Quasi-hereditary Skew Group Algebras

Anna Rodriguez

November 24, 2023

Introduction

Quasi-hereditary algebras are algebras equipped with a partial order on the isomorphism classes of simples which fulfills certain additional properties. They were first introduced by Scott in [25], and then became a central notion in the theory of highest weight categories, initiated by Cline, Parshall and Scott in [10]. The primary motivation of [10] came from the theory of representations of semisimple algebraic groups. Among natural examples of quasi-hereditary algebras arising from this area are the Schur algebras of symmetric groups and algebras underlying blocks of Bernstein-Gelfand-Gelfand category $\mathcal{O}$ associated to a semisimple complex Lie algebra $\mathfrak{g}$.

Many other families of quasi-hereditary algebras of significant interest come from representation theory of finite-dimensional algebras itself. Among these are all finite-dimensional algebras of global dimension less than or equal to two [13, Theorem 2], in particular path algebras of quivers and Auslander algebras.

Again in analogy to Bernstein-Gelfand-Gelfand category $\mathcal{O}$, given a quasi-hereditary algebra A, there is a set of A-modules known as the *standard modules* over A, which mimics the structure and properties of Verma modules over a semisimple complex Lie algebra $\mathfrak{g}$. Similarly to the setting of category $\mathcal{O}$, the category $\mathcal{F}(\Delta)$ of A-modules admitting a filtration by standard modules is of particular interest. By the Dlab-Ringel reconstruction theorem [14], the algebra A together with its quasi-hereditary structure can be reconstructed from $\mathcal{F}(\Delta)$.

Recall that Verma modules over $\mathfrak{g}$ are defined by induction from simple finite-dimensional modules over a Borel subalgebra. In analogy, Koenig introduced in [17] the concept of an exact Borel subalgebra B of a quasi-hereditary algebra A, which is a directed subalgebra of A such that, in particular, the induction from B-modules to A-modules maps simple B-modules to standard A-modules, so that we obtain a bijection between isomorphism classes of simple B-modules and standard A-modules. Additionally, one requires the induction functor to be exact, whence the name, which enables us to transfer homological information from $\operatorname{mod} B$ to $\mathcal{F}(\Delta)$. Together, these conditions allow us to describe the structure of $\mathcal{F}(\Delta)$ using an exact Borel subalgebra.

A fundamental theorem in the study of exact Borel subalgebras of quasi-hereditary algebras is that of Koenig, Külshammer and Ovsienko, proved in [18], which states that every quasi-hereditary algebra is Morita equivalent to a quasi-hereditary algebra with an exact Borel subalgebra.

Quasi-hereditary algebras also feature in the work of Chuang and Kessar in [6], which was later used by Chuang and Rouquier [7] in their proof of Broué's Abelian Defect Group conjecture for symmetric groups. There, the quasi-hereditary algebras that appear are Schur algebras corresponding to blocks of the group algebra of the symmetric group. Of central interest in this setting are the RoCK blocks. These are blocks of a given weight w which are Morita equivalent to the wreath product of the principal block with the symmetric group S_w. The Schur algebras corresponding to these blocks are then also Morita equivalent to the wreath product of the Schur algebra corresponding to the principal block with the symmetric group, see [8, Theorem 5.1]. This enables Chuang and Rouquier to use previous results by Chuang and Tan in [9] on the wreath products of quasi-hereditary algebras with symmetric groups, something which was later again studied by Chan in [5].

Wreath products of quasi-hereditary algebras appear again in a more recent article by Evseev and Kleshchev [15], which generalizes the result of Chuang and Rouquier from the group algebra of

the symmetric group to arbitrary Hecke algebras. Here, the quasi-hereditary algebras considered are zigzag algebras.
Recall that the wreath product algebra $A \wr S_n$ of an algebra A with a symmetric group S_n is isomorphic to the skew group algebra $A^{\otimes n} * S_n$. Thus, one can hope that, after additional investigation of the structure of tensor products of quasi-hereditary algebras, results on skew group algebras of quasi-hereditary algebras may be applied to wreath product algebras of quasi-hereditary algebras.
A skew group algebra $A * G$ is an algebra constructed from an algebra A with an action by a group G in the following way:

- As a k-vector space, $A * G := A \otimes_{\mathrm{k}} \mathrm{k}\, G$.
- Multiplication is given by
$$(a \otimes g) \cdot (a' \otimes g') := ag(a') \otimes gg'.$$

The structure of skew group algebras, including their Morita equivalence class, their Hochschild cohomology and their Yoneda algebra, has been studied extensively, see for example [23], [11], [20], [26], [19]. The preservation of various structural properties of A under the skew group construction, such as global dimension, the property of being an Auslander algebra, or the property of being Calabi-Yau, has also been investigated by many authors, including [27], [21], [23].
In this article, we examine the relation between possible quasi-hereditary structures on A and those on $A * G$. Further, we study the relation between the exact Borel subalgebras of the two.
Assuming a natural compatibility of the group action with the partial order, we show that $\leq_A$ induces a partial order $\leq_{A*G}$ on the isomorphism classes of simple $A * G$-modules, and we obtain the following theorem:

Theorem (Theorem 3.14)**.** *The algebra $(A, \leq_A)$ is quasi-hereditary if and only if $(A * G, \leq_{A*G})$ is quasi-hereditary .*

Moreover, again assuming compatibility with the G-action, we can also relate exact Borel subalgebras of A and $A * G$.

Theorem (Theorem 3.17)**.** *Let $B \subseteq A$ be a subalgebra of A such that $g(B) = B$ for every $g \in G$. Then $(B, \leq_B)$ is an exact Borel subalgebra of $(A, \leq_A)$ if and only if $(B * G, \leq_{B*G})$ is an exact Borel subalgebra of $(A * G, \leq_{A*G})$.*

The structure of the article is as follows. Section 1 contains a brief account of skew group algebras, including a description of the simple $A * G$-modules in terms of simple A-modules and irreducible representations of certain subgroups of G. In Section 2, we recall some of the central results about quasi-hereditary algebras and exact Borel subalgebras. Section 3 is dedicated to the synthesis of the two concepts and contains our main results. Finally, in Section 4 we describe some exact Borel subalgebras of Auslander algebras of certain Nakayama algebras, exemplifying our methods from preceding sections.

Notation

Let k be an algebraically closed field. All algebras are assumed to be finite-dimensional k-algebras, and all modules are assumed to be finite-dimensional as k-vector spaces. Unless otherwise stated, all modules are assumed to be left modules. Tensor products, if not otherwise indicated, are tensor products over k. We denote by $D := \mathrm{Hom}_{\mathrm{k}}(-, \mathrm{k})$ the usual k-duality.
For a module M and an indecomposable module N over some algebra A we write $N|M$ if and only if N is isomorphic to a direct summand of M.
We denote by $\mathrm{Sim}(A)$ a set of representatives of the isomorphism classes of the simple A-modules, and for $S \in \mathrm{Sim}(A)$ we write $[M : S]$ for the multiplicity of S in M. Moreover, for any module M we pick a projective cover P_M.
For any algebra A, we denote by rad the functor
$$\mathrm{rad} : \mathrm{mod}\, A \to \mathrm{mod}\, A$$

mapping a module to its radical and by top the functor

$$\mathrm{top} : \operatorname{mod} A \to \operatorname{mod} A$$

mapping a module to its top. Recall that $\mathrm{top}(M) = M/\operatorname{rad}(M)$ for every $M \in \operatorname{mod} A$.

1 Skew Group Algebras

Throughout, let A be a finite-dimensional algebra over k and G be a finite group acting on A such that $|G|$ does not divide the characteristic of k. In this chapter, we will repeat some basic definitions and results about skew group algebras. For a more detailed introduction see for example [22] and [23].

Definition 1.1. *For an A-module M we define $gM := M$ as a* k*-vector spaces together with the multiplication*

$$a \cdot_{gM} m := g^{-1}(a)m.$$

Moreover, for an A-linear map $f : M \to N$ we define $g(f)(m) := f(m)$.
In this way, every $g \in G$ gives rise to an autoequivalence

$$\operatorname{mod} A \to \operatorname{mod} A, M \mapsto gM, f \mapsto g(f)$$

such that the map $G \to \operatorname{Aut}(\operatorname{mod} A)$ is a group homomorphism.

The module gM is also sometimes denoted ${}^{g}M$, see for example [23, p. 235].
However, we have chosen this notation, so that we may identify gM with the set of formal products $\{gm : m \in M\}$ and then be able to write

$$a \cdot gm = gg^{-1}(a)m.$$

Definition 1.2. *[23, p. 224] The skew group algebra $A * G$ is defined as*

$$A * G := A \otimes \mathrm{k}\, G$$

as a k*-vector space together with the multiplication*

$$(a \otimes g) \cdot (a' \otimes g') := ag(a') \otimes gg'.$$

Definition 1.3. *Let M be an A-module. We say that M has a G-action if there are isomorphisms of A-modules*

$$\mathrm{tr}_g^M : gM \to M$$

such that

$$\mathrm{tr}_g^M \circ g(\mathrm{tr}_h^M) = \mathrm{tr}_{gh}^M$$

for all $g, h \in M$.

Remark 1.4. *Let M, N be two A-modules with a G-action. Then G acts on $\operatorname{Hom}_A(M, N)$ via*

$$g \cdot f = \mathrm{tr}_g^N \circ g(f) \circ (\mathrm{tr}_g^M)^{-1}.$$

We call a homomorphism $f \in \operatorname{Hom}_A(M, N)$ compatible with the actions on M and N, if it is a fixed point of the induced action on $\operatorname{Hom}_A(M, N)$.
Moreover, note that if $M = N$, then the action of G on $\operatorname{End}_A(M)$ is an action of algebra automorphisms.

Proposition 1.5. *[22, Proposition 4.8] There is a one-to-one correspondence between modules $(M, (\mathrm{tr}_g^M)_{g\in G})$ with a G-action and $A * G$-modules given by*

$$g \cdot m := \mathrm{tr}_g^M(m).$$

which induces an equivalence of categories between the $A * G$-modules and the A-modules with a G-action together with the A-linear maps compatible with this action.

Remark 1.6. *Note that with this identification, the G-action on $\mathrm{Hom}_A(M, N)$ for two $A * G$-modules defined in Remark 1.4 can be written as*

$$g \cdot f(m) = g(f(g^{-1}m)).$$

Definition 1.7. *Let M be an A-module. Then we define an $A * G$-module $\mathrm{k}\, G \otimes M$ via*

$$g' \cdot (g \otimes m) := g'g \otimes m$$
$$a(g \otimes m) := g \otimes g^{-1}(a)m$$

for $g, g' \in G, m \in M, a \in A$.
Moreover, if H is a subgroup of G and M is an $A*H$ module, then M is in particular a $\mathrm{k}\, H$-module, so that we can define an $A * G$-module $\mathrm{k}\, G \otimes_{\mathrm{k}\, H} M$ in the same way.

Remark 1.8. *Note that if H is a normal subgroup of G, then the G-action on A induces a G-action on $A * H$ via*

$$g(a \otimes h) = g(a) \otimes ghg^{-1}.$$

Definition 1.9. *Let M be an $A*G$-module, V be a $\mathrm{k}\, G$-module. Then we define an $A*G$-module $M \otimes V$ via*

$$g \cdot (m \otimes v) := gm \otimes gv$$
$$a(m \otimes v) := am \otimes v$$

for $g \in G, m \in M, v \in V$ and $a \in A$.
If $f : M \to N$ is a homomorphism of $A * G$-modules, then

$$f \otimes \mathrm{id}_V : M \otimes V \to N \otimes V$$

is a homomorphism of $A * G$-modules, and

$$- \otimes V : \mathrm{mod}\, A * G \to \mathrm{mod}\, A * G, M \mapsto M \otimes V, f \mapsto f \otimes \mathrm{id}_V$$

defines an additive functor.
Moreover, note that $- \otimes (V \oplus V') \cong - \otimes V \oplus - \otimes V'$ and $- \otimes (V \otimes V') \cong (- \otimes V) \otimes V'$.

Definition 1.10. *We denote by*

$$I_G : \mathrm{mod}\, A \to \mathrm{mod}\, A * G, M \mapsto \mathrm{k}\, G \otimes M, f \mapsto \mathrm{id}_{\mathrm{k}\, G} \otimes f$$

the induction functor along G.
We denote by

$$R_G : \mathrm{mod}\, A * G \to \mathrm{mod}\, A, M \mapsto_{A|} M$$

the canonical restriction functor. Moreover, if H is a subgroup of G, we denote by

$$I_{G/H} : \mathrm{mod}\, A * H \to \mathrm{mod}\, A * G, M \mapsto \mathrm{k}\, G \otimes_{\mathrm{k}\, H} M, f \mapsto \mathrm{id}_{\mathrm{k}\, G} \otimes f$$

the induction functor from $A * H$ to $A * G$ and by

$$R_{G/H} : \mathrm{mod}\, A * G \to \mathrm{mod}\, A * H, M \mapsto_{A*H|} M$$

the canonical restriction functor.

Lemma 1.11. *[23, Theorem 1.1 C] We have* $\mathrm{rad}(A * G) = \mathrm{rad}(A) \otimes \mathrm{k}\, G$.

The content of the following proposition is essentially a compilation of results in [23]. However, for the sake of convenience we will give a quick proof.

Proposition 1.12. *Let V be an indecomposable $\mathrm{k}\, G$-module. Then the following statements hold:*

1. *For any $A * G$-module M*

$$I_G R_G(M) \cong M \otimes \mathrm{k}\, G,$$

in other words,

$$\mathrm{k}\, G \otimes M \cong M \otimes \mathrm{k}\, G.$$

More precisely, there is a natural equivalence

$$I_G R_G \cong - \otimes \mathrm{k}\, G$$

2. *For any A-module M*

$$R_G I_G(M) \cong \bigoplus_{g \in G} gM.$$

More precisely, there is a natural equivalence

$$R_G I_G \cong \bigoplus_{g \in G} g$$

3. *R_G, I_G and $- \otimes V$ are additive.*

4. *R_G, I_G and $- \otimes V$ are exact and reflect exact sequences.*

5. *R_G, I_G and $- \otimes V$ preserve and reflect projective modules.*

6. *R_G, I_G and $- \otimes V$ preserve and reflect injective modules.*

7. *R_G, I_G and $- \otimes V$ preserve and reflect semisimple modules.*

8. *We have natural isomorphisms $I_G \circ \mathrm{top} \cong \mathrm{top} \circ I_G$, $R_G \circ \mathrm{top} \cong \mathrm{top} \circ R_G$ and $(- \otimes V) \circ \mathrm{top} \cong \mathrm{top} \circ (- \otimes V)$.*

Proof.

1. Let M be an $A * G$-module. Then we define an isomorphism

$$\alpha_M : I_G R_G(M) = \mathrm{k}\, G \otimes_{A|} M \to M \otimes \mathrm{k}\, G, g \otimes m \mapsto gm \otimes g.$$

It is easy to check that this is an isomorphism of $A * G$-modules and that $\alpha = (\alpha_M)_M$ defines a natural isomorphism.

2. Let M be an A-module. Then we define an isomorphism

$$\beta_M : R_G I_G(M) =_{A|} \mathrm{k}\, G \otimes M \to \bigoplus_{g \in G} gM, g \otimes m \mapsto (\delta_{gg'} m)_{g' \in G},$$

where $\delta_{gg'}$ is the Kronecker delta of g and g'. It is easy to check that this is an isomorphism of A-modules and that $\beta = (\beta_M)_M$ defines a natural isomorphism.

3. This is obvious.

4. R_G is a restriction functor and thus exact. Moreover, since tensor products over k are exact, I_G and $- \otimes V$ are exact.

5. Since A is an $A*G$-module via $g \cdot a = g(a)$,

$$I_G(A) \cong A \otimes \mathrm{k}\,G = A * G.$$

Since the projective modules in $\operatorname{mod} A$ are exactly the modules isomorphic to direct sums of direct summands of A, and the projective modules in $\operatorname{mod} A*G$ are exactly the modules isomorphic to direct sums of direct summands of A, this implies that I_G preserves projectives. On the other hand, if M is an A-module such that $I_G(M)$ is projective, then so is $R_G I_G(M) \cong \bigoplus_{g\in G} gM$ and since $M = eM$ is a direct summand of $\bigoplus_{g\in G} gM$ this implies that M is projective. Similarly,

$$R_G(A*G) \cong \bigoplus_{g\in G} gA \cong |G|A.$$

Since the projective modules in $\operatorname{mod} A$ are exactly the modules isomorphic to direct sums of direct summands of A, and the projective modules in $\operatorname{mod} A*G$ are exactly the modules isomorphic to direct sums of direct summands of A, this implies that R_G preserves projectives. Moreover, if M is an $A*G$-module such that $R_G M$ is projective, then so is $I_G R_G(M) \cong M \otimes \mathrm{k}\,G$ and since $\mathrm{k} \,|\, \mathrm{k}\,G$,

$$M \cong M \otimes \mathrm{k}\,|M \otimes \mathrm{k}\,G.$$

This implies that M is projective.
Finally, since V is an indecomposable projective $\mathrm{k}\,G$-module, it is isomorphic to a direct summand of $\mathrm{k}\,G$, so that

$$-\otimes V| - \otimes\, \mathrm{k}\,G \cong I_G R_G,$$

and thus $-\otimes V$ preserves and reflects projectives.

6. Since $\mathrm{D}\,\mathrm{k}\,G \cong \mathrm{k}\,G$ as a $\mathrm{k}\,G$-module, this is analogous to the previous statement replacing A by $\mathrm{D}\,A$ and $A*G$ by $\mathrm{D}\,A*G$

7. Let S be a semisimple A-module. Then $\operatorname{rad}(A)S = (0)$ and hence

$$\begin{aligned} \operatorname{rad}(A*G)I_G(S) &= (\operatorname{rad}(A)\otimes \mathrm{k}\,G)\,\mathrm{k}\,G \otimes S = \operatorname{rad}(A)(\mathrm{k}\,G \otimes S) \\ &= \sum_{g\in G} \mathrm{k}\,G \otimes (g(\operatorname{rad}(A))S) = \sum_{g\in G} \mathrm{k}\,G \otimes (\operatorname{rad}(A)S) = (0) \end{aligned}$$

where the first equality follows from Lemma 1.11 and $g(\operatorname{rad}(A)) = \operatorname{rad}(A)$ since G acts on A via algebra automorphisms. Thus $I_G(S)$ is semisimple.
On the other hand, if S is a simple $A*G$ module, then $\operatorname{rad}(A*G)S = (0)$ and hence

$$\operatorname{rad}(A)R_G(S) = \operatorname{rad}(A)S \subseteq \operatorname{rad}(A*G)S = (0)$$

so that $R_G(S)$ is semisimple. Thus R_G preserves and reflects semisimple modules.
Since for every $A*G$-module M we have

$$R_G(M \otimes V) \cong R_G(M) \otimes V \cong \dim_{\mathrm{k}}(V) R_G(M),$$

this implies that $-\otimes V$ also preserves and reflects semisimple modules.
Moreover, if S is an A-module such that $I_G(S)$ is semisimple, then so is $R_G I_G(S)$ and hence S, since S is a direct summand of $R_G I_G(S) \cong \bigoplus_{g\in G} gS$; and if S is an $A*G$-module such that $R_G(S)$ is semisimple, then so is $I_G R_G(S)$ and hence S, since S is a direct summand of $I_G R_G(S) \cong S \otimes \mathrm{k}\,G$.

8. Let M be an A-module. Then by Lemma 1.11,

$$\operatorname{rad}(I_G M) = \operatorname{rad}(A*G) I_G M = (\operatorname{rad}(A)\otimes \mathrm{k}\,G)(\mathrm{k}\,G \otimes M).$$

Since G acts on A via algebra automorphisms, we have $g(\mathrm{rad}(A)) = \mathrm{rad}(A)$ for every $g \in G$. Thus

$$(\mathrm{rad}(A) \otimes \mathrm{k}\, G)(\mathrm{k}\, G \otimes M) = \mathrm{k}\, G \otimes \mathrm{rad}(A)M = \mathrm{k}\, G \otimes \mathrm{rad}(M) = I_G(\mathrm{rad}(M))$$

Since I_G is exact, we thus have $I_G \circ \mathrm{top} \cong \mathrm{top} \circ I_G$.
On the other hand, let M be an $A * G$-module. Then

$$R_G(\mathrm{rad}(M)) = R_G((\mathrm{rad}(A) \otimes G)M) = R_G(\mathrm{rad}(A)M) = \mathrm{rad}(A)R_G(M),$$

so that, since R_G is exact, we have $R_G \circ \mathrm{top} \cong \mathrm{top} \circ R_G$.
Finally, let M be an $A * G$-module. Then by Lemma 1.11 we have

$$\mathrm{rad}(M \otimes V) = \mathrm{rad}(A * G)(M \otimes V) = (\mathrm{rad}(A) \otimes \mathrm{k}\, G)(M \otimes V) = (\mathrm{rad}(A * G)M) \otimes V = \mathrm{rad}(M) \otimes V.$$

Thus, since $- \otimes V$ is exact, we have $(- \otimes V) \circ \mathrm{top} \cong \mathrm{top} \circ (- \otimes V)$.

