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Energy based methods applied in mechanics by using the extended Noether's formalism

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Physical systems are modeled by field equations; these are coupled, partial differential equations in space and time. Field equations are often given by balance equations and constitutive equations, where the former are axiomatically given and the latter are thermodynamically derived. This approach is useful in thermomechanics and electromagnetism, yet challenges arise once we apply it in damage mechanics for generalized continua. For deriving governing equations, an alternative method is based on a variational framework known as the extended NOETHER's formalism. Its formal introduction relies on mathematical concepts limiting its use in applied mechanics as a field theory. In this work, we demonstrate the power of extended NOETHER's formalism by using tensor algebra and usual continuum mechanics nomenclature. We demonstrate derivation of field equations in damage mechanics for generalized continua, specifically in the case of strain gradient elasticity.

1 | INTRODUCTION

In rational continuum mechanics, we axiomatically start with the balance equations in a formal manner established in [1]. For thermomechanics there are balance equations of mass, momenta, and energy. For electromagnetism there are FARADAY relation (balance of magnetic flux) and balance of electric charge, they lead to MAXWELL's equations. These balance equations are called "universal" since they hold for all materials. For a specific application with known materials, equations for stress, heat flux, (internal or free) energy, electric current, electromagnetic force, charge and current potentials are yet to be defined by constitutive equations in order to close the system of equations. The constitutive equations are needed for establishing the material specific behavior into the system.

Starting with [2–4], the thermodynamics of irreversible processes has been used for obtaining the constitutive equations in a formal way. There are ample methods in the literature, we may name at least four famous approaches: The COLEMAN–NOLL procedure [5], MÜLLER's rational thermodynamics [6], non-equilibrium thermodynamics [7], and extended thermodynamics [8]. Thermodynamical approaches consider bulk properties; for surface phenomena such as crack propagation or surface polarization, additional assumptions or models need to be suggested. In the case of surfaces, additional axiomatic balance equations may be used for singular surfaces [9]. When it comes to edge effects, further complications arise and there is simply not one method covering all systems in generalized continua. In short, for applications involving first space derivatives (gradients) of unknowns—the method is useful [10], where we start directly with balance equations and derive all the rest. If an application demands higher gradients to cover surface and edge effects (and even beyond) then we need a generalization of this method, technical challenges arise.

The formal difficulties aside, definition of internal or free energy is challenging as well. Energy is a directly measurable quantity and a theory based on energy is known as the variational formulation with roots over many centuries long [11, 12]. In continuum mechanics, variational formulation has been suggested in different settings, in hydrodynamics [13, 14], fluid-structure interactions [15, 16], multiphysics [17, 18], dynamics [19, 20], mechanics with dissipation [21, 22], and for discrete structures [23]. Specifically for crack propagation, variational approaches exist in quasi-static cases [24]. This approach uses an energy concept and may be extended to generalized mechanics, yet it causes technical difficulties in the case of applied surface or volume forces [25]. In classical mechanics, energy integral in time is called action. Invariance of action is postulated by using a LAGRANGE function. Although the method is useful, its motivation is formal [26]. This LAGRANGE function leads to EULER–LAGRANGE equations that are the field equations for the bulk. In the case of bulk quantities, classical mechanics [27–29] is utilized for displacement in "regular" systems, i.e. with no singularities.

Topological methods by using geometric formulations [30, 31] have been exploited for developing a geometric continuum mechanics [32–35], which are applied in multiphysics as well as in multiscale theories [36, 37]. Such formulations are developed for regular domains, in simpler terms, the underlying mathematical structure needs smooth functions that is only possible if the continuum body has no voids or cracks. A crack may be seen as a singular surface in the sense that the displacement value on both sides of the surface is different. Thus, there is a jump in the displacement field across this "singular" surface. We call this singularity a "defect" or "fracture" in the material.

In elasticity, a propagating crack is modeled by forces that are acting on this defect [38]. These so-called material (or configurational) forces are conceptually beneficial for generating a so-called material stress and an energy-momentum tensor as introduced in [39]. This method provides a model for the surface related phenomena. The energy-momentum tensor is written by using a space-time continuum. This approach has been proven to be adequate in the special and then general relativity [40], where a spatial space is used in the formulation. A spatial space is coordinates pointing at locations with or without massive particles occupying this locations. This concept is well-known in fluid dynamics with a control volume, also called a EULERian approach, where space coordinates are fixed in the physical space (where we live). We call it a laboratory frame. Motion of coordinates are not related to the motion of massive particles. Often we visualize that the laboratory frame is fixed (not moving). In continuum mechanics, we use a material frame where the space coordinates point to the same massive particle (material point or matter). Motion of coordinates means the motion of massive particles. As the matter may move, a continuum body is deforming, we may think of a co-moving frame for space. The four-dimensional notation has been used in the ESHELBY energy-momentum tensor by means of a material frame. In non-relativistic approaches, we skip a distinction between time defined on a laboratory or co-moving frame. By decomposing the energy-momentum tensor into a time and space part, the so-called ESHELBY stress tensor is obtained and used in connection with crack modeling in different formulations [41–45] and also in applications with numerical examples [46–51]. For the history of ESHELBY stress tensor and its use in variational formulation, we refer to [52]. For technical details in surface phenomena [53] and ESHELBY stress tensor's relation to fracture mechanics, we refer to [54].

Displacement is the primitive function that we calculate in mechanics. In continuum mechanics, functions are regular without singularities. Crack is indeed causing a jump of the displacement, since the bond between material particles has been lost, and its thickness is at a smaller length-scale such that we consider this singularity as a fictitious surface (without its own mass). The ESHELBY stress tensor is used in [55] in a systematic study in order to define an entropy production on the singular surface. This excellent idea leads naturally to constitutive equations for the velocity of the singular surface that is tantamount to the crack in physics; in the literature, there is no such consideration for suggesting a relation and its experimental verification. Another novel approach is presented in [56], where MÜLLER's rational thermodynamics is used to obtain a balance equation with the necessary constitutive equation for the damage parameter. A differential equation related to the ESHELBY stress tensor is without doubt a possible modeling tool for crack propagation [57, 58].

In a first-order continuum, in the case of elasticity, we know the governing equations including ESHELBY stress tensor. When it comes to a generalization, for example higher-order theories or in multiphysics, it becomes challenging to motivate if and how ESHELBY stress tensor needs to be modified. Therefore, we follow another approach and derive field equations by using NOETHER's formalism in continuum mechanics. In this approach, all governing equations, including ESHELBY stress tensor, are acquired by applying NOETHER's formalism. Herein we exploit this approach by using a continuum mechanics jargon known as RICCI calculus. The manuscript comprises a detailed motivation of the formalism:

1. In Section 2, we discuss about variation of fields, which is the core of the subject and is discussed in an abstract setting. We give examples to well-known transformation rules for clarifying their role in this formulation. For the sake of consistency, we include possibly well known concepts from variational calculus and tensor algebra.

2. In Section 3, we explain the extended NOETHER's theorem for fields. We emphasize that we demonstrate (discrete) systems as in dynamics depending on time; and also (continuous) systems in space and time. It is of utmost importance to distinguish that many variational formulations utilize a spatial space. Historically, NOETHER's theorem has been applied for such systems mainly in electromagnetism, since there, the fields propagate with or without matter. But herein, we focus on a continuum body, where space is defined as attached to and co-moving with the matter. Therefore, we may call it a "configurational" space, see [59] for the history of this distinction. We obtain the so-called RUND–TRAUTMAN identity in this configurational space.

3. In Section 4, we obtain the usual EULER–LAGRANGE equations. We provide the derivation of NOETHER's current leading to conservation (balance) laws, by using EULER–LAGRANGE equations and RUND–TRAUTMAN identity. The formalism is utilized in an abstract manner in order to employ for generalized continuum.

An important outcome of this work is the clear differing in governing equations and their derivations. In this way, we enable a more broader use of this formalism. The NOETHER's formalism leads to EULER–LAGRANGE equations. This approach is often used in classical mechanics for discrete systems, but we use it herein for obtaining field equations. The balance laws are acquired with this formalism. Usually, after acquiring the balance laws, the NOETHER's formalism is taken aside. We do the contrary and continue with the extended NOETHER's formalism that leads to RUND–TRAUTMAN identity. This identity contains not only balance laws yet also ESHELBY stress tensor, where the latter leads to a so-called J-integral used in damage mechanics. Therefore, we exploit the extended NOETHER's formalism in order to obtain field equations in damage mechanics, first, for classical elasticity, then, for strain gradient elasticity:

1. In Section 5, we give examples of well-known elastodynamics by using the extended NOETHER's formalism. In this way, it is clarified that the balance of momentum is obtained from the EULER–LAGRANGE equations, balance of energy is acquired from the NOETHER's current, and the ESHELBY stress tensor is attained by the RUND–TRAUTMAN identity.

2. In Section 6, we apply the same approach for generalized continua. In addition to balance laws for generalized elasticity, we acquire ESHELBY stress tensor (and J-integral) in the case of strain gradient elasticity.

2 | VARIATION OF FIELDS

We use standard continuum mechanics notation and understand a summation over all repeated indices called EINSTEIN summation convention. Consider a body composed of massive particles. Particle is defined as the smallest observable piece of material consisting of an infinitesimal volume element. This volume element is, however, large enough to neglect any quantum phenomena. The body's image is the continuum domain in the space R^3, with (contravariant) coordinates x^i, where $i = 1, 2, 3$. Coordinates label the particles of the body; in other words, we use the particle coordinates analogous as in [60]. We use time, t, and space, (x^1, x^2, x^3); the numerical values of coordinates do not change as they denote the same massive particle. As the numerical values of x^i do not change for the same particle, the particle rests in this "material" frame. Since the equilibrium is defined by referring the material frame, we call it the *reference* frame. It may be called the *inertial* frame with respect to which the dynamics of the system is prescribed. An inertial frame is basically the coordinate system labeling particles, since the definition of the inertial frame is the one where the mass rests,[1] and thus, does not carry any inertial forces [61].

Motion is modeled by means of two distinct processes. First, the particle may deviate from its equilibrium position, this displacement is fully recoverable. The best example is in elasticity. An elastic deformation is such that the particles change their positions by deviating from the reference frame (equilibrium); after unloading, the particles turn to the same equilibrium position. Reference frame itself—equivalently the equilibrium position of the continuum body—may be altered irreversibly. If the motion of the body is such that the system obtains another equilibrium position upon unloading, as in plasticity, then we need different field equations defining the deviation from the equilibrium and the motion (evolution) of the reference frame.

We aim for a general formulation that employs transformation properties of tensors. We start with the continuous transformation that forms the basis of the HAMILTON–JACOBI equation in the NOETHERian formalism. This variational principle leads to governing equations as introduced formally by [62] and [63] we refer to [64] for the historical remarks about NOETHER's theorem. The variational principle is based on the variational calculus that is the algebra of functionals and their extremal values. Suppose that we define a (tensor of rank 0) function $L(t; x^i, \partial x^i/\partial t)$ depending on an

independent variable, t, and on variables depending on this independent variable, $x^i = x^i(t)$ and $\partial x^i/\partial t = \frac{\partial x^i}{\partial t}(t)$. Its integral over the independent variable is called action:

$$\mathcal{I} = \int L \, \mathrm{d}t \,, \tag{1}$$

which becomes a form by rewriting,

$$\mathcal{I} = \int \frac{\mathrm{d}U}{\mathrm{d}t} \, \mathrm{d}t = \int \, \mathrm{d}U \,. \tag{2}$$

This so-called first integral in U means that the path of the integral does not matter, transformation in t leaves its value unchanged. According to the theory of invariants the differential form $\mathrm{d}U$ is an invariant: its transformation in x^i fails to change its numerical value. In short, action is invariant under transformation of t and x^i. We discuss this formal property going back to [65] in more detail in what follows.

