Daniel Panizo Pérez

Blowing Bubbles
from String Theory
Abstract

Since the discovery of the accelerated expansion of the universe at the end of the nineties, physicists have tried to describe this cosmological behaviour from theories that are complete in the ultraviolet regime, i.e. theories that are well defined at arbitrarily high energies. The most fundamental property that any proposed model must reproduce is that of a de Sitter space in the lower-dimensional description. In other words, an accelerated expanding four-dimensional cosmology driven by an unknown energy dubbed as dark energy.

String theory seems to be the most promising candidate for a theory of quantum gravity, as it naturally accommodates the graviton, the quantum propagator of gravity and fulfills all the requirements to be an ultraviolet-complete theory. However, despite all the efforts of recent decades, no lower-dimensional effective field theory description with the desired positive cosmological constant has yet been obtained. This suspicious conspiracy has motivated the physics community to take a different approach; to investigate what criteria lower-dimensional effective field theories of gauge fields coupled to gravity should satisfy to be derived from string theory rather than aiming to derive four-dimensional effective field theories directly from string theory. One of the most striking conclusions that can be drawn from these criteria is that four-dimensional de Sitter cosmology may belong to the swampland, the set of effective field theories that cannot be obtained from string theory.

In the first part of this monograph we will review both classical and quantum approaches to four-dimensional cosmology and then introduce the main aspects and issues faced by the aforementioned string cosmology constructions. The second part of this thesis presents an alternative and novel realisation of a four-dimensional expanding cosmology from string theory. This low-dimensional universe rides a three-dimensional brane that mediates the decay between two different five-dimensional and non-supersymmetric anti-de Sitter vacua. This proposal, known as the Dark Bubble model, turns into advantages some of the liabilities found within the swampland programme, granting an innovative way of realising dark energy from string theory. In this work, we will discuss the top-down construction of the Dark Bubble model from ten-dimensional supergravity, the higher-dimensional origin of four-dimensional dark energy and the new intrinsic hierarchy of energy scales that emerges from this stringy construction. Furthermore, we will also show how four-dimensional matter, radiation and both gravitational and electromagnetic waves will arise naturally from features of the higher-dimensional spacetime.

Keywords: String theory; Supergravity; String Cosmology; String Phenomenology; Braneworlds; Dark Bubbles; Cosmology; Gravity in Higher Dimensions

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URN urn:nbn:se:uu:diva-525876 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-525876)
To my aunt, Paloma.
Nunca perseguí la gloria ni dejar en la memoria de los hombres mi canción; yo amo los mundos sutiles, ingrávidos y gentiles como pompas de jabón.

Me gusta verlos pintarse de sol y grana, volar bajo el cielo azul, temblar súbitamente y quebrarse.
Poem Illustration: Rebeca Alzaga Romo (IG), @reicudeiro/ @irezumi_rei
Poem: Cantar I, Proverbios y Cántares, Don Antonio Machado
Translation:

Never have I aimed for glory,
nor endeavored that my story
be for Memory destined.
I have loved my worlds appeasing,
subtly fleeting, gently pleasing,
all with bubbles of a kind.

How I like to watch them topping,
glowing out of ever hue,
soaring up toward the blue,
then abruptly trembling, popping.
List of Publications

This monograph is based on the following papers, which are referred to in the introduction by their roman numerals.


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1. Begin at the Beginning

*In the beginning the Universe was created. This had made many people very angry and has been widely regarded as a bad move.*

Douglas Adams, *The Restaurant at the End of the Universe*

They have always been with us. Anywhere. At any time. A few words of gossip between neighbours. Two friends sharing their respective experiences of the weekend. A ballet. An impossible relationship across the stars. Through a poem. Around a campfire, with torches as witnesses. In a trial. Even during the silence of the people at some of the stations of the Via Crucis. They represent the easiest way to share culture and history. To help form an identity. To connect individuals emotionally. To educate and to learn and to inspire and to create. In short, they have always been humanity’s best tool for navigating life and its mysteries.

This chapter also aims to tell a story. Neither the most heroic nor the saddest. Certainly not the story of the universe. That is somehow left for other chapters in this work. It will simply tell the story of the researchers who explored the unknown territories of the universe. Equipped with their own intelligence and diligence, pen and paper or powerful telescopes, they sought to understand the rules that govern the cosmos. A legend about how far they reached in their own ways to shed new light on human knowledge. Perhaps this myth ends up sparking our own curiosity. Perhaps this tale will inspire us to pack up and set off along these beaten paths to reach the point where they stopped, so that we can continue charting unknown territories. But, dear reader, let us ask first:

1.1 Have you ever heard about...?

Our story begins with Einstein’s publication on *The Foundation of the General Theory of Relativity* in 1916 [1, 2]. These developments provided a solid framework for studying gravity from a new point of view. Not as a force, as the Newtonian paradigm proposed, but as a result of the curvature of spacetime, bent by the presence of massive objects and energy. This powerful machinery seemed to be the correct one to describe the large-scale
behaviour of the cosmos. In fact, it was Albert himself who started this task [3].

At that time, it was not clear how big the cosmos was. Some people claimed that it was just our Milky Way galaxy, while others said that it was bigger, a place where many other objects like our galaxy could fit in. This did not change the fact that early observations, dressed up with some beliefs, pointed to a static universe that avoided gravitational collapse, as predicted by the principle of relativity. This was noticed by Albert and, biased towards the idea of such an "eternal" universe, he introduced a constant into his equation, $\Lambda$. This cosmological constant could act as a counterbalance to the aforementioned gravitational collapse, pushing the cosmos "out" in the right way to keep it static. Eternal. Beautiful.

But Einstein was not the only one who was having a good time with general relativity. By 1919, a German mathematician, Theodor Kaluza, shared with Einstein a novel idea of his; what if we tried to describe Maxwell’s Electromagnetic theory and General Relativity using the same formalism? The proposal, which was later published in 1921 [4], seemed to work but at an interesting cost: An extra dimension was needed to accommodate Maxwell’s and Einstein's equations into the same framework... This was of little interest to Sir Arthur Stanley Eddington, who was interested in photography. He wanted to take some good shots of the eclipse that took place 29th May that same year. Of course, his intention was not to post them on Instagram, but to find evidence that Einstein's theory worked. In a series of papers written by the Dutch gentlemen Willem de Sitter [5, 6], it was shown that according to general relativity, that any massive body would deform the geometry of spacetime, creating a gravitational lens that forces light rays to "bend" as they pass close to the heavy object causing the deformation. This led Arthur to initiate the expedition, taking pictures where the light rays of stars, easily visible in the darkness of the eclipse, were bent as predicted by Einstein's theory. First goal of the game for Einstein.

It was 1920 and Einstein's theory was still the talk of the town and people carried on using the framework to unravel the mysteries of the cosmos. Do you remember that people were not quite sure how big the cosmos was? Well, this confrontation had its climax during the debate held on 26th April 1920 at the Smithsonian Museum of Natural History, Washington. In one corner of the ring, Harlow Sharpley. Opposite him, Heber Curtis. While the former defended that only the Milky Way was the whole of the universe, the

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1In this introduction, we will use this word to refer to the concept of everything outside planet Earth that they had in that time. The word universe will be saved reserved for something "bigger". As we do not want to spoil our dear reader, please, continue reading to understand this subtle difference.
latter argued that some observed nebulae\(^2\) were in fact separate galaxies located far beyond the Milky Way. Sharpley’s argument relied heavily on his observations of the distribution of stars within the Milky Way. On the other hand, Curtis defended that the luminosity measures of novae\(^3\) suggested that those bright spots were too distant to be part of our galaxy. However, sufficient evidence could not be produced, so the debate ended in a draw, awaiting observational signatures that could prove one hypothesis or the other.

Although Curtis’ view of the universe was gigantic, there were people who wanted to make it even bigger. This was the case of the Soviet physicist Alexander Friedmann, who introduced the idea of an expanding universe with generalised geometries (open, flat and closed) in 1922 \([7]\). The main motivation for this project was that the static solution proposed by Einstein seemed unstable; the slightest disturbance would cause the universe collapse or expand. However, Einstein did not like this idea and instead worked on revising his equations to support the static, eternal universe.

Do you remember the great debate introduced above? Well, the verdict had to wait until 1924, when Edwin Hubble identified Cepheid variables in several nebulae. Cepheid stars are considered to be a standard candle\(^4\) discovered by Henrietta Leavitt \([8]\). The distance to these objects can be determined by comparing their apparent luminosity with their intrinsic luminosity. His observations proved conclusively that these nebulae containing Cepheid variables were too distant to be part of the Milky Way. Thus, they were entire galaxies outside our own. Winner of the battle: Curtis.

The second half of the twenties became quite exciting regarding our understanding of the largest and smallest scales in the universe. For a start, a Swedish theoretical physicist called Oskar Klein proposed a quantised version of Kaluza’s five-dimensional formalism. Motivated by the recent developments in quantum mechanics\(^5\) by Heisenberg, Schrödinger, Dirac et al. \([9]\), Klein introduced the hypothesis that the fifth dimension was rolled up and microscopic. He suggested that the geometry of the extra fifth dimension could take the form of a circle, with a radius as big as \(10^{-30}\) cm \([10]\).

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\(^2\)Plural of nebula. It means ”cloud” in Latin. They got this name in the past because the image quality that old telescopes could provide was not good enough to identify their features. They appeared as ”foggy” not well defined objects. The term is currently used to describe clouds of gas and dust inside galaxies.

\(^3\)Plural of nova in Latin. It means new and refers to ”new” stars appearing in the sky. In reality, it is the bright death of an old star that causes a newer and brighter point of light to appear in the sky.

\(^4\)A standard candle is a celestial object of known intrinsic luminosity.

\(^5\)Quantum mechanics is a fundamental theory of physics that describes the behaviour of nature at and below the scale of atoms. At these regimes, the deterministic capabilities of classical theories must be replaced by probabilities of finding particles in a given state.
This proposal laid the foundations for one of the main features of compactifications in string theory: Any extra inaccessible dimension must be small and rolled in itself.

The visible dimensions (the three spatial ones plus time) were the main concern of Georges Lemaître, a Belgian priest and physicist, who in 1927 proposed a model of an expanding universe, similar to that of Friedmann [11, 12]. When Lemaître met Einstein at a conference in the same year, they discussed his latest proposal of the former. Einstein told him that Friedmann had suggested something similar five years earlier, but he was still sceptical about this idea. It is believed that Einstein shared the following thoughts with Lemaître: "Vos calculs sont corrects, mais votre physique est abominable" ⁶ [13]. Good old Albert had to retract these words two years later, when Hubble showed that the absorption spectra of most of the galaxies were redshifted [14]. According to the Doppler effect, ⁷ this meant that they were "moving away" from us! Friedmann was right! It was at this point that Einstein remarked that introducing the constant term to keep the universe in a static state was his biggest blunder. In the following papers of his cosmological research, he discarded this term, claiming that it was "theoretically unsatisfactory in any case " [15, 16].

However, there was a problem with Hubble's discovery. The physics community did not want to accept that galaxies were moving away from us, as it would mean that our galaxy sits at a very special place in the universe, kinda like the centre of it. This idea was disproved by Lemaître, who managed to show that galaxies are not moving away from us, but that the fabric of space-time itself is expanding! Galaxies remain static at a point in space, while this stretches in all directions. These and other results were collected in several papers between 1927 and 1933 , when Lemaître published a final summary with a new controversial idea [11, 12, 17]. In this paper, he proposed that if one extrapolated this cosmic expansion backwards in time, the universe would shrink and shrink with less and less matter, describing an increasingly concentrated cosmos preceded by a singularity in which the classical concepts of space and time lose their meaning. Lemaître called this special state the Primeval Atom, a state with initial density and temperature that is well beyond humanity's ability to replicate, where the laws of quantum mechanics would apply [17]. The physics community received this proposal with concern, as they thought that Lemaître's idea was biased by his religious beliefs and had been manipulated to find evidence for a Godly Creation.

⁶In French, "your calculations are correct but your physics is abominable".

⁷Conceptually we all know what this is. When an ambulance or police car drives past with its alarm on, the pitch of the sound increases as it approaches and decreases as it moves away from us. The electromagnetic version is also used by speed cameras.
As we all know, the late thirties and early fourties were awful years for the whole world, but science did not sleep. Especially with the progress in nuclear physics thanks to Oppenheimer, Fermi, Bethe et al. to develop something that had to do with a little boy and a fat man.\(^8\) In any case, it was not until 1948 that cosmology returned to the forefront of research. Ralph Alpher, Hans Bethe\(^9\) and George Gamow published on April 1st a paper in which they showed that if Lemaître's primeval atom was the right way to go, then all the light elements (i.e. hydrogen, helium and lithium) decorating the universe should have been created during the rapid expansion of the primeval atom state\(^{[18]}\). This process of element creation is known as nucleosynthesis. They also predicted the existence of a remnant glow of radiation from this initial hot state that would permeate the entire universe\(^{[18]}\). However, similarly to what happened with Lemaître's work, this proposal was also ridiculed, partly because of the playfulness of the author's names, and partly because of the lack of detailed mathematical descriptions. The English astronomer Fred Hoyle, a firm defender of the steady state of the universe, pejoratively referred to this proposal as the "Big Bang theory" during a BBC radio broadcast in 1949. The reader is probably familiar with this name...

The decade of the fifties brought no major breakthrough regarding cosmology but in the development of quantum mechanics applied to classical fields. Some of the knowledge gained in nuclear physics during the previous decade paid off in this decade, when Richard Feynmann, Julian Schwinger, Freeman Dyson and Shinichiro Tomonaga managed to win the "war against infinities" by developing the method of renormalisation.\(^{10}\) In addition, Feynmann introduced the path integral formulation\(^{11}\) and his famous diagrams, a visual tool to intuitively organise and to help calculate terms in the aforementioned perturbative expansion. All these improvements led to Quantum Field Theory (QFT) as a complete theoretical framework.