□

Corollary 1.13. *Let H be a subgroup of G and let $Z \subseteq G$ be a set of representatives of G/H. Then the following statements hold:*

1. *$I_{G/H} \circ I_H$ is naturally equivalent to I_G.*
2. *$R_H \circ R_{G/H}$ is naturally equivalent to R_G.*
3. *$R_{G/H}$ and $I_{G/H}$ are additive.*
4. *$R_{G/H}$ and $I_{G/H}$ are exact and refelct exact sequences.*
5. *$R_{G/H}$ and $I_{G/H}$ preserve and reflect projective modules.*
6. *$R_{G/H}$ and $I_{G/H}$ preserve and reflect injective modules.*
7. *$R_{G/H}$ and $I_{G/H}$ preserve and reflect semisimple modules.*

Moreover, if H is a normal subgroup, then we additionally have

8. *For any $A * H$-module M*

$$R_{G/H} I_{G/H}(M) \cong \bigoplus_{z \in Z} zM.$$

More precisely, there is a natural equivalence

$$R_{G/H} I_{G/H} \cong \bigoplus_{z \in Z} z$$

Proof. 1.-3. These are obvious.

4. $R_{G/H}$ is a restriction functor and thus exact. Moreover, $\mathrm{k}\, H$ is semisimple, so that tensoring over $\mathrm{k}\, H$ is exact.

5.-7. Since

$$I_{G/H} | I_{G/H} \circ (- \otimes \mathrm{k}\, H) \cong I_{G/H} \circ I_H \circ R_H \cong I_G \circ R_H$$

and

$$R_{G/H} | (- \otimes \mathrm{k}\, H) \circ R_{G/H} \cong I_H \circ R_H \circ R_{G/H} \cong I_H \circ R_G,$$

this follows from Proposition 1.12[5.-7.] for H and G.

8. This follows analogously to 2. in Proposition 1.12

□

Corollary 1.14. *Let L' be a simple $A * G$-module. Then there is a simple A-module L such that $L'|I_G L$. On the other hand, if L is a simple A-module, then there is a simple $A * G$-module L' such that $L|R_G L'$.*

Proof. Let L' be a simple $A * G$-module. Then $R_G L' \cong \bigoplus_{L \in \mathrm{Sim}(A)} [R_G L' : L] L$ is semisimple and L' is a summand of

$$L' \otimes \mathrm{k}\, G \cong I_G R_G L' \cong \bigoplus_{L \in \mathrm{Sim}(A)} [R_G L' : L] I_G L.$$

Since L' is simple, this implies that there is some $L \in \mathrm{Sim}(A)$ such that $L'|I_G L$.
The second statement is analogous. □

1.1 An explicit description of the simples

Note that Definition 1.1 tells us that G acts on $\mathrm{mod}\, A$ via autoequivalences. In particular, gL is simple for any simple A-module L, so that we obtain an induced action of G on $\mathrm{Sim}(A)$. For $L \in \mathrm{Sim}(A)$ denote by H_L the stabilizer of the isomorphism class of L in G and let Z_L be a set of representatives of G/H_L.
In this subsection, we will give an explicit description of the simples of $A * G$ in terms of simple A-modules L and simple representations of the corresponding stabilizers H_L, rectifying a result in [19].
This description is not needed for our main results, but will make it possible to obtain an explicit description of the standard modules of $A * G$, see Lemma 3.11.

Lemma 1.15. *For every isomorphism class of simple A-modules, there exists a representative L which is H_L-equivariant.*

Proof. Clearly, we can assume $G = H_L$. Moreover, since $\mathrm{rad}(A)$ acts as zero on L, we can assume that A is semisimple. In this case A is a direct product of matrix rings. Again, the matrix rings not corresponding to L act via zero, so we can assume $A = \mathrm{Mat}_n(\mathrm{k})$ and $L = \mathrm{k}^n$.
Now G acts on A via automorphisms, but since A is a matrix ring, all of these are inner, so we obtain a group homomorphism $\varphi : G \to \mathrm{Gl}_n(\mathrm{k})$. Hence L obtains the structure of an $A * H_L$-module via $\mathrm{tr}_g : gL \to L, x \mapsto \varphi(g)x$. □

The following proposition is a rectification of Lemma 2 in [19].

Proposition 1.16. *The simple modules of $A * G$ are exactly the modules of the form*

$$\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$$

for some irreducible $\mathrm{k}\, H_L$-module V and an H_L-equivariant simple A-module L. Two modules $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$ and $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_{L'}} (L' \otimes W)$ of this form are isomorphic if and only if there is $g \in G$ such that $gL \cong L'$ and $gV \cong W$.

Proof. First we show that a module of the form $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$ is indecomposable if V is indecomposable. First note that, using Corollary 1.13[8], we have an isomorphism

$$\mathrm{End}_{A*H_L}(\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)) \to \bigoplus_{z \in Z_L} \mathrm{Hom}_{A*H_L}(z(L \otimes V), \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)),$$
$$f \mapsto (f_z : z(L \otimes V) \to \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V), zx \otimes v \mapsto f(zx \otimes v))_{z \in Z}$$

of vector spaces. It is easy to see that this is in fact an isomorphism of $\mathrm{k}\,G$-modules, where where the G-action on the left is given by conjugation, and the G-action on the right is given by

$$\begin{aligned} &g(f_z : z(L\otimes V) \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))_{z\in Z} \\ &\qquad\qquad = (f'_z : z(L\otimes V) \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V), z(x\otimes v) \mapsto g(f_{g^{-1}z}(g^{-1}z(x\otimes v))))_{z\in Z}. \end{aligned}$$

Moreover, since $\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V) \cong g(\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))$ as an $A * H_L$-module, we have an isomorphism

$$\bigoplus_{z\in Z_L} \mathrm{Hom}_{A*H_L}(z(L\otimes V), \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)) \to \bigoplus_{z\in Z_L} \mathrm{Hom}_{A*H_L}(z(L\otimes V), z(\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))),$$
$$(f_z : z(L\otimes V) \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))_{z\in Z} \mapsto (f'_z : z(L\otimes V) \to z\,\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V), x \mapsto z(z^{-1}f_z(x))).$$

This is again an isomorphism of $\mathrm{k}\,G$-modules, where the G-action on the right is given by

$$g(f_z : z(L\otimes V) \to z\,\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))_{z\in Z} = (gf_{g^{-1}z} : z(L\otimes V) \to z\,\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))_{z\in Z}.$$

Furthermore, we have an isomorphism

$$\bigoplus_{z\in Z_L} \mathrm{Hom}_{A*H_L}(z(L\otimes V), z(\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))) \to \bigoplus_{z\in Z_L} z\,\mathrm{Hom}_{A*H_L}(L\otimes V, \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))$$
$$(f_z : z(L\otimes V) \to z(\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))_{z\in Z} \mapsto (z(z^{-1}f_z : L\otimes V \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)))_{z\in Z},$$

which is again an isomorphism of $\mathrm{k}\,G$-modules, where the G-action on the right is given by

$$g(z(f_z : L\otimes V \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)))_{z\in Z} = (z(f_{g^{-1}z} : L\otimes V \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)))_{z\in Z}.$$

Finally, we have an isomorphism

$$\begin{aligned} \bigoplus_{z\in Z_L} z\,\mathrm{Hom}_{A*H_L}(L\otimes V, \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)) &\to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} \mathrm{Hom}_{A*H_L}(L\otimes V, \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} L\otimes V), \\ (z(f_z : L\otimes V \to \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)))_{z\in Z} &\mapsto \sum_{z\in Z} z\otimes f_z \end{aligned}$$

of $\mathrm{k}\,G$-modules, where the G-action on the right is given by left multiplication in $\mathrm{k}\,G$. Now if $-^G$ denotes the fix point functor under the G-action

$$\begin{aligned} &\mathrm{End}_{A*G}(\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)) = \mathrm{End}_{A*H_L}(\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V))^G \\ &\cong (\mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} \mathrm{Hom}_{A*H_L}(L\otimes V, \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)))^G \cong \mathrm{Hom}_{A*H_L}(L\otimes V, \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)). \end{aligned}$$

Note that for $z \notin H_L$,

$$\mathrm{Hom}_A(L\otimes V, z(L\otimes V)) \cong \dim(V)^2\,\mathrm{Hom}_A(L, zL) = 0$$

so that for $z \notin H_L$

$$\mathrm{Hom}_{A*H_L}(L\otimes V, z(L\otimes V)) \subseteq \mathrm{Hom}_A(L\otimes V, z(L\otimes V)) = (0).$$

Thus,

$$\begin{aligned} \mathrm{Hom}_{A*H_L}(L\otimes V, \mathrm{k}\,G \otimes_{\mathrm{k}\,H_L} (L\otimes V)) &\cong \bigoplus_{z\in Z_L} \mathrm{Hom}_{A*H_L}(L\otimes V, z(L\otimes V)) \\ &\cong \mathrm{End}_{A*H_L}(L\otimes V) \cong \mathrm{End}_A(L\otimes V)^{H_L} \cong (\mathrm{End}_A(L)\otimes \mathrm{End}_{\mathrm{k}}(V))^{H_L}. \end{aligned}$$

Since $\mathrm{k} = \mathrm{End}_{A*H_L}(L) = \mathrm{End}_A(L)^{H_L}$, we have $\mathrm{End}_A(L) = \mathrm{k}$ with the trivial G-action, so the above is isomorphic to

$$(\mathrm{k}\otimes \mathrm{End}_{\mathrm{k}}(V))^{H_L} \cong \mathrm{End}_{\mathrm{k}\,H_L}(V) \cong \mathrm{k}\,.$$

Hence $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} L \otimes V$ is indecomposable. Since it is semisimple by Corollary 1.13[7], it is thus simple.
To see that these are up to isomorphism all simple $A * G$-modules, note that by Corollary 1.14, every simple $A * G$-module is a summand of $\mathrm{k}\, G \otimes L$ for some $L \in \mathrm{Sim}(A)$ and

$$\mathrm{k}\, G \otimes L \cong \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} \mathrm{k}\, H_L \otimes L \cong \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes \mathrm{k}\, H_L).$$

So decomposing $\mathrm{k}\, H_L$ into indecomposable summands yields the claim.
Clearly, we have isomorphisms of $A * G$-modules

$$\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} L \otimes V \to \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} gL \otimes gV, h \otimes x \otimes v \mapsto hg \otimes x \otimes v.$$

Finally, suppose we have an isomorphism of $A * G$-modules

$$\varphi : \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V) \to \mathrm{k}\, G \otimes_{\mathrm{k}\, H_{L'}} (L' \otimes W).$$

Then, restricting to A, we obtain an isomorphism

$$R_G(\varphi) : R_G(\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)) \to R_G(\mathrm{k}\, G \otimes_{\mathrm{k}\, H_{L'}} (L' \otimes W)).$$

Since

$$\begin{aligned} R_G(\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)) &\cong \bigoplus_{z \in Z_L} \dim_{\mathrm{k}}(V) zL \\ \text{and } R_G(\mathrm{k}\, G \otimes_{\mathrm{k}\, H_{L'}} (L' \otimes W)) &\cong \bigoplus_{z' \in Z_{L'}} \dim_{\mathrm{k}}(W) zL', \end{aligned}$$

the theorem of Krull-Remak-Schmidt thus yields a $g \in G$ such that $gL \cong L'$. In particular, $H_L = H_{L'}$ and, since $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} L \otimes g^{-1}W \cong \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} gL \otimes W$ we have an isomorphism

$$\varphi' : \mathrm{k}\, G \otimes_{H_L} (L \otimes V) \to \mathrm{k}\, G \otimes_{H_L} (L \otimes g^{-1}W)$$

We can restrict this to $A * H_L$ to obtain an isomorphism

$$\bigoplus_{z \in Z_L} z(L \otimes V) \to \bigoplus_{z \in Z_L} z(L \otimes g^{-1}W).$$

Since $L \otimes V$, $L \otimes g^{-1}W$ are simple $A * H_L$-modules by the above, we conclude that we have an isomorphism

$$\varphi'' : L \otimes V \cong L \otimes g^{-1}W$$

of $A * H_L$-modules. Now note that

$$\begin{aligned} &\mathrm{Hom}_{A*H_L}(L \otimes V, L \otimes g^{-1}W) \cong \mathrm{Hom}_A(L \otimes V, L \otimes g^{-1}W)^{H_L} \\ \cong &(\mathrm{Hom}_A(L, L) \otimes \mathrm{Hom}_{\mathrm{k}}(V, g^{-1}W))^{H_L} \cong (\mathrm{Hom}_{\mathrm{k}}(V, g^{-1}W))^{H_L} \cong \mathrm{Hom}_{\mathrm{k}\, H_L}(V, g^{-1}W). \end{aligned}$$

Hence $gV \cong W$. $\square$

The following is a counterexample to Lemma 2 in [19], which claims that every simple $A * G$-module is isomorphic to a module of the form $S \otimes V$ where S is a simple $A * G$ submodule of the socle of A and V is an irreducible representation of G. The error in their proof lies in the erroneous assumption that if S is a simple $A * G$ submodule of the socle of A, then $\mathrm{Hom}_A(S, S) \cong \mathrm{k}$.
While this is in general false, it holds if, for example, all simple $\mathrm{k}\, G$-modules are one-dimensional, i.e. if G is commutative. The reason for this is that in this case we have for any simple $A*G$-module S and any irreducible representation V of G isomorphisms

$$\begin{aligned} &\mathrm{End}_{A*G}(S \otimes V) \cong \mathrm{End}_A(S \otimes V)^G \cong (\mathrm{End}_A(S) \otimes \mathrm{End}_{\mathrm{k}}(V))^G \\ &\cong (\mathrm{End}_A(S) \otimes \mathrm{k})^G \cong \mathrm{End}_{A*G}(S) \cong \mathrm{k} \end{aligned}$$

where G acts trivially on $\mathrm{End}_{\mathrm{k}}(V)$ as $\mathrm{End}_{\mathrm{k}}(V) \cong \mathrm{k} \cong \mathrm{End}_{\mathrm{k}\, G}(V) = \mathrm{End}_{\mathrm{k}}(V)^G$, so that $S \otimes V$ is irreducible. Thus the result in [19, Lemma 2] holds in particular if G is commutative.

Example 1.17. *Consider* $\mathrm{k} = \mathbb{C}$, $A := \mathrm{k}^5$, $G := S_5$ *acting on* A *via permutations of the entries.* A *is semisimple and basic, and* $H_1 := H_{L_1} \cong S_4$, *where* L_1 *is the simple corresponding to the first copy of* k, *so that by the above proposition we have a simple* $A*G$*-module* $\mathrm{k}\, S_5 \otimes_{\mathrm{k}\, S_4} (\mathrm{k} \times \{0\}^4 \otimes V)$ *for every irreducible representation* V *of* S_4. *In particular, since* S_4 *has an irreducible representation* V *of dimension 3,* $A * G$ *has a simple module of dimension*

$$\dim_{\mathrm{k}}(\mathrm{k}\, S_5 \otimes_{\mathrm{k}\, S_4} (\mathrm{k} \times \{0\}^4 \otimes V)) = \dim_{\mathrm{k}}(\mathrm{k}\, S_5/S_4) \dim_{\mathrm{k}} V = 5 \cdot 3 = 15.$$

Moreover, note that A *is a simple* $A * G$*-module, and if* W *is an irreducible representation of* S_5,

$$\dim_{\mathrm{k}}(A \otimes W) = 5 \dim_{\mathrm{k}} W.$$

Since S_5 *has no irreducible representations of dimension 3, this implies that not all simple* $A * G$*-modules are of the form* $A \otimes W$. *Additionally, note that*

$$\mathrm{Hom}_A(A, A) \cong A \ncong \mathrm{k}\,.$$

Corollary 1.18. *The indecomposable projective* $A * G$*-modules are exactly of the form*

$$\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (P_L \otimes V)$$

for some irreducible $\mathrm{k}\, H_L$*-module* V. *They are isomorphic if and only if there is* $g \in G$ *such that* $gL = L', gV = W$.

Proof. Since $V | \mathrm{k}\, H_L$, $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (P_L \otimes V)$ is a direct summand of $\mathrm{k}\, G \otimes P_L$ as above, and is therefore projective. Moreover, by Proposition 1.12[8]

$$\begin{aligned} \mathrm{top}(\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (P_L \otimes V)) &= \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} \mathrm{top}(P_L \otimes V) \\ &= \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (\mathrm{top}(P_L) \otimes V) \\ &= \mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V). \end{aligned}$$

By Proposition 1.16, this is simple, so that $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (P_L \otimes V)$ is indecomposable.
For any finite-dimensional algebra, we have a bijection between the isomorphism classes of projective indecomposable modules and the isomorphism classes of simple modules given by

$$[P] \mapsto [\mathrm{top}(P)],$$

Now since by Proposition 1.16 every simple $A * G$-module is isomorphic to a module of the form $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$, and $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (P_L \otimes V)$ is a projective indecomposable with top $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$, every projective indecomposable $A * G$-module is isomorphic to a module of the form $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$. Moreover, since two simple $A * G$-modules $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (L \otimes V)$ and $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_{L'}} (L' \otimes V')$ are isomorphic if and only if there is $g \in G$ such that $gL = L', gV = W$, their projective covers $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_L} (P_L \otimes V)$ and $\mathrm{k}\, G \otimes_{\mathrm{k}\, H_{L'}} (P_{L'} \otimes V')$ are also isomorphic if and only if there is $g \in G$ such that $gL = L', gV = W$. $\square$

2 Quasi-Hereditary Algebras

In this section, we shall repeat some standard definitions and results about quasi-hereditary algebras, as introduced by [25] and [10]. For an introduction to quasi-hereditary algebras see for example [14].
Let A be an algebra. Denote by $\mathrm{Sim}(A)$ the set of simple A-modules and suppose $\leq$ is a partial order on $\mathrm{Sim}(A)$.

Remark 2.1. *Suppose* $\leq$ *is a partial order on the set* $\mathrm{Sim}(A)$ *of simple* A*-modules. Then this induces a partial order on its additive closure* $\mathrm{add}(\mathrm{Sim}(A))$ *via*

$$S \leq S' \Leftrightarrow L \leq L' \text{ for all } L|S, L'|S'.$$

Thus, if $\leq$ *is a partial order on* $\mathrm{Sim}(A)$, *we will also use it to compare semisimple modules.*

Definition 2.2. *[14, p. 3] We call a partial order $\leq$ on* $\mathrm{Sim}(A)$ *adapted, if all $M \in \operatorname{mod} A$ with simple top* $\mathrm{top}(M) = L$ *and simple socle* $\mathrm{soc}(M) = L'$*, such that L and L' are incomparable with respect to $\leq$, have a composition factor L'' such that $L'' > L$ and $L'' > L'$.*

Lemma 2.3.

1. *Let M be a module with a composition factor L'. Then there is a factor module M' of M with socle* $\mathrm{soc}(M') \cong L'$.

2. *Let M be a module with a composition factor L'. Then there is a submodule M' of M with top* $\mathrm{top}(M') \cong L'$.

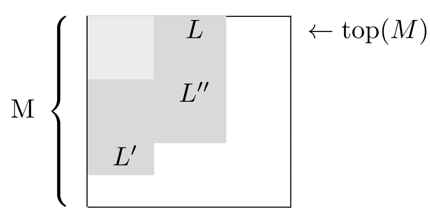
Proof.