An important side note is the invariants and their use in mechanics. For example in hyperelasticity, deformation energy is modeled by invariants (of strain) since they do not change under coordinate transformations. The values of the (strain) tensor change by following the transformation rules. Although the components of a tensor change, the invariants remain the same. Therefore, by using the invariants, we obtain an energy definition independent of coordinate systems.

The transformation is arbitrary[2] and L is called Lagrangean. In order to understand, what the transformation means, we briefly introduce a continuous transformation. We use a function taking the variable x^i and transforms along ε, as follows:

$$x^{i'} = \zeta^i(\boldsymbol{x}, \varepsilon) \,. \tag{3}$$

We provide some examples in the Appendix. The general framework is used for an arbitrary but linear transformation. When we expand ζ in Taylor series, around $\varepsilon = 0$, then a first order approximation for the transformation reads

$$x^{i'} = x^i + \varepsilon \frac{\partial \zeta^i}{\partial \varepsilon}\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2) \approx x^i + \varepsilon \xi^i(\boldsymbol{x}) \,, \tag{4}$$

where the coefficients to the first power, ξ^i, are called the *generators* of the transformation [68, Section 4.1]. As at the limit $\varepsilon \to 0$, the approximation becomes accurate, ε is often introduced as a "small" parameter. If we define a tensor of rank one and weight zero, $\boldsymbol{A}$, in the space R^3, then the variation of the tensor's (contravariant) components becomes, due to the change of (contravariant) coordinates,

$$x^{i'} = x^i + \varepsilon \xi^i \,,$$
$$A^i\Big|_{x^{j'}} = A^i\Big|_{x^j} + \delta A^i\Big|_{x^j} = \frac{\partial x^{i'}}{\partial x^j} A^j\Big|_{x^j} = \left(\delta^i_j + \varepsilon \frac{\partial \xi^i}{\partial x^j}\right) A^j\Big|_{x^j} \,, \tag{5}$$
$$A^i + \delta A^i = \left(\delta^i_j + \varepsilon \frac{\partial \xi^i}{\partial x^j}\right) A^j \,, \quad \delta A^i = \varepsilon \frac{\partial \xi^i}{\partial x^j} A^j \,.$$

In other words, the variation of A^i means

$$\delta A^i = A^i\Big|_{x^{j'}} - A^i\Big|_{x^j} = A^i(x^{j'}) - A^i(x^j) \,. \tag{6}$$

Standard tensor calculus rules apply; the transformation is invertible,

$$\frac{\partial x^{i'}}{\partial x^j} \frac{\partial x^j}{\partial x^{k'}} = \delta^{i'}_{k'} = \delta^i_k \,. \tag{7}$$

We find the variation of covariant components of the same tensor of rank one [67, Section 23] in a straight-forward manner

$$x^i = x^{i'} - \varepsilon \xi^i \,,$$
$$A_i\Big|_{x^{j'}} = A_i\Big|_{x^j} + \delta A_i\Big|_{x^j} = A_j \frac{\partial x^j}{\partial x^{i'}} = A_j \left(\delta^j_i - \varepsilon \frac{\partial \xi^j}{\partial x^{i'}}\right) \,, \quad \delta A_i = -\varepsilon A_j \frac{\partial \xi^j}{\partial x^i} \,. \tag{8}$$

Analogously, for a tensor of rank two and weight zero, we obtain

$$\delta A^{ik} = \varepsilon \frac{\partial \xi^i}{\partial x^j} A^{jk} + \varepsilon \frac{\partial \xi^k}{\partial x^j} A^{ij},$$

$$\delta A^i{}_k = \varepsilon \frac{\partial \xi^i}{\partial x^j} A^j{}_k - \varepsilon \frac{\partial \xi^j}{\partial x^k} A^i{}_j,$$

$$\delta A_{ik} = -\varepsilon \frac{\partial \xi^j}{\partial x^i} A_{jk} - \varepsilon \frac{\partial \xi^j}{\partial x^k} A_{ij}. \tag{9}$$

As we have determined the transformation of co- and contravariant components of tensors, we start applying the formalism regarding transformation of LAGRANGEan.

3 | EXTENDED NOETHER'S THEOREM

Suppose that $L(t; \chi^i, \partial \chi^i / \partial t)$ is defined by an independent variable, $t \in [a, b]$, by variables, $\chi^i = \chi^i(t)$, and their derivatives, $\partial \chi^i / \partial t = \frac{\partial \chi^i}{\partial t}(t)$, with respect to the independent variable. We define the transformation, $t' = t + \varepsilon \tau$ and $\chi^{i'} = \chi^i + \varepsilon \xi^i$, with the corresponding generators, τ, ξ^i. If L is a scalar, its numerical value does not change—it is invariant—with respect to a particular choice of an infinitesimal transformation with the generators, τ, ξ^i, as follows:

$$L(t; \chi^i, \partial \chi^i / \partial t) = L(t'; \chi^{i'}, \partial \chi^{i'} / \partial t'). \tag{10}$$

We use a simplified notation $L' = L(t'; \chi^{i'}, \partial \chi^{i'} / \partial t')$ and rewrite $L = L'$. The (action) functional, $\mathcal{I} = \int L \, dt$, has its variation

$$\delta \mathcal{I} = \mathcal{I}' - \mathcal{I} = \int L\left(t'; \chi^{i'}, \frac{\partial \chi^{i'}}{\partial t'}\right) dt' - \int L\left(t; \chi^i, \frac{\partial \chi^i}{\partial t}\right) dt$$

$$= \int \left(L\left(t'; \chi^{i'}, \frac{\partial \chi^{i'}}{\partial t'}\right) \frac{dt'}{dt} - L\left(t; \chi^i, \frac{\partial \chi^i}{\partial t}\right) \right) dt. \tag{11}$$

As $L = L'$ or in other words, $\delta L = 0$, we expect that $\delta \mathcal{I}$ is independent of t, χ^i. The trivial way is to choose, $\delta \mathcal{I} = 0$. As the theory is at first order, that is in ε^1, owing to Equation (4), we acquire $\delta \mathcal{I} \to 0$ faster than $\varepsilon \to 0$. By choosing a small ε, linear in ε is an accurate theory in first order. Therefore, we may have a linear in ε difference,

$$\delta \mathcal{I} = \int \varepsilon \, dF = \int \varepsilon \frac{dF}{dt} \, dt, \tag{12}$$

where F is an arbitrary (but given) function. The latter brings an important equation:

$$L\left(t'; \chi^{i'}, \frac{\partial \chi^{i'}}{\partial t'}\right) \frac{dt'}{dt} - L\left(t; \chi^i, \frac{\partial \chi^i}{\partial t}\right) = \varepsilon \frac{dF}{dt}. \tag{13}$$

This abstract transformation that produces a source term has been used extensively in [69, 70] to derive the *principle of least action*. A more general form for the right-hand side fails to exist, since the transformation is linear in ε. Although not written in this way, the formalism goes back to MAUPERTIUS and has been used for solving differential equations. This formalism is often called a *canonical transformation*.[3] We transform the functional in the parameter t that produces a right-hand side, if L is in unit of energy then the induced F is in momentum times position.

Now, we generalize the procedure by defining (t, x^i) as independent variables and $\phi^k = \phi^k(t, x^i)$ as variables depending on the independent variables. A LAGRANGEan density (per space-time) depends on them and their derivatives,

$$\mathcal{L} = \mathcal{L}\left(t, x^i; \phi^k, \frac{\partial \phi^k}{\partial t}, \frac{\partial \phi^k}{\partial x^i}\right), \quad \mathcal{I} = \int \mathcal{L} \, d\Sigma, \tag{14}$$

where $\mathcal{L}$ is in unit of energy per volume. We use an infinitesimal space-time element, $d\Sigma = \sqrt{g}\,dx\,dt$, in a frame with the metric tensor, $g_{\mu\nu}$, and its determinant, $g = \det(g_{\mu\nu})$. As aforementioned, this frame may be chosen as laboratory (fixed) or reference (co-moving) frame. Herein, we choose it as the reference frame (material system) and use a standard mapping to an arbitrary frame by means of a transformation. In continuum mechanics, such a mapping is introduced from reference to current frame. Herein, we use a general formulation by using an arbitrary (linear) transformation. We define the reference frame in a Cartesian system, leading to $dx \equiv dx^1\,dx^2\,dx^3$, and thus, the transformed frame is oblique but not curvilinear, $dx' = \varepsilon^{ijk}\,dx^1_{i'}\,dx^2_{j'}\,dx^3_{k'}$, by using the LEVI-CIVITA symbol, ε^{ijk}. By constructing a four-dimensional space for space-time with its metric (for continuous transformation), $g_{\mu\nu}$, where $\mu, \nu \in \{1, 2, 3, 4\}$, we obtain

$$X^{\mu'}_{\mu} = \delta^{\mu'}_{\mu} + \varepsilon \frac{\partial(\tau, \xi^1, \xi^2, \xi^3)^{\mu'}}{\partial(t, x^1, x^2, x^3)^{\mu}} = \begin{pmatrix} 1 + \varepsilon\frac{\partial\tau}{\partial t} & \varepsilon\frac{\partial\tau}{\partial x^1} & \varepsilon\frac{\partial\tau}{\partial x^2} & \varepsilon\frac{\partial\tau}{\partial x^3} \\ \varepsilon\frac{\partial\xi^1}{\partial t} & 1 + \varepsilon\frac{\partial\xi^1}{\partial x^1} & \varepsilon\frac{\partial\xi^1}{\partial x^2} & \varepsilon\frac{\partial\xi^1}{\partial x^3} \\ \varepsilon\frac{\partial\xi^2}{\partial t} & \varepsilon\frac{\partial\xi^2}{\partial x^1} & 1 + \varepsilon\frac{\partial\xi^2}{\partial x^2} & \varepsilon\frac{\partial\xi^2}{\partial x^3} \\ \varepsilon\frac{\partial\xi^3}{\partial t} & \varepsilon\frac{\partial\xi^3}{\partial x^1} & \varepsilon\frac{\partial\xi^3}{\partial x^2} & 1 + \varepsilon\frac{\partial\xi^3}{\partial x^3} \end{pmatrix}, \tag{15}$$

$$J = \det(X^{\mu'}_{\mu}) = 1 + \varepsilon\frac{\partial\tau}{\partial t} + \varepsilon\frac{\partial\xi^i}{\partial x^i} + \mathcal{O}(\varepsilon^2)\,,$$

and therefore $d\Sigma' = J\,d\Sigma$ or equivalently $dx'\,dt' = J\,dx\,dt$. Moreover, we have

$$g_{\mu\nu} = X^{\mu'}_{\mu} X^{\nu'}_{\nu} g_{\mu'\nu'}\,, \quad g' = \det(g_{\mu'\nu'})\,, \quad g = \det(g_{\mu\nu})\,, \tag{16}$$

by taking the determinant of the first equation, we obtain

$$g = JJg'\,,$$
$$\sqrt{g} = J\sqrt{g'}\,. \tag{17}$$

Since the continuous transformations are written as a set with an identity element, they have unique inverse transformations; therefore, they form a LIE group according to the group theory. Of course, we want to have mathematical relations transforming as the coordinate system transforms. Therefore, we use tensors in formulating such relations. A tensor of rank 0 is a scalar. We begin with an assertion that the LAGRANGE density is such a proper scalar, $\mathcal{L} = \mathcal{L}'$, and observe

$$\mathcal{I} = \int \mathcal{L}\sqrt{g}\,dx\,dt = \int \mathcal{L}J\sqrt{g'}\,dx\,dt = \int \mathcal{L}\sqrt{g'}\,dx'\,dt' = \int \mathcal{L}'\sqrt{g'}\,dx'\,dt' = \mathcal{I}'\,. \tag{18}$$

Equivalently, we may begin with a scalar, $\mathcal{I} = \mathcal{I}'$, for a given transformation, and obtain that the LAGRANGEan is an invariant. This invariance is exploited for the principle of least action as aforementioned, and this formalism is called NOETHER's theorem. We continue by building on this formalism and demonstrate an extension of this formalism.

