade objekt. En algebra kan representeras med hjälp av vektorrum och linjära avbildningar. Eftersom linjär algebra är ett mycket välutvecklat område är det en klar förenkling gentemot att studera algebror direkt. Samtidigt kan representationerna ge mycket information om själva algebran. Centralt i denna avhandling är *bimoduler*, vilket är vektorrum med två kompatibla modulstrukturer över två algebror. När vi studerar representationsteori kan målet vara att klassificera så kallade *odelbara* moduler. Som namnet antyder kan de inte delas upp i mindre moduler; vi kan illustrera alla moduler som ett antal odelbara moduler som ligger bredvid varandra. En särskild klass av odelbara moduler är de *enkla* modulerna. Dessa är de minsta byggstenarna, eller atomerna, bland modulerna.

Kategoriteori tar abstraktionen ännu ett steg längre. En kategori består kortfattat av objekt, och morfier (pilar) mellan objekten som kan sammansättas enligt vissa regler. Funktioner mellan kategorier som bevarar strukturen kallas för funktorer. Kategorier och funktorer generaliserar en rad matematiska koncept, bland annat algebror och representationer av algebror. Kategoriteori kan fungera som ett verktyg för att beskriva vitt skilda matematiska koncept på samma språk – precis som samma typ av ekvation kan beskriva fenomen från olika vetenskapsområden, kan samma kategoriteoritiska konstruktion beskriva olika matematiska fenomen. En annan intressant användning för kategoriteori är kategorifiering, det vill säga att hitta en motsvarighet i kategorivärlden till ett objekt vi vill studera. Kategorifieringen är i regel ”svårare”, men har å andra sidan mer struktur att få information från. Det var från sådana exempel intresset för birepresentationer av bikategorier kom när det först började studeras systematiskt av Mazorchuk och Miemietz i [15].

2-kategorier generaliserar kategorier genom att tillåta två nivåer av struktur. De består av objekt, 1-morfier mellan objekten, och 2-morfier mellan 1-morfierna. När vi använder diagram för att illustrera strukturerna är ett typsikt diagram i en (1-)kategori respektive en 2-kategori som nedan.

$$\begin{tikzcd} \heartsuit \arrow[r] & \Diamond & \heartsuit \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \Diamond \end{tikzcd}$$

De flesta intressanta exempel är inte 2-kategorier utan den mindre strikta varianten *bikategori*. I många fall kan en bikategori dock ”striktifieras” till en 2-kategori. För merparten av den här avhandlingen, som behandlar representationer av 2- eller bikategorier, är distinktionen mellan koncepten irrelevant.

Precis som i den klassiska representationsteorin är idén med birepresentationer av bikategorier att studera ett objekt via andra, ”enklare” objekt. I fallet med bikategorier kan de enklare objekten vara en samling välstuderade kategorier, och passande funktorer mellan dessa kategorier. Enkla moduler kategorifieras av *enkla transitiva birepresentationer*, vilka alltså utgör de ”minsta” birepresentationerna.

En *bikategori av bimoduler* är en bikategori vars 1-morfier är bimoduler över algebror, och sammansättningen av 1-morfier är tensorprodukt. Tensorprodukten är en sorts multiplikation av bimoduler. I två av artiklarna – Artikel I och Artikel IV – studerar vi den kombinatoriska strukturen av vissa bimoduler under tensorprodukt. Frågan vi ställer oss kan beskrivas som följer: om vi fixerar en bimodul M och tar dess tensorprodukt med andra bimoduler, vilka bimoduler kan vi då få? Eftersom tensorprodukten inte är kommutativ – ordningen spelar roll – måste vi ställa denna fråga när vi tensorerar M från vänster, höger, och båda sidor. Vi kan också vända på frågan: vilka bimoduler kan vi tensorera och få M som resultat? Genom att beskriva tensorkombinatoriken utifrån den första frågan får vi den så kallade cellstrukturen med vänster-, höger-, och tvåsidiga celler. Cellstrukturen kan definieras för 1-morfier i mer generella bikategorier och är viktig när vi studerar birepresentationer. Varje enkel transitiv birepresentation av en bikategori kan associeras till en unik tvåsidig cell, birepresentationens apex. Vidare ger varje vänstercell upphov till en enkel transitiv birepresentation – en cellbirepresentation. I Artikel II och III studerar vi klassifikationen av enkla transtiva birepresentationer av en familj av bikategorier av bimoduler. Många av de bikategorier vars representationsteori tidigare studerats har varit så kallat fiab, vilket innebär att en rad tekniska villkor uppfylls. De bikategorier vi studerar här är inte fiab, varför de resultat om deras birepresentationer som presenteras här utgör intressanta tillskott till teorin.