1. By definition of a composition factor, there is a factor module M'' of M such that we have an embedding $\iota : L' \to M''$.
   Now let N be a maximal submodule of M'' subject to $N \cap \mathrm{im}(\iota) = (0)$. Then $M' := M''/N$ is also a factor module of M. Denote by
   $$\pi : M'' \to M'$$
   the canonical projection. Then $\pi \circ \iota : L' \to M'$ is injective, so that $L' | \mathrm{soc}(M')$. Write $\mathrm{soc}(M') = \pi \circ \iota(L') \oplus S$. Then $N \subseteq \pi^{-1}(S)$ and $\iota(L') \cap \pi^{-1}(S) = (0)$. Hence by maximality of N, $\pi^{-1}(S) = N$, so that $S = (0)$.

2. This is dual to 1.

$\square$

Lemma 2.4. *The partial order $\leq$ is adapted if and only if every module M which has a composition factor L' such that L' is incomparable to every summand L of its top* $\mathrm{top}(M)$ *has a composition factor L'' and a composition factor $L | \mathrm{top}(M)$ which is a summand of the top, such that $L'' > L$.*



Proof. Suppose $\leq$ is adapted and let M be a module with a composition factor L' such that L' is incomparable to every summand L of its top $\mathrm{top}(M)$.
By Lemma 2.3, M has a factor module M' with simple socle L'. Let L be any summand of $\mathrm{top}(M')$. Then, since M' is a factor module of M, we have $\mathrm{top}(M') | \mathrm{top}(M)$. Hence L is also a summand of $\mathrm{top}(M)$, and thus in particular incomparable to L'. Moreover, we can again apply Lemma 2.3 to obtain a submodule M'' of M' with simple top $\mathrm{top}(M'') \cong L$. As M'' is a submodule of M', $\mathrm{soc}(M'') | \mathrm{soc}(M') \cong L'$, so that M'' has simple socle isomorphic to L'. Since L and L' are incomparable, M'' thus has a composition factor $L'' > L, L'$, and since L'' is a composition factor of M'', which is a submodule of a factor module of M, L'' is also a composition factor of M. Hence this proves the first implication.
On the other hand, suppose every module M which has a composition factor L' such that L' is incomparable to every summand of its top has a composition factor L'' and a composition factor $L | \mathrm{top}(M)$ such that $L'' > L$, and let M be a module with simple top L and simple socle L' such that L and L' are incomparable.
Then by assumption, M has a composition factor L'' such that $L'' > L$. Without loss of generality we can choose L'' maximal with respect to $L'' > L$. Then by Lemma 2.3, M has a submodule M' with simple top $\mathrm{top}(M') \cong L''$. Since M has simple socle L', M' also has simple socle L'. Now if L' and L'' were incomparable, then by assumption M' would have a composition factor $L''' > L''$, which is a contradiction to the maximality of L''. Hence L' and L'' are comparable. Since $L'' > L$ and L' and L are incomparable, this implies $L' < L''$. $\square$

Definition 2.5. *Let $\leq$ be a partial order on* $\mathrm{Sim}(A)$*. Then for every simple A-module L we define*

$$\Delta_L := P_L / \left(\sum_{L' \nleq L, \varphi \in \mathrm{Hom}_A(P_{L'}, P_L)} \mathrm{im}(\varphi) \right)$$

and

$$\widehat{\Delta}_L := P_L / \left(\sum_{L' > L, \varphi \in \mathrm{Hom}_A(P_{L'}, P_L)} \mathrm{im}(\varphi) \right).$$

Denote by $\pi_L : P_L \to \Delta_L$ and $\widehat{\pi}_L : P_L \to \widehat{\Delta}'_L$ the canonical projection. Moreover, write

$$\Delta := \bigoplus_{L \in \mathrm{Sim}(A)} \Delta_L$$

and

$$\widehat{\Delta} := \bigoplus_{L \in \mathrm{Sim}(A)} \widehat{\Delta}_L$$

and call $(\Delta_L)_{L \in \mathrm{Sim}(A)}$ the collection of standard modules and $(\widehat{\Delta}_L)_{L \in \mathrm{Sim}(A)}$ the collection of pseudostandard modules for $(A, \leq)$.

Later, in case more than one algebra is involved, we will sometimes add a superscript to Δ, $\widehat{\Delta}$, Δ_L and $\widehat{\Delta}_L$ indicating the respective algebra.

Lemma 2.6. *Let $L \in \mathrm{Sim}(A)$. Then the following statements hold:*

1. *$L' \nleq L$ for every summand L' of $\mathrm{top}(\ker(\pi_L))$ and for every epimorphism $f : P_L \to M$ such that $L' \nleq L$ for every summand L' of $\mathrm{top}(\ker(f))$ we have an epimorphism $g : M \to \Delta_L$ such that $\pi_L = g \circ f$.*

2. *$L' \leq L$ for every composition factor L' of Δ_L and for every homomorphism $f : P_L \to M$ such that $L' \leq L$ for every composition factor L' of M we have a homomorphism $g : \Delta_L \to M$ such that $f = g \circ \pi_L$.*

3. *We have $\mathrm{top}(\ker(\widehat{\pi}_L)) > L$ and for every epimorphism $f : P_L \to M$ such that $\mathrm{top}(\ker(f)) > L$ we have an epimorphism $g : M \to \widehat{\Delta}_L$ such that $\widehat{\pi}_L = g \circ f$.*

4. *$L' \ngtr L$ for every composition factor L' of $\widehat{\Delta}'_L$ and for every homomorphism $f : P_L \to M$ such that $L' \ngtr L$ for every composition factor L' of M we have a homomorphism $g : \widehat{\Delta}'_L \to M$ such that $f = g \circ \widehat{\pi}_L$.*

Proof. 1. We have $L' \nleq L$ for every summand L' of $\mathrm{top}(\ker(\pi_L))$ by definition. So let $f : P_L \to M$ be an epimorphism such that $L' \nleq L$ for every summand L' of $\mathrm{top}(\ker(f))$. Then we have a projection $\pi : \bigoplus_{L' \nleq L} n_{L'} P_{L'} \to \ker(f)$ for some $n_{L'} \in \mathbb{N}_0$. Composing with the embedding yields that

$$\ker(f) \subseteq \sum_{L' \nleq L, \varphi \in \mathrm{Hom}_A(P_{L'}, P_L)} \mathrm{im}(\varphi) = \ker(\pi_L).$$

Hence π_L factors through f.

2. By definition, $L' \leq L$ for every composition factor L' of Δ_L. So let $f : P_L \to M$ such that $L' \nleq L$ for every composition factor L' of M. Let $L' \nleq L$ and let $\varphi : P_{L'} \to P_L$. Then, since all composition factors of M are less than or equal to L, $f \circ \varphi = 0$. Hence $\mathrm{im}(\varphi) \subseteq \ker(f)$, so that $\ker(\pi_L) \subseteq \ker(f)$ and thus f factors through π_L.

3. This is analogous to 1.

4. This is analogous to 2.

$\square$

Lemma 2.7. *For every $L, L' \in \mathrm{Sim}(A)$ such that $\mathrm{Ext}^1(\widehat{\Delta}_{L'}, \Delta_L) \neq (0)$ we have $L' < L$*

Proof. By definition, the module $\widehat{\Delta}_{L'}$ has a projective presentation

$$\bigoplus_{L''>L'} n_{L''} P_{L''} \longrightarrow P'_L \longrightarrow \widehat{\Delta}'_L \longrightarrow (0)$$

for some integers $n_{L''} \in \mathbb{N}_0$. Suppose $\mathrm{Ext}^1_A(\widehat{\Delta}_{L'}, \Delta_L) \neq (0)$. Then

$$\mathrm{Hom}_A(\bigoplus_{L''>L'} n_{L''} P_{L''}, \Delta_L) \neq (0),$$

so that for some $L'' > L'$

$$\mathrm{Hom}_A(P_{L''}, \Delta_L) \neq (0).$$

Thus L'' is a composition factor of Δ_L. Now since every composition factor of Δ_L is less than or equal to L, this implies $L' < L'' \leq L$. $\square$

The following definition is due to [14]; it resembles the definition of an exceptional collection, originating in the work of Beilinson [1, 2], developed in [16] and [3], the only difference being that condition 3. is here only required for Ext^1 instead of for Ext^n for all $n \geq 1$.

Definition 2.8. *Let $\leq$ be a partial order on $\mathrm{Sim}(A)$. Then a standardisable set for $(A, \leq)$ is a family of modules $M = (M_L)_{L \in \mathrm{Sim}(A)}$ such that*

1. $\mathrm{top}(M_L) \cong L$.
2. $\mathrm{Hom}_A(M_L, M_{L'}) \neq (0) \Rightarrow L \leq L'$.
3. $\mathrm{Ext}^1_A(M_L, M_{L'}) \neq (0) \Rightarrow L < L'$.

Remark 2.9. *Note that $(\Delta_L)_L$ fulfills condition 1. and 2. in the above definition. Thus by Lemma 2.7, if $\widehat{\Delta}_L = \Delta_L$ for every $L \in \mathrm{Sim}(A)$, then $(\Delta_L)_L = (\widehat{\Delta}_L)_L$ is a standardisable set.*

The following lemma tells us that $(\Delta_L)_L = (\widehat{\Delta}_L)_L$ if and only if $\leq$ is adapted. Moreover, the former is the case if and only if any refinement of our partial order will give rise to the same set of standard modules. Thus, being adapted means that our partial order is, in a sense, fine enough.

Lemma 2.10. *The following statements are equivalent:*

1. $\Delta_L = \widehat{\Delta}_L$ *for every* $L \in \mathrm{Sim}(A)$.
2. $\leq$ *is adapted.*
3. $(\widehat{\Delta}_L)_L$ *is a standardisable set.*
4. $\mathrm{Hom}_A(\widehat{\Delta}_{L'}, \widehat{\Delta}_L) \neq (0) \Rightarrow L' \leq L$.

Proof. $1 \Rightarrow 2$ Suppose $\Delta_L = \widehat{\Delta}_L$ for every $L \in \mathrm{Sim}(A)$ and let M be an A-module with simple top L and socle L'. Suppose no composition factor L'' of M is bigger than L. Since $L = \mathrm{top}(M)$ there is an epimorphism $\pi_M : P_L \to M$. Now since no composition factor L'' of M is bigger than L, Lemma 2.6 yields a homomorphism $g : \widehat{\Delta}_L \to M$ such that $\pi_M = g \circ \widehat{\pi}_L$. In particular, g is an epimorphism, so that, since every composition factor of $\widehat{\Delta}_L = \Delta_L$ is less than or equal to L, every composition factor of M is less than or equal to L. In particular L' and L are comparable.

$2 \Rightarrow 3$ This holds by Remark 2.9.

$3 \Rightarrow 4$ This holds by definition.

$4 \Rightarrow 1$ Suppose there is some $L \in \mathrm{Sim}(A)$ such that $\Delta_L \neq \widehat{\Delta}_L$. Then $\widehat{\Delta}_L$ has some composition factor $L' \nleq L$. Let L' be a maximal such composition factor. Then there is a non-zero homomorphism $f : P_{L'} \to \widehat{\Delta}_L$. Moreover, since L' is a maximal composition factor of $\widehat{\Delta}_L$, we have $L'' \ngtr L'$ for every composition factor L'' of $\widehat{\Delta}_L$. Thus Lemma 2.6 yields a homomorphism $g : \widehat{\Delta}_{L'} \to \widehat{\Delta}_L$ such that $f = g \circ \widehat{\pi}_{L'}$. In particular,

$$\mathrm{Hom}_A(\widehat{\Delta}_{L'}, \widehat{\Delta}_L) \neq (0).$$

□

Example 2.11. *Consider the algebra A given by the quiver*

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c$$

with relations $\langle \beta\alpha \rangle = J^2$ and the partial order on $\mathrm{Sim}(A) = \{L_a, L_b, L_c\}$ given by $L_c < L_a$ and $L_c < L_b$. Then the indecomposable projective modules are given by

$$P_a = \begin{pmatrix} a \\ b \end{pmatrix}, \; P_b = \begin{pmatrix} b \\ c \end{pmatrix} \text{ and } P_c = (c),$$

so $\widehat{\Delta}_i = P_i$ for every $i \in \{a, b, c\}$. In particular, $\mathrm{Hom}_A(\widehat{\Delta}_b, \widehat{\Delta}_a) \neq (0)$, so that $(\widehat{\Delta}_i)_i$, is not standardisable.
On the other hand $\Delta_a = L_a$, $\Delta_b = P_b$ and $\Delta_c = P_c$, so that

$$\mathrm{Ext}^1_A(\Delta_j, \Delta_i) = (0)$$

for all $n \geq 1$, $i \in \{a, b, c\}$ and $j \in \{b, c\}$. Moreover

$$(0) \longrightarrow P_c \xrightarrow{r_\beta} P_b \xrightarrow{r_\alpha} P_a \xrightarrow{\pi_a} L_a \longrightarrow (0)$$

is a projective resolution of L_a, where r_α and r_β denote right multiplication by α resp. β and π_a is the canonical projection. Since $\mathrm{Hom}_A(P_b, \Delta_a) = \mathrm{Hom}_A(P_b, \Delta_c) = (0)$, this implies that

$$\mathrm{Ext}^1_A(\Delta_a, \Delta_a) = (0) = \mathrm{Ext}^1_A(\Delta_a, \Delta_c).$$

Finally, since

$$r_\beta^* : \mathrm{Hom}_A(P_b, \Delta_b) \to \mathrm{Hom}_A(P_c, \Delta_b)$$

is injective, we also obtain that $\mathrm{Ext}^1_A(\Delta_a, \Delta_b) = (0)$. Thus $(\Delta_i)_i$ is standardisable.

The following definition is an adaptation of the definition of an exceptional sheaf which can be found in [16], where, as in definition 2.8, we replace the requirement on Ext^n for $n \geq 0$ by a requirement only on Ext^1.

Definition 2.12. *An A-module M is called exceptional if $\mathrm{Ext}^1_A(M, M) = (0)$ and $\mathrm{End}_A(M) \cong \mathrm{k}$.*

Lemma 2.13. *The following statements are equivalent:*

1. *$\leq$ is adapted and $\Delta_L = \widehat{\Delta}_L$ is exceptional for every $L \in \mathrm{Sim}(A)$.*
2. *Every composition factor L' of $\mathrm{rad}(\widehat{\Delta}_L)$ fulfills $L' < L$.*
3. *$\mathrm{Hom}_A(\widehat{\Delta}_{L'}, \mathrm{rad}(\widehat{\Delta}_L)) \neq (0) \Rightarrow L' < L$.*

Proof. 1 $\Rightarrow$ 2 Recall that if $\leq$ is adapted, then $(\Delta_L)_L = (\widehat{\Delta}_L)_L$. Let L' be a composition factor of $\mathrm{rad}(\widehat{\Delta}_L) = \mathrm{rad}(\Delta_L)$. Then $L' \leq L$. If $L = L'$ then this induces a non-zero homomorphism $P_L \to \mathrm{rad}(\Delta_L)$ and thus, by Lemma 2.6, an endomorphism $\Delta_L \to \mathrm{rad}(\Delta_L) \to \Delta_L$ which is neither zero nor invertible. This is a contradiction. Thus $L' < L$.

2 $\Rightarrow$ 3 Suppose $\mathrm{Hom}_A(\widehat{\Delta}_{L'}, \mathrm{rad}(\widehat{\Delta}_L)) \neq (0)$. Then, since $\widehat{\Delta}_{L'}$ has simple top L', L' is a composition factor of $\mathrm{rad}(\widehat{\Delta}_L)$ and thus $L' < L$.

3 $\Rightarrow$ 1 If $L \neq L'$ then

$$\mathrm{Hom}_A(\widehat{\Delta}_{L'}, \mathrm{rad}(\widehat{\Delta}_L)) = (0) \Leftrightarrow \mathrm{Hom}_A(\widehat{\Delta}_{L'}, \widehat{\Delta}_L) = (0).$$

Thus

$$\mathrm{Hom}_A(\widehat{\Delta}_{L'}, \widehat{\Delta}_L) \neq (0) \Rightarrow L' \leq L,$$

so that $\leq$ is adapted, $\widehat{\Delta}_L = \Delta_L$ and $(\Delta_L)_L$ is a standardisable set by Lemma 2.10.
In particular, $\mathrm{Ext}^1(\Delta_L, \Delta_L) = (0)$.
Moreover, $\mathrm{Hom}_A(\widehat{\Delta}_L, \mathrm{rad}(\widehat{\Delta}_L)) = (0)$, so that any endomorphism of $\widehat{\Delta}_L$ is either surjective or zero. Thus $\mathrm{End}_A(\Delta_L) = \mathrm{End}_A(\widehat{\Delta}_L) \cong \mathrm{k}$. □

Definition 2.14. *We denote by $\mathcal{F}(\Delta)$ the full subcategory of* $\mathrm{mod}\, A$ *which contains all A-modules admitting a filtration by the Δ_L, $L \in \mathrm{Sim}(A)$. In other words, an A-module M is in $\mathcal{F}(\Delta)$ if and only if there is an integer $m \geq 0$ and an ascending sequence of submodules*

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

such that for every $1 \leq i \leq m$ there is an $L_i \in \mathrm{Sim}(A)$ such that $M_i/M_{i-1} \cong \Delta_{L_i}$.

Proposition 2.15. *[14, Lemma 1.4 and Lemma 1.5] The subcategory $\mathcal{F}(\Delta)$ is closed under direct sums, direct summands and extensions.*

Definition 2.16. *[10, 24, 12] An algebra A together with an adapted partial order $\leq$ on* $\mathrm{Sim}(A)$ *is called*

1. *left standardly stratified, if $A \in \mathcal{F}(\Delta)$*
2. *quasi-hereditary, if additionally Δ_L is exceptional for all $L \in \mathrm{Sim}(A)$*
3. *strongly quasi-hereditary, if additionally every Δ_L has projective dimension one.*
4. *directed, if $\Delta_L \cong L$ for all $L \in \mathrm{Sim}(A)$.*

Definition 2.17. *[17, p. 405][4, Definition 3.4] Let $(A, \leq)$ be a quasi-hereditary algebra. Then a subalgebra $B \subseteq A$ is called an exact Borel subalgebra if there is a bijection $i : \mathrm{Sim}(B) \to \mathrm{Sim}(A)$ such that*

1. *A is projective as a right B-module,*
2. *B is directed*
3. *$A \otimes_B L = \Delta_{i(L)}$ for all $L \in \mathrm{Sim}(B)$.*

Moreover, it is called

1. *a strong exact Borel subalgebra if there is a maximal semisimple subalgebra of A which is also a semisimple subalgebra of B;*
2. *a homological exact Borel subalgebra if the induced maps*

$$A \otimes_B - : \mathrm{Ext}^*_B(L, L') \to \mathrm{Ext}^*_A(\Delta_L, \Delta_{L'})$$

are isomorphisms in degree greater than or equal to two and epimorphisms in degree one for all $L, L' \in \mathrm{Sim}(B)$;

3. *a normal exact Borel subalgebra, if there is a splitting of the inclusion $\iota : B \to A$ whose kernel is a right ideal in A;*

4. *and a regular exact Borel subalgebra if it is normal and the induced maps*

$$A \otimes_B - : \mathrm{Ext}^*_B(L, L') \to \mathrm{Ext}^*_A(\Delta_{i(L)}, \Delta_{i(L')})$$

are isomorphisms in degree greater than or equal to one for all $L, L' \in \mathrm{Sim}(B)$.

Remark 2.18. *Let A be a finite-dimensional algebra, $L^{(A)} := A/\operatorname{rad}(A)$ and $\pi_A : A \to A/\operatorname{rad}(A)$ be the canonical projection. By definition, $L^{(A)}$ is the maximal semisimple quotient of A. However, recall that by the Wedderburn-Malcev theorem, there is an embedding*

$$\iota_A : L^{(A)} \to A$$

such that $\pi_A \circ \iota_A = \mathrm{id}_{L^{(A)}}$, which turns $L^{(A)}$ into a maximal semisimple subalgebra of A. Moreover, by the same result, any two maximal semisimple subalgebras of A are conjugated, so that, in particular, $L^{(A)}$ is up to isomorphism the unique semisimple subalgebra of A.