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\(^8\) Names of the atomic bombs dropped on Hiroshima and Nagasaki on 6\(^{th}\) and 9\(^{th}\) of August 1945, respectively.

\(^9\) Funnily enough, Hans Bethe was added as an author without his permission, so that the paper could be called the \(\alpha\beta\gamma\)-paper. Although he did not actually contribute to this paper, he was a pioneer in the subsequent research of the subject.

\(^10\) It refers to the infinite quantities that appear in the framework when higher order corrections are considered in the perturbative approach to the theory. By introducing the method of renormalisation, i.e. introducing a finite set of well-tuned counter-terms to cancel out these divergences, it is possible to obtain results that are compatible with observations\(^{[19]}\).

\(^11\) See chapter 3 for further information.
Although this great progress continue shinning with the theoretical development of the Standard Model of particle physics\textsuperscript{12} during the decades of the sixties and seventies, it was until 1964 that Lemaître and the "\(\alpha\beta\gamma\) gentlemen" got their redemption. By that year, Robert Wilson and Arno Penzias were conducting some tests and experiments with the Holmdel Horn Antenna at Bell Labs in Holmdel Township, New Jersey. They discovered a source of noise in the atmosphere that they could not explain. First they thought it was a radio signal coming from New York city. Other possibilities for the source of the noise were some after-effects of an atomic bomb test that had been conducted over the Pacific several years earlier. Or even worse; pigeon droppings. After a whole year of back and forth testing, the scientists concluded that they had detected the cosmic background radiation (CMB) \textsuperscript{21}, the remnant glow of radiation emitted by the universe in the early times after the Big Bang, as predicted by the "\(\alpha\beta\gamma\)-gentlemen". This confirmed their proposal for nucleosynthesis. And by extension, our favourite Belgian priest was proved right. The universe had started in a very hot and dense state. Very soon after it had started expanding, and cooling down, light elements had been created.

Similarly, in the seventies, Stephen Hawking and Roger Penrose managed to show that from the perspective of general relativity, a singularity in the early universe was inevitable \textsuperscript{22}, providing further mathematical evidence for the Big Bang theory. The clues were clear: a quantum formal approach to cosmology was needed, as the early universe had been in a state where general relativity breaks down and quantum effects become important. In principle, one can try to quantise general relativity by following the formalism of quantum mechanics and applying the path integral perturbation developed by Feynmann in the fifties. However, this would start a new war of infinities that could never be won. The number of counter-terms would be infinite, introducing too many variables into the game and losing control of the theory \textsuperscript{23, 24}. This did not stop Hartle, Hawking and Vilenkin, among others, from trying a semi-quantum approach to cosmology \textsuperscript{25,26}, where the state of the universe is described by its wave function. However, they soon realised that one needs to specify a good choice of boundary conditions for the universe to determine which solution, coming from a Schrödinger-like equation \textsuperscript{27}, describes the state that we observe. But how can one choose the right boundary conditions without leaving the system to be described?

Albeit the boundary condition problem left quantum cosmologists in a everlasting discussion about the right interpretation, all hope was not lost. In the early seventies, in an attempt to describe some of the fundamental interactions of constituents in the atomic nucleus, Yoichiro Nambu, Hol-

\textsuperscript{12}This model successfully describes all the fundamental interactions except gravity, and its many predictions have received remarkable experimental confirmation the decades since \textsuperscript{20}.
ger Nielsen, and Leonard Susskind presented a new physical interpretation of these interactions [28]. In this approach, inspired by the work of Gabriel Veneziano [29, 30], the nuclear forces could be described as vibrating, one-dimensional strings. However, this string description of the strong force made many predictions that were in direct contradiction to experimental results, making this proposal irrelevant ... until 1974. That year, John Schwarz, Joël Scherk and Taimiki Yonega found that one of the vibrational modes of these close one-dimensional strings could have properties exactly matching those of the graviton, the hypothetical messenger particle of the gravitational force [31,32]. Could this new theory be the way to make quantum gravity a reality?

This discovery became the starting point for the two main revolutions in string theory that took place in the eighties and nineties. During these two decades, theoretical physicists made progress in unraveling the mysteries of this promising theory, capable of describing not only quantum gravity, but also all elementary particles (in the form of different vibrational modes of the strings) and all the interactions between them. A theory to explain general relativity and the standard model of particle physics using the same formalism. The price to pay was six times greater than Kaluza-Klein’s work; A total of ten dimensions were needed for the theory to make sense [33]. This was of little concern to Philip Candelas, Gary Horowitz, Andrew Strominger and Edward Witten,13 who inspired by the seminal work of Kaluza and Klein in the twenties, showed that the six small extra dimensions need to be compactified14 on a Calabi-Yau manifold [34].

Among all the lines of string theory research that appeared during these two decades, there was one called string cosmology. The main goal of this line is to reproduce the observed four-dimensional cosmological features from the higher-dimensional framework provided by string theory. The idea was simple: take the higher-dimensional theory and hide some extra dimensions by compactifying them on a Calabi-Yau manifold. The resulting four-dimensional theory would have a vacuum energy associated with the cosmological constant that Einstein introduced into his equations. However, this was not a major problem, as the general consensus and observations since the thirties pointed to an expanding cosmology and not a static universe. So this value had to be zero, right?

This changed abruptly at the end of the Millenium. Two independent projects studying supernovae15 found that the redshift of these explosions did not match the expansion rate detected by Hubble [14]. In fact, all the

13This gentleman is included in the frontcover of this thesis. Can you identify him?
14To be small and curled on themselves.
15A powerful luminous explosion of a star which is a standard candle, i.e. it has similar properties to the Cepheid variables discussed some pages above.
data pointed to an accelerated expansion rate [35, 36]. In other words; the confirmed expansion of the universe, once proposed by Lemaître, was getting faster and faster every day! From a theoretical point of view, it was time to dust off Einstein’s cosmological constant $\Lambda$. An accelerated expansion of the universe could be explained by a positive and really small value of the cosmological constant $\Lambda$. This would be equivalent to a vacuum energy with a positive value. As the nature of this energy was completely unknown, it could not be detected but only inferred from the observation of the expansion, it was called "dark energy". The rules of the string cosmology game had changed after this discovery. The vacuum energy was no longer trivial, but small, really small and positive [36]. In this way, one could provide an explanation on the dynamics of the universe from a microscopic point view, i.e. that of a theory of quantum gravity. This major change drew the attention of a large part of the theoretical physics community to achieve something that seemed naively simple; to obtain dark energy from string theory.

It is 2024 and we are still looking for a single model capable of reproducing dark energy from string theory. What could have gone wrong?

1.2 Organisation of this thesis

Dear reader, this work is divided into two main informal blocks. The first block of this thesis aims to review some historical and recently developed concepts within the field of (string) cosmology. This block will present the physical details and subtleties of the path taken by those great researchers whose stories we have narrated in the previous section. These will serve as motivation and foundation for the dark bubble model that will be presented and discussed in the second block. This new construction is an innovative proposal to realise four-dimensional dark energy $\rho_\Lambda$ from string theory [37, 38].

We would like to warmly invite you to join us in this exciting expedition to shed light on the problem of four-dimensional effective field theories equipped with a tiny positive cosmological constant $\Lambda$. As the journey can be long and exhausting, we would like to share with you our plan of four parts/stages for this journey, so that stops along the way can be found to recover some energy:

- **Stage I, Our Shire**, reviews both classical and quantum formalisms of four-dimensional cosmology, their most relevant key features and also limitations.
- **Stage II, Higher-Dimensional Lands**, which is a summary of string theory, its low-energy limit supergravity and the two main mecha-
nisms that hide the existence of extra dimensions: Braneworld Constructions and String Compactifications. Furthermore, a brief introduction to the swampland and some of its relevant conjectures is also included in this stage.

• Stage III, The (Dark) Bubble Universe, will be the core of this work and it is strongly based on [37–47]. Here we will introduce the dark bubble model, a new way to realise a four-dimensional positive cosmological constant \( \Lambda \) from string theory. As this territory of the journey may be completely unknown to our reader, we have decided to add bullet points at the beginning of each chapter with the most relevant contents. They look like this:

In Stage III you will learn...

• what the dark bubble model is and how dark energy is induced on a \( D_3 \) brane (chapter 6)

• how to embed this model in string theory, what the ten-dimensional origin of four-dimensional dark energy is and about energy scales associated with the dark bubble embedding (chapter 7, based on papers III, V)

• how the dark bubble model relates to the choice of boundary conditions in four-dimensional quantum cosmology (chapter 8, based on paper I)

• what the role and behaviour of gravity is and the importance of backreaction (chapter 9, partially based on paper IV)

• how to decorate the induced four-dimensional cosmos on the brane with matter, radiation and waves (chapter 10, partially based on papers II, IV)

• Stage IV, To Infinity and Beyond, will summarise our adventures along the journey and propose new expeditions to chart new lands of the dark bubble construction. In addition, we will find here a powerful Swiss army knife in the form of four appendices \( \{A, B, C, D\} \), useful for navigating through computations and concepts during our journey. If any material from the appendices is required, it will be specified before the content is introduced in the current chapter.

As you can see, this work is a pilgrimage in itself. In the first block of the thesis, inspired by the (hi)story introduced in section 1.1, we will depart from our cozy abode, the four-dimensional cosmology and then start walking towards breathtaking spots of the higher dimensional lands: Meadows, mountains, valleys and swamps. This first block will simply be a walk along
beaten path of those wiser researchers whose adventures are recounted in section 1.1.

However, the second block presents the challenge of walking and exploring on our own. In these swamps, which we will find in stage II, we will discover nucleating bubbles into which we will jump. On these bubble universes we will learn about the rich folklore they are decorated with: gastronomy, myths and legends, music and painting. Finally, although not arrived at destination, as this is a never-ending journey, we will elucidate on what steps we can take next. Does it sound appealing? Dear reader, You are most welcome to come and walk with us! As every pilgrim should be wished:

*Buen Camino!* 16

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16 In Spanish, have a good way. It refers to the most common wish to pilgrims that walk the famous pilgrimage route in Spain, the Camino de Santiago (Way of St. James or le Chemin de Saint-Jacques-de-Compostelle). According to the tradition, the remains of Santiago the apostle are buried in Santiago de Compostela, the final destination of the pilgrimage.
Part I:
Our Shire: Four-dimensional Classical and Quantum Cosmology
2. Home, Sweet Home: Classical Cosmology

Had the Lord, the Almighty, consulted me before embarking in the creation, I would have recommended something simpler.

Alfonso X, the wise, King of Castilla

Several satellites that we have sent out there,\(^1\) into the empty and cold darkness of space, have provided enough data to prove what Hubble already stated in 1929 [14]: Every galaxy far away (and not so far away) from us is in the process of getting further away. What is more, their distance is increasing even faster today that it was yesterday [35, 36]. It could be, perhaps, due to the fact that the Universe itself sees human beings as a potential plague, and wants to avoid us. Or could be that some sort of colossal multidimensional being has decided to stretch the fabric of spacetime itself, just for fun. Nonsense aside, the most widely accepted reason for the accelerating expansion of the universe today is **Dark Energy**. And what is **Dark Energy**? Where does it come from? What it is and its origin are still open questions. But let us first go back more than a century ago, to understand the synthesis of our current comprehension of the cosmos.

As we saw in the introduction, Einstein published his theory of **General Relativity** (GR) [1, 2] in 1915. The world did not become a better place due to this, but at least we were provided with a ridiculous powerful tool to describe gravitational events. In years following its publication, Friedmann, Hubble, Lemaître [7, 14, 48] (among many others) used general relativity technology to describe the Universe as a whole. Their work set the foundations for what we call today **The Standard Model of Cosmology** (a.k.a. \(\Lambda\)-CDM model) [35, 49, 50].\(^2\) There are three main components that give name to this model:

- **Dark energy**, an unknown form of energy related to the vacuum expectation value, that causes spacetime to expand at an accelerated rate. The cosmological constant \(\Lambda\), which Einstein introduced in his equations has been reassigned to represent the dark energy density.

- **Cold Dark Matter**, a hypothetical non-relativistic type of matter with no known interactions with regular matter other than gravitational.

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\(^1\)Plus all evidence collected also from the Earth’s surface.

\(^2\)Worthy of a Nobel prize, awarded to Jim Peebles in 2019.
Its presence can be inferred by the measuring rotational velocities of spiral galaxies [51, 52].

• Regular relativistic and non-relativistic ordinary matter, such as electromagnetic radiation and the atoms that form observable structures like ourselves.

This model can provide a reasonably good account of the following observed features of the universe:

• The existence of the Cosmic Microwave Background (CMB), remnant radiation from the early universe that was released as the early hot universe cooled down, creating a homogenous wall of radiation, "visible" at radio frequencies (see the CMB temperature distribution in figure 2.1).

• The observed abundances of light elements such as hydrogen, helium and lithium. Predictions of these distributions can be compared with the measured power spectrum and anisotropies of the CMB, which contain observational evidence for these abundances produced during the nucleosynthesis process.3

• Large-scale structures in the distribution of galaxies.

• The observed accelerating expansion of the universe at large scales.

This model is good enough to describe our current observations of the Universe. And not only that. If we reversed the observed expansion back to very close to the beginning of everything, we could still have a really good description of the events happening in the almost newborn Universe. This model has four really simple foundations:

1. **Copernican**: Our planet occupies no special position in the Universe.

2. **GR + Expansion**: The assumption that the gravitational dynamics of the universe are correctly described by Einstein equation.

3. **Hubble’s discovery**: The observations performed during 1929 by Edwin Hubble showed that the greater the distance between any two galaxies, the greater their relative speed of separation [14]. As Lemaître later showed, it is not that galaxies are moving away from each other, but spacetime stretching, i.e. the universe is expanding [11].

4. **Perfect fluidity**: We can assume that all the contents of the Universe behave like a perfect fluid, i.e. they are not sticky.

Of course this model has its flaws, but we leave these downsides for future lines in section 2.3. At really big scales, Copernican principle holds. No

---

3The production of light nuclei other than the hydrogen during the early phases of the universe.
point in space has a special position. Wherever you sit at and look at, everything is more or less the same. In technical words, this means homogeneity and isotropy. The next step is to find a reliable way to measure distances so that we can describe the geometry of spacetime. This is given by the line invariant, which takes the famous FLRW\(^4\) form, adequate to describe a Lorentzian signature spacetime with a high degree of symmetry like the one we seem to live in. This can be written as:

\[
d s^2 = g_{ab}dx^a dx^b = -N^2(t)dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin[\theta]^2 d\phi^2) \right),
\]

(2.1)

where \(N(t)\) is a lapse function\(^5\) and \(a(t)\) is the scale factor that describes the size of the three-dimensional spatial slices. We will see later that the behaviour of \(a(t)\) it is tightly constrained by any type of energy densities that the geometry may contain.

Before we start talking about the spatial properties of the previous line invariant (2.1), let us first discuss about the lapse function \(N(t)\). This function is responsible for time reparametrisation invariance. As we do not want to overcomplicate our computation, the most useful and convenient choices for \(N(t)\) are:

- \(N(t) = 1\): This is the choice of global time, \(t\). Any clock measuring this choice of time is moving along the Hubble flow, which is just the motion of astronomical objects just due to the expansion of the universe.

- \(N(t) = a(t)\): This is the so-called conformal time, \(\eta\). The conformal time is the amount of time it would take a photon to travel from where we are located at to the furthest observable distance.

The extra parameter we have not talked about yet is \(k\). This has the power to change the topology of the spatial sections in the line invariant. It comes in three different flavours [7]:

- \(k = 0\): A rather boring case. No curvature, where the spatial sections of the geometry are flat, like \(\mathbb{R}^3\).

- \(k = 1\): At this value, the spatial sections are closed, as in \(S^3\).

- \(k = -1\): Spatial sections are open, as in the hyperboloid \(H^3\).

So the line invariant (2.1) allows us to describe a dynamical universe with different types of spatial curvature. What are we supposed to do with this tool? How do we get a specific equation(s) that explicitly describe the evolu-

\(^4\)Friedmann-Lemaître-Robertson-Walker.

\(^5\)A general function depending on time and other coordinates. The "duration" of the measured time intervals depends on the choice of this function.
tion of the universe in terms of its curvature and content? It is at this point where Einstein’s equation comes in.

2.1 Friedmann equations, an useful household set

Einstein’s equations are *Equation of Motions* (EOM) that can be used to describe the dynamics of the universe. These equations can be obtained by extremising the Einstein-Hilbert action

$$S_{[g_{MN}, \phi_i]} = \int d^D X \sqrt{|g|} \left( \frac{R^{(D)}}{2\kappa_D} + \mathcal{L}_{\text{mat}}(\phi_i, d\phi_i) \right), \quad (2.2)$$

with $\sqrt{|g|}$ the square root of the minus determinant of the metric, $\mathcal{L}_{\text{mat}}$ as the matter lagrangian of some fields $\phi_i$, coupled to gravity and $R$ is the Ricci scalar, which carries geometrical information. $\kappa_D$ encodes information about the $D$-dimensional Newton’s gravitational constant as:

$$\kappa_D = 8\pi G_D = M^{2-D}_{\text{Pl}} = \ell^{-2}_\text{Pl}, \quad (2.3)$$

where $M_{\text{Pl}}$ and $\ell_{\text{Pl}}$ are the Planck mass and length for $D$ dimensions. As we are working out the classical cosmology scenario, so far we will stick to $D = 4$. To obtain the Einstein equations, we need only to vary the action (2.2) with respect to $g_{MN}$ (now that we fix the dimensions to four) to obtain:

$$R_{ab} - \frac{1}{2} g_{ab} R = \kappa_4 \left( \frac{\mathcal{L}_{\text{mat}} g_{ab}}{\delta g_{ab}} - 2 \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g_{ab}} \right) = \kappa_4 T_{ab}. \quad (2.4)$$

The left-hand side of equation (2.4) condenses pure geometry information in the Einstein tensor $G_{ab}$, while the right-hand side represents the matter field contribution. This is packed into a rank two tensor called the energy-momentum tensor $T_{ab}$. As we assume that the universe is homogenous, isotropic and its contents can be described as a perfect fluid, the energy-momentum (EM) tensor takes the simple form:

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}, \quad (2.5)$$

where $u_a = (-N(t),0,0,0)$ is the fluid four-velocity, and $(\rho, p)$ its energy density and pressure. These can be used to describe pressureless matter (dust).

---

6Note that we will work with $\hbar = c = 1$. This will always be the case unless otherwise specified.

7Planck scales makes refer to scales at which quantum effects of gravity become significant. When this energy scale is reached, gravity cannot be ignored in other fundamental interactions.

8See appendix A for further information about the notations and conventions used in this work.
or relativistic one (radiation), among others. As energy is a conserved quantity, the EM tensor must also be conserved, i.e. $\nabla_a T^{ab} = 0$.

From the clear relationship between geometry and matter content in Einstein equation (2.4), Wheeler once stated: "Spacetime tells matter how to move; matter tells spacetime how to curve". If we choose $N(t) = 1$ (i.e. global time coordinate) for the FLRW metric (2.1) described above, the Einstein’s equation yields two equations\(^9\) of the form:

\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G_4}{3} \sum_i \rho_i - \frac{k}{a^2},
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G_4}{3} \left( \sum_i \rho_i + 3p_i \right),
\]

where $\frac{\dot{a}}{a}$ is the Hubble rate $H$, which measures the expansion of the universe (static when $H = 0$). These are the Friedmann equations.

\(^9\)One for the $tt$-component and three identical copies for the spatial ones.
In order to obtain information about what may be responsible for this behaviour, we need to characterise different types of content. This is done by the *equation of state*, which relates pressure to energy density as \( \omega = \frac{p}{\rho} \). Any type of matter with \( \omega > -\frac{1}{3} \), will be responsible for any decelerated expansion of the universe. On the other hand, any content with \( \omega < -\frac{1}{3} \) will accelerate the expansion.\(^{10}\) We clearly observe that the universe is expanding in this era, but there is nothing that we have (yet) discovered that can be identified with this equation of state. As the universe is a dark place, and this unknown energy density seems to be the source of the accelerated expansion, it has been given the famous name of *dark energy*.\(^{11}\) In general, the value of the state parameter for any energy density can be determined from basic principles such as:

- In the case we consider non-relativistic matter, the energy density \( \rho_m \) is dominated by the rest mass energy \( E = mc^2 \), as the momentum, and hence, the exerted pressure are negligible compared to \( \rho_m \). Thus, to a good approximation, \( \omega \approx 0 \).

- For radiation, we are now dealing with relativistic effects. Following the same reasoning as before, the pressure, which is proportional to the velocity \( v \approx c \) will be relevant. Assuming an isotropic distribution of the pressure across the three spatial dimensions, this can be described by \( \omega = \frac{1}{3} \).

- Finally, dark energy seems to exert a repulsive force, that causes the universe to expand. This is described by a state parameter \( \omega = -1 \). This value can be constrained observationally by data extracted from *Barionic Accustic Oscillations* (BAO), perturbations of the hot plasma of the early universe that imprinted in the *Cosmic Microwave Background* (CMB). The most recent value (2018) is \( \omega_\Lambda = -1.028 \pm 0.031 \)\(^{55}\).

Equipped with this information, we are one step closer to understanding the true power of Friedmann’s equations (2.6, 2.7). But one step at a time, cosmic hitchhikers! If the universe is evolving, so does the energy density of its contents and vice versa. This is governed by the conservation of energy. The explicit expression for the covariant derivative of the EM tensor (2.5) in a FRLW geometry (2.1) gives a relation between the energy density \( \rho_i \) of any

\(^{10}\)This follows from the Null Energy Condition (NEC) [53]. This condition states that the matter energy-momentum tensor \( T_{MN} \) obeys

\[
T_{MN} n^N M^M > 0, \tag{2.8}
\]

for any null or light-like vector \( n^M \), i.e. for any vector satisfying \( g_{MN} M^M N^N = 0 \). This implies that \( \omega \) is bounded as \( -1 \leq \omega \leq 1 \).

\(^{11}\)The first appearance of the term "dark energy" can be found in [54].
Table 2.2. Observed Cosmological parameters [55]. $\Omega$ is the normalised fraction between $\rho_i$ and $\rho_{\text{crit}} = 3H_0^2 / 8\pi G_4$. It is the sum of the normalised densities $\sum_i \Omega_i = 1$ that gives the famous distribution of the energy content of the universe: (Roughly) 70% of dark energy, 25% of dark matter and 5% of ordinary matter and radiation.

<table>
<thead>
<tr>
<th>Energy content</th>
<th>Scaling</th>
<th>Present value ((\Omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matter</td>
<td>$\rho_M \sim \rho_{M0} a^{-3}$</td>
<td>0.3103 ± 0.0057</td>
</tr>
<tr>
<td>Radiation</td>
<td>$\rho_\gamma \sim \rho_{\gamma0} a^{-4}$</td>
<td>~ 0</td>
</tr>
<tr>
<td>Dark energy</td>
<td>$\rho_\Lambda \sim \rho_{\Lambda0}$</td>
<td>0.6897 ± 0.0057</td>
</tr>
<tr>
<td>Curvature</td>
<td>$-$</td>
<td>0.0005 ± 0.0039</td>
</tr>
</tbody>
</table>

content and the scale factor $a(t)$ of the form:

$$\rho \propto a^{-(3(1+\omega))}, \quad (2.9)$$

One of the most interesting features of relation (2.9) is that $\rho_\Lambda$ remains the same yesterday, today and tomorrow. It is not diluted by the expansion of spacetime. This can be interpreted as an intrinsic property of spacetime itself, which can be captured by an additional EM tensor of the form:

$$T_\Lambda = -\rho_\Lambda g_{ab} = -\frac{\Lambda}{\kappa_4} g_{ab}, \quad (2.10)$$

Where $\Lambda$ is a constant of dimension [Length]$^{-2}$. This is the famous *cosmological constant* (CC),\(^{12}\) which encodes the information of a non-varying energy density throughout the expanding cosmos. As this can be interpreted as an intrinsic geometrical property, it is perhaps more appropriate to move it to the LHS of Einstein’s equation (2.4) as:

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = \kappa_4 T_{ab}, \quad (2.11)$$

which is the usual Einstein equation for cosmology. Now that we have fully identified all the characters in this cosmic play, let us come back to the Friedmann equation (2.6). Taking into account all possible types of energy densities $\rho_i$, we obtain the following expression:

$$a^2 = \frac{8\pi G_4}{3} \left( \frac{\rho_{\text{mat}}}{a} + \frac{\rho_\gamma}{a^2} + \rho_\Lambda a^2 - \frac{3k}{8\pi G_4 + \rho_c} \right), \quad (2.12)$$

where we have rewritten the curvature contribution in a more appropriate notation, i.e. $\rho_c$. A simple look at expression (2.12) is enough to see that it can be written in a quite familiar conservation equation as $T + V = 0$, where the potential $V(a)$ corresponds to the RHS of (2.12), with the opposite sign.

\(^{12}\)The famous one that *Albert* introduced in his equation to obtain a static universe.
Our current observations suggest the fact that our universe has entered region C in figure 2.3 some years ago (about $4 \times 10^9$ years ago). We can also see that the gradient of this potential in region C is negative, indicating an accelerated expansion of the universe, in agreement with observations. It is also important to note the divergence of the potential $V(a)$ when $a \to 0$, which causes a singularity at $t = 0$ (conventionally chosen [22]). What does this mean? From a classical cosmology point of view, this corresponds to the Big bang. A singular state that we cannot understand with the tools at hand, i.e. general relativity and classical cosmology, where all the contents of the universe were compressed into a single point at extremely high temperatures [17]. This should already make us bat an eye. We are trying to apply a classical framework to a system whose energy regime very well requires a quantum description of all the forces involved. It should come as no surprise that we end up with singularities in our description if we are using a sledgehammer to crack a nut.

But we cannot be so conformist. If we cannot understand through the lens of a theory such as general relativity, this may indicate that our toolbox lacks some important tools. Before we go to our favourite DIY (Do it yourself) store to buy these tools, we must first understand what and how we want to fix our lack of understanding. This will be the main topic covered in section 2.3. But let us first discuss how to explicitly decorate the universe with two extremely relevant fields, as a warm-up to understanding its higher-dimensional embedding in chapter 10.

---

The most prominent absence is that of a quantum description of gravity.
2.2 Decorating our abode with waves

As we have seen above, the universe is not empty. Although almost to 95% of its content is made up of the unknown (dark energy and dark matter), there is still a tiny fraction that we know very well how to describe dynamically. In this part of the chapter, we will focus on two radiation fields that fill the spacetime: These are gravitational and electromagnetic waves. As the aim of this thesis is to be as pedagogical as possible, the purpose of this section is doublefold:

1. To present these fields acting and propagating in a well-known environment, where the dynamics can be easily computed and imagined.

2. To present important techniques and methods that will be used later in this thesis.

These fields will be again revisited when the higher-dimensional approach to cosmology is studied in chapter 10.

2.2.1 Shake it!

When gravity, in its full glory, is introduced in a course on general relativity for the first time, it is normally presented at leading order. The geometry of spacetime is presented pure, without any possible little perturbations that may mess up in our computations. But the universe is not so pure. Albert was already aware of this fact back in 1916, when he presented his general theory of relativity [1, 2]. One shocking prediction of GR was the existence of gravitational waves (GW): Ripples in the fabric of spacetime, causing ondulations that would propagate at the speed of light $c$ in all directions, away from the source causing it. The culprits of these events could be cataclysmic events, such as supernovae or other extremely massive accelerating objects, such as a pair of black holes orbiting around each other. Even the very beginning of the universe could have left imprinted these waves in its space-time, but extremely weak and diluted to be detected by the current technology. It took 100 years to get observational evidence of gravitational waves, but it was very well worthed the Nobel prize for Barry Barish, Kip Thorne and Rainer Weiss as leaders of the LIGO project that detect them for the first time [56].

We will describe gravitational perturbations in general relativity\(^{14}\) as a perturbation of the metric $g_{ab}$ as

$$g_{ab} = g_{ab}^{(0)} + \epsilon g_{ab}^{(1)} + \epsilon^2 g_{ab}^{(2)} + O(\epsilon^3),$$

(2.13)

with $|\epsilon| \ll 1$ as the formal expansion parameter. The conventional procedure is to solve Einstein’s equation order by order in $\epsilon$. If one plugs Eq. (2.13)

\(^{14}\)In four dimensions. Generalisations to $D$ dimensions are straightforward.
into Einstein equation, it yields:

\[
G_{ab} + \Lambda g_{ab} = \epsilon^0 \left( G_{ab}^{(0)}[g^{(0)}] + \Lambda g_{ab}^{(0)} \right) \\
+ \epsilon^1 \left( G_{ab}^{(1)}[g^{(1)}] + \Lambda g_{ab}^{(1)} \right) \\
+ \epsilon^2 \left( G_{ab}^{(1)}[g^{(2)}] + G_{ab}^{(2)}[g^{(1)}] + \Lambda g_{ab}^{(2)} \right) + \mathcal{O}(\epsilon^3) = 0,
\]

(2.14)

where \( G_{ab}^{(i)}[g^{(j)}] \) denotes the \( i \)-th order variation of the Einstein tensor evaluated on the \( j \)-th order metric perturbation. Formally this is a quantity of order \( \max \{i, j\} \) in \( \epsilon \).

At zeroth order, Einstein’s equation simply yields the background geometry \( g_{ab}^{(0)} \). This is the usual result, already discussed in section 2.1. Gravitational waves appear at first order in \( \epsilon \) through the linearized Einstein equation (the GW equation), which can be schematically understood as solutions to

\[
G_{ab}^{(1)}[g^{(1)}] + \Lambda g_{ab}^{(1)} = 0.
\]

(2.15)

The second order has a more subtle meaning. It can be written as:

\[
G_{ab}^{(1)}[g^{(2)}] + \Lambda g_{ab}^{(2)} = -G_{ab}^{(2)}[g^{(1)}],
\]

(2.16)

where we have moved the quadratic term \( G_{ab}^{(2)}[g^{(1)}] \) to the right hand side. In this way, this term can be interpreted as the energy-momentum tensor \( T_{ab} \) generated by the presence of the waves that fill the spacetime geometry. These perturbations will "decorate" the cosmos with specific energy densities representing its nature.

One could then continue with higher and higher order corrections, exploring how further sub-leading perturbations would create further contributions to the energy-momentum tensor and how these would affect the whole geometry up to any order in the perturbation series. In order to keep our reader’s sanity in its finest, we will just stick up to the second order correction.

Let us then consider gravitational waves in an expanding FLRW cosmology with spherical spatial sections. We will describe gravitational waves using the metric

\[
\text{d} s^2 = g_{ab}^{(0)} + \epsilon g_{ab}^{(1)} = a(\eta)^2 \left( - \text{d} \eta^2 + \text{d} \Omega_3^2 \right) + \epsilon a(\eta)^2 p_{ij}(\eta, [\theta]) \text{d} \Omega_3^2,
\]

(2.17)

where \( \eta \) represents the conformal time,

\[
\eta = \int \frac{\text{d} t}{a(t)},
\]

(2.18)
\[ \frac{d^2 \zeta}{d \eta^2} + 2H \frac{d \zeta}{d \eta} + (n^2 - 1)\zeta = 0. \] (2.20)

\( H \) is the conformal Hubble parameter, given by\(^\text{16}\)
\[ H = \frac{1}{a} \frac{da}{d\eta} = -\cot \eta. \] (2.22)

To simplify the integration of Eq. (2.20), it is useful to redefine the time coordinate to
\[ v = \cos \eta, \quad \text{with } v \in [0, 1). \] (2.23)

Inserting this definition in Eq. (2.20) yields
\[ (1 - v^2) \frac{d^2 \zeta}{dv^2} + v \frac{d \zeta}{dv} + (n^2 - 1)\zeta = 0. \] (2.24)

Three regular singular points: \((v = -1, +1, \infty)\)... This smells like an hypergeometric differential equation (B.1). Solutions can be written in the form (B.2)
\[ \zeta(v) = 2F_1\left(-\frac{n+1}{2}, \frac{n-1}{2}; \frac{1}{2}; 1-v^2\right), \] (2.25a)
\[ \tilde{\zeta}(v) = (1-v^2)^{3/2} 2F_1\left(1-\frac{n}{2}, 1+\frac{n}{2}; \frac{5}{2}; 1-v^2\right). \] (2.25b)

Luckily, as the wave number \( n \) is an integer, these hypergeometrics can be rewritten in terms of the Chebyshev polynomials \( \{T_n, U_n\} \) (B.4, B.5):
\[ \zeta(v) = v T_n(v) - \frac{n}{n+1} T_{n+1}(v), \] (2.26a)
\[ \tilde{\zeta}(v) = \sqrt{1-v^2} \left[ v U_{n-1}(v) - \frac{n}{n+1} U_n(v) \right]. \] (2.26b)

\(^\text{15}\)In summary, every vector and tensor in the spatial sections can be decomposed into spherical harmonics. Further information can be found in appendix B.

\(^\text{16}\)One can obtain an explicit expression for the scale factor \( a(\eta) \) by solving the first Friedmann equation of (2.17). This is
\[ a(\eta) = -\frac{1}{H\sin \eta}, \] (2.21)
with \( \eta \in [-\pi/2, 0). \)
Restoring the conformal time \( \eta \) coordinate, it finally yields:

\[
\zeta(\eta) = \frac{1}{n+1} \cos ((n+1)\eta) + \sin \eta \sin (n\eta) \tag{2.26c}
\]

\[
\tilde{\zeta}(\eta) = \frac{1}{n+1} \sin ((n+1)\eta) - \sin \eta \cos (n\eta), \tag{2.26d}
\]

Note that at late times, i.e. \( \eta \to 0 \), gravitational waves with large wave number \( n \) become like plane waves in a flat universe as \(^{17}\)

\[
\zeta(\eta) = -\eta \cos (n\eta) + \frac{1}{n} \sin (n\eta). \tag{2.27}
\]

This means that the wave will freeze out to a constant at very late times. We will also be able to read this from the energy-momentum tensor \( T_{ab} \) for these waves. For the sake of simplicity let us consider a uniform background of gravitational waves.\(^{18}\) This implies a combination of waves that are homogeneously and isotropically distributed at late time cosmologies and with large wave number \( n \). This can be done in three easy steps:

1. Consideration of ingoing and outgoing waves through a given spatial direction.
2. Averaging the previous contribution over all possible phases that the wave may have, integrating as in Eq. (B.15).
3. Finally, isotropising, i.e. taking into account all possible directions in the spatial sections, i.e. applying Eq. (B.16).

By following the previous recipe, one finds the isotropic energy-momentum tensor (2.16) to be:\(^{19}\)

\[
\langle T^a_b \rangle_{iso} = \frac{7}{8\kappa_4} \frac{1}{a^2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3}
\end{pmatrix} + \frac{n^2}{4\kappa_4 H^2} \frac{1}{a^4} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3}
\end{pmatrix}. \tag{2.28}
\]

Note that the first term corresponds to the curvature! This can be seen as its energy is diluted as \( \rho \sim 1/a^2 \). It can also be read from the trace\(^{20}\) its equation of state \( p = -\rho/3 \). The second term has the equation of state \( p = \rho/3 \)

\(^{17}\)Late time cosmologies take place at large scale factor \( a \) in an expanding universe. This implies that the curvature term in the first Friedmann Eq. (2.6) can be neglected and spatial sections can be seen as flat.

\(^{18}\)This convenience will manifest itself in chapter 10.

\(^{19}\)Note that we have expanded the Einstein equation up to second order in \( \epsilon \), but the perturbative metric (2.17) only sum up to first order. Hence, the first term on the left hand side of Eq. (2.16) is 0!

\(^{20}\)We will mainly display the energy-momentum tensor and the related tensor with one index up and one down along this thesis. See appendix A for more information.
and a dilution $\rho \sim 1/a^4$, which corresponds to radiation. If the wavelength of the gravitational waves is larger than the horizon (hence the wave number is smaller) it becomes frozen, and the curvature component is all that remains.

It is interesting to see that an isotropic and homogeneous distribution of perturbations in the spacetime geometry will end up contributing with both radiation and curvature to the content of the universe. Let us now see if this is also the case for electromagnetic waves.

2.2.2 Charge it!

We have already presented the Einstein-Hilbert action (2.2) in section 2.1, which can account for matter and/or fields within the geometry of spacetime. If the field of our choice is that of electromagnetism, this is described by a four-dimensional Einstein-Hilbert-Maxwell theory:

$$S[g_{ab}, A_a] = \int d^4x \sqrt{|g|} \left( R - 2\Lambda - \frac{1}{4g^2} F_{ab} F^{ab} \right),$$ (2.29)

where $F_{ab}$ is the field strength associated with the $U(1)$ gauge field $A_a$, and $g$ is the gauge coupling constant. For convenience, we assume this field is sourceless: it is just there, permeating the spacetime, without no nearby source emitting it. Electromagnetic wave solutions in this background are solutions to sourceless Maxwell’s equations, which can be formulated in three equivalent ways:

1. The first way is simply to vary the action with respect to the gauge field $A_a$, which yields the standard Maxwell equations:

$$\ast d \ast F = 0 = \nabla_b F^{ab}, \quad dF = 0 = \nabla_{[a} F_{bc]}.$$ (2.30)

2. One can go further and combine both previous equations to obtain a wave equation for the field strength $F_{ab}$ as:

$$\nabla_i \nabla^i F_{ab} = -2 R_{abcd} F^{cd} + R_a^c F_{cb} + F_a^c R_{cb},$$ (2.31)

where relation (B.17) has been used. This points to a remarkable feature of curved spacetime: electromagnetic waves couple to the curvature of spacetime. This implies that the curvature can act as a source of electromagnetic waves: $F_{ab}$ can be decomposed along the world line of an observer (see [57, 58] for further details)

$$F_{ab} = 2 u_{[a} E_{b]} + \epsilon_{abc} B^c,$$ (2.32)

where $u^a = dx^a/d\tau$ is the 4-velocity tangent to the observer’s world-line. This implies that the electric and magnetic fields observed by our
witness are given by the projections \( E_a = F_{ab} u^b \), and \( H_a = \epsilon_{abc} F^{bc} / 2 \). Projecting (2.31) along \( u^a \) gives the wave equation for the electric field \( E^a \), and its dual gives the wave equation for the magnetic field.

3. Finally, Maxwell’s equations can also be written for the gauge field \( A_a \) instead of the field strength \( F_{ab} \). Eq. (2.31) This becomes:

\[
\nabla_i \nabla^i A^a - \nabla^a \nabla_i A^i - R^a_{\ i} A^i = 0.
\] (2.33)

The extra curvature term comes again from the commutation of the covariant derivatives (B.17). This can be further simplified if imposing the Lorenz gauge \( \nabla_i A^i = 0 \). With this choice, the left-hand side becomes the de Rham operator \( \Delta_{(dR)} \), which is the ordinary Laplacian supplemented by the Ricci curvature. The resulting wave equation is also referred to as the de Rham equation,

\[
\Delta_{(dR)} A^a = \nabla_i \nabla^i A^a - R^a_{\ i} A^i = 0.
\] (2.34)

Although the three formulations above are equivalent in content, choosing wisely which one to use can save us pain and time. Note that Eqs. (2.31) and (2.33) are inherently second order in the field strength and the gauge field respectively. On the other hand, the first formulation in (2.30) involves only a single derivative on \( F_{ab} \). This will be the approach taken in this section.

Let us solve for electromagnetic waves in a homogeneous and isotropic universe with flat geometry described by

\[
ds^2 = a(\eta)^2 \left(-d\eta^2 + dx_i dx^i\right),
\] (2.35)

with \( x^i = \{x, y, z\} \). In case we want to solve for a closed geometry, one only needs to start from the non-perturbed part of Eq. (2.17), decompose the vector field \( A^a \) into vector harmonics (see appendix B) and solve. But the main interest of this thesis is to explain late time cosmologies, hence a conformally flat geometry like (2.35) is enough. The scale factor associated to this geometry is given by

\[
a(\eta) = -\frac{1}{H\eta}.
\] (2.36)

We can then decompose \( F_{ab} \) in a basis of plane waves, under the following Ansantz:

\[
F_{ab} = e^{in_\lambda x^\lambda} Y^{ab}(\eta)
\] (2.37)

where \( n_\lambda \) is the wave vector and \( Y \) is the tensorial part with conformal time dependence. Again, we will compute what the solution of one monochromatic wave (i.e. a single wave number \( n \)) is by solving Eqs. (2.30) for a wave travelling in the z-direction. This yields:

\[
Y^t, z = \eta \left( \eta Y^t, \eta - 4Y^{\eta, i} - i n \eta Y^{i, z} - \eta Y^{t, z} + 4Y^{i, z} + i n \eta Y^{\eta, i} \right) = 0,
\] (2.38)
where a \( Y' \) denotes a conformal time derivative and \( x^i = \{x, y\} \)-directions. Inserting this into the Ansatz (2.37) we get

\[
\begin{align*}
F^{\eta, x} &= \frac{e^{inz}}{a^4} \left( c_1 \cos(n \eta) + c_2 \sin(n \eta) \right), \\
F^{\eta, y} &= \frac{e^{inz}}{a^4} \left( c_3 \cos(n \eta) + c_4 \sin(n \eta) \right), \\
F^{x, z} &= \frac{e^{inz}}{a^4} \left( c_2 \cos(n \eta) - c_1 \sin(n \eta) \right), \\
F^{y, z} &= \frac{e^{inz}}{a^4} \left( c_4 \cos(n \eta) - c_3 \sin(n \eta) \right).
\end{align*}
\]

The energy momentum tensor carried by the electromagnetic wave is obtained by varying the action (2.29) with respect to \( A^a \), resulting in

\[
T^a_b = F^{ac} F_{bc} - \frac{1}{4} \delta^a_b F^{cd} F_{cd}.
\]

Finally, in the same spirit as for the gravitational waves in section 2.2.1, a uniform background of electromagnetic radiation requires an average of all possible phases and directions of the waves with different polarizations. This results in an isotropic energy momentum tensor of the form:

\[
\langle T^a_b \rangle_{\text{iso}} = \frac{\mathcal{E}^2}{2a^4} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3}
\end{pmatrix},
\]

where \( \mathcal{E} \) accounts for the amplitude of the wave, encoded by the constants \( c_i \) in eq (2.39). Note that, in contrast to the gravitational case in the previous sections, electromagnetic waves contribute only with a radiation piece with \( \rho \sim a^{-4} \). Furthermore, no electromagnetic wave can become frozen, no matter how large its wavelength is.

With these results, we close this section. They will return later in chapter 10, acting as "boundary conditions" for the higher-dimensional view of cosmology to be presented.

2.3 No classical plumber will fix the \( \Lambda \)-CDM Model

We talked about some limitations of the standard model of cosmology at the end of section 2.1. In this section we will look at them in more detail. Let us start with two of its most well known issues.
A lack of causal contact (Horizon problem)

To understand this fundamental problem which the standard model of cosmology cannot explain, we must first define what to be in contact is. There is nothing faster than light in the universe [59]. This imposes an upper bound on the size of a region where two points could have been in causal contact, i.e. information between the points could have travelled in a period of time less or equal to the time it would take light to travel the distance separating them. This region is called the light cone. Massless particles, as photons,

![Figure 2.4. Light cone. Photons travel along world lines of zero proper time, $ds^2 = 0$, called null geodesics (both lines at 45 degrees). The set of all null geodesics passing through a given event P in spacetime is called the light cone. The interior of the light cone is defined as the region of spacetime causally related to that event P. Causally disconnected events (such as Q) are outside the lightcone related to the event P.]

follow null geodesic trajectories, which imply $ds^2 = 0$. Furthermore, as the universe is isotropic, only the radial coordinate will be relevant for computations. Calculating in conformal time $\eta$, we can extract the following relationship from the light-like particle line invariant:

$$ds^2 = a^2(\eta) [-d\eta^2 + d\chi^2] = 0 \quad \Rightarrow \quad \chi(\eta) = \eta_f - \eta_i = \int_{t_i}^{t_f} \frac{dt}{a(t)}, \quad (2.42)$$

where $\chi$ is a generic radial coordinate that accounts for all possible topologies [60]. When $\eta_i = 0$, $\chi(\eta)$ represents the comoving particle horizon, which is the maximum distance light can travel between time 0 and any other time $t$. One can then further rewrite expression (2.42) in terms of the comoving Hubble radius $R_H = (aH)^{-1}$, which is the radius of the observable universe at a given time $t$. Then:

$$\eta_f = \int_0^{t_f} \frac{dt}{a(t)} = \int_0^a \frac{da}{Ha^2} = \int_0^a d\ln(a) \frac{1}{aH}. \quad (2.43)$$

We would like to make contact with today’s data, so we can rewrite the Hubble radius at any moment to today’s by massaging the first Friedmann equa-
tion (2.6) together with the energy density evolution (2.9) to see that:

\[ R_H = R_{H_0} a^{1/2(1+3\omega)}. \]  

(2.44)

It is important to note here the dependence of the exponent, which depends on the state parameter \( \omega \). This leaves us with a dependence of \( \eta \) as:

\[ \eta \propto a^{1/2(1+3\omega)}, \]  

(2.45)

which means that the comoving horizon grows monotonically with time.

This implies that some of the scales that are entering our horizon today were outside the Causal contact horizon in the past (The universe was dominated by radiation and matter, as can be seen in (2.3)). This can be seen in the CMB. Its homogeneity suggests that the universe was highly homogeneous at that time, but we have just seen that these homogenous regions could not have been in Causal contact in the past. How is this possible? What key concept is missing from our interpretations?

Wrinkles after ironing (Flatness problem)

Imagine that you are doing your laundry. Normally, when you take your clothes out of the washing machine, they are full of wrinkles. Sometimes they get even worse after they dry. Then, you patiently iron them down to smooth the surface. Now, try to imagine the opposite case; that your clothes were extremely smooth and flat when you took them out from the washing machine and after ironing, they are still smooth, but not as nicely wrinkle-free as they were just when you opened the washing machine door. Makes no sense, does it? Something similar is what our universe shows.

Let us rewrite the first Friedmann’s equation (2.6) as:

\[ \left( \Omega(a)^{-1} - 1 \right) \sum \rho_i a^2 = - \frac{3k}{8\pi G_4}, \]  

(2.46)

where \( \Omega(a) = \sum \rho_i a^{-3(1+\omega)}/\rho_{\text{crit}} \), with \( \rho_{\text{crit}} = 3H^2/(8\pi G_4) \). Note that the RHS of the previous expression is a constant quantity, \(^{21}\) while the LHS evolves with time. The density \( \rho \) will decrease with time, counterbalanced by the increase in the scale factor \( a \). As the universe was dominated by radiation and matter in the past, this implies that the combination \( \rho a^2 \) has decreased with time. In order to keep the behaviour constant, this requires that \( \Omega(a)^{-1} - 1 \) has increased in the same way. This constrains the value of \( \Omega(a) \approx 1 \) further and further as we go back in time. This points to an extreme fine-tuning of \( \Omega(a) \) at early times, to deviations from one of no more than \( \delta(10^{-16}) \) during the nucleosynthesis period, for example. Any deviation greater than such ridiculously small values would not have allowed the flat universe that we observe today.

\(^{21}\) In fact, 0, if we account that \( \Omega_k \approx 0 \) as we saw in figure 2.2.
The Hubble tension

The Hubble constant today $H_0$, which is defined as the ratio between the speed and the distance of an object in space ($H_0 = \frac{v}{d}$), can be measured in two different ways:

- You can find the speed of the object (usually a galaxy) by looking at its redshift. For the distance, the most reliable way is to make use of standard candles; Type Ia supernovae whose peak luminosity is the same, no matter where they are in the universe. Given the relationship between relative and absolute magnitudes, it is possible to extract the distance to these objects with good accuracy. This method is used to study nearby celestial objects, i.e. not looking back much into the past. The current value for $H_0$ using this method is $H_0 = 72.3 \pm 1.3 \text{ km s}^{-1}\text{Mpc}^{-1}$ [55].

- An alternative way of obtaining $H_0$ from observations with our current technology is to study the CMB, i.e. to look far into the past. Theoretical predictions for the CMB can be obtained by tuning certain parameters in the $\Lambda$-CDM model, such as curvature, energy densities, etc. These predictions can then be compared with the observations, to decide which one is the best fit. The best fit so far yields a $H_0$ value of $67.4 \pm 0.5 \text{ km s}^{-1}\text{Mpc}^{-1}$ [61].

As we can see, there is an apparent tension between these two results which has been confirmed by technological development over the years. What could be the reason for obtaining two different values for the expansion rate of the universe at two different eras? Several proposals have been discussed, without observational confirmation or agreement in the community yet (see [62]). In any case, observations point to an existing difference that cannot (yet) be explained by the standard model of cosmology.

The cosmological constant problem

We have previously talked about different observations of the Hubble constant using two different methods of measurement. However, this is not the only dichotomy that exists in $\Lambda$-CDM model.

Let us quantise matter fields appearing in the Einstein-Hilbert action (2.2) as proposed by Weinberg [63]. At the end of the day, the remaining physical interactions (electromagnetism, weak and strong nuclear forces) are quantised in the standard model of particle physics. Each of these interactions, has a vacuum energy, corresponding to the minimum background

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22It could be that our technological limitations were the reason for the difference, but as technology has developed, this difference has been confirmed with better and more reliable measurements.
energy that space itself has. In the absence of curved spacetime, this vacuum energy can somehow be ignored. But, this is not the case if we want to study any of these fields in the presence of gravity. We know from the equivalence principle that gravity couples to all possible forms of energy, including that of the vacuum.

The quantisation of matter fields will result in an additional constant energy density contribution to the energy-momentum tensor $T_{ab}$. As this term is constant and with the same dimensionality as the cosmological constant (i.e. $[\text{Length}]^{-2}$), we can add it to $\Lambda$ to obtain an effective cosmological constant as:

$$\Lambda_{\text{eff}} = \Lambda + \kappa_4 \rho_{\text{vac}},$$  \hspace{1cm} (2.47)

which is in fact the cosmological constant that we are able to measure from observations. Let us now identify each of its contributions numerically. According to [55], the effective value of the cosmological constant is:

$$\Lambda_{\text{eff}} \sim 3 \times 10^{-122} M_{\text{Pl}}^2,$$ \hspace{1cm} (2.48)

while any first-order correction due to quantum fluctuations of the matter fields is:

$$\Lambda_{\text{vac}} \sim \frac{M_{\text{Pl}}^2}{16\pi^2} \sim 6.3 \times 10^{-3} M_{\text{Pl}}^2.$$ \hspace{1cm} (2.49)

This is a difference of about $10^{120}$ orders of magnitude (in Planck units), as often stated in the literature. However, previous computations of the vacuum energy are slightly handwavy, because the renormalisation scheme does not respect Lorentz invariant. As noted in [64], an accurate renormalisation will return a mismatch of 54 orders of magnitude instead of 120. In both cases, this means that the bare cosmological constant $\Lambda$ has to be extremely fine tuned to cancel out in such an almost perfect way that the small effective one we can observe remains. A recent discussion about this issue can be found in [66]. This last required coincidence is what gives title to this section.

The first two previously discussed problems of the $\Lambda$-CDM model (i.e. the horizon and flatness problems) may find possible solutions in the proposed inflation mechanism.\footnote{This additional contribution comes from the closed-loop Feynmann diagrams (i.e. quantum fluctuations around the vacuum state) of each of the matter fields involved. We will not include a discussion of this computation in these notes, but we refer the reader to [64, 65] for further reading and details.} However, there is another underlying problem, which is probably related to all previously discussed problems, and

\footnote{This mechanism proposes that the universe went through a de Sitter phase (i.e. a phase in which dark energy dominates the dynamics, giving rise to an exponential behaviour of the scale factor $a \sim e^{Ht}$) moments after the Big Bang. As this work will be mainly oriented towards the understanding of late-time cosmologies from higher-dimensional construc-}
which was already pointed out by Lemaître in [17]; General Relativity is a classical and effective\textsuperscript{25} field theory of spacetime itself. This means that it cannot provide a description of gravity in the short-range regime. There is a cut-off, an upper bound, the Planck length $\ell_{\text{Pl}}$ at which quantum effects, hence new degrees of freedom, will become important. In this regime, the classical description of spacetime breaks down, pointing to the need for a more fundamental theory that is completed in the \textit{ultraviolet} regime, i.e. a theory that is well defined at arbitrary high energies and without free parameters. A theory of \textbf{quantum} gravity. At this point the question arises:

\textit{How and where can one acquire such powerful tool as an UV-complete theory of gravity?}

Perhaps it is time to leave the comfort of our cottage and explore the boundaries of our knowledge, in search of a clue that will bring us closer to the desired fully consistent theory of quantum gravity. It is well known that the best possible candidate we have so far is string theory, which will be reviewed and discussed in stage II of this work. However, it could be instructive to first explore what the intricacies and subtleties of a quantised version of classical four-dimensional cosmology are.

\textsuperscript{25}An effective field theory is one that provides simple a description of phenomena when applied at the right scale. It includes the appropriate number of degrees of freedom to describe physical phenomena occurring at a chosen length scale or energy scale, and ignores substructure and degrees of freedom that may occur at shorter distances.

\textsuperscript{tions, we will not touch much on the concept of inflation. However, we leave some references here for the curious reader [60,67–69].}
The previous chapter was devoted to presenting the observable universe and its dynamics from a classical perspective. General relativity seemed to work like a charm in the usual regimes where the $\Lambda$-CDM can be applied. However, we have also seen in section 2.3 that it has its flaws. Without delving into too much detail, some of these issues can be solved assuming that the universe underwent a period of exponential expansion called inflation immediately after the Big Bang [67, 69–73]. Although this proposal elegantly solves some of the $\Lambda$-CDM problems such as the horizon or flatness problem, it raises new complicated questions hard to answer. For example, one may wonder what the initial conditions for any fields controlling such exponential enlargement were. This will undoubtedly lead us towards the fine tuning problem [74]. Moreover, it is still necessary to find reliable observational evidence which can prove that such process took place at the beginning [75, 76].

But there is an even more fundamental problem lying behind that of initial conditions. As Lemaître already pointed out in 1933 [11] and discussed in chapter 2, the effective potential divergence in Eq. (2.12) when $a \to 0$ signals a breakdown of our classical understanding of cosmology. The primeval atom proposed by the priest in [48] could be defined as a metastable and quantum state where the notion of spacetime ceases to exist. The key word to put the focus on here is quantum.

As we saw in the introduction 1, one of the main priorities of the physics community has been to find a formalism to describe all the fundamental interactions within the same framework. While electromagnetism and both the strong and weak nuclear forces can be studied within the framework of the standard model of particle physics [19, 77], gravity remains as a troublemaker to our wishes, unleashing infinities that cannot be tamed. The unified description of the three forces mentioned above can work out in energetic regimes where gravity is not strong enough to affect their interactions. However, this was not what the weather looked like at the very beginning of the universe ($\sim 10^{30} K$). In this regime, the Compton wavelength of a particle is more or less equal to its gravitational (Schwarzschild) radius. Hence,
any quantum fluctuation would "blur" the classical concept of spacetime, pointing to a breakdown of the classical description of gravity.

Although the dream is to achieve a consistent theory of quantum gravity, the relentless efforts of the physics community have not yet provided the desired result. String theory, the best candidate we have for this title, will be discussed in stage II of this work. In this chapter, we will prepare our minds for this task by taking a more modest approach. Instead of quantising a non-perturbative renormalisable theory as gravity, we will try to describe the whole state of the universe with a semi-classical description, via the canonical quantisation of General Relativity. The premise is that this semi-classical approximation should coincide with the semi-classical low-energy description of the yet-to-be found theory of quantum gravity. This is the main aim of Quantum Cosmology [78, 79].

The idea is somehow simple: One takes one's favorite universe, described by the rules of General Relativity and proceeds to quantise canonically by following the Dirac's method [80] as if it was an usual quantum mechanics system. This implies identifying what the canonical variables are and to introduce a quantum wavefunction $\psi$ (i.e. a quantum state $|\psi\rangle$ living in a Hilbert space) to represent the state of the universe. When the canonical variables and their conjugated counterparts have been promoted to operators, an Schrödinger-like equation can be defined to describe the evolution of the state of the universe. Finally, one would need to solve for the specific quantum state $|\psi\rangle$ that solves the aforementioned equation. This requires us to provide the right set of boundary conditions. An interesting question to ask here would be: What is the right choice of boundary conditions? How can we define a "boundary" for the quantum system to be studied if we have never left it? We will see in this chapter that this still remains as a source of debate.

In this chapter we will not provide a complete review of the current state of quantum cosmology, but we will introduce the basic notions and framework that will be latter discussed (from a higher dimensional point of view) in this thesis. We will start with a discussion of the quantisation procedure for the most general four-dimensional cosmological configuration in section 3.1. We will then realise that the amount of information to be handled is overwhelming, which will require us to drastically reduce the number of variables controlling the system. A simple, yet powerful toy model will be considered in section 3.2, where we will also discuss about the physical implications of the most well-known boundary condition proposals.
3.1 Scene reconstruction of the quantum cosmos

Before we start with the contents of this section, we would like to invite our dear reader to enjoy the hypersurface discussion in appendix C, so that the lecture of this chapter will be a more pleasant experience after having acquired some familiarity with the to be used geometrical notation.

Let us start by choosing a Lorentzian manifold $M$ that accepts a global time coordinate. This type of manifold always accepts time-orientability, which allows us to simplify the computation by decomposing the spacetime components. This will consist in separating the space slices from the global time coordinate $t$. Each spacelike hypersurface of constant time $t$ will be denoted by $\Sigma_t$. This is is the ADM formalism, named after Arnowitt, Deser and Misner [81].

The set of coordinates used to describe the decomposition foliation is given by:

$$x^a = (t, x^i),$$

and the most general expression of the metric on the manifold $\mathcal{M}$ in these coordinates is:

$$ds^2 = g_{ab}dx^a dx^b = -\left(N^2 - N_i N^i\right) dt^2 + 2N_i dx^i dt + \gamma_{ij}(t, x) dx^i dx^j,$$

where $N \equiv N(t)$ represents the lapse function, as discussed in chapter 2. The function $N_i$ is called the shift function, and measures the path difference between the same point $p$ on the hypersurface $\Sigma_t$ at different "slices" of time $t$. When $N_i = 0$, one recovers the usual description in comoving spatial coordinates. We will see that these two functions will play the role of constraints when we study the dynamics of the system. The metric $h_{ij}$ represents the spatial sections of the four-dimensional geometry, i.e. the metric induced on them.

The dynamics of these slices will be controlled by the classical Einstein-Hilbert action (2.2) enhanced by the Gibbons-Hawking-York boundary term [82,83]. This extra piece accounts for any extrinsic contribution, i.e. how the $\Sigma_t$ slices is embedded in the whole four-dimensional space. The total action is given by:

$$S[g, h, \Phi] = \frac{1}{2\kappa^4} \int_\mathcal{M} d^4 x \sqrt{|g|} \left((^{(4)} R - 2\Lambda_4\right) + \frac{\epsilon}{\kappa^4} \int_{\partial \mathcal{M}} d^3 x \sqrt{|h|} K + S_m[\Phi],$$

where $|h| = \det h_{ab}$ the determinant of the induced metric on the boundary $\partial \mathcal{M}$. $\epsilon$ represents the norm of the normal vector $n_a$ and $K$ is the trace of the extrinsic curvature (see appendix C). Finally, any matter fields are encoded in the action term $S_m[\Phi] = S_m[\phi_0, \cdots, \phi_n]$. This four-dimensional action can be broken down into its $3 + 1$ slice decomposition. By use of Eq. (C.21) one
can write:

\[ S[h, \Phi] = \int dL = \frac{1}{2\kappa_4} \int d^3x \sqrt{|h|} N \left( (3)R - K_{ij}K^{ij} - K^2 - 2\Lambda \right) + S_m[\Phi], \quad (3.4) \]

with the extrinsic curvature explicitly given by:

\[ K_{ij} = \frac{1}{2N} \left( \partial_t h_{ij} + \nabla_i N_j - \nabla_j N_i \right). \quad (3.5) \]

Although it may seem appealing to compute the equations of motion directly from the action (3.4), it will be more illustrative to perform a Legendre transformation to the Lagrangian and obtain the Hamiltonian, as proposed by the ADM formalism. This requires us to identify the canonical coordinates in the system: \{h_{ij}, N, N_i, \Phi\}. Hence, the canonical momenta can be computed in the standard way [84]:

\[ \pi_{ij} = \frac{\delta L}{\delta \dot{h}_{ij}} = -\sqrt{|h|} \left( K_{ij} - h_{ij}K \right), \quad \pi_i = \frac{\delta L}{\delta \dot{N}_i} = 0, \]

\[ \pi_{\phi_n} = \frac{\delta L}{\delta \dot{\phi}_n} = \frac{\sqrt{|h|}}{N} \left( \dot{\phi}_n - N^i \partial_i \phi_n \right), \quad \pi_N = \frac{\delta L}{\delta \dot{N}} = 0. \quad (3.6) \]

Note that the conjugated momenta associated to the lapse \(N\) and shift \(N_i\) functions are zero. This implies that we are dealing with Dirac’s primary constraints [85]. Perhaps our reader has never heard of such constraints. Let us rephrase them in a more "peasant" language. To do this, we then write the Hamiltonian as:

\[ S = \int d^4x \left( \pi_N \dot{N} + \pi_i \dot{N}_i - N \mathcal{H} - N_i \mathcal{H}_i \right), \quad (3.7) \]

where \(\mathcal{H}_m\) represents the Hamiltonian piece for the matter fields \(\phi_i\) and

\[ \mathcal{H} = 2\kappa_4 G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{h}}{2\kappa_4} \left( (3)R - 2\Lambda_4 \right) + \mathcal{H}_m, \]

\[ \mathcal{H}_i = -2\nabla^j \pi_{ij} + \partial_i \mathcal{H}_m. \quad (3.8) \]

Note that the derivatives of Eq. (3.7) with respect to the lapse \(N\) and shift \(N_i\) act as Lagrange multipliers, which will result in severe constraints as:

\[ \mathcal{H}_i = 0, \quad \mathcal{H} = 0. \quad (3.9) \]

From now on, we will refer to \(\mathcal{H}\) as The Hamiltonian. This Hamiltonian will govern the evolution of the state of the universe along the space of configurations it can have. In order to have a good notion of distances and the

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1Note that the last term in Eq. (C.21) is a total derivative.
2Note that the choice of canonical variables is not unique. Different choices will lead to different quantum theories upon quantisation. Here we will stick to the ADM choice [81].
geometry of such territory, we will define $G_{ijkl}$. This object receives the name of DeWitt metric [27] and it is formulated as

$$G_{ijkl} = \frac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}),$$

(3.10)

which characterises the geometry of the superspace. Formally, one can define this space by:

$$S(\Sigma) \equiv \{ h_{ij}(x), \phi(x) \mid x \in \Sigma \} / \text{Diff}_0(\Sigma).$$

(3.11)

As mentioned above, the superspace contains all possible different metrics $h_{ij}$ and matter field configurations that the universe can have. It is infinitely dimensional, with a finite number of coordinates $\{ h_{ij}(x), \Phi(x) \}$ at each point $x$ of the three-dimensional surface $\Sigma$.

As we know from our quantum mechanics course, the quantum state of a system is represented by the wave function $\psi$ associated to it [86]. This object is a functional $\psi[h_{ij}, \Phi]$ of the superspace, which provides the amplitude to find a particular hypersurface $\Sigma_t$ of the universe with a given three-dimensional metric $h_{ij}$ and matter field configuration $\Phi$. Note the absence of any explicit time dependence. This should not be a cause of concern; At the end of the day, we are aiming to quantise general relativity, so there is an implicit time dependence in the spatial $h_{ij}$ information.

Let us now quantise the system. According to Dirac’s quantisation procedure [87], substituting the canonical momenta (3.6) \footnote{Note that we have neither $\pi_i$ nor $\pi_N$. This is because the to-be operator version of the constraints should act like them, which is the case of the lapse and shift functions. This implies that the wavefunction $\psi$ is independent of them.} \footnote{Note that we have neither $\pi_i$ nor $\pi_N$. This is because the to-be operator version of the constraints should act like them, which is the case of the lapse and shift functions. This implies that the wavefunction $\psi$ is independent of them.} by operators

$$\pi^{ij} = -i \frac{\delta}{\delta h_{ij}}, \quad \pi_n = -i \frac{\delta}{\delta \phi_n},$$

(3.12)

which yields the following equations for $\psi$:

$$\mathcal{H} i \psi = \mathcal{H} \psi = 0. \quad (3.13)$$

The first constraint forces the wavefunction $\psi$ to be invariant under any three-dimensional diffeomorphisms. This will not be of much relevance in this work, as we will restrict ourselves to comoving frame (So $N_i = 0$). For the second constraint we especially have:

$$\left[ 2\kappa^4 G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \frac{\sqrt{\hbar}}{2\kappa^4} \left( \frac{3}{2} R - 2\Lambda \right) - \mathcal{H}_m \right] \psi = 0. \quad (3.14)$$

This equation is known as the Wheeler-DeWitt equation [27] and it will be the main object of study in quantum cosmology. It describes the dynamical evolution of the wavefunction of the universe, hence its state. It
also ensures the explicit **time independence** of the "timeless" wavefunction $\psi$. This equation can then be thought of a zero-energy analogue of the Schrödinger equation [88] due to its similarities. At the end of the day, it describes the temporal evolution of a quantum system.

We would not like to finish this section of the chapter without commenting on one of the most studied and reliable forms of solving Schrödinger-like equations in quantum mechanics: The path integral [19, 89]. This object provides the probability amplitude for a system (i.e. a particle or a universe) to move between two different states within a given time interval. The trajectory it will follow will not be deterministic, as the uncertainty principle will limit our precision in calculating pairs of physical quantities [90]. In this case, we are required then to sum over all possible configurations that interpolate between the final and initial states of study. Moreover, as our object of study is a gravitational system with potential topological changes, this would require us to account for them along the path. This can be cast as:

$$Z = \langle h^{(f)}_{ij}, \phi_{f}; \Sigma_f | h^{(i)}_{ij}, \phi_{i}; \Sigma_i \rangle = \sum_m \int Dg D\phi e^{iS[g,\phi]}, \quad (3.15)$$

where the sum $\sum_m$ takes into account all possible topologies that the four-dimensional geometry can have. The integration is performed over all possible $g_{ab}$ and $\phi_i$ configurations, represented by $\{Dg, D\phi\}$. The action $S[g,\phi]$ is that described in Eq. (3.3). Note that this would imply a strongly oscillating integrand which could suffer from convergence issues when integrating. One might, in principle, think that an analytic continuation to the Euclidean description (i.e. $r = it$) would tame such a problem. Nothing further from reality. Divergences will continue appearing due to the non-renormalisable nature of gravity [23, 24, 91]. Furthermore, the non-perturbative aspect of this force will lead to an unbounded from below action (3.3) [92, 93].

All in all, despite the difficulties, the path integral of gravity has been proved to be an extremely useful tool in the semi-classical (i.e. the quantum cosmology) approximation. In this regime, the path integral is a weighted sum over all solutions that extremise the action (3.3). This ease the computation, and allows us to define the wavefunction $\psi$ describing the state of the universe as:

$$\psi[h_{ij}, \phi; \Sigma] = \int Dg D\phi e^{iS[g,\phi]}.$$  

This general wavefunction $\psi$ satisfies the Wheeler-DeWitt equation (3.14). However, there is a subtle catch here; the path integral formalism does not provide a specific initial configuration state $| h_{ij}, \phi; \Sigma \rangle$. This brings us back to the initial condition issue discussed at the beginning of this chapter. We have solved for the most general solution of the wavefunction (3.16) and in

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4 Recall that we consider all possible intermediate states.
order to select the specific wave that describes the evolution of the universe, we need to impose a set of boundary conditions on the contour of integration. From the perspective of a quantum mechanics course, this is easy. You are given a potential with some boundary conditions, impose them and pick out the solution. As an external observer of the system, you have an idea of the "shape" of the studied system. Nevertheless, within the framework of quantum cosmology, where the observer is part of the system, the choice of boundary conditions to be imposed is not so clear. Ideally, such a choice should be provided by the physics of the system. However, from a four-dimensional quantum cosmology point of view proposals and debates about the choice of boundary conditions are all we have to work with. This will change when we approach this problem from a string theory point of view in chapter 8, using the dark bubble framework [43].

Before embarking ourselves on the study of the two most common proposals for the aforementioned discussion, let us first drastically reduce the number of degrees of freedom and limit ourselves to a reduced set of them to have a concrete description of a simple wavefunction describing the evolution of the universe. Then, we will be able to easily impose the two different boundary choices and delve into their physical implications.

3.2 When you have eliminated all that is impossible

Let us start by simplifying our superspace. Its infinite dimensionality does not help with computations. So the best way to proceed is to do what physicists do best; to approximate the system with a toy model. In this case, we will restrict our attention to just a few individual variables of the superspace and freeze any other degrees of freedom. The resulting configuration of the superspace is called minisuperspace. This simplification will allow us to have a tractable set of degrees of freedom, which will facilitate any explicit computation.

The toy model that we have proposed considers the quantisation of an empty four-dimensional universe, with closed spatial section and a positive cosmological constant $\Lambda_4$. As we saw in chapter 2, this can be described by a Friedmann-Robertson-Leimatre-Walker metric (2.1) with $k = 1$. As our aim is to obtain the dynamics controlling its evolution, we need to substitute Eq. (2.1) in the action (3.3) to obtain the Lagrangian:

$$S = \frac{\text{Vol}_3}{\kappa_4} \int dtN \left( -\frac{3a\dot{a}^2}{N^2} + 3a - \Lambda_4 a^3 \right), \quad (3.17)$$

---

5 Another way of thinking about this is to try to explain the concept of phase transition from gas to liquid, but when the only conceptual physical understanding available is that of the liquid phase.

6 No, physicists are not the best at naming things.
where \( \text{Vol}_{S^3} = 2\pi^2 \) appears after integrating over the closed spatial directions \( x = \{\alpha, \beta, \gamma\} \). Note that the only dynamical variable present in the previous action is the scale factor \( a \). This implies that we have reduced the minisuperspace to only one dimension. The canonical coordinates of the Lagrangian (3.17) are then given by \((a, \pi_a)\), where \( \pi_a \) is the conjugated momentum as

\[
\pi_a = \frac{\delta L}{\delta \dot{a}} = -\frac{6\text{Vol}_{S^3}}{\kappa_4 N} a \dot{a}.
\]

The corresponding Legendre transformation will yield The classical Hamiltonian for this toy model, which is:

\[
\mathcal{H} = -\frac{\kappa_4}{\text{Vol}_{S^3}} \frac{\pi_a^2}{12a} + \frac{\text{Vol}_{S^3}^2}{\kappa_4^2} \frac{(6a^2(3 - \Lambda_4 a^2))}{V(a)}.
\]

If we now quantise the system as described in section 3.1, we need to replace the conjugated momentum \( \pi_a \) by \(-i\partial_a\) and The Hamiltonian constraint \( \mathcal{H} = 0 \) by the Wheeler-DeWitt equation (3.14). Simplifying and re-arranging terms so that the above equation resembles that of Schrödinger with an effective potential \( V(a) \), we get:

\[
\left[ -\frac{1}{2} \frac{\delta^2}{\delta a^2} + \frac{6\text{Vol}_{S^3}^2}{\kappa_4^2} \frac{(3 - \Lambda_4 a^2)}{V(a)} \right] \psi_{4D} = 0.
\]

Note that the effective potential in figure 3.1 has two roots at \( a_0 = 0 \) and \( a_* = 3/\sqrt{\Lambda_4} \). Returning to the Schrödinger equation analogy, we can think of our cosmos system as being driven by the effective potential \( V(a) \), which has two clear regions separated by the turning point \( a_* \) when \( \Lambda_4 > 0 \). These would be called quantum region when \( V(a) > 0 \) and the classical region when \( V(a) < 0 \).

Given the "tameness" of the potential \( V(a) \) in the Wheeler-DeWitt equation (3.20), the wavefunction solution can be found using the semi-classical Wentzel-Kramers-Brillouin (WKB) approximation [88]. For simplicity of notation, let us define:

\[
S(a_f, a_i) = \frac{\text{Vol}_{S^3}}{\kappa_4} \int_{a_i}^{a_f} da' \sqrt{2|V(a')|},
\]

which is the argument of the exponents in the wavefunction solution:

\[
\psi(a) = \begin{cases} 
\mathcal{A} e^{S(a,0)} + \mathcal{B} e^{-S(a,0)} & \text{if } a < a_*, \\
\mathcal{C} e^{iS(a,a_*)} + \mathcal{D} e^{-iS(a,a_*)} & \text{if } a > a_*.
\end{cases}
\]

The pairs \( \{\mathcal{A}, \mathcal{B}\} \) and \( \{\mathcal{C}, \mathcal{D}\} \in \mathbb{C} \) and can be related by the WKB formulas as:

\[
\begin{align*}
\mathcal{A} &= \frac{1}{2} e^{-S(a,0)} \left( e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}} \right), \\
\mathcal{B} &= e^{S(a,0)} \left( e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}} \right), \\
\mathcal{C} &= \frac{1}{2} e^{-S_0 + i\frac{\pi}{4}} + \mathcal{A} e^{S_0 - i\frac{\pi}{4}}, \\
\mathcal{D} &= \frac{1}{2} e^{-S_0 - i\frac{\pi}{4}} + \mathcal{A} e^{S_0 + i\frac{\pi}{4}}.
\end{align*}
\]
Note that the wavefunction Eq. (3.22) and its undetermined coefficients \( \{A, B, C, D\} \) clearly point to those infinitely many possible solutions to the Wheeler-DeWitt Eq. (3.20). Here we will use the two most common boundary condition proposals introduced in section 3.1. 

![Figure 3.1](image_url)  
*Figure 3.1. Plot of the potential controlling the dynamics of the wavefunction describing the evolution of the universe. The real parts of the Vilenkin and Hartle-Hawking wavefunctions are also shown. Region I corresponds to the quantum or Euclidean region, while Region II represents the classical part of the potential.*

**No-boundary proposal**

This proposal, argued by Hartle and Hawking [25, 94], suggests that the Euclidean version of the wavefunction (3.16) should be restricted to the integration on compact four-dimensional Euclidean manifolds. This implies that the slice \( \Sigma \) where \( \psi \) is defined is the **only** boundary to the system. From a more physical perspective, this can be translated into the claim that the universe had **no** singular **boundary** in the past. Hence the name of the proposal.

The interpretation of the no-boundary wavefunction is that the geometry arises from nothing. Translated this to the initial condition problem this would lead to conditions on \( h_{ij}(x) \) and \( \phi(x) \) and its derivatives in the imaginary time component. The full discussion of how to fix these restrictions can be found in [79]. This implies a choice of the coefficients for region I to be such that only the increasing exponential part of the wave exists, i.e. \( (A, B) = (1, 0) \). Making use of the relations (3.23) one can also obtain the parametrical dependence in region II. The entire wavefunction (3.22) un-
der the no-boundary condition proposal reads:

\[
\psi_{\text{HH}}(a) = \frac{1}{|V(a)|^{1/4}} \begin{cases} 
    e^{S(a,0)} & \text{Region I} \\
    2e^{S(a_*,0)} \cos \left( S(a, a_*) - \frac{\pi}{4} \right) & \text{Region II}
\end{cases}
\]  

(3.24)

**Tunneling**

The second well-known boundary proposal is that of Vilenkin [26, 95, 96]. This proposal requires the wavefunction \( \psi \) to be everywhere bounded, and at singular boundaries of superspace, \( \psi \) includes only outgoing modes. This can be thought of as an analogy to quantum tunneling in quantum mechanics. The boundary condition imposed there is an statement about outgoing modes at \( \infty \). From a more physical point of view, the idea behind this proposal is that any possible state described by \( \psi \) should not include universe’s states contracting down from an infinite size, i.e. only expanding states from "nothing". Given the simplicity of the minisuperspace and behaviour of the wavefunction solution in the classical region, it is easy to see that \( (C, D) = (0, 1) \). Using relations (3.23), and imposing \( \mathcal{A} \sim 0 \), as it is exponentially suppressed, we find:

\[
\psi_{\text{V}}(a) \approx \frac{1}{|V(R)|^{1/4}} \begin{cases} 
    e^{S(a_*,0)} e^{-S(a,0) + \frac{i\pi}{4}} & \text{Region I} \\
    e^{-iS(a,a_*)} & \text{Region II}
\end{cases}
\]  

(3.25)

The explicit form of both the Hartle-Hawking and Vilenkin wavefunction expressions (3.24) and (3.25) allow us to extract what the nucleation probability of a universe with a cosmological constant \( \Lambda_4 \) is. This is no more than the amplitude of the wave under the quantum region as:

\[
P_{\text{HH}} \propto \exp \left( + \frac{24\pi^2}{\kappa_4 \Lambda_4} \right), \quad P_{\text{V}} \propto \exp \left( - \frac{24\pi^2}{\kappa_4 \Lambda_4} \right).
\]  

(3.26)

Note how the Hartle-Hawking probability favours the nucleation of universes with small positive cosmological constant \( \Lambda_4 \), while the opposite is true for the Vilenkin case. However, the question remains; what is the best set of boundary conditions to describe the beginning of our cosmos? Although recent developments and claims [97] and counterclaims [98] have been made in the last years, we will take a different approach to settle this discussion. This will involve using string theory and its extra dimensions, which means a higher-dimensional view of the problem and letting the UV-complete theory operating in that description to fix the boundary conditions in the four-dimensional interpretation for us. We will then revisit this boundary choice discussion, but from a string theory perspective, using the dark bubble construction [43] in chapter 8. But first we will venture into the realm of string theory and its extra dimensions in stage II.
Part II:
Higher-dimensional Lands: Meadows, Cliffs, Gorges and Bogs
4. Stringy Kingdom: Braneworlds and the Landscape

- Look, Simba. Everything the light string theory touches is our Kingdom.
- Wow! Everything the light string theory touches... What about the shadowy place?
- That is beyond our borders. You must never go there, Simba.

Adapted Mufasa and Simba dialogue, The Lion King

In the fifth century BC, an ancient Greek philosopher called Democritus, proposed that all matter was made up of small indivisible particles called "atoms". Democritus believed that atoms were too small for the human senses to detect and that they were infinitely many with infinitely many shapes. He thought that these atoms floated in the void (i.e. vacuum) and that they were constantly moving and colliding with each other [99].

This idea, although tweaked and mutated, has survived through the centuries up to our modern times in the form of different formalisms for defining the fundamental constituents of matter. Along similar lines to those of Democritus, string theory proposes the existence of one-dimensional fundamental objects, i.e. strings, as the indivisible constituents of our reality. The vibrational modes of these strings are interpreted as particles, in the same way that we understand musical notes in a guitar and field excitations in quantum field theory. These strings exist in two different configurations: Open and closed. What makes string theory so appealing is that the lowest vibrational mode of closed strings would correspond to a massless spin-two particle. This state represents the graviton, the propagator of gravity in its quantum description [31–33]. This means that string theory naturally leads to quantum gravity, making it the leading candidate for a UV-complete theory, i.e. a theory that is not effective, but well defined at arbitrarily high energies.

String theory has many other appealing properties, such as having no free parameters beyond the string length scale \(\ell_s\) or being pertubatively renormalisable, necessary features to be a theory of everything. However, and specifically for superstring theory, conformal anomalies are cancelled when spacetime is ten-dimensional.\(^1\) Details of the subtleties of this and

\(^1\)This is called the critical dimension. If the dimensionality results to be non-critical, an additional contribution proportional to the string mass will appear in the effective action. As
other requirements of string theory can be found in its classical literature, for example [102–105]. In the following, we will "zoom out" and lose the resolution of the strings, i.e. we will lower the energy and reach the classical limit of string theory, treating the theory in a perturbative way.

The critical superstring theory reduces to ten-dimensional supergravity in the limit of vanishing string coupling $g_s$ and arbitrarily small spacetime curvatures and field gradients, i.e. the aforementioned low energy classical limit\(^{2}\) plus quantum corrections, controlled by such coupling constant $g_s$ and curvature corrections, controlled by $\alpha' = \sqrt{\ell_s}$, which is the string tension. The whole string theory action can be understood schematically as:

$$ST = \underbrace{\text{supergravity}}_{\text{perturbative}} + \sum_n c_n g_s^n + \sum_n \left(\frac{\ell_s}{L}\right)^n d_n + e^{-1/g_s(\ldots)} + e^{-L/\ell_s(\ldots)} + \ldots \quad (4.1)$$

where we distinguished between perturbative and non-perturbative corrections. In this work, we will mainly discuss the supergravity truncation, although we will discuss both non-perturbative and perturbative corrections in sections 4.2 and 7.6 respectively.

Although supergravity can exist in different disguises after the perturbative truncation, we will focus on the so-called type II A/B supergravity theories [102–105]. The ten-dimensional action for these descriptions is given schematically by

$$S_{10} = S_{\text{NS–NS}} + S_{\text{RR}} + S_{\text{source}} + S_{\text{CS}}, \quad (4.2)$$

where $S_{\text{NS–NS}}$ represents the Neveu-Schwarz Neveu-Schwarz sector of superstring theory while the piece $S_{\text{RR}}$ represents the Ramond-Ramond sector, which contains a set of massless p-form fields. The piece $S_{\text{source}}$ describes localised sources, as fundamental objects with dimension $p$ where open strings end. These objects are called $D_p$-branes and we will later comment on them. Finally, $S_{\text{CS}}$ refers to the Chern-Simons piece, a topological term without metric dependence.

The Neveu-Schwarz Neveu-Schwarz sector is given by:\(^3\)

$$S_{\text{NS–NS}} = \frac{1}{2\kappa_{10}} \int d^{10}X \sqrt{|g_{(10)}|} \left(R_{10} - \frac{1}{2}(\partial \phi)^2 - \frac{1}{2} e^{-\phi} |\partial |^2 \right), \quad (4.3)$$

the authors of [100, 101] mention that it seems difficult to understand whether there really is any perturbative control, so for that reason we will not discuss them further.

\(^2\)Supersymmetric gauge fields coupled to the usual gravity.

\(^3\)This is the very first higher dimensional expression to be written in this work. In order to be able to navigate through the different spacetime dimensionalities, we refer the reader to the appendix A, where basic information can be found on the coordinate conventions used in this work.
where the ten-dimensional Newton’s constant $\kappa_{10}$ (see Eq. (2.3)) is related to the string length $\ell_s$ as
\[
\frac{1}{2\kappa_{10}} = \frac{2\pi}{(2\pi\ell_s)^8 \frac{g_s^2}{8}}. 
\tag{4.4}
\]

The field strength $\mathcal{F}$ is related to the antisymmetric two-form $\mathcal{B}_{MN}$, i.e. the gauge form under which open strings are electrically charged (see appendix A for the conventions of index contraction of forms.). The scalar field $\phi$ represents the dilaton. The action (4.3), which can be derived from the beta functions of the stringy action in the worldsheet [103, 104] also contains a ten-dimensional Ricci scalar $R_{10}$. It is in this natural way that a consistent quantum theory of strings describes Einstein’s theory of gravity in its low-energy classical regime.

The Ramond-Ramond sector is represented by the second piece of the action (4.2). It has field strengths $\mathcal{F}_p$ where $p$ is the degree of this differential form. Its action is represented by
\[
S_{RR} = -\frac{1}{2\kappa_{10}} \int d^{10}X \sqrt{|g_{(10)}|} \sum_{p \leq 10} \frac{1}{4} e^{\frac{5-p}{2} \phi} |\mathcal{F}_p|^2, 
\tag{4.5}
\]

with $p$ is even or odd for IIA/B. Note the $p$-dependent exponential term before each field strength. This will appear after performing a Weyl rescaling, so that the action is given in the Einstein frame, i.e. the Ricci scalar $R_{(D)}$ is not multiplied by a scalar field or function. The Ramond-Ramond field strengths $\mathcal{F}_p$ are written as:
\[
\mathcal{F}_1 = dC_0, \\
\mathcal{F}_p = dC_{p-1} - B_2 \wedge \mathcal{F}_{p-2}, \quad \text{for} \quad p \geq 2, 
\tag{4.6}
\]

where $C_p$ is a $p$-form gauge potential and $\wedge$ the wedge product.\(^5\) In order to understand the role of these forms and fields, we can use the analogy of Maxwell’s electromagnetism. In this theory there is a two-form $F_{\mu\nu}$ representing the field strength, which is intrinsically related to the one-form $A_\mu$, under which point particles (i.e. 0-dimensional objects), are charged. Similarly, one-dimensional strings are charged under the two-form $\mathcal{B}_{MN}$, which gives rise to the field strength $\mathcal{F}_{MNP}$. $D_p$-branes are the next iteration of this logic, i.e. they are $p$-dimensional ($p + 1$ if we take the time component into account) objects where open strings end, sourcing $U(1)$ gauge theories to the $D_p$-brane worldvolume.

\(^4\) $\mathcal{F}_0$ would represent the so-called Roman mass in type IIA. This term will not be relevant in this work.

\(^5\) Further information on the notation can be found in appendix A.
The general action to describe the dynamics of this type of hypersurface is given by:

\[ S_{Dp} = -T_{Dp} \int d\xi^{p+1} e^{\frac{p-3}{2} \phi} \sqrt{|g_{MN} + 2\pi \alpha' \hat{F}_{MN}|} + T_{Dp} \int [\mathcal{C} e^{\hat{F}}]_{p+1}, \quad (4.7) \]

where \( \xi \) represents the choice of induced coordinates on the hypersurface. The field strength \( \hat{F} \) remains gauge invariant as a combination of the field strengths associated with the \( U(1) \) gauge fields and the antisymmetric two-form \( \mathcal{B}_{MN} \) as

\[ 2\pi \alpha' \hat{F} = 2\pi \alpha' F + \mathcal{B}. \quad (4.8) \]

The tension of the brane \( T_{Dp} \) is

\[ T_{Dp} = \frac{2\pi}{g_s (2\pi \ell_s)^{p+1}}. \quad (4.9) \]

The action (4.7) is a combination of pieces:

- The Dirac-Born-Infeld (DBI) action, which takes into account the geometry of the brane and how this is charged under the \( U(1) \) fields discussed above.

- The Wess-Zumino (WZ) term, which couples the brane to the gauge potential. This implies that the \( D_p \)-brane electrically sources the \( \hat{F}_{p+2} \) field strength. Here we use a polyform notation \( \mathcal{C} = \sum \mathcal{C}_p \), where we sum over (odd)even forms depending on the chosen supergravity type. The subscript \( p + 1 \) means that one should extract the \( (p + 1) \)-form piece of the integrand.

Finally, the topological term in the action (4.2) differs between type IIA and IIB theories as follows:

\[ S_{CS,IIA} = \frac{1}{4\kappa_{10}} \int \mathcal{C}_3 \wedge \hat{F}_3 \wedge \hat{F}_4, \quad S_{CS,IIB} = \frac{1}{4\kappa_{10}} \int \mathcal{C}_4 \wedge \hat{F}_3 \wedge \hat{F}_3. \quad (4.10) \]

This concludes our brief introduction to the low-energy classical description of string theory, supergravity, and some of its components. As discussed above, this limit requires a ten-dimensional spacetime to be free of anomalies. This corresponds to six more dimensions than we currently perceive in our universe. This would require us to find mechanisms to "hide" or to confine our cosmos to hypersurfaces in this higher-dimensional spacetime. If such techniques exist, then we should find a way to reproduce the most important characteristic of our cosmos, its accelerated expansion, from this higher-dimensional framework. In other words:

To explain how a lower-dimensional dark energy \( \rho_\Lambda \) could arise from such a ten-dimensional background.
In this chapter we will discuss two different regions of our Stringy Kingdom to achieve this. In section 4.1, we will explore the plains of the braneworld approach, i.e. our universe confined to a brane that can move in the higher-dimensional space. The second approach to be discussed in section 4.2, which is the standard one in the string theory lore, would force us to search in the mountains and valleys of string compactifications, where extra dimensions are rolled up into in closed and small manifolds. The geometry of these spaces will then determine the properties of the remaining open and large dimensions.

4.1 Brane heaths

One of the best known mechanisms for solving the issue of extra dimensions is to consider our universe as a three-dimensional hypersurface (brane) embedded in a higher-dimensional space. Our universe could then move freely along those extra dimensions, but any content of it would be confined to its volume, with no possibility of accessing the higher-dimensional picture.

First proposals of this type of construction date back to [106, 107]. In this toy model, where gravitational effects were not taken into account, matter was confined to a domain wall, the brane, which could move freely along an extra infinite dimension. Higher-dimensional equations of motion admit solutions close enough to the wall. However, gravity was missing, which could be a big problem when one wants to find a consistent theory of gravity which the graviton can propagate freely throughout the full spacetime.

The first "braneworld" model which aimed to include gravity was discussed in [108]. This proposal consists of a three-dimensional membrane, in which all matter is confined and a set of extra dimensions which are required to be not infinitely large, but flat, closed (i.e. $y \sim y + 1$) and small.\(^6\) In this sense, matter is still localised on the brane, while now propagating throughout all spacetime. Although this model includes gravity, it does not take into account a subtle but important detail; As we saw in section 2.2, any content in the universe will cause a backreaction in its geometry. In the same way, the presence of the brane carrying the matter in this model should cause changes in the higher-dimensional spacetime that contains it. This is known as warping and it describes how the brane deforms as it moves through the extra dimensions.

The first model to take into account both the presence of gravity and any possible deformations of the membrane which is embedded in the higher-dimensional space was that of Randall and Sundrum [109, 110]. This string

\(^6\)If extra dimensions were large in this configuration, the force of gravity could leak into them. This would cause modifications in Newton's law and lead to contradiction with experiments.
theory inspired braneworld proposal, not only took into account those subtleties forgotten in the previously discussed models, but also proposed a solution to the gauge hierarchy problem in particle physics (i.e. why the gap between the electroweak scale and that of Planck is so big). This model requires two different branes, with positive and negative tensions respectively, both embedded in a five-dimensional Anti-de Sitter space. Thanks to the backreaction, which translates into curvature, i.e. warping in the extra dimensions, any mass trapped on the negative tension brane is much smaller than the Planck scale. This is known as RS-I model. In a later paper [110], a second model was proposed, called RS-II.7

Firstly, this second RS model does not require the presence of the negative tension brane. This model involves a single positive tension brane. This brane, as discussed above, will warp the geometry and it only involves an "infinitely" large extra dimension.8 Although this may seem counter-productive, generating possible contradictions in Newtonian gravity at low energies on the brane, it is not. Randall and Sundrum managed to avoid this problem by forcing gravity to be localised on the brane. Let us review some of the key concepts of this construction.

Following [111], let us work in a five-dimensional effective theory of Einstein gravity with matter as

\[
S = \frac{1}{2\kappa_5} \int d^5x \sqrt{|g(5)|} \left( R(5) - 2\Lambda_5 + \mathcal{L}_m \right) + \frac{1}{\kappa_5} \int d^4x \sqrt{|h|} K, \quad (4.11)
\]

where \(\kappa_5\) is the Newton’s constant (2.3) in five dimensions and the five-dimensional cosmological constant \(\Lambda_5 < 0\). The matter Lagrangian is taken to be \(\mathcal{L}_m\). As discussed in appendix 3, the extrinsic curvature \(K\) contains information about how the matter, if localised on a hypersurface \(\Sigma\), is embedded in the higher-dimensional spacetime. If we choose \(\mathcal{L}_m = 0\), it can be shown that the metric Ansatz that solves the equation of motion is that of AdS5 space9

\[
d_{\text{AdS}_5}^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2, \quad (4.12)
\]

where the aforementioned warp is reflected in the factor \(A(r) = \pm kr\) and \(k = \sqrt{-\Lambda_5/6}\) is the Anti-de Sitter curvature. The AdS5 radial coordinate, i.e. the holographic coordinate \(r \in (-\infty, \infty)\) increases monotonically from the centre to the boundary of the Anti-de Sitter space.

Let us now imagine a (mem)brane with tension \(\sigma\) placed at \(r = 0\).10 This shell of matter will divide the five-dimensional space into two regions and

---

7 Again, physicists lack the imagination to nominate.
8 In fact, this is an artefact of the choice of Poincaré coordinates. The presence of the \(Z_2\) and the warp factor indicate a finite volume, as we will see later.
9 Some of these steps will be discussed in greater detail in chapter 6.
10 This object is not a fundamental one coming from string theory, hence the change of notation to \(\sigma\).
it will hence generate a jump in the extrinsic curvature $K_{\mu\nu}$ on both sides of the brane (see appendix C for further information). This can be denoted in the warp factor as:

$$A(r) = \epsilon_+ k_+ \Theta(r) r + \epsilon_- k_- \Theta(-r) r,$$

(4.13)

where $\epsilon_\pm$ represents the normalisation and orientation choice of the normal to the hypersurface $\Sigma$, $\Theta(r)$ is the Heaviside function and $k_\pm$ represent the curvatures of each side of both vacua. One of the most important key fea-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_1.png}
\caption{Figure 4.1. Left: A pictorial representation of the configuration under discussion (inside/inside). The presence of $Z_2$ and the choice of normal orientations will have important consequences for this construction. Right: The presence of the "mirror" will cause the warp factor $A(r)$ to peak on the brane and to fade to 0 when $r \to \pm\infty$.}
\end{figure}

tures of the Randall-Sundrum model is that the brane will act as a "mirror", i.e. a $Z_2$ symmetry is imposed at $r = 0$. This forces us to identify both five-dimensional vacua across the brane;\footnote{Hence the choice of $k_-$ on either sides of the brane in figure 4.1.} In addition, the Randall-Sundrum model makes a specific choice of normal orientations, which, together with the aforementioned $Z_2$ translate into:

$$\epsilon_- = -\epsilon_+ = 1 \quad \text{and} \quad k_+ = k_-.$$  

(4.14)

This identification is often referred to as inside-inside construction. This results in a matter Lagrangian defined as:\footnote{This can be obtained from the formal Ansatz}

$$\mathcal{L}_m = \frac{6}{\kappa_5} k \delta(r) = \sigma \delta(r).$$

(4.16)

Imposing this matter configuration in the junction condition (C.35), while restricting to the case where $\epsilon_- = -\epsilon_+ = 1$ and $k_+ = k_- \quad \text{and} \quad \text{leaving the warp factor} A(r) \quad \text{undetermined,} \footnote{There will be a backreaction of the brane in the embedding space. The new warping factor is not relevant to the current discussion.} \quad \text{one can then induce an effective four-}$$

What happens when we remove the $Z_2$ symmetry and choose the normal vectors to point in the direction of always increasing transverse volume is left to be discussed in chapter 6.
dimensional cosmological constant on the brane which reads:

\[ \Lambda_4 = \frac{\kappa_5^2}{12} \sigma^2 - 3 k^2. \] (4.17)

The first thing to note is the absence of a cosmological constant when \( \sigma = \frