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From the EFT of spinning gravitating objects to Poincaré and gauge invariance at the 4.5PN precision frontier

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Abstract: We confirm the generalized actions of the complete NLO cubic-in-spin interactions for generic compact binaries which were first tackled via an extension of the EFT of spinning gravitating objects. We first reduce these generalized actions to standard actions with spins, where the interaction potentials are found to consist of 6 independent sectors, including a new unique sector that is proportional to the square of the quadrupolar deformation parameter, C_{ES^2}. We derive the general Hamiltonians in an arbitrary reference frame, and for generic kinematic configurations. With these most general Hamiltonians we construct the full Poincaré algebra of all the sectors at the fourth and a half post-Newtonian (4.5PN) order, including the third subleading spin-orbit sector, recently accomplished uniquely via our framework, thus proving the Poincaré invariance of all relevant sectors. We then derive the binding energies with gauge-invariant relations useful for gravitational-wave applications. Finally, we also derive the extrapolated scattering angles in the aligned-spins configuration for the scattering problem. Yet, as made clear already as of quadratic-in-spin sectors, the aligned-spins simplification inherent to the scattering-angle observable, entails a great loss of physical information, that is only growing with higher-spin sectors. Our completion of the full Poincaré algebra at the present 4.5PN order provides strong confidence that this new precision frontier in PN theory has now been established.

Keywords: Effective Field Theories, Black Holes, Space-Time Symmetries, Scattering Amplitudes

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Contents

1 Introduction

Within few years of gravitational-wave (GW) astronomy, we already have an impressive catalogue of measurements of 90 accumulated confirmed GW signals [1–3]. This data collected by current second-generation GW detectors Advanced LIGO [4], Advanced VIRGO [5], and KAGRA [6], includes as sources inspirals of compact binaries of black holes (BHs) [7], neutron stars (NSs) [8], and even of mixed binaries of a BH and a NS. [9] as sources. These binaries evolve virtually all of their lifetime through non-relativistic (NR) motion in weak gravity, and thus their evolution has been studied analytically using the post-Newtonian (PN) approximation of General Relativity (GR), see Blanchet's Living Review for the comprehensive progress and status of PN theory [10]. Building on PN theory, and interpolating over the swift phase of merger, in which the binary is subject to strong gravity fields, the effective-one-body (EOB) approach [11] has consistently enabled the generation of theoretical gravitational waveforms from such sources, against which measured signals are compared.

To that end efforts have been increasing in recent years to push the state of the art of PN theory. For the conservative dynamics of generic compact binaries the high fifth PN (5PN) accuracy in the two-body potentials has been tackled in both traditional GR [12–14], and effective field theory (EFT) approaches [15–17]. These 5PN precision results have been obtained in the point-mass sector, where finite-size effects, and thus the internal structure of the individual components of the binary first kicks in at this high order. The third subleading spin-orbit sector at the 4.5PN order (the PN counting of sectors with spin is always evaluated for maximally-rotating objects) was also approached following the approach in [12, 13] via traditional GR [18, 19], but was fully accomplished including general Hamiltonians via the EFT of spinning gravitating objects in [20–23]. Similar to the point-mass sector the spin-orbit sector, which is linear in the spins, is also uniquely simple in that finite-size effects first enter at an even higher PN order than the point-mass sector.

Yet, such finite-size effects hold valuable information to better our understanding of strong gravity and QCD theories. For higher-spin sectors, namely as of the quadratic order in the spins of rotating compact components of the binary, such finite-size effects enter already at the 2PN order [24], and thus have to be consistently tackled in order to push the PN precision frontier. This has required to formulate a theory for higher-spin orders in gravity, which was introduced in [20, 25], and has fascinating links to higher-spin field theories, see e.g. [26]. In particular for the 4.5PN accuracy, the next-to-leading (NLO) cubic-in spin sectors for generic compact binaries also need to be completed.

This paper is aimed at the completion of such results in the NLO cubic-in spin sectors at the 4.5PN order, following on our EFT computation of the generalized action of these sectors [27]. The latter work has built on the EFT of spinning gravitating objects [20, 28], the `EFTofPNG` public code [21], and a series of works in this approach that accomplished the present state of the art in sectors with spins at the 4PN order [25, 29–32] including their general Hamiltonians. Recently, this EFT approach has also enabled the completion of the third subleading ($\mathrm{N^3LO}$) quadratic-in-spin sectors at the 5PN order in [33–36], including the general Hamiltonians in [35] (and then also in [37]), and their full Poincaré in [36]. In this paper we reduce the generalized actions of the NLO cubic-in spin sectors from [27] to standard actions with spins, and derive the general Hamiltonians for an arbitrary reference frame, and for generic kinematic configurations. From them we derive the binding energies with gauge-invariant relations useful for GW applications.

This paper is also aimed at validating the results of all conservative sectors at the 4.5PN accuracy in order to establish this as the new precision frontier. We accomplish this objective through the construction of the full Poincaré algebra at this PN order, which includes: 1. The NLO cubic-in-spin sectors from [27], with the general Hamiltonians obtained in the present paper. 2. The N^{3}LO spin-orbit sector with the general Hamiltonians obtained in our [23]. The Poincaré algebra in phase space provides the most stringent consistency check for the full general PN Hamiltonians by way of proving their Poincaré invariance, and it is sensitive to the smallest deviations from proper canonical Hamiltonians. Thus in addition to providing the global Poincaré invariants of the system, the completion of the Poincaré algebra at the 4.5PN order provides a powerful validation of this new PN precision frontier.

In the scattering problem the NLO cubic-in-spin sectors in the weak-field or so-called "post-Minkowskian" (PM) approximation have also been approached. In [38] scattering angles for BHs in the aligned-spins case were first approached. In [39] the NLO PM Hamiltonians in the center-of-mass (COM) frame for BHs were presented, and in [40] similar NLO COM Hamiltonians for generic compact binaries were approached. All of these scattering studies built on the higher-spin theory introduced and formulated in our EFT of spinning objects [20, 25] as their basis, and thus they all put forward derivations that are inherently dependent on our framework. Moreover, scattering angles simply make for poor input at higher-spin sectors, since they are always inherently restricted to the aligned-spins simplification, where there is a growing loss of physical information that is only increasing with spin orders, as of the quadratic order in spins. Furthermore, when Hamiltonians are provided in these scattering studies, which thus far seems to be feasible only at low loop orders and spin orders, which are already known in PN theory, they are always restricted to the COM frame. This fact also does not allow to study the Poincaré algebra of the system, which could in turn also provide a critical check for the validity of such results.

In these scattering-amplitudes derivations, quantum degrees of freedom (DOFs) are unnecessarily invoked, that then need to be laboriously removed from the meaningful classical results. Moreover the scattering results should be linked to the bound inspiral setup, which requires more work, and becomes an obstacle as of third subleading loop or spin orders. In our EFT approach, there are only classical DOFs, directly set up in the bound problem, and thus our approach readily gets at the necessary results for GW measurements. Moreover, our approach provides the general arbitrary reference-frame Hamiltonians, which form part of the full Poincaré algebra. Our EFT approach is thus instrumental to high-precision GW measurements, as well as critical to guide such efforts to attempt at diverse derivations in the related scattering problem. To that end, we also derive from our generic PN Hamiltonians the extrapolated scattering angles in the aligned-spins configuration for the scattering problem.

This paper is organized as follows. In section 2 we review our EFT of higher-spin in gravity, which contains two main formal ingredients in the theory, that contribute to all orders in spin: spin-gauge invariance and spin-induced couplings [20, 25, 28]. In section 3 we confirm the generalized actions that were evaluated via an EFT computation in [27], and reduce them to the final actions of 6 independent subsectors from which the equations of motion (EOMs) for both the position and spin can be obtained directly and simply. In section 4 we derive the full general Hamiltonians in an arbitrary reference frame, and then gradually specialize to the COM frame, and to the aligned-spins configuration, where it is shown how significant is the loss of physical information in these simplifications, notably growing with higher orders in spin. In section 5 we construct the full Poincaré algebra with the general Hamiltonians of the NLO cubic-in-spin sectors, and N^{3}LO spin-orbit sector, that make up the 4.5PN precision, thus proving their Poincaré invariance and establishing this PN order as the new precision frontier. In section 6 we derive useful observables and gauge-invariant relations for GW applications. We also derive the extrapolated scattering angles in the aligned-spins configuration of the scattering problem for guidance of scattering-amplitudes derivations. Finally in our appendices we include: a brief note on typos in [27] in appendix A, explicit results for the new redefinitions, final actions, and general Hamiltonians of the NLO cubic-in-spin sectors, in appendices B, C, and D, respectively, and the Poincaré COM generator of the N^{3}LO spin-orbit sector in appendix E. All the corresponding results are also provided in the supplementary material attached to this publication.

2 EFT of higher spin in gravity

To complete the precision frontier at the 4.5PN order, we need to consider carefully the EFT of spinning gravitating objects [20], which was originally formulated to establish the 3PN order as the state of the art, and to then obtain the present state of the art at the 4PN order, via an EFT approach for spins in gravity. In particular the 4.5PN order requires to tackle the NLO sectors that are cubic in spin, and thus to extend the EFT at higher orders in spin with greater attention and rigour [27, 41]. Let us review our theory that was presented in [20, 28], which we then built on.

For the conservative interactions of the compact binary inspiral we consider an effective action that captures a two-particle system in a weak gravity field at the orbital scale of binary separation [15]:

$$S_{\text{eff}} = S_{\text{gr}}[g_{\mu\nu}] + \sum_{a=1}^{2} S_{\text{pp}}(\lambda_a). \tag{2.1}$$

S_{gr} is the purely gravitational action in some classical theory of gravity, e.g. GR, and it is supplemented by an infinite tower of interactions between the gravitational field and the worldline degrees of freedom (DOFs), representing each a-th compact object. These interactions make up the point-particle action, S_{pp}, localized on the worldlines, parametrized by λ_a. The challenge for rotating objects is then to bootstrap the effective action of a spinning particle.

First, for a spinning object the action of a point-particle can be written as [20, 42–44]:

$$S_{\text{pp}}\left[g_{\mu\nu}, y^\mu, e^\mu_A\right] = \int d\lambda \left[-m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{\text{SI}}\left[u^\mu, S_{\mu\nu}, g_{\mu\nu}\left(y^\mu\right)\right]\right], \tag{2.2}$$

where $u^\mu \equiv dy^\mu/d\lambda$, and y^μ, e^μ_A are the particle worldline coordinate and tetrad DOFs, respectively. From the worldline tetrad, $\eta^{AB} e_A{}^\mu(\lambda) e_B{}^\nu(\lambda) = g^{\mu\nu}$, the angular velocity is defined as $\Omega^{\mu\nu}(\lambda) \equiv e^\mu_A \frac{De^{A\nu}}{D\lambda}$, and then its conjugate, the worldline spin, $S_{\mu\nu}(\lambda) \equiv -2\frac{\partial L}{\partial \Omega^{\mu\nu}}$, is added as another explicit DOF to the action. L_{SI} denotes the non-minimal coupling part of the Lagrangian that is induced due to the presence of spin.

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While the minimal coupling in the form of eq. (2.2) is fixed only from the symmetries of general covariance and reparametrization invariance [42, 43], it turns out that it conceals additional symmetries related with the rotational DOFs, the worldline tetrad and the spin: as we have shown in [20] this form of the action in fact already assumes the Tulczyjew gauge for the spin [45]. Related with that hidden symmetry is the fact that the particle worldline coordinate can in general be shifted from the position that represents the rotating object's "center". As to the non-minimal coupling part of the action, the symmetries of parity and SO(3) invariance play a major role in constraining it.

Indeed, these were the 2 fundamental challenges tackled successfully in bootstrapping the effective action of a spinning particle in our EFT formulated in [20]: 1. Making the spin gauge invariance manifest in the action as of minimal coupling. Notice that this contributes to all orders in spin. 2. Fixing the leading non-minimal couplings to all orders in spin. In sections 2.1 and 2.2 below we go over these 2 major formal developments accomplished in [20].

2.1 Spin gauge invariance

The key observation here is the symmetries related with the worldline tetrad. There is an SO(3) invariance of the worldline spatial triad, and then what we refer to as "spin gauge invariance", which is some freedom to complete the timelike component of the worldline spatial triad to a tetrad [20]. This gauge choice will fix both tetrad and spin variables. To make the gauge freedom of the rotational variables manifest in the effective action, we applied a 4-dimensional covariant boost-like transformation on the worldline tetrad, introducing new gauge DOFs, $\hat{e}_{[0]\mu} = w_\mu$, for the timelike vector of the tetrad. This leads to a generic gauge condition for the spin (traditionally called "SSC") [20]:

$$\hat{S}^{\mu\nu}\left(p_\nu + \sqrt{p^2}\hat{e}_{[0]\nu}\right) = 0, \tag{2.3}$$

which removes the redundant DOFs from both the angular velocity and the spin. From the minimal coupling term in eq. (2.2) we then obtain [20]:

$$\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} = \frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} + \frac{\hat{S}^{\mu\nu}p_\nu}{p^2}\frac{Dp_\mu}{D\sigma}, \tag{2.4}$$

where a new general term emerges in the action, which was not accounted for in past formulations of spin in gravity, including in Yee and Bander [46], which was later adopted in [47]. This kinematic term, essentially Thomas precession as elaborated in [20], originates from minimal coupling, and thus is clearly not preceded by any Wilson coefficient, though it contributes from leading order in spin — to finite-size effects at all orders in spin.

Using the worldline Lorentz matrices, $\eta^{AB}\Lambda_A{}^a(\lambda)\Lambda_B{}^b(\lambda) = \eta^{ab}$, we can write the locally-flat angular velocity, $\hat{\Omega}^{ab}_{\text{flat}} = \hat{\Lambda}^{Aa}\frac{d\hat{\Lambda}_A{}^b}{d\lambda}$, and the conjugate local spin, $\hat{S}_{ab} = \tilde{e}^\mu_a\tilde{e}^\nu_b\hat{S}_{\mu\nu}$, with the local tetrad field, $\eta^{ab}\tilde{e}_a{}^\mu(x)\tilde{e}_b{}^\nu(x) = g^{\mu\nu}(x)$. Then, using the Ricci rotation coefficients, $\omega_\mu{}^{ab} \equiv \tilde{e}^b{}_\nu D_\mu \tilde{e}^{a\nu}$, the first term on the r.h.s. in eq. (2.4) can be rewritten as [20, 48]:

$$\frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} = \frac{1}{2}\hat{S}_{ab}\hat{\Omega}^{ab}_{\text{flat}} + \frac{1}{2}\hat{S}_{ab}\omega_\mu{}^{ab}u^\mu. \tag{2.5}$$

At this point we note that we fix the gauge of the rotational variables so as to fully disentangle the field from the worldline DOFs, as was first put forward in [49]. We fix the gauge of rotational variables to the canonical gauge for curved spacetime that we generalized from flat spacetime Pryce-Newton-Wigner SSC [50, 51].

2.2 Higher-spin coupling

Based on the full set of symmetries that we noted, the key element in bootstrapping the non-minimal coupling of spin to gravity was to invoke the classical analogue of the Pauli-Lubanski vector, S^μ, as the building block for the action [20, 25]. Focusing on parity and SO(3) invariance and through a full rigorous analysis, the leading non-minimal couplings to all orders in spin were presented in [20]:

$$\begin{aligned} L_{\rm SI} = & \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{{\rm ES}^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n-1}} S^{\mu_{2n}} \\ & + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{{\rm BS}^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n-1}} S^{\mu_{2n}} S^{\mu_{2n+1}}, \end{aligned} \tag{2.6}$$

with a new infinite set of Wilson coefficients that correspond to "multipolar deformation parameters" in traditional GR. This infinite tower of operators contains definite-parity curvature components, either the electric or magnetic, $E_{\mu\nu}$ or $B_{\mu\nu}$, respectively. For the present sectors, we only need to pull out the first two terms of this infinite series [20, 25]:

$$L_{{\rm ES}^2} \equiv -\frac{C_{{\rm ES}^2}}{2m} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu, \tag{2.7}$$

$$L_{{\rm BS}^3} \equiv -\frac{C_{{\rm BS}^3}}{6m^2} \frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu S^\lambda, \tag{2.8}$$

which correspond to the quadrupolar [24, 44] and octupolar [25] deformations.

Following the EFT of spinning gravitating objects introduced in [20, 25], and as reviewed above, various scattering-amplitudes approaches tackled the gravitational scattering problem with massive higher-spin particles, including [38–40]. In particular, the infinite tower of S^l couplings in eq. (2.6) has been used for the corresponding 3-point amplitudes with massive particles of spin $s = l/2$, that make up the building blocks to derive any scattering amplitude. Furthermore, all these approaches used input from implementing the EFT of spinning gravitating objects [20, 25] for the specific case of BHs within traditional GR [38] — as a critical guide to their derivations. In particular, the dependence of [38] in our worldline theory for higher-spin [20, 25] should be noted here, as it was omitted in [38].

As to non-minimal couplings that are quadratic in the curvature, an extension of the action that covers the cubic order in spin was introduced in [41]. Similar to the spin-orbit sector, it is found that such cubic-in-spin operators enter only at the 6.5PN order, and thus are not relevant to the present sectors. At this point it should be noted that the spin which is used in the construction of non-minimal couplings is in the Tulczyjew gauge to begin with, and thus in order to switch to a generic spin variable as in section 2.1 above, the following relation should be used:

$$S_{\mu\nu} = \hat{S}_{\mu\nu} - \frac{\hat{S}_{\mu\rho} p^\rho p_\nu}{p^2} + \frac{\hat{S}_{\nu\rho} p^\rho p_\mu}{p^2}. \tag{2.9}$$

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3 Effective actions

Using our effective theory for higher spin in gravity reviewed in the previous section, we carried out an EFT evaluation of the NLO cubic-in-spin sectors in [27]. The evaluation of the relevant interactions involved 53 unique Feynman graphs [27]. The printed values for 5 of these graphs contained typos, which we note in appendix A below. These copying errors in the individual values of graphs in the printed manuscript are arbitrary typos, and did not affect the total sum of the graphs, that was provided in [27]. The generalized actions of the NLO cubic-in-spin interactions are then written as [27]:

$$L^{\rm NLO}_{\rm S^3} = L^{\rm NLO}_{\rm S_1^2S_2} + L^{\rm NLO}_{\rm S_1^3} + (1 \leftrightarrow 2), \tag{3.1}$$

where we identify the following distinct pieces:

$$\begin{aligned} L^{\rm NLO}_{\rm S_1^2S_2} = &+ \frac{G^2}{r^5} L_{(1)} + C_{1(\rm ES^2)} \frac{G}{r^4}\frac{1}{m_1} L_{(2)} + C_{1(\rm ES^2)} \frac{G^2}{r^5} L_{(3)} + C_{1(\rm ES^2)} \frac{G^2 m_2}{r^5 m_1} L_{(4)} \\ &+ \frac{G^2}{r^4} L_{(5)} + C_{1(\rm ES^2)} \frac{G}{r^3}\frac{1}{m_1} L_{(6)} + C_{1(\rm ES^2)} \frac{G^2}{r^4} L_{(7)} + C_{1(\rm ES^2)} \frac{G^2 m_2}{r^4 m_1} L_{(8)} \\ &+ C_{1(\rm ES^2)} \frac{G}{r^2}\frac{1}{m_1} L_{(9)} + C_{1(\rm ES^2)} \frac{G}{r}\frac{1}{m_1} L_{(10)}, \end{aligned} \tag{3.2}$$

as well as the following ones:

$$\begin{aligned} L^{\rm NLO}_{\rm S_1^3} = &\; C_{1(\rm ES^2)} \frac{G^2 m_2}{r^5 m_1} L_{[1]} + C_{1(\rm ES^2)} \frac{G^2 m_2^2}{r^5 m_1^2} L_{[2]} + C_{1(\rm BS^3)} \frac{G m_2}{r^4 m_1^2} L_{[3]} \\ &+ C_{1(\rm BS^3)} \frac{G^2 m_2}{r^5 m_1} L_{[4]} + C_{1(\rm BS^3)} \frac{G^2 m_2^2}{r^5 m_1^2} L_{[5]} + C_{1(\rm ES^2)} \frac{G m_2}{r^3 m_1^2} L_{[6]} \\ &+ C_{1(\rm ES^2)} \frac{G^2 m_2}{r^4 m_1} L_{[7]} + C_{1(\rm BS^3)} \frac{G m_2}{r^3 m_1^2} L_{[8]} + C_{1(\rm BS^3)} \frac{G m_2}{r^2 m_1^2} L_{[9]}, \end{aligned} \tag{3.3}$$

where we also provide these generalized actions in machine-readable format in the supplementary material attached to this paper. Let us stress that the computer files of [27], also included in the supplementary material attached to this paper, contain the correct results. Note that in eq. (3.3) there can also exist in principle a piece of the form $C_{1(\rm ES^2)} m_2/(m_1^2 r^2)$, which is absorbed into $L_{[6]}$, eq. (5.19) in [27], by a total time derivative.

3.1 Redefinition of actions

As noted in [27] the generalized actions that are obtained from the EFT computation need to be reduced to "standard" actions with spin variables, namely which do not contain higher-order time derivatives beyond velocity and spin. The reduction procedure via formal redefinitions that we show here was introduced in [52] to include rotational variables, and we build here on the derivations shown in [23, 35]. Table 1 summarizes the redefinitions that need to be applied gradually to the relevant sectors that contribute to the present sectors, in increasing PN order, even those that do not require any reduction in themselves.

Based on the spin and PN power-counting of the various redefinitions [52], we first note that similar to [20], for higher-spin sectors as of the NLO, we need to apply position shifts

l \ n	(N^0)LO	$\mathrm{N}^{(1)}$LO
S^0		
S^1	+	++
S^2		++
S^3	+	++

Table 1. The shorthand notation of sectors, (n, l), and the general formula $n + l + \mathrm{Parity}(l)/2$ for their PN counting was introduced in [27], where n is the subleading order (or highest n-loop order), and l is the highest order in spins of each of the sectors, and the parity is 0 or 1 for even or odd l, respectively. The 8 sectors that contribute to the present NLO cubic-in-spin sectors, $(1, 3)$, through redefinition of variables. "+" marks sectors that require only position shifts or redefinitions of rotational variables to be fixed, and "++" marks sectors that require redefinition of both position and rotational variables.

beyond linear order. On the other hand, also according to the extension of the procedure beyond linear order in the rotational variables which we carried out in [23], here we only need to apply redefinitions of the rotational variables to linear order. For the present sectors we need to take into account redefinitions that are fixed in 5 sectors, as shown in table 1, and we follow the detailed presentation in [23, 35].