Let us choose a specific type of generators for the arbitrary transformation, say, LORENTZ transformation; $\mathcal{I}$ is called a LORENTZ scalar. We may give an analogy to continuum mechanics by neglecting shortly the time integral; consider that $\mathcal{I} = \mathcal{I}'$ means that the energy is the same in both frames. This fact makes sense, but how do we know that the energy density (per volume) is also the same in both frames such that $\mathcal{L} = \mathcal{L}'$ is also fulfilled? Indeed, we may want that the energy density is the same for corresponding material particles, but how do we enforce this case? Now by using this analogy, we ask to answer the question: What if we do have a scalar, $\mathcal{I}' = \mathcal{I}$; however, not a proper scalar, $\mathcal{L} \neq \mathcal{L}'$, by definition, what are the additional conditions to be fulfilled in order to generate a proper scalar, $\mathcal{L} = \mathcal{L}'$?

For a formulation with field equations, we may incorporate the source term in Equation (13) (right-hand side) in two steps. First, we build up the procedure for fields with the LAGRANGEan in Equation (14) for arbitrary transformations in space and time. All the transformation is achieved with different generators but one-parameter ε. The variation reads in domain Ω for fields

$$\delta\mathcal{I} = \int_{\Omega'} \mathcal{L}'\,d\Sigma' - \int_{\Omega} \mathcal{L}\,d\Sigma = \int_{\Omega} (\mathcal{L}'J - \mathcal{L})\,d\Sigma\,, \tag{19}$$

since $\mathrm{d}\Sigma' = J\,\mathrm{d}\Sigma$. Second, from [71] we know that a transformation may produce a non-conservative force. In order to implement this concept, we need a clear distinction between conservative and non-conservative forces. A conservative force is derivable from a proper scalar, S, which is an invariant, as follows:

$$F_i^{\text{cons.}} = \frac{\partial S}{\partial x^i}\,. \tag{20}$$

A non-conservative force does not have this special property; thus, in the general case, we may generate a tensor rank 1 from a tensor rank 2, S_i^k, by using its derivative

$$F_i^{\text{non-cons.}} = \frac{\partial S_i^k}{\partial x^k}\,. \tag{21}$$

This property is a general identity, because in tensor calculus, we may always reduce the rank by a derivative.

In a transformation from one frame (with prime) onto another frame (without prime), a (non-conservative) force may be generated. This fundamental property may be physically understood as a (virtual) work in case of such a transformation. If we suppose that this transformation is between frames like the current and reference one, the deviation is a virtual displacement: $\delta u^i = x^{i'} - x^i$. For this transformation, we need to supply an energy, that is a virtual work into the system, $\delta A = F_i \delta u^i$, where this non-conservative force is measurable on the reference frame. The work is virtual since the transformation between the frames is nothing physical. The choice of frame, where the fields are described and equations are evaluated, has nothing to do with the system itself. Thus, the work caused by the transformation is virtual. However, the force is real and measurable. The direction of transformation from the reference to current gives a minus sign

$$(t, x^i) \rightarrow (t, x^{i'})\,,$$

$$x^{i'} = x^i + \delta u^i\,, \quad J = 1 + \frac{\partial \delta u^i}{\partial x^i}\,,$$

$$\mathcal{L} = \mathcal{L}'J + \delta A \Rightarrow \delta \mathcal{I} = \int_\Omega \left(\mathcal{L}'J - \mathcal{L}\right)\mathrm{d}\Sigma = -\int \delta A\,\mathrm{d}\Sigma\,. \tag{22}$$

This notion is used in elasticity. But the shown principle is applicable for a transformation between laboratory and reference frame as well. Then a transformation from the fixed laboratory frame (control volume) to the co-moving reference frame reads as a positive virtual work on the right-hand side. We are going to use $\pm$ in front of this term, in order to emphasize that both choices are adequate depending on the chosen frame to transform to. This right-hand side is introduced for the first time in [72] as $\delta \mathcal{I} = \varepsilon \frac{\partial F^\mu}{\partial x^\mu}$ that is a simplification and is called a *divergence invariance*. We stick to the more general form with the virtual work. We emphasize that this virtual work is given by a non-conservative force such that it is possible to begin with $\delta \mathcal{I} = \pm \delta \mathcal{R}$, where $\delta \mathcal{R}$ is a non-conservative (dissipative) work, also called RAYLEIGH dissipation. Indeed, the names work, force, and displacement are in harmony with mechanics, but the relation holds true in multiphysics, as also known from dynamics in discrete systems, and they may be called "generalized" force or work-conjugate term. In thermodynamics, one often calls them "thermodynamical" forces and fluxes.

The invariance of the LAGRANGEan leads to the RUND–TRAUTMAN identity [68, Section 6.5]. The LAGRANGEan density $\mathcal{L}' = \mathcal{L}(t', x^{i'}; \phi^{k'}, \frac{\partial \phi^{k'}}{\partial t'}, \frac{\partial \phi^{k'}}{\partial x^{i'}})$ depends on the independent variables (t, x^i) and primitive variables $(\phi^k, \frac{\partial \phi^k}{\partial t}, \frac{\partial \phi^k}{\partial x^i})$, which depend on the independent variables. We axiomatically assume that primitive variables exist. We stress that the frame is oblique, hence, we skip distinguishing between covariant and partial space derivatives. The same holds for the time derivative since we measure this in the reference frame. The linear transformations read

$$t' = t + \varepsilon\tau\,, \quad x^{i'} = x^i + \varepsilon\xi^i\,, \quad \phi^{k'} = \phi^k + \varepsilon\varphi^k\,, \tag{23}$$

where all of them are along the same parameter ε. Since the invariance property $L = L'$ asserts the condition to satisfy, $\mathcal{L}'J - \mathcal{L} = \mp \delta A$, we can set the variation along one-parameter ε vanish such as:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\mathcal{L}'J - \mathcal{L} \pm \delta A\right)\Big|_{\varepsilon=0} = 0\,, \tag{24}$$

where we utilize a directed differentiation often called a GATEUX derivative. Although we have introduced the virtual work as a physical quantity, $\delta A = F_i \delta u^i$, with an analogy in mechanics, in the general case, each primitive variable causes virtual work, $\delta A = F_i \varepsilon \varphi^i$. The linear transformations depend on ε, and thus, $\mathcal{L}'$ as well as J depend on ε. However, the term $\mathcal{L}$ does not, therefore, we obtain

$$\left(\frac{\mathrm{d}\mathcal{L}'}{\mathrm{d}\varepsilon} J \right)\Big|_{\varepsilon=0} + \left(\mathcal{L}' \frac{\mathrm{d}J}{\mathrm{d}\varepsilon} \right)\Big|_{\varepsilon=0} = \mp F_k \varphi^k . \tag{25}$$

By using Equation (15), we obtain

$$\mathcal{L}'\big|_{\varepsilon=0} = \mathcal{L} \ , J = 1 + \varepsilon \frac{\partial \tau}{\partial t} + \varepsilon \frac{\partial \xi^i}{\partial x^i} \Rightarrow J\big|_{\varepsilon=0} = 1 \ , \frac{\mathrm{d}J}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial t} + \frac{\partial \xi^i}{\partial x^i} \ , \tag{26}$$

and thus,

$$\frac{\mathrm{d}\mathcal{L}'}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} + \mathcal{L}\left(\frac{\partial \tau}{\partial t} + \frac{\partial \xi^i}{\partial x^i} \right) = \mp F_k \varphi^k . \tag{27}$$

By using a short notation $()^{\bullet} = \partial()/\partial t$ and $()_{,i} = \partial()/\partial x^i$, we calculate

$$\begin{aligned}
\frac{\partial \phi^{k'}}{\partial t'}\Big|_{\varepsilon=0} &= \frac{\partial(\phi^k + \varepsilon\varphi^k)}{\partial t} \frac{\partial t}{\partial t'}\Big|_{\varepsilon=0} = \frac{\partial(\phi^k + \varepsilon\varphi^k)}{\partial t} \frac{\partial(t' - \varepsilon\tau)}{\partial t'}\Big|_{\varepsilon=0} \\
&= \left(\frac{\partial \phi^k}{\partial t} + \varepsilon \frac{\partial \varphi^k}{\partial t} \right)\left(1 - \varepsilon \frac{\partial \tau}{\partial(t + \varepsilon\tau)} \right)\Big|_{\varepsilon=0} = \frac{\partial \phi^k}{\partial t} = (\phi^k)^{\bullet} ,
\end{aligned} \tag{28}$$

where $t' = t + \varepsilon\tau$ has been explicitly inserted, and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \frac{\partial \phi^{k'}}{\partial t'}\Big| = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\left(\frac{\partial \phi^k}{\partial t} + \varepsilon \frac{\partial \varphi^k}{\partial t} \right)\left(1 - \varepsilon \frac{\partial \tau}{\partial(t + \varepsilon\tau)} \right) \right)\Big|_{\varepsilon=0} = -\frac{\partial \phi^k}{\partial t} \frac{\partial \tau}{\partial t} + \frac{\partial \varphi^k}{\partial t} = (\varphi^k)^{\bullet} - (\phi^k)^{\bullet}\tau^{\bullet} . \tag{29}$$

Analogously, we acquire

$$\begin{aligned}
\frac{\partial \phi^{k'}}{\partial x^{i'}}\Big|_{\varepsilon=0} &= \frac{\partial(\phi^k + \varepsilon\varphi^k)}{\partial x^j} \frac{\partial x^j}{\partial x^{i'}}\Big|_{\varepsilon=0} = \frac{\partial(\phi^k + \varepsilon\varphi^k)}{\partial x^j} \frac{\partial(x^{j'} - \varepsilon\xi^j)}{\partial x^{i'}}\Big|_{\varepsilon=0} \\
&= \left(\frac{\partial \phi^k}{\partial x^j} + \varepsilon \frac{\partial \varphi^k}{\partial x^j} \right)\left(\delta_i^j - \varepsilon \frac{\partial \xi^j}{\partial(x^i + \varepsilon\xi^i)} \right)\Big|_{\varepsilon=0} = \frac{\partial \phi^k}{\partial x^i} = \phi^k_{,i} ,
\end{aligned} \tag{30}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \frac{\partial \phi^{k'}}{\partial x^{i'}}\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\left(\frac{\partial \phi^k}{\partial x^j} + \varepsilon \frac{\partial \varphi^k}{\partial x^j} \right)\left(\delta_i^j - \varepsilon \frac{\partial \xi^j}{\partial(x^i + \varepsilon\xi^i)} \right) \right)\Big|_{\varepsilon=0} = -\frac{\partial \phi^k}{\partial x^j} \frac{\partial \xi^j}{\partial x^i} + \frac{\partial \varphi^k}{\partial x^i} = \varphi^k_{,i} - \phi^k_{,j}\xi^j_{,i} . \tag{31}$$