4.2 Sammanfattning av avhandlingens resultat

$\Bbbk$ betecknar en kropp. I Artikel I, II och III antar vi att $\Bbbk$ är algebraiskt sluten och har karakteristik 0. I alla artiklarna förekommer vägalgebran över följande koger.

$$Q_1 : \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd} \qquad Q_n,\ n \geq 2 : \begin{tikzcd} & & n \arrow[dll] & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \dots \arrow[r] & n-1 \arrow[ull] \end{tikzcd}$$

I Artikel I undersöker vi tensorkombinatoriken av bimoduler över algebror från en särskild familj. Fixera $n \geq 1$ och låt Λ_n vara vägalgebran över Q_n modulo idealet som genereras av alla vägar av längd 2. Då är Λ_n en Nakayamaalgebra med radikalkvadrat noll. Omslutningsalgebran $\Lambda_n \otimes_{\Bbbk} \Lambda_n^{\mathrm{op}}$ är en så kallad speciell biseriell algebra, så dess vänstermoduler – alltså bimoduler över Λ_n – kan klassificeras med hjälp av resultat från [2, 23]. De kan delas in i tre grupper: projektiva-injektiva, strängbimoduler och bandbimoduler.

Artikelns huvudresultat är en explicit beskrivining av tensorkombinatoriken av de odelbara Λ_n-Λ_n-bimodulerna i termer av vänster-, höger-, och tvåsidiga celler. De tvåsidiga cellerna betecknas $\mathscr{J}_{\mathrm{split}}$, $\mathscr{J}_{M_0}$, $\mathscr{J}_{\mathrm{band}}$ och $\mathscr{J}_k$ för $k \geq 1$. De är linjärt ordnade med avseende på den tvåsidiga preordningen $\geq_J$:

$$\mathscr{J}_{\mathrm{split}} >_J \mathscr{J}_{M_0} >_J \mathscr{J}_1 >_J \mathscr{J}_2 >_J \dots >_J \mathscr{J}_{\mathrm{band}}.$$

Inom varje tvåsidig cell utom $\mathscr{J}_{\mathrm{band}}$ gäller dessutom att snittet mellan en vänster- och en högercell alltid innehåller exakt ett element.

I Artikel II studerar vi enkla transtiva birepresentationer av bikategorin av bimoduler över de duala talen $D \simeq \Lambda_1$. Beteckna med $\mathscr{D}$ bikategorin av bimoduler över de duala talen. Från Artikel I känner vi till cellstrukturen för de odelbara D-D-bimodulerna, och gör en falluppdelning beroende på vilken tvåsidig cell som är apex för representationen. Vår huvudsats ger information om enkla transitiva birepresentationer av $\mathscr{D}$ för alla möjliga ändliga apex. Om apex är den maximala cellen $\mathscr{J}_{\mathrm{split}}$ är varje enkel transitiv birepresentation ekvivalent med en cellbirepresentation. För varje annat val av ändligt apex gäller att en invariant kallad *rang* är 1 eller 2. Vidare gäller för dessa val av apex att en birepresentation av rang 2 är ekvivalent med en cellbirepresentation. Vi visar också att det existerar en enkel transitiv birepresentation med rang 1 som har apex $\mathscr{J}_1$. Med stöd i våra beräkningar formulerar vi en förmodan om att det, för alla $k \geq 1$, existerar en unik enkel transitiv birepresentation av rang 1 med apex $\mathscr{J}_k$.