Lemma 2.19. *Suppose B is an exact Borel subalgebra of A. Then B is a strong exact Borel subalgebra of A if and only if $A\operatorname{rad}(B) \subseteq \operatorname{rad}(A)$.*

Proof. Let $L^{(A)} := A/\operatorname{rad}(A)$ and $L^{(B)} := B/\operatorname{rad}(B)$. By Remark 2.18 $L^{(A)}$ and $L^{(B)}$ are up to isomorphism the unique maximal semisimple subalgebras of A and B respectively.
In particular, a semisimple subalgebra of A is a maximal semisimple subalgebra if and only if it has the same vector space dimension as $L^{(A)}$. Since any semisimple subalgebra of B is also a semisimple subalgebra of A, this means that B contains a maximal semisimple subalgebra of A if and only if $\dim_{\mathrm{k}} L^{(A)} = \dim_{\mathrm{k}} L^{(B)}$.
Thus B is a strong exact Borel subalgebra of A if and only if $\dim_{\mathrm{k}} L^{(B)} = \dim_{\mathrm{k}} L^{(A)}$.
Let $L_1^B, \ldots, L_n^B$ be a set of representative of the simple B-modules. Write $\Delta_i^A := A \otimes_B L_i^B$ and $L_i^A := \operatorname{top}(A \otimes_B L_i^B)$. Since B is an exact Borel subalgebra of A, $L_1^A, \ldots, L_n^A$ is a set of representatives of the simple A-modules, and $\Delta_1^A, \ldots, \Delta_n^A$ are the corresponding standard modules.
Let $X := A\operatorname{rad}(B)/(\operatorname{rad}(A) \cap A\operatorname{rad}(B))$. Then, as A-modules,

$$\begin{aligned} L^{(A)} = A/\operatorname{rad}(A) &\cong ((\operatorname{rad}(A) + A\operatorname{rad}(B))/\operatorname{rad}(A)) \oplus A/(\operatorname{rad}(A) + A\operatorname{rad}(B)) \\ &\cong A\operatorname{rad}(B)/(\operatorname{rad}(A) \cap A\operatorname{rad}(B)) \oplus \operatorname{top}(A/A\operatorname{rad}(B)) \\ &\cong X \oplus \operatorname{top}\left(A \otimes_B B/\operatorname{rad}(B)\right) \\ &\cong X \oplus \operatorname{top}\left(\bigoplus_i [\operatorname{top} B : L_i^B] \Delta_{L_i^A}\right) \\ &\cong X \oplus \bigoplus_i [L^{(B)} : L_i^B] L_i^A. \end{aligned}$$

Recall that for any $n \in \mathbb{N}$ we have $\dim_{\mathrm{k}} \mathrm{k}^n = n = [\mathrm{Mat}_n(\mathrm{k}) : \mathrm{k}^n]$. Moreover, by the Wedderburn-Artin theorem, any finite-dimensional semisimple algebra over k is isomorphic to a direct sum of matrix rings, and for any two finite dimensional algebras R and R', $\mathrm{Sim}(R \oplus R')$ is the disjoint union of $\mathrm{Sim}(R)$ and $\mathrm{Sim}(R')$ where R acts on $L' \in \mathrm{Sim}(R')$ via zero and analogously for $L \in \mathrm{Sim}(R)$. Hence the equation above generalizes to any finite-dimensional semisimple algebra over k, so that we have

$$\dim_{\mathrm{k}} L_i^A = [L^{(A)} : L_i^A]$$

for every $1 \leq i \leq n$. Thus,

$$\dim_{\mathrm{k}} L_i^A = [L^{(A)} : L_i^A] = [X : L_i^A] + [L^{(B)} : L_i^B] = [X : L_i^A] + \dim_{\mathrm{k}} L_i^B,$$

so that

$$\begin{aligned}\dim_{\mathrm{k}} L^{(A)} &= \dim_{\mathrm{k}} X + \sum_i [L^{(B)} : L_i^B] \dim_{\mathrm{k}} L_i^A \\ &= \dim_{\mathrm{k}} X + \sum_i [L^{(B)} : L_i^B]([X : L_i^A] + \dim_{\mathrm{k}}(L_i^B)) \\ &= \dim_{\mathrm{k}} X + \sum_i [X : L_i^A][L^{(B)} : L_i^B] + \dim_{\mathrm{k}} L^B.\end{aligned}$$

Hence B is a strong exact Borel subalgebra of A if and only if $X = (0)$, i.e. if $A\,\mathrm{rad}(B) \subseteq B$. $\square$

3 Quasi-Hereditary Algebras with a Group Action

Throughout, let, as before, A be a finite-dimensional k-algebra and G be a group acting on A via automorphisms such that the order of G does not divide the characteristic of k.

Definition 3.1. *A partial order $\leq_A$ on $\mathrm{Sim}(A)$ is called G-equivariant if*

$$L <_A L' \Leftrightarrow gL <_A hL' \text{ for all } g, h \in G.$$

On the other hand, a partial order $\leq_{A*G}$ on $\mathrm{Sim}(A*G)$ is called G-stable if

$$S <_{A*G} S' \Leftrightarrow S \otimes V <_{A*G} S' \otimes W$$

for all $\mathrm{k}\,G$-modules V, W.

Definition 3.2. *1. Let $\leq$ be a G-equivariant partial order on $\mathrm{Sim}(A)$. Then we define a partial order $\leq_G$ on $\mathrm{Sim}(A*G)$ via*

$$S <_G S' :\Leftrightarrow_{A|} S <_{|A} S'.$$

2. Let $\leq'$ be a G-stable partial order on $\mathrm{Sim}(A*G)$. Then we define a partial order $\leq'_{|G}$ on $\mathrm{Sim}(A)$ via

$$L <'_{|G} L' :\Leftrightarrow \mathrm{k}\,G \otimes L <' \mathrm{k}\,G \otimes L',$$

, where we extend the partial orders from the simple modules to the semisimple modules as in Remark 2.1.

Hence we define a strict partial order $(<_A)_G$ as the pullback of a strict partial order $<_A$ along the map

$$\mathrm{add}\,\mathrm{Sim}(A*G) \to \mathrm{add}(\mathrm{Sim}(A)), M \mapsto {}_{|A}M,$$

and similarly a strict partial order $(<_{A*G})_{|G}$ as the pullback of a strict partial order $<_{A*G}$ along the map

$$\mathrm{add}\,\mathrm{Sim}(A) \to \mathrm{add}(\mathrm{Sim}(A*G)), M \mapsto I_G M.$$

Note that since the above maps are not necessarily injective, the pullbacks of $\leq_A$ resp. $\leq_{A*G}$ along these maps are not necessarily partial orders. For example, if we consider the case $A = \mathrm{k}$, $G = \mathbb{Z}/2\mathbb{Z}$ with the trivial action, and the unique partial order $\leq_A$ on $\mathrm{Sim}(A)$, then we have $A*G \cong \mathrm{k}^2$ with simple modules L_1 and L_2 corresponding to the first and the second copy of k, respectively. In this setting, if we tried to define $L_i \leq_{A*G} L_j \Leftrightarrow R_G L_i \leq_A R_G L_j$ we would obtain that $L_1 \leq L_2$ and $L_2 \leq L_1$ since $R_G L_1 \cong R_g L_2$, that is, the asymmetry of $\leq_{A*G}$ would be violated, so that it would not be a partial order.
On the other hand, the pullback of a strict partial order along any map is always a strict partial

order.
In particular, the above pullbacks yield well-defined partial orders $\leq_G$ and $\leq'_{|G}$ even for a not necessarily G-equivariant partial order $\leq$ and a not necessarily G-stable partial order $\leq'$.
However, if e.g. $\leq$ is not G-equivariant, then there may be some simple $A*G$ modules $S \neq S'$ such that $L <_A L'$, but $L'' \not< L'''$ for some simple summands of L, L'' of S and L', L''' of S', so S and S' would be incomparable with respect to $\leq_A$, even though all summands of ${}_{|A}S$ and ${}_{|A}S'$ may be comparable. Hence even a total order $\leq$ might result in an empty order $\leq_G$, and so there is little hope to conclude adaptedness of $\leq_G$ from adaptedness of $\leq$.
Similar considerations hold for a non G-stable partial order $\leq'$ and its induced partial order $<'_{|G}$.

Proposition 3.3. *1. Let $\leq$ be a G-equivariant partial order on* $\mathrm{Sim}(A)$. *Then $\leq_G$ is the unique G-stable partial order such that*

$$L < L' \Leftrightarrow \mathrm{k}\,G \otimes L <_G \mathrm{k}\,G \otimes L'.$$

2. Let $\leq'$ be a G-stable partial order on $\mathrm{Sim}(A*G)$. *Then $\leq'_{|G}$ is the unique G-equivariant partial order on* $\mathrm{Sim}(A)$ *such that*

$$S <' S' \Leftrightarrow_{A|} S(<'_{|G})_{A|}S'.$$

Proof. 1. Clearly, $\leq_G$ is G-stable, and

$$L < L' \Leftrightarrow \mathrm{k}\,G \otimes L <_G \mathrm{k}\,G \otimes L'.$$

Now suppose that $\leq'$ is another G-stable partial order on $\mathrm{Sim}(A*G)$ such that

$$L < L' \Leftrightarrow \mathrm{k}\,G \otimes L <' \mathrm{k}\,G \otimes L'.$$

Let $S, S' \in \mathrm{Sim}(A*G)$. Then there are $L, L' \in \mathrm{Sim}(A)$ such that $L|_{A|}S$, $L'|_{A|}S'$. Thus $\mathrm{k}\,G \otimes L|\,\mathrm{k}\,G \otimes {}_{A|}S \cong S \otimes \mathrm{k}\,G$ and $\mathrm{k}\,G \otimes L'|\,\mathrm{k}\,G \otimes {}_{A|}S' \cong S' \otimes \mathrm{k}\,G$.
Moreover, by Corollary 1.14, there are $L'', L''' \in \mathrm{Sim}(A)$ such that $S|\,\mathrm{k}\,G \otimes L'', S'|\,\mathrm{k}\,G \otimes L'''$. In particular, $L|_{A|}\,\mathrm{k}\,G \otimes L''$ and $L'|_{A|}\,\mathrm{k}\,G \otimes L'''$, so that $L \cong gL''$ and $L' \cong hL'''$ for some $g, h \in G$. Hence $\mathrm{k}\,G \otimes L \cong \mathrm{k}\,G \otimes L''$ and $\mathrm{k}\,G \otimes L' \cong \mathrm{k}\,G \otimes L'''$. Thus

$$\begin{aligned} S <' S' \Leftrightarrow S \otimes \mathrm{k}\,G <' S' \otimes \mathrm{k}\,G &\Rightarrow \mathrm{k}\,G \otimes L <' \mathrm{k}\,G \otimes L' \\ &\Leftrightarrow L < L' \Leftrightarrow \mathrm{k}\,G \otimes L <_G \mathrm{k}\,G \otimes L' \Rightarrow S <_G S', \end{aligned}$$

and analogously $S <_G S' \Rightarrow S <' S'$.

2. Clearly, $<'_{|G}$ is G-equivariant and

$$\begin{aligned} {}_{A|}S <'_{|G} {}_{A|}S' &\Leftrightarrow \mathrm{k}\,G \otimes_{A|} S <' \mathrm{k}\,G \otimes_{A|} S' \\ &\Leftrightarrow S \otimes \mathrm{k}\,G <' S' \otimes \mathrm{k}\,G \Leftrightarrow S < S'. \end{aligned}$$

Moreover, if $\leq$ is another G-equivariant partial order on $\mathrm{Sim}(A)$ such that

$$S <' S' \Leftrightarrow_{A|} S <_{A|} S'.$$

then

$$L < L' \Leftrightarrow \bigoplus_{g \in G} gL < \bigoplus_{g \in G} gL' \Leftrightarrow_{A|} \mathrm{k}\,G \otimes L <_{A|} \mathrm{k}\,G \otimes L' \Leftrightarrow \mathrm{k}\,G \otimes L <' \mathrm{k}\,G \otimes L' \Leftrightarrow L <'_G L'$$

for all $L, L' \in \mathrm{Sim}(A)$.

$\square$

Corollary 3.4. *Let $\leq$ be a G-equivariant partial order on* $\mathrm{Sim}(A)$*. Then $\leq$ coincides with $(\leq_G)_{|G}$. Let $\leq'$ be a G-stable partial order on* $\mathrm{Sim}(A*G)$*. Then $\leq'$ coincides with $(<'_{|G})_G$. Thus the assignments*

$$\begin{aligned}\{G\text{-equivariant partial orders on }\mathrm{Sim}(A)\}&\\ \to \{G\text{-stable partial orders on }\mathrm{Sim}(A*G)\},&\\ \leq \mapsto \leq_G&\end{aligned}$$

and

$$\begin{aligned}\{G\text{-stable partial orders on }\mathrm{Sim}(A*G)\}&\\ \to \{G\text{-equivariant partial orders on }\mathrm{Sim}(A)\},&\\ \leq' \mapsto \leq'_{|G}&\end{aligned}$$

are mutually inverse bijections.

From now on, let $\leq_A$ be a G-equivariant partial order on $\mathrm{Sim}(A)$ and $\leq_{A*G}$ the corresponding induced order on $\mathrm{Sim}(A*G)$, or the other way around.

Lemma 3.5. *For every $g \in G$, $L \in \mathrm{Sim}(A)$ we have $g\Delta_L \cong \Delta_{g(L)}$ and $g\,\widehat{\Delta}_L \cong \widehat{\Delta}_{g(L)}$*

Proof. This is a direct consequence of the fact that g induces an order preserving automorphism of $\mathrm{mod}\,A$. □

Proposition 3.6. *For every simple A-module $L \in \mathrm{Sim}(A)$*

$$\mathrm{k}\,G \otimes \widehat{\Delta}_L \cong \bigoplus_{S\in\mathrm{Sim}(A*G)} [\mathrm{k}\,G \otimes L : S]\,\widehat{\Delta}_S$$

where $[\mathrm{k}\,G \otimes L : S]$ denotes the multiplicity of the simple summand S in $\mathrm{k}\,G \otimes L$.

Proof. Since $\mathrm{k}\,G \otimes P_L$ is projective with top

$$\mathrm{top}(\mathrm{k}\,G \otimes P_L) \cong \mathrm{k}\,G \otimes \mathrm{top}(P_L) = \mathrm{k}\,G \otimes L$$

by Proposition 1.12[8], there is an isomorphism

$$\varphi'_L : \bigoplus_{S\in\mathrm{Sim}(A*G)} [\mathrm{k}\,G \otimes L : S]P_S \to \mathrm{k}\,G \otimes P_L.$$

For the sake of notation, fix a set $\mathcal{S}_L$ of simple $A*G$-modules such that for any $S \in \mathrm{Sim}(A*G)$ there are exactly $[\mathrm{k}\,G \otimes L : S]$ modules in $\mathcal{S}_L$ which are isomorphic to S and consider instead the isomorphism

$$\varphi_L : \bigoplus_{S\in\mathcal{S}_L} P_S \to \mathrm{k}\,G \otimes P_L.$$

For every $S \in \mathcal{S}_L$ consider the map

$$f_S := (\mathrm{k}\,G \otimes \widehat{\pi}_L) \circ \varphi_L \circ \iota_{P_S} : P_S \to \mathrm{k}\,G \otimes \widehat{\Delta}_L$$

where $\iota_{P_S} : P_S \to \bigoplus_{S'\in\mathcal{S}_L} P_{S'}$ is the canonical embedding.
Then since $\widehat{\Delta}_L$ has a filtration by $L' \not>_A L$ and $\mathrm{k}\,G \otimes -$ is exact, $\mathrm{k}\,G \otimes \widehat{\Delta}_L$ has a filtration by $\mathrm{k}\,G \otimes L'$ such that $L' \not>_A L$. Since for every $L' \not>_A L$ any two simple summands $S | \mathrm{k}\,G \otimes L$ and $S' | \mathrm{k}\,G \otimes L'$ fulfill $S' \not>_A S$, this means that no composition factor S' of $\mathrm{k}\,G \otimes \widehat{\Delta}_L$ is greater than any summand S of $\mathrm{k}\,G \otimes L$.

Hence Lemma 2.6 implies that for any $S \in \mathcal{S}_L$ there is a homomorphism $\gamma_S : \widehat{\Delta}_S \to \mathrm{k}\, G \otimes \widehat{\Delta}_L$ such that $f_S = \gamma_S \circ \widehat{\pi}_S$. Thus we obtain a homomorphism

$$\gamma : \bigoplus_{S \in \mathcal{S}_L} \widehat{\Delta}_S \to \mathrm{k}\, G \otimes \widehat{\Delta}_L, (x_S)_S \mapsto \sum_S \gamma_S(x_S)$$

such that

$$\begin{aligned} \gamma \circ (\widehat{\pi}_S)_S((x_S)_S) &= \sum_S \gamma_S \circ \widehat{\pi}_S(x_S) = \sum_S f_S(x_S) \\ &= \sum_S (\mathrm{k}\, G \otimes \widehat{\pi}_L) \circ \varphi_L \circ \iota_{P_S}(x_S) = (\mathrm{k}\, G \otimes \widehat{\pi}_L) \circ \varphi_L((x_S)_S). \end{aligned}$$

In other words, the diagram

$$\begin{tikzcd} \bigoplus_S P_S \arrow[r, "(\widehat{\pi}_S)_S"] \arrow[d, "\varphi_L"'] & \bigoplus_S \widehat{\Delta}_S \arrow[d, "\gamma"] \\ \mathrm{k}\, G \otimes P_L \arrow[r, "\mathrm{k}\, G \otimes \widehat{\pi}_L"'] & \mathrm{k}\, G \otimes \widehat{\Delta}_L \end{tikzcd}$$

commutes. On the other hand, note that by Proposition 1.12[2] and Lemma 3.5 we have a commutative diagram

$$\begin{tikzcd} R_G(\mathrm{k}\, G \otimes P_L) \arrow[r] \arrow[d, "R_G(\mathrm{k}\, G \otimes \widehat{\pi}_L)"] & \bigoplus_{g \in G} gP_L \arrow[r] \arrow[d, "\bigoplus_{g \in G} g(\widehat{\pi}_L)"] & \bigoplus_{g \in G} P_{gL} \arrow[d, "\bigoplus_{g \in G} \widehat{\pi}_{gL}"] \\ R_G(\mathrm{k}\, G \otimes \widehat{\Delta}_L) \arrow[r] & \bigoplus_{g \in G} g\, \widehat{\Delta}_L \arrow[r] & \bigoplus_{g \in G} \widehat{\Delta}_{gL} \end{tikzcd}$$

where the horizontal arrows are isomorphisms. Since $R_G(\widehat{\Delta}_S)$ has a filtration by $R_G(S')$ where $S' \in \mathrm{Sim}(A * G)$ such that $S' \not>_{A*G} S$ and any simple summand L' of $R_G(S')$ fulfills $S' | \mathrm{k}\, G \otimes L'$, we have for any composition factor L' of $R_G(\widehat{\Delta}_S)$ that $\mathrm{k}\, G \otimes L' \not> S$. Thus for $S | \mathrm{k}\, G \otimes L$, $R_G(\widehat{\Delta}_S)$ has no composition factor L' such that $\mathrm{k}\, G \otimes L'$ is greater than $\mathrm{k}\, G \otimes L$ and thus no composition factor L' which is greater than L.
Hence, analogously to the construction of γ, we can construct an A-linear map $\gamma' : R_G(\mathrm{k}\, G \otimes \widehat{\Delta}_L) \to R_G(\bigoplus_S \widehat{\Delta}_S)$ such that the diagram

$$\begin{tikzcd} R_G(\mathrm{k}\, G \otimes P_L) \arrow[r, "R_G(\mathrm{k}\, G \otimes \widehat{\pi}_L)"] \arrow[d, "R_G(\varphi_L)^{-1}"] & R_G(\mathrm{k}\, G \otimes \widehat{\Delta}_L) \arrow[d, "\gamma'"'] \\ R_G(\bigoplus_S P_S) \arrow[r, "R_G((\widehat{\pi}_S)_S)"] & R_G(\bigoplus_S \widehat{\Delta}_S) \end{tikzcd}$$

commutes.
We obtain a diagram

$$\begin{tikzcd} R_G(\mathrm{k}\, G \otimes P_L) \arrow[r, "R_G(\mathrm{k}\, G \otimes \widehat{\pi}_L)"] \arrow[d, shift left, "R_G(\varphi_L)^{-1}"] & R_G(\mathrm{k}\, G \otimes \widehat{\Delta}_L) \arrow[d, shift left, "\gamma'"] \\ R_G(\bigoplus_S P_S) \arrow[r, "R_G((\widehat{\pi}_S)_S)"] \arrow[u, shift left, "R_G(\varphi_L)"] & R_G(\bigoplus_S \widehat{\Delta}_S) \arrow[u, shift left, "R_G(\gamma)"] \end{tikzcd}$$

where both squares commute, i.e.

$$\begin{aligned} \gamma' \circ R_G(\mathrm{k}\, G \otimes \widehat{\pi}_L) &= R_G((\widehat{\pi}_S)_S) \circ R_G(\varphi_L)^{-1} \\ \text{and } R_G(\gamma) \circ R_G((\widehat{\pi}_S)_S) &= R_G(\mathrm{k}\, G \otimes \widehat{\pi}_L) \circ R_G(\varphi_L). \end{aligned}$$

In particular,

$$\begin{aligned} R_G(\gamma)\circ\gamma'\circ R_G(\mathrm{k}\,G\otimes\widehat{\pi}_L) &= R_G(\gamma)\circ R_G((\widehat{\pi}_S)_S)\circ R_G(\varphi_L)^{-1} \\ &= R_G(\mathrm{k}\,G\otimes\widehat{\pi}_L)\circ R_G(\varphi_L)\circ R_G(\varphi_L)^{-1} \\ &= R_G(\mathrm{k}\,G\otimes\widehat{\pi}_L) \end{aligned}$$

and

$$\begin{aligned} \gamma'\circ R_G(\gamma)\circ R_G((\widehat{\pi}_S)_S) &= \gamma'\circ R_G(\mathrm{k}\,G\otimes\widehat{\pi}_L)\circ R_G(\varphi_L) \\ &= R_G((\widehat{\pi}_S)_S)\circ R_G(\varphi_L)^{-1}\circ R_G(\varphi_L) \\ &= R_G((\widehat{\pi}_S)_S). \end{aligned}$$

Since $R_G(\mathrm{k}\,G\otimes\widehat{\pi}_L)$ and $R_G((\widehat{\pi}_S)_S)$ are epimorphisms, this implies that $\gamma' = R_G(\gamma)^{-1}$. In particular, γ is bijective and hence an isomorphism.
Thus

$$\mathrm{k}\,G\otimes\widehat{\Delta}_L \cong \bigoplus_{S\in\mathcal{S}_L}\widehat{\Delta}_S \cong \bigoplus_{S\in\mathrm{Sim}(A*G)} [\mathrm{k}\,G\otimes L : S]\,\widehat{\Delta}_S\,.$$

□

Corollary 3.7. *For every simple $A*G$-module $S\in\mathrm{Sim}(A*G)$*

$${}_{A|}\,\widehat{\Delta}_S \cong \bigoplus_{L\in\mathrm{Sim}(A)} [{}_{A|}S : L]\,\widehat{\Delta}_L$$

where $[{}_{A|}S : L]$ denotes the multiplicity of the simple summand L in ${}_{A|}S$.

Proof. By Corollary 1.14, there is a simple A-module L such that $[\mathrm{k}\,G\otimes L : S]\neq 0$. By Proposition 3.6, this implies that $\widehat{\Delta}_S$ is a direct summand of $\mathrm{k}\,G\otimes\widehat{\Delta}_L$. Hence ${}_{A|}\,\widehat{\Delta}_S$ is a direct summand of

$$R_G(\mathrm{k}\,G\otimes\widehat{\Delta}_L)\cong\bigoplus_{g\in G} g\,\widehat{\Delta}_L\cong\bigoplus_{g\in G}\widehat{\Delta}_{gL}$$

where the last isomorphism follows from Lemma 3.5. In particular, ${}_{A|}\,\widehat{\Delta}_S$ is a direct sum of pseudostandard modules. Moreover, by Proposition 1.12[8],

$$\mathrm{top}({}_{A|}\,\widehat{\Delta}_S)\cong {}_{A|}\,\mathrm{top}(\widehat{\Delta}_S)\cong {}_{A|}S$$

so that, since every pseudostandard module $\widehat{\Delta}_L$ has simple top L, we obtain

$${}_{A|}\,\widehat{\Delta}_S \cong \bigoplus_{L\in\mathrm{Sim}(A)} [{}_{A|}S : L]\,\widehat{\Delta}_L\,.$$

□

Corollary 3.8. $\widehat{\Delta}^{A*G}\,|\,\mathrm{k}\,G\otimes\widehat{\Delta}^{A}$ *and* $\widehat{\Delta}^{A}\,|R_G(\widehat{\Delta}^{A*G})$.

Proof. The first claim follows directly from Proposition 3.6 and Corollary 1.14, while the second claim follows from Corollary 3.7 and Corollary 1.14. □

Example 3.9. *Let Q be the quiver*

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and $A := \mathrm{k}\,Q/J^2$, where J denotes the arrow ideal. Consider the partial order $\leq$ on $\mathrm{Sim}(A) = \{L_1, L_2\}$ given by an antichain, i.e. all distinct elements are incomparable. Then the standard modules of A are simple. Moreover, the group $G = \{e, g\}\cong\mathbb{Z}/2\mathbb{Z}$ acts on A via the k*-linear map defined via $g(e_1) = e_2$, $g(e_2) = e_1$, $g(\alpha) = \beta$ and $g(\beta) = \alpha$, and the G action preserves the partial order. Now by [23, 2.3], $A*G$ is Morita equivalent to $\mathrm{k}[x]/(x^2)$, hence $\Delta_{\mathrm{k}} = \mathrm{k}[x]/(x^2)$ is not simple. In particular, neither Proposition 3.6 nor Corollary 3.8 hold for $(\Delta_L)_L$ instead of $(\widehat{\Delta}_L)_L$.*

Using Corollary 3.8 and Proposition 1.16, we also obtain a concrete description of $(\widehat{\Delta}_S^{A*G})_S$. We denote, as before, by H_L the stabilizer of the isomorphism class of L and, using Lemma 1.15, choose an H_L-equivariant representative L of this class. Moreover, we let $P_L^{A*H_L}$ be a projective cover of L as an $A*H_L$-module. We endow $\mathrm{Sim}(A*H_L)$ with the partial order induced by the partial order on $\mathrm{Sim}(A)$ as in Definition 3.2 and let $\widehat{\Delta}_L^{A*H_L}$ be the pseudostandard module of $A*H_L$ corresponding to L with respect to this partial order. Then by Corollary 3.8, the restriction ${}_{A|}\widehat{\Delta}_L^{A*H_L}$ is isomorphic to a direct sum of standard modules with top

$$\mathrm{top}({}_{A|}\widehat{\Delta}_L^{A*H_L}) \cong {}_{A|}\mathrm{top}(\widehat{\Delta}_L^{A*H_L}) \cong {}_{A|}L,$$

so that ${}_{A|}\widehat{\Delta}_L^{A*H_L} \cong \widehat{\Delta}_L^{A}$. Thus we obtain the following corollary:

Corollary 3.10. *$\widehat{\Delta}_L^A$ has the structure of an $A*H_L$-module.*

Lemma 3.11. *The pseudostandard modules $\widehat{\Delta}_S^{A*G}$ for $A*G$ are of the form $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(\widehat{\Delta}_L\otimes V)$ where V is an irreducible representation of H_L. Morevoer, two such modules $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(\widehat{\Delta}_L\otimes V)$ and $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_{L'}}(\widehat{\Delta}_{L'}\otimes W)$ of this form are isomorphic if and only if there is $g\in G$ such that $gL\cong L'$ and $gV\cong W$.*

Proof. By Proposition 3.6 and Corollary 1.13[1]

$$\begin{aligned}\bigoplus_{S\in\mathrm{Sim}(A*G)}[\mathrm{k}\,G\otimes L:S]\,\widehat{\Delta}_S &\cong \mathrm{k}\,G\otimes\widehat{\Delta}_L\cong \mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}\mathrm{k}\,H_L\otimes\widehat{\Delta}_L\\ &\cong \mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(\widehat{\Delta}_L\otimes\mathrm{k}\,H_L)\cong\bigoplus_{V\in\mathrm{Sim}(\mathrm{k}\,H_L)}[\mathrm{k}\,H_L:V]\,\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(\widehat{\Delta}_L\otimes V).\end{aligned}$$

Moreover, $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(\widehat{\Delta}_L\otimes V)$ has simple top $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(L\otimes V)$ and is in particular indecomposable. Thus it is isomorphic to $\widehat{\Delta}_{\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(L\otimes V)}$.
Moreover, $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(\widehat{\Delta}_L\otimes V)\cong\widehat{\Delta}_{\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(L\otimes V)}$ and $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_{L'}}(\widehat{\Delta}_{L'}\otimes W)\cong\widehat{\Delta}_{\mathrm{k}\,G\otimes_{\mathrm{k}\,H_{L'}}(L'\otimes W)}$ are isomorphic if and only if $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_L}(L\otimes V)$ and $\mathrm{k}\,G\otimes_{\mathrm{k}\,H_{L'}}(L'\otimes W)$ are isomorphic, which, by Proposition 1.16, is the case if and only if there is a $g\in G$ such that $gL\cong L'$ and $gV\cong W$. □

Now we use the description of the pseudostandard modules to compare properties of $(A,\leq_A)$ to $(A*G,\leq_{A*G})$.

Lemma 3.12. *The following statements are equivalent:*

1. *$\leq_A$ is adapted and (Δ_L^A) is exceptional.*
2. *$\leq_{A*G}$ is adapted and (Δ_S^{A*G}) is exceptional.*

Proof. By Lemma 2.13 and Proposition 3.3, $\leq_A$ is adapted and $(\Delta_L^A)_L$ is exceptional if and only if

$$\mathrm{Hom}_A(\widehat{\Delta}_L,\mathrm{rad}(\widehat{\Delta}_{L'}))\neq(0)\Rightarrow L<_A L'$$

and $\leq_{A*G}$ is adapted and $(\Delta_S^{A*G})_S$ is exceptional if and only if

$$\begin{aligned}&\mathrm{Hom}_{A*G}(\widehat{\Delta}_S,\mathrm{rad}(\widehat{\Delta}_{S'}))\neq(0)\\ \Rightarrow\,& L<_A L' \text{ for some (equivalently all) } L,L' \text{ such that } S|\,\mathrm{k}\,G\otimes L \text{ and } S'|\,\mathrm{k}\,G\otimes L'\end{aligned}$$

Now if $\leq_A$ is adapted and $(\Delta_L^A)_L$ exceptional, then for all $S,S'\in\mathrm{Sim}(A*G)$ and $L,L'\in\mathrm{Sim}(A)$ such that $S|\,\mathrm{k}\,G\otimes L, S'|\,\mathrm{k}\,G\otimes L'$ we have

$$\begin{aligned}\mathrm{Hom}_{A*G}(\widehat{\Delta}_S,\mathrm{rad}(\widehat{\Delta}_S')) &\subseteq \mathrm{Hom}_{A*G}(\mathrm{k}\,G\otimes\widehat{\Delta}_L,\mathrm{rad}(\mathrm{k}\,G\otimes\widehat{\Delta}_{L'}))\\ &\subseteq \mathrm{Hom}_A(\mathrm{k}\,G\otimes\widehat{\Delta}_L,\mathrm{rad}(\mathrm{k}\,G\otimes\widehat{\Delta}_{L'}))\cong\bigoplus_{g,g'\in G}\mathrm{Hom}_A(\widehat{\Delta}_{g(L)},\mathrm{rad}(\widehat{\Delta}_{g'(L')})).\end{aligned}$$

Thus if $\mathrm{Hom}_{A*G}(\widehat{\Delta}_S, \mathrm{rad}(\widehat{\Delta}_{S'})) \neq (0)$, there exist $g, g' \in G$ such that

$$\mathrm{Hom}_A(\widehat{\Delta}_{g(L)}, \mathrm{rad}(\widehat{\Delta}_{g'(L')})) \neq (0).$$

By assumption, this implies $gL <_A g'L$, and hence, since $\leq_A$ is G-equivariant, $L <_A L'$. Thus $\mathrm{k}\, G \otimes L <_{A*G} \mathrm{k}\, G \otimes L'$, so that $S <_{A*G} S'$.
On the other hand, if $\leq_{A*G}$ is adapted and (Δ_S^{A*G}) exceptional, then

$$\begin{aligned}
&\mathrm{Hom}_A(\widehat{\Delta}_L, \mathrm{rad}(\widehat{\Delta}_{L'})) \neq (0)\\
\Rightarrow\, &\mathrm{Hom}_{A*G}(\mathrm{k}\, G \otimes \widehat{\Delta}_L, \mathrm{rad}(\mathrm{k}\, G \otimes \widehat{\Delta}_{L'})) \neq (0)\\
\Rightarrow\, &\bigoplus_{S|\,\mathrm{k}\, G\otimes L, S'|\,\mathrm{k}\, G \otimes L'} \mathrm{Hom}_{A*G}(\widehat{\Delta}_S, \mathrm{rad}(\widehat{\Delta}_{S'})) \neq (0)\\
\Rightarrow\, &\exists S|\,\mathrm{k}\, G \otimes L, S'|\,\mathrm{k}\, G \otimes L' : \mathrm{Hom}_{A*G}(\widehat{\Delta}_S, \mathrm{rad}(\widehat{\Delta}_{S'})) \neq (0)\\
\Rightarrow\, &\exists S|\,\mathrm{k}\, G \otimes L, S'|\,\mathrm{k}\, G \otimes L' : S <_{A*G} S'\\
\Leftrightarrow\, &L <_A L'.
\end{aligned}$$

$\square$

Example 3.13. *It is not true that $\leq_{A*G}$ is adapted if and only if $\leq_A$ is adapted. Consider for example $A := k[x]/(x^2)$ and $G = \{1, g\} \cong \mathbb{Z}/2\mathbb{Z}$ with $g(x) = -x$. Then A has a unique simple* k *and the unique order is clearly G-equivariant and adapted. However, $A * G$ is given by the quiver*

$$\begin{tikzcd}
L_1 \cong \mathrm{k}(1/\sqrt{2}(1+g)) \arrow[r, bend left, "\alpha = r_{-gx}"] & L_2 \cong \mathrm{k}(1/\sqrt{2}(1-g)) \arrow[l, bend left, "\beta = r_{gx}"]
\end{tikzcd}$$

with $\alpha\beta = 0 = \beta\alpha$ and $\leq_{A*G}$ being an antichain. Hence $\Delta_1' = P_1$ has socle L_2 and top L_1 and L_2 is incomparable to L_1.

Theorem 3.14. *The following statements hold:*

1. *$(A, \leq_A)$ is quasi-hereditary if and only if $(A * G, \leq_{A*G})$ is quasi-hereditary .*
2. *$(A, \leq_A)$ is strongly quasi-hereditary if and only if $(A * G, \leq_{A*G})$ is strongly quasi-hereditary.*
3. *$(A, \leq_A)$ is directed if and only if $(A * G, \leq_{A*G})$ is directed.*

Proof. 1. Suppose $(A, \leq_A)$ is quasi-hereditary . Then $\leq_A$ is adapted and (Δ_L^A) is exceptional. So by Lemma 3.12 $\leq_{A*G}$ is adapted and (Δ_S^{A*G}) is exceptional. Moreover, A has a filtration by the $\Delta_L = \widehat{\Delta}_L$. Hence $A * G \cong \mathrm{k}\, G \otimes A$ has a filtration by $\mathrm{k}\, G \otimes \widehat{\Delta}_L$. Since these decompose into a direct sum of $\widehat{\Delta}_S$, where $S|\,\mathrm{k}\, G \otimes L$, by Proposition 3.6 and $\Delta_S = \widehat{\Delta}_S$ by Lemma 2.10, this implies that $A * G$ has a filtration by standard modules $\Delta_S = \widehat{\Delta}_S$. Hence $A * G \in F(\Delta^{A*G})$.
On the other hand, suppose $A * G$ is quasi-hereditary . Then $\leq_{A*G}$ is adapted and (Δ_S^{A*G}) is exceptional, so that $\leq_A$ is adapted and (Δ_L^A) is exceptional. Moreover, $A * G$ has a filtration as an $A * G$-module by standard modules. Hence as an A-module $A * G \cong \bigoplus_{g \in G} gA \cong |G|A$ has a filtration by the $R_G(\Delta_S)$, and by Proposition 1.12[2], Lemma 2.10 and Lemma 3.5,

$$R_G(\Delta_S) = R_G(\widehat{\Delta}_S) | R_G(\mathrm{k}\, G \otimes \widehat{\Delta}_L) \cong \bigoplus_{g \in G} \widehat{\Delta}_{g(L)} = \bigoplus_{g \in G} \Delta_{g(L)}$$

for $S|\,\mathrm{k}\, G \otimes L'$, so that, since the standard modules of A are indecomposable, $R_G(\Delta_S)$ is a direct sum of standard modules and hence $|G|A \in \mathcal{F}(\Delta)$. Since by Proposition 2.15, $\mathcal{F}(\Delta)$ is closed under direct summands, $A \in \mathcal{F}(\Delta)$.

2. By 1., $(A, \leq_A)$ is quasi-hereditary if and only if $(A*G, \leq_{A*G})$ is quasi-hereditary . Moreover, if the projective dimension of Δ^A is less than or equal to one, so is the projective dimension of $\Delta^{A*G} = \widehat{\Delta}^{A*G} | \mathrm{k}\, G \otimes \widehat{\Delta}^A = \mathrm{k}\, G \otimes \Delta^A$, and if the projective dimension of Δ^{A*G} is less than or equal to one, so is the projective dimension of $\Delta^A = \widehat{\Delta}^A | R_G(\widehat{\Delta}^{A*G}) = R_G(\Delta^{A*G})$.

3. By 1., $(A, \leq_A)$ is quasi-hereditary if and only if $(A*G, \leq_{A*G})$ is quasi-hereditary . Moreover, if Δ^A is semisimple, so is $\Delta^{A*G} = \widehat{\Delta}^{A*G} | \mathrm{k}\, G \otimes \widehat{\Delta}^A = \mathrm{k}\, G \otimes \Delta^A$, and if Δ_{A*G} is semisimple, so is $\Delta^A = \widehat{\Delta}^A | R_G(\widehat{\Delta}^{A*G}) = R_G(\Delta^{A*G})$.

□

Remark 3.15. *Example 3.13 shows that it is not in general true that if $(A, \leq_A)$ is standardly stratified so is $(A * G, \leq_{A*G})$.*

The following lemma can essentially be found in [19, Lemma 8]. However, we have included a proof, since the assumptions there slightly vary from our assumptions.

Lemma 3.16. *Let M be an $A*G$-module. Then G acts on $\mathrm{End}_A(R_G M)$ via algebra automorphism as in Remark 1.4, and we have an algebra isomorphism*

$$\begin{aligned} \theta_M : \mathrm{End}_A(R_G M) * G &\to \mathrm{End}_{A*G}(\mathrm{k}\, G \otimes M), \\ f \otimes g &\mapsto (h \otimes m \mapsto hg^{-1} \otimes f(gm)) \end{aligned}$$

Proof. It is easy to see that the action given by Remark 1.4 is an action of algebra automorphisms. Let $f \otimes g \in \mathrm{End}_A(R_G M) * G$. Then for any $h, h' \in G$, $a \in A$, $m \in M$ we have

$$\begin{aligned} \theta_M(f \otimes g)(a \otimes h')(h \otimes m)) &= \theta_M(f \otimes g)(h'h \otimes (hh')^{-1}(a)m) \\ &= h'hg^{-1} \otimes f(g(hh')^{-1}(a)m)) \\ &= h'hg^{-1} \otimes ((hh'g^{-1})^{-1}(a)f(gm) \\ &= (a \otimes h')(hg^{-1} \otimes f(gm)) \\ &= (a \otimes h')\theta_M(f \otimes g)(h \otimes m). \end{aligned}$$

Thus $\theta_M(f \otimes g)$ is $A*G$-linear, so that θ_M is well-defined. Moreover, for any $f, f' \in \mathrm{End}_A(R_G M)$, $g, g', h \in G$, $m \in M$ we have

$$\begin{aligned} \theta_M(f \otimes g) \circ \theta_M(f' \otimes g')(h \otimes m) &= \theta_M(f \otimes g)(h(g')^{-1} \otimes f'(g'm))) \\ &= h(g')^{-1}g^{-1} \otimes f(gf'(g'm)) \\ &= h(gg')^{-1} \otimes f \circ (g \cdot f')(gg'm) \\ &= \theta_M(f \circ (g \cdot f') \otimes gg')(h \otimes m) \\ &= \theta_M((f \otimes g)(f' \otimes g'))(h \otimes m) \end{aligned}$$

Thus θ_M is an algebra homomorphism. Let $t \in \mathrm{End}_{A*G}((\mathrm{k}\, G \otimes M))$. Then we can identify $R_G t \in \mathrm{End}_A(\bigoplus_{g \in G} gM)$ with a matrix $R_G t = (t_{g,h})_{g,h \in G}$ where

$$t_{g,h} : gM \to hM$$

and since t is $A * G$-linear we have

$$t_{g,h} = t_{g'g, g'h}$$

for all $g, g', h \in G$. For $g \in G$ let

$$f_g(t) : M \to M, f_g(t) := t_{g,e} \circ (\mathrm{tr}_g^M)^{-1}$$

and set

$$\tau_M(t) := \sum_{g\in G} f_g \otimes g.$$

Then for all $m \in M, h \in G$ we have

$$\theta_M(\tau_M(t))(h\otimes m) = \sum_{g\in G} \theta_M(f_g(t)\otimes g)(h\otimes m)$$
$$= \sum_{g\in G} hg^{-1} \otimes f_g(t)(gm) = \sum_{g\in G} hg^{-1} = \sum_{g\in G} hg^{-1} \otimes t_{g,e} \circ (\mathrm{tr}_g^M)^{-1}(gm)$$
$$= \sum_{g\in G} hg^{-1} \otimes t_{g,e}(m) = \sum_{g\in G} hg^{-1} \otimes t_{h,hg^{-1}}(m)$$
$$= \sum_{g\in G} g \otimes t_{h,g}(m) = t(h\otimes m).$$

On the other hand, for $f \otimes g \in \mathrm{End}_A(R_G M * G$ we have

$$f_h(\theta_M(f\otimes g)) = 0$$

unless $h = g$ and

$$f_g(\theta_M(f\otimes g)) = f$$

Thus $\tau_M(\theta_M(f\otimes g)) = f\otimes g$, so that $\tau_M = \theta_M^{-1}$, and θ_M is an isomorphism. $\square$

Theorem 3.17. *Let $(A, \leq_A)$ be quasi-hereditary and let B be a subalgebra of A such that $g(B) = B$ for all $g \in G$. Then there is a partial order $\leq_B$ on $\mathrm{Sim}(B)$ such that $(B, \leq_B)$ is an exact Borel subalgebra of $(A, \leq_A)$ if and only if there is a partial order $\leq_{B*G}$ on $\mathrm{Sim}(B * G)$ such that $(B * G, \leq_{B*G})$ is an exact Borel subalgebra of $(A * G, \leq_{A*G})$.*

Proof. First of all, note that $B * G$ is a well-defined subalgebra of $A * G$, since $g(B) \subseteq B$ for all $g \in G$, and that $A * G$ is quasi-hereditary by Theorem 3.14.
Let $(L_i^B)_{1\leq i\leq n}$ be a set of representatives of the isomorphism classes of simple B-modules and let $(L_j^{B*G})_{1\leq i\leq m}$ be a set of representatives of the isomorphism classes of simple $B * G$-modules.
Suppose first that $(B, \leq_B)$ is an exact Borel subalgebra of $(A, \leq_A)$.
Recall that R_G both preserves and reflects short exact sequences, and note that we have a natural isomorphism

$$(A \otimes_B -) \circ R_G \to R_G \circ ((A * G) \otimes_{B*G} -) \tag{1}$$

given by

$$A \otimes_B R_G M \to R_G((A * G) \otimes_{B*G} M), a \otimes m \mapsto (a \otimes 1_G) \otimes m$$

with inverse given by

$$R_G((A * G) \otimes_{B*G} M) \to A \otimes_B R_G M, (a \otimes g) \otimes m \mapsto a \otimes gm.$$

Thus if $A \otimes_B -$ is exact, so is $(A * G) \otimes_{B*G} -$.
By assumption we have a bijection

$$\mathrm{Sim}(B) \to \mathrm{Sim}(A), L_i^B \mapsto L_i^A := \mathrm{top}(A \otimes_B L_i^B)$$

such that $L_i^B \leq_B L_j^B$ if and only if $L_i^A \leq_A L_j^A$. In particular, since

$$g(\mathrm{top}(A \otimes_B L_i^B)) = \mathrm{top}(g(A \otimes_B L_B)) \cong \mathrm{top}(A \otimes_B gL_B))$$

and $\leq_A$ is G-invariant, so is $\leq_B$. Thus it induces a partial order $\leq_{B*G}$ on $\mathrm{Sim}(B*G)$ according to Proposition 3.3.
Next we want to show that there is a bijection between isomorphism classes of simple modules given by

$$\mathrm{Sim}(B*G) \to \mathrm{Sim}(A*G), L_j^{B*G} \mapsto L_j^{A*G} := \mathrm{top}((A*G) \otimes_{B*G} L_j^{B*G}).$$

Note that for this, it suffices to show that for any semisimple $B*G$-module S, the induced map

$$s_2 : \mathrm{End}_{B*G}(S) \to \mathrm{End}_{A*G}(\mathrm{top}((A*G) \otimes_{B*G} S)), f \mapsto \mathrm{top}(\mathrm{id}_{A*G} \otimes f)$$

is a bijection. Recall from Proposition 1.5 that $\mathrm{End}_{B*G}(S) = \mathrm{End}_B(R_G S)^G$ and $\mathrm{End}_{A*G}(\mathrm{top}((A* G) \otimes_{B*G} S)) = \mathrm{End}_A(R_G \,\mathrm{top}((A*G) \otimes_{B*G} S))^G$ where the G-action is given as in Remark 1.6. Moreover, note that by Proposition 1.12[8] and 1, we have a natural isomorphism

$$\begin{aligned} \alpha : R_G \circ \mathrm{top} \circ ((A*G) \otimes_{B*G} -) &\to \mathrm{top} \circ (A \otimes_B -) \circ R_G, \\ \alpha_M : R_G \,\mathrm{top}((A*G) \otimes_{B*G} M) &\to \mathrm{top}(A \otimes_B R_G M), \\ (a \otimes g) \otimes m + \mathrm{rad}(A*G) \otimes M &\mapsto a \otimes gm + \mathrm{rad}(A) \otimes M \end{aligned}$$

with inverse given by

$$\alpha_M^{-1}(a \otimes m + \mathrm{rad}(A) \otimes M) = (a \otimes 1_G) \otimes m + \mathrm{rad}(A*G) \otimes M.$$

Transporting the $A*G$-module structure of $R_G \,\mathrm{top}((A*G) \otimes_{B*G} S)$ along α_S gives rise to an $A*G$-module structure on $\mathrm{top}(A \otimes_B R_G S)$, which is given by

$$g(a \otimes s + \mathrm{rad}(A) \otimes S) := g(a) \otimes gs + \mathrm{rad}(A) \otimes S.$$

According to Remark 1.4 this induces a G-action on $\mathrm{End}_A(\mathrm{top}(A \otimes_B R_G S))$. With this G-action, the homomorphism

$$\begin{aligned} s_1 : \mathrm{End}_B(R_G S) &\to \mathrm{End}_A(\mathrm{top}(A \otimes_B R_G S)), \\ f &\mapsto \mathrm{top}(\mathrm{id}_A \otimes f) \end{aligned}$$

becomes G-equivariant, since we have

$$\begin{aligned} \mathrm{top}(\mathrm{id}_A \otimes (g \cdot f))(a \otimes s + \mathrm{rad}(A) \otimes S) &= a \otimes (g \cdot f)(s) + \mathrm{rad}(A) \otimes S \\ = a \otimes g(f(g^{-1}s)) + \mathrm{rad}(A) \otimes S &= g(\mathrm{id}_A(g^{-1}(a))) \otimes g(f(g^{-1}s)) + \mathrm{rad}(A) \otimes S \\ = g(\mathrm{top}(\mathrm{id}_A \otimes f)(g^{-1}(s \otimes a + \mathrm{rad}(A) \otimes S))) &= g \cdot (\mathrm{top}(\mathrm{id}_A \otimes f). \end{aligned}$$

Additionally, since B is an exact Borel subalgebra of A, s_1 is an isomorphism. Hence it restricts to an isomorphism

$$\begin{aligned} s_1^G : \mathrm{End}_B(R_G S)^G &\to \mathrm{End}_A(\mathrm{top}(A \otimes_B R_G S))^G, \\ f &\mapsto \mathrm{top}(\mathrm{id}_A \otimes f). \end{aligned}$$

Moreover, for any $f \in \mathrm{End}_B(R_G S)^G$ we have

$$\begin{aligned} \alpha_S^{-1} \circ \mathrm{top}(\mathrm{id}_A \otimes f) \circ \alpha_S((a \otimes g) \otimes s + \mathrm{rad}(A*G) \otimes S) &= \alpha_S^{-1} \circ \mathrm{top}(\mathrm{id}_A \otimes f)(a \otimes g(s) + \mathrm{rad}(A) \otimes S) \\ = \alpha_S^{-1}(a \otimes f(g(s)) + \mathrm{rad}(A) \otimes S) &= \alpha_S^{-1}(a \otimes g(f(s)) + \mathrm{rad}(A) \otimes S) \\ = (a \otimes 1) \otimes g(f(s)) + \mathrm{rad}(A*G) \otimes S &= (a \otimes g) \otimes f(s) + \mathrm{rad}(A*G) \otimes S = \mathrm{top}(\mathrm{id}_{A*G} \otimes f). \end{aligned}$$

Hence conjugation of s_1^G by α_S gives the desired isomorphism s_2. Thus we have a bijection

$$\begin{aligned} \mathrm{Sim}(B*G) &\to \mathrm{Sim}(A*G) \\ L_j^{B*G} \mapsto L_j^{A*G} &= \mathrm{top}((A*G) \otimes_{B*G} L_j^{B*G}). \end{aligned}$$

Moreover, for every $L_i^{B*G}, L_j^{B*G} \in \mathrm{Sim}(B*G)$ we have, using Definition 3.2, Proposition 3.3 and the fact that B is an exact Borel subalgebra of A, that

$$\begin{aligned}L_i^{B*G} <_{B*G} L_j^{B*G} \Leftrightarrow R_G L_i^{B*G} <_B R_G L_j^{B*G} \Leftrightarrow \mathrm{top}(A \otimes_B R_G L_i^{B*G}) <_A \mathrm{top}(A \otimes_B R_G L_j^{B*G}) \\ \Leftrightarrow R_G(\mathrm{top}((A*G) \otimes_{B*G} L_i^{B*G})) <_A R_G(\mathrm{top}((A*G) \otimes_{B*G} L_j^{B*G})) \\ \Leftrightarrow \mathrm{top}((A*G) \otimes_{B*G} L_i^{B*G}) <_{A*G} \mathrm{top}((A*G) \otimes_{B*G} L_j^{B*G}) \Leftrightarrow L_i^{A*G} < L_j^{A*G}.\end{aligned}$$

Finally, by Proposition 3.6,

$$\begin{aligned}A*G \otimes_{B*G} (\mathrm{k}\, G \otimes L^B) &\cong \mathrm{k}\, G \otimes (A \otimes_B L^B) \\ &\cong \mathrm{k}\, G \otimes \bigoplus_{L_i^A \in \mathrm{Sim}(A)} \Delta_{L_i^A} \\ &\cong \bigoplus_{L_i^A \in \mathrm{Sim}(A)} \bigoplus_{L_j^{A*G} \in \mathrm{Sim}(A*G)} [\mathrm{k}\, G \otimes L_i^A : L_j^{A*G}] \Delta_{L_j^{A*G}}.\end{aligned}$$

Since for every simple $B*G$-module L_j^{B*G} we have that $A*G \otimes_{B*G} L_j^{B*G}$ is a summand of $A*G \otimes_{B*G} \mathrm{k}\, G \otimes L^B$, the module $A*G \otimes_{B*G} L_j^{B*G}$ is isomorphic to a direct sum of standard modules. Since it has indecomposable top L_j^{A*G}, this implies that $A*G \otimes_{B*G} L_j^{B*G} \cong \Delta_{L_j^A *G}$. Thus $(B*G, \leq_{B*G})$ is an exact Borel subalgebra of $(A*G, \leq_{A*G})$.

On the other hand, suppose that $(B*G, \leq_{B*G})$ is an exact Borel subalgebra of $(A*G, \leq_{A*G})$. Recall that I_G both preserves and reflects short exact sequences, and note that we have a natural isomorphism

$$((A*G) \otimes_{B*G} -) \circ I_G \to I_G \circ (A \otimes_B -). \tag{2}$$

given by

$$(A*G) \otimes_{B*G} \mathrm{k}\, G \otimes M \to \mathrm{k}\, G \otimes A \otimes_B M, (a \otimes g) \otimes h \otimes m \mapsto gh \otimes gh(a) \otimes m$$

with inverse given by

$$\mathrm{k}\, G \otimes A \otimes_B M \to (A*G) \otimes_{B*G} \mathrm{k}\, G \otimes M, g \otimes a \otimes m \mapsto (g^{-1}(a) \otimes 1) \otimes g \otimes m.$$

Thus if $(A*G) \otimes_{B*G} -$ is exact so is $A \otimes_B -$.
By assumption we have a bijection

$$\mathrm{Sim}(B*G) \to \mathrm{Sim}(A*G), L_j^{B*G} \mapsto L_j^{A*G} := \mathrm{top}((A*G) \otimes_{B*G} L_j^{B*G})$$

such that $L_i^{B*G} \leq_{B*G} L_j^{B*G}$ if and only if $L_i^{A*G} \leq_{A*G} L_j^{A*G}$. In particular, since

$$(A*G) \otimes_{B*G} (L_j^{B*G} \otimes V) \cong ((A*G) \otimes_{B*G} L_j^{B*G}) \otimes V)$$

for every irreducible representation V of G, and since $\leq_{A*G}$ is G-stable, so is $\leq_{B*G}$. Thus it induces a partial order $\leq_B$ on $\mathrm{Sim}(B)$ according to Proposition 3.3.
Next we want to show that there is a bijection between isomoprhism classes of simple modules given by

$$\mathrm{Sim}(B) \to \mathrm{Sim}(A), L_i^B \mapsto L_i^A := \mathrm{top}(A \otimes_B L_i^B).$$

Note that for this, it suffices to show that for any semisimple B-module S, the induced map

$$s_1 : \mathrm{End}_B(S) \to \mathrm{End}_A(\mathrm{top}(A \otimes_B S)), f \mapsto \mathrm{top}(\mathrm{id}_A \otimes f) \tag{3}$$

is a bijection, and, in fact, this holds as soon as it holds for some semisimple B-module S such that $[S : L_i^B] \neq 0$ for all $L_i^B \in \mathrm{Sim}(B)$. By Corollary 1.14, it thus suffices to consider the case $S = R_G L^{B*G}$, where $L^{B*G} = \bigoplus_{L_j^{B*G} \in \mathrm{Sim}(B*G)} L_j^{B*G}$. By Lemma 3.16, we have an isomorphism

$$\theta_{L^{B*G}} : \mathrm{End}_B(R_G L^{B*G}) * G \to \mathrm{End}_{B*G}(I_G R_G L^{B*G}), f \otimes g \mapsto (h \otimes x \mapsto hg^{-1} \otimes f(gx))$$

as well as an isomorphism

$$\begin{aligned}\theta_{(A*G)\otimes_{B*G}L^{B*G}} : \mathrm{End}_B(R_G(A*G) \otimes_{B*G} L^{B*G}) * G \to \mathrm{End}_{A*G}(I_G R_G (A*G) \otimes_{B*G} L^{B*G}),\\ f \otimes g \mapsto (h \otimes (a \otimes g) \otimes m \mapsto hg^{-1} \otimes f(g \cdot ((a \otimes g) \otimes m)))\end{aligned}$$

Moreover, since $B*G$ is a regular exact Borel subalgebra of $A*G$, we have an isomorphism

$$s_2 : \mathrm{End}_{B*G}(I_G R_G L^{B*G}) \to \mathrm{End}_{A*G}(\mathrm{top}((A*G) \otimes_{B*G} I_G R_G L^{B*G})) f \mapsto \mathrm{top}(\mathrm{id}_{A*G} \otimes f)$$

Additionally, by Proposition 1.12[8] and 2, there are natural isomorphisms

$$\begin{aligned}\phi_1 : \mathrm{top} \circ (A \otimes_B -) \circ R_G \to R_G \circ \mathrm{top} \circ ((A*G) \otimes_{B*G} -),\\ \phi_1^M : \mathrm{top}(A \otimes_B R_G M) \to R_G(\mathrm{top}((A*G) \otimes_{B*G} M)),\\ a \otimes m + \mathrm{rad}(A) \otimes R_G M \mapsto (a \otimes 1_G) \otimes m + \mathrm{rad}(A*G) \otimes M\end{aligned}$$

and

$$\begin{aligned}\phi_2 : I_G \circ R_G \circ \mathrm{top} \circ ((A*G) \otimes_{B*G} -) \to \mathrm{top} \circ ((A*G) \otimes_{B*G} -) \circ I_G \circ R_G\\ \phi_2^M : I_G R_G \,\mathrm{top}(((A*G) \otimes_{B*G} M)) \to \mathrm{top}(A*G) \otimes_{B*G} I_G R_G M,\\ h \otimes ((a \otimes g) \otimes m + I_G R_G \,\mathrm{rad}(A*G) \otimes M) \mapsto (h(a) \otimes h) \otimes (1_G \otimes g(m)) + \mathrm{rad}(A*G) \otimes I_G R_G M,\end{aligned}$$

which give rise to isomorphisms

$$\varphi_1 : \mathrm{End}_A(\mathrm{top}(A \otimes_B R_G L^{B*G})) \to \mathrm{End}_A(R_G(\mathrm{top}((A*G) \otimes_{B*G} L^{B*G}))).$$

and

$$\varphi_2 : \mathrm{End}_{A*G}(\mathrm{top}((A*G) \otimes_{B*G} I_G R_G L^{B*G})) \to \mathrm{End}_{A*G}(I_G R_G \,\mathrm{top}((A*G) \otimes_{B*G} L^{B*G})).$$

Now consider the diagram

$$\begin{tikzcd}
\mathrm{End}_B(R_G L^{B*G}) * G \arrow[r, "\theta_{L^{B*G}}"] \arrow[d, "s_1 \otimes \mathrm{id}_{\Bbbk G}"'] & \mathrm{End}_{B*G}(I_G R_G L^{B*G}) \arrow[d, "s_2"] \\
\mathrm{End}_A(\mathrm{top}(A \otimes_B R_G L^{B*G})) * G \arrow[d, "\varphi_2 \otimes \mathrm{id}_{\Bbbk G}"'] & \mathrm{End}_{A*G}(\mathrm{top}((A*G) \otimes_{B*G} I_G R_G L^{B*G})) \arrow[d, "\varphi_1"] \\
\mathrm{End}_A(R_G(\mathrm{top}((A*G) \otimes_{B*G} L^{B*G})) * G \arrow[r, "\theta_{\mathrm{top}((A*G)\otimes_{B*G}L^{B*G})}"'] & \mathrm{End}_{A*G}(I_G R_G \,\mathrm{top}((A*G) \otimes_{B*G} L^{B*G}))
\end{tikzcd}$$