The first term in Equation (27) reads

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}'}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} &= \frac{\mathrm{d}\mathcal{L}\left(t', x^{i'}; \phi^{k'}, \frac{\partial \phi^{k'}}{\partial t'}, \frac{\partial \phi^{k'}}{\partial x^{i'}} \right)}{\mathrm{d}\varepsilon}\Bigg|_{\varepsilon=0} \\
&= \left(\frac{\partial \mathcal{L}}{\partial t'}\tau + \frac{\partial \mathcal{L}}{\partial x^{i'}}\xi^i + \frac{\partial \mathcal{L}}{\partial \phi^{k'}}\varphi^k + \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi^{k'}}{\partial t'}} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\partial \phi^{k'}}{\partial t'} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi^{k'}}{\partial x^{i'}}} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\partial \phi^{k'}}{\partial x^{i'}} \right)\Bigg|_{\varepsilon=0}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\partial \mathcal{L}}{\partial(t + \varepsilon\tau)}\tau + \frac{\partial \mathcal{L}}{\partial(x^i + \varepsilon\xi^i)}\xi^i + \frac{\partial \mathcal{L}}{\partial(\phi^k + \varepsilon\varphi^k)}\varphi^k \right)\bigg|_{\varepsilon=0} \\
&\quad + \frac{\partial \mathcal{L}}{\partial(\phi^k)^\bullet}\big((\varphi^k)^\bullet - (\phi^k)^\bullet\tau^\bullet\big) + \frac{\partial \mathcal{L}}{\partial\phi^k_{,i}}\big(\varphi^k_{,j}\delta^j_i - \phi^k_{,j}\xi^j_{,i}\big) \\
&= \frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^i}\xi^i + \frac{\partial \mathcal{L}}{\partial\phi^k}\varphi^k + \frac{\partial \mathcal{L}}{\partial(\phi^k)^\bullet}\big((\varphi^k)^\bullet - (\phi^k)^\bullet\tau^\bullet\big) + \frac{\partial \mathcal{L}}{\partial\phi^k_{,i}}\big(\varphi^k_{,i} - \phi^k_{,j}\xi^j_{,i}\big).
\end{aligned} \tag{32}$$

Now by inserting the latter into Equation (27), we find the RUND–TRAUTMAN identity:

$$\tau\frac{\partial \mathcal{L}}{\partial t} + \xi^i\frac{\partial \mathcal{L}}{\partial x^i} + \varphi^k\frac{\partial \mathcal{L}}{\partial\phi^k} + \tau^\bullet\left(\mathcal{L} - (\phi^k)^\bullet\frac{\partial \mathcal{L}}{\partial(\phi^k)^\bullet}\right) + \xi^j_{,i}\left(\mathcal{L}\delta^i_j - \phi^k_{,j}\frac{\partial \mathcal{L}}{\partial\phi^k_{,i}}\right) + (\varphi^k)^\bullet\frac{\partial \mathcal{L}}{\partial(\phi^k)^\bullet} + \varphi^k_{,i}\frac{\partial \mathcal{L}}{\partial\phi^k_{,i}} = \mp F_k\varphi^k. \tag{33}$$

This RUND–TRAUTMAN identity [73–75] is general. In the following, we analyze this identity term-by-term and discuss some simplifications leading to well-known scenarios:

1. First term: When LAGRANGEan does not depend on time, $\partial\mathcal{L}/\partial t = 0$, and the right-hand side vanishes, $F_k = 0$, then arbitrary transformations in time are allowed and $\mathcal{L}$ is conserved. This property is known as "constant energy."
2. Second term: In the case of homogeneity—LAGRANGEan is constant in space, x^k, thus, the second term vanishes— arbitrary transformation in space is allowed in the case of vanishing right-hand side, $F_k = 0$. The simple example is a free motion of a rigid body.
3. Third term: If LAGRANGEan does not depend on primitive fields, ϕ^k, leading to, $\partial\mathcal{L}/\partial\phi^k = 0$, then no supply or volumetric terms apply. In mechanics, supply is because of gravitational or electromagnetic fields.
4. Fourth and fifth terms: Whenever the generator of time transformation depends on time or the generator of space transformation on space, we have a term called energy-momentum tensor if time and space are written together. We discuss these terms in the following in more detail. In mechanics, this term is often simplified and not discussed.
5. The terms $\partial\mathcal{L}/\partial(\phi^k)^\bullet$ and $\partial\mathcal{L}/\partial\phi^k_{,i}$ are called the conjugated momenta, however, in case of fields this name may be misleading.

The RUND–TRAUTMAN identity is an extension to the classical NOETHERian approach, we may claim that this identity is one step before obtaining "conservation laws." Especially the fourth term is important to notice: multiplied by a minus, it is often introduced as HAMILTONian of the system

$$H = -\mathcal{L} + (\phi^k)^\bullet\frac{\partial \mathcal{L}}{\partial(\phi^k)^\bullet}. \tag{34}$$

Herein, we see that the term is motivated by the RUND–TRAUTMAN identity. The latter definition for the HAMILTONian is numerically equal to the canonical HAMILTONian in a LAGRANGEan formulation [76]. We emphasize that these concepts of LAGRANGEan and HAMILTONian are used for systems with t as the only independent variable. We continue the NOETHERian formalism in the following with time and space as independent variables. First, we combine all of the independent variables together as a set $a^\nu = \{t, x^1, x^2, x^3\}$. We may even think of $\nu = 0, 1, 2, 3$ in order to have $a^0 = t$ and $a^1 = x^1, a^2 = x^2, a^3 = x^3$, for a simpler analogy. Second, the linear transformation is rewritten, $a^{\nu'} = a^\nu + \varepsilon\alpha^\nu$. Third, we write the RUND–TRAUTMAN identity once more for $\mathcal{L}(a^\nu; \phi^k(a^\nu), \phi^k_{,\mu}(a^\nu))$ in this notation,

$$\alpha^\nu\mathcal{L}_{,\nu} + \varphi^k\frac{\partial \mathcal{L}}{\partial\phi^k} + \alpha^\mu_{,\nu}\left(\mathcal{L}\delta^\nu_\mu - \phi^k_{,\mu}\frac{\partial \mathcal{L}}{\partial\phi^k_{,\nu}}\right) + \varphi^k_{,\mu}\frac{\partial \mathcal{L}}{\partial\phi^k_{,\mu}} = \mp F_k\varphi^k, \tag{35}$$

where we use $()_{,\nu} = \partial()/\partial a^\nu$. Now we define the energy-momentum tensor just by rewriting the third term in the latter,

$$\mathcal{H}_\mu{}^\nu = -\mathcal{L}\delta^\nu_\mu + \phi^k_{,\mu}\frac{\partial \mathcal{L}}{\partial\phi^k_{,\nu}}. \tag{36}$$

We observe that the LAGRANGE function, $\mathcal{L}$, and its invariance is more fundamental than a theory based on the HAMILTON function, $H = \mathcal{H}_0^{\,0}$, because it fails to have an invariance property in general. Indeed the "energy-momentum" tensor in Equation (36) is a generalization of the HAMILTONian in Equation (34). The energy-momentum tensor is composed of HAMILTONian in its time (scalar) part, H, and ESHELBY stress tensor in its space part $\mathcal{H}_j^{\,i}$. The term $\mathscr{P}_k^{\,\nu} = \partial\mathcal{L}/\partial\phi_{,\mu}^k$ is called a canonical momentum in the case of the rigid body motion, where $\mathcal{L}$ incorporates the kinetic energy without deformation energy. We prefer to call it a conjugate term, for example, in elasticity, stress is energetic conjugate of strain that is the space derivative of the primitive variable, which is displacement. In this analogy, the space part of the energy-momentum tensor, $\mathcal{H}_j^{\,i}$, is often introduced by a so-called LEGENDRE transformation. Herein, we realize that the same term is directly generated by the NOETHER formalism. The result has been obtained in [60]; however, the condition of functional $\mathcal{I}$ being extremal has also been used. The procedure herein is different and less restrictive since the energy-momentum tensor asserts only the invariance of the LAGRANGEan but not its density. The invariance and extremality are separate properties, until now, we have only used invariance.

4 | CONSERVATION LAWS

We will derive conservation laws in two steps, first, we obtain the so-called EULER–LAGRANGE equations, second, we use them in the RUND–TRAUTMAN identity in order to obtain the conservation laws. They are the balance laws derived by using the formalism herein. We begin with $\alpha^\nu \equiv 0$, where this restriction means that we neglect, for the moment, any shift of the independent variables, thus no time and space variation. First and third terms vanish in Equation (35) and we obtain

$$\varphi^k \frac{\partial\mathcal{L}}{\partial\phi^k} + \varphi_{,\mu}^k \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}^k} = \mp F_k \varphi^k \, . \tag{37}$$

This identity is rewritten in an integral form over a space-time domain Ω and integrated by parts,

$$\int_\Omega \left(\varphi^k \frac{\partial\mathcal{L}}{\partial\phi^k} + \varphi_{,\mu}^k \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}^k} \pm F_k \varphi^k \right) \mathrm{d}\Sigma = 0 \, ,$$

$$\int_\Omega \left(\varphi^k \frac{\partial\mathcal{L}}{\partial\phi^k} - \varphi^k \left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}^k} \right)_{,\mu} \pm F_k \varphi^k \right) \mathrm{d}\Sigma + \oint_{\partial\Omega} \varphi^k \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}^k} \, \mathrm{d}S_\mu = 0 \, , \tag{38}$$

where the surface integral, $\mathrm{d}S_\mu = n_\mu \, \mathrm{d}S$, is computed on the boundaries with surface normal, n_μ, this normal comprises space—on boundaries of the continuum body, n_i is known as the outer normal—and time (initial and end time). We are interested in a differential equation within the domain, in other words, values at boundaries are given; no variation is needed, $\varphi^k = 0$, $\forall a^\mu \in \partial\Omega$. Thus the last integral vanishes and the well-known EULER–LAGRANGE equations appear

$$\frac{\partial\mathcal{L}}{\partial\phi^k} - \left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}^k} \right)_{,\mu} = \mp F_k \, . \tag{39}$$

This equation is well-known, for example given as the integral LAGRANGE–D'ALEMBERT principle in [77], Definition 7.8.4 for discrete systems. We assert it herein as an additional condition to the RUND–TRAUTMAN identity. It is also called an extremal principle, since in Equation (24), the directional derivative vanishes, such that the condition, $\mathcal{L}'J - \mathcal{L} \pm \delta A$, is evaluated at its extremum (minimum, maximum, or saddle point). We realize that for a specific case of no space and time translation, all methods are identical, which we will discuss further in an application. Often, the right-hand side is neglected in fields, we refer to [78] for a connection of the right-hand side with RAYLEIGH dissipation function. We emphasize that we have neglected a reference frame evolution that is an irreversible phenomenon; for an example, we refer to the Appendix with an application. This EULER–LAGRANGE equation is obtained by using an invariant but it does not mean that the EULER–LAGRANGE equation remains the same under coordinate transformations. Their form changes; balance of momentum is a typical example in mechanics, under a proper coordinate transformation, additional terms emerge.