I fallet med fiab bikategorier innehåller varje vänstercell en speciell 1-morfi, cellens sÃ¥ kallade Duflo-1-morfi. Denna är viktig för birepresentationsteorin. Vi definierar en motsvarigeht i mer generella bikategorier, och bevisar att vår definition sammanfaller med den ursprungliga i fiabfallet. Vi visar också att vänsterceller i (relevanta delkvoter av) $\mathscr{D}$ innehåller en sådan generaliserad Duflo-1-morfi, eller en dual version. För fiab bikategorier kan enkla transitiva birepresentationer också beskrivas som kategorier av högerkomoduler över koalgebra-1-morfier i bikategorin enligt [13]. Vi påvisar existensen av både koalgebra-1-morfier och algebra-1-morfier i $\mathscr{D}$, men visar även att kategorier av (ko)moduler inte kan ge upphov till enkla transitiva birepresentationer av $\mathscr{D}$ av rang 1.

I Artikel III bevisar vi förmodan från Artikel II, och generaliserar resultaten i Artikel II till bikategorier av bimoduler över Λ_n för godtyckligt n.

Låt $\mathscr{D}_n$ beteckna bikategorin av Λ_n-Λ_n-bimoduler. Särskilt är då $\mathscr{D}_1$ bikategorin $\mathscr{D}$ från Artikel II. Vår huvudsats är en fullständig klassifikation av enkla transitiva birepresentationer av $\mathscr{D}_n$ med ändligt apex.

Sats. *Fixera heltal $n, k \geq 1$.*

(i) Varje enkel transitiv birepresentation av $\mathscr{D}_n$ med apex $\mathscr{J}_{\mathrm{split}}$ är ekvivalent med en cellbirepresentation.

(ii) Varje enkel transitiv birepresentation av $\mathscr{D}_n$ med apex $\mathscr{J}_k$ har rang mellan n och $2n$.

(iii) För varje $0 \leq j \leq n$ finns exakt $\binom{n}{j}$ enkla transitiva birepresentationer av $\mathscr{D}_n$ som har rang $n + j$ och apex $\mathscr{J}_k$.

En viktig komponent för beviset av satsen är lokalisering av birepresentationer. Vi definierar detta som en birepresentationsteoretisk variant av klassisk lokalisering i kategorier (eller ringar). Vi bevisar också att lokalisering en enkel transitiv birepresentation av en bikategori ger upphov till en ny birepresentation, som också den är enkel transitiv.

Artikel IV – J-equivalence for associative algebras – behandlar en betydligt större bikategori än de andra artiklarna, nämligen bikategorin $\mathscr{BM}_{\Bbbk}$ av alla bimoduler över $\Bbbk$-algebror, där $\Bbbk$ är någon fixerad kropp. Objekten i $\mathscr{BM}_{\Bbbk}$ är $\Bbbk$-algebror. 1-morfierna från A till B är B-A-bimoduler, sammansättningen är tensorprodukt, och 2-morfierna är bimodulhomomorfier. Identitets-1-morfin för ett objekt A är den reguljära A-A-bimodulen ${}_AA_A$, och vi studerar den tvåsidiga relationen, alltså J-relationen, mellan identitets-1-morfierna. Givet algebror A och B säger vi $A \geq_J B$ om det finns bimoduler ${}_AM_B$ och ${}_BN_A$ så att A är en direkt summand av $M \otimes_B N$ som A-A-bimodul.

Vi bevisar ett antal resultat om konstruktioner som ger upphov till J-relation. Bland annat surjektiva algebrahomomorfier, separabla algebrautvidgningar, skevgruppsalgebror och delalgebran av invarianter under en gruppverkan struderas.