The redefinitions in 3 of these sectors, below cubic in spin, are detailed in [23, 35], and thus here we need to further consider the 2 sectors that are cubic in the spin, shown in tables 2–3, whose structure was explained in [23]. The algorithm used for the reduction is similar to that we used in [23, 35], only that it implements higher-order position shifts as seen in table 3. Thus, we now go through the relevant sectors according to their PN order, with the unreduced actions always computed with the `EFTofPNG` code [21]. For the LO and NLO spin-orbit, and NLO quadratic-in-spin sectors, our unreduced actions and redefinitions can be found in [23], and [35], respectively.

Now we can approach the LO cubic-in-spin sectors as shown in table 2, where we will not conform to the choices of unreduced actions and redefinitions of our original derivation of these sectors in [25]. Thus the unreduced potential is:

$$\begin{aligned}
V^{\mathrm{LO}}_{\mathrm{S}^3} = \frac{3GC_{1\mathrm{ES}^2}}{m_1 r^4}\Big[& S_1^2 \vec{S}_2\times\vec{n}\cdot\vec{v}_1 - 2\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1 - S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2 \\
&+ 2\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2 - 5\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2 + 5\vec{S}_2\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big] \\
&- \frac{3GC_{1\mathrm{ES}^2}}{m_1 r^3}\Big(\dot{\vec{S}}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2 + \vec{S}_1\cdot\vec{n}\dot{\vec{S}}_1\times\vec{n}\cdot\vec{S}_2\Big) \\
&+ \frac{GC_{1\mathrm{BS}^3} m_2}{{m_1}^2 r^4}\Big[S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1 - S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2 - 5\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2 \\
&+ 5\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big].
\end{aligned} \tag{3.4}$$

There is no new position shift in this sector, but a new redefinition for the rotational variables that is fixed as:

$$\left(\omega^{ij}\right)^{\mathrm{LO}}_{\mathrm{S}^3} = \frac{3GC_{1\mathrm{ES}^2}}{m_1 r^3}\Big[\vec{S}_2\cdot\vec{n}S_1^i n^j - 2\vec{S}_1\cdot\vec{n}S_2^i n^j + S_2^i S_1^j\Big] - (i\leftrightarrow j). \tag{3.5}$$

from \ to	(0P)N	LO S^2
LO S^1		$\Delta\vec{x}$
LO S^3	$\Delta\vec{S}$	

Table 2. Contributions to the LO S^3 sectors from position shifts and spin redefinitions in lower-order sectors.

from \ to	(0P)N	LO S^1	LO S^2	NLO S^2	LO S^3
LO S^1	$(\Delta\vec{x})^3$	$(\Delta\vec{x})^2$		$\Delta\vec{x}$	
NLO S^1			$\Delta\vec{x}$		$\Delta\vec{S}$
NLO S^2		$\Delta\vec{x}$	$\Delta\vec{S}$		
LO S^3		$\Delta\vec{S}$			
NLO S^3	$\Delta\vec{x}$, $\Delta\vec{S}$				

Table 3. Contributions to the NLO S^3 sectors from position shifts and spin redefinitions in lower-order sectors.

We can now consider the redefinitions at the present NLO cubic-in-spin sectors, as detailed in table 3.

The new position shifts and redefinitions of rotational variables fixed in the present sectors can be written as:

$$(\Delta\vec{x}_1)^{\rm NLO}_{\rm S^3} = (\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1^3} + (\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1^2S_2} + (\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1S_2^2} + (\Delta\vec{x}_1)^{\rm NLO}_{\rm S_2^3}\,, \tag{3.6}$$

$$\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S^3} = \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^3} + \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^2S_2} + \left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1S_2^2} - (i \leftrightarrow j), \tag{3.7}$$

where the explicit redefinitions are presented in appendix B, and we also provide them in machine-readable format in the supplementary material attached to this paper.

3.2 Final actions

As explained in [20] we can already obtain the EOMs for the position and spin from the generalized actions before reduction due to our use of the generalized canonical gauge formulated in [20]. However, it is much easier to derive the EOMs with the more compact actions obtained after the reduction that we have shown in the previous section. The final potentials that we obtain for the NLO cubic-in-sectors, comprise the following 6 distinct sectors:

$$\begin{aligned} V^{\rm NLO}_{\rm S^3} &= V^{\rm NLO}_{\rm S_1^3} + C_{\rm 1ES^2} V^{\rm NLO}_{\rm (ES_1^2)S_1} + C_{\rm 1ES^2}^2 V^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3} + C_{\rm 1BS^3} V^{\rm NLO}_{\rm BS_1^3} + V^{\rm NLO}_{\rm S_1^2S_2} + C_{\rm 1ES^2} V^{\rm NLO}_{\rm (ES_1^2)S_2} \\ &\quad + (1 \leftrightarrow 2), \end{aligned} \tag{3.8}$$

where the explicit actions are presented in appendix C, and we also provide them in machine-readable format in the supplementary material attached to this paper.

Notice that a new "self-induced" cubic-in-spin potential arises in eq. (C.4), that is proportional to the square of the quadrupolar-deformation parameter for generic compact binaries. As we shall see in our construction of the Poincaré algebra of the sectors at this PN order in section 5 below, and more specifically from eq. (5.27) there, this new sector is actually imposed by Poincaré invariance. It can be seen as arising from the precession of spin due to the leading quadrupolar deformation, as it enters and affects in turn on higher orders of the spin-induced quadrupolar potential. The interference of misaligned quadrupole effects effectively gives rise to this new self-induced octupole potential. Of course, when the simplification, that all spins are aligned, is assumed, then this new effect drops out.

From these final potentials for the NLO cubic-in-sectors the consequent EOMs for the position and spin were derived in [53].

4 Hamiltonians

From the generalized canonical gauge for the rotational variables that is included in our formulation [20], the derivation of full general Hamiltonians is straightforward, via a Legendre transform of the final actions only with respect to the position variables. This Legendre transform involves all the sectors that lead up to the present ones as noted in table 1. It should also be highlighted that the Hamiltonians which we derive in our approach hold for an arbitrary reference frame, and thus are the most general ones, and in particular more general than various specialized Hamiltonians provided in all other methods, such as the COM, EOB, or aligned-spins Hamiltonians, which are all — to begin with — already restricted to the COM frame.

The general Hamiltonians already have important applications both formally and phenomenologically. With the general Hamiltonians the Poincaré algebra of the conserved integrals of motion can be uniquely uncovered, which also provides a stringent kinematic consistency-check for the validity of Hamiltonians obtained, as will be discussed in detail in section 5 below. Phenomenologically from the Hamiltonians one can also construct various possible EOB models for the present sectors, and study how they perform. Finally, the general Hamiltonians can be specialized to certain simplified kinematic configurations, see section 4.2 below, in which gauge-invariant observables, notably the binding energies as function of the GW frequencies, can be extracted, as will be detailed in section 6 below.

4.1 Full Hamiltonians

Similar to the final potentials presented in section 3.2, our full general Hamiltonian for the present NLO cubic-in-spin sectors is comprised of 6 distinct sectors:

$$H^{\rm NLO}_{\rm S^3} = H^{\rm NLO}_{\rm S_1^3} + C_{\rm 1ES^2} H^{\rm NLO}_{\rm (ES_1^2)S_1} + C^2_{\rm 1ES^2} H^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3} + C_{\rm 1BS^3} H^{\rm NLO}_{\rm BS_1^3} + H^{\rm NLO}_{\rm S_1^2 S_2} + C_{\rm 1ES^2} H^{\rm NLO}_{\rm (ES_1^2)S_2}$$
$$+ (1 \leftrightarrow 2), \tag{4.1}$$

where the explicit Hamiltonians are presented in appendix D, and we also provide them in machine-readable format in the supplementary material attached to this paper.

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4.2 Specialized Hamiltonians

In order to express specialized Hamiltonians, we use various binary-mass conventions:

$$m \equiv m_1 + m_2, \quad q \equiv m_1/m_2, \quad \mu \equiv m_1 m_2/m, \tag{4.2}$$

$$\nu \equiv m_1 m_2/m^2 = q/(1+q)^2 = \mu/m, \tag{4.3}$$

where the latter is the dimensionless symmetric mass-ratio. We further transform all variables to be dimensionless using Gm and μ to rescale length and mass, respectively, and denote all dimensionless variables with a tilde.

First, the Hamiltonians are specified in the center-of-mass (COM) frame, with $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$. Using $\vec{p}$, the orbital angular momentum is defined as $\vec{L} \equiv r\vec{n} \times \vec{p}$. Then the COM Hamiltonians of the NLO cubic-in-spin sectors are written in the form:

$$\begin{aligned}\tilde{H}^{\rm NLO}_{\rm S^3} = {} & \tilde{H}^{\rm NLO}_{\rm S_1^3} + C_{\rm 1ES^2}\tilde{H}^{\rm NLO}_{\rm (ES_1^2)S_1} + C^2_{\rm 1ES^2}\tilde{H}^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3} + C_{\rm 1BS^3}\tilde{H}^{\rm NLO}_{\rm BS_1^3} + \tilde{H}^{\rm NLO}_{\rm S_1^2S_2} + C_{\rm 1ES^2}\tilde{H}^{\rm NLO}_{\rm (ES_1^2)S_2} \\ & + (1 \leftrightarrow 2),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\tilde{H}^{\rm NLO}_{\rm S_1^3} = {} & \frac{\nu^2 \tilde{\vec{L}}\cdot\tilde{\vec{S}}_1 \tilde{S}_1^2}{\tilde{r}^6}\left[-3\nu - \frac{9}{4} - \frac{9}{4}\frac{\tilde{L}^2}{\tilde{r}} + \tilde{p}_r^2\tilde{r}\left(\frac{33}{16} - \frac{39\nu}{8}\right)\right. \\ & \left. + \frac{1}{\nu q}\left(-3\nu^2 - \frac{3\nu}{2} + \frac{9}{4} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9}{4} - \frac{9\nu}{2}\right) + \tilde{p}_r^2\tilde{r}\left(-\frac{75\nu^2}{16} + 9\nu - \frac{33}{16}\right)\right)\right] \\ & + \frac{\nu^2(\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1)^3}{\tilde{r}^7}\left[2\nu + \frac{7}{8} + \frac{1}{\nu q}\left(\frac{15\nu^2}{8} - \frac{\nu}{4} - \frac{7}{8}\right)\right] \\ & + \frac{\nu^2 \tilde{\vec{L}}\cdot\tilde{\vec{S}}_1(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[\frac{15\nu}{2} + \frac{15}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{69}{16} - \frac{9\nu}{8}\right) + \tilde{p}_r^2\tilde{r}\left(\frac{39\nu}{8} + \frac{111}{16}\right)\right. \\ & + \frac{1}{\nu q}\left(\frac{15\nu^2}{2} + \frac{15\nu}{2} - \frac{15}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{15\nu^2}{16} + \frac{39\nu}{4} - \frac{69}{16}\right)\right. \\ & \left.\left. + \tilde{p}_r^2\tilde{r}\left(\frac{75\nu^2}{16} + 9\nu - \frac{111}{16}\right)\right)\right] \\ & + \frac{\nu^2 \tilde{p}_r \tilde{\vec{L}}\cdot\tilde{\vec{S}}_1 \tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[6\nu + \frac{21}{8} + \frac{1}{\nu q}\left(\frac{45\nu^2}{8} - \frac{3\nu}{4} - \frac{21}{8}\right)\right],\end{aligned} \tag{4.5}$$

$$\begin{aligned}\tilde{H}^{\rm NLO}_{\rm (ES_1^2)S_1} = {} & \frac{\nu^2 \tilde{\vec{L}}\cdot\tilde{\vec{S}}_1 \tilde{S}_1^2}{\tilde{r}^6}\left[2\nu - \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9\nu}{8} + \frac{15}{16}\right) + \tilde{p}_r^2\tilde{r}\left(\frac{9\nu}{8} + \frac{63}{16}\right)\right. \\ & \left. + \frac{1}{\nu q}\left(2\nu^2 - \frac{17\nu}{2} + \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}}\left(\frac{27\nu^2}{16} + \frac{3\nu}{4} - \frac{15}{16}\right) + \tilde{p}_r^2\tilde{r}\left(\frac{57\nu^2}{16} + \frac{27\nu}{4} - \frac{63}{16}\right)\right)\right] \\ & + \frac{\nu^2(\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1)^3}{\tilde{r}^7}\left[-\frac{3}{4} + \frac{1}{\nu q}\left(\frac{3}{4} - \frac{3\nu}{2}\right)\right] \\ & + \frac{\nu^2 \tilde{\vec{L}}\cdot\tilde{\vec{S}}_1(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[24 - \frac{27\nu}{4} + \frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{45\nu}{8} - \frac{87}{16}\right) + \tilde{p}_r^2\tilde{r}\left(-\frac{45\nu}{8} - \frac{123}{16}\right)\right.\end{aligned}$$

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$$
+\frac{1}{\nu q}\left(-\frac{27\nu^2}{4}+\frac{273\nu}{4}-24+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{135\nu^2}{16}-\frac{21\nu}{4}+\frac{87}{16}\right)\right.
$$

$$
\left.\left.+\tilde{p}_r^2\tilde{r}\left(-\frac{285\nu^2}{16}-\frac{39\nu}{4}+\frac{123}{16}\right)\right)\right]
$$

$$
+\frac{\nu^2\tilde{p}_r\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{9}{4}+\frac{1}{\nu q}\left(\frac{15\nu^2}{4}-\frac{9\nu}{2}+\frac{9}{4}\right)\right], \tag{4.6}
$$

$$
\tilde{H}^{\rm NLO}_{C^2_{\rm ES_1^2}S_1^3}=\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[-\frac{3\nu}{2}-3+\frac{1}{\nu q}\left(-\frac{3\nu^2}{2}-\frac{9\nu}{2}+3\right)\right], \tag{4.7}
$$

$$
[8pt]\tilde{H}^{\rm NLO}_{\rm BS_1^3}=\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{S}_1^2}{\tilde{r}^6}\left[\nu+7+\frac{\tilde{L}^2}{\tilde{r}}(1-\nu)-\frac{7\nu}{2}\tilde{p}_r^2\tilde{r}\right.
$$

$$
\left.+\frac{1}{\nu q}\left(\nu^2+\frac{21\nu}{2}-7+\frac{\tilde{L}^2}{\tilde{r}}\left(-\nu^2+3\nu-1\right)+\tilde{p}_r^2\tilde{r}\left(\frac{7\nu}{2}-\frac{7\nu^2}{2}\right)\right)\right]
$$

$$
+\frac{\nu^2(\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1)^3}{\tilde{r}^7}\left[-1+\frac{1}{\nu q}(1-2\nu)\right]
$$

$$
+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[-5\nu-36+\frac{\tilde{L}^2}{\tilde{r}}(5\nu-1)+\tilde{p}_r^2\tilde{r}\left(\frac{35\nu}{2}-4\right)\right.
$$

$$
\left.+\frac{1}{\nu q}\left(-5\nu^2-\frac{109\nu}{2}+36+\frac{\tilde{L}^2}{\tilde{r}}\left(5\nu^2-7\nu+1\right)+\tilde{p}_r^2\tilde{r}\left(\frac{35\nu^2}{2}-\frac{51\nu}{2}+4\right)\right)\right]
$$

$$
+\frac{\nu^2\tilde{p}_r\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-5\nu-3+\frac{1}{\nu q}\left(-5\nu^2-\nu+3\right)\right], \tag{4.8}
$$

$$
\tilde{H}^{\rm NLO}_{\rm S_1^2S_2}=\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[-\frac{\nu}{4}-\frac{21}{2}-\frac{9\nu}{8}\frac{\tilde{L}^2}{\tilde{r}}+\frac{9\nu}{8}\tilde{p}_r^2\tilde{r}\right.
$$

$$
\left.+\frac{1}{q}\left(5-\frac{\nu}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{3}{8}-\frac{9\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(-\frac{33\nu}{8}-\frac{45}{8}\right)\right)\right]
$$

$$
+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2\tilde{S}_1^2}{\tilde{r}^6}\left[-\frac{5\nu}{4}-\frac{21}{4}-\frac{15\nu}{4}\tilde{p}_r^2\tilde{r}+\frac{1}{q}\left(-\frac{5\nu}{4}-\frac{7}{2}+\tilde{p}_r^2\tilde{r}\left(\frac{3}{2}-\frac{45\nu}{16}\right)\right)\right]
$$

$$
+\frac{\nu^2(\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1)^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2}{\tilde{r}^7}\left[\frac{9\nu}{4}+\frac{1}{q}\left(\frac{15\nu}{8}-\frac{3}{4}\right)\right]
$$

$$
+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[\frac{45}{2}-\frac{15\nu}{4}+3\nu\frac{\tilde{L}^2}{\tilde{r}}-9\nu\tilde{p}_r^2\tilde{r}\right.
$$

$$
\left.+\frac{1}{q}\left(-\frac{15\nu}{4}-\frac{71}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{63\nu}{8}+\frac{15}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{93\nu}{8}+\frac{9}{2}\right)\right)\right]
$$

$$
+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\left[\frac{33}{4}-\frac{3\nu}{4}+\frac{9\nu}{8}\frac{\tilde{L}^2}{\tilde{r}}+\frac{9\nu}{8}\tilde{p}_r^2\tilde{r}\right.
$$

$$
\left.+\frac{1}{q}\left(-\frac{3\nu}{4}-\frac{7}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{39\nu}{16}+\frac{9}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{69\nu}{16}-\frac{15}{8}\right)\right)\right]
$$

$$
\begin{aligned}
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^5}\Bigg[\frac{15\nu}{4}-3-\frac{51\nu}{8}\frac{\tilde{L}^2}{\tilde{r}}+\frac{21\nu}{8}\tilde{p}_r^2\tilde{r}\\
&+\frac{1}{q}\left(\frac{15\nu}{4}+\frac{37}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{27\nu}{8}-\frac{45}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{3}{8}-\frac{3\nu}{2}\right)\right)\Bigg]\\
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[\frac{21\nu}{4}+\frac{1}{q}\left(\frac{15\nu}{8}-\frac{21}{4}\right)\right]\\
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[9\nu+\frac{1}{q}\left(\frac{15\nu}{4}+3\right)\right], \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}^{\rm NLO}_{({\rm ES}_1^2){\rm S}_2}=&\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\left[\frac{3\nu}{4}+\frac{1}{2}-3\nu\tilde{p}_r^2\tilde{r}+\frac{1}{q}\left(\frac{3\nu}{4}-8+\tilde{p}_r^2\tilde{r}\left(-\frac{15\nu}{4}-3\right)\right)\right]\\
&+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2\tilde{S}_1^2}{\tilde{r}^6}\Bigg[-\frac{\nu}{4}-\frac{41}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15\nu}{8}-\frac{15}{16}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{99\nu}{8}-\frac{15}{16}\right)\\
&+\frac{1}{q}\left(-\frac{\nu}{4}-16+\frac{39\nu}{16}\frac{\tilde{L}^2}{\tilde{r}}+\tilde{p}_r^2\tilde{r}\left(\frac{249\nu}{16}-9\right)\right)\Bigg]+\frac{3\nu^2(\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1)^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2}{q\tilde{r}^7}\\
&+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\Bigg[14-\frac{15\nu}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15}{8}-\frac{15\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{15}{8}-\frac{51\nu}{4}\right)\\
&+\frac{1}{q}\left(55-\frac{15\nu}{4}-\frac{39\nu}{8}\frac{\tilde{L}^2}{\tilde{r}}+\tilde{p}_r^2\tilde{r}\left(6-\frac{129\nu}{8}\right)\right)\Bigg]\\
&+\frac{\nu^2\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\Bigg[\frac{3\nu}{2}+\frac{121}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{45}{16}-\frac{45\nu}{8}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{45}{16}-\frac{273\nu}{8}\right)\\
&+\frac{1}{q}\left(\frac{3\nu}{2}+44+\frac{\tilde{L}^2}{\tilde{r}}\left(3-\frac{117\nu}{16}\right)+\tilde{p}_r^2\tilde{r}\left(6-\frac{687\nu}{16}\right)\right)\Bigg]\\
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^5}\Bigg[\frac{31}{2}-\frac{3\nu}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15}{8}-\frac{15\nu}{4}\right)+\tilde{p}_r^2\tilde{r}\left(\frac{15}{8}-\frac{63\nu}{4}\right)\\
&+\frac{1}{q}\left(-\frac{3\nu}{4}-11-\frac{39\nu}{8}\frac{\tilde{L}^2}{\tilde{r}}+\tilde{p}_r^2\tilde{r}\left(3-\frac{159\nu}{8}\right)\right)\Bigg]\\
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{L}}\cdot\tilde{\vec{S}}_1\tilde{\vec{S}}_2\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[3\nu+\frac{1}{q}\left(\frac{15\nu}{4}+3\right)\right]\\
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{L}}\cdot\tilde{\vec{S}}_2\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[12\nu+\frac{1}{q}(15\nu+6)\right]. \qquad (4.10)
\end{aligned}
$$

As can be seen, the COM specification already results significant simplification of the general Hamiltonians. Yet the COM simplification has been the most detailed form of results provided in recent popular work via scattering-amplitudes methods which treat the unbound scattering problem.

Comparison of COM results from two scattering-amplitudes works. Chen et al. in [40], and Bern et al. in [41], presented COM Hamiltonians for BHs, and for generic compact objects, respectively. To start with, their results are discrepant with each other in several ways, and already at the initial level of scattering amplitudes, from which their COM Hamiltonians are derived. As we note below discrepancies of the two works with our results, we also touch on the discrepancies between them.