If there is a coordinate transformation leaving the EULER–LAGRANGE equation unchanged then this transformation is called *invariant transformation*. Herein, we use a formalism for arbitrary transformations.

As the right-hand side is a (non-conservative) force, it is rank 1. Analogous to the aforementioned case, without introducing any assumption or reduction, we may deduce a rank 1 tensor from a rank 2 tensor by (space and time) derivative

$$\frac{\partial \mathcal{L}}{\partial \phi^k} - \mathscr{P}_{k,\mu}^{\ \mu} = S_{k,\mu}^{\ \mu}\,, \quad \mathscr{P}_k^{\ \mu} = \frac{\partial \mathcal{L}}{\partial \phi^k_{,\mu}}\,. \tag{40}$$

For the derivation, we have used an oblique but not curvilinear coordinate system (we refer to [67], Section 43], [40] for a generic derivation with CHRISTOFFEL symbols). Now we start with Equation (36) and obtain, by using $\mathcal{L} = \mathcal{L}(a^\mu; \phi^k(a^\mu), \phi^k_{,\nu}(a^\mu))$ and chain rule,

$$\mathcal{H}_{\nu,\rho}^{\ \rho} = \left(-\mathcal{L}\delta_\nu^\rho + \phi^k_{,\nu}\mathscr{P}_k^{\ \rho}\right)_{,\rho} = -\mathcal{L}_{,\nu} - \phi^k_{,\nu}\frac{\partial \mathcal{L}}{\partial \phi^k} - \phi^k_{,\mu\nu}\mathscr{P}_k^{\ \mu} + \left(\phi^k_{,\nu}\mathscr{P}_k^{\ \rho}\right)_{,\rho} = -\mathcal{L}_{,\nu} - \phi^k_{,\nu}\frac{\partial \mathcal{L}}{\partial \phi^k} + \phi^k_{,\nu}\mathscr{P}_{k,\rho}^{\ \rho}\,. \tag{41}$$

By inserting the latter in the RUND–TRAUTMAN identity in Equation (35), we acquire

$$\alpha^\nu \mathcal{L}_{,\nu} + \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k} - \alpha^\mu_{,\nu}\mathcal{H}_\mu^{\ \nu} + \varphi^k_{,\mu}\mathscr{P}_k^{\ \mu} = \varphi^k S_{k,\mu}^{\ \mu}\,,$$

$$\alpha^\nu\left(\mathcal{L}_{,\nu} + \mathcal{H}_{\nu,\rho}^{\ \rho}\right) + \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k} - \left(\alpha^\mu \mathcal{H}_\mu^{\ \nu}\right)_{,\nu} + \left(\varphi^k \mathscr{P}_k^{\ \mu}\right)_{,\mu} - \varphi^k \mathscr{P}_{k,\mu}^{\ \mu} = \varphi^k S_{k,\mu}^{\ \mu}\,, \tag{42}$$

$$-\alpha^\nu \phi^k_{,\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \mathscr{P}_{k,\rho}^{\ \rho}\right) + \varphi^k\left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \mathscr{P}_{k,\mu}^{\ \mu} - S_{k,\mu}^{\ \mu}\right) + \left(\varphi^k \mathscr{P}_k^{\ \nu} - \alpha^\mu \mathcal{H}_\mu^{\ \nu}\right)_{,\nu} = 0\,.$$

By assuming that EULER–LAGRANGE equations hold, we insert Equation (40) and obtain

$$\left(\varphi^k \mathscr{P}_k^{\ \nu} - \alpha^\mu \mathcal{H}_\mu^{\ \nu}\right)_{,\nu} = \alpha^\nu \phi^k_{,\nu} S_{k,\mu}^{\ \mu}\,. \tag{43}$$

By renaming the left-hand side as NOETHER's current, $\mathcal{J}^\mu$, we obtain the corresponding equation:

$$\mathcal{J}^\mu_{,\mu} = \alpha^\nu \phi^k_{,\nu} S_{k,\mu}^{\ \mu}\,, \quad \mathcal{J}^\mu = \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k_{,\mu}} - \alpha^\nu \mathcal{H}_\nu^{\ \mu}\,. \tag{44}$$

In the case of vanishing right-hand side, $S_{k,\mu}^{\ \mu} = \pm F_k = 0$, the latter "balance" equations are called conservation laws, $\mathcal{J}^\mu_{,\mu} = 0$.

5 | ELASTODYNAMICS

In order to demonstrate the meaning of conservation laws, we give an example in elastodynamics. The primitive variables, ϕ^k, are the components of the displacement field, u_1, u_2, u_3, expressed in Cartesian coordinates. Reference frame is not evolving such that we have a reversible process. Let us use the following LAGRANGEan density:

$$\mathcal{L} = \frac{1}{2}\tilde{C}_{i\mu k\nu}u_{i,\mu}u_{k,\nu}\,, \quad \tilde{C}_{i\mu k\nu} = \begin{cases} \rho_{\text{ref.}}\delta_{ik} & \text{if } \mu = 0, \nu = 0\,, \\ 0 & \text{if } \mu = 0, \nu \neq 0 \text{ or } \mu \neq 0, \nu = 0\,, \\ -C_{ijkl} & \text{if } \mu = j, \nu = l\,, \end{cases} \tag{45}$$

where the stiffness tensor, C_{ijkl}, is given for the corresponding material. Mass density, $\rho_{\text{ref.}}$, is defined on the reference frame, thus, it is constant in time. In the case of linear and isotropic material, the stiffness tensor reads $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk}$, where the so-called LAME parameters are given by engineering constants; YOUNG's modulus, E, and POISSON's ratio, ν, as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,, \quad \mu = \frac{E}{2(1+\nu)}\,. \tag{46}$$

We separate time, $\mu = 0$, and space, $\mu = \{1, 2, 3\} = i$, for a direct analogy with linear elasticity theory. Hence, the aforementioned LAGRANGEan density in space and time reads

$$\mathcal{L} = \frac{1}{2}\rho_{\text{ref.}}u_i^{\bullet}u_i^{\bullet} - w \,, \quad w = \frac{1}{2}u_{i,j}C_{ijkl}u_{k,l} \,, \tag{47}$$

with the deformation (or stored) energy density, w in J/m^3. Instead of starting with Equation(45), we may introduce LAGRANGEan density as kinetic energy density minus deformation energy density. Albeit not immediately obvious, we use a linear strain measure, $E_{ij} = 1/2(u_{i,j} + u_{j,i})$, thus, the stored energy density, $w = 1/2E_{ij}C_{ijkl}E_{kl}$, is objective and reduces to the quadratic energy, $w = 1/2u_{i,j}C_{ijkl}u_{k,l}$, effected by the minor symmetries, $C_{ijkl} = C_{jikl} = C_{ijlk}$. The formulation is analogous for finite strain formulation, for demonstrating this, we use a stress measure in the following. By choosing the deformation energy differently, for example as known in hyperelasticity, stress is calculated by its derivative with respect to displacement gradient.

In the case of vanishing viscous effects, $F_k = 0$, we rewrite the conservation laws in Equation (44), as follows:

$$\mathcal{J}_0^{\bullet} + \mathcal{J}_{i,i} = 0 \,, \quad \mathcal{J}_0 = \varphi_k \frac{\partial \mathcal{L}}{\partial u_k^{\bullet}} - \tau \mathcal{H}_{00} - \xi_i \mathcal{H}_{i0} \,, \quad \mathcal{J}_i = \varphi_k \frac{\partial \mathcal{L}}{\partial u_{k,i}} - \tau \mathcal{H}_{0i} - \xi_j \mathcal{H}_{ji} \,. \tag{48}$$

The whole formulation is in material frame, although for simplicity, we ignore the difference between reference and current frame. The energy-momentum tensor reads with a time part called HAMILTONian and a space part called ESHELBY stress tensor

$$\mathcal{H}_{\alpha\beta} = -\mathcal{L}\delta_{\alpha\beta} + u_{k,\alpha}\frac{\partial \mathcal{L}}{\partial u_{k,\beta}} \,. \tag{49}$$

We use its counterpart in space and time

$$\begin{aligned}
\mathcal{H}_{00} &= -\mathcal{L} + u_k^{\bullet}\frac{\partial \mathcal{L}}{\partial u_k^{\bullet}} = -\frac{1}{2}\rho_{\text{ref.}}u_i^{\bullet}u_i^{\bullet} + w + u_k^{\bullet}\rho_{\text{ref.}}u_k^{\bullet} = \frac{1}{2}\rho_{\text{ref.}}u_i^{\bullet}u_i^{\bullet} + w \,, \\
\mathcal{H}_{0i} &= u_k^{\bullet}\frac{\partial \mathcal{L}}{\partial u_{k,i}} = u_k^{\bullet}\mathcal{P}_{ki} \,, \quad \mathcal{P}_{ki} = \frac{\partial \mathcal{L}}{\partial u_{k,i}} = -\frac{\partial w}{\partial u_{k,i}} = -C_{kilm}u_{l,m} = -\sigma_{ki} \,, \\
\mathcal{H}_{i0} &= u_{k,i}\frac{\partial \mathcal{L}}{\partial u_k^{\bullet}} = u_{k,i}\rho_{\text{ref.}}u_k^{\bullet} \,, \\
\mathcal{H}_{ij} &= -\mathcal{L}\delta_{ij} + u_{k,i}\frac{\partial \mathcal{L}}{\partial u_{k,j}} = -\frac{1}{2}\rho_{\text{ref.}}u_k^{\bullet}u_k^{\bullet}\delta_{ij} + w\delta_{ij} + u_{k,i}\mathcal{P}_{kj} \,.
\end{aligned} \tag{50}$$

NOETHER currents become

$$\begin{aligned}
\mathcal{J}_0 &= \varphi_k\rho_{\text{ref.}}u_k^{\bullet} - \frac{\tau}{2}\rho_{\text{ref.}}u_i^{\bullet}u_i^{\bullet} - \tau w - \xi_i u_{k,i}\rho_{\text{ref.}}u_k^{\bullet} \,, \\
\mathcal{J}_i &= \varphi_k\mathcal{P}_{ki} - \tau u_k^{\bullet}\mathcal{P}_{ki} + \xi_j\left(\frac{1}{2}\rho_{\text{ref.}}u_k^{\bullet}u_k^{\bullet}\delta_{ji} - w\delta_{ji} - u_{k,j}\mathcal{P}_{ki}\right) \,.
\end{aligned} \tag{51}$$

Technically, $\mathcal{P}_{ij}$ is the minus nominal stress or minus transpose of PIOLA stress, indeed, in small strain assumption that corresponds to the minus CAUCHY stress. The conservation laws,

$$\begin{aligned}
0 &= \mathcal{J}_0^{\bullet} + \mathcal{J}_{i,i} \\
0 &= \left(\varphi_k\rho_{\text{ref.}}u_k^{\bullet} - \frac{\tau}{2}\rho_{\text{ref.}}u_i^{\bullet}u_i^{\bullet} - \tau w - \xi_i u_{k,i}\rho_{\text{ref.}}u_k^{\bullet}\right)^{\bullet} \\
&\quad + \left(\varphi_k\mathcal{P}_{ki} - \tau u_k^{\bullet}\mathcal{P}_{ki} + \xi_j\left(\frac{1}{2}\rho_{\text{ref.}}u_k^{\bullet}u_k^{\bullet}\delta_{ji} - w\delta_{ji} - u_{k,j}\mathcal{P}_{ki}\right)\right)_{,i}
\end{aligned} \tag{52}$$

may be rewritten as follows, as they hold for arbitrary transformations,

$$
\begin{aligned}
0 =& \varphi_k \big(\rho_{\text{ref.}} u_k^{\bullet\bullet} + \mathscr{P}_{ki,i} \big) - \tau \left(\left(\frac{1}{2} \rho_{\text{ref.}} u_i^\bullet u_i^\bullet + w \right)^\bullet + \big(u_k^\bullet \mathscr{P}ki \big)_{,i} \right) \\
&+ \xi_j \left(- \big(\rho_{\text{ref.}} u_k^\bullet u_{k,j} \big)^\bullet + \left(\frac{1}{2} \rho_{\text{ref.}} u_k^\bullet u_k^\bullet \delta_{ji} - w \delta_{ji} - u_{k,j} \mathscr{P}_{ki} \right)_{,i} \right) \\
&+ \varphi_k^\bullet \rho_{\text{ref.}} u_k^\bullet - \tau^\bullet \left(\frac{1}{2} \rho_{\text{ref.}} u_i^\bullet u_i^\bullet + w \right) - \xi_j^\bullet \rho_{\text{ref.}} u_k^\bullet u_{k,j} + \varphi_{k,i} \mathscr{P}_{ki} \\
&- \tau_{,i} u_k^\bullet \mathscr{P}_{ki} + \xi_{j,i} \left(\frac{1}{2} \rho_{\text{ref.}} u_k^\bullet u_k^\bullet \delta_{ji} - w \delta_{ji} - u_{k,j} \mathscr{P}_{ki} \right) .
\end{aligned}
\tag{53}
$$

We stress that the transformations in Equation (23) may be chosen in such a way that the above equation holds. In this way, it is possible to formulate an inverse problem and search for possible transformations by solving so-called KILLING equations from the latter. A direct problem is to test different transformations and find out the consequence of invariance and extremality of $\mathcal{I}$. The former has brought us the RUND–TRAUTMAN identity and the latter EULER–LAGRANGE equations. By using both of them, we have obtained NOETHER's currents leading to Equation (53). If we examine a transformation in displacement field, $\varphi_k = $ const, and insert it into the latter, we obtain the balance of linear momentum,

$$
0 = \varphi_k \big(\rho_{\text{ref.}} u_k^{\bullet\bullet} + \mathscr{P}_{ki,i} \big) .
\tag{54}
$$

We emphasize that $\mathscr{P}_{ij} = -\sigma_{ji}$ in small deformation assumption. Since the transformation is constant in space and time, such a displacement is called a rigid body motion. We may write the result on material frame for a domain $\mathcal{B}$, with its closure (smooth boundary) $\partial \mathcal{B}$, after applying GAUSS–OSTROGRADSKIY's theorem,

$$
\left(\int_{\mathcal{B}} \rho_{\text{ref.}} u_k^\bullet \, \mathrm{d}V \right)^\bullet = \int_{\partial \mathcal{B}} \bar{t}_k \, \mathrm{d}A , \quad \bar{t}_k = -\mathscr{P}_{ki} n_i = \sigma_{ik} n_i ,
\tag{55}
$$

where the traction vector $\bar{t}_i$ is defined on the boundary. Without using CAUCHY's tetrahedron argumentation or usual balance equations' argumentation, we obtain the result as a consequence of the transformation rule. The justification is obvious that we search for laws holding under rigid body translations. It is possible to call that a chosen symmetry, $\varphi_k = $ const, generates a balance law. This result is not an additional balance law, since we have to satisfy EULER–LAGRANGE equations,

$$
\frac{\partial \mathcal{L}}{\partial u_k} - \mathscr{P}_{ki,i} - \mathscr{P}_{k0}^\bullet = 0 , \quad \frac{\partial \mathcal{L}}{\partial u_k} = 0 , \quad \mathscr{P}_{k0} = \rho_{\text{ref.}} u_k^\bullet , \quad \mathscr{P}_{ki} = -\frac{\partial w}{\partial u_{k,i}} = -\sigma_{ki} ,
\tag{56}
$$

leading to the same governing equation. This consequence is in fact the aforementioned relation that RUND–TRAUTMAN identity reduces to the EULER–LAGRANGE equations for the case of no space and time variations. More interestingly, now we may easily examine other transformation rules in order to acquire additional governing equations.

Analogously, we may examine a time translation, $\tau = $ const, in order to obtain the balance of energy with the total specific (per mass) energy, e in J/kg,

$$
(\rho_{\text{ref.}} e \, \mathrm{d}V)^\bullet = \int_{\partial \mathcal{B}} u_i^\bullet \bar{t}_i \, \mathrm{d}A , \quad e = \frac{1}{2} u_i^\bullet u_i^\bullet + \frac{w}{\rho_{\text{ref.}}} .
\tag{57}
$$

A GALILEIan transformation, for example $\xi_i = $ const, reads

$$
\left(\int_{\mathcal{B}} \rho_{\text{ref.}} u_k^\bullet u_{k,j} \, \mathrm{d}V \right)^\bullet = \int_{\partial \mathcal{B}} \big(n_j \mathcal{L} + u_{k,j} \bar{t}_k \big) \, \mathrm{d}A .
\tag{58}
$$

This new balance law reduces to the well-known J-integral with the ESHELBY stress tensor,

$$
0 = \int_{\partial \mathcal{B}} \big(-n_j w + u_{k,j} \bar{t}_k \big) \, \mathrm{d}A ,
\tag{59}
$$

for the stationary case. If the integral is taken around a crack tip, the value of this integral is seen as an energy release rate for forming a discontinuity (propagating a crack) [79–81]. Herein, its dynamical counterpart is acquired by the NOETHER formalism, so its interpretation and use are more obvious. In general, we may skip this balance law and hope that it is fulfilled, but actually, adding an additional restriction is a remedy in numerical accuracy related problems [82]. More different transformations may be examined, for example scaling or rotation (leading to the balance of angular momentum) is studied in [83]. A generalization of this formalism for thermoelasticity is possible as in [84].

We have obtained two governing equations to be fulfilled in an isothermal case. One is the term multiplied by φ_k and the other is the term multiplied by ξ_i.

6 | GENERALIZED CONTINUA

Without repeating all the analysis, we now address the case if the LAGRANGEan depends also on the second derivative $\mathcal{L}(a^\nu; \phi^k(a^\nu), \phi^k_{,\mu}(a^\nu), \phi^k_{,\mu\gamma}(a^\nu))$. As the transformation is still linear, we have extra terms in Equation (32) appearing for the term multiplied by α^μ leading to the RUND–TRAUTMAN identity for generalized continua,

$$\alpha^\nu \mathcal{L}_{,\nu} + \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k} + \alpha^\mu_{,\nu}\left(\mathcal{L}\delta^\nu_\mu - \phi^k_{,\mu}\frac{\partial \mathcal{L}}{\partial \phi^k_{,\nu}} - \phi^k_{,\gamma\mu}\frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\nu}}\right) + \varphi^k_{,\mu}\frac{\partial \mathcal{L}}{\partial \phi^k_{,\mu}} + \varphi^k_{,\gamma\mu}\frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\mu}} = \varphi^k S^\mu_{k,\mu}, \tag{60}$$

Therefore, the energy-momentum tensor is renewed

$$\mathcal{H}_\mu^\nu = -\mathcal{L}\delta^\nu_\mu + \phi^k_{,\mu}\frac{\partial \mathcal{L}}{\partial \phi^k_{,\nu}} + \phi^k_{,\gamma\mu}\frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\nu}}. \tag{61}$$

With the same integral form as in Equation (38) and in this round twice-integrating by parts, we obtain EULER–LAGRANGE equations in generalized continua

$$\frac{\partial \mathcal{L}}{\partial \phi^k} - \left(\frac{\partial \mathcal{L}}{\partial \phi^k_{,\mu}}\right)_{,\mu} + \left(\frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\mu}}\right)_{,\gamma\mu} = S^\mu_{k,\mu}. \tag{62}$$

For a direct analogy, we utilize the same notation as in the previous section with an extension,

$$\mathcal{P}_k^{\ \mu} = \frac{\partial \mathcal{L}}{\partial \phi^k_{,\mu}}, \quad \mathcal{R}_k^{\gamma\mu} = \frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\mu}}, \tag{63}$$

and rewrite the EULER–LAGRANGE equations:

$$\frac{\partial \mathcal{L}}{\partial \phi^k} - \mathcal{P}_{k\ ,\mu}^{\ \mu} + \mathcal{R}_{k\ ,\gamma\mu}^{\gamma\mu} = S^\mu_{k,\mu}. \tag{64}$$

We repeat the same procedure

$$\begin{aligned}
\mathcal{H}_{\nu,\rho}^{\ \rho} &= \left(-\mathcal{L}\delta^\rho_\nu + \phi^k_{,\nu}\mathcal{P}_k^{\ \rho} + \phi^k_{,\gamma\nu}\mathcal{R}_k^{\gamma\rho}\right)_{,\rho} \\
&= -\mathcal{L}_{,\nu} - \phi^k_{,\nu}\frac{\partial \mathcal{L}}{\partial \phi^k} - \phi^k_{,\mu\nu}\mathcal{P}_k^{\ \mu} - \phi^k_{,\gamma\mu\nu}\mathcal{R}_k^{\gamma\mu} + \left(\phi^k_{,\nu}\mathcal{P}_k^{\ \rho} + \phi^k_{,\gamma\nu}\mathcal{R}_k^{\ \gamma\rho}\right)_{,\rho} \\
&= -\mathcal{L}_{,\nu} - \phi^k_{,\nu}\frac{\partial \mathcal{L}}{\partial \phi^k} + \phi^k_{,\nu}\mathcal{P}_{k,\rho}^{\ \rho} + \phi^k_{,\gamma\nu}\mathcal{R}_{k\ ,\rho}^{\gamma\rho} \\
&= -\mathcal{L}_{,\nu} - \phi^k_{,\nu}\frac{\partial \mathcal{L}}{\partial \phi^k} + \phi^k_{,\nu}\mathcal{P}_{k,\rho}^{\ \rho} + \left(\phi^k_{,\nu}\mathcal{R}_{k\ ,\rho}^{\gamma\rho}\right)_{,\gamma} - \phi^k_{,\nu}\mathcal{R}_{k\ ,\gamma\rho}^{\gamma\rho}.
\end{aligned} \tag{65}$$

By using the latter in the RUND–TRAUTMAN identity in Equation (60), we obtain

$$\alpha^\nu \mathcal{L}_{,\nu} + \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k} - \alpha^\mu_{,\nu} \mathcal{H}^\nu_\mu + \varphi^k_{,\mu} \mathcal{P}^\mu_k + \varphi^k_{,\gamma\mu} \mathcal{R}^{\gamma\mu}_k = \varphi^k S^\mu_{k,\mu} \, ,$$

$$\alpha^\nu \left(\mathcal{L}_{,\nu} + \mathcal{H}^\rho_{\nu,\rho} \right) + \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k} - \left(\alpha^\mu \mathcal{H}^\nu_\mu \right)_{,\nu} + \left(\varphi^k \mathcal{P}^\mu_k \right)_{,\mu} - \varphi^k \mathcal{P}^\mu_{k,\mu} +$$
$$+ \left(\varphi^k_{,\gamma} \mathcal{R}^{\gamma\mu}_k \right)_{,\mu} - \varphi^k_{,\gamma} \mathcal{R}^{\gamma\mu}_{k,\mu} = \varphi^k S^\mu_{k,\mu} \, ,$$

$$\alpha^\nu \left(\mathcal{L}_{,\nu} + \mathcal{H}^\rho_{\nu,\rho} \right) + \varphi^k \frac{\partial \mathcal{L}}{\partial \phi^k} - \left(\alpha^\mu \mathcal{H}^\nu_\mu \right)_{,\nu} + \left(\varphi^k \mathcal{P}^\mu_k \right)_{,\mu} - \varphi^k \mathcal{P}^\mu_{k,\mu} +$$
$$+ \left(\varphi^k_{,\gamma} \mathcal{R}^{\gamma\mu}_k \right)_{,\mu} - \left(\varphi^k \mathcal{R}^{\gamma\mu}_{k,\mu} \right)_{,\gamma} + \varphi^k \mathcal{R}^{\gamma\mu}_{k,\gamma\mu} = \varphi^k S^\mu_{k,\mu} \qquad (66)$$

which is rewritten, as follows:

$$- \alpha^\nu \phi^k_{,\nu} \left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \mathcal{P}^\rho_{k,\rho} + \mathcal{R}^{\gamma\rho}_{k,\gamma\rho} \right) + \varphi^k \left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \mathcal{P}^\mu_{k,\mu} + \mathcal{R}^{\gamma\mu}_{k,\gamma\mu} - S^\mu_{k,\mu} \right) +$$
$$+ \left(\varphi^k \left(\mathcal{P}^\nu_k - \mathcal{R}^{\nu\mu}_{k,\mu} \right) + \varphi^k_{,\gamma} \mathcal{R}^{\gamma\nu}_k - \alpha^\mu \mathcal{H}^\nu_\mu \right)_{,\nu} = 0 \, . \qquad (67)$$

Into the latter, we insert the generalized EULER–LAGRANGE equations in Equation (64) and obtain generalized NOETHER's current,

$$\left(\varphi^k \left(\mathcal{P}^\nu_k - \mathcal{R}^{\nu\mu}_{k,\mu} \right) + \varphi^k_{,\gamma} \mathcal{R}^{\gamma\nu}_k - \alpha^\mu \mathcal{H}^\nu_\mu \right)_{,\nu} = \alpha^\nu \phi^k_{,\nu} S^\mu_{k,\mu} \, ,$$

$$\mathcal{J}^\mu_{,\mu} = \alpha^\nu \phi^k_{,\nu} S^\mu_{k,\mu} \, , \quad \mathcal{J}^\mu = \varphi^k \left(\frac{\partial \mathcal{L}}{\partial \phi^k_{,\mu}} - \left(\frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\mu}} \right)_{,\gamma} \right) + \varphi^k_{,\gamma} \frac{\partial \mathcal{L}}{\partial \phi^k_{,\gamma\mu}} - \alpha^\nu \mathcal{H}^\mu_\nu \, . \qquad (68)$$

We follow the same guidelines and generalize to higher order continua for metamaterials. The primitive variable is again "only" the displacement, $\phi^k = u_k$, expressed in Cartesian coordinates. But now the second derivative plays a role as well, so we use the following LAGRANGEan density:

$$\mathcal{L} = \frac{1}{2} \tilde{C}_{i\mu k\nu} u_{i,\mu} u_{k,\nu} + \frac{1}{2} \tilde{D}_{i\mu\gamma l\nu\eta} u_{i,\mu\gamma} u_{l,\nu\eta} + \tilde{G}_{i\mu k\nu\eta} u_{i,\mu} u_{k,\nu\eta} \, ,$$

$$\tilde{C}_{i\mu k\nu} = \begin{cases} \rho_{\text{ref.}} \delta_{ik} & \text{if } \mu = \nu = 0 \, , \\ -C_{ijkl} & \text{if } \mu = j, \nu = l \, , \, , \\ 0 & \text{else} \, , \end{cases}$$

$$\tilde{D}_{i\mu\gamma l\nu\eta} = \begin{cases} \rho_{\text{ref.}} \tau^2_{\text{ref.}} \delta_{il} & \text{if } \mu = \nu = 0, \gamma = \eta = 0 \, , \\ \rho_{\text{ref.}} d^2_{\text{ref.}} \delta_{il} \delta_{jk} & \text{if } \mu = \nu = 0, \gamma = j, \eta = k \, , \\ -D_{ijklmn} & \text{if } \mu = j, \gamma = k, \nu = m, \eta = n \, , \\ 0 & \text{else} \, , \end{cases} \qquad (69)$$

$$\tilde{G}_{i\mu k\nu\eta} = \begin{cases} -G_{ijklm} & \text{if } \mu = j, \nu = l, \eta = m \, , \\ 0 & \text{else} \, , \end{cases}$$

where rank 4, 5, 6 tensors, C_{ijkl}, D_{ijklmn}, G_{ijklm}, are given for the corresponding metamaterial. Their measurement seems to be challenging [85–87], yet there exist different homogenization methods [88–93] that calculate these parameters [94–98]. Inertia related terms, $\rho_{\text{ref.}}$, $d_{\text{ref.}}$, $\tau_{\text{ref.}}$, are all defined on the reference frame. We redo the same analysis as

before and separate time and space, in order to obtain

$$\mathcal{L} = \frac{1}{2}\rho_{\text{ref.}}\left(u_i^{\bullet}u_i^{\bullet} + \tau_{\text{ref.}}^2 u_i^{\bullet\bullet}u_i^{\bullet\bullet} + d_{\text{ref.}}^2 u_{i,j}^{\bullet}u_{i,j}^{\bullet}\right) - w ,$$

$$w = \frac{1}{2}u_{i,j}C_{ijkl}u_{k,l} + \frac{1}{2}u_{i,jk}D_{ijklmn}u_{l,mn} + G_{ijklm}u_{i,j}u_{k,lm} . \tag{70}$$

In the case of elasticity, $S_{k,\mu}^{\mu} = 0$, we acquire the conservation laws in Equation (44), as follows:

$$\mathcal{J}_0^{\bullet} + \mathcal{J}_{i,i} = 0 ,$$

$$\mathcal{J}_0 = \varphi_k\left(\frac{\partial\mathcal{L}}{\partial u_k^{\bullet}} - \left(\frac{\partial\mathcal{L}}{\partial u_k^{\bullet\bullet}}\right)^{\bullet} - \left(\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\bullet}}\right)_{,i}\right) + \varphi_k^{\bullet}\frac{\partial\mathcal{L}}{\partial u_k^{\bullet\bullet}} + \varphi_{k,i}\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\bullet}} - \tau\mathcal{H}_{00} - \xi_i\mathcal{H}_{i0} ,$$

$$\mathcal{J}_i = \varphi_k\left(\frac{\partial\mathcal{L}}{\partial u_{k,i}} - \left(\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\bullet}}\right)^{\bullet} - \left(\frac{\partial\mathcal{L}}{\partial u_{k,ij}}\right)_{,j}\right) + \varphi_k^{\bullet}\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\bullet}} + \varphi_{k,j}\frac{\partial\mathcal{L}}{\partial u_{k,ij}} - \tau\mathcal{H}_{0i} - \xi_j\mathcal{H}_{ji} . \tag{71}$$

Since we are interested in the terms multiplied by φ_k and ξ_i, only the following terms are sought after

$$\varphi_k\left(\left(\frac{\partial\mathcal{L}}{\partial u_k^{\bullet}} - \left(\frac{\partial\mathcal{L}}{\partial u_k^{\bullet\bullet}}\right)^{\bullet} - \left(\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\bullet}}\right)_{,i}\right)^{\bullet} + \left(\frac{\partial\mathcal{L}}{\partial u_{k,i}} - \left(\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\bullet}}\right)^{\bullet} - \left(\frac{\partial\mathcal{L}}{\partial u_{k,ij}}\right)_{,j}\right)_{,i}\right) = 0 \tag{72}$$

and

$$-\xi_i\mathcal{H}_{i0}^{\bullet} - \xi_j\mathcal{H}_{ji,i} = 0 ,$$
$$\xi_i\left(\mathcal{H}_{i0}^{\bullet} + \mathcal{H}_{ij,j}\right) = 0 . \tag{73}$$

For the generalized elasticity, from the terms multiplied by φ_k, we acquire

$$\left(\rho_{\text{ref.}}u_k^{\bullet} - (\rho_{\text{ref.}}\tau_{\text{ref.}}^2 u_k^{\bullet\bullet})^{\bullet} - \left(\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,i}^{\bullet}\right)_{,i}\right)^{\bullet} + \left(-\frac{\partial w}{\partial u_{k,i}} - \left(\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,i}^{\bullet}\right)^{\bullet} + \left(\frac{\partial w}{\partial u_{k,ij}}\right)_{,j}\right)_{,i} = 0 ,$$

$$\rho_{\text{ref.}}u_k^{\bullet\bullet} - \rho_{\text{ref.}}\tau_{\text{ref.}}^2 u_k^{\bullet\bullet\bullet\bullet} - \left(\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,i}^{\bullet\bullet}\right)_{,i} + \left(-(C_{kijl}u_{j,l} + G_{kijlm}u_{j,lm}) - \rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,i}^{\bullet\bullet} + \right.$$

$$\left. +\left(D_{kijlmn}u_{l,mn} + u_{m,n}G_{mnkij}\right)_{,j}\right)_{,i} = 0 , \tag{74}$$

In the case of a homogeneous material, where $\rho_{\text{ref.}}$ is constant in space, and a centro-symmetric metamaterial, $G_{ijklm} = 0$, analogous to the previous case, we may use stress $\sigma_{kj} = C_{kjlm}u_{l,m}$ and double stress $m_{kij} = D_{kijlmn}u_{l,mn}$, in order to obtain

$$\rho_{\text{ref.}}u_k^{\bullet\bullet} - \rho_{\text{ref.}}\tau_{\text{ref.}}^2 u_k^{\bullet\bullet\bullet\bullet} - 2\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,ii}^{\bullet\bullet} - \sigma_{ki,i} + m_{kij,ji} = 0 . \tag{75}$$

For the generalized mechanics, the energy-momentum tensor becomes

$$\mathcal{H}_{\alpha\beta} = -\mathcal{L}\delta_{\alpha\beta} + u_{k,\alpha}\frac{\partial\mathcal{L}}{\partial u_{k,\beta}} + u_{k,\gamma\alpha}\frac{\partial\mathcal{L}}{\partial u_{k,\gamma\beta}} , \tag{76}$$

we write it in terms of space and time,

$$\mathcal{H}_{00} = -\mathcal{L} + u_k^{\cdot}\frac{\partial\mathcal{L}}{\partial u_k^{\cdot}} + u_k^{\cdot\cdot}\frac{\partial\mathcal{L}}{\partial u_k^{\cdot\cdot}} + u_{k,i}^{\cdot}\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\cdot}} = \frac{1}{2}\rho_{\text{ref.}}\left(u_i^{\cdot}u_i^{\cdot} + \tau_{\text{ref.}}^2 u_i^{\cdot\cdot}u_i^{\cdot\cdot} + d_{\text{ref.}}^2 u_{i,j}^{\cdot}u_{i,j}^{\cdot}\right) + w\,,$$

$$\mathcal{H}_{0i} = u_k^{\cdot}\frac{\partial\mathcal{L}}{\partial u_{k,i}} + u_k^{\cdot\cdot}\frac{\partial\mathcal{L}}{\partial u_{k,i}^{\cdot}} + u_{k,j}^{\cdot}\frac{\partial\mathcal{L}}{\partial u_{k,ji}}$$

$$= -u_k^{\cdot}\left(C_{kilm}u_{l,m} + G_{kijlm}u_{j,lm}\right) + u_k^{\cdot\cdot}\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,i}^{\cdot} - u_{k,j}^{\cdot}\left(D_{kjilmn}u_{l,mn} + u_{l,m}G_{lmkji}\right),$$

$$\mathcal{H}_{i0} = u_{k,i}\frac{\partial\mathcal{L}}{\partial u_k^{\cdot}} + u_{k,i}^{\cdot}\frac{\partial\mathcal{L}}{\partial u_k^{\cdot\cdot}} + u_{k,ji}\frac{\partial\mathcal{L}}{\partial u_{k,j}^{\cdot}} = u_{k,i}\rho_{\text{ref.}}u_k^{\cdot} + u_{k,i}^{\cdot}\rho_{\text{ref.}}\tau_{\text{ref.}}^2 u_k^{\cdot\cdot} + u_{k,ji}\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,j}^{\cdot}\,, \quad (77)$$

$$\mathcal{H}_{ij} = -\mathcal{L}\delta_{ij} + u_{k,i}\frac{\partial\mathcal{L}}{\partial u_{k,j}} + u_{k,i}^{\cdot}\frac{\partial\mathcal{L}}{\partial u_{k,j}^{\cdot}} + u_{k,li}\frac{\partial\mathcal{L}}{\partial u_{k,lj}}$$

$$= -\frac{1}{2}\rho_{\text{ref.}}\left(u_k^{\cdot}u_k^{\cdot} + \tau_{\text{ref.}}^2 u_k^{\cdot\cdot}u_k^{\cdot\cdot} + d_{\text{ref.}}^2 u_{k,l}^{\cdot}u_{k,l}^{\cdot}\right)\delta_{ij} + w\delta_{ij} - u_{k,i}\left(C_{kjlm}u_{l,m} + G_{kjlmn}u_{l,mn}\right) +$$

$$+ u_{k,i}^{\cdot}\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,j}^{\cdot} - u_{k,li}\left(D_{kljmno}u_{m,no} + u_{m,n}G_{mnklj}\right).$$

By using the latter, we obtain the generalized J-integral

$$\int_{\mathcal{B}}\left(\mathcal{H}_{i0}^{\cdot} + \mathcal{H}_{ij,j}\right)\mathrm{d}V = 0$$

$$\left(\int_{\mathcal{B}}\mathcal{H}_{i0}\,\mathrm{d}V\right)^{\cdot} = -\int_{\partial\mathcal{B}}n_j\mathcal{H}_{ij}\,\mathrm{d}A\,, \quad (78)$$

as follows:

$$\left(\int_{\mathcal{B}}\left(u_{k,i}\rho_{\text{ref.}}u_k^{\cdot} + u_{k,i}^{\cdot}\rho_{\text{ref.}}\tau_{\text{ref.}}^2 u_k^{\cdot\cdot} + u_{k,ji}\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,j}^{\cdot}\right)\mathrm{d}V\right)^{\cdot}$$

$$= \int_{\partial\mathcal{B}}\left(n_i\mathcal{L} + n_j\left(u_{k,i}\left(C_{kjlm}u_{l,m} + G_{kjlmn}u_{l,mn}\right) - u_{k,i}^{\cdot}\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,j}^{\cdot} + \right.\right. \quad (79)$$

$$\left.\left. + u_{k,li}\left(D_{kljmno}u_{m,no} + u_{m,n}G_{mnklj}\right)\right)\right)\mathrm{d}A\,.$$

We emphasize that inertial terms arise on the surface integral. Such a result is challenging to obtain without a formal structure as presented herein. In the case of the stationary case, for a centro-symmetric material, $G_{ijklm} = 0$, we obtain

$$0 = \int_{\partial\mathcal{B}}\left(-n_i w + n_j\left(u_{k,i}C_{kjlm}u_{l,m} + u_{k,li}D_{kljmno}u_{m,no}\right)\right)\mathrm{d}A\,. \quad (80)$$

By utilizing $\sigma_{kj} = C_{kjlm}u_{l,m}$ and $m_{klj} = D_{kljmno}u_{m,no}$ for obtaining traction $\bar{t}_k = \sigma_{kj}n_j$ and double traction $\bar{s}_{kl} = m_{klj}n_j$, the generalized J-integral reads

$$0 = \int_{\partial\mathcal{B}}\left(-n_i w + u_{k,i}\bar{t}_k + u_{k,li}\bar{s}_{kl}\right)\mathrm{d}A\,. \quad (81)$$

Hence, we understand that even a stable crack propagation is steered by not only traction but also double traction. The latter term may be proposed with the help of its structure, but the term from the kinetic energy $u_{k,i}^{\cdot}\rho_{\text{ref.}}d_{\text{ref.}}^2 u_{k,j}^{\cdot}n_j$ would be missed easily. It is rather difficult to predict its significance in the formulation. We stress that $d_{\text{ref.}}$ is challenging to measure. In the literature, there are different simplifications for reducing the number of inertial terms [99, 100]. We refer to [101], for a numerical analysis with an experimental comparison for the role of $d_{\text{ref.}}$,

7 | CONCLUSION

We have revisited the extended NOETHER's formalism in continuum mechanics by using tensor algebra and applied directly to elastodynamics. Apart the well-known balance equations, we have observed how the *J*-integral is obtained with the ESHELBY stress tensor, which is of importance in modeling damage mechanics. In this way, we understand that this formalism includes all necessary information for a theory. Hence, it seems to be useful in order to generalize the conventional mechanics. A necessity for generalization may be explained by introducing dissipation as a reason of non-local interaction among particles [102], for its English translation, see [103]. Such effects may be modeled by generalized continua [104–107], especially at smaller length-scales, where the continuum length-scale converges to the microstructure [108]. By using a variational method, generalized mechanics is acquired in a straight-forward manner [109, 110]. However, its generalization to damage mechanics has difficulties, since damage mechanics is not acquired directly from the variational formalism. One possible approach is a hemivariational approach [111–113] but its extension in multiphysics is challenging.

With this work, we expect to shed some light on this formalism and motivate to develop numerical methods based on a purely variational formulation [114]. In this manner, possible explanations arise for difficult concepts such as contact formulations [115]. We understand that the additional equations from the extended NOETHER formalism is necessary to fulfill in order obtain physically correct models in fracture mechanics [116, 117]. These configurational forces are employed to compute the crack propagation in linear elasticity [118] and elasto-plasticity [119]. Herein, we have derived the analogous equations for the generalized mechanics.

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ENDNOTE
[1] From a philosophical point of view, the co-moving frame shall not be called an inertial frame since we cannot distinguish between the forces due to the gravitational forces and due to the acceleration (inertial).

[2] Often, its study is conducted by using differential forms [66]. We will not make much use of this so called exterior calculus and use the fact that tensors in oblique coordinate systems under (affine) transformations produce same calculus as the invariant theory of (differential) forms [67, Section 9].

[3] In solving partial differential equations canonical transformation has another meaning of bringing the set of equations into a JORDAN normal form. Here the same name is used for a different formalism.

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APPENDIX

Transformation examples

Suppose that $x^i = (x, y, z)$ refer to physical coordinates expressed in a Cartesian system and the transformation is the orthogonal rotation around $+z$-axis,

$$x' = x \cos(\varepsilon) + y \sin(\varepsilon), \quad y' = -x \sin(\varepsilon) + y \cos(\varepsilon), \quad z' = z, \tag{A1}$$

where ε is the rotation angle. Suppose that $x^i = (t, x, y, z)$ refer to space-time where the transformation is to a moving system in the direction of x. Between inertial systems, the following is called a GALILEIan transformation:

$$t' = t, \quad x' = x - \varepsilon t, \quad y' = y, \quad z' = z, \tag{A2}$$

where ε is the constant velocity. Suppose now that $x^i = (x, y, z, ct)$ is space-time in a MINKOWSKIan system and a possible special LORENTZ transformation reads

$$x' = x \cosh(\varepsilon) - ct \sinh(\varepsilon), \quad y' = y, \quad z' = z, \quad ct' = -x \sinh(\varepsilon) + ct \cosh(\varepsilon), \tag{A3}$$

where ε is called rapidity of the transformation and the speed of light, c, is a universal constant, thus we have used $(ct)' = ct'$.

Measure's role in irreversibility

We demonstrate in a simplified form how the measure gets a role in the irreversibility. Although the given example below is out of our scope in mechanics, it is beneficial to see this relation. Probably, this application is the only physical example, where the metric evolution is known. In cosmology [120–122] the universe is expanding with a (known) parameter $a = \bar{a}(t)$ everywhere the same, leading to the metric for that expanding universe:

$$g_{ij} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}, \quad g = \det(g_{ij}) = a^6, \quad \sqrt{g} = a^3. \tag{A4}$$

Thus, the infinitesimal volume element reads $dV = \sqrt{g}\, dx = a^3\, dx$ and is time dependent. The rigid motion of galaxies, for example in one direction, χ, will be calculated. We use the same short notation, $\chi^\bullet = \partial \chi / \partial t$, and build the LAGRANGEan in that time dependent (expanding) metric

$$L = \int \mathcal{L}\, dx = \int \left(\frac{1}{2} \rho \chi^\bullet \chi^\bullet - \mathcal{V} \right) a^3\, dx. \tag{A5}$$

The latter gives the energy density with the ground state $\mathcal{V} = \rho U \chi$ depending solely on the motion χ, the ground state has different names in the literature: dark energy, vacuum energy, as well as cosmological constant. We plug in the LAGRANGE

density into the EULER–LAGRANGE equations, use the material frame, $\rho^{\bullet} = 0$, and obtain

$$\frac{\partial \mathcal{L}}{\partial \chi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \chi^{\bullet}} = 0 \,,$$

$$-\rho U a^3 - \frac{\partial}{\partial t}\left(\rho \chi^{\bullet} a^3\right) = 0 \,,$$

$$U a^3 + \chi^{\bullet\bullet} a^3 + 3 a^2 a^{\bullet} \chi^{\bullet} = 0$$

$$U + \chi^{\bullet\bullet} + 3 h \chi^{\bullet} = 0 \,,$$

(A6)

where we have used the so-called HUBBLE constant $h = a^{\bullet}/a$. Obviously, due to the expansion with velocity related to h, there is a damping in this partial differential equation such that the process is irreversible. We use this analogy and understand plasticity in mechanics as an irreversible change of the reference frame (herein the metric). Of course, the situation is far more difficult since additional governing equations need to be solved. In this simple example from cosmology, the HUBBLE constant is a given constant.