For $f \in \mathrm{End}_B(R_G L^{B*G})$, $g, h \in G$, $a \in A$ and $x \in L^{B*G}$ we have

$$\begin{aligned}
&\varphi_2 \circ s_2 \circ \theta_{R_G L^{B*G}}(f \otimes g)(1_G \otimes ((1_A \otimes 1_G) \otimes x + \mathrm{rad}(A * G) \otimes L^{B*G}))\\
&= (\phi_2^{L^{B*G}})^{-1} \circ (s_2 \circ \theta_{R_G L^{B*G}})(f \otimes g) \circ \phi_2^{L^{B*G}}(1_G \otimes ((1_A \otimes 1_G) \otimes x + \mathrm{rad}(A * G) \otimes L^{B*G}))\\
&= (\phi_2^{L^{B*G}})^{-1} \circ (s_2 \circ \theta_{R_G L^{B*G}})(f \otimes g)((1_A \otimes 1_G) \otimes 1_G \otimes x + \mathrm{rad}(A * G) \otimes I_G R_G L^{B*G})\\
&= (\phi_2^{L^{B*G}})^{-1} \circ (\mathrm{top}(\mathrm{id}_{A*G} \otimes \theta_{R_G L^{B*G}}(f \otimes g)))((1_A \otimes 1_G) \otimes 1_G \otimes x + \mathrm{rad}(A * G) \otimes I_G R_G L^{B*G}\\
&= (\phi_2^{L^{B*G}})^{-1}((1_A \otimes 1_G) \otimes \theta_{R_G L^{B*G}}(f \otimes g)(1_G \otimes x) + \mathrm{rad}(A * G) \otimes I_G R_G L^{B*G})\\
&= (\phi_2^{L^{B*G}})^{-1}((1_A \otimes 1_G) \otimes g^{-1} \otimes f(gx)) + \mathrm{rad}(A * G) \otimes I_G R_G L^{B*G})\\
&= (\phi_2^{L^{B*G}})^{-1}((g^{-1}(1_A) \otimes g^{-1}) \otimes 1_G \otimes f(gx) + \mathrm{rad}(A * G) \otimes I_G R_G L^{B*G})\\
&= g^{-1} \otimes ((1_A \otimes 1_G) \otimes f(g(x) + \mathrm{rad}(A * G) \otimes L^{B*G}))
\end{aligned}$$

and

$$\begin{aligned}
&\theta_{\mathrm{top}((A*G)\otimes_{B*G} L^{B*G})} \circ (\varphi_1 \otimes \mathrm{id}_{\Bbbk G}) \circ (s_1 \otimes \mathrm{id}_{\Bbbk G})(f \otimes g)(1_G \otimes (1_A \otimes 1_G) \otimes x + I_G R_G \,\mathrm{rad}(A * G) \otimes L^{B*G})\\
&= \theta_{\mathrm{top}((A*G)\otimes_{B*G} L^{B*G})}((\varphi_1 \circ s_1)(f) \otimes g)(1_G \otimes ((1_A \otimes 1_G) \otimes x + I_G R_G \,\mathrm{rad}(A * G) \otimes L^{B*G}))\\
&= g^{-1} \otimes ((\varphi_1 \circ s_1)(f))(g((1_A \otimes 1_G) \otimes x + \mathrm{rad}(A * G) \otimes L^{B*G}))\\
&= g^{-1} \otimes \phi_1^{L^{B*G}} \circ s_1(f) \circ (\phi_1^{L^{B*G}})^{-1}(g((1_A \otimes 1_G) \otimes x + \mathrm{rad}(A * G) \otimes L^{B*G}))\\
&= g^{-1} \otimes \phi_1^{L^{B*G}} \circ s_1(f)(1_A \otimes gx + \mathrm{rad}(A) \otimes R_G L^{B*G})\\
&= g^{-1} \otimes \phi_1^{L^{B*G}} \circ \mathrm{top}(\mathrm{id}_A \otimes f)(1_A \otimes gx + \mathrm{rad}(A) \otimes R_G L^{B*G})\\
&= g^{-1} \otimes \phi_1^{L^{B*G}}(1_A \otimes f(gx) + \mathrm{rad}(A) \otimes R_G L^{B*G})\\
&= g^{-1} \otimes ((1_A \otimes 1_G) \otimes f(gx) + \mathrm{rad}(A * G) \otimes L^{B*G}).
\end{aligned}$$

As all maps are $A*G$-linear, and elements of the form $1_G \otimes ((1_A \otimes 1_G) \otimes x + I_G R_G \,\mathrm{rad}(A*G) \otimes L^{B*G})$ generate $I_G R_G \,\mathrm{top}((A*G)\otimes_{B*G} L^{B*G})$ as an $A*G$-module, this proves that the diagram commutes. Since we know all maps except $s_1 \otimes \mathrm{id}_{\Bbbk G}$ to be isomorphisms, we can conclude that $s_1 \otimes \mathrm{id}_{\Bbbk G}$, and thus s_1 is an isomorphism. This shows that we have a bijection between the isomorphism classes of simple modules given by

$$\mathrm{Sim}(B) \to \mathrm{Sim}(A), L_i^B \mapsto L_i^A := \mathrm{top}(A \otimes_B L_i^B).$$

Moreover, using Definition 3.2, Proposition 3.3 and the fact that $B*G$ is an exact Borel subalgebra of $A * G$, we have that

$$\begin{aligned}
L_i^B <_B L_j^B &\Leftrightarrow I_G L_i^B <_{B*G} I_G L_j^B\\
&\Leftrightarrow \mathrm{top}((A * G) \otimes_{B*G} I_G L_i^B) <_{A*G} \mathrm{top}((A * G) \otimes_{B*G} L_j^B)\\
&\Leftrightarrow I_G(\mathrm{top}(A \otimes_B L_i^B)) <_{A*G} I_G(\mathrm{top}(A \otimes_B L_j^B))\\
&\Leftrightarrow \mathrm{top}(A \otimes_B L_i^B) <_A \mathrm{top}(A \otimes_B L_j^B) \Leftrightarrow L_i^A <_A L_j^A.
\end{aligned}$$

Finally, by Corollary 3.7,

$$A \otimes_B R_G L^{B*G} \cong R_G((A * G) \otimes_{B*G} L^{B*G}) \cong R_G(\Delta^{A*G}) \cong \bigoplus_{i=1}^{n} [L^{A*G} : L_i^A] \Delta^A_{L_i^A}.$$

Since for every simple B-module L_i^B we have that $A \otimes_B L_i^B$ is a summand of $A \otimes_B R_G L^{B*G}$, the module $A \otimes_B L_i^B$ is isomorphic to a direct sum of standard modules. Since it has indecomposable top L_i^A, this implies that $A \otimes_B L_i^B \cong \Delta_{L_i^A}$.
Thus $(B, \leq_B)$ is an exact Borel subalgebra of $(A, \leq_A)$. □

Proposition 3.18. *Let $(A, \leq_A)$ be quasi-hereditary and let B be a subalgebra of A such that $g(B) = B$ for all $g \in G$. Let $(B * G, \leq_{B*G})$ be the corresponding exact Borel subalgebra of $(A * G, \leq_{A*G})$ Then the following statements hold:*

1. *B is a strong exact Borel subalgebra if and only if $B * G$ is a strong exact Borel subalgebra.*
2. *B is a normal exact Borel subalgebra if and only if $B * G$ is a normal exact Borel subalgebra.*
3. *B is a homological exact Borel subalgebra if and only if $B * G$ is a homological exact Borel subalgebra.*
4. *B is a regular exact Borel subalgebra if and only if $B * G$ is a regular exact Borel subalgebra.*

Proof. 1. Suppose B is a strong exact Borel subalgebra. Then by Lemma 2.19, $A\,\mathrm{rad}(B) \subseteq \mathrm{rad}(A)$. Hence $A * G\,\mathrm{rad}(B * G) \subseteq \mathrm{rad}(A * G)$ by Lemma 1.11, so that again by Lemma 2.19 $B * G$ is a strong Borel subalgebra of $A * G$. On the other hand, suppose that $B * G$ is a strong Borel subalgebra of $A * G$. Then $A * G\,\mathrm{rad}(B * G) \subseteq \mathrm{rad}(A * G)$, so that again by Lemma 1.11

$$A\,\mathrm{rad}(B) \subseteq (A * G\,\mathrm{rad}(B * G)) \cap A \subseteq \mathrm{rad}(A * G) \cap A = \mathrm{rad}(A).$$

2. Suppose that B is normal. Then the inclusion $\iota : B \to A$ has a splitting $\pi : A \to B$ as right B-modules whose kernel is a right ideal of A. Since tensoring over k is exact, the inclusion $\iota \otimes \mathrm{id}_{\mathrm{k}\,G}$ has the splitting $\pi \otimes \mathrm{id}_{\mathrm{k}\,G}$ which is a right $B * G$-module homomorphism whose kernel is a right ideal of $A * G$.
On the other hand, suppose that $B * G$ is normal. Then $\iota \otimes \mathrm{id}_{\mathrm{k}\,G}$ has a splitting $\pi' : A * G \to B * G$ of right $B * G$-modules whose kernel is a right ideal in $A * G$. Since the fixed point functor $-^G$ for the G-action given by left multiplication is exact by [19, Lemma 3], $(\pi')^G$ is a splitting of the embedding $(\iota \otimes \mathrm{id}_{\mathrm{k}\,G})^G$ as right $(B * G)^G$-modules such that its kernel is a right ideal in $(A * G)^G$. Now since the upwards arrows in the commutative diagram

$$\begin{tikzcd} (B * G)^G \arrow[r, "(\iota \otimes \mathrm{id}_{\mathrm{k}\,G})^G"] & (A * G)^G \\ B \arrow[u, "b \mapsto \frac{1}{|G|}\sum_{g \in G} g(b) \otimes g"] \arrow[r, "\iota"] & A \arrow[u, "a \mapsto \frac{1}{|G|}\sum_{g \in G} g(a) \otimes g"'] \end{tikzcd}$$

are isomorphisms of algebras, $(\pi')^G$ induces a splitting of ι as right B-modules such that its kernel is a right ideal in A.

3.+4. Assume that B is homological, resp. regular. In the latter case, we have already seen that $B * G$ is normal. Let P^B be a projective resolution of L^B. Then $A \otimes_B P^B$ is a projective resolution of Δ^A, $\mathrm{k}\,G \otimes P^B$ is a projective resolution of $\mathrm{k}\,G \otimes L^B$, $\mathrm{k}\,G \otimes (A \otimes_B P^B)$ is a projective resolution of $\mathrm{k}\,G \otimes (A \otimes_B L^B)$, and similarly for the restriction of the induction. Now G acts on $\mathrm{End}_B(R_G(\mathrm{k}\,G \otimes P^B))$ and on $\mathrm{End}_A(R_G(\mathrm{k}\,G \otimes (A \otimes_B P^B)))$, and since $R_G(\mathrm{k}\,G \otimes L^B)$ is semisimple, the map

$$\mathrm{End}_B(R_G(\mathrm{k}\,G \otimes P^B)) \to \mathrm{End}_A(A \otimes_B R_G(\mathrm{k}\,G \otimes P^B)), f \mapsto \mathrm{id}_A \otimes f$$

is an epimorphism in homology of degree one and an isomorphism in homology of degree strictly greater than one, resp. an isomorphism in homology of degree strictly greater than zero.
As in Theorem 3.17 conjugating with the isomorphism 1

$$A \otimes_B R_G(\mathrm{k}\,G \otimes P^B) \cong R_G((A * G) \otimes_{B*G} (\mathrm{k}\,G \otimes P^B))$$

yields an isomorphism

$$\mathrm{End}_B(R_G(\mathrm{k}\, G \otimes P^B)) \to \mathrm{End}_A(R_G(A * G) \otimes_{B*G} (\mathrm{k}\, G \otimes P^B)), f \mapsto \mathrm{id}_{A*G} \otimes f,$$

which is, as before, G-equivariant, so that it induces a homomorphism

$$\begin{aligned}&\mathrm{End}_{B*G}(\mathrm{k}\, G \otimes P^B) = \mathrm{End}_B(R_G(\mathrm{k}\, G \otimes P^B))^G \\ \to &\mathrm{End}_{A*G}((A * G) \otimes_{B*G} (\mathrm{k}\, G \otimes P^B)) = \mathrm{End}_A(R_G(A * G) \otimes_{B*G} (\mathrm{k}\, G \otimes P^B))^G, \\ &f \mapsto \mathrm{id}_{A*G} \otimes f.\end{aligned}$$

Since the fixed point functor $-^G$ is exact, this is an epimorphism in homology of degree one and an isomorphism in homology of degree strictly greater than one, resp. an isomorphism in homology of degree strictly greater than zero.
Thus we obtain an epimorphism in degree one and an isomorphism in degree strictly greater than one, resp. an isomorphism in degree strictly greater than zero

$$\begin{aligned}\mathrm{Ext}^*_{B*G}(\mathrm{k}\, G \otimes L^B, \mathrm{k}\, G \otimes L^B) \to \mathrm{Ext}^*_{A*G}((A * G) \otimes_{B*G} L^B, (A * G) \otimes_{B*G} L^B),\\ [f] \mapsto [\mathrm{id}_{A*G} \otimes f].\end{aligned}$$

The result now follows from Corollary 1.14.
On the other hand, suppose $B * G$ is homological resp. regular. In the latter case, we have already seen that B is normal. Let P^{B*G} be a projective resolution of L^{B*G}. Then $(A * G) \otimes_{B*G} P^{B*G}$ is a projective resolution of Δ_{A*G}, $R_G P^{B*G}$ is a projective resolution of $R_G(L^{B*G})$, and $R_G((A * G) \otimes_{B*G} P^{B*G}) \cong A \otimes_B R_G(P^{B*G})$ is a projective resolution of $R_G(A * G) \otimes_{B*G} L^{B*G}) \cong A \otimes_B R_G(L^{B*G})$ and similarly for the induction of the restriction. Since $L^{B*G} \otimes \mathrm{k}\, G$ is semisimple, we have that by assumption

$$\mathrm{Ext}^*_{B*G}(L^{B*G} \otimes \mathrm{k}\, G, L^{B*G} \otimes \mathrm{k}\, G) \to \mathrm{Ext}^*_{A*G}(\Delta^{A*G} \otimes \mathrm{k}\, G, \Delta^{A*G} \otimes \mathrm{k}\, G)$$

is an epimorphism in degree one and an isomorphism in degree strictly greater than one, resp. an isomorphism in degree strictly greater than zero.
Moreover, G acts on $\mathrm{End}_B(R_G(P^{B*G}))$ and on $\mathrm{End}_A(R_G((A * G) \otimes_{B*G} P^{B*G}))$ via $g \cdot f = gf(g^{-1}-)$ and, arguing as in Theorem 3.17, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_B(R_G P^{B*G}) * G \arrow[r, "\theta_{P^{B*G}}"] \arrow[d, "s_1 \otimes \mathrm{id}_{\mathrm{k}\, G}"'] & \mathrm{End}_{B*G}(I_G R_G P^{B*G}) \arrow[d, "s_2"] \\
\mathrm{End}_A(A \otimes_B R_G P^{B*G}) * G \arrow[d, "\varphi_2 \otimes \mathrm{id}_{\mathrm{k}\, G}"'] & \mathrm{End}_{A*G}((A * G) \otimes_{B*G} I_G R_G P^{B*G}) \arrow[d, "\varphi_1"] \\
\mathrm{End}_A(R_G((A * G) \otimes_{B*G} P^{B*G})) * G \arrow[r, "\theta_{(A*G) \otimes_{B*G} P^{B*G}}"'] & \mathrm{End}_{A*G}(I_G R_G (A * G) \otimes_{B*G} P^{B*G})
\end{tikzcd}$$

where $\theta_{P^{B*G}}$ and $\theta_{P^{B*G}}$ and $\theta_{(A*G) \otimes_{B*G} P^{B*G}}$ are the isomorphisms from Lemma 3.16,

$$\begin{aligned}s_1 &: \mathrm{End}_B(R_G P^{B*G}) \to \mathrm{End}_A(A \otimes_B P^{B*G}), f \mapsto \mathrm{id}_A \otimes f \\ s_2 &: \mathrm{End}_{B*G}(I_G R_G P^{B*G}) \to \mathrm{End}_{A*G}((A * G) \otimes_{B*G} I_G R_G P^{B*G}), f \mapsto \mathrm{id}_{A*G} \otimes f\end{aligned}$$

are the maps obtained from the induction functor, and φ_1 and φ_2 are the isomorphisms arising from the natural isomorphisms 1 and 2.
Note that all maps except $s_1 \otimes \mathrm{id}_{\mathrm{k}\, G}$ and s_2 are isomorphisms and that by assumption s_2 is an epimorphism in homology degree one and an isomorphism in homology in degree strictly greater than one, resp. an isomorphism in homology in degree strictly greater than zero.

Thus, $\mathrm{id}_{\mathrm{k}\,G} \otimes s_1$ is an an epimorphism in homology degree one and an isomorphism in homology in degree strictly greater than one, resp. an isomorphism in homology in degree strictly greater than zero.
This implies that s_1 is in homology degree one and an isomorphism in homology in degree strictly greater than one, resp. an isomorphism in homology in degree strictly greater than zero.
We obtain that the vector space homomorphism

$$H^*(s_1) : \mathrm{Ext}^*_B(R_G L^{B*G}, R_G L^{B*G}) \to \mathrm{Ext}^*_A(A \otimes_B L^{B*G}, A \otimes_B L^{B*G}), [f] \mapsto [\mathrm{id}_A \otimes f]$$

is an epimorphism in degree one and an isomorphism in degree strictly greater than one, resp. an isomorphism in degree strictly greater than zero.
Moreover, since $A \otimes_B L_i^B \cong \Delta_{L_i^A}$ for every $L_i^B \in \mathrm{Sim}(B)$ and

$$R_G(L^{B*G}) \cong \bigoplus_{L_i^B \in \mathrm{Sim}(B)} [R_G(L^{B*G}) : L_i^B] L_i^B$$

by Proposition 1.12[7], this induces for every $L_i^B, L_j^B \in \mathrm{Sim}(B)$ a vector space homomorphism

$$\begin{aligned}[][R_G(L^{B*G}) : L_i^B][R_G(L^{B*G}) : L_j^B] \,\mathrm{Ext}^*_B(L_i^B, L_j^B) & \\ \to [R_G(L^{B*G}) : L_i^B][R_G(L^{B*G}) : L_j^B] \,\mathrm{Ext}^*_A(\Delta_{L_i^A}, \Delta_{L_j^A}), & \\ f \mapsto \mathrm{id}_A \otimes f, &\end{aligned}$$

which is an epimorphism in degree one and an isomorphisms in degree strictly greater than one, resp. an isomorphism in degree strictly greater than zero. Since by Corollary 1.14 $[R_G(L^{B*G}) : L_i^B] \neq 0$ for all i, this implies that $(B, \leq_B)$ is homological resp. regular. □

4 Auslander algebras of Nakayama algebras

In this section, we will give, at some length, an example of the above. However, before that, we will need two more general statements.

Lemma 4.1. *Let A be a finite-dimensional algebra and suppose G is a commutative group acting on A via automorphisms. Let $[X]$ be an isomorphism class of indecomposable A-modules and let $H_{[X]}$ be the stabilizer of $[X]$ in G. Then there is a representative $Y \in [X]$ such that Y has an $H_{[X]}$-action.*

Proof. Let $n := |H_{[X]}|$. Let X be any representative of $[X]$ and consider the $A * H_{[X]}$-module $\mathrm{k}\,H_{[X]} \otimes X$. As an A-module, this is isomorphic to the direct sum nX, so that via this isomorphism nX obtains likewise the structure of an $A * H_{[X]}$-module. Thus we obtain a group homomorphism $\varphi : H_{[X]} \to \mathrm{Mat}_n(\mathrm{End}_A(X))^*$, where $\mathrm{Mat}_n(\mathrm{End}_A(X))^*$ denote the invertible elements of $\mathrm{Mat}_n(\mathrm{End}_A(X))^*$. Since $\mathrm{End}_A(X)$ is local with residue field k, this induces a group homomorphism $\varphi' : H_{[X]} \to \mathrm{Gl}_n(\mathrm{k}) \cong \mathrm{Gl}_n(\mathrm{Aut}(X))$. Since G and thus $H_{[X]}$ is commutative, the matrices in the image have a common eigenvector. This corresponds to a summand $Y|nX$, $Y \cong X$ which is stable under the action by $H_{[X]}$ on nX defined by φ', and thus Y has an $H_{[X]}$-action. □

The following proposition is related to [23, Theorem 1.3 (c) iii].

Proposition 4.2. *Let G be a group acting via automorphisms on an algebra D of finite representation type. Then there is an induced G-action on an algebra A' Morita equivalent to the Auslander algebra A of D such that $A' * G$ is Morita equivalent to the Auslander algebra of $D * G$. Moreover, if G is commutative, there is even an induced action on the Auslander algebra A of D such that $A * G$ is Morita equivalent to the Auslander algebra of $D * G$.*

Proof. Let $\{X\}$ be a set of representatives of the isomorphism classes of indecomposable D-modules, and let $M := \bigoplus_X X$ and $N := I_G M = \mathrm{k}\, G \otimes M$. Note that for every indecomposable $A{*}G$-module Y, $R_G Y$ is a direct summand of M, so that $I_G R_G Y$ is a direct summand of $I_G M = N$. Since $Y | I_G R_G Y$ by Proposition 1.12[2], this implies that Y is a summand of N. By definition, $A' := \mathrm{End}_D(R_G N)$ is Morita equivalent to the Auslander algebra A of D. Moreover, as N is a $D * G$-module, G acts on $A' = \mathrm{End}_D(R_G N)^{\mathrm{op}}$ via conjugation. Now by Lemma 3.16,

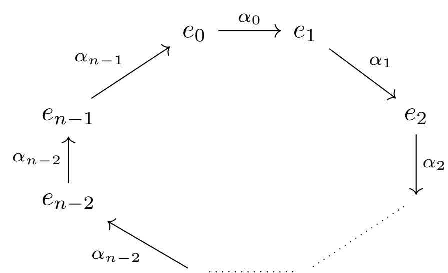
$$A' * G = \mathrm{End}_D(R_G N)^{\mathrm{op}} * G \cong (\mathrm{End}_D(R_G N) * G)^{\mathrm{op}} \cong \mathrm{End}_{D*G}(N \otimes \mathrm{k}\, G)^{\mathrm{op}}.$$

As every indecomposable $D * G$-module is a summand of N, the latter is Morita equivalent to the Auslander algebra of $D * G$.
Now if G is commutative, choose instead a set of representatives $\{Y\}$ for the orbits of the isomorphism classes of indecomposable D-modules under the G-action which are equipped with a $H_{[Y]}$-action, according to Lemma 4.1.
Then the module $N' := \bigoplus_{Y \in \{Y\}} \mathrm{k}\, G \otimes_{\mathrm{k}\, H_{[Y]}} Y$ obtains the structure of a $D * G$-module. Moreover, for any $Y \in \{Y\}$ let $Z_{[Y]}$ be a set of representatives of $G/H_{[Y]}$. Let X be an indecomposable D-module. Then by assumption there is a $g \in G$ and a $Y \in \{Y\}$ such that $X \cong gY$. Since $hY \cong Y$ for every $h \in H_{[Y]}$ we can assume without loss of generality that $g = z \in Z_{[Y]}$. Since by Corollary 1.13[8] $R_G N' \cong \bigoplus_{Y \in \{Y\}} \bigoplus_{z \in Z_{[Y]}} zY$, we have that $X | R_G N$. Thus $A := \mathrm{End}_D(N')$ is isomorphic to the Auslander algebra of D. As before, we can use Lemma 3.16 to see that

$$A * G \cong \mathrm{End}_D(R_G N')^{\mathrm{op}} * G \cong (\mathrm{End}_D(R_G N') * G)^{\mathrm{op}} \cong \mathrm{End}_{D*G}(I_G(R_G N'))^{\mathrm{op}},$$

which is Morita equivalent to the Auslander algebra of $D * G$, arguing as before that since every indecomposable D-module is isomorphic to a summand of $R_G N'$, every indecomposable $D * G$-module is a summand of $I_G R_G N$. $\square$

The example we consider arises as follows. Let $D := \mathrm{k}[x]/(x^N)$ and $G = \langle g | g^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$ be a cyclic group with n elements and generator $g \in G$.
Consider the G-action on D given by $gx = \xi x$ where ξ is a primitive n-th root of unity. Then by [23, p.241-244] $D * G$ is a self-injective Nakayama algebra with quiver Q



and relations given by all paths of length N, i.e. $D{*}G = kQ/J^N$. The idempotent e_j for $1 \leq i \leq n$ here is given by

$$e_j = \frac{1}{n} \sum_{k=0}^{n-1} \xi^{-kj} \otimes g^k$$

and α_j corresponds to the element

$$e_{j+1} x e_j = 1/n \sum_{k=0}^{n-1} \xi^{-jk} x \otimes g^k = x e_j = e_{j+1} x.$$

In particular, $g e_j = \xi^j e_j$ and

$$\alpha_{j+j'-1} \ldots \alpha_j = x e_{j+j'-1} x e_{j+j'-2} \ldots x e_{j+1} x e_j = x^{j'} e_j = x^{j'} e_j = \sum_{k=0}^{n-1} \xi^{-jk} x^{j'} \otimes g^k$$

so that

$$g\alpha_{j+j'}\dots\alpha_j = g\sum_{k=0}^{n-1}\xi^{-jk}x^{j'}\otimes g^k = \sum_{k=0}^{n-1}\xi^{-jk+j'}x^{j'}\otimes g^{k+1} = \xi^{j+j'}\sum_{k=0}^{n-1}\xi^{-jk}x^{j'}\otimes g^k$$

.
The indecomposable modules of D are given by $M_i = D/(x^{N-i})$ for $0 \le i \le N-1$, and the irreducible maps between them are given by the canonical projections and embeddings

$$\pi_i : M_i \to M_{i+1}, d + (x^{N-i}) \mapsto d + (x^{N-i-1}) \text{ for } 0 \le i \le N-2$$
$$\iota_i : M_i \to M_{i-1}, d + (x^{N-i}) \mapsto dx + (x^{N-i+1}) \text{ for } 1 \le i \le N-1$$

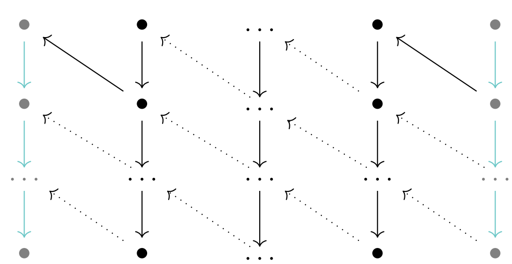
with relations

$$\pi_{i-1}\circ\iota_i = \iota_{i+1}\circ\pi_i \text{ for } 1 < i \le N-1$$
$$\pi_{N-2}\circ\iota_{N-1} = 0.$$

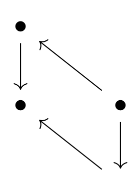
So the Auslander algebra A of D has quiver Q'

$$\begin{tikzcd} M_0 \arrow[r, bend left, "\pi_0"] & M_1 \arrow[l, bend left, "\iota_1"] \arrow[r, dotted, bend left] & \dots \arrow[l, dotted, bend left] \arrow[r, dotted, bend left] & M_{N-2} \arrow[l, dotted, bend left] \arrow[r, bend left, "\pi_{N-2}"] & M_{N-1} \arrow[l, bend left, "\iota_{N-1}"] \end{tikzcd}$$

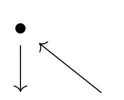
with commutator relations at every middle point $M_1, \dots M_{N-2}$ and a zero relation at M_{N-1}. Moreover, $gM_i = M_i$ for all $0 \le i \le N_1$, so in Proposition 4.2 we may chose $N' := \sum_{i=0}^{N-1} M_i$ and obtain that $A = \mathrm{End}_A(N')^{\mathrm{op}}$ is the Auslander algebra of D, and G acts on it via $g(\mathrm{id}_{M_i}) = \mathrm{id}_{M_i}$, $g(\pi_i) = \pi_i$ and $g(\iota_i) = \xi\iota_i$.
Note that since G acts trivially on the primitive idempotents, $A*G$ is basic, so that it is isomorphic to the Auslander algebra of $D * G$.
By [23, p.241-244], the algebra $A * G$ has Gabriel quiver Q' given by



where the last column is identified with the first column, so that Q' becomes a cylinder, with relations given by commutator relations in every parallelogram



and a zero relation given between neighbouring points in the last row



More concretely, the simples correspond to the $D*G$ modules

$$M_{i,j} = P_j^{D*G}/(x^{N-i}P_j^{D*G}) = ((D*G)e_j)/(x^{N-i}(D*G)e_j).$$

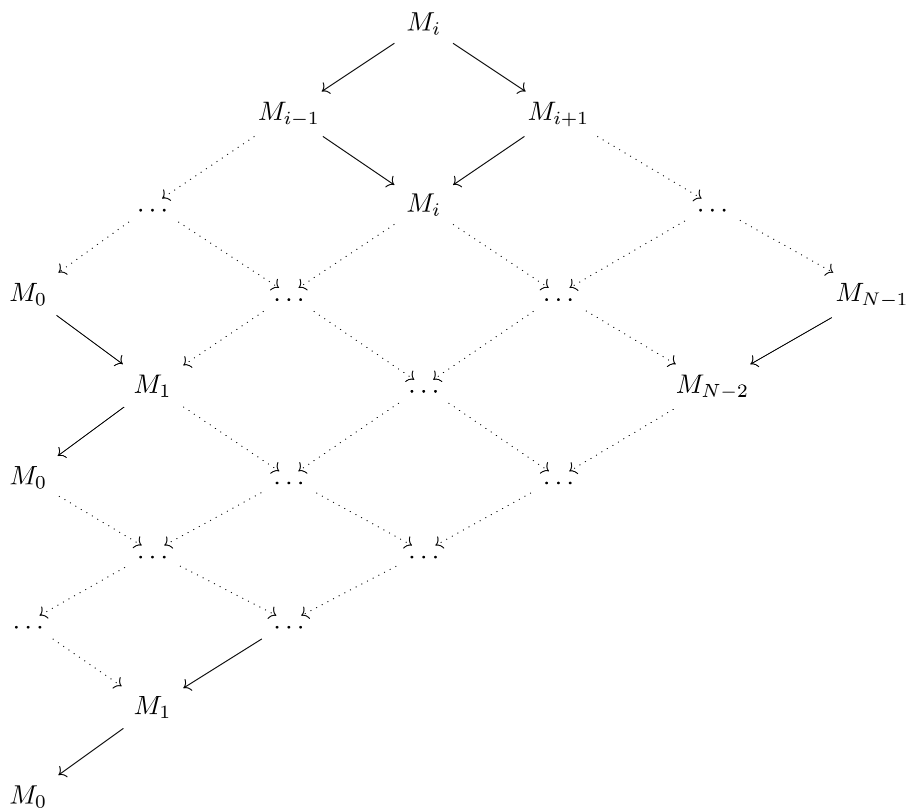
for $1 \leq i \leq N$, $1 \leq j \leq n$, which is given by the D-module M_i together with the multiplication

$$g \cdot (x^{j'}e_j + x^{N-i}P_j^{D*G}) = g \cdot (\alpha_{j+j'-1}\dots\alpha_j) = \xi^{j+j'}x^{j'}e_j + x^{N-i}P_j^{D*G},$$

and the extensions correspond to the maps

$$\iota_{i,j} : M_{i,j} \to M_{i-1,j-1}, a + x^{N-i}P_j^{D*G} \mapsto axe_{j-1} + x^{N-i+1}P_{j-1}^{D*G} = a\alpha_{j-1} + x^{N-i+1}P_{j-1}^{D*G}$$
$$\pi_{i,j} : M_{i,j} \to M_{i+1,j}, a + x^{N-i}P_j^{D*G} \mapsto a + x^{N-i-1}P_j^{D*G}$$

Now consider A. The projective P_i^A at M_i is given by



If we define the partial order $M_i \leq M_j$ if and only if $i \leq j$, then we obtain that A is quasi-hereditary with standard modules given by

$$\begin{tikzcd} M_i \arrow[r, "\pi_i"] & M_{i-1} \arrow[r, dotted] & \dots \arrow[r, dotted] & M_1 \arrow[r, "\pi_2"] & M_0 \end{tikzcd}$$

where P_i^A has standard filtration

$$\begin{tikzcd}
\Delta_i^A \arrow[dr] & & & \\
& \Delta_{i+1}^A \arrow[dr, dotted] & & \\
& & \dots \arrow[dr, dotted] & \\
& & & \Delta_n^A
\end{tikzcd}$$

Moreover, note that the subalgebra B of A given by the quiver

$$M_0 \xrightarrow[\pi_0]{} M_1 \dashrightarrow \dots \dashrightarrow M_{N-1}$$

is directed. Additionally A has a vector space basis consisting of

$$\iota_{i-k} \circ \dots \circ \iota_i \circ \mathrm{id}_{M_i} \circ \pi_{i-1} \circ \dots \circ \pi_{i-j} \text{ for } 0 \leq i \leq N-1; 0 \leq j, k \leq i.$$

Thus, as a right B-module

$$A_{|B} = \bigoplus_{i=0}^{N-1} \bigoplus_{k=0}^{i} \iota_{i-k} \circ \dots \circ \iota_i \circ \mathrm{id}_{M_i} B \cong \bigoplus_{i=0}^{N-1} \bigoplus_{j=i}^{N-1} e_i B$$

is projective, and

$$A \otimes_B L_{M_i}^B \cong A \,\mathrm{id}_{M_i} / (A \,\mathrm{rad}(B) \,\mathrm{id}_{M_i}) \cong P_{M_i}^A / A\pi_i = \Delta_i^A.$$

Thus B is an exact Borel subalgebra of A.
Now, since G acts trivially on the simples of A, $\leq_A$ is automatically G-equivariant. Hence we obtain by Proposition 3.3 an induced partial order $\leq_{A*G}$ on $A * G$, given by $M_{i,j} <_{A*G} M_{i',j'} \Leftrightarrow i < i'$, such that by Theorem 3.14 $(A * G, \leq_{A*G})$ is quasi-hereditary . Moreover, by Theorem 3.17 it has an exact Borel subalgebra given by $B * G$. Using our explicit description of B, and the fact that G acts trivially on B, we obtain that $B * G$ is the subalgebra of $A * G$ given by the subquiver

$$\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[d] & \dots \arrow[d, dotted] & \bullet \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[d] & \bullet \arrow[d] & \dots \arrow[d, dotted] & \bullet \arrow[d] & \bullet \arrow[d] \\
\dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d, dotted] & \dots \arrow[d] & \dots \arrow[d] \\
\bullet & \bullet & \dots & \bullet & \bullet
\end{tikzcd}$$

of Q'. Moreover, if $\iota_B : B \to A$ is the canonical embedding, $I \subset A$ is the ideal generated by $\iota_1, \dots, \iota_{N-1}$ and $\pi_I : A \to A/I$ is the canonical projection, then $\pi_I \circ \iota_B$ is an isomorphism, so that ι_B admits a splitting with kernel I. Hence B is normal in A, so that $B * G$ is normal in $A * G$ by Theorem 3.17.
However, note that for $N \geq 3$ the extension of Δ_1^A and Δ_2^A given by

$$\begin{tikzcd}
& M_1 \arrow[dl] \arrow[dr] & \\
M_0 \arrow[dr] & & M_2 \arrow[dl] \\
& M_1 \arrow[dl] & \\
M_0 & &
\end{tikzcd}$$

has a submodule X

$$\begin{tikzcd}
M_0 \arrow[dr] & & M_2 \arrow[dl] \\
& M_1 \arrow[dl] & \\
M_0 & &
\end{tikzcd}$$

which is an extension of Δ_0^A and Δ_2^A. Hence $\mathrm{Ext}^1_A(\Delta_0^A, \Delta_2^A) \neq (0) = \mathrm{Ext}^1_B(L^B_{M_0}, L^B_{M_2})$, so that B is not regular. Similarly, we can see that $B * G$ is not regular by considering tha $A * G$-module

$$\begin{tikzcd}
M_{0,1} \arrow[dr] & & M_{2,2} \arrow[dl] \\
& M_{1,1} \arrow[dl] & \\
M_{0,0} & &
\end{tikzcd}$$

which is an extension of $\Delta_{0,1}^{A*G}$ and $\Delta_{2,2}^{A*G}$, while $\mathrm{Ext}^1_{B*G}(L^{B*G}_{M_{0,1}}, L^{B*G}_{M_{2,2}}) = (0)$ For the case $N = 2$, B is a regular exact Borel subalgebra as seen in [18, Example A.1].

References

[1] Alexander Beilinson. Coherent Sheaves on P^n and Problems of Linear Algebra. Functional Analysis and its Applications, 12:214–216, 1978.

[2] Alexander Beilinson. The Derived Category of Coherent Sheaves on P^n. Selecta Mathematica Sovietica, 3:233–237, 1984.

[3] A I Bondal. Representation of Assotiative Algebras and Coherent Sheaves. Mathematics of the USSR-Izvestiya, 34(1):23, 1990.

[4] Tomasz Brzeziński, Steffen Koenig, and Julian Külshammer. From Quasi-Hereditary Algebras with Exact Borel Subalgebras to Directed Bocses. Bulletin of the London Mathematical Society, 52(2):367–378, 2020.

[5] Aaron Chan. Some Homological Properties of Tensor and Wreath Products of Quasi-Hereditary Algebras. Communications in Algebra, 42:2368 – 2379, 2012.

[6] Joseph Chuang and Radha Kessar. Symmetric Groups, Wreath Products, Morita Equivalences, and Broué's Abelian Defect Group Conjecture. Bulletin of the London Mathematical Society, 34, 2002.

[7] Joseph Chuang and Raphaël Rouquier. Derived Equivalences for Symmetric Groups and sl_2-Categorification. Annals of Mathematics, 167:245–298, 2008.

[8] Joseph Chuang and Kai Meng Tan. Filtrations in Rouquier Blocks of Symmetric Groups and Schur Algebras. Proceedings of the London Mathematical Society, 86(3):685–706, 2003.

[9] Joseph Chuang and Kai Meng Tan. Representations of Wreath Products of Algebras. Mathematical Proceedings of the Cambridge Philosophical Society, 3(135), 2003.

[10] Edward Cline, Brian Parshall, and Leonard Scott. Finite Dimensional Algebras and Highest Weight Categories. Journal für die reine und angewandte Mathematik, 391:85–99, 1988.

[11] Laurent Demonet. Skew Group Algebras of Path Algebras and Preprojective Algebras. Journal of Algebra, 323(4):1052–1059, 2010.

[12] Vlastimil Dlab. Quasi-hereditary Algebras Revisited. In An. Stiin., volume 4, pages 43–54. Univ. Ovidius Constantza, 1996.

[13] Vlastimil Dlab and Claus Michael Ringel. Quasi-hereditary Algebras. Illinois Journal of Mathematics, 33, 1989.

[14] Vlastimil Dlab and Claus Michael Ringel. The Module Theoretical Approach to Quasi-hereditary Algebras. In Representations of Algebras and Related Topics, volume 168 of London Mathematical Society Lecture Note Series. Cambridge University Press, 1992.

[15] Anton Evseev and Alexander Kleshchev. Blocks of Symmetric Groups, Semicuspidal KLR Algebras and Zigzag Schur-Weyl Duality. Annals of Mathematics, 118, 2016.

[16] A. L. Gorodentsev and Alexei N. Rudakov. Exceptional Vector Bundles on Projective Spaces. Duke Mathematical Journal, 54:115–130, 1987.

[17] Steffen Koenig. Exact Borel Subalgebras of Quasi-Hereditary Algebras, I. Mathematische Zeitschrift, 220:399–426, 1995.

[18] Steffen Koenig, Julian Külshammer, and Sergiy Ovsienko. Quasi-Hereditary Algebras, Exact Borel Subalgebras, A-∞-Categories and Boxes. Advances in Mathematics, 262:546–592, 2014.

[19] Roberto Martínez-Villa. Skew Group Algebras and their Yoneda Algebras. Mathematical Journal of Okayama University, 43, 2001.

[20] Patrick Le Meur. On the Morita Reduced Versions of Skew Group Algebras of Path Algebras. The Quarterly Journal of Mathematics, 2018.

[21] Patrick Le Meur. Crossed products of calabi-yau algebras by finite groups. Journal of Pure and Applied Algebra, 224(10):106394, 2020.

[22] Edward Poon. Skew Group Rings. `https://alistairsavage.ca/pubs/Poon-Skew_group_rings.pdf`, 2016. Student Project.

[23] Idun Reiten and Christine Riedtmann. Skew Group Algebras in the Representation Theory of Artin Algebras. Journal of Algebra, 92(1):224–282, 1985.

[24] Claus Michael Ringel. Iyama's Finiteness Theorem via Strongly Quasi-Hereditary Algebras. Journal of Pure and Applied Algebra, 214:1687–1692, 2009.

[25] Leonard Scott. Simulating Algebraic Geometry with Algebra, I: The Algebraic Theory of Derived Categories. In The Arcata Conference on Representations of Finite Groups, volume 47 of Proc. Symp. Pure Math, pages 271–281. Amer. Math. Soc., 1987.

[26] Sarah Witherspoon. Hochschild Cohomology for Algebras. Graduate studies in Mathematics 204. American Mathematical Society, 2019.

[27] Quanshui Wu and Can Zhu. Skew Group Algebras of Calabi–Yau Algebras. Journal of Algebra, 340(1):53–76, 2011.