1. **Chen et al.** In [39] COM Hamiltonians for the case of BHs were presented in eqs. (3.96), (3.97), for the $\mathrm{S}_1^2\mathrm{S}_2$, and S_1^3 parts of the potential, respectively. Both of these parts are discrepant with our results. As can be seen in their section 3.4, [39] could not succeed in relating their results to ours via canonical transformations, beyond the quadratic-in-spin sectors [20], which were already well-confirmed then. As noted in version 2 of [39], ref. [53] with our final results here was fully shared on summer 2021 with Chen et al. upon their request, and this discrepancy with our results has been known prior to version 1 of [39].

   Moreover, [39] presented in appendix B their preliminary scattering amplitudes for generic compact objects, in which they were notably missing the new contribution proportional to $C_{\mathrm{ES}^2}^2$, that would correspond to our eq. (4.7) in the Hamiltonian. As this new sector contributes even when results are specified to the case of BHs, it is also clearly missing from their eq. (3.97) in [39] for the Hamiltonian. As we noted at the end of section 3.2, and as we shall see in section 5.1 below, this new sector is also imposed by Poincaré invariance, and therefore the Hamiltonian of [39] is generally not Poincarè-invariant, and thus is not only canonically inequivalent with our Hamiltonian, but is also inequivalent with any physically correct Hamiltonian.

   More generally, it should also be noted that the results in [39] are discrepant already at the level of scattering amplitudes, even when restricted to the case of BHs, with various corresponding amplitudes, as in [40], including the absence in [39] of a part that is proportional to $C_{\mathrm{ES}^2}^2$.

2. **Bern et al.** The COM Hamiltonians for generic compact objects found in the ancillary files of [40] are discrepant with ours, even at the LO, and even when specified to the simpler case of BHs. As noted in [40], their Hamiltonians contain, as of the cubic order in spin, new unfamiliar singularities in the COM momenta, $1/p^2$, where it was claimed there, that these singularities drop out for the case of BHs. However, it is easy to verify already at the LO cubic-in-spin sectors in [40], that these singularities remain, even after restricting to the case of BHs, namely when the 2 Wilson coefficients in these leading sectors are specified to unity, $C_{\mathrm{ES}^2} = C_{\mathrm{BS}^3} = 1$, as stipulated in our [20]. Thus the results in [40] are discrepant with our LO results in [25] from 2014, where those have since been well-confirmed via numerous independent methodologies, including in traditional GR methods.

   At the NLO the COM Hamiltonians in [40] for the cubic-in-spin sectors display similar singularities, even for BHs. Moreover, at the NLO the work in [40] included contributions with a claimed new Wilson coefficient, H_2, that they stipulated in their formulation. Such an extra free parameter violates spin-gauge invariance [20], and is also discrepant and absent in other corresponding scattering amplitudes, as in [39].

Finally, the singularities that appear in [40] as of the LO, as well as the above noted extra independent piece at the NLO in [40], cannot be generated by canonical transformations, and therefore the Hamiltonians in [40] are not canonically equivalent to our well-verified Hamiltonians at the LO [25], nor to our new present Hamiltonians at the NLO, which we verified via Poincaré invariance, as shall be seen in section 5.1 below.

Next, we can further restrict the Hamiltonians to the aligned-spins configuration, in which the spins satisfy $\vec{S}_a \cdot \vec{n} = \vec{S}_a \cdot \vec{p} = 0$, namely they are both aligned with the orbital angular momentum. It should be highlighted that the higher in spin the sectors are, the more dramatically they are affected by this simplification, with a greater loss of physical information as a consequence. This is in contrast to the simple spin-orbit sector, in which the single spin and the angular momentum are still trivially coupled [19]. In the present higher-spin sectors, applying the aligned-spins constraints to our COM Hamiltonians yields:

$$
\begin{aligned}
\tilde{H}^{\text{NLO}}_{\text{S}^3} = \frac{\nu^2 \tilde{L}\tilde{S}_1^3}{\tilde{r}^6} \Bigg[& -3\nu - \frac{9}{4} + \frac{\tilde{L}^2}{\tilde{r}} \left(2\nu - \frac{11}{8}\right) + \tilde{p}_r^2 \tilde{r} \left(\frac{33}{16} - \frac{39\nu}{8}\right) \\
& + \frac{1}{\nu q} \left(-3\nu^2 - \frac{3\nu}{2} + \frac{9}{4} + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{15\nu^2}{8} - \frac{19\nu}{4} + \frac{11}{8}\right) + \tilde{p}_r^2 \tilde{r} \left(-\frac{75\nu^2}{16} + 9\nu - \frac{33}{16}\right)\right) \\
& + C_{1(\text{ES}^2)} \Bigg(2\nu - \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{9\nu}{8} + \frac{3}{16}\right) + \tilde{p}_r^2 \tilde{r} \left(\frac{9\nu}{8} + \frac{63}{16}\right) \\
& + \frac{1}{\nu q} \left(2\nu^2 - \frac{17\nu}{2} + \frac{3}{2} + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{27\nu^2}{16} - \frac{3\nu}{4} - \frac{3}{16}\right) + \tilde{p}_r^2 \tilde{r} \left(\frac{57\nu^2}{16} + \frac{27\nu}{4} - \frac{63}{16}\right)\right)\Bigg) \\
& + C_{1(\text{BS}^3)} \Bigg(\nu + 7 - \nu \frac{\tilde{L}^2}{\tilde{r}} - \frac{7\nu}{2} \tilde{p}_r^2 \tilde{r} + \frac{1}{\nu q} \Bigg(\nu^2 + \frac{21\nu}{2} - 7 + \frac{\tilde{L}^2}{\tilde{r}} \left(\nu - \nu^2\right) \\
& + \tilde{p}_r^2 \tilde{r} \left(\frac{7\nu}{2} - \frac{7\nu^2}{2}\right)\Bigg)\Bigg)\Bigg] + \frac{\nu^2 \tilde{L}\tilde{S}_1^2 \tilde{S}_2}{\tilde{r}^6} \Bigg[-\frac{3\nu}{2} - \frac{63}{4} + \frac{9\nu}{8} \frac{\tilde{L}^2}{\tilde{r}} - \frac{21\nu}{8} \tilde{p}_r^2 \tilde{r} \\
& + \frac{1}{q} \left(\frac{3}{2} - \frac{3\nu}{2} + \frac{\tilde{L}^2}{\tilde{r}} \left(-\frac{3\nu}{8} - \frac{3}{8}\right) + \tilde{p}_r^2 \tilde{r} \left(-\frac{111\nu}{16} - \frac{33}{8}\right)\right) \\
& + C_{1(\text{ES}^2)} \Bigg(\frac{\nu}{2} - \frac{39}{4} + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{15\nu}{8} - \frac{15}{16}\right) + \tilde{p}_r^2 \tilde{r} \left(\frac{75\nu}{8} - \frac{15}{16}\right) \\
& + \frac{1}{q} \left(\frac{\nu}{2} - 24 + \frac{\tilde{L}^2}{\tilde{r}} \left(\frac{39\nu}{16} + 3\right) + \tilde{p}_r^2 \tilde{r} \left(\frac{189\nu}{16} - 12\right)\right)\Bigg)\Bigg] + (1 \leftrightarrow 2), \qquad (4.11)
\end{aligned}
$$

where as noted above, the significant loss of physical information in higher-spin sectors, due to the aligned-spins simplification, even in comparison with the already simplified COM Hamiltonian, is evident. Moreover, notably one of the 6 sectors of the potential drops upon this simplification — the new distinct sector that appears in eqs. (C.4), (D.4), (4.7), with the $C^2_{\text{ES}^2}$ prefactor. Accordingly, we see now that this new unique feature will not show up in any of the familiar observables, be it for GWs, or in the scattering problem, which are all in the aligned-spins simplified kinematic configuration.

A final simplification appropriate for the inspiral phase, where the orbit is quasi-circular, is that the necessary circular-orbit condition, $p_r \equiv \vec{p} \cdot \vec{n} = 0 \Rightarrow p^2 = p_r^2 + L^2/r^2 \rightarrow L^2/r^2$, is satisfied. Further applying this condition to our aligned-spins Hamiltonians yields:

$$\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}^3} = \frac{\nu^2 \tilde{L}\tilde{S}_1^3}{\tilde{r}^6}\Bigg[&-3\nu-\frac{9}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(2\nu-\frac{11}{8}\right)+\frac{1}{\nu q}\left(-3\nu^2-\frac{3\nu}{2}+\frac{9}{4}\right.\\
&\left.+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15\nu^2}{8}-\frac{19\nu}{4}+\frac{11}{8}\right)\right)+C_{1(\mathrm{ES}^2)}\left(2\nu-\frac{3}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{9\nu}{8}+\frac{3}{16}\right)\right.\\
&\left.+\frac{1}{\nu q}\left(2\nu^2-\frac{17\nu}{2}+\frac{3}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{27\nu^2}{16}-\frac{3\nu}{4}-\frac{3}{16}\right)\right)\right)\\
&+C_{1(\mathrm{BS}^3)}\left(\nu+7-\nu\frac{\tilde{L}^2}{\tilde{r}}+\frac{1}{\nu q}\left(\nu^2+\frac{21\nu}{2}-7+\frac{\tilde{L}^2}{\tilde{r}}\left(\nu-\nu^2\right)\right)\right)\Bigg]\\
&+\frac{\nu^2\tilde{L}\tilde{S}_1^2\tilde{S}_2}{\tilde{r}^6}\Bigg[-\frac{3\nu}{2}-\frac{63}{4}+\frac{9\nu}{8}\frac{\tilde{L}^2}{\tilde{r}}+\frac{1}{q}\left(\frac{3}{2}-\frac{3\nu}{2}+\frac{\tilde{L}^2}{\tilde{r}}\left(-\frac{3\nu}{8}-\frac{3}{8}\right)\right)\\
&+C_{1(\mathrm{ES}^2)}\left(\frac{\nu}{2}-\frac{39}{4}+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{15\nu}{8}-\frac{15}{16}\right)+\frac{1}{q}\left(\frac{\nu}{2}-24+\frac{\tilde{L}^2}{\tilde{r}}\left(\frac{39\nu}{16}+3\right)\right)\right)\Bigg]\\
&+(1\leftrightarrow 2). \qquad (4.12)
\end{aligned}$$

5 Poincaré algebra

The global Poincaré symmetry of isolated N-body systems in GR provides a powerful self-consistency check for the validity of general PN Hamiltonians in an arbitrary reference frame. From Noether's theorem, this global symmetry implies the existence of conserved integrals of motion, which form a representation of the Poincaré algebra in phase space. That is, the generators of Poincaré transformations satisfy the algebra that reads:

$$\{P_i,P_j\}=\{P_i,H\}=\{J_i,H\}=0,\quad \{J_i,J_j\}=\epsilon_{ijk}J_k,\quad \{J_i,P_j\}=\epsilon_{ijk}P_k, \qquad (5.1)$$

$$\{G_i,P_j\}=\delta_{ij}H,\quad \{G_i,H\}=P_i,\quad \{G_i,G_j\}=-\epsilon_{ijk}J_k,\quad \{J_i,G_j\}=\epsilon_{ijk}G_k, \qquad (5.2)$$

with $\vec{P}$ the total linear momentum, H the Hamiltonian, $\vec{J}$ the total angular momentum, and $\vec{K}$ the boost generator, which is traded here for $\vec{G}$, the generalized relativistic "center-of-mass" (henceforth center-of-mass), using $\vec{K}\equiv\vec{G}-t\vec{P}$. Thus, G_i/H is the center-of-mass that forms a canonical pair with the total linear momentum, P_i, but note that this center-of-mass does not satisfy the vanishing Poisson brackets of Newtonian center-of-mass vectors, but rather the relativistic Wigner rotation, $\{G_i,G_j\}=-\epsilon_{ijk}J_k$. Notice that the Poisson brackets in the sectors with spins are extended to include spin variables via the generalization [42, 54]:

$$\{f,g\}\equiv\{f,g\}_x+\{f,g\}_S, \qquad (5.3)$$

with

$$\{f,g\}_x=\sum_{I=1}^{2}\left(\frac{\partial f}{\partial x_I}\cdot\frac{\partial g}{\partial p_I}-\frac{\partial f}{\partial p_I}\cdot\frac{\partial g}{\partial x_I}\right), \qquad (5.4)$$

$$\{f,g\}_S=\sum_{I=1}^{2}S_I\times\frac{\partial f}{\partial S_I}\cdot\frac{\partial g}{\partial S_I}, \qquad (5.5)$$

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where all terms on the r.h.s. are understood to be vectors in scalar or triple products. By construction H satisfies translation and rotation invariance, and thus the Poisson brackets in eq. (5.1) are trivially satisfied. However, it is far from trivial to solve for the center-of-mass, $\vec{G}$, which should satisfy the Poisson brackets in eq. (5.2) with:

$$\vec{P} = \vec{p}_1 + \vec{p}_2, \quad \vec{J} = \sum_{I=1}^{2} \left(\vec{x}_I \times \vec{p}_I + \vec{S}_I \right), \tag{5.6}$$

for our binary system, and thus complete the full Poincaré algebra. Let us then turn to accomplish this ambitious task.

First, for $\vec{G}$ to satisfy:

$$\{G_i, P_j\} = \delta_{ij} H, \tag{5.7}$$

it should have the following form:

$$\vec{G} = h_1 \vec{x}_1 + h_2 \vec{x}_2 + \vec{Y}, \quad h_1 + h_2 = H, \tag{5.8}$$

where h_I and $\vec{Y}$ are translation-invariant, namely:

$$\{h_I, P_i\} = \{Y_i, P_j\} = 0, \tag{5.9}$$

which is equivalent to requiring that the dependence of h_I and $\vec{Y}$ on the position variables should be only through $\vec{x}_1 - \vec{x}_2$, i.e. in terms of $\vec{n}$ and r. $\vec{G}$ can then be uniquely solved from the constraint:

$$\{G_i, H\} = P_i. \tag{5.10}$$

As $\vec{P}$ and H are symmetric under the exchange of worldline labels, $1 \leftrightarrow 2$, $\vec{G}$ should be symmetric under this exchange as well. This means that $\vec{Y}$ needs to be symmetric under $1 \leftrightarrow 2$, and h_I can be written as:

$$h_I = \frac{H}{2} + h_I^{\text{AS}}, \tag{5.11}$$

where $h_1^{AS} = -h_2^{AS}$ needs to be antisymmetric under $1 \leftrightarrow 2$. Since this just means that $h_1 \vec{x}_1 + h_2 \vec{x}_2 = H(\vec{x}_1 + \vec{x}_2)/2 + h_1^{AS} r \vec{n}$, we can just recast $\vec{G}$ as:

$$\vec{G} = H(\vec{x}_1 + \vec{x}_2)/2 + \vec{Y}, \tag{5.12}$$

where $h_1^{AS} r \vec{n}$ is just contained in $\vec{Y}$. Our task thus boils down to constructing $\vec{Y}$ from the most general constrained ansatz, and finding the unique solution for it, using eq. (5.10).

Let us then list the considerations for the construction of $\vec{G}$. First, the solution of $\vec{G}$ is decomposed into different sectors classified according to their PN order, spin order in S_1 and S_2, as well as possible factors of Wilson coefficients. The building blocks for $\vec{Y}$ are the vectors: $\vec{n}$, $\vec{p}_I$, $\vec{S}_I$, and we use dimensional analysis and Euclidean covariance, including parity invariance and time-reversal. The constraint in eq. (5.2), $\{J_i, G_j\} = \epsilon_{ijk} G_k$, is automatically satisfied as long as $\vec{G}$ is constructed from vectors, $\vec{x}_I$, $\vec{p}_I$ and $\vec{S}_I$, to satisfy Euclidean covariance, such that $\vec{G}$ behaves as a vector under rotation. One subtle point though is that at 3 dimensions every 4 vectors are dependent through the general identity:

$$\vec{A}(\vec{B} \cdot \vec{C} \times \vec{D}) - \vec{B}(\vec{C} \cdot \vec{D} \times \vec{A}) + \vec{C}(\vec{D} \cdot \vec{A} \times \vec{B}) - \vec{D}(\vec{A} \cdot \vec{B} \times \vec{C}) = 0, \tag{5.13}$$

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which can hide a certain redundancy in a general ansatz for sectors that contain more than 3 vectors, as in any of the sectors with spins.

Finally, in flat spacetime, i.e. where $G_N \to 0$, the relativistic COM generator can be written in the following closed form [54]:

$$\vec{G}_{\text{flat}} = \sum_{I=1}^{2} \left(\gamma_I m_I \vec{x}_I - \frac{\vec{S}_I \times \vec{p}_I}{m_I(1+\gamma_I)} \right), \tag{5.14}$$

where $\gamma_I = \sqrt{1 + p_I^2/m_I^2}$. Eq. (5.14) is then used to fix the $\mathcal{O}(G_N^0)$ terms in $\vec{G}$ to agree with the special-relativistic limit. If the latter is used to constrain $\vec{G}$, then the remainder critical Poisson brackets involving the Wigner rotation, $\{G_i, G_j\} = -\epsilon_{ijk} J_k$, are also automatically satisfied in the solution for $\vec{G}$.

Following the various considerations above, we proceeded to solve for the full Poincaré algebra of the new complete precision-frontier at the 4.5PN order, which includes only sectors with spins: the NLO cubic-in-spin sectors from our [27] with the full general Hamiltonian first presented in section 4 above, and the N^{3}LO spin-orbit sector with the general Hamiltonian provided first in our [23]. It should be highlighted that for the latter sector the general ansatz to solve for, contains an order of $\sim 10^3$ free dimensionless coefficients. Therefore we needed to scale the solution of this problem, even compared to the most advanced Poincaré algebra at the 4PN order, that we provided in [32]. Note that the comprehensive construction of the Poincaré algebra is particularly strong as a consistency check of all sectors at the 4.5PN order: since there are no new Wilson coefficients introduced in any of the sectors at this order, the fulfilment of the Poincaré algebra is non-trivial for each of the relevant subsectors, so that they are all tested by the requirement of Poincaré invariance. This comprehensive check will thus establish the 4.5PN order as the new precision frontier.

5.1 NLO cubic-in-spin sectors

Let us then enumerate all the sectors in H and $\vec{G}$ relevant to the solution of $\vec{G}$ at the present NLO cubic-in-spin sectors. For the Hamiltonian we have:

$$H = H_{\text{N}} + H_{\text{1PN}} + H_{\text{SO}}^{\text{LO}} + H_{\text{S}^2}^{\text{LO}} + H_{\text{SO}}^{\text{NLO}} + H_{\text{S}^2}^{\text{NLO}} + H_{\text{S}^3}^{\text{LO}} + H_{\text{S}^3}^{\text{NLO}}, \tag{5.15}$$

with

$$H_{\text{SO}}^{\text{LO}} = H_{\text{S}_1}^{\text{LO}} + (1 \leftrightarrow 2), \qquad H_{\text{SO}}^{\text{NLO}} = H_{\text{S}_1}^{\text{NLO}} + (1 \leftrightarrow 2), \tag{5.16}$$

$$H_{\text{S}^2}^{\text{LO}} = C_{1\text{ES}^2} H_{\text{ES}_1^2}^{\text{LO}} + H_{\text{S}_1\text{S}_2}^{\text{LO}} + (1 \leftrightarrow 2), \tag{5.17}$$

$$H_{\text{S}^2}^{\text{NLO}} = H_{\text{S}_1^2}^{\text{NLO}} + C_{1\text{ES}^2} H_{\text{ES}_1^2}^{\text{NLO}} + H_{\text{S}_1\text{S}_2}^{\text{NLO}} + (1 \leftrightarrow 2), \tag{5.18}$$

$$H_{\text{S}^3}^{\text{LO}} = C_{1\text{ES}^2} H_{(\text{ES}_1^2)\text{S}_1}^{\text{LO}} + C_{1\text{BS}^3} H_{\text{BS}_1^3}^{\text{LO}} + H_{\text{S}_1^2\text{S}_2}^{\text{LO}} + C_{1\text{ES}^2} H_{(\text{ES}_1^2)\text{S}_2}^{\text{LO}} + (1 \leftrightarrow 2), \tag{5.19}$$

while $H_{\text{S}^3}^{\text{NLO}}$ is given in eq. (D.1). For the generalized COM, $\vec{G}$, we have:

$$\vec{G} = \vec{G}_{\text{N}} + \vec{G}_{\text{1PN}} + \vec{G}_{\text{SO}}^{\text{LO}} + \vec{G}_{\text{SO}}^{\text{NLO}} + \vec{G}_{\text{S}^2}^{\text{NLO}} + \vec{G}_{\text{S}^3}^{\text{NLO}}, \tag{5.20}$$

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with

$$\vec{G}^{\rm LO}_{\rm SO} = \vec{G}^{\rm LO}_{\rm S_1} + (1 \leftrightarrow 2), \qquad \vec{G}^{\rm NLO}_{\rm SO} = \vec{G}^{\rm NLO}_{\rm S_1} + (1 \leftrightarrow 2), \tag{5.21}$$

$$\vec{G}^{\rm LO}_{\rm S^2} = \vec{G}^{\rm LO}_{\rm S^3} = \vec{0}, \tag{5.22}$$

$$\vec{G}^{\rm NLO}_{\rm S^2} = \vec{G}^{\rm NLO}_{\rm S_1^2} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm ES_1^2} + \vec{G}^{\rm NLO}_{\rm S_1S_2} + (1 \leftrightarrow 2), \tag{5.23}$$

and we need to solve for:

$$\begin{aligned}\vec{G}^{\rm NLO}_{\rm S^3} =& \vec{G}^{\rm NLO}_{\rm S_1^3} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1} + C^2_{\rm 1ES^2}\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3} + C_{\rm 1BS^3}\vec{G}^{\rm NLO}_{\rm BS_1^3} + \vec{G}^{\rm NLO}_{\rm S_1^2S_2} + C_{\rm 1ES^2}\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_2} \\ &+ (1 \leftrightarrow 2). \end{aligned}\tag{5.24}$$

Note that all generators for LO sectors with spins (beyond the spin-orbit sector) are vanishing.

The decomposition above determines which Poisson brackets contribute in eq. (5.10) to solve for a given sector. $\vec{G}^{\rm NLO}_{\rm S_1^3}$ is solved by:

$$0 = \{\vec{G}^{\rm NLO}_{\rm S_1^3}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm S_1^3}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm NLO}_{\rm S_1^2}\}_x + \{\vec{G}^{\rm NLO}_{\rm S_1^2}, H^{\rm LO}_{\rm S_1}\}_x. \tag{5.25}$$

$\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1}$ is solved by:

$$\begin{aligned}0 =& \{\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_1}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm (ES_1^2)S_1}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm NLO}_{\rm ES_1^2}\}_x + \{\vec{G}_{\rm 1PN}, H^{\rm LO}_{\rm (ES_1^2)S_1}\}_x \\ &+ \{\vec{G}^{\rm NLO}_{\rm S_1}, H^{\rm LO}_{\rm ES_1^2}\}_x + \{\vec{G}^{\rm NLO}_{\rm ES_1^2}, H^{\rm LO}_{\rm S_1}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm (ES_1^2)S_1}\}_S + \{\vec{G}^{\rm NLO}_{\rm S_1^2}, H^{\rm LO}_{\rm ES_1^2}\}_S. \end{aligned}\tag{5.26}$$

$\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3}$ is solved by:

$$0 = \{\vec{G}^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3}\}_x + \{\vec{G}^{\rm NLO}_{\rm ES_1^2}, H^{\rm LO}_{\rm ES_1^2}\}_S. \tag{5.27}$$

$\vec{G}^{\rm NLO}_{\rm BS_1^3}$ is solved by:

$$0 = \{\vec{G}^{\rm NLO}_{\rm BS_1^3}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm BS_1^3}\}_x + \{\vec{G}_{\rm 1PN}, H^{\rm LO}_{\rm BS_1^3}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm BS_1^3}\}_S. \tag{5.28}$$

$\vec{G}^{\rm NLO}_{\rm S_1^2S_2}$ is solved by:

$$\begin{aligned}0 =& \{\vec{G}^{\rm NLO}_{\rm S_1^2S_2}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm S_1^2S_2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_2}, H^{\rm NLO}_{\rm S_1^2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_1}, 2H^{\rm NLO}_{\rm S_1S_2}\}_x \\ &+ \{\vec{G}_{\rm 1PN}, H^{\rm LO}_{\rm S_1^2S_2}\}_x + \{\vec{G}^{\rm NLO}_{\rm S_1}, 2H^{\rm LO}_{\rm S_1S_2}\}_x + \{2\vec{G}^{\rm NLO}_{\rm S_1S_2}, H^{\rm LO}_{\rm S_1}\}_x + \{\vec{G}^{\rm NLO}_{\rm S_1^2}, H^{\rm LO}_{\rm S_2}\}_x \\ &+ \{\vec{G}^{\rm LO}_{\rm S_2}, H^{\rm LO}_{\rm S_1^2S_2}\}_S + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm S_1^2S_2}\}_S + \Big[\{2\vec{G}^{\rm NLO}_{\rm S_1S_2}, 2H^{\rm LO}_{\rm S_1S_2}\}_S\Big]_{\rm S_1^2S_2} + \{\vec{G}^{\rm NLO}_{\rm S_1^2}, 2H^{\rm LO}_{\rm S_1S_2}\}_S, \end{aligned}\tag{5.29}$$

where $[f]_{\rm S_1^2S_2}$ means extracting the part of f that is quadratic in S_1 and linear in S_2. Finally, $\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_2}$ is solved by:

$$\begin{aligned}0 =& \{\vec{G}^{\rm NLO}_{\rm (ES_1^2)S_2}, H_{\rm N}\}_x + \{\vec{G}_{\rm N}, H^{\rm NLO}_{\rm (ES_1^2)S_2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_2}, H^{\rm NLO}_{\rm ES_1^2}\}_x + \{\vec{G}_{\rm 1PN}, H^{\rm LO}_{\rm (ES_1^2)S_2}\}_x \\ &+ \{\vec{G}^{\rm NLO}_{\rm S_2}, H^{\rm LO}_{\rm ES_1^2}\}_x + \{\vec{G}^{\rm NLO}_{\rm ES_1^2}, H^{\rm LO}_{\rm S_2}\}_x + \{\vec{G}^{\rm LO}_{\rm S_2}, H^{\rm LO}_{\rm (ES_1^2)S_2}\}_S + \{\vec{G}^{\rm LO}_{\rm S_1}, H^{\rm LO}_{\rm (ES_1^2)S_2}\}_S \\ &+ \{2\vec{G}^{\rm NLO}_{\rm S_1S_2}, H^{\rm LO}_{\rm ES_1^2}\}_S + \{\vec{G}^{\rm NLO}_{\rm ES_1^2}, 2H^{\rm LO}_{\rm S_1S_2}\}_S. \end{aligned}\tag{5.30}$$

Each of the above 6 equations corresponds to each of the 6 subsectors that we saw in the final actions of section 3.2. Notice that none of these equations is trivial, since the NLO cubic-in-spin sectors do not contain any new Wilson coefficients. Even the new subsector that is proportional to $C^2_{\mathrm{ES}_1^2}$ depends on the already-encountered Wilson coefficient $C_{\mathrm{ES}_1^2}$, and therefore, as can be seen, eq. (5.27) is non-trivial, since it contains a third term from the spin Poisson brackets of two lower-order sectors. Thus even though, as we shall see below in eq. (5.32), the solution for the COM generator of $C^2_{\mathrm{ES}_1^2}$ does not contribute to the general COM generator of the present NLO cubic-in-spin sectors, eq. (5.27) is only fulfilled in a non-trivial manner, which actually proves that the new subsector is inevitable due to the requirement of Poincaré invariance.

We recall that the solution for $\vec{G}^{\mathrm{NLO}}_{\mathrm{S}^3}$ is written as:

$$\vec{G}^{\mathrm{NLO}}_{\mathrm{S}^3} = H^{\mathrm{LO}}_{\mathrm{S}^3}\frac{(\vec{x}_1+\vec{x}_2)}{2} + \left(\vec{Y}^{\mathrm{NLO}}_{\mathrm{S}_1^3} + \vec{Y}^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2} + (1\leftrightarrow 2)\right), \tag{5.31}$$

and we find:

$$\begin{aligned}
\vec{Y}^{\mathrm{NLO}}_{\mathrm{S}_1^3} = &-\frac{GC_{1\mathrm{ES}^2}m_2}{8{m_1}^3r^3}\Big[15\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2\vec{n} - 15\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{n}(\vec{S}_1\cdot\vec{n})^2 \\
&- 12\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{S}_1 + 2S_1^2\vec{S}_1\times\vec{p}_1 + 6(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{p}_1\Big] \\
&+ \frac{GC_{1\mathrm{BS}^3}m_2}{6{m_1}^3r^3}\Big[6\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{S}_1 + S_1^2\vec{S}_1\times\vec{p}_1 - 3(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{p}_1\Big] \\
&+ \frac{Gm_2}{4{m_1}^3r^3}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2\vec{n} - 9S_1^2\vec{S}_1\times\vec{n}\vec{p}_1\cdot\vec{n} - 3\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{S}_1 \\
&+ 2S_1^2\vec{S}_1\times\vec{p}_1 + 6(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{p}_1\Big] + \frac{3GC_{1\mathrm{ES}^2}}{8{m_1}^2r^3}\Big[S_1^2\vec{S}_1\times\vec{n}\vec{p}_2\cdot\vec{n} \\
&+ 5\vec{S}_1\times\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 - 2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{n} - 3S_1^2\vec{S}_1\times\vec{p}_2 \\
&+ 7(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{p}_2\Big] - \frac{GC_{1\mathrm{BS}^3}}{6{m_1}^2r^3}\Big[6\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{S}_1 + S_1^2\vec{S}_1\times\vec{p}_2 \\
&- 3(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{p}_2\Big] - \frac{G}{4{m_1}^2r^3}\Big[3S_1^2\vec{S}_1\times\vec{n}\vec{p}_2\cdot\vec{n} - 12\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{n} \\
&- 5S_1^2\vec{S}_1\times\vec{p}_2 + 12(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{p}_2\Big],
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
\vec{Y}^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2} = &-\frac{G}{4{m_1}^2r^3}\Big[15\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{S}_2\vec{n} - 9\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{n} \\
&- 15\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{n}\vec{S}_2\cdot\vec{n} + 6\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_2\cdot\vec{n}\vec{S}_1 + 4\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{S}_1 \\
&- 18\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{S}_2 - 6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{p}_1 + 6\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{p}_1\Big] \\
&- \frac{GC_{1\mathrm{ES}^2}}{8{m_1}^2r^3}\Big[24\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2\vec{n} - 6\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{S}_2\vec{n} + 27\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{n} \\
&+ 21\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\vec{p}_1\cdot\vec{n} + 3\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\vec{p}_1\cdot\vec{n} + 24\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\vec{p}_1\cdot\vec{n} \\
&- 15\vec{S}_2\times\vec{n}\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 - 18\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_2\cdot\vec{n}\vec{S}_1 - 31\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{S}_1 \\
&- 27\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{p}_1 + 15\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{p}_1 - 19\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{S}_2\Big] \\
&+ \frac{GC_{1\mathrm{ES}^2}}{8m_1m_2r^3}\Big[21\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2\vec{n} + 24\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}\vec{n}
\end{aligned}$$

$$
\begin{aligned}
&+24\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\vec{p}_2\cdot\vec{n}+30\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\vec{p}_2\cdot\vec{n}-24\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{n}\vec{S}_2\cdot\vec{n}\\
&+39\vec{S}_2\times\vec{n}\cdot\vec{p}_2\vec{n}(\vec{S}_1\cdot\vec{n})^2-24\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_2\cdot\vec{n}\vec{S}_1-34\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2\vec{S}_1\\
&-30\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{p}_2+18\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{p}_2-18\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{S}_2\Big]\\
&+\frac{G}{4m_1m_2r^3}\Big[12\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2\vec{n}+9\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{S}_2\vec{n}\\
&+15\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}\vec{n}+12\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\vec{p}_2\cdot\vec{n}-3\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\vec{p}_2\cdot\vec{n}\\
&+21\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\vec{p}_2\cdot\vec{n}+15\vec{S}_2\times\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-12\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_2\cdot\vec{n}\vec{S}_1\\
&-12\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2\vec{S}_1-9\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{S}_2-9\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{p}_2\\
&+10\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{p}_2-7\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{S}_2\Big].
\end{aligned}\tag{5.33}
$$

To recap, we solved for the complete Poincaré algebra of the present NLO cubic-in-spin sectors, which provides a strong confirmation for the validity of our new general Hamiltonian presented in eq. (D.1).

5.2 N^{3}LO spin-orbit sector

To complete the Poincaré algebra at the 4.5PN order we proceed to solve for the N^{3}LO spin-orbit sector. Again, we enumerate all the sectors in H and $\vec{G}$ relevant to the solution of $\vec{G}$ at the N^{3}LO spin-orbit sector. For the Hamiltonian we have:

$$H=H_{\rm N}+H_{\rm 1PN}+H_{\rm SO}^{\rm LO}+H_{\rm 2PN}+H_{\rm SO}^{\rm NLO}+H_{\rm 3PN}+H_{\rm SO}^{\rm N^2LO}+H_{\rm SO}^{\rm N^3LO},\tag{5.34}$$

with

$$H_{\rm SO}^{\rm LO}=H_{\rm S_1}^{\rm LO}+(1\leftrightarrow2),\qquad H_{\rm SO}^{\rm NLO}=H_{\rm S_1}^{\rm NLO}+(1\leftrightarrow2),\tag{5.35}$$
$$H_{\rm SO}^{\rm N^2LO}=H_{\rm S_1}^{\rm N^2LO}+(1\leftrightarrow2),\qquad H_{\rm SO}^{\rm N^3LO}=H_{\rm S_1}^{\rm N^3LO}+(1\leftrightarrow2),\tag{5.36}$$

where $H_{\rm SO}^{\rm N^3LO}$ is given in our [23]. For the generalized COM, $\vec{G}$, we have:

$$\vec{G}=\vec{G}_{\rm N}+\vec{G}_{\rm 1PN}+\vec{G}_{\rm SO}^{\rm LO}+\vec{G}_{\rm 2PN}+\vec{G}_{\rm SO}^{\rm NLO}+\vec{G}_{\rm 3PN}+\vec{G}_{\rm SO}^{\rm N^2LO}+\vec{G}_{\rm SO}^{\rm N^3LO},\tag{5.37}$$

with

$$\vec{G}_{\rm SO}^{\rm LO}=\vec{G}_{\rm S_1}^{\rm LO}+(1\leftrightarrow2),\qquad \vec{G}_{\rm SO}^{\rm NLO}=\vec{G}_{\rm S_1}^{\rm NLO}+(1\leftrightarrow2),\tag{5.38}$$
$$\vec{G}_{\rm SO}^{\rm N^2LO}=\vec{G}_{\rm S_1}^{\rm N^2LO}+(1\leftrightarrow2),\tag{5.39}$$

and we need to solve for:

$$\vec{G}_{\rm SO}^{\rm N^3LO}=\vec{G}_{\rm S_1}^{\rm N^3LO}+(1\leftrightarrow2).\tag{5.40}$$

$\vec{G}_{\rm S_1}^{\rm N^3LO}$ is solved by:

$$
\begin{aligned}
0=&\{\vec{G}_{\rm S_1}^{\rm N^3LO},H_{\rm N}\}_x+\{\vec{G}_{\rm N},H_{\rm S_1}^{\rm N^3LO}\}_x+\{\vec{G}_{\rm S_1}^{\rm LO},H_{\rm 3PN}^{\rm N^3LO}\}_x+\{\vec{G}_{\rm 1PN},H_{\rm S_1}^{\rm N^2LO}\}_x\\
&+\{\vec{G}_{\rm S_1}^{\rm NLO},H_{\rm 2PN}^{\rm N^2LO}\}_x+\{\vec{G}_{\rm 2PN}^{\rm N^2LO},H_{\rm S_1}^{\rm NLO}\}_x+\{\vec{G}_{\rm S_1}^{\rm N^2LO},H_{\rm 1PN}\}_x+\{\vec{G}_{\rm 3PN}^{\rm N^3LO},H_{\rm S_1}^{\rm LO}\}_x\\
&+\{\vec{G}_{\rm S_1}^{\rm LO},H_{\rm S_1}^{\rm N^2LO}\}_S+\{\vec{G}_{\rm S_1}^{\rm NLO},H_{\rm S_1}^{\rm NLO}\}_S+\{\vec{G}_{\rm S_1}^{\rm N^2LO},H_{\rm S_1}^{\rm LO}\}_S.
\end{aligned}\tag{5.41}
$$

We recall that the solution for $\vec{G}_{\rm SO}^{\rm N^3LO}$ is written as:

$$\vec{G}_{\rm SO}^{\rm N^3LO} = H_{\rm SO}^{\rm N^2LO} \frac{(\vec{x}_1 + \vec{x}_2)}{2} + \left(\vec{Y}_{\rm S_1}^{\rm N^3LO} + (1 \leftrightarrow 2)\right), \tag{5.42}$$

and we find $\vec{Y}_{\rm S_1}^{\rm N^3LO}$, which we provide in appendix E due to its large volume. Thus, we found the complete Poincaré algebra of the N^{3}LO spin-orbit sector, which provides a strong confirmation for the validity of the full general Hamiltonian first presented in our [23].

To illustrate how stringent the consistency check of the Poincaré algebra is for PN Hamiltonians, let us note that we have checked that the N^{3}LO spin-orbit Hamiltonian of Mandal et al. [55] fails to pass the Poincaré-algebra consistency-check, namely the Hamiltonian of [55] is not Poincaré invariant (beware that version 1 of [55] also incorrectly assumed that an insertion of the EOM in their section 4.1 was equivalent to the necessary redefinitions). This is in contrast with our corresponding result, earlier derived in [23], which we proved to be Poincaré invariant in this section. Thus, clearly, as we also verified independently, the result of [55] is not canonically related to ours in [23].

6 GW and scattering observables

The full Lagrangians and general Hamiltonians are both useful to derive the EOMs or the Poincaré invariance, and to construct EOB models essential for GW templates. Despite their wealth in general physical information, they are all gauge-dependent. Accordingly they are bulky and leave some room for possible ambiguities, which dramatically multiply when going to higher-order sectors, such as the present ones. For these reasons, it is crucial to also obtain some handy observables in various restricted kinematic configurations, which can be readily compared in GW measurements carried out in LIGO, Virgo, or KAGRA. With kinematic constraints, as outlined in section 4.2 above for simplified Hamiltonians, one can define the associated binding energies, e, using $e \equiv \tilde{H}$, and relate them to observables and gauge-invariant quantities. In this section we provide such meaningful gauge-invariant relations expressed via the measured frequency of GWs, which have been critical in the construction of GW templates.

We also derive extrapolated scattering angles that are specific to aligned spins for the guidance of recent popular studies of the scattering problem in the weak-field approximation.

6.1 Binding energies and gauge-invariants

The gauge-invariant relations in this section are all derived under the condition of circular orbit, which is very fitting for the inspiral phase of the binary. Using $\dot{p}_r = -\partial\tilde{H}(\tilde{r},\tilde{L})/\partial\tilde{r} = 0$, on top of the constraints listed in section 4.2, enables to eliminate the coordinate dependence from the specialized Hamiltonians, and obtain the binding energy for the present sectors as a function of the total angular momentum:

$$
\begin{aligned}
(e)^{\rm NLO}_{\rm S^3}(\tilde{L}) = \frac{\nu^2\tilde{S}_1^3}{\tilde{L}^{11}}\bigg[&\frac{167\nu}{4}-\frac{389}{8}+\left(11\nu-\frac{927}{8}\right)C_{1\rm ES^2}-13C_{1\rm BS^3}\\
&+\frac{1}{\nu q}\left(\frac{291\nu^2}{8}+29\nu+\frac{389}{8}+\left(\frac{103\nu^2}{8}-\frac{169\nu}{4}+\frac{927}{8}\right)C_{1\rm ES^2}\right.\\
&\left.+\left(13-\frac{17\nu}{2}\right)C_{1\rm BS^3}\right)\bigg]+\frac{\nu^2\tilde{S}_1^2\tilde{S}_2}{\tilde{L}^{11}}\bigg[\frac{201\nu}{4}+\frac{2913}{4}+\left(5\nu+\frac{1437}{8}\right)C_{1\rm ES^2}\\
&+\frac{1}{q}\left(\frac{243\nu}{8}+\frac{1917}{4}+\left(\frac{55\nu}{8}+222\right)C_{1\rm ES^2}\right)\bigg]+(1\leftrightarrow 2).
\end{aligned} \tag{6.1}
$$

Using the PN parameter, $x\equiv\tilde{\omega}^{2/3}$, for the gauge-invariant frequency, in Hamilton's equation for the orbital phase, $d\phi/d\tilde{t}\equiv\tilde{\omega}=\partial\tilde{H}(\tilde{r},\tilde{L})/\partial\tilde{L}=0$, provides the relation of total angular momentum to the GW frequency:

$$
\begin{aligned}
\frac{1}{\tilde{L}^2}\supset \nu^2 x^{11/2}\tilde{S}_1^3\bigg[&\frac{247\nu}{81}-\frac{757}{6}+\left(-\frac{443\nu}{9}-\frac{361}{4}\right)C_{1\rm ES^2}+\left(14\nu+\frac{92}{3}\right)C_{1\rm BS^3}\\
&+\frac{1}{\nu q}\left(\frac{59\nu^2}{18}-\frac{202\nu}{3}+\frac{757}{6}+\left(-\frac{617\nu^2}{12}-\frac{293\nu}{2}+\frac{361}{4}\right)C_{1\rm ES^2}\right.\\
&\left.+\left(14\nu^2+\frac{149\nu}{3}-\frac{92}{3}\right)C_{1\rm BS^3}\right)\bigg]+\nu^2x^{11/2}\tilde{S}_1^2\tilde{S}_2\bigg[\frac{4835\nu}{54}+\frac{9329}{18}\\
&+\left(\frac{\nu}{9}-\frac{83}{12}\right)C_{1\rm ES^2}+\frac{1}{q}\left(\frac{1703\nu}{18}+\frac{3083}{6}+\left(\frac{40}{3}-\frac{25\nu}{12}\right)C_{1\rm ES^2}\right)\bigg]\\
&+(1\leftrightarrow 2).
\end{aligned} \tag{6.2}
$$

Then, by using the two previous relations, the binding energy can also be expressed in terms of the frequency:

$$
\begin{aligned}
(e)^{\rm NLO}_{\rm S^3}(x) = \nu^2x^{11/2}\tilde{S}_1^3\bigg[&\frac{4}{3}-\frac{128\nu}{81}+\left(2-\frac{20\nu}{9}\right)C_{1\rm ES^2}+\left(-4\nu-\frac{4}{3}\right)C_{1\rm BS^3}\\
&+\frac{1}{\nu q}\left(-\frac{8\nu^2}{9}+\frac{8\nu}{3}-\frac{4}{3}+\left(\frac{2\nu^2}{3}+16\nu-2\right)C_{1\rm ES^2}\right.\\
&\left.+\left(-4\nu^2-\frac{28\nu}{3}+\frac{4}{3}\right)C_{1\rm BS^3}\right)\bigg]+\nu^2x^{11/2}\tilde{S}_1^2\tilde{S}_2\bigg[\frac{4\nu}{27}+\frac{82}{9}\\
&+\left(\frac{64\nu}{9}-\frac{32}{3}\right)C_{1\rm ES^2}+\frac{1}{q}\left(\frac{28}{3}-\frac{32\nu}{9}+\left(10\nu-\frac{32}{3}\right)C_{1\rm ES^2}\right)\bigg]\\
&+(1\leftrightarrow 2).
\end{aligned} \tag{6.3}
$$

These results for the binding energy which we provide here for the first time match those which were initially derived in [53].

6.2 Extrapolated scattering angles

In the so-called post-Minkowskian (PM) approximation for a weak gravitational field, where scattering events are studied in a perturbative expansion in G, the common observable is the scattering angle, defined in the COM frame for the simplified case of aligned spins. The extrapolated scattering angle can be computed at low perturbative orders, namely where logarithms do not show up yet, in the overlap of PN and PM approximations. For the PM approximation this link is not feasible as of the third subleading order, which amounts to reaching results only up to the 2PN order. Yet recently a unique novel approach was put forward in [56], which capitalizes on amplitudes methods directly in the bound problem of the binary inspiral. Thus the approach in [56] faces none of the obstructions that are common to all other amplitudes-based approaches that are set on the scattering problem.

At the present NLO sectors thus, the link with scattering can be achieved starting the computation from our aligned-spins Hamiltonians in eq. (4.11) by extending the binding energy of our PN Hamiltonian of a binary inspiral to the kinetic energy of scattering. These Hamiltonians are not specified to the "quasi-isotropic" gauge, as in all other scattering-based works, and we simply use the integration considerations outlined in [57]. Our scattering angles are thus computed similarly to our [23, 35], where here we just need to truncate our final expansion in G at $\mathcal{O}(G^2)$.

We remind some conventional notation [35]:

$$p_\infty = \frac{m_1 m_2}{E}\sqrt{\gamma^2-1}, \quad E = \sqrt{m_1^2+m_2^2+2m_1m_2\gamma}, \quad \gamma = \frac{1}{\sqrt{1-v_\infty^2}}, \tag{6.4}$$

and:

$$\tilde{b} = \frac{v_\infty^2}{Gm}b, \qquad \tilde{v} = \frac{v_\infty}{c}, \qquad \tilde{a}_i = \frac{S_i}{bm_ic}, \qquad \Gamma = \frac{E}{mc^2} = \sqrt{1+2\nu(\gamma-1)}, \tag{6.5}$$

and thus we find that our consequent scattering angles in the present sectors are given by:

$$\theta^{\rm NLO}_{\rm S^3} = \theta^{\rm NLO}_{\rm S_1^3} + \theta^{\rm NLO}_{\rm S_1^2 S_2} + (1 \leftrightarrow 2), \tag{6.6}$$

where

$$\begin{aligned}
\frac{\theta^{\rm NLO}_{\rm S_1^3}}{\Gamma} = \tilde{v}\tilde{a}_1^3 \Bigg[& -\frac{4}{\tilde{b}} C_{1{\rm BS}^3} + \frac{\pi}{\tilde{b}^2}\left(\frac{15\nu}{4}\tilde{v}^2 + \left(3\nu - 6 - \left(\frac{3\nu}{2}+\frac{27}{4}\right)\tilde{v}^2\right) C_{1{\rm ES}^2}\right. \\
& + \left(-6 + \left(\frac{27\nu}{4} - \frac{33}{4}\right)\tilde{v}^2\right) C_{1{\rm BS}^3} \\
& \left. + \frac{\nu}{q}\left(\frac{15}{4}\tilde{v}^2 + \left(3 - \frac{3}{2}\tilde{v}^2\right) C_{1{\rm ES}^2} + \frac{27}{4}\tilde{v}^2 C_{1{\rm BS}^3}\right)\right)\Bigg],
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
\frac{\theta^{\rm NLO}_{\rm S_1^2 S_2}}{\Gamma} = \tilde{v}\tilde{a}_1^2\tilde{a}_2 \Bigg[& -\frac{12}{\tilde{b}} C_{1{\rm ES}^2} + \frac{\pi}{\tilde{b}^2}\left(6\nu - 12 + \tilde{v}^2\left(\frac{27\nu}{8} - \frac{99}{8}\right)\right. \\
& + \left(-3\nu + \left(\frac{39\nu}{8} - \frac{207}{8}\right)\tilde{v}^2 - 21\right) C_{1{\rm ES}^2} \\
& \left. + \frac{\nu}{q}\left(6 + \frac{27}{8}\tilde{v}^2 + \left(-3 + \frac{39}{8}\tilde{v}^2\right) C_{1{\rm ES}^2}\right)\right)\Bigg].
\end{aligned} \tag{6.8}$$

Our scattering angles agree with the NLO PM ones derived for the case of BHs in [38], namely when the Wilson coefficients for both of the objects are specified to unity, $C_{\rm ES^2} = C_{\rm BS^3} = 1$, as we prescribed in [20]. Note that the derivations in [38] built on our higher-spin worldline theory, and results presented in [20, 25], as this dependence was omitted in [38]. Thus the limited results in [38] are inherently dependent on our self-contained worldline framework. It should also be highlighted that our results in this work — in contrast with those in [38] — are also not limited to the aligned-spins constraint, which is a significant and ever-growing simplification when going to higher-spin sectors. The latter was already clearly demonstrated at the quadratic-in-spin sectors in [35], in section 4.2 above, and in [36]. The aligned-spins constraint entails a growing loss of physical information that is always absent from the scattering-angle observable.

7 Conclusions

We confirmed the generalized actions of the complete NLO cubic-in-spin interactions for generic compact binaries which were tackled first in [27] via an extension of the EFT of spinning gravitating objects [20] and the public `EFTofPNG` code [21]. These higher-spin sectors enter at the 4.5PN order, and are at the present precision frontier in PN theory. The interaction potentials are made up of 6 independent sectors, including a new unique sector that is proportional to the square of the quadrupolar deformation parameter, C_{ES^2}. From these actions the EOMs of both the position and spin can be directly obtained via straightforward variation [20, 53]. We derived the full general Hamiltonians in an arbitrary reference frame and in generic kinematic configurations. Such general Hamiltonians uniquely enable to study the full global Poincaré algebra in phase space, which also provides a critical consistency check of state-of-the-art PN theory.

We carried out such a complete study of the Poincaré algebra for all of the sectors at the 4.5PN precision frontier, including the N^{3}LO spin-orbit sector that we presented for the first time in [23], in order to establish the new precision frontier at this order. We fully solved for the Poincaré algebra of both the NLO cubic-in-spin sectors from our [27], and the N^{3}LO spin-orbit sector from [22, 23]. We note that to accomplish the latter it was crucial in particular to extend the formal procedure of redefinitions of rotational variables, which was first introduced in [52]. This extension was indeed carried out in [23] beyond linear order, but was critically missed in version 1 of [55], and contributed to their incorrect result. It should thus be highlighted that the Poincaré construction, which serves as a strong consistency-check, was especially critical in this case, since a valid general Hamiltonian of the N^{3}LO spin-orbit sector was obtained only in [23]: even the corrected Hamiltonian in the present revised version 2 of [55] failed to pass the Poincaré consistency-check, and thus clearly, is also not canonically related to ours in [23].

We note that similar to the N^{3}LO spin-orbit sector, the NLO cubic-in-spin sectors do not involve new operators/Wilson coefficients with respect to the LO ones at the 3.5PN order (though the EFT evaluation of the NLO is obviously much more intricate). This is why the construction of the Poincaré algebra is particularly strong as a consistency check of all sectors at the 4.5PN order: since there are no new sectors or Wilson coefficients introduced in any of the sectors at this order, the fulfilment of the Poincaré algebra is non-trivial for each of the relevant subsectors, so that they are all tested by Poincaré invariance. Nevertheless, the Wilson coefficients that appear only in such higher-order higher-spin sectors are critical to learn on strong gravity and QCD theories.

Subsequently we derived simplified Hamiltonians under restricted kinematic constraints, where it is seen that the COM aligned-spins Hamiltonians get significantly less informative at higher-spin sectors. In particular the new potential proportional to $C_{\mathrm{ES}^2}^2$ vanishes in the aligned-spins simplification. From these simplified Hamiltonians we derived the observable binding energies in terms of their gauge-invariant relations to the angular momentum and the frequency, which are critical for GW applications. We also derived the extrapolated scattering angles defined for the aligned-spins configuration in the scattering problem. We found agreement with the angles derived for the scattering of BHs via scattering-amplitudes methods, that built on our higher-spin theory, and are thus also dependent on our self-contained framework. Finally, our completion of the Poincaré algebra at the 4.5PN order provides strong confidence that this new precision frontier for GW measurements has now been established.

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A Generalized actions from EFT evaluation

We collect below typos from manually printing results from our computer files for 5 graphs in the journal version of [27], where the correct values are noted in boldface (after an arrow):

Figure 2(a1) $\supset$

$$\begin{aligned}
&-C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big((2\rightarrow\mathbf{3})\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n}(S_1^2-5(S_1\cdot\vec{n})^2)-\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\big)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big((S_1^2-5(\vec{S}_1\cdot\vec{n})^2)\vec{v}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n})\\
&-\vec{S}_1\cdot\vec{v}_2(\vec{S}_1\cdot\vec{v}_1-5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n})\Big)+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(\frac{1}{2}S_1^2(v_1^2+v_2^2)\\
&-\vec{S}_1\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n})-\frac{5}{2}(\vec{S}_1\cdot\vec{n})^2(v_1^2+v_2^2)\Big)\\
&-\frac{1}{2}\vec{v}_1\cdot\vec{v}_2\times\vec{n}\ \vec{S}_1\cdot\vec{v}_1(S_1^2-5(\vec{S}_1\cdot\vec{n})^2)\Big],
\end{aligned}\tag{A.1}$$

Figure 2(a2) $\supset$

$$\begin{aligned}
&\frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\ \Big[\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(S_1^2(v_1^2+3v_2^2)-2\vec{S}_1\cdot(\vec{n}\rightarrow\tilde{\mathbf{v}}_\mathbf{1})(\vec{S}_1\cdot\vec{v}_2-5\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n})\\
&-5(\vec{S}_1\cdot\vec{n})^2(v_1^2+3v_2^2)\Big)-2\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\ \vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\Big],
\end{aligned}\tag{A.2}$$

Figure 2(a4) $\supset$

$$+C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[2\ \vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ (\vec{S}_1\cdot\vec{a}_1+\dot{\vec{S}}_1\cdot\vec{v}_1)+2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{v}_1$$
$$+4\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{S}_1\ +(2\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\ S_1^2\!\rightarrow\mathbf{0})\ +\vec{S}_2\cdot\vec{v}_2\times\vec{a}_2(S_1^2-3(\vec{S}_1\cdot\vec{n})^2)$$
$$-6\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big((S_1^2\ \vec{a}_1\cdot\vec{n}\!\rightarrow\mathbf{0})-2\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}$$
$$+\vec{S}_1\cdot\vec{a}_1\ \vec{S}_1\cdot\vec{n}\Big)\Big], \tag{A.3}$$

Figure 2(a9) $\supset$

$$-\frac{3}{2}C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}\ +\vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}$$
$$+\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2\ \ -5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\ \Big)-\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big((+\!\rightarrow-)S_1^2\vec{v}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}$$
$$-5(\vec{S}_1\cdot\vec{n})^2\vec{v}_2\cdot\vec{n}\Big)-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\ \Big(S_1^2(\vec{v}_1\cdot\vec{v}_2\ \ -5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n})-2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2$$
$$+10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}+10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big)$$
$$+5(\vec{S}_1\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2\ \ -7\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n})\Big)\Big], \tag{A.4}$$

Figure 3(b6) $\supset$

$$C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\Big[-(23\!\rightarrow\mathbf{39})\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+13\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}$$
$$-\vec{S}_2\cdot\vec{v}_1\times\vec{n}((31\!\rightarrow\mathbf{15})S_1^2-66(\vec{S}_1\cdot\vec{n})^2)-\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(10\vec{S}_1\cdot\vec{v}_1-(51\!\rightarrow\mathbf{63})\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\Big)$$
$$+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(11\vec{S}_1\cdot\vec{v}_2-(54\!\rightarrow\mathbf{66})\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]. \tag{A.5}$$

As we noted in [27] and section 3 above the generalized actions of the NLO cubic-in-spin interactions are confirmed (eqs. (5.2) and (5.13) there). There were 2 independent typos in print (correction marked boldface after an arrow) compared with our computer files. In the last term of $L_{(2)}$ (eq. (5.4) in [27]):

$$L_{(2)}\supset\ +3\ \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{S}_2\ (v_1^2-5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n})-\vec{S}_1\cdot\vec{v}_1\ \vec{S}_2\cdot\vec{v}_1$$
$$+(15\!\rightarrow\mathbf{5})\ \vec{S}_1\cdot\vec{v}_2\ \vec{S}_2\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big), \tag{A.6}$$

and in $L_{(6)}$ (eq. (5.8) in [27]):

$$L_{(6)}\supset\ +\frac{1}{2}\ \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \Big(\vec{S}_1\cdot\vec{v}_1-(1\!\rightarrow\mathbf{2})\vec{S}_1\cdot\vec{v}_2-3\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-3\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big). \tag{A.7}$$

Let us stress that the computer files of [27], also included in the supplementary material attached to this paper, contain the correct results.

B New redefinitions at the NLO cubic-in-spin sectors

The new position shifts fixed in the present sectors can be written as:

$$(\Delta\vec{x}_1)^{\rm NLO}_{\rm S^3}=(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1^3}+(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1^2S_2}+(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1S_2^2}+(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_2^3}\,, \tag{B.1}$$

where

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1^3} =& -\frac{3Gm_2}{16{m_1}^3r^3}\Big[17S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{n}+4S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{n}-S_1^2\vec{S}_1\times\vec{n}\big(7\vec{v}_1\cdot\vec{n}\\
&+5\vec{v}_2\cdot\vec{n}\big)-5\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1-4\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{S}_1\\
&+12\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}+8S_1^2\vec{S}_1\times\vec{v}_1+3S_1^2\vec{S}_1\times\vec{v}_2\Big]\\
&-\frac{GC_{\rm 1ES^2}m_2}{4{m_1}^3r^3}\Big[6S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{n}+12S_1^2\vec{S}_1\times\vec{n}\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)\\
&-9\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1-24\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}+24\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{n}\\
&-7S_1^2\vec{S}_1\times\vec{v}_1+15(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{v}_1+12S_1^2\vec{S}_1\times\vec{v}_2-24(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{v}_2\Big]\\
&-\frac{GC_{\rm 1BS^3}m_2}{6{m_1}^3r^3}\Big[3S_1^2\vec{S}_1\times\vec{n}\vec{v}_2\cdot\vec{n}-15\vec{S}_1\times\vec{n}\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1+6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2\vec{S}_1-6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}\\
&+S_1^2\vec{S}_1\times\vec{v}_2-3(\vec{S}_1\cdot\vec{n})^2\vec{S}_1\times\vec{v}_2\Big]\\
&+\frac{GC_{\rm 1ES^2}m_2}{2{m_1}^3r^2}\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{S}}_1\vec{S}_1-\vec{S}_1\cdot\dot{\vec{S}}_1\vec{S}_1\times\vec{n}\Big]+\frac{Gm_2}{16{m_1}^3r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\dot{\vec{S}}_1\vec{S}_1\\
&+3\vec{S}_1\cdot\dot{\vec{S}}_1\vec{S}_1\times\vec{n}-5\vec{S}_1\cdot\vec{n}\vec{S}_1\times\dot{\vec{S}}_1\Big]\\
&+\frac{1}{8{m_1}^3}\Big[(2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\ddot{\vec{S}}_1-2\vec{S}_1\cdot\ddot{\vec{S}}_1\vec{S}_1\times\vec{v}_1)+(2\vec{S}_1\times\dot{\vec{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_1\\
&-\dot{\vec{S}}_1\cdot\vec{v}_1\vec{S}_1\times\dot{\vec{S}}_1+\dot{S}_1^2\vec{S}_1\times\vec{v}_1)-S_1^2\vec{S}_1\times\dot{\vec{a}}_1+(\vec{S}_1\times\dot{\vec{S}}_1\cdot\vec{a}_1\vec{S}_1\\
&+2\vec{S}_1\cdot\vec{a}_1\vec{S}_1\times\dot{\vec{S}}_1-3\vec{S}_1\cdot\dot{\vec{S}}_1\vec{S}_1\times\vec{a}_1)-\vec{S}_1\cdot\dot{\vec{S}}_1\dot{\vec{S}}_1\times\vec{v}_1-S_1^2\dot{\vec{S}}_1\times\vec{a}_1\Big],
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1^2S_2} =& -\frac{G}{16{m_1}^2r^3}\Big[12\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{n}-12\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{n}\\
&-15S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}+12\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\big(4\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)-12S_1^2\vec{S}_2\times\vec{n}\vec{v}_1\cdot\vec{n}\\
&-12\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)+12\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_1\vec{S}_1\\
&+24\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\vec{S}_1+15\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{S}_1+12\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_2\\
&-36\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{S}_1\times\vec{n}+12\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\times\vec{n}-8\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{S}_2\\
&+\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{S}_2-16S_1^2\vec{S}_2\times\vec{v}_1-\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{v}_2\Big]\\
&+\frac{GC_{\rm 1ES^2}}{8{m_1}^2r^3}\Big[24\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\big(\vec{v}_1-\vec{v}_2\big)-12\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\vec{n}\\
&-24S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1\vec{n}+24\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_2\vec{n}+27S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}\\
&+6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{n}-24\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\vec{v}_2\cdot\vec{n}-12\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\big(\vec{v}_1\cdot\vec{n}\\
&-2\vec{v}_2\cdot\vec{n}\big)-12S_1^2\vec{S}_2\times\vec{n}\big(\vec{v}_1\cdot\vec{n}+3\vec{v}_2\cdot\vec{n}\big)+12\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\vec{v}_1\cdot\vec{n}\\
&+15\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}(\vec{S}_1\cdot\vec{n})^2+60\vec{S}_2\times\vec{n}\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2+12\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_1\vec{S}_1\\
&-18\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{S}_1+8\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{S}_1+12\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\times\vec{n}\\
&-6\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{S}_2+14\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{v}_2+7S_1^2\vec{S}_2\times\vec{v}_2-33(\vec{S}_1\cdot\vec{n})^2\vec{S}_2\times\vec{v}_2\Big]\\
&-\frac{GC_{\rm 1ES^2}}{2{m_1}^2r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\dot{\vec{S}}_2\vec{S}_1-2\vec{S}_1\cdot\dot{\vec{S}}_2\vec{S}_1\times\vec{n}+S_1^2\dot{\vec{S}}_2\times\vec{n}-3(\vec{S}_1\cdot\vec{n})^2\dot{\vec{S}}_2\times\vec{n}\Big]\\
&+\frac{G}{4{m_1}^2r^2}\Big[\vec{S}_1\cdot\dot{\vec{S}}_2\vec{S}_1\times\vec{n}-\vec{S}_1\cdot\vec{n}\vec{S}_1\times\dot{\vec{S}}_2\Big],
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1S_2^2} = {} & \frac{GC_{\rm 2ES^2}}{4m_1m_2r^3}\Big[12\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_2\vec{v}_2\cdot\vec{n}-6S_2^2\vec{S}_1\times\vec{n}\vec{v}_2\cdot\vec{n}+12\vec{S}_1\cdot\vec{S}_2\vec{S}_2\times\vec{n}\vec{v}_2\cdot\vec{n}\\
& +30\vec{S}_1\times\vec{n}\vec{v}_2\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2+4\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{S}_2-12\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\vec{S}_1\times\vec{n}\\
& +S_2^2\vec{S}_1\times\vec{v}_1-3(\vec{S}_2\cdot\vec{n})^2\vec{S}_1\times\vec{v}_1-6S_2^2\vec{S}_1\times\vec{v}_2+18(\vec{S}_2\cdot\vec{n})^2\vec{S}_1\times\vec{v}_2\\
& -4\vec{S}_1\cdot\vec{S}_2\vec{S}_2\times\vec{v}_2\Big]-\frac{G}{4m_1m_2r^3}\Big[3\vec{S}_1\cdot\vec{S}_2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}-3\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{n}\\
& -3S_2^2\vec{S}_1\times\vec{n}\vec{v}_2\cdot\vec{n}-6\vec{S}_1\cdot\vec{S}_2\vec{S}_2\times\vec{n}\vec{v}_2\cdot\vec{n}-15\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\\
& +3\vec{S}_2\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{S}_1-3\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{S}_2-\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{S}_2\\
& +3\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\vec{S}_1\times\vec{n}+6\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_2\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\vec{S}_1\times\vec{S}_2+S_2^2\vec{S}_1\times\vec{v}_2\\
& +3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_2\times\vec{v}_2+\vec{S}_1\cdot\vec{S}_2\vec{S}_2\times\vec{v}_2\Big]\\
& -\frac{GC_{\rm 2ES^2}}{m_1m_2r^2}\Big[\vec{S}_1\times\vec{n}\cdot\dot{\vec{S}}_2\vec{S}_2+\vec{S}_1\times\vec{n}\cdot\vec{S}_2\dot{\vec{S}}_2-3\vec{S}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\\
& +\vec{S}_2\cdot\dot{\vec{S}}_2\vec{S}_1\times\vec{n}+\vec{S}_1\cdot\dot{\vec{S}}_2\vec{S}_2\times\vec{n}+\vec{S}_1\cdot\vec{S}_2\dot{\vec{S}}_2\times\vec{n}\Big],
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_2^3} = {} & \frac{G}{4m_2{}^2r^3}\Big[9S_2^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}-12S_2^2\vec{S}_2\times\vec{n}\vec{v}_2\cdot\vec{n}+12\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\vec{S}_2\times\vec{n}\\
& +7S_2^2\vec{S}_2\times\vec{v}_2\Big]-\frac{GC_{\rm 2BS^3}}{6m_2{}^2r^3}\Big[3S_2^2\vec{S}_2\times\vec{n}\vec{v}_2\cdot\vec{n}-15\vec{S}_2\times\vec{n}\vec{v}_2\cdot\vec{n}(\vec{S}_2\cdot\vec{n})^2\\
& +6\vec{S}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\vec{S}_2\times\vec{n}-S_2^2\vec{S}_2\times\vec{v}_2+3(\vec{S}_2\cdot\vec{n})^2\vec{S}_2\times\vec{v}_2\Big]\\
& +\frac{3GC_{\rm 2ES^2}}{8m_2{}^2r^3}\Big[S_2^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}+5\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{n}(\vec{S}_2\cdot\vec{n})^2-2\vec{S}_2\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\vec{S}_2\\
& +3S_2^2\vec{S}_2\times\vec{v}_2-7(\vec{S}_2\cdot\vec{n})^2\vec{S}_2\times\vec{v}_2\Big]\\
& +\frac{GC_{\rm 2BS^3}}{6m_2{}^2r^2}\Big[6\vec{S}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{S}_2\times\vec{n}-2\vec{S}_2\cdot\dot{\vec{S}}_2\vec{S}_2\times\vec{n}-S_2^2\dot{\vec{S}}_2\times\vec{n}\\
& +3(\vec{S}_2\cdot\vec{n})^2\dot{\vec{S}}_2\times\vec{n}\Big].
\end{aligned}
\tag{B.5}
$$

The new redefinitions of rotational variables fixed in the present sectors can be written as:

$$
\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S^3}=\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^3}+\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^2S_2}+\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1S_2^2}-(i\leftrightarrow j),
\tag{B.6}
$$

where

$$
\begin{aligned}
\left(\omega_1^{ij}\right)^{\rm NLO}_{\rm S_1^3} = {} & -\frac{GC_{\rm 1BS^3}m_2}{6m_1{}^2r^3}\Big[3S_1^2\big(v_2^iv_1^j-3\vec{v}_2\cdot\vec{n}v_1^in^j+3\vec{v}_2\cdot\vec{n}v_2^in^j\big)+18\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(v_1^in^j\\
& -v_2^in^j\big)-6\vec{S}_1\cdot\vec{n}S_1^i\big(v_1^2n^j-\vec{v}_1\cdot\vec{v}_2n^j+5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}n^j-5(\vec{v}_2\cdot\vec{n})^2n^j\big)\\
& -6\vec{S}_1\cdot\vec{v}_1S_1^i\big(\vec{v}_1\cdot\vec{n}n^j-2\vec{v}_2\cdot\vec{n}n^j\big)-6\vec{S}_1\cdot\vec{v}_2S_1^i\vec{v}_2\cdot\vec{n}n^j-6\vec{S}_1\cdot\vec{n}S_1^j\big(\vec{v}_1\cdot\vec{n}v_1^i\\
& +5\vec{v}_2\cdot\vec{n}v_1^i-2\vec{v}_1\cdot\vec{n}v_2^i-4\vec{v}_2\cdot\vec{n}v_2^i\big)-2\vec{S}_1\cdot\vec{v}_1S_1^j\big(3v_1^i-2v_2^i\big)+2\vec{S}_1\cdot\vec{v}_2S_1^jv_1^i\\
& -9\big(v_2^iv_1^j-5\vec{v}_2\cdot\vec{n}v_1^in^j+5\vec{v}_2\cdot\vec{n}v_2^in^j\big)(\vec{S}_1\cdot\vec{n})^2\Big]\\
& -\frac{Gm_2}{8m_1{}^2r^3}\Big[2S_1^2\big(16v_2^iv_1^j-9\vec{v}_1\cdot\vec{n}v_1^in^j+24\vec{v}_2\cdot\vec{n}v_1^in^j-30\vec{v}_1\cdot\vec{n}v_2^in^j\big)
\end{aligned}
$$

$$
\begin{aligned}
&- 6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(7v_1^i n^j - 10v_2^i n^j\big) + 12\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2 v_1^i n^j + 9\vec{S}_1\cdot\vec{n}S_1^i\big(3v_1^2 n^j \\
&- 4\vec{v}_1\cdot\vec{v}_2 n^j\big) + 9\vec{S}_1\cdot\vec{v}_1 S_1^i\big(2\vec{v}_1\cdot\vec{n}n^j - 5\vec{v}_2\cdot\vec{n}n^j\big) + 36\vec{S}_1\cdot\vec{v}_2 S_1^i\vec{v}_1\cdot\vec{n}n^j \\
&+ 15\vec{S}_1\cdot\vec{n}S_1^j\big(3\vec{v}_1\cdot\vec{n}v_1^i - 2\vec{v}_2\cdot\vec{n}v_1^i\big) + 5\vec{S}_1\cdot\vec{v}_1 S_1^j\big(v_1^i - 7v_2^i\big) + 18\vec{S}_1\cdot\vec{v}_2 S_1^j v_1^i \\
&+ 12v_2^i v_1^j(\vec{S}_1\cdot\vec{n})^2\Big] + \frac{3GC_{1\mathrm{ES}^2}m_2}{8m_1{}^2r^3}\Big[4S_1^2\big(7v_2^i v_1^j + 3\vec{v}_1\cdot\vec{n}v_1^i n^j + 7\vec{v}_2\cdot\vec{n}v_1^i n^j \\
&- 10\vec{v}_1\cdot\vec{n}v_2^i n^j\big) - 5\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(5v_1^i n^j - 8v_2^i n^j\big) - 16\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2 v_1^i n^j \\
&+ 8\vec{S}_1\cdot\vec{n}S_1^i\big(v_1^2 n^j - \vec{v}_1\cdot\vec{v}_2 n^j\big) - 32\vec{S}_1\cdot\vec{v}_1 S_1^i\vec{v}_2\cdot\vec{n}n^j + 32\vec{S}_1\cdot\vec{v}_2 S_1^i\vec{v}_1\cdot\vec{n}n^j \\
&+ \vec{S}_1\cdot\vec{n}S_1^j\big(\vec{v}_1\cdot\vec{n}v_1^i - 24\vec{v}_2\cdot\vec{n}v_1^i + 24\vec{v}_1\cdot\vec{n}v_2^i\big) + 8\vec{S}_1\cdot\vec{v}_1 S_1^j\big(v_1^i - 4v_2^i\big) \\
&+ 24\vec{S}_1\cdot\vec{v}_2 S_1^j v_1^i - 16v_2^i v_1^j(\vec{S}_1\cdot\vec{n})^2\Big] \\
&- \frac{6G^2C_{1\mathrm{ES}^2}m_2}{m_1r^4}\vec{S}_1\cdot\vec{n}S_1^i n^j - \frac{5G^2m_2{}^2}{8m_1{}^2r^4}\vec{S}_1\cdot\vec{n}S_1^i n^j - \frac{2G^2C_{1\mathrm{ES}^2}m_2{}^2}{m_1{}^2r^4}\vec{S}_1\cdot\vec{n}S_1^i n^j \\
&- \frac{GC_{1\mathrm{ES}^2}m_2}{2m_1{}^2r^2}\Big[S_1^2 a_1^i n^j - \vec{S}_1\cdot\vec{n}S_1^j a_1^i\Big] + \frac{Gm_2}{8m_1{}^2r^2}\Big[5S_1^2 a_1^i n^j - 6\vec{S}_1\cdot\vec{a}_1 S_1^i n^j \\
&+ \vec{S}_1\cdot\vec{n}S_1^j a_1^i\Big] - \frac{Gm_2}{16m_1{}^2r^2}\Big[8\vec{S}_1\cdot\dot{\vec{S}}_1 v_1^i n^j - 6\dot{\vec{S}}_1\cdot\vec{v}_1 S_1^i n^j + 7\vec{S}_1\cdot\vec{v}_1\dot{S}_1^i n^j \\
&- 2\dot{\vec{S}}_1\cdot\vec{n}S_1^j v_1^i - 9\dot{S}_1^i S_1^j\vec{v}_1\cdot\vec{n} - 16\vec{S}_1\cdot\vec{n}\dot{S}_1^j v_1^i\Big] - \frac{GC_{1\mathrm{ES}^2}m_2}{4m_1{}^2r^2}\Big[2\vec{S}_1\cdot\vec{v}_1\dot{S}_1^i n^j \\
&- \dot{S}_1^i S_1^j\vec{v}_1\cdot\vec{n} - \vec{S}_1\cdot\vec{n}\dot{S}_1^j v_1^i\Big] \\
&- \frac{1}{8m_1{}^2}\Big[(\vec{S}_1\cdot\vec{v}_1\dot{S}_1^j a_1^i + \vec{S}_1\cdot\vec{a}_1\dot{S}_1^j v_1^i) + \vec{S}_1\cdot\vec{v}_1\ddot{S}_1^j v_1^i + \dot{\vec{S}}_1\cdot\vec{v}_1\dot{S}_1^j v_1^i\Big], \qquad \text{(B.7)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\omega_1^{ij}\right)^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2} = {}& \frac{G}{8m_1r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}v_2^i v_1^j + \vec{S}_1\cdot\vec{S}_2\big(v_2^i v_1^j + 12\vec{v}_1\cdot\vec{n}v_1^i n^j - 15\vec{v}_2\cdot\vec{n}v_1^i n^j \\
&- 21\vec{v}_1\cdot\vec{n}v_2^i n^j\big) + 9\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1 v_1^i n^j + 21\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1 v_2^i n^j \\
&+ 3\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2 v_1^i n^j - 12\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2 v_1^i n^j - 12\vec{S}_2\cdot\vec{v}_1 S_1^i\vec{v}_1\cdot\vec{n}n^j \\
&+ 12\vec{S}_2\cdot\vec{v}_2 S_1^i\vec{v}_1\cdot\vec{n}n^j - 3\vec{S}_1\cdot\vec{n}S_2^i\big(4v_1^2 n^j - \vec{v}_1\cdot\vec{v}_2 n^j\big) + 12\vec{S}_1\cdot\vec{v}_1 S_2^i\vec{v}_1\cdot\vec{n}n^j \\
&- 3\vec{S}_1\cdot\vec{v}_2 S_2^i\vec{v}_1\cdot\vec{n}n^j + 3\vec{S}_2\cdot\vec{n}S_1^j\vec{v}_1\cdot\vec{n}v_1^i - \vec{S}_2\cdot\vec{v}_1 S_1^j\big(5v_1^i + 7v_2^i\big) \\
&+ 8\vec{S}_2\cdot\vec{v}_2 S_1^j v_1^i + S_2^i S_1^j\big(8v_1^2 - 5\vec{v}_1\cdot\vec{v}_2 + 12\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n} - 12(\vec{v}_1\cdot\vec{n})^2\big) \\
&- 3\vec{S}_1\cdot\vec{n}S_2^j\big(4\vec{v}_1\cdot\vec{n}v_1^i - 9\vec{v}_2\cdot\vec{n}v_1^i\big) + 5\vec{S}_1\cdot\vec{v}_1 S_2^j v_1^i - 11\vec{S}_1\cdot\vec{v}_2 S_2^j v_1^i\Big] \\
&+ \frac{GC_{1\mathrm{ES}^2}}{4m_1r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(3v_2^i v_1^j - 20\vec{v}_2\cdot\vec{n}v_1^i n^j - 5\vec{v}_1\cdot\vec{n}v_2^i n^j + 15\vec{v}_2\cdot\vec{n}v_2^i n^j\big) \\
&- 3\vec{S}_1\cdot\vec{S}_2\big(7v_2^i v_1^j - 8\vec{v}_1\cdot\vec{n}v_1^i n^j + 10\vec{v}_2\cdot\vec{n}v_1^i n^j - 5\vec{v}_1\cdot\vec{n}v_2^i n^j + \vec{v}_2\cdot\vec{n}v_2^i n^j\big) \\
&- 3\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(2v_1^i n^j + 11v_2^i n^j\big) - 3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\big(2v_1^i n^j - 3v_2^i n^j\big) \\
&+ 3\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(14v_1^i n^j - v_2^i n^j\big) - 3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2 v_2^i n^j + 6\vec{S}_2\cdot\vec{n}S_1^i\big(v_1^2 n^j \\
&- 2\vec{v}_1\cdot\vec{v}_2 n^j + v_2^2 n^j - 5(\vec{v}_2\cdot\vec{n})^2 n^j\big) - 6\vec{S}_2\cdot\vec{v}_1 S_1^i\vec{v}_1\cdot\vec{n}n^j \\
&+ 12\vec{S}_2\cdot\vec{v}_2 S_1^i\vec{v}_1\cdot\vec{n}n^j - 3\vec{S}_1\cdot\vec{n}S_2^i\big(2v_1^2 n^j - 5\vec{v}_1\cdot\vec{v}_2 n^j \\
&+ 3v_2^2 n^j - 25\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}n^j - 5(\vec{v}_2\cdot\vec{n})^2 n^j\big) - 3\vec{S}_1\cdot\vec{v}_1 S_2^i\big(2\vec{v}_1\cdot\vec{n}n^j \\
&- 7\vec{v}_2\cdot\vec{n}n^j\big) - 3\vec{S}_1\cdot\vec{v}_2 S_2^i\big(13\vec{v}_1\cdot\vec{n}n^j - 2\vec{v}_2\cdot\vec{n}n^j\big) + 3\vec{S}_2\cdot\vec{n}S_1^j\big(4\vec{v}_2\cdot\vec{n}v_1^i
\end{aligned}
$$

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$$
\begin{aligned}
&+ \vec{v}_1 \cdot \vec{n} v_2^i - 5\vec{v}_2 \cdot \vec{n} v_2^i) - 3\vec{S}_2 \cdot \vec{v}_1 S_1^j v_2^i + \vec{S}_2 \cdot \vec{v}_2 S_1^j (4v_1^i - v_2^i) + S_2^i S_1^j (6v_1^2 \\
&- 13\vec{v}_1 \cdot \vec{v}_2 + 7v_2^2 - 3\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 6(\vec{v}_1 \cdot \vec{n})^2 - 15(\vec{v}_2 \cdot \vec{n})^2) \\
&- 3\vec{S}_1 \cdot \vec{n} S_2^j (6\vec{v}_1 \cdot \vec{n} v_1^i - 11\vec{v}_2 \cdot \vec{n} v_1^i - 6\vec{v}_1 \cdot \vec{n} v_2^i + 3\vec{v}_2 \cdot \vec{n} v_2^i) + 6\vec{S}_1 \cdot \vec{v}_1 S_2^j (v_1^i \\
&+ 3v_2^i) - \vec{S}_1 \cdot \vec{v}_2 S_2^j (29v_1^i - 5v_2^i)\Big] \\
&- \frac{2G^2 C_{1\mathrm{ES}^2}}{r^4}\Big[\vec{S}_2 \cdot \vec{n} S_1^i n^j - 2\vec{S}_1 \cdot \vec{n} S_2^i n^j + S_2^i S_1^j\Big] + \frac{4G^2 C_{1\mathrm{ES}^2} m_2}{m_1 r^4}\Big[\vec{S}_2 \cdot \vec{n} S_1^i n^j \\
&- 2\vec{S}_1 \cdot \vec{n} S_2^i n^j + S_2^i S_1^j\Big] - \frac{3G^2}{4r^4}\Big[3\vec{S}_2 \cdot \vec{n} S_1^i n^j + 2S_2^i S_1^j\Big] + \frac{3G^2 m_2}{m_1 r^4}\vec{S}_1 \cdot \vec{n} S_2^i n^j \\
&- \frac{G C_{1\mathrm{ES}^2}}{4m_1 r^2}\Big[3\vec{S}_1 \cdot \vec{n}\vec{S}_2 \cdot \vec{n} a_2^i n^j - 11\vec{S}_1 \cdot \vec{S}_2 a_2^i n^j + 6\vec{S}_2 \cdot \vec{n} S_1^i \vec{a}_2 \cdot \vec{n} n^j \\
&+ 2\vec{S}_2 \cdot \vec{a}_2 S_1^i n^j - 15\vec{S}_1 \cdot \vec{n} S_2^i \vec{a}_2 \cdot \vec{n} n^j + 7\vec{S}_1 \cdot \vec{a}_2 S_2^i n^j + 5\vec{S}_2 \cdot \vec{n} S_1^j a_2^i \\
&+ 3S_2^i S_1^j \vec{a}_2 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{n} S_2^j a_2^i\Big] - \frac{G}{4m_1 r^2}\Big[2\vec{S}_1 \cdot \dot{\vec{S}}_2 v_1^i n^j - \dot{\vec{S}}_2 \cdot \vec{v}_1 S_1^i n^j \\
&- \dot{S}_2^i S_1^j \vec{v}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{n}\dot{S}_2^j v_1^i\Big] - \frac{G C_{1\mathrm{ES}^2}}{4m_1 r^2}\Big[3\vec{S}_1 \cdot \vec{n}\dot{\vec{S}}_2 \cdot \vec{n}(2v_1^i n^j - v_2^i n^j) \\
&+ \vec{S}_1 \cdot \dot{\vec{S}}_2 (2v_1^i n^j - 5v_2^i n^j) - 3\dot{\vec{S}}_2 \cdot \vec{n} S_1^i (\vec{v}_1 \cdot \vec{n} n^j - 4\vec{v}_2 \cdot \vec{n} n^j) + \dot{\vec{S}}_2 \cdot \vec{v}_1 S_1^i n^j \\
&- 21\vec{S}_1 \cdot \vec{n}\dot{S}_2^i \vec{v}_2 \cdot \vec{n} n^j - 4\vec{S}_1 \cdot \vec{v}_1 \dot{S}_2^i n^j + 5\vec{S}_1 \cdot \vec{v}_2 \dot{S}_2^i n^j - \dot{\vec{S}}_2 \cdot \vec{n} S_1^j (3v_1^i - 5v_2^i) \\
&+ \dot{S}_2^i S_1^j (\vec{v}_1 \cdot \vec{n} + 7\vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \cdot \vec{n}\dot{S}_2^j (v_1^i + v_2^i)\Big] \\
&- \frac{G C_{1\mathrm{ES}^2}}{6m_1 r}\Big[(3\ddot{\vec{S}}_2 \cdot \vec{n} S_1^i n^j - 6\vec{S}_1 \cdot \vec{n}\ddot{S}_2^i n^j + 3\ddot{S}_2^i S_1^j) + (\dot{\vec{S}}_2 \cdot \vec{n}\dot{S}_1^i n^j \\
&- 2\dot{\vec{S}}_1 \cdot \vec{n}\dot{S}_2^i n^j + \dot{S}_2^i \dot{S}_1^j)\Big], \qquad \text{(B.8)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\omega_1^{ij}\right)^{\mathrm{NLO}}_{\mathrm{S_1 S_2^2}} = &- \frac{G C_{2\mathrm{ES}^2}}{2m_2 r^3}\Big[S_2^2 (5v_2^i v_1^j + 3\vec{v}_2 \cdot \vec{n} v_1^i n^j - 3\vec{v}_2 \cdot \vec{n} v_2^i n^j) - 6\vec{S}_2 \cdot \vec{n}\vec{S}_2 \cdot \vec{v}_2 (v_1^i n^j \\
&- v_2^i n^j) - 6\vec{S}_2 \cdot \vec{n} S_2^j (\vec{v}_2 \cdot \vec{n} v_1^i - \vec{v}_2 \cdot \vec{n} v_2^i) + 2\vec{S}_2 \cdot \vec{v}_2 S_2^j (v_1^i - v_2^i) \\
&- 3(3v_2^i v_1^j - 5\vec{v}_2 \cdot \vec{n} v_1^i n^j + 5\vec{v}_2 \cdot \vec{n} v_2^i n^j)(\vec{S}_2 \cdot \vec{n})^2\Big] - \frac{G}{4m_2 r^3}\Big[S_2^2 (2v_2^i v_1^j \\
&+ 9\vec{v}_2 \cdot \vec{n} v_1^i n^j - 3\vec{v}_1 \cdot \vec{n} v_2^i n^j - 9\vec{v}_2 \cdot \vec{n} v_2^i n^j) - 3\vec{S}_2 \cdot \vec{n}\vec{S}_2 \cdot \vec{v}_2 (3v_1^i n^j - 2v_2^i n^j) \\
&+ 3\vec{S}_2 \cdot \vec{n} S_2^i (3\vec{v}_1 \cdot \vec{v}_2 n^j - 3v_2^2 n^j - 5\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} n^j - 5(\vec{v}_2 \cdot \vec{n})^2 n^j) \\
&- 9\vec{S}_2 \cdot \vec{v}_1 S_2^i \vec{v}_2 \cdot \vec{n} n^j + 3\vec{S}_2 \cdot \vec{v}_2 S_2^i (\vec{v}_1 \cdot \vec{n} n^j + 4\vec{v}_2 \cdot \vec{n} n^j) + 3\vec{S}_2 \cdot \vec{n} S_2^j (\vec{v}_2 \cdot \vec{n} v_1^i \\
&- 3\vec{v}_1 \cdot \vec{n} v_2^i - 4\vec{v}_2 \cdot \vec{n} v_2^i) - 3\vec{S}_2 \cdot \vec{v}_1 S_2^j v_2^i + 2\vec{S}_2 \cdot \vec{v}_2 S_2^j (v_1^i + v_2^i) + 3(v_2^i v_1^j \\
&+ 5\vec{v}_1 \cdot \vec{n} v_2^i n^j + 5\vec{v}_2 \cdot \vec{n} v_2^i n^j)(\vec{S}_2 \cdot \vec{n})^2\Big] \\
&+ \frac{2G^2}{r^4}\vec{S}_2 \cdot \vec{n} S_2^i n^j - \frac{G^2 m_1}{2m_2 r^4}\vec{S}_2 \cdot \vec{n} S_2^i n^j + \frac{39G^2 C_{2\mathrm{ES}^2} m_1}{4m_2 r^4}\vec{S}_2 \cdot \vec{n} S_2^i n^j \\
&- \frac{G}{4m_2 r^2}\Big[S_2^2 a_2^i n^j - 3\vec{S}_2 \cdot \vec{n} S_2^i \vec{a}_2 \cdot \vec{n} n^j - \vec{S}_2 \cdot \vec{a}_2 S_2^i n^j - 3\vec{S}_2 \cdot \vec{n} S_2^j a_2^i \\
&+ 3a_2^i n^j (\vec{S}_2 \cdot \vec{n})^2\Big] - \frac{3G}{4m_2 r^2}\Big[2\vec{S}_2 \cdot \vec{n}\dot{\vec{S}}_2 \cdot \vec{n} v_2^i n^j - \dot{\vec{S}}_2 \cdot \vec{n} S_2^i \vec{v}_2 \cdot \vec{n} n^j
\end{aligned}
$$

$$
\begin{aligned}
&-\vec{S}_2\cdot\vec{n}\dot{S}_2^i\vec{v}_2\cdot\vec{n}n^j-\dot{\vec{S}}_2\cdot\vec{n}S_2^jv_2^i-\vec{S}_2\cdot\vec{n}\dot{S}_2^jv_2^i\Big]-\frac{GC_{2\mathrm{ES}^2}}{4m_2r^2}\Big[6\vec{S}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}(v_1^in^j\\
&-v_2^in^j)-2\vec{S}_2\cdot\dot{\vec{S}}_2(3v_1^in^j+v_2^in^j)-3\dot{\vec{S}}_2\cdot\vec{n}S_2^i\vec{v}_2\cdot\vec{n}n^j+\dot{\vec{S}}_2\cdot\vec{v}_2S_2^in^j\\
&-3\vec{S}_2\cdot\vec{n}\dot{S}_2^i\vec{v}_2\cdot\vec{n}n^j+\vec{S}_2\cdot\vec{v}_2\dot{S}_2^in^j+\dot{\vec{S}}_2\cdot\vec{n}S_2^j(2v_1^i+v_2^i)+\vec{S}_2\cdot\vec{n}\dot{S}_2^j(2v_1^i\\
&+v_2^i)\Big]\\
&+\frac{GC_{2\mathrm{ES}^2}}{3m_2r}\dot{\vec{S}}_2\cdot\vec{n}\dot{S}_2^in^j.
\end{aligned}
\tag{B.9}
$$

C Final actions

The final potentials that we obtain for the NLO cubic-in-sectors, comprise the following 6 distinct sectors:

$$
\begin{aligned}
V_{\mathrm{S}^3}^{\mathrm{NLO}}=&V_{\mathrm{S}_1^3}^{\mathrm{NLO}}+C_{1\mathrm{ES}^2}V_{(\mathrm{ES}_1^2)\mathrm{S}_1}^{\mathrm{NLO}}+C_{1\mathrm{ES}^2}^2V_{C_{\mathrm{ES}_1^2}^2\mathrm{S}_1^3}^{\mathrm{NLO}}+C_{1\mathrm{BS}^3}V_{\mathrm{BS}_1^3}^{\mathrm{NLO}}+V_{\mathrm{S}_1^2\mathrm{S}_2}^{\mathrm{NLO}}+C_{1\mathrm{ES}^2}V_{(\mathrm{ES}_1^2)\mathrm{S}_2}^{\mathrm{NLO}}\\
&+(1\leftrightarrow2),
\end{aligned}
\tag{C.1}
$$

where

$$
\begin{aligned}
V_{\mathrm{S}_1^3}^{\mathrm{NLO}}=&\frac{Gm_2}{16{m_1}^2r^4}\Big[S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1(22v_1^2-12\vec{v}_1\cdot\vec{v}_2+36v_2^2+180\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-55(\vec{v}_1\cdot\vec{n})^2\\
&-180(\vec{v}_2\cdot\vec{n})^2)-12S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2(4\vec{v}_1\cdot\vec{v}_2-20\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+15(\vec{v}_1\cdot\vec{n})^2)\\
&+12S_1^2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2(11\vec{v}_1\cdot\vec{n}-10\vec{v}_2\cdot\vec{n})-10\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(7\vec{v}_1\cdot\vec{n}\\
&-18\vec{v}_2\cdot\vec{n})\vec{S}_1\times\vec{n}\cdot\vec{v}_1-60\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+120\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}\\
&-2\vec{v}_2\cdot\vec{n})\vec{S}_1\times\vec{n}\cdot\vec{v}_2+48\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2-12\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\\
&-55\vec{S}_1\times\vec{n}\cdot\vec{v}_1v_1^2(\vec{S}_1\cdot\vec{n})^2+60\vec{S}_1\times\vec{n}\cdot\vec{v}_2v_1^2(\vec{S}_1\cdot\vec{n})^2+14\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_1)^2\Big]\\
&-\frac{G^2m_2}{2m_1r^5}\Big[S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1+6\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^2{m_2}^2}{8{m_1}^2r^5}\Big[12S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\\
&+4S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2-3\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2-12\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
V_{(\mathrm{ES}_1^2)\mathrm{S}_1}^{\mathrm{NLO}}=&-\frac{3Gm_2}{16{m_1}^2r^4}\Big[S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1(3v_1^2-8\vec{v}_1\cdot\vec{v}_2+4v_2^2-40\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+20(\vec{v}_1\cdot\vec{n})^2\\
&+10(\vec{v}_2\cdot\vec{n})^2)-4S_1^2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2(6\vec{v}_1\cdot\vec{n}-5\vec{v}_2\cdot\vec{n})-20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n}\\
&-2\vec{v}_2\cdot\vec{n})\vec{S}_1\times\vec{n}\cdot\vec{v}_1-40\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2-8\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&-24\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2+24\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2-5\vec{S}_1\times\vec{n}\cdot\vec{v}_1(7v_1^2\\
&-16\vec{v}_1\cdot\vec{v}_2+8v_2^2-14(\vec{v}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2+20\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2(4\vec{v}_1\cdot\vec{n}\\
&-3\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+4\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_1)^2+4\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&+\frac{3G^2{m_2}^2}{2{m_1}^2r^5}\Big[2S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1+3S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2-13\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2\\
&-11\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^2m_2}{4m_1r^5}\Big[23S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1+2S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&-102\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2-18\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{C.3}
$$

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$$
V^{\rm NLO}_{C^2_{\rm ES^2_1}{\rm S}^3_1}=-\frac{3G^2{m_2}^2}{2{m_1}^2r^5}\Big[2\vec{S}_1\times\vec{v}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-3\vec{S}_1\times\vec{v}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\Big], \tag{C.4}
$$

$$
\begin{aligned}
V^{\rm NLO}_{\rm BS^3_1}=\frac{Gm_2}{2{m_1}^2r^4}\Big[&S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-\vec{v}_1\cdot\vec{v}_2+v_2^2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)-S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2\\
&-\vec{v}_1\cdot\vec{v}_2+v_2^2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+S_1^2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)\\
&+5\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(2\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)\vec{S}_1\times\vec{n}\cdot\vec{v}_1-10\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\\
&+\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(3\vec{v}_1\cdot\vec{n}-2\vec{v}_2\cdot\vec{n}\big)\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&+10\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2-2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&+6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2-5\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(v_1^2-\vec{v}_1\cdot\vec{v}_2\\
&+v_2^2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2+5\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(v_1^2-\vec{v}_1\cdot\vec{v}_2\\
&+v_2^2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2-5\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2\\
&-2\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_1)^2+3\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{v}_1)^2\Big]\\
&+\frac{G^2m_2}{6m_1r^5}\Big[4S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1-3S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2-18\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2\\
&+15\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^2{m_2}^2}{6{m_1}^2r^5}\Big[24S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_1-25S_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&-126\vec{S}_1\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2+129\vec{S}_1\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned} \tag{C.5}
$$

$$
\begin{aligned}
V^{\rm NLO}_{\rm S^2_1S_2}=\frac{3G}{16m_1r^4}\Big[&4\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\big(v_1^2-4\vec{v}_1\cdot\vec{v}_2+2v_2^2-10(\vec{v}_2\cdot\vec{n})^2\big)\\
&+2\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_1\big(11v_1^2-5\vec{v}_1\cdot\vec{v}_2-2v_2^2+40\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-30(\vec{v}_1\cdot\vec{n})^2\\
&+20(\vec{v}_2\cdot\vec{n})^2\big)-4S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1\big(7v_1^2+6\vec{v}_1\cdot\vec{v}_2-2v_2^2-10(\vec{v}_1\cdot\vec{n})^2+10(\vec{v}_2\cdot\vec{n})^2\big)\\
&+2\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\big(v_1^2-4\vec{v}_1\cdot\vec{v}_2+2v_2^2-10(\vec{v}_2\cdot\vec{n})^2\big)\\
&+8\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}+4\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\big(6v_1^2+3\vec{v}_1\cdot\vec{v}_2-10\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)\\
&+S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\big(16v_1^2-12\vec{v}_1\cdot\vec{v}_2-5(\vec{v}_1\cdot\vec{n})^2\big)-12\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\\
&+4\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\big(4\vec{v}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n}\big)-16S_1^2\vec{S}_2\times\vec{v}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\\
&+10\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(v_1^2-4\vec{v}_1\cdot\vec{v}_2+2v_2^2-14(\vec{v}_2\cdot\vec{n})^2\big)\vec{S}_1\times\vec{n}\cdot\vec{v}_1\\
&+20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\big(3\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big)\vec{S}_1\times\vec{n}\cdot\vec{v}_1-28\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_1\\
&-80\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_1+20\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\\
&+26\vec{S}_2\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2+8\vec{S}_1\cdot\vec{v}_1\vec{S}_2\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&+40\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2+8\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\\
&-8\vec{S}_1\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2-40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}v_1^2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&+40\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2+40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&-24\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1\vec{S}_1\times\vec{n}\cdot\vec{v}_2-12\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&+30\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2-20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2\\
&+12\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{S}_2\times\vec{n}\cdot\vec{v}_2-40\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2
\end{aligned}
$$

$$
\begin{aligned}
&+8\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2-4\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{v}_1\cdot\vec{v}_2\\
&+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\times\vec{v}_1\cdot\vec{v}_2+40\vec{S}_2\times\vec{n}\cdot\vec{v}_1v_1^2(\vec{S}_1\cdot\vec{n})^2-5\vec{S}_2\times\vec{n}\cdot\vec{v}_2\big(5v_1^2\\
&-4\vec{v}_1\cdot\vec{v}_2\big)(\vec{S}_1\cdot\vec{n})^2+32\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_1)^2-16\vec{S}_2\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{v}_1)^2\Big]\\
&-\frac{G^2m_2}{8m_1r^5}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\big(13\vec{v}_1\cdot\vec{n}-10\vec{v}_2\cdot\vec{n}\big)+12\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\\
&+78\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1-6\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_1+10S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1\\
&+40\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1+200\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_2+456\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&-312\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2+194S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2+69\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\\
&+6\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2+3\vec{S}_2\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^2}{4r^5}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\big(5\vec{v}_1\cdot\vec{n}\\
&-3\vec{v}_2\cdot\vec{n}\big)-99\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1-240\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_1\\
&+136\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_1-98S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1-19\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\\
&-26\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_2-72\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{v}_2+48\vec{S}_1\cdot\vec{S}_2\vec{S}_1\times\vec{n}\cdot\vec{v}_2\\
&-25S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2-40\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2+33\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2\\
&-27\vec{S}_2\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
V^{\rm NLO}_{({\rm ES}_1^2){\rm S}_2}=&\frac{3G}{16m_1r^4}\Big[8S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1\big(3v_1^2-3\vec{v}_1\cdot\vec{v}_2+5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-10(\vec{v}_1\cdot\vec{n})^2\big)\\
&-16\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\big(v_1^2-\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)+16\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{n}\\
&-\vec{v}_2\cdot\vec{n}\big)-16\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}-S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\big(16v_1^2-16\vec{v}_1\cdot\vec{v}_2+v_2^2\\
&+40\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-70(\vec{v}_1\cdot\vec{n})^2\big)+2\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\big(v_2^2-40\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&+10(\vec{v}_1\cdot\vec{n})^2\big)-4\vec{S}_1\cdot\vec{v}_1\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\big(5\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big)\\
&+16\vec{S}_1\cdot\vec{v}_2\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}-4S_1^2\vec{S}_2\times\vec{v}_1\cdot\vec{v}_2\big(9\vec{v}_1\cdot\vec{n}-8\vec{v}_2\cdot\vec{n}\big)\\
&+80\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)\vec{S}_2\times\vec{n}\cdot\vec{v}_1-80\vec{S}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_1\vec{v}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\\
&+16\vec{S}_1\cdot\vec{v}_1\vec{S}_2\times\vec{n}\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(5\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big)\vec{S}_2\times\vec{n}\cdot\vec{v}_2\\
&+80\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\vec{S}_2\times\vec{n}\cdot\vec{v}_2-16\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\\
&-28\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\vec{S}_2\times\vec{v}_1\cdot\vec{v}_2+32\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{S}_2\times\vec{v}_1\cdot\vec{v}_2-40\vec{S}_2\times\vec{n}\cdot\vec{v}_1\big(v_1^2\\
&-\vec{v}_1\cdot\vec{v}_2-7\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2+5\vec{S}_2\times\vec{n}\cdot\vec{v}_2\big(v_2^2-56\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&+14(\vec{v}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2+20\vec{S}_2\times\vec{v}_1\cdot\vec{v}_2\big(5\vec{v}_1\cdot\vec{n}-4\vec{v}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2\\
&-16\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{v}_1)^2+16\vec{S}_2\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{v}_1)^2\Big]\\
&+\frac{G^2m_2}{4m_1r^5}\Big[252\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)-24\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1\\
&-40S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1+116\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1+18\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_2-3S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2\\
&-47\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2+180\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^2}{4r^5}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\big(\vec{v}_1\cdot\vec{n}\\
&-\vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_1+31S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_1-40\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_1
\end{aligned}
$$

$$+2\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{S}_1\cdot\vec{v}_2-3S_1^2\vec{S}_2\times\vec{n}\cdot\vec{v}_2+4\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{S}_2\cdot\vec{v}_2$$
$$-123\vec{S}_2\times\vec{n}\cdot\vec{v}_1(\vec{S}_1\cdot\vec{n})^2+15\vec{S}_2\times\vec{n}\cdot\vec{v}_2(\vec{S}_1\cdot\vec{n})^2\Big]. \tag{C.7}$$

D General Hamiltonians

our full general Hamiltonian for the present NLO cubic-in-spin sectors is comprised of 6 distinct sectors:

$$H^{\rm NLO}_{\rm S^3} = H^{\rm NLO}_{\rm S_1^3} + C_{\rm 1ES^2} H^{\rm NLO}_{\rm (ES_1^2)S_1} + C^2_{\rm 1ES^2} H^{\rm NLO}_{C^2_{\rm ES_1^2}\rm S_1^3} + C_{\rm 1BS^3} H^{\rm NLO}_{\rm BS_1^3} + H^{\rm NLO}_{\rm S_1^2S_2} + C_{\rm 1ES^2} H^{\rm NLO}_{\rm (ES_1^2)S_2}$$
$$+ (1\leftrightarrow 2), \tag{D.1}$$

where

$$\begin{aligned}
H^{\rm NLO}_{\rm S_1^3} = {}& \frac{3G}{4{m_1}^3 m_2 r^4}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2\big(p_2^2-5(\vec{p}_2\cdot\vec{n})^2\big) \\
& - 4\vec{S}_1\times\vec{n}\cdot\vec{p}_2 S_1^2\big(\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big) - 10\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2 S_1^2\vec{p}_2\cdot\vec{n} \\
& - 20\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 + 4\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\Big] \\
& + \frac{Gm_2}{16{m_1}^5 r^4}\Big[11\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2\big(2p_1^2-5(\vec{p}_1\cdot\vec{n})^2\big) - 70\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& - 55\vec{S}_1\times\vec{n}\cdot\vec{p}_1 p_1^2(\vec{S}_1\cdot\vec{n})^2 + 14\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{p}_1\cdot\vec{S}_1)^2\Big] \\
& - \frac{3G}{4{m_1}^4 r^4}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2\big(\vec{p}_1\cdot\vec{p}_2-15\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big) + 15\vec{S}_1\times\vec{n}\cdot\vec{p}_2 S_1^2(\vec{p}_1\cdot\vec{n})^2 \\
& - 11\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2 S_1^2\vec{p}_1\cdot\vec{n} - 15\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& - 10\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 + \vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& + 5\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 - 5\vec{S}_1\times\vec{n}\cdot\vec{p}_2 p_1^2(\vec{S}_1\cdot\vec{n})^2\Big] \\
& + \frac{3G^2{m_2}^2}{4{m_1}^3 r^5}\Big[3\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2 - 10\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2\Big] + \frac{G^2 m_2}{4{m_1}^2 r^5}\Big[10\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2 \\
& - 11\vec{S}_1\times\vec{n}\cdot\vec{p}_2 S_1^2 - 36\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2 + 24\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
H^{\rm NLO}_{\rm (ES_1^2)S_1} = {}& -\frac{3Gm_2}{16{m_1}^5 r^4}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2\big(p_1^2+20(\vec{p}_1\cdot\vec{n})^2\big) - 20\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& - 25\vec{S}_1\times\vec{n}\cdot\vec{p}_1 p_1^2(\vec{S}_1\cdot\vec{n})^2 + 4\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{p}_1\cdot\vec{S}_1)^2\Big] \\
& + \frac{3G}{2{m_1}^4 r^4}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2\big(\vec{p}_1\cdot\vec{p}_2+5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big) + 3\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2 S_1^2\vec{p}_1\cdot\vec{n} \\
& - 5\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 + 3\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& + \vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 - 10\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 \\
& - 10\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\Big] - \frac{3G}{8{m_1}^3 m_2 r^4}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1 S_1^2\big(2p_2^2+5(\vec{p}_2\cdot\vec{n})^2\big) \\
& + 10\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2 S_1^2\vec{p}_2\cdot\vec{n} - 20\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 \\
& + 12\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 - 5\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(4p_2^2-7(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2 \\
& - 30\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 + 2\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{p}_2\cdot\vec{S}_1)^2\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3G^2{m_2}^2}{2{m_1}^3r^5}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2-16\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^2}{2m_1r^5}\Big[7\vec{S}_1\times\vec{n}\cdot\vec{p}_2S_1^2\\
&-27\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^2m_2}{4{m_1}^2r^5}\Big[5\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2+21\vec{S}_1\times\vec{n}\cdot\vec{p}_2S_1^2\\
&-60\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2-45\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{D.3}
$$

$$
H^{\mathrm{NLO}}_{C^2_{\mathrm{ES}^2_1}\mathrm{S}^3_1}=-\frac{9G^2m_2}{2{m_1}^2r^5}\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2+\frac{3G^2{m_2}^2}{{m_1}^3r^5}\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2,
\tag{D.4}
$$

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{BS}^3_1}=&\frac{Gm_2}{{m_1}^5r^4}\Big[5\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1-\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&-\frac{G}{2{m_1}^4r^4}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2(\vec{p}_1\cdot\vec{p}_2+5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})+\vec{S}_1\times\vec{n}\cdot\vec{p}_2S_1^2p_1^2\\
&-\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2S_1^2\vec{p}_1\cdot\vec{n}+5\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\\
&+15\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1-6\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\\
&+10\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1-\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-5\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{p}_1\cdot\vec{p}_2\\
&+7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2-5\vec{S}_1\times\vec{n}\cdot\vec{p}_2p_1^2(\vec{S}_1\cdot\vec{n})^2+5\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\\
&-3\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{p}_1\cdot\vec{S}_1)^2\Big]+\frac{G}{2{m_1}^3m_2r^4}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2p_2^2+\vec{S}_1\times\vec{n}\cdot\vec{p}_2S_1^2(\vec{p}_1\cdot\vec{p}_2\\
&+5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})-\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2S_1^2\vec{p}_2\cdot\vec{n}+10\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\\
&+10\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1-2\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\\
&-2\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-5\vec{S}_1\times\vec{n}\cdot\vec{p}_1p_2^2(\vec{S}_1\cdot\vec{n})^2-5\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{p}_1\cdot\vec{p}_2\\
&+7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2+5\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{G^2{m_2}^2}{{m_1}^3r^5}\Big[7\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2-36\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{5G^2}{2m_1r^5}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_2S_1^2\\
&-5\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^2m_2}{6{m_1}^2r^5}\Big[17\vec{S}_1\times\vec{n}\cdot\vec{p}_1S_1^2-46\vec{S}_1\times\vec{n}\cdot\vec{p}_2S_1^2\\
&-87\vec{S}_1\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2+234\vec{S}_1\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}^2_1\mathrm{S}_2}=&-\frac{3G}{8{m_1}^4r^4}\Big[2\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2\big(9p_1^2-10(\vec{p}_1\cdot\vec{n})^2\big)\\
&-\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{S}_2\big(17p_1^2-30(\vec{p}_1\cdot\vec{n})^2\big)+2\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{n}\cdot\vec{S}_2p_1^2\\
&-\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2p_1^2+40\vec{S}_2\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\\
&+5\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}p_1^2\vec{S}_2\cdot\vec{n}-30\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\\
&+14\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2-20\vec{S}_2\times\vec{n}\cdot\vec{p}_1p_1^2(\vec{S}_1\cdot\vec{n})^2-16\vec{S}_2\times\vec{n}\cdot\vec{p}_1(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{3G}{16{m_1}^3m_2r^4}\Big[8\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2\big(2\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)\\
&+\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2\big(16p_1^2-5(\vec{p}_1\cdot\vec{n})^2\big)-16\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2S_1^2\vec{p}_1\cdot\vec{n}\\
&-10\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{S}_2\big(\vec{p}_1\cdot\vec{p}_2-8\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)
\end{aligned}
$$

