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Superspace expansion of the 11D linearized superfields in the pure spinor formalism, and the covariant vertex operator

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Abstract: 11D pure spinors have been shown to successfully describe 11D supergravity in a manifestly super-Poincaré covariant manner. The feasibility of its actual usage for scattering amplitude computations requires an efficient manipulation of the superfields defining linearized 11D supergravity. In this paper, we directly address this problem by finding the superspace expansions of these superfields, at all orders in θ, from recursive relations their equations of motion obey in Harnad-Shnider-like gauges. After introducing the 11D analogue of the 10D $\mathcal{ABC}$ superparticle, we construct, for the first time, a fully covariant vertex operator for 11D supergravity by making use of the linearized 11D superfields. Notably, we show that this vertex reproduces the Green-Gutperle-Kwon 11D operators in light-cone gauge.

Keywords: M-Theory, Supergravity Models, Superspaces

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Contents

1 Introduction

The pure spinor formalism for the superstring [1] has been shown to be tremendously useful for efficient computation of scattering amplitudes involving bosonic and fermionic string states at tree- and loop-level [2–6]. Its respective field-theory limit, namely the 10D pure spinor superparticle [7], has also been proved to be convenient for studying and computing 10D super-Yang-Mills interactions, as well as for analyzing the high-energy behavior of the theory through the use of simple arguments based on zero mode counting and pure spinor algebraic properties [8, 9].

Soon after the discovery of his new superstring formalism, Berkovits introduced the pure spinor versions of the 11D superparticle and supermembrane in [10]. In this work, it was remarkably shown how the full field content of the Batalin-Vilkovisky description of linearized 11D supergravity can elegantly be described by the 11D pure spinor BRST cohomology. Although this fact gives the pure spinor formalism a privileged place as the appropriate framework for a consistent covariant quantization scheme in 11D, no explicit scattering amplitude computation has been carried out to date. This is mainly due to the lack of understanding of the building blocks needed for evaluating pure spinor correlators, including vertex operators of lower ghost numbers and 11D pure spinor identities.

Over the past few years, this 11D pure spinor program has been revived, and some significant progress has been made. For instance, one of the authors recently introduced the ghost number one and two pure spinor vertex operators, and developed a new prescription for computing tree-level 11D pure spinor correlators [11–13]. Likewise, some technical subtleties were found when trying to use a standard descent equation and define a ghost number zero vertex operator [14], a fundamental piece for the calculation of four- and higher-point interactions in 11D supergravity. These results provide the toolbox needed for calculating three-particle scattering processes from pure spinor superspace expressions, and demand a revision or more careful analysis of the ghost number zero vertex operator.

In this paper we start the study of both issues mentioned above. As in 10D, the explicit computation of 11D pure spinor correlators requires the exact knowledge of the superspace expansions of all the superfields defining the 11D pure spinor vertex operators, namely the linearized 11D supergravity superfields. For this purpose, we find the complete set of equations of motion of linearized 11D supergravity in superspace, from the linearization of the 11D supergeometry and the four form field-strength of 11D supergravity. The use of Harnad-Shnider-like gauges [15] on the lowest-dimensional components of the 11D superfields will be shown to give rise to a solvable system of recursive relations yielding every coefficient of the superspace expansions of all the linearized 11D superfields. The originality of our method relies on its feasibility and effectiveness within the pure spinor worldline framework. Indeed, our approach is pretty much exclusive and convenient for studying the specific forms of the 11D superfields involved in the construction of pure spinor vertex operators in 11D. This is in contrast to the general analysis carried out in [16], where superfields not directly relevant to the pure spinor formalism are studied. In this sense, the results of the first part of our paper will have a transcendent and direct significance for the development of the pure spinor program in 11D.

The second part of our paper discusses the construction of a covariant vertex operator for 11D supergravity. This idea is strongly inspired by the relationship found between the vertex operators of the 10D $\mathcal{ABC}$ [17] and Brink-Schwarz [18] superparticles in light-cone gauge [19]. The 11D light-cone gauge vertex operators were introduced by Green, Gutperle and Kwon in [20]. In order to reproduce these vertices from a covariant expression, we will first define the 11D analogue of the 10D $\mathcal{ABC}$ superparticle, and show it contains the same physical degrees of freedom as the standard 11D superparticle [20]. Next, we construct a covariant vertex operator by making exclusive use of supersymmetric quantities, as well as the linearized 11D superfields. The superspace expansions found in the first part of this work will allow us to show that this covariant operator exactly reproduces the Green-Gutperle-Kwon vertices in light-cone gauge.

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The paper is organized as follows. In section 2 we review the pure spinor formulation of the 11D superparticle, and discuss how the 11D supergravity physical states emerge from the cohomology of the pure spinor BRST charge. Section 3 motivates the definition of the linearized 11D superfields relevant to the definition of pure spinor vertex operators, and constructs the full set of equations of motion and gauge transformations satisfied by these. In section 4, we systematically solve the system of recursive relations found from the previous set of equations when superfields are subject to Harnad-Shnider-like gauges, and show they are self-consistent. Section 5 introduces the 11D $\mathcal{ABC}$ superparticle, and presents an 11D covariant vertex operator made out of supersymmetric worldline fields and the linearized 11D superfields, which is shown to reduce to the Green-Gutperle-Kwon vertex operators in light-cone gauge. Section 6 closes with discussions and future perspectives. We collect our conventions for gamma matrices in appendix A, and briefly review the 10D $\mathcal{ABC}$ superparticle vertex operator and its relation to the light-cone gauge Brink-Schwarz operators in appendix B. In appendix C we compare our equations of motion and superfields to the ones in [16].

2 11D pure spinor superparticle

The 11D pure spinor superparticle action is defined by [10, 21]

$$S = \int d\tau \left[P^a \partial_\tau X_a + p_\alpha \partial_\tau \theta^\alpha + w_\alpha \partial_\tau \lambda^\alpha - \frac{1}{2} P^2 \right] . \tag{2.1}$$

We use letters from the beginning of the Greek/Latin alphabet to denote spinor/vector SO(1, 10) indices. The variables (P_a, p_α) are the conjugate momenta associated to the usual 11D superspace coordinates (X^a, θ^α). The bosonic spinor λ^α satisfies the 11D pure spinor constraint, i.e. $(\lambda\gamma^a\lambda) = 0$, and thus its respective conjugate momentum w_α is only defined up to the gauge transformation $\delta w_\alpha = (\gamma^a\lambda)_\alpha \sigma_a$, for any vector σ_a. Due to their wrong statistics, $(\lambda^\alpha, w_\beta)$ will be referred to as ghost variables, and assigned to carry ghost charges 1 and -1, respectively. The 11D gamma matrices will be represented by $(\gamma^a)_{\alpha\beta}$, $(\gamma^a)^{\alpha\beta}$, and they satisfy the Clifford algebra: $(\gamma^a)_{\alpha\beta}(\gamma^b)^{\beta\delta} + (\gamma^b)_{\alpha\beta}(\gamma^a)^{\beta\delta} = 2\eta^{ab}\delta_\alpha^\delta$. We will raise and lower spinor indices by using the antisymmetric charge conjugation matrix $C_{\alpha\beta}$ and its inverse $C^{\alpha\beta}$, which obey the relation $C_{\alpha\beta}C^{\beta\delta} = \delta_\alpha^\delta$, so that $(\gamma^a)^{\alpha\beta} = C^{\alpha\epsilon}C^{\beta\delta}(\gamma^a)_{\epsilon\delta}$ for example (see appendix A for more details).

As is well-known, the space of physical states is defined by the cohomology of the BRST operator $Q = \lambda^\alpha d_\alpha$, where $d_\alpha = p_\alpha - \frac{1}{2}(\gamma^a\theta)_\alpha P_a$ is the familiar primary constraint of the superparticle [20]. Such a cohomology can be shown to be non-trivial up to ghost number 7, describing the 11D supergravity states in its Batalin-Vilkovisky formulation. More explicitly, the ghost number 0, 1, 2 and 3 sectors respectively accommodate the gauge symmetry ghost-for-ghost-for-ghost; the gauge symmetry ghost-for-ghost; the supersymmetry, diffeomorphism and gauge symmetry ghosts; and the 11D supergravity physical fields. The higher ghost number sectors form a mirror of the fields described above, and correspond to the 11D supergravity antifields. One can easily see this by analyzing the ghost number three sector,

$U^{(3)} = \lambda^\alpha\lambda^\beta\lambda^\delta A_{\alpha\beta\delta}$. The BRST-closedness condition implies that

$$Q\Psi = 0 \to D_{(\alpha}A_{\beta\delta\epsilon)} = (\gamma^a)_{(\alpha\beta}A_{a\delta\epsilon)}\,, \tag{2.2}$$

and the BRST-exactness restriction imposes that

$$\delta\Psi = Q\Lambda \to \delta A_{\alpha\beta\delta} = D_{(\alpha}\Lambda_{\beta\delta)}\,, \tag{2.3}$$

where $\Lambda = \lambda^\alpha\lambda^\beta\Lambda_{\alpha\beta}$, and $\Lambda_{\alpha\beta}$ is a gauge parameter. These equations match the linearized equations of motion of 11D supergravity in superspace [22], after making the identification $A_{\alpha\beta\delta} = C_{\alpha\beta\delta}$, where $C_{\alpha\beta\delta}$ is the linearized version of the lowest-dimensional component of the 11D supergravity super three form. As we will see later on, in a particular gauge, one can show that $U^{(3)}$ has the θ-expansion,

$$\begin{aligned}U^{(3)} = &-\frac{3}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)\epsilon^{b_2b_3} - \frac{1}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)c^{b_1b_2b_3}\\ &+\frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\gamma^{b_3}\Psi^{b_2}) - \frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\gamma^{b_1b_2}\Psi^{b_3})\\ &+O(\theta^5)\,,\end{aligned} \tag{2.4}$$

with c_{abc}, ϵ_{ab}, Ψ^a_α being the three form, graviton and gravitino of 11D supergravity. Indeed, they can be shown to satisfy the linearized equations of motion

$$\partial^d\partial_{[d}c_{abc]} = 0\,, \quad \Box\epsilon_{bc} - 2\partial^a\partial_{(b}\epsilon_{c)a} + \partial_b\partial_c(\eta^{ad}\epsilon_{ad}) = 0\,, \quad (\gamma^{abc})_{\alpha\beta}\partial_b\Psi^\beta_c = 0\,, \tag{2.5}$$

and gauge transformations

$$\delta c_{abc} = \partial_{[a}s_{bc]}\,, \quad \delta\epsilon_{ab} = \partial_{(a}t_{b)}\,, \quad \delta\Psi^\alpha_a = \partial_a\kappa^\beta\,, \tag{2.6}$$

where s_{ab}, t_b and κ^β are arbitrary gauge parameters.