Vi generaliserar också vissa resultat av Peacock gällande separabel division, det vill säga J-relation inducerad av höger-vänster-projektiva bimoduler. Dessa resultat ger att J-relation mellan två algebror A och B bevaras om vi skapar tensoralgebrorna $A \otimes_{\Bbbk} C$ och $B \otimes_{\Bbbk} C$ med någon tredje algebra C. Dessutom bevarar J-ekvivalens representationstyp.

Vi ger också ett antal exempel på J-relaterade algebror. Beteckna med $\Lambda_n^{(k)}$ vägalgebran över Q_n modulo idealet som genereras av alla vägar av längd k. Låt A_n vara vägalgebran av kogret $1 \to 2 \to \ldots \to n$ modulo idealet som genereras av alla vägar av längd 2. Slutligen, låt A_3' vara vägalgebran av kogret $1 \to 2 \leftarrow 3$, och Θ Kronoeckeralgebra. Våra viktigaste resultat sammanfattas här.

Sats. *Låt $\Bbbk$ vara en algebraiskt sluten kropp av karakteristik 0.*

(i) $A_2 >_J A_3 >_J \ldots >_J \Bbbk[x]/(x^2) >_J \Bbbk[x]/(x^3) >_J \Bbbk[x]/(x^4) >_J \Bbbk[x]/(x^5)$.

(ii) För alla n och alla $k \geq 2$, $\Lambda_n^{(k)} \sim_J \Bbbk[x]/(x^k)$.

(iii) $\Bbbk[x]/(x^2) >_J \Theta \geq_J \Bbbk[x]/(x^2, xy, y^2)$.

(iv) $A_3' >_J \Theta$.

Följande påstående tycks intuitivt troligt, och motsägs in av något av våra exempel eller resultat. Låt $\ell\ell$ beteckna Loewylängd.

Förmodan. *Om $A \geq_J B$ så gäller $\ell\ell(A) \leq \ell\ell(B)$.*

5. Acknowledgements

First and foremost I thank my supervisor Walter Mazorchuk. Thank you for your help and your expectations in me; for guidance towards a way of thinking and working. Thank you for helping me trust my intuition and formalize it.

I also want to thank my second supervisor Martin Herschend. Knowing that your door is always open has been very reassuring, and your enthusiasm is unprecedented.

Throughout my years of study at Uppsala University I have had many inspiring teachers, for which I am very grateful. In particular I thank Ernst Dieterich, who supervised my bachelor thesis, and who shaped my views of mathematics in many ways.

During the roller-coaster that doctoral studies is, I am grateful for the colleagues that shared the ride with me. Andrea, Andreas, Carmina, Darius, Filipe, Joel, Johan A, Johan R, Jonathan, Marcus, Sam, Yu, and many others - thanks for all the lunches, walks and fikas. It has been invaluable. Christoffer and Aron, you have both been wonderful office mates at different times. Christoffer, thanks for always keeping the spirit up, and Aron, thanks for making my desk seem relatively tidy. Markus, doing maths and talking about maths/life with you has been great. Thank you for being outspoken and funny, and having such a refreshing distance to academia.

My academic big brother Jakob, thank you for all the support, encouragement and jokes. Having you as a friend meant the world to me as I started my journey into the 2-world. My academic little brother Matti, doing research and going to conferences with you has been a joy, and I will always brag about you. Thank you for being so incredibly smart and knowledgeable and, most of all, kind. Elin, my academic twin sister, I cannot list all that I want to thank you for, but nothing – in particular not this thesis – would be the same without you. Thank you for hearing me out and questioning all my ideas, good and bad. Finally, Linnéa, who seemed to magically appear to join me as I took my first steps towards the goal I have now almost reached: imagining this journey without you is impossible. I am lucky to share my passion for mathematics with a friend like you.

Thanks to the administrative staff of the department of mathematics for always sorting everything out, and for being so friendly.

Thanks to the meteorologists (in the extended sense) for being my gang throughout many years of study. I also want to thank some other special friends: Frida, for always making me feel like I could do anything, and Hanna, Joella, Lovisa, Moremi, and Pontus, for making me feel like what I do is worth

something although you don't understand it. Most importantly, if it wasn't for all the times I enjoyed life with you, mathematics would have been unbearable.

I am grateful to my parents for the support they have offered me. Thank you for always encouraging me to follow my own path. I am also grateful to my brother Erik. Thank you for helping me understand the chain rule, and for telling me that I'm the smart one when I couldn't imagine anyone smarter than you. Likewise, I am grateful to my brother Aron for immeasurable love.

Lastly, with all my heart, I thank Viktor, who has held me in has arms, and Majken, whom I get to hold in mine. Thank you for making me forget all about mathematics.

References

[1] I. Assem, A. Skowronski, and D. Simson. *Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory*, volume 1 of *London Mathematical Society Student Texts*. Cambridge University Press, 2006.

[2] M.C.R. Butler and C.M. Ringel. Auslander-reiten sequences with few middle terms and applications to string algebras. *Comm. Alg.*, 15(1-2):145–179, 1987.

[3] A. Chan and V. Mazorchuk. Diagrams and discrete extensions for finitary 2-representations. *Math. Proc. Camb. Phil. Soc.*, 166:325–352, 2019.

[4] J. Chuang and R. Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math.*, 167(1):245–298, 2008.

[5] Yu. Drozd. Tame and wild matrix problems. In V. Dlab and P. Gabriel, editors, *Representation Theory II*, pages 242–258, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.

[6] Yu. Drozd and V. V. Kirichenko. *Finite dimensional algebras*. Springer, 1994. Translated by V. Dlab.

[7] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2015.

[8] J. A. Green. On the structure of semigroups. *Ann. of Math.*, 2(54):163–172, 1951.

[9] P. A. Grillet. *Abstract algebra*, volume 242 of *Graduate texts in mathematics*. Springer, 2 edition, 2007.

[10] L. Kadison. Uniquely separable extensions. *Exp. Math.*, 38(2):217–231, 2020.

[11] M. Khovanov. A categorification of the jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[12] Z. Leszczyński. On the representation type of tensor product algebras. *Fundamenta Mathematicae*, 144:143–161, 1994.

[13] M. Mackaay, V. Mazorchuk, V. Miemietz, and D. Tubbenhauer. Simple transitive 2-representations via (co)algebra 1-morphisms. *Ind. Univ. Math. J.*, 68(1):1–33, 2019.

[14] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer, and X. Zhang. Finitary birepresentations of finitary bicategories. *Forum Math.*, 33(5):1261–1320, 2021.

[15] V. Mazorchuk and V. Miemietz. Cell 2-representations of finitary 2-categories. *Composito Math.*, 147:1519–1545, 2011.

[16] V. Mazorchuk and V. Miemietz. Transitive 2-representations of finitary 2-categories. *Transactions of the American Mathematical Society*, 368:7623–7644, 2014.

[17] V. Mazorchuk, V. Miemietz, and X. Zhang. Characterisation and applications of k-split bimodules. *Math. Scand.*, 124:161–178, 2019.

[18] V. Mazorchuk, V. Miemietz, and X. Zhang. Pyramids and 2-representations. *Rev. Mat. Iberoam.*, 36(2):387–405, 2020.

[19] S.F. Peacock. Separable equivalence, complexity and representation type. *J. Alg.*, 490:219–240, 2017.

[20] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of artin algebras. *J. Alg.*, 92:224–282, 1985.

[21] M. Stroiński. Weighted colimits of 2-representations and star algebras. *Appl. Categ. Struct.*, 31(29), 2023.

[22] E. Thörnblad and J. Zimmermann. Counting quasi-idempotent irreducible integer matrices. *J. Integer Seq.*, 28(18.4.8), 2017.

[23] B. Wald and J. Waschbusch. Tame biserial algebras. *J. Alg.*, 95(2):480–500, 1985.

[24] X. Zhang. Duflo involutions for 2-categories associated to tree quivers. *J. Algebra Appl.*, 15(3), 2016.

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