$$
\begin{aligned}
&- 8\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 \big(2p_1^2 - 5(\vec{p}_1 \cdot \vec{n})^2\big) + 16\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 \vec{p}_1 \cdot \vec{n} \\
&- 16\vec{p}_1 \cdot \vec{S}_1 \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \vec{p}_1 \cdot \vec{p}_2 + 8\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{p}_1 \cdot \vec{S}_2 \vec{p}_1 \cdot \vec{p}_2 \\
&- 8\vec{p}_2 \cdot \vec{S}_1 \vec{S}_1 \times \vec{p}_1 \cdot \vec{S}_2 \vec{p}_1 \cdot \vec{n} + 12\vec{p}_1 \cdot \vec{S}_1 \vec{S}_1 \times \vec{p}_2 \cdot \vec{S}_2 \vec{p}_1 \cdot \vec{n} \\
&+ 40\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_1 + 30\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_1 \\
&+ 4\vec{S}_2 \times \vec{p}_1 \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_1 + 40\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{S}_1 \\
&- 32\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 \vec{p}_1 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{S}_1 - 40\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 \vec{S}_2 \cdot \vec{n} \\
&- 40\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} + 8\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{p}_1 \cdot \vec{S}_1 \vec{S}_2 \cdot \vec{n} \\
&- 80\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 - 40\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 \\
&+ 16\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{p}_1 \cdot \vec{S}_1 \vec{p}_1 \cdot \vec{S}_2 + 26\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{p}_2 \cdot \vec{S}_1 \vec{p}_1 \cdot \vec{S}_2 \\
&+ 8\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{p}_1 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{S}_2 - 40\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 \vec{p}_1 \cdot \vec{p}_2 (\vec{S}_1 \cdot \vec{n})^2 \\
&- 25\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 p_1^2 (\vec{S}_1 \cdot \vec{n})^2 - 16\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 (\vec{p}_1 \cdot \vec{S}_1)^2 \Big] \\
&+ \frac{3G}{4{m_1}^2{m_2}^2 r^4}\Big[2\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 S_1^2 \big(p_2^2 - 5(\vec{p}_2 \cdot \vec{n})^2\big) - 3\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 S_1^2 \vec{p}_1 \cdot \vec{p}_2 \\
&- \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{S}_2 \big(p_2^2 - 10(\vec{p}_2 \cdot \vec{n})^2\big) \\
&+ \vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 \big(3\vec{p}_1 \cdot \vec{p}_2 - 10\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n}\big) + \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 \vec{p}_2 \cdot \vec{n} \\
&+ 2\vec{p}_1 \cdot \vec{S}_1 \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \big(p_2^2 - 5(\vec{p}_2 \cdot \vec{n})^2\big) - \vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{p}_1 \cdot \vec{S}_2 \big(p_2^2 - 5(\vec{p}_2 \cdot \vec{n})^2\big) \\
&- 5\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{S}_1 + 3\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 \vec{p}_1 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{S}_1 \\
&+ 5\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \big(p_2^2 - 7(\vec{p}_2 \cdot \vec{n})^2\big) \vec{S}_2 \cdot \vec{n} + 5\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{p}_2 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \\
&- \vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{p}_2 \cdot \vec{S}_1 \vec{S}_2 \cdot \vec{n} + 10\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 \\
&- 3\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{p}_2 \cdot \vec{S}_1 \vec{p}_1 \cdot \vec{S}_2 + 10\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{p}_2 \cdot \vec{S}_2 \\
&- 2\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{p}_2 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{S}_2 + 5\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 \vec{p}_1 \cdot \vec{p}_2 (\vec{S}_1 \cdot \vec{n})^2 \Big] \\
&- \frac{G^2 m_2}{2{m_1}^2 r^5}\Big[27\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \vec{p}_1 \cdot \vec{n} + 42\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 S_1^2 + 81\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \\
&- 45\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{S}_2 + 35\vec{p}_1 \cdot \vec{S}_1 \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 - 25\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{p}_1 \cdot \vec{S}_2 \\
&- 3\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 (\vec{S}_1 \cdot \vec{n})^2 \Big] - \frac{3G^2}{4m_2 r^5}\Big[6\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \vec{p}_2 \cdot \vec{n} - \vec{S}_2 \times \vec{n} \cdot \vec{p}_2 S_1^2 \\
&+ 28\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} - 20\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 + 6\vec{p}_2 \cdot \vec{S}_1 \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \\
&- 8\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{p}_2 \cdot \vec{S}_2 + 13\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 (\vec{S}_1 \cdot \vec{n})^2 \Big] \\
&+ \frac{G^2}{4m_1 r^5}\Big[6\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \big(3\vec{p}_1 \cdot \vec{n} + 8\vec{p}_2 \cdot \vec{n}\big) - 145\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 S_1^2 \\
&- 28\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 S_1^2 - 240\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} - 126\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \\
&+ 144\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \vec{S}_1 \cdot \vec{S}_2 + 90\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 - 126\vec{p}_1 \cdot \vec{S}_1 \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 \\
&- 49\vec{p}_2 \cdot \vec{S}_1 \vec{S}_1 \times \vec{n} \cdot \vec{S}_2 + 33\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{p}_1 \cdot \vec{S}_2 + 3\vec{S}_1 \cdot \vec{n} \vec{S}_1 \times \vec{p}_2 \cdot \vec{S}_2 \\
&+ 57\vec{S}_2 \times \vec{n} \cdot \vec{p}_1 (\vec{S}_1 \cdot \vec{n})^2 - 6\vec{S}_2 \times \vec{n} \cdot \vec{p}_2 (\vec{S}_1 \cdot \vec{n})^2 \Big], \qquad \text{(D.6)}
\end{aligned}
$$

$$
\begin{aligned}
H^{\mathrm{NLO}}_{(\mathrm{ES}_1^2)\mathrm{S}_2} = {} & \frac{15G}{16m_1{m_2}^3r^4}\Big[\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2p_2^2+2\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2p_2^2-5\vec{S}_2\times\vec{n}\cdot\vec{p}_2p_2^2(\vec{S}_1\cdot\vec{n})^2\Big] \\
& +\frac{3G}{{m_1}^4r^4}\Big[\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2\big(p_1^2-5(\vec{p}_1\cdot\vec{n})^2\big)-\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n} \\
& +5\vec{S}_2\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1-\vec{S}_2\times\vec{n}\cdot\vec{p}_1(\vec{p}_1\cdot\vec{S}_1)^2\Big] \\
& -\frac{3G}{8{m_1}^3m_2r^4}\Big[4\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2\big(3\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big) \\
& +\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2\big(8p_1^2-35(\vec{p}_1\cdot\vec{n})^2\big)+18\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2S_1^2\vec{p}_1\cdot\vec{n} \\
& +8\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\big(\vec{p}_1\cdot\vec{p}_2+5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)-8\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{n} \\
& -8\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}+10\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2(\vec{p}_1\cdot\vec{n})^2 \\
& -10\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}+40\vec{S}_2\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& +50\vec{S}_2\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1+14\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& +40\vec{S}_2\times\vec{n}\cdot\vec{p}_1\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1-8\vec{S}_2\times\vec{n}\cdot\vec{p}_1\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 \\
& -20\vec{S}_2\times\vec{n}\cdot\vec{p}_1\big(\vec{p}_1\cdot\vec{p}_2+7\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)(\vec{S}_1\cdot\vec{n})^2-35\vec{S}_2\times\vec{n}\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2 \\
& -50\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-8\vec{S}_2\times\vec{n}\cdot\vec{p}_2(\vec{p}_1\cdot\vec{S}_1)^2\Big] \\
& +\frac{3G}{2{m_1}^2{m_2}^2r^4}\Big[\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2\big(2\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)+4\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2S_1^2\vec{p}_2\cdot\vec{n} \\
& +10\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-2\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2\vec{p}_2\cdot\vec{n} \\
& -2\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}+10\vec{S}_2\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
& +10\vec{S}_2\times\vec{n}\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1+4\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 \\
& -2\vec{S}_2\times\vec{n}\cdot\vec{p}_2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-35\vec{S}_2\times\vec{n}\cdot\vec{p}_2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 \\
& -10\vec{S}_2\times\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2\Big] \\
& +\frac{G^2m_2}{{m_1}^2r^5}\Big[66\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}-24\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2-8\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{n}\cdot\vec{S}_2 \\
& -47\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2+99\vec{S}_2\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2\Big] \\
& +\frac{G^2}{4m_2r^5}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}+39\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2-2\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{n}\cdot\vec{S}_2 \\
& +58\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2-177\vec{S}_2\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big] \\
& -\frac{G^2}{4m_1r^5}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{n}\cdot\vec{S}_2\big(\vec{p}_1\cdot\vec{n}+42\vec{p}_2\cdot\vec{n}\big)+76\vec{S}_2\times\vec{n}\cdot\vec{p}_1S_1^2 \\
& -57\vec{S}_2\times\vec{n}\cdot\vec{p}_2S_1^2+5\vec{p}_1\cdot\vec{S}_1\vec{S}_1\times\vec{n}\cdot\vec{S}_2-22\vec{p}_2\cdot\vec{S}_1\vec{S}_1\times\vec{n}\cdot\vec{S}_2 \\
& +103\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_1\cdot\vec{S}_2-131\vec{S}_1\cdot\vec{n}\vec{S}_1\times\vec{p}_2\cdot\vec{S}_2-312\vec{S}_2\times\vec{n}\cdot\vec{p}_1(\vec{S}_1\cdot\vec{n})^2 \\
& +252\vec{S}_2\times\vec{n}\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big].
\end{aligned}
\tag{D.7}
$$

E COM generator of the N^{3}LO spin-orbit sector

We recall that the solution for $\vec{G}_{\rm SO}^{\rm N^3LO}$ is written as:

$$\vec{G}_{\rm SO}^{\rm N^3LO} = H_{\rm SO}^{\rm N^2LO}\frac{(\vec{x}_1+\vec{x}_2)}{2} + \left(\vec{Y}_{\rm S_1}^{\rm N^3LO} + (1\leftrightarrow 2)\right), \tag{E.1}$$

and we find:

$$\begin{aligned}
\vec{Y}_{\rm S_1}^{\rm N^3LO} = {} & \frac{Gm_2}{32{m_1}^5 r}\Big[7\vec{S}_1\times\vec{n}\cdot\vec{p}_1 p_1^4\vec{n} + 6\vec{S}_1\times\vec{p}_1 p_1^4\Big] - \frac{G}{32{m_1}^3 m_2 r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(11p_1^2p_2^2\vec{n} \\
& - 16(\vec{p}_1\cdot\vec{p}_2)^2\vec{n} + 42\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1 - 39\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_1 - 2p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_2 \\
& + 24\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_2 - 12p_1^2(\vec{p}_2\cdot\vec{n})^2\vec{n} + 21\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_1 - 24\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\big) \\
& + 8\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(3p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n} - p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1 - 2\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\big) + 2\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(13\vec{p}_1\cdot\vec{p}_2\vec{p}_1 \\
& - 5p_1^2\vec{p}_2 - 9\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1 + 16(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\big) - 2\vec{S}_1\times\vec{n}\big(6p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2 \\
& - 13\vec{p}_1\cdot\vec{n}p_1^2p_2^2 + 3\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2\big) + \vec{S}_1\times\vec{p}_1\big(2p_1^2p_2^2 - 46(\vec{p}_1\cdot\vec{p}_2)^2 \\
& + 68\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2 - 29p_2^2(\vec{p}_1\cdot\vec{n})^2 - 12p_1^2(\vec{p}_2\cdot\vec{n})^2 - 9(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big) \\
& + 4\vec{S}_1\times\vec{p}_2\big(11p_1^2\vec{p}_1\cdot\vec{p}_2 - 5\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\big)\Big] + \frac{G}{32{m_1}^4 r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(6p_1^2\vec{p}_1\cdot\vec{p}_2\vec{n} \\
& - 24p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1 + 7\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2 + 3\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{n}\big) + 15\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(-p_1^4\vec{n} \\
& + \vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\big) - 4\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2p_1^2\vec{p}_1 - \vec{S}_1\times\vec{n}\big(15\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2 - 22\vec{p}_2\cdot\vec{n}p_1^4\big) \\
& + 4\vec{S}_1\times\vec{p}_1\big(8p_1^2\vec{p}_1\cdot\vec{p}_2 - 3\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\big) - \vec{S}_1\times\vec{p}_2\big(14p_1^4 - 15p_1^2(\vec{p}_1\cdot\vec{n})^2\big)\Big] \\
& + \frac{G}{32{m_1}^2{m_2}^2 r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(22\vec{p}_1\cdot\vec{p}_2p_2^2\vec{n} - \vec{p}_2\cdot\vec{n}p_2^2\vec{p}_1 - 2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_2 \\
& + 3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2\vec{n} - 63\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\vec{n} + 12(\vec{p}_2\cdot\vec{n})^3\vec{p}_1 - 3\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_2\big) \\
& - \vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(38p_1^2p_2^2\vec{n} - 32(\vec{p}_1\cdot\vec{p}_2)^2\vec{n} + 34\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1 - 30\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_1 \\
& - 42p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_2 + 32\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_2 + 3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{n} - 15p_1^2(\vec{p}_2\cdot\vec{n})^2\vec{n} \\
& - 9\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_1 + 21\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\vec{p}_2\big) - \vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(43p_2^2\vec{p}_1 \\
& - 32\vec{p}_1\cdot\vec{p}_2\vec{p}_2 - 54\vec{p}_1\cdot\vec{n}p_2^2\vec{n} - 54(\vec{p}_2\cdot\vec{n})^2\vec{p}_1 + 21\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2 \\
& + 60\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{n}\big) + \vec{S}_1\times\vec{n}\big(3p_1^2\vec{p}_2\cdot\vec{n}p_2^2 - 8p_1^2(\vec{p}_2\cdot\vec{n})^3 - 5(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^3\big) \\
& - \vec{S}_1\times\vec{p}_1\big(23\vec{p}_1\cdot\vec{p}_2p_2^2 - 4\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2 - 8\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2 - 11\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big) \\
& - \vec{S}_1\times\vec{p}_2\big(41p_1^2p_2^2 - 42(\vec{p}_1\cdot\vec{p}_2)^2 + 57\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2 - 41p_2^2(\vec{p}_1\cdot\vec{n})^2 \\
& - 46p_1^2(\vec{p}_2\cdot\vec{n})^2 + 21(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)\Big] + \frac{G}{32m_1{m_2}^3 r}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(-2p_2^4\vec{n} \\
& + 2\vec{p}_2\cdot\vec{n}p_2^2\vec{p}_2 + 33p_2^2(\vec{p}_2\cdot\vec{n})^2\vec{n} - 9(\vec{p}_2\cdot\vec{n})^3\vec{p}_2 - 15(\vec{p}_2\cdot\vec{n})^4\vec{n}\big) \\
& + \vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(2\vec{p}_1\cdot\vec{p}_2p_2^2\vec{n} + 8\vec{p}_2\cdot\vec{n}p_2^2\vec{p}_1 - 10\vec{p}_1\cdot\vec{n}p_2^2\vec{p}_2 - 9\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\vec{n} \\
& + 9\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\vec{p}_2\big) + \vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(10p_2^2\vec{p}_2 - 18\vec{p}_2\cdot\vec{n}p_2^2\vec{n} - 9(\vec{p}_2\cdot\vec{n})^2\vec{p}_2 \\
& + 45(\vec{p}_2\cdot\vec{n})^3\vec{n}\big) - \vec{S}_1\times\vec{n}\big(11\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^3 + 5\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^4\big) - 2\vec{S}_1\times\vec{p}_1\big(2p_2^4 \\
& + 4p_2^2(\vec{p}_2\cdot\vec{n})^2 - 5(\vec{p}_2\cdot\vec{n})^4\big) - \vec{S}_1\times\vec{p}_2\big(8\vec{p}_1\cdot\vec{p}_2p_2^2 + 16\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2 \\
& + 21\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2 - 13\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)\Big] + \frac{G}{32{m_2}^4 r}\Big[\vec{S}_1\times\vec{n}\big(3\vec{p}_2\cdot\vec{n}p_2^4
\end{aligned}$$

$$
\begin{aligned}
&+10p_2^2(\vec{p}_2\cdot\vec{n})^3-5(\vec{p}_2\cdot\vec{n})^5\big)+\vec{S}_1\times\vec{p}_2\big(5p_2^4+6p_2^2(\vec{p}_2\cdot\vec{n})^2-19(\vec{p}_2\cdot\vec{n})^4\big)\Big]\\
&-\frac{G^2{m_2}^2}{48{m_1}^3r^2}\Big[2\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(p_1^2\vec{n}-28\vec{p}_1\cdot\vec{n}\vec{p}_1-4(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)+30\vec{S}_1\times\vec{n}\vec{p}_1\cdot\vec{n}p_1^2\\
&+\vec{S}_1\times\vec{p}_1\big(355p_1^2+16(\vec{p}_1\cdot\vec{n})^2\big)\Big]+\frac{G^2m_2}{96{m_1}^2r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(1626p_1^2\vec{n}+632\vec{p}_1\cdot\vec{p}_2\vec{n}\\
&+282\vec{p}_1\cdot\vec{n}\vec{p}_1+426\vec{p}_2\cdot\vec{n}\vec{p}_1+196\vec{p}_1\cdot\vec{n}\vec{p}_2-670\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}+1287(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)\\
&-\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(155p_1^2\vec{n}-176\vec{p}_1\cdot\vec{n}\vec{p}_1+32(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)\\
&+2\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(256\vec{p}_1-241\vec{p}_1\cdot\vec{n}\vec{n}\big)+\vec{S}_1\times\vec{n}\big(1929\vec{p}_1\cdot\vec{n}p_1^2\\
&+384p_1^2\vec{p}_2\cdot\vec{n}-47(\vec{p}_1\cdot\vec{n})^3-320\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{p}_1\big(219p_1^2-3136\vec{p}_1\cdot\vec{p}_2\\
&+358\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+744(\vec{p}_1\cdot\vec{n})^2\big)+8\vec{S}_1\times\vec{p}_2\big(49p_1^2+38(\vec{p}_1\cdot\vec{n})^2\big)\Big]\\
&-\frac{G^2}{96m_1r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(6300\vec{p}_1\cdot\vec{p}_2\vec{n}+2894p_2^2\vec{n}-4262\vec{p}_2\cdot\vec{n}\vec{p}_1-1372\vec{p}_1\cdot\vec{n}\vec{p}_2\\
&+6343\vec{p}_2\cdot\vec{n}\vec{p}_2+2512\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}-111(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)-\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(3384p_1^2\vec{n}\\
&+1413\vec{p}_1\cdot\vec{p}_2\vec{n}-3888\vec{p}_1\cdot\vec{n}\vec{p}_1+2332\vec{p}_2\cdot\vec{n}\vec{p}_1+1073\vec{p}_1\cdot\vec{n}\vec{p}_2-1960\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}\\
&-224(\vec{p}_1\cdot\vec{n})^2\vec{n}\big)-\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(4972\vec{p}_1-1323\vec{p}_2-5458\vec{p}_1\cdot\vec{n}\vec{n}+1283\vec{p}_2\cdot\vec{n}\vec{n}\big)\\
&+\vec{S}_1\times\vec{n}\big(3282p_1^2\vec{p}_2\cdot\vec{n}-3105\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+28\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\\
&+1951\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)+2\vec{S}_1\times\vec{p}_1\big(789\vec{p}_1\cdot\vec{p}_2+1034p_2^2-1993\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-108(\vec{p}_2\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{p}_2\big(3276p_1^2-3429\vec{p}_1\cdot\vec{p}_2+941\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-2420(\vec{p}_1\cdot\vec{n})^2\big)\Big]-\frac{G^2m_1}{48{m_2}^2r^2}\Big[2\vec{S}_1\times\vec{n}\big(24\vec{p}_2\cdot\vec{n}p_2^2-83(\vec{p}_2\cdot\vec{n})^3\big)\\
&-15\vec{S}_1\times\vec{p}_2\big(13p_2^2-15(\vec{p}_2\cdot\vec{n})^2\big)\Big]+\frac{G^2}{96m_2r^2}\Big[\vec{S}_1\times\vec{n}\cdot\vec{p}_1\big(3561p_2^2\vec{n}\\
&-1827\vec{p}_2\cdot\vec{n}\vec{p}_2-284(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)-\vec{S}_1\times\vec{n}\cdot\vec{p}_2\big(2343\vec{p}_1\cdot\vec{p}_2\vec{n}-984p_2^2\vec{n}\\
&+320\vec{p}_2\cdot\vec{n}\vec{p}_1-935\vec{p}_1\cdot\vec{n}\vec{p}_2-3180\vec{p}_2\cdot\vec{n}\vec{p}_2-560\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{n}-900(\vec{p}_2\cdot\vec{n})^2\vec{n}\big)\\
&-\vec{S}_1\times\vec{p}_1\cdot\vec{p}_2\big(1459\vec{p}_2-2773\vec{p}_2\cdot\vec{n}\vec{n}\big)+\vec{S}_1\times\vec{n}\big(1713\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&-3057\vec{p}_2\cdot\vec{n}p_2^2-1453\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2+583(\vec{p}_2\cdot\vec{n})^3\big)\\
&+2\vec{S}_1\times\vec{p}_1\big(516p_2^2-631(\vec{p}_2\cdot\vec{n})^2\big)-\vec{S}_1\times\vec{p}_2\big(2805\vec{p}_1\cdot\vec{p}_2\\
&-2589p_2^2-1313\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+777(\vec{p}_2\cdot\vec{n})^2\big)\Big]-\frac{G^3{m_2}^3}{1800m_1r^3}\Big[17349\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{n}\\
&+42474\vec{S}_1\times\vec{n}\vec{p}_1\cdot\vec{n}-15950\vec{S}_1\times\vec{p}_1\Big]+\frac{G^3{m_1}^2}{720r^3}\Big[9744\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{n}\\
&+13449\vec{S}_1\times\vec{n}\vec{p}_2\cdot\vec{n}-9665\vec{S}_1\times\vec{p}_2\Big]-\frac{G^3{m_2}^2}{14400r^3}\Big[150\vec{S}_1\times\vec{n}\cdot\vec{p}_1(479-657\pi^2)\vec{n}\\
&+484176\vec{S}_1\times\vec{n}\cdot\vec{p}_2\vec{n}+3\vec{S}_1\times\vec{n}\big((182400-20025\pi^2)\vec{p}_1\cdot\vec{n}+101392\vec{p}_2\cdot\vec{n}\big)\\
&-25(21320+513\pi^2)\vec{S}_1\times\vec{p}_1+151800\vec{S}_1\times\vec{p}_2\Big]-\frac{G^3m_1m_2}{14400r^3}\Big[667716\vec{S}_1\times\vec{n}\cdot\vec{p}_1\vec{n}\\
&+150\vec{S}_1\times\vec{n}\cdot\vec{p}_2(1001-639\pi^2)\vec{n}+3\vec{S}_1\times\vec{n}\big(239372\vec{p}_1\cdot\vec{n}\\
&-(107500+44775\pi^2)\vec{p}_2\cdot\vec{n}\big)-226500\vec{S}_1\times\vec{p}_1+25(21896+513\pi^2)\vec{S}_1\times\vec{p}_2\Big].
\end{aligned}
\tag{E.2}
$$

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