As shown in [14], it is also possible to describe the physical fields of linearized 11D supergravity through a ghost number one vertex operator involving momentum variables. Unlike the ghost number three operator, this alternative operator describes the 11D supergravity three form gauge field through its field strength. Next, we review this construction and extend the analysis elaborated in [14] to find a complete set of superspace equations of motion giving rise to linearized 11D supergravity.

3 Linearized 11D supergravity equations of motion

Let us first set some notation. We will use capital letters from the beginning/middle of the Latin alphabet to represent tangent/curved superspace indices, and lowercase letters from the beginning (middle) of the Latin/Greek alphabet to denote tangent (curved) space vector/spinor indices. The 11D supergeometry is then defined by the one form superfields E^A and $\Omega_B{}^C$, referred to as the vielbein and spin-connection, and the super-Bianchi identities

$$\mathcal{D}T^A = E^B R_B{}^A\,, \qquad \mathcal{D}R_A{}^B = 0\,, \tag{3.1}$$

where $T^A = \mathcal{D}E^A$ is the super-torsion, $R_A{}^B = \mathcal{D}\Omega_A{}^B$ is the super-curvature, and $\mathcal{D} = E^A\nabla_A$ is the super-covariant derivative. Its action on an arbitrary tensor $\mathcal{F}_{A_1\ldots A_m}{}^{B_1\ldots B_n}$ is,

$$\begin{aligned}\mathcal{D}\mathcal{F}_{A_1\ldots A_m}{}^{B_1\ldots B_n} = d\mathcal{F}_{A_1\ldots A_m}{}^{B_1\ldots B_n} + \Omega_{A_1}{}^C\mathcal{F}_{CA_2\ldots A_m}{}^{B_1\ldots B_n} + \ldots \\ - \mathcal{F}_{A_1\ldots A_m}{}^{C\ldots B_n}\Omega_C{}^{B_1} - \ldots\,, \end{aligned} \tag{3.2}$$

where d is the ordinary exterior derivative. As is well-known, the following relations hold in Lorentz superspace

$$\Omega_{A\beta}{}^\delta = \frac{1}{4}(\gamma^{bc})_\alpha{}^\beta\Omega_{Abc}\,, \tag{3.3}$$

$$R_{AB,\alpha}{}^\beta = \frac{1}{4}(\gamma^{cd})_\alpha{}^\beta R_{AB,cd}\,. \tag{3.4}$$

3.1 Review of the ghost number one vertex operator

As discussed in [14], a simple way of defining a ghost number one vertex operator in the BRST-cohomology, is via a linear perturbation of the BRST charge $Q = \lambda^\alpha d_\alpha$, by $Q \to Q + U^{(1)}$. The nilpotency requirement of the deformed charge then automatically implies that $\{Q, U^{(1)}\} = 0$. This perturbation can readily be obtained from coupling the pure spinor superparticle (2.1) to a curved background. When doing so, the BRST charge can be shown to be defined as $Q = \lambda^\alpha E_\alpha{}^M(P_M + \Omega_{M\beta}{}^\delta\lambda^\beta w_\delta)$, where P_M denotes the curved space supermomentum. Therefore, $U^{(1)}$ is given by

$$U^{(1)} = \lambda^\alpha(P_a h_\alpha{}^a + d_\beta h_\alpha{}^\beta - \Omega_{\alpha\beta}{}^\delta\lambda^\beta w_\delta)\,, \tag{3.5}$$

where $h_A{}^B = \hat{E}_A{}^M E_M^{(1)B} = -E_A^{(1)M}\hat{E}_M{}^B$, $(\hat{E}_A{}^M,\, \hat{E}_M{}^B)$ are the background values of the vielbeins, and $(E_A^{(1)M},\, E_M^{(1)A})$ are their corresponding first order perturbations.

As a check, one can explicitly compute $\{Q, U^{(1)}\} = 0$, to find the following relations

$$\lambda^\alpha\lambda^\beta P_a\Big[D_\alpha h_\beta{}^a - h_\alpha{}^\delta(\gamma^a)_{\beta\delta}\Big] = 0\,, \tag{3.6}$$

$$\lambda^\alpha\lambda^\beta d_\delta\Big[D_\alpha h_\beta{}^\delta - \Omega_{\alpha\beta}{}^\delta\Big] = 0\,, \tag{3.7}$$

$$\lambda^\alpha\lambda^\beta\lambda^\delta w_\epsilon R_{\alpha\beta,\delta}{}^\epsilon = 0\,. \tag{3.8}$$

As we will see below, these equations become identities after plugging the superspace constraints of 11D supergravity.

3.2 Full set of equations of motion

Eqs. (3.1) imply the familiar relations

$$[\nabla_A, \nabla_B\} = -T_{AB}{}^C\nabla_C - 2\Omega_{[AB\}}{}^C\nabla_C, \tag{3.9}$$

$$R_{AB,C}{}^D = 2\nabla_{[A}\Omega_{B\}C}{}^D + T_{AB}{}^F\Omega_{FC}{}^D - \Omega_{[A|C|}{}^F\Omega_{B\}F}{}^D\,, \tag{3.10}$$

where $[\,,\,\}$ means graded commutator. The spectrum of 11D supergravity contains a three form gauge field which can be promoted to the three form superfield $F = E^C E^B E^A F_{ABC}$, satisfying the gauge transformation $\delta F = dL$, for any two form superfield L. Its field strength takes the form $G = dF$, and it satisfies the Bianchi identity $dG = 0$. In order to write down the full set of equations of motion of linearized 11D supergravity, one first expresses the covariant derivative $\nabla_A = E_A{}^M \partial_M$ at linear order as

$$\nabla_A = D_A - h_A{}^B D_B \,, \tag{3.11}$$

where $D_A = \hat{E}_A{}^M \partial_M$. The dynamical constraints $T_{\alpha\beta}{}^a = (\gamma^a)_{\alpha\beta}$, $G_{\alpha\beta ab} = (\gamma_{ab})_{\alpha\beta}$, along with the conventional constraints $T_{\alpha\beta}{}^\delta = T_{a\alpha}{}^c = T_{ab}{}^c = G_{\alpha\beta\delta\epsilon} = G_{a\alpha\beta\delta} = G_{abc\alpha} = 0$ [22], then imply the following set of equations of motion

$$2D_{(\alpha}h_{\beta)}{}^a - 2h_{(\alpha}{}^\delta(\gamma^a)_{\beta)\delta} + h_b{}^a(\gamma^b)_{\alpha\beta} = 0\,, \tag{3.12}$$
$$2D_{(\alpha}h_{\beta)}{}^\delta - 2\Omega_{(\alpha\beta)}{}^\delta + (\gamma^a)_{\alpha\beta}h_a{}^\delta = 0\,, \tag{3.13}$$
$$\partial_a h_\alpha{}^\beta - D_\alpha h_a{}^\beta - T_{a\alpha}{}^\beta - \Omega_{a\alpha}{}^\beta = 0\,, \tag{3.14}$$
$$\partial_a h_\alpha{}^b - D_\alpha h_a{}^b - h_a{}^\beta(\gamma^b)_{\beta\alpha} + \Omega_{\alpha a}{}^b = 0\,, \tag{3.15}$$
$$\partial_a h_b{}^\alpha - \partial_b h_a{}^\alpha - T_{ab}{}^\alpha = 0\,, \tag{3.16}$$
$$\partial_a h_b{}^c - \partial_b h_a{}^c - 2\Omega_{[ab]}{}^c = 0\,. \tag{3.17}$$

Notice that eqs. (3.12), (3.13) immediately imply eqs. (3.6), (3.7), respectively. Using these constraints, one can also show that (see [11] for a detailed discussion)

$$R_{(\alpha\beta,\delta)}{}^\epsilon + (\gamma^a)_{(\alpha\beta}T_{a\delta)}{}^\epsilon = 0\,, \tag{3.18}$$
$$R_{(\alpha\beta),b}{}^c + 2(\gamma^c)_{\gamma(\beta}T_{|b|\alpha)}{}^\gamma = 0\,, \tag{3.19}$$
$$R_{(\alpha\beta),c}{}^d - 2D_{(\alpha}\Omega_{\beta)c}{}^d - (\gamma^a)_{\alpha\beta}\Omega_{ac}{}^d = 0\,, \tag{3.20}$$

where $T_{a\delta}{}^\epsilon$ is defined by the four form field strength G via

$$T_{a\alpha}{}^\beta = (\mathcal{T}_a{}^{bcde})_\alpha{}^\beta H_{bcde}\,, \tag{3.21}$$

and[1]

$$(\mathcal{T}_a{}^{bcde})_\alpha{}^\beta = \frac{1}{36}\left[\delta_a^{[b}(\gamma^{cde]})_\alpha{}^\beta + \frac{1}{8}(\gamma_a{}^{bcde})_\alpha{}^\beta\right]. \tag{3.22}$$

Eq. (3.18) immediately shows the validity of eq. (3.8).

For later use, it will be convenient to rewrite $R_{\alpha\beta,b}{}^c$ in terms of H_{abcd}. This can readily be done through the use of eq. (3.19). Explicitly,

$$R_{\alpha\beta,b}{}^c = (\mathcal{R}_b{}^{cdefg})_{\alpha\beta}H_{defg}\,, \tag{3.23}$$

with

$$(\mathcal{R}_{bc}{}^{defg})_{\alpha\beta} = \frac{1}{6}\left[\delta_b^{[d}\delta_c^e(\gamma^{fg]})_{\alpha\beta} + \frac{1}{24}(\gamma_{bc}{}^{defg})_{\alpha\beta}\right]. \tag{3.24}$$

[1]The tensors $\mathcal{T}$ and $\mathcal{R}$ defined in eqs. (3.22) and (3.24), respectively, differ from those in [16] by factors of -1 or i. See appendix C for a more detailed discussion on this point.

Likewise, eq. (3.17) is automatically satisfied by the relation

$$\Omega_{abc} = \partial_{[a} h_{b]c} - \partial_{[a} h_{c]b} + \partial_{[c} h_{b]a} \,. \tag{3.25}$$

The equations of motion associated to the components of the linearized version of the three form superfield F can directly be deduced from a four form superfield H defined from the field strength G as [23]

$$H_{ABCD} = (-1)^{Q(P+N+M+C+B+A)+P(N+M+B+A)+N(M+A)} \hat{E}_{[D}{}^{Q} \hat{E}_{C}{}^{P} \hat{E}_{B}{}^{N} \hat{E}_{A\}}{}^{M} G_{MNPQ} \,, \tag{3.26}$$

which can equivalently be written as $H_{ABCD} = 4D_{[A}C_{BCD\}} + 6\hat{T}_{[AB}{}^{E}C_{ECD\}}$, where $C_{ABC} = (-1)^{P(N+M+B+A)+N(M+A)} \hat{E}_{[C}{}^{P} \hat{E}_{B}{}^{N} \hat{E}_{A\}}{}^{M} F_{MNP}$, and $\hat{T}^{A}$ is the flat space torsion. The letters in the exponents denote the degree of the index to which it is associated, so it takes the value of 1 if the index is spinorial, and 0 if it is vectorial. The expansion of (3.26) then yields

$$4D_{(\alpha} C_{\beta\delta\epsilon)} + 6(\gamma^{a})_{(\alpha\beta} C_{a\delta\epsilon)} = 0 \,, \tag{3.27}$$

$$\partial_{a} C_{\alpha\beta\delta} - 3D_{(\alpha} C_{a\beta\delta)} + 3(\gamma^{b})_{(\alpha\beta} C_{ba\delta)} = -3(\gamma_{ab})_{(\alpha\beta} h_{\delta)}{}^{b} \,, \tag{3.28}$$

$$2\partial_{[a} C_{b]\alpha\beta} + 2D_{(\alpha} C_{\beta)ab} + (\gamma^{c})_{\alpha\beta} C_{cab} = 2(\gamma_{[b}{}^{c})_{\alpha\beta} h_{a]c} - 2(\gamma_{ab})_{\delta(\alpha} h_{\beta)}{}^{\delta} \,, \tag{3.29}$$

$$3\partial_{[a} C_{bc]\alpha} - D_{\alpha} C_{abc} = -3(\gamma_{[ab})_{\alpha\beta} h_{c]}{}^{\beta} \,. \tag{3.30}$$

The equations of motion displayed in (3.12)–(3.17) are invariant under the gauge transformations

$$\begin{aligned} &\delta h_{\alpha}{}^{a} = D_{\alpha}\Lambda^{a} + (\gamma^{a})_{\alpha\beta}\Lambda^{\beta} \,, && \delta h_{\alpha}{}^{\beta} = D_{\alpha}\Lambda^{\beta} + \Lambda_{\alpha}{}^{\beta} \,, && \delta\Omega_{\alpha\beta}{}^{\epsilon} = D_{\alpha}\Lambda_{\beta}{}^{\epsilon} \,, \\ &\delta h_{a}{}^{b} = \partial_{a}\Lambda^{b} + \Lambda_{a}{}^{b} \,, && \delta h_{a}{}^{\beta} = \partial_{a}\Lambda^{\beta} \,, && \delta\Omega_{a\alpha}{}^{\beta} = \partial_{a}\Lambda_{\alpha}{}^{\beta} \,, \end{aligned} \tag{3.31}$$

where Λ^{a}, Λ^{α}, $\Lambda_{\alpha}{}^{\beta} = \frac{1}{4}(\gamma^{ab})_{\alpha}{}^{\beta}\Lambda_{ab}$ are arbitrary gauge parameters. Similarly, the gauge transformations acting on the components of the superfield C, which leave the equations of motion listed in (3.27)–(3.30) invariant, take the form

$$\delta C_{\alpha\beta\epsilon} = D_{(\alpha}\Lambda_{\beta\epsilon)} + (\gamma^{a})_{(\alpha\beta}\Lambda_{a\epsilon)} \,, \tag{3.32}$$

$$\delta C_{a\alpha\epsilon} = \frac{1}{3}\partial_{a}\Lambda_{\alpha\epsilon} + \frac{2}{3}D_{(\alpha}\Lambda_{\epsilon)a} + \frac{1}{3}(\gamma^{b})_{\alpha\epsilon}\Lambda_{ba} + (\gamma_{ab})_{\alpha\epsilon}\Lambda^{b} \,, \tag{3.33}$$

$$\delta C_{ab\alpha} = \frac{2}{3}\partial_{[a}\Lambda_{b]\alpha} + \frac{1}{3}D_{\alpha}\Lambda_{ab} - (\gamma_{ab})_{\alpha\beta}\Lambda^{\beta} \,, \tag{3.34}$$

$$\delta C_{abc} = \partial_{[a}\Lambda_{bc]} \,. \tag{3.35}$$

Next we use these transformations to conveniently fix the lowest-dimensional components of the h-, Ω- and C-superfields to specific values. This gauge fixing will allow us to find a system of recursive relations, which will be systematically and explicitly solved to obtain the full θ-expansions of the linearized 11D superfields.

4 Superspace expansions of the h-, Ω- and C-superfields

In this section, we will show the system of equations defined by the relations (3.12)–(3.17) and (3.27)–(3.30) is closed, by explicitly solving it in the Harnad-Shnider-like gauges.

4.1 Harnad-Shnider-like gauges

As done in 10D [15], one can use the gauge transformations listed in (3.31)–(3.35) to impose the so-called Harnad-Shnider-like gauges

$$\theta^\alpha h_\alpha{}^A = 0\,, \qquad \theta^\alpha \Omega_{\alpha A}{}^B = 0\,, \qquad \theta^\alpha C_{\alpha AB} = 0\,. \tag{4.1}$$

After contracting both sides of the C-field equations of motion (3.27)–(3.30) with θ^α, and introducing the D-operator $D = \theta^\alpha \partial_\alpha$, this gauge choice implies

$$(D+3)C_{\beta\delta\epsilon} + 3\theta^\alpha(\gamma^a)_{\alpha(\beta}C_{a\delta\epsilon)} = 0\,, \tag{4.2}$$

$$(D+2)C_{a\beta\delta} - 2\theta^\alpha(\gamma^b)_{\alpha(\beta}C_{ba\delta)} = 2\theta^\alpha(\gamma_{ab})_{\alpha(\beta}h_{\delta)}{}^b\,, \tag{4.3}$$

$$(D+1)C_{\beta ab} + \theta^\alpha(\gamma^c)_{\alpha\beta}C_{cab} = 2\theta^\alpha(\gamma_{[b}{}^c)_{\alpha\beta}h_{a]c} - \theta^\alpha(\gamma_{ab})_{\alpha\delta}h_\beta{}^\delta\,, \tag{4.4}$$

$$DC_{abc} = 3(\theta\gamma_{[ab})_\beta h_{c]}{}^\beta\,. \tag{4.5}$$

Analogously, after contracting both sides of the equations of motion (3.12)–(3.15) with θ^α, one gets for the h-fields

$$(D+1)h_\beta{}^a - h_\beta{}^\delta(\theta\gamma^a)_\delta + h_b{}^a(\theta\gamma^b)_\beta = 0\,, \tag{4.6}$$

$$(D+1)h_\beta{}^\delta - \frac{1}{4}(\theta\gamma^{bc})^\delta\Omega_{\beta bc} + (\theta\gamma^a)_\beta h_a{}^\delta = 0\,, \tag{4.7}$$

$$Dh_a{}^\beta + 4\theta^\alpha(\mathcal{T}_a{}^{bcde})_\alpha{}^\beta\partial_b C_{cde} + \frac{1}{4}(\theta\gamma^{bc})^\beta\Omega_{abc} = 0\,, \tag{4.8}$$

$$Dh_a{}^b + (\theta\gamma^b)_\beta h_a{}^\beta = 0\,, \tag{4.9}$$

where we used eqs. (3.3), (3.21) and (3.25). Likewise, eqs. (3.19) and (3.20) together give

$$(1+D)\Omega_{\beta c}{}^d = (\theta\mathcal{R}_c{}^{defgl})_\beta H_{efgl} - (\theta\gamma^a)_\beta\Omega_{ac}{}^d = 0\,. \tag{4.10}$$

The θ-expansions of the superfields can now be obtained by recursively solving the equations above. The first step is to input the zeroth order in θ for the fields C_{abc}, $h_a{}^\alpha$ and h_{ab},

$$C_{abc} = c_{abc} + \mathcal{O}(\theta)\,, \qquad h_a{}^\alpha = -\Psi_a^\alpha + \mathcal{O}(\theta)\,, \qquad h_{ab} = -\epsilon_{ab} + \mathcal{O}(\theta)\,, \tag{4.11}$$

then eqs. (4.5) and (4.9) can be used to find the $\mathcal{O}(\theta)$ terms in C_{abc} and h_{ab} respectively. Similarly, eq. (4.9) gives the θ^n term of $h_a{}^\alpha$ in terms of θ^{n-1} terms in C_{abc} and h_{ab}. From these three fields the θ-expansions of all other fields can be determined, as depicted in figure 1. The interpretation of the figure is as follows: the θ^{n+1} terms in $\Omega_{\alpha ab}$ are determined by θ^n of C_{abc} and h_{ab}. Then the newly determined $\Omega_{\alpha ab}$ and $h_a{}^\alpha$ give the components of $h_\alpha{}^\beta$. Similarly, the components of $h_\alpha{}^a$ and $C_{\alpha ab}$ are obtained from $h_\alpha{}^\beta$, h_{ab}, and C_{abc}. Finally, $C_{\alpha ab}$ and $h_\alpha{}^a$ determine $C_{\alpha\beta a}$, which gives the superfield expansion of $C_{\alpha\beta\gamma}$. We will see this explicitly in the next section where we give the recursive equations and display the superfield expansions.

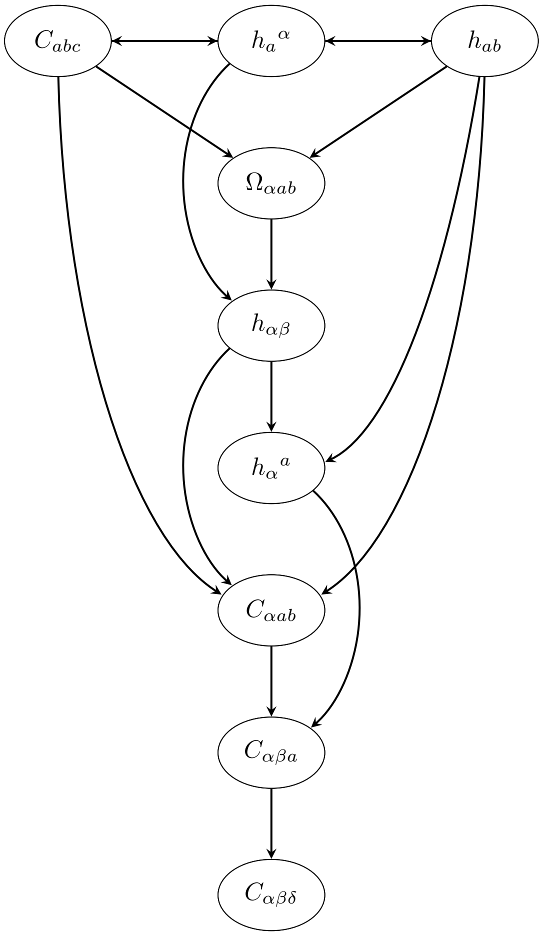

Figure 1. Schematic representation of the equations of motion contracted with θ^α. Arrows indicate θ-expansion dependency, for example, the arrow pointing from C_{abc} to $\Omega_{\alpha ab}$ indicates that components of order θ^n in the former contribute to components of order θ^{n+1} in the latter.

4.2 Example expansions

Here we provide the superfield expansion of $h_a{}^b$ and $h_a{}^\alpha$, which will be important in the next section where we provide a covariant vertex operator in 11D. In addition, we provide the ghost number three vertex operator for the pure spinor superparticle up to θ^5. Starting with (4.11) we can obtain the first order in θ of the higher-dimensional component of the C-field using eq. (4.5). For higher orders in θ, the D-operator just becomes a (non-zero) multiplicative factor and so it can be inverted. Then (4.8), (4.9) and (4.5) can be solved, giving the initial set of recursion relations

$$h_a{}^\beta|_{\theta^n} = -\frac{4}{n}\theta^\alpha(\mathcal{T}_a{}^{bcde})_\alpha{}^\beta\partial_b C_{cde}|_{\theta^{n-1}} - \frac{1}{4n}(\theta\gamma^{bc})^\beta\Omega_{abc}|_{\theta^{n-1}}\,, \tag{4.12}$$

$$h_a{}^b|_{\theta^n} = -\frac{1}{n}(\theta\gamma^b)_\beta h_a{}^\beta|_{\theta^{n-1}}\,, \tag{4.13}$$

$$C_{abc}|_{\theta^n} = \frac{3}{n}\theta^\alpha(\gamma_{[ab})_{\alpha\beta}h_{c]}{}^\beta|_{\theta^{n-1}}\,. \tag{4.14}$$

The expansions of the remaining fields are then obtained from these initial three, using the remaining equations of motion. In order of dependence, they are

$$\Omega_{\alpha c}{}^{d}\big|_{\theta^n} = \frac{4}{n+1}(\theta\mathcal{R}_c{}^{defgl})_\beta \partial_e C_{fgl}\big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^a)_\alpha \Omega_{ac}{}^{d}\big|_{\theta^{n-1}}\,, \tag{4.15}$$

$$h_\beta{}^\delta\big|_{\theta^n} = \frac{1}{4(n+1)}(\theta\gamma^{bc})^\delta \Omega_{\beta bc}\big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^a)_\beta h_a{}^\delta\big|_{\theta^{n-1}}\,, \tag{4.16}$$

$$h_\beta{}^a\big|_{\theta^n} = \frac{1}{n+1}(\theta\gamma^a)_\delta h_\beta{}^\delta\big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^b)_\beta h_b{}^a\big|_{\theta^{n-1}}\,, \tag{4.17}$$

$$C_{\beta ab}\big|_{\theta^n} = \frac{2}{n+1}(\theta\gamma_{[b}{}^c)_\beta h_{a]c}\big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma^c)_\beta C_{cab}\big|_{\theta^{n-1}} - \frac{1}{n+1}(\theta\gamma_{ab})_\delta h_\beta{}^\delta\big|_{\theta^{n-1}} \tag{4.18}$$

$$C_{a\beta\delta}\big|_{\theta^n} = \frac{2}{n+2}(\theta\gamma^b)_{(\beta}C_{\delta)ba}\big|_{\theta^{n-1}} + \frac{2}{n+2}(\theta\gamma_{ab})_{(\beta}h_{\delta)}{}^b\big|_{\theta^{n-1}}\,, \tag{4.19}$$

$$C_{\beta\delta\epsilon}\big|_{\theta^n} = -\frac{3}{n+3}(\theta\gamma^a)_{(\beta}C_{\delta\epsilon)a}\big|_{\theta^{n-1}}\,. \tag{4.20}$$

We remind the reader that the tensors $\mathcal{T}$ and $\mathcal{R}$ can be found in eqs. (3.22) and (3.24). To simplify the expansions we replace $\partial_a \to k_a$, and introduce the Schoonschip notation, where vectors contracted with a tensor appear as indices. For example, $\gamma^a k_a \to \gamma^k$. Keeping terms up to θ^4 in h_{ab}, we find the superfield expansion

$$h^{a_1a_2}\big|_{\theta^0} = -\,\epsilon^{a_1a_2} \tag{4.21}$$

$$h^{a_1a_2}\big|_{\theta^1} = +\,(\theta\gamma^{a_2}\Psi^{a_1}) \tag{4.22}$$

$$\begin{aligned} h^{a_1a_2}\big|_{\theta^2} = &+\frac{1}{4}(\theta\gamma^{a_2b_1k}\theta)\epsilon^{a_1b_1} \\ &-\frac{1}{2}(\theta\mathcal{T}^{a_1b_1b_2b_3b_4}\gamma^{a_2}\theta)h^{b_1b_2b_3b_4} \end{aligned} \tag{4.23}$$

$$\begin{aligned} h^{a_1a_2}\big|_{\theta^3} = &-\frac{1}{24}(\theta\gamma^{a_2b_1k}\theta)(\theta\gamma^{a_1}\Psi^{b_1}) \\ &-\frac{1}{24}(\theta\gamma^{a_2b_1k}\theta)(\theta\gamma^{b_1}\Psi^{a_1}) \\ &+\frac{1}{24}(\theta\gamma^{a_2b_1b_2}\theta)(\theta\gamma^{b_1}\Psi^{b_2})k^{a_1} \\ &-2(\theta\mathcal{T}^{a_1kb_1b_2b_3}\gamma^{a_2}\theta)(\theta\gamma^{b_1b_2}\Psi^{b_3}) \end{aligned} \tag{4.24}$$

$$\begin{aligned} h^{a_1a_2}\big|_{\theta^4} = &-\frac{1}{192}(\theta\gamma^{a_1b_1k}\theta)(\theta\gamma^{a_2b_2k}\theta)\epsilon^{b_1b_2} \\ &-\frac{1}{192}(\theta\gamma^{a_2b_1k}\theta)(\theta\gamma^{b_1b_2k}\theta)\epsilon^{a_1b_2} \\ &+\frac{1}{192}(\theta\gamma^{a_2b_1b_2}\theta)(\theta\gamma^{b_1b_3k}\theta)\epsilon^{b_2b_3}k^{a_1} \\ &-\frac{1}{4}(\theta\mathcal{T}^{a_1kb_1b_2b_3}\gamma^{a_2}\theta)(\theta\gamma^{b_1b_2b_4k}\theta)\epsilon^{b_3b_4} \\ &-\frac{1}{2}(\theta\mathcal{T}^{a_1kb_1b_2b_3}\gamma^{a_2}\theta)(\theta\mathcal{T}^{b_1b_4b_5b_6b_7}\gamma^{b_2b_3}\theta)h^{b_4b_5b_6b_7} \\ &+\frac{1}{96}(\theta\mathcal{T}^{a_1b_1b_2b_3b_4}\gamma^{b_5}\theta)(\theta\gamma^{a_2b_5k}\theta)h^{b_1b_2b_3b_4} \\ &+\frac{1}{96}(\theta\mathcal{T}^{b_1b_2b_3b_4b_5}\gamma^{a_1}\theta)(\theta\gamma^{a_2b_1k}\theta)h^{b_2b_3b_4b_5} \\ &+\frac{1}{96}(\theta\mathcal{T}^{b_1b_2b_3b_4b_5}\gamma^{b_6}\theta)(\theta\gamma^{a_2b_1b_6}\theta)h^{b_2b_3b_4b_5}k^{a_1} \\ &+\mathcal{O}(\theta^5)\,, \end{aligned} \tag{4.25}$$

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and keeping terms up to θ^3 for $h_a{}^\alpha$ we have

$$h_{a_1}^{\ \alpha}\big|_{\theta^0} = -\psi_a^\alpha \tag{4.26}$$

$$\begin{aligned} h_{a_1}^{\ \alpha}\big|_{\theta^1} = &+\frac{1}{2}(\gamma^{b_1k}\theta)^\alpha\epsilon^{a_1b_1} \\ &-(\theta\mathcal{T}^{a_1b_1b_2b_3b_4})^\alpha h^{b_1b_2b_3b_4} \end{aligned} \tag{4.27}$$

$$\begin{aligned} h_{a_1}^{\ \alpha}\big|_{\theta^2} = &-\frac{1}{8}(\gamma^{b_1k}\theta)^\alpha(\theta\gamma^{a_1}\Psi^{b_1}) \\ &-\frac{1}{8}(\gamma^{b_1k}\theta)^\alpha(\theta\gamma^{b_1}\Psi^{a_1}) \\ &+\frac{1}{8}(\gamma^{b_1b_2}\theta)^\alpha(\theta\gamma^{b_1}\Psi^{b_2})k^{a_1} \\ &-6(\theta\mathcal{T}^{a_1kb_1b_2b_3})^\alpha(\theta\gamma^{b_1b_2}\Psi^{b_3}) \end{aligned} \tag{4.28}$$

$$\begin{aligned} h_{a_1}^{\ \alpha}\big|_{\theta^3} = &-\frac{1}{48}(\gamma^{b_1k}\theta)^\alpha(\theta\gamma^{a_1b_2k}\theta)\epsilon^{b_1b_2} \\ &-\frac{1}{48}(\gamma^{b_1k}\theta)^\alpha(\theta\gamma^{b_1b_2k}\theta)\epsilon^{a_1b_2} \\ &+\frac{1}{48}(\gamma^{b_1b_2}\theta)^\alpha(\theta\gamma^{b_1b_3k}\theta)\epsilon^{b_2b_3}k^{a_1} \\ &+\frac{1}{24}(\gamma^{b_1k}\theta)^\alpha(\theta\mathcal{T}^{a_1b_2b_3b_4b_5}\gamma^{b_1}\theta)h^{b_2b_3b_4b_5} \\ &+\frac{1}{24}(\gamma^{b_1k}\theta)^\alpha(\theta\mathcal{T}^{b_1b_2b_3b_4b_5}\gamma^{a_1}\theta)h^{b_2b_3b_4b_5} \\ &+\frac{1}{24}(\gamma^{b_1b_2}\theta)^\alpha(\theta\mathcal{T}^{b_1b_3b_4b_5b_6}\gamma^{b_2}\theta)h^{b_3b_4b_5b_6}k^{a_1} \\ &-(\theta\mathcal{T}^{a_1kb_1b_2b_3})^\alpha(\theta\gamma^{b_1b_2b_4k}\theta)\epsilon^{b_3b_4} \\ &-2(\theta\mathcal{T}^{a_1kb_1b_2b_3})^\alpha(\theta\mathcal{T}^{b_1b_4b_5b_6b_7}\gamma^{b_2b_3}\theta)h^{b_4b_5b_6b_7} \\ &+\mathcal{O}(\theta^4)\,. \end{aligned} \tag{4.29}$$

In addition we present the θ-expansion of the ghost number three vertex operator $U^{(3)} = \lambda^\alpha\lambda^\beta\lambda^\gamma C_{\alpha\beta\gamma}$, which describes the physical fields of 11D supergravity in the cohomology of the pure spinor BRST charge [10]. Up to θ^5, it takes the form

$$\begin{aligned} C_{\lambda\lambda\lambda}\big|_{\theta^3} = &-\frac{3}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)\epsilon^{b_2b_3} \\ &-\frac{1}{8}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)c^{b_1b_2b_3} \end{aligned} \tag{4.30}$$

$$\begin{aligned} C_{\lambda\lambda\lambda}\big|_{\theta^4} = &+\frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\gamma^{b_3}\Psi^{b_2}) \\ &-\frac{1}{5}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\gamma^{b_1b_2}\Psi^{b_3}) \end{aligned} \tag{4.31}$$

$$\begin{aligned} C_{\lambda\lambda\lambda}\big|_{\theta^5} = &+\frac{1}{32}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\gamma^{b_3b_4k}\theta)\epsilon^{b_2b_4} \\ &-\frac{1}{32}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\gamma^{b_1b_2b_4k}\theta)\epsilon^{b_3b_4} \\ &-\frac{11}{192}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_1b_3}\theta)(\theta\mathcal{T}^{b_2b_4b_5b_6b_7}\gamma^{b_3}\theta)h^{b_4b_5b_6b_7} \\ &-\frac{11}{192}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\gamma^{b_3}\theta)(\theta\mathcal{T}^{b_1b_4b_5b_6b_7}\gamma^{b_2b_3}\theta)h^{b_4b_5b_6b_7} \end{aligned}$$

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$$
\begin{aligned}
&-\frac{1}{1536}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\mathcal{R}^{b_1b_2b_3b_4b_5b_6}\theta)(\theta\theta)h^{b_3b_4b_5b_6}\\
&+\frac{1}{3072}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_1b_2}\theta)(\lambda\mathcal{R}^{b_3b_4b_5b_6b_7b_8}\theta)(\theta\gamma^{b_2b_3b_4}\theta)h^{b_5b_6b_7b_8}\\
&+\frac{1}{3072}(\lambda\gamma^{b_1}\theta)(\lambda\gamma^{b_2}\theta)(\lambda\mathcal{R}^{b_3b_4b_5b_6b_7b_8}\theta)(\theta\gamma^{b_1b_2b_3b_4}\theta)h^{b_5b_6b_7b_8}\\
&+\mathcal{O}(\theta^6)\,.
\end{aligned}
\tag{4.32}
$$

The definitinos of the tensors $\mathcal{T}$ and $\mathcal{R}$ can be found in (3.22) and (3.24). In these expansions we introduced the four-form field strength,

$$
h_{abcd} = 4\partial_{[a}c_{bcd]}\,, \tag{4.33}
$$

which makes gauge invariance under $c_{abc} \to \partial_{[a}\omega_{bc]}$ manifest. In the supplementary material attached to this paper we include expansions of all the fields up to θ^5, both as they appear above and also after expanding the tensors $\mathcal{T}$, $\mathcal{R}$, and all resulting gamma matrix products.[2] These were obtained by implementing the recursion relations in FORM [25] and manipulating the gamma matrix products using routines presented in [26].

5 A covariant vertex operator for 11D supergravity

This section introduces, for the first time, a covariant vertex operator for 11D supergravity in ordinary superspace. To this end, we first construct the 11D analogue of the $\mathcal{ABC}$ superparticle [17].

5.1 The 11D $\mathcal{ABC}$ superparticle

It is well-known that the 11D superparticle possesses first- and second-class constraints, which cannot be easily separated out in a manifestly Lorentz covariant manner. As will be shown below (see appendix B for the 10D analogue), one can overcome this difficulty by writing an alternative fully first-order framework, subject to a specific set of first-class constraints. The resulting theory, which we will refer to as the 11D $\mathcal{ABC}$ superparticle, will then be shown to be physically equivalent to the original 11D superparticle [20].

The 11D $\mathcal{ABC}$ superparticle action will be defined as

$$
S = \int d\tau[P^a\partial_\tau X_a + p_\alpha\partial_\tau\theta^\alpha + \rho\mathcal{A} + \xi_\alpha\mathcal{B}^\alpha + \iota^{\alpha\beta}\mathcal{C}_{\alpha\beta}] \tag{5.1}
$$

where ρ, ξ_α, $\iota^{\alpha\beta}$ are the Lagrange multipliers associated to the constraints

$$
\mathcal{A} = P^aP_a\,, \qquad \mathcal{B}^\alpha = (\gamma^a d)^\alpha P_a\,, \qquad \mathcal{C}_{\alpha\beta} = d_{[\alpha}d_{\beta]} \tag{5.2}
$$

and d_α is defined as in section 2. The only non-zero (anti)commutators describing the constraint algebra are given by

$$
\{\mathcal{B}^\alpha,\mathcal{B}^\beta\} = -(\gamma^a)^{\alpha\beta}P_a\mathcal{A}\,, \quad [\mathcal{C}_{\alpha\beta},\mathcal{C}_{\delta\epsilon}] = -4(\gamma^a)_{\underline{\beta}\bar{\delta}}P_a\mathcal{C}_{\underline{\alpha}\bar{\epsilon}}\,, \quad [\mathcal{B}^\alpha,\mathcal{C}_{\beta\delta}] = 2\delta^\alpha_{[\delta}d_{\beta]}\mathcal{A} \tag{5.3}
$$

[2] Since the 3-point function in 11D pure spinor superspace involves the superfields $\Phi^a = \lambda^\alpha h_\alpha{}^a$ and $U^{(3)}$, see [11, 23, 24], we provide the θ-expansion of $U^{(3)}$ up to θ^7.

where we are using barred and underlined letters to denote index antisymmetrization. In order to show that the physical degrees of freedom of the model (5.1) match those of the 11D superparticle, we need to define the so-called light-cone gauge.

5.2 Light-cone gauge

We begin by defining the light-cone directions

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{10})\,, \tag{5.4}$$

and transverse directions are denoted by i, j, l. We take gamma matrices to be represented by

$$\gamma^i_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i_{A\dot{A}} \\ -\sigma^i_{\dot{B}B} & 0 \end{pmatrix}, \quad \gamma^+_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -i\sqrt{2}I_{\dot{B}\dot{A}} \end{pmatrix}, \quad \gamma^-_{\alpha\beta} = \begin{pmatrix} -i\sqrt{2}I_{AB} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.5}$$

where σ^i are SO(9) Pauli matrices and $A, \dot{A}$ are SO(9) spinor indices. Although we use dotted index notation as in 10D, in 11D these indices can be contracted using the charge conjugation matrix,

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -I_{\dot{A}A} \\ I_{B\dot{B}} & 0 \end{pmatrix}. \tag{5.6}$$

As usual, one can use the constraint $\mathcal{A} = P^a P_a$ to fix X^+, and to determine P^-,

$$X^+ = x^+ + \tau P^+\,, \qquad P^- = \frac{P^i P^i}{2P^+}\,. \tag{5.7}$$

Moreover, one can use the constraint $\mathcal{B}^\alpha$ to fix one of the SO(9) components of θ^α and d_α

$$(\gamma^+\theta)_\alpha = 0\,, \qquad (\gamma^- d)^\alpha = \frac{1}{P^+}[-(\gamma^+ d)^\alpha P^- + (\gamma^i d)^\alpha P_i]\,. \tag{5.8}$$

The remaining variables are then given by $(X^-, X^i, P^+, P^i, \theta^A, p_B)$. For convenience, instead of (θ^A, p_B) we will use the pair of variables (d^A, q_B), where $q_\alpha = p_\alpha + \frac{1}{2}(\gamma^a\theta)_\alpha P_a$ is the supersymmetric charge. It is not hard to check that,

$$\{d_\alpha, d_\beta\} = -(\gamma^a)_{\alpha\beta} P_a\,, \qquad \{d_\alpha, q_\beta\} = 0\,, \qquad \{q_\alpha, q_\beta\} = (\gamma^a)_{\alpha\beta} P_a\,, \tag{5.9}$$

which implies that

$$\{d_A, d_B\} = -\delta_{AB}\,, \qquad \{d_A, q_B\} = 0\,, \qquad \{q_A, q_B\} = \delta_{AB}\,, \tag{5.10}$$

where we used the redefinition $d_A \to \sqrt{\sqrt{2}iP^+} d_A$. In this manner, the constraint $\mathcal{C}_{\alpha\beta}$ requires that d_A obeys the following gauge transformation,

$$\delta d_A = m_A{}^B d_B\,, \tag{5.11}$$

where m_{AB} is a completely antisymmetric matrix. The constraint $\mathcal{C}_{\alpha\beta} = 0$ requires that $d_A = \chi y_A$, where χ is a fermionic constant, and y_A is a bosonic SO(9) spinor, and so eq. (5.11) allows us to set $d_A = (d_1, 0, \ldots, 0)$. Using that $\mathcal{C}_{\alpha\beta}$ is invariant under $d_\alpha \to -d_\alpha$, one can fix the eigenvalue of $d_1^2 = -1$, and therefore the only dynamical variables are defined by $(X^-, X^i, P^+, P^i, q_A)$. This is exactly the same number and type of variables describing the light-cone gauge 11D superparticle [20]. Indeed, the Hilbert space is spanned by the vector space realizing the q-algebra in (5.10), i.e. it is described by $2^{\frac{16}{2}} = 256$ states, the number of the 11D supergravity physical states.

5.3 Vertex operator

The covariant vertex operator will be made out of supersymmetric quantities, and the linearized 11D supergravity superfields studied in previous sections. As in 10D, see appendix B for a short review, after adequately imposing the light-cone gauge conditions and solving the $\mathcal{C}$-constraint of (5.2), the covariant operator will be shown to coincide with the Green-Gutperle-Kwon vertices from the 11D superparticle [20].

Concretely, we define the 11D covariant vertex operator as

$$V = P^a P^b h_{ab} + P^a h_a{}^\alpha d_\alpha\,, \tag{5.12}$$

where h_{ab}, $h_a{}^\alpha$ are the linearized superfields of section 3. This vertex is the 11D analogue of the Siegel vertex operator for 10D super-Yang-Mills [17], see eq. B.2.

In addition to the constraints fixed for the world-line variables in the previous section, we also need to gauge fix the physical fields. We take light-cone gauge fixing conditions

$$\epsilon_a{}^+ = 0\,, \qquad \Psi^{\alpha+} = 0\,, \qquad c_{ab}{}^+ = 0\,. \tag{5.13}$$

We assume that the momentum carried by a physical state satisfies $k^+ = 0$, and that k^- is non-infinite. This means that the k^i components need to be complex, in order to maintain the massless condition $k^2 = 0$. Using residual gauge freedom and the conditions in (5.13) we can further fix

$$\epsilon_{ij} k^i = 0\,, \qquad \epsilon_i{}^i = 0\,. \tag{5.14}$$

For the three form c_{abc} we have simply

$$c_{iab} k^i = 0\,. \tag{5.15}$$

Additional constraints can be imposed on the gravitino but for this section we only focus on the bosonic sector, so we ignore it from now.

To proceed with the comparison to [20] we sectorize the vertex operator into parts containing ϵ^{--}, ϵ^{-i}, ϵ^{ij}, $c_{ij}{}^-$ and c_{ijl}. Keeping only terms with ϵ^{--} and inserting the superfield expansions of the previous section into the vertex operator (5.12) we find that all terms containing θ vanish due to the light-cone gauge conditions and the fact that the one-form γ^a and two-form γ^{ab} are symmetric in their spinor indices. For example, we have

$$(\theta\gamma^{a+b}\theta) = \frac{1}{3}(\theta\gamma^a\gamma^+\gamma^b\theta) = 0\,, \tag{5.16}$$

and similarly

$$(\theta\gamma^{a+bc}\theta) = 0\,. \tag{5.17}$$

Due to this, the vertex operator for ϵ^{--} becomes

$$V\big|_{\epsilon^{--}} = -P^+ P^+ \epsilon^{--}\,. \tag{5.18}$$

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For the ϵ^{i-} components of the graviton, one also needs to make use of

$$(\theta\gamma^{i_1\dots i_n}\theta) \propto (\theta\gamma^{i_1\dots i_n}\gamma^+\gamma^-\theta) = 0\,, \tag{5.19}$$

as well as the substitution

$$\theta^\alpha = \frac{1}{2P^+}(\gamma^+)^{\alpha\beta}(q_\beta - d_\beta)\,, \tag{5.20}$$

which are both valid in light-cone gauge. The vertex operator then becomes

$$\begin{aligned} V\big|_{\epsilon^{i-}} &= \epsilon^{i-}P_iP^+ - \frac{1}{16}(q\gamma^{+ik}q)\epsilon^{i-} + \frac{1}{16}(d\gamma^{+ik_1}d)\epsilon^{i-} \\ &= \epsilon^{i-}P_iP^+ - \frac{1}{16}(q\gamma^{+ik}q)\epsilon^{i-}\,. \end{aligned} \tag{5.21}$$

The second equality comes about because the term with two d_α vanishes due to the $C_{\alpha\beta}$ constraint. In fact, any term with more than one d_α will automatically vanish by using Fierz identities and the $C_{\alpha\beta}$ constraint in (5.2). Next, the term containing two q_α charges can be mapped to SO(9) rotation generators introduced in [20]. Defining

$$R^{ij} = \frac{1}{16P^+}(q\gamma^{+ij}q)\,, \tag{5.22}$$

they obey the algebra

$$[R^{ij}, R_{lk}] = \delta^{[i}_{[k}\delta^{j]}_{l]}\,, \tag{5.23}$$

and in terms of these generators the final form of the vertex operator is,

$$V\big|_{\epsilon^{i-}} = \epsilon^{i-}P_iP^+ - R^{ik}P^+\epsilon^{i-}\,. \tag{5.24}$$

For the transverse components ϵ_{ij} we use the identities (5.16), (5.17), (5.19), as well as the replacement for θ variables in terms of q and d with (5.20) to find the vertex operator

$$\begin{aligned} V\big|_{\epsilon^{ij}} &= -\epsilon_{ij}P^iP^j + \frac{1}{8}(q\gamma^{+ik}q)\epsilon_{ij}P^j(P^+)^{-1} \\ &\quad + \frac{1}{768}(q\gamma^{+aik}q)(q\gamma^{+ajk}q)\epsilon_{ij}(P^+)^{-2} - \frac{1}{768}(q\gamma^{+ik}q)(q\gamma^{+jk}q)\epsilon_{ij}(P^+)^{-2}\,. \end{aligned} \tag{5.25}$$

For SO(9) we have the Fierz identity

$$q_Aq_B = q^2\delta_{AB} + \frac{1}{32}(\sigma_{ij})_{AB}(q\sigma^{ij}q) + \frac{1}{96}(\sigma_{ijl})_{AB}(q\sigma^{ijl}q)\,. \tag{5.26}$$

It is not hard to show that this equation implies the identity

$$(q\gamma^{+(i|ja}q)(q\gamma^{+|l)jb}q)k_ak_bh_{il} = 5(q\gamma^{+(i|a}q)(q\gamma^{+|l)b}q)k_ak_bh_{il}\,. \tag{5.27}$$

To show this, recall that γ^+ is a projector, and that the expressions above are proportional to SO(9) spinor expressions, for example $(q\gamma^{+ijl}q) \propto (q\sigma^{ijl}q)$. Plugging eq. (5.27) in (5.25) then gives the result

$$\begin{aligned} V\big|_{\epsilon^{ij}} &= -\epsilon_{ij}P^iP^j + \frac{1}{8}(q\gamma^{+ik}q)\epsilon_{ij}P^j(P^+)^{-1} - \frac{1}{128}(q\gamma^{+ik}q)(q\gamma^{+jk}q)\epsilon_{ij}(P^+)^{-2}\,, \\ &= -\epsilon_{ij}P^iP^j + 2R^{ik}\epsilon_{ij}P^j - 2R^{ik}R^{jk}\epsilon_{ij}\,, \end{aligned} \tag{5.28}$$

which is in complete agreement with [20].

For the three form vertex operators we work along similar lines and find

$$V\big|_{c_{ij}{}^-} = \frac{1}{96}(q\gamma^{+ijl}q)H_{ijl}{}^- \,, \tag{5.29}$$

$$V\big|_{c_{ijl}} = H_{i_1i_2i_3i_4}\Big(P^{i_1} - \frac{1}{24}(q\gamma^{+i_1k}q)\Big)\frac{1}{96}(q\gamma^{+i_2i_3i_4}q)\,. \tag{5.30}$$

Identifying $R^{ijl} = \frac{1}{96}(q\gamma^{+ijl}q)$, these components of the vertex operator become

$$V\big|_{c_{ij}{}^-} = R^{ijl}H_{ijl}{}^- \,, \tag{5.31}$$

$$V\big|_{c_{ijl}} = H_{i_1i_2i_3i_4}\Big(P^{i_1} - \frac{2}{3}R^{i_1k}\Big)R^{i_2i_3i_4} \,, \tag{5.32}$$

which is once again in agreement with [20].

The analysis for the fermionic states immediately follows from supersymmetry arguments.

6 Discussions

In this work, we have found a compact and straightforward list of recursive relations, (4.12)–(4.20), which determine the superspace expansions of all the superfields describing linearized 11D supergravity, and which are relevant to the pure spinor formalism. These results possess a variety of applications including the computation of three-particle interactions with manifest supersymmetry, the construction of a new pure spinor twistor transform describing 11D supergravity along the lines of [27–29], the superspace expansion of multiparticle superfields relevant to perturbiner methods,[3] among others. We plan to explore these directions further in the near future. In particular, the 3-point correlator has been found in [24] from a field-theory perspective, and in [11] from a worldline approach. The component amplitudes should then immediately follow from an appropriate projection procedure, and the θ-expansions presented in this paper. We plan to tackle this problem in the near future, as well as to extend the state-of-the-art amplitude prescription, so that supergravity interactions involving an arbitrary number of external bosonic and fermionic states could directly be obtained from pure spinor superspace expressions.

Furthermore, we have introduced the so-called 11D $\mathcal{ABC}$ superparticle, and have shown that the respective covariant vertex operator (5.12) reduces to the Green-Gutperle-Kwon operators after imposing the light-cone gauge. This result also gives rise to several follow-up ideas. For instance, although the 10D $\mathcal{ABC}$ superparticle is equivalent to the Brink-Schwarz superparticle in light-cone gauge [31], its covariant quantization fails in describing 10D super-Yang-Mills [32]. One possible way of fixing this issue is by introducing an extra constraint, which defines the so-called first-ilk or $\mathcal{ABCD}$ superparticle. It is the BRST quantization of this model which reproduces the right physical spectrum [33]. It would be interesting to investigate if the same phenomenon occurs in 11D, as well as to explore the possible modifications one needs to make to the 11D $\mathcal{ABC}$ superparticle here proposed, so that the BRST treatment of the resulting model reproduces the physical degrees of freedom

[3] See [30] for the 10D analogue of this statement.

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of 11D supergravity. It is worthwhile mentioning that the infinite tower of ghosts in the BRST-closed vertex operator of the 10D $\mathcal{ABCD}$ superparticle can effectively be described by the pure spinor sector of the ghost number zero vertex operator in the pure spinor worldline formalism [34, 35]. This means that the computation of the BRST-closed operator in the 11D $\mathcal{ABCD}$ superparticle will provide extremely important information about the structure of the ghost sector in the pure spinor vertex operator of ghost number zero. We leave the study of this issue and related topics for future work.

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A Gamma matrix conventions

A.1 10D gamma matrices

We define the 16×16 gamma matrices,

$$\gamma^0_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

$$\gamma^9_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

$$\gamma^{i=1,\ldots,8}_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i \\ (\sigma^i)^T & 0 \end{pmatrix}, \tag{A.3}$$

where the superscript T refers to transposition, and the σ matrices are defined by,

$$\sigma^1 = \tau^2 \otimes \tau^2 \otimes \tau^2\,, \tag{A.4}$$

$$\sigma^2 = 1 \otimes \tau^1 \otimes \tau^2\,, \tag{A.5}$$

$$\sigma^3 = 1 \otimes \tau^3 \otimes \tau^2\,, \tag{A.6}$$

$$\sigma^4 = \tau^1 \otimes \tau^2 \otimes 1\,, \tag{A.7}$$

$$\sigma^5 = \tau^3 \otimes \tau^2 \otimes 1\,, \tag{A.8}$$

$$\sigma^6 = \tau^2 \otimes 1 \otimes \tau^1\,, \tag{A.9}$$

$$\sigma^7 = \tau^2 \otimes 1 \otimes \tau^3\,, \tag{A.10}$$

$$\sigma^8 = 1 \otimes 1 \otimes 1\,. \tag{A.11}$$

The τ matrices are partly rescaled Pauli matrices,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.12}$$

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A.2 11D gamma matrices

We abuse notation by referring to both the 11D and 10D gamma matrices with γ. In 11D we make the initial definition of the 32×32 gamma matrices,

$$\gamma^{10}_{\alpha\beta} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \tag{A.13}$$

$$\gamma^{i=0,\ldots,9}_{\alpha\beta} = \begin{pmatrix} i\gamma^i & 0 \\ 0 & -i\gamma^i \end{pmatrix} . \tag{A.14}$$

Next we redefine $\gamma^9 \leftrightarrow \gamma^{10}$ in order to have the light-cone directions defined as in eq. (5.4). Additionally in order to have the block form for $\gamma^{\pm}$, we rotate all gamma matrices by $\gamma \to R \cdot \gamma \cdot R^T$ where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{A.15}$$

Finally in 11D we have a charge conjugation matrix,

$$C^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{A.16}$$

Our conventions for products of gamma matrices keep expressions as similar as possible to 10D gamma matrix products. So our definitions of forms are, for example, $(\gamma^{ab})_\alpha{}^\beta = (\gamma^{[a})_{\alpha\delta}(\gamma^{b]})^{\delta\beta}$. At times we have to raise and lower spinor indices using the charge conjugation matrix. We always raise or lower the right-most index in gamma matrix products, for example

$$(\gamma^{ab})^{\alpha\beta} = C^{\beta\gamma}(\gamma^{ab})^\alpha{}_\gamma = -(\gamma^{ab}C)^{\alpha\beta} , \tag{A.17}$$

where in the last equality we used that the charge conjugation matrix is antisymmetric. Spinor products are written such that, if present, the charge conjugation is contracted in to the right-most spinor, so for instance

$$(\lambda\gamma^{ab}\lambda) = (\lambda\gamma^{ab}C\lambda) , \tag{A.18}$$

with ordinary matrix multiplication inside the parenthesis.

B The Siegel vertex operator for 10D super-Yang-Mills

In search for an alternative manifestly supersymmetric description of superstring theory free of the quantization problems presented by the Green-Schwarz superstring, Siegel proposed a completely first-order formulation for the particle-limit of the latter [17]. The worldline variables of this proposal consist of the coordinates (X^m, θ^α), and their respective conjugate momenta (P^m, p_α), subject to the constraints

$$\mathcal{A} = P^m P_m , \qquad \mathcal{B} = (\gamma^m d)_\alpha P_m , \qquad \mathcal{C}^{mnp} = (d\gamma^{mnp} d) , \tag{B.1}$$

where d_α is the familiar fermionic constraint of the Brink-Schwarz superparticle. Throughout this appendix we will use letters from the middle/beginning of the Latin/Greek alphabet to denote SO$(1,9)$ vector/spinor indices. This new superparticle model was later shown to correctly reproduce the massless states of the open superstring when quantized in light-cone gauge [31]. The super-Yang-Mills vertex operator was thus found to be described by

$$V = P^m A_m + d_\alpha W^\alpha \,. \tag{B.2}$$

The objects A_m, W^α in (B.2) are the familiar 10D super-Yang-Mills superfields associated to the gluon and gluino states, respectively. They satisfy the superspace equations of motion

$$\begin{aligned} D_\alpha A_\beta + D_\beta A_\alpha &= (\gamma^m)_{\alpha\beta} A_m \,, & D_\alpha A_m &= \partial_m A_\alpha + (\gamma_m W)_\alpha \,, \\ D_\alpha W^\beta &= -\frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn} \,, & D_\alpha F_{mn} &= 2(\gamma_{[m}\partial_{n]} W)_\alpha \,, \end{aligned} \tag{B.3}$$

where A_α is the lowest-dimensional component of the super-gauge connection, and F_{mn} is the field-strength superfield. After fixing the Harnad-Shnider gauge $\theta^\alpha A_\alpha = 0$, these equations provide a solvable system of recursive relations which yield the superspace expansion coefficients of all the 10D super-Yang-Mills superfields, at all order in θ [36, 37].

Next we study the vertex operator (B.2) in light-cone gauge, and show it matches the light-cone gauge operators of the Brink-Schwarz superparticle.

B.1 Light-cone gauge

The light-cone gauge conditions on the worldline fields read

$$X^+ = x_0^+ + P^+\tau \,, \tag{B.4}$$

$$(\gamma^+\theta)_\alpha = 0 \,. \tag{B.5}$$

Using the SO(8) splitting $\theta^\alpha = (\theta^a, \bar\theta^{\dot a})$, where a, $\dot a$ are respectively SO(8) chiral and antichiral spinor indices, one can write eq. (B.5) in the equivalent form $\bar\theta^{\dot a} = 0$. This is easily seen to be the case in the basis where the gamma matrices are represented as in appendix A.

The supersymmetric derivative and charge are denoted by d_α and q_α respectively, and defined as

$$d_\alpha = p_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha P_m \,, \tag{B.6}$$

$$q_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha P_m \,. \tag{B.7}$$

Their forms in light-cone gauge are given by

$$\bar d_{\dot a} = \bar p_{\dot a} + \frac{1}{2}(\sigma^i)_{\dot a a}\theta^a P_i \,, \qquad d_a = p_a - \frac{\sqrt{2}}{2}\theta_a P^+ \,, \tag{B.8}$$

$$\bar q_{\dot a} = \bar p_{\dot a} - \frac{1}{2}(\sigma^i)_{\dot a a}\theta^a P_i \,, \qquad q_a = p_a + \frac{\sqrt{2}}{2}\theta_a P^+ \,, \tag{B.9}$$

where $d_\alpha = (d_a, \bar{d}_{\dot{a}})$, $q_\alpha = (q_a, \bar{q}_{\dot{a}})$ and we use i, j, k for SO(8) vector indices. Likewise, the non-vanishing components of the $\mathcal{C}$-constraint in the light-cone frame read

$$\mathcal{C}^{+ij} = \sqrt{2}\, d_a (\sigma^{ij})^{ab} d_b \,, \qquad \mathcal{C}^{ijk} = 2\, d_a (\sigma^{ijk})^{a\dot{a}} \bar{d}_{\dot{a}} \,. \tag{B.10}$$

Moreover, the SO(8) spinor variables S^a of the light-cone gauge Brink-Schwarz superparticle, are related to the supersymmetric charge via

$$S^a = \frac{q^a}{\sqrt{\sqrt{2}P^+}} \,. \tag{B.11}$$

As usual, we will assume $k^+ \to 0$. In order for the component k^- to remain finite, one must have $k^i k^i \to 0$. This means that the momentum will be taken to be complex, and be restricted to take real values again in our final formulae. The use of this configuration and the gauge symmetry $\delta\epsilon_m = \partial_m \lambda$, allows one to set $\epsilon^+ \to 0$. Similarly, the transversality condition requires $\epsilon^i k^i \to 0$, and thus ϵ^- is finite. Analogously, the equation of motion of the gluino imposes that its SO(8) components are related to each other. All in all, one has

$$k^+ = 0 \,, \quad k^- = \frac{k^i k^i}{2k^+} \,, \epsilon^+ = 0 \,, \quad \epsilon^- = \frac{\epsilon^i k^i}{k^+} \,, \quad \chi_a = -\frac{1}{\sqrt{2}} (\sigma^i)_{a\dot{a}} k_i \bar{\xi}^{\dot{a}} \,, \quad \bar{\xi}^{\dot{a}} = \frac{\bar{\chi}^{\dot{a}}}{k^+} \,, \tag{B.12}$$

where $\chi^\alpha = (\chi^a, \bar{\chi}^{\dot{a}})$ is the gluino field.

Let us now analyze the covariant vertex operator (B.2) in light-cone gauge. To this end, we first list the θ-expansions of the superfields A_m, W^α (see [36, 37]),

$$\begin{aligned} A_m &= \epsilon_m - (\chi \gamma_m \theta) - \frac{1}{8} (\theta \gamma_m \gamma^{pq} \theta) f_{pq} + \frac{1}{12} (\theta \gamma_m \gamma^{pq} \theta)(\partial_p \chi \gamma_q \theta) \\ &\quad + \frac{1}{192} (\theta \gamma_{mrs} \theta)(\theta \gamma^{spq} \theta) \partial^r f_{pq} + O(\theta^5) \,, \end{aligned} \tag{B.13}$$

$$\begin{aligned} W^\alpha &= \chi^\alpha - \frac{1}{4} (\gamma^{mn} \theta)^\alpha f_{mn} + \frac{1}{4} (\gamma^{mn} \theta)^\alpha (\partial_m \chi \gamma_n \theta) + \frac{1}{48} (\gamma^{mn} \theta)^\alpha (\theta \gamma_n \gamma^{pq} \theta) \partial_m f_{pq} \\ &\quad - \frac{1}{96} (\gamma^{mn} \theta)^\alpha (\theta \gamma_n \gamma^{pq} \theta)(\partial_m \partial_p \chi \gamma_q \theta) + O(\theta^5) \,. \end{aligned} \tag{B.14}$$

It is not hard to see that $A^+ = 0$ in light-cone gauge. Therefore, the vertex operator (B.2) can be written as

$$V = -P^+ A^- + P^i A^i + d_a W^a + \bar{d}_{\dot{a}} \bar{W}^{\dot{a}} \,. \tag{B.15}$$

For simplicity, let us focus on the bosonic sector. The fermionic counterpart directly follows from supersymmetry. Using eqs. (B.13), (B.14), one finds that all the terms of the expansions vanish except for those linear and quadratic in θ^α. Explicitly,

$$\begin{aligned} V &= -P^+ \left[\epsilon^- + \frac{\sqrt{2}}{4} (\theta \sigma^{ij} \theta) k_i \epsilon_j \right] + P^i \epsilon^i + d_a \left[-\frac{1}{2} (\sigma^{ij} \theta)^a k_i \epsilon_j \right] \\ &= -P^+ \epsilon^- + P^i \epsilon^i - \frac{\sqrt{2}}{4} (\theta \sigma^{ij} \theta) k_i \epsilon_j P^+ - \frac{1}{2} (d \sigma^{ij} \theta) k_i \epsilon_j \,. \end{aligned} \tag{B.16}$$

The relations (B.8), (B.9) define θ^a by the simple formula

$$\theta^a = \frac{1}{\sqrt{2}P^+}(q^a - d^a)\,. \tag{B.17}$$

After plugging (B.17) into (B.16), and using (B.10), one is left with

$$\begin{aligned} V &= -P^+\epsilon^- + P^i\epsilon^i - \frac{\sqrt{2}}{8P^+}(q\sigma^{ij}q)k_i\epsilon_j + \frac{\sqrt{2}}{4P^+}(q\sigma^{ij}d)k_i\epsilon_j - \frac{1}{2\sqrt{2}P^+}(d\sigma^{ij}q)k_i\epsilon_j \\ &= -P^+\epsilon^- + P^i\epsilon^i - \frac{\sqrt{2}}{8P^+}(q\sigma^{ij}q)k_i\epsilon_j\,. \end{aligned} \tag{B.18}$$

Finally, the use of eq. (B.11) allows one to conclude that

$$V = -P^+\epsilon^- + P^i\epsilon^i - \frac{1}{4}(S\sigma^{ij}S)k_i\epsilon_j\,, \tag{B.19}$$

which is exactly the gluon vertex operator in the light-cone gauge Brink-Schwarz worldline framework.

C Comparison with literature

Here we compare our recursion relations with those in [16]. Starting from eq. (4.14) we can multiply both sides by 4 and take the exterior derivative giving

$$H_{abcd}|_{\theta^n} = \frac{6}{n}\theta^\alpha(\gamma_{[ab})_{\alpha\beta}T_{cd]}{}^\beta|_{\theta^{n-1}}\,, \tag{C.1}$$

where we used (3.16), $2\partial_{[a}h_{b]}{}^\alpha - \partial_b h_a{}^\alpha = T_{ab}{}^\alpha$ in the right-hand side. This agrees with the first part of eq. (53) in ref. [16], after making the redefinitions $T_{ab}{}^\alpha \to -T_{ab}{}^\alpha$, $H_{ab\alpha\beta} \to -i(\gamma_{ab})_{\alpha\beta}$. Taking an exterior derivative of (4.12) we now find

$$T_{ab}{}^\beta|_{\theta^n} = \frac{2}{n}\theta^\alpha(\mathcal{T}_{[a}{}^{cdef})_\alpha{}^\beta\partial_{b]}H_{cdef}|_{\theta^{n-1}} - \frac{1}{4n}(\theta\gamma^{cd})^\beta R_{abcd}|_{\theta^{n-1}}\,, \tag{C.2}$$

where $R_{abcd} = 2\partial_{[a}\Omega_{b]cd}$ to linearized level. Using the fact our $\mathcal{T}$ only differs by a sign from that in [16], eq. (C.2) agrees with the second part of eq. (53) in [16], after redefining $T_{ab}{}^\alpha \to -T_{ab}{}^\alpha$. Finally by taking two derivatives of (4.13) and (anti)symmetrizing to construct a linearized Riemann tensor we find

$$R_{abcd}|_{\theta^n} = -\frac{2}{n}(\theta\mathcal{S}_{[a|cd]}{}^{ef}\partial_{|b]}T_{ef})|_{\theta^{n-1}}\,, \tag{C.3}$$

where $\mathcal{S}_{bcd}{}^{ef} = (\gamma_b\delta_c^{[e}\delta_d^{f]} + \gamma_c\delta_b^{[e}\delta_d^{f]} - \gamma_d\delta_b^{[e}\delta_c^{f]})/2$, once again in agreement with [16] after redefining the torsion, and realizing that our $\mathcal{S}$ differs from that in [16] by a factor of i.

Since the recursion relations for our C fields do not appear similar to those in [16], we compare some components of the superfield expansions. We begin by recalling the definition,

$$C_{\alpha\beta\gamma} = (-1)^{N(P+Q)+P(Q+1)}\hat{E}_\alpha{}^N\hat{E}_\beta{}^P\hat{E}_\gamma{}^Q F_{NPQ}\Big|_{\text{lin}}\,, \tag{C.4}$$

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where the hatted fields are zeroth order in the fields, the F field is linear in the perturbations, and the indices in the exponent are 1 if the index is spinorial and zero otherwise.

At order θ^3 the graviton terms of the ghost-number three vertex operator $U^{(3)} = \lambda^\alpha\lambda^\beta\lambda^\gamma C_{\alpha\beta\gamma}$ are

$$\lambda^\mu\lambda^\nu\lambda^\alpha \hat{E}_\alpha^{(1)s} F^{(2)}_{\mu\nu s} = \frac{i}{8}(\lambda\gamma^a\theta)(\lambda\gamma_{bc}\theta)(\lambda\gamma^c\theta)h^b_a, \tag{C.5}$$

$$\lambda^\sigma\lambda^\alpha \hat{E}_\alpha^{(1)m}\lambda^\beta \hat{E}_\beta^{(1)n} F^{(1)}_{\sigma mn} = -\frac{i}{4}(\lambda\gamma^a\theta)(\lambda\gamma_{bc}\theta)(\lambda\gamma^c\theta)h^b_a\,, \tag{C.6}$$

where we used equation (95) of [16], ${e_m}^a = \delta^a_m + h^a_m$. Use the definition (C.4) we have to sum over all ways where the spinor and vector indices can appear, which gives three identical contributions. Overall we find the coefficient of the graviton is $-\frac{3i}{8}$, in agreement with (4.30) up to a factor of i.

At $\mathcal{O}(\theta^5)$ terms that contain the graviton in the superfields of [16] are

$$\begin{aligned}\lambda^\mu\lambda^\nu\lambda^\alpha \hat{E}_\alpha^{(1)s} F^{(4)}_{\mu\nu s} = &\frac{1}{48}(\lambda\gamma^s\theta)\Big((\lambda\gamma^a\theta)(\lambda\gamma^b\theta)\theta^2\omega_{sab} - \frac{1}{2}(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\theta\gamma_{abcd}\theta){\omega_s}^{cd} \\ &-(\lambda\gamma_{ab}\theta)(\lambda\gamma^\theta)(\theta\gamma^{bef}\theta)\omega_{sef}\Big)\,, \end{aligned}\tag{C.7}$$

$$\begin{aligned}\lambda^\sigma\lambda^\beta \hat{E}_\beta^{(1)m}\lambda^\delta \hat{E}_\delta^{(1)n} F^{(3)}_{\sigma mn} = &(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)\Big(\frac{1}{64}(\theta{\gamma^{ghe}}_m\theta)(\lambda\gamma^e\theta)\omega_{ngh} - \frac{1}{32}\theta^2(\lambda\gamma^e\theta)\omega_{mne} \\ &-\frac{1}{64}(\theta\gamma^{ghe}\theta)(\theta\gamma_{ne}\lambda)\omega_{mgh}\Big)\,, \end{aligned}\tag{C.8}$$

$$\lambda^\alpha\lambda^\beta\lambda^\delta \hat{E}_\alpha^{(1)m}\hat{E}_\beta^{(1)n}\hat{E}_\delta^{(1)p} F^{(3)}_{mnp} = \frac{3}{32}(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)\left(\theta^2\omega_{mnp} - \frac{1}{2}(\theta{\gamma^{gh}}_{mn}\theta)\omega_{pgh}\right). \tag{C.9}$$

Due to the sum over permutations of indices both (C.7) and (C.8) get a factor 3. Summing up all of these terms we find exactly the same contribution as in (4.32). We have additionally checked the three-form and gravitino at θ^3 and θ^4 respectively, and found they match our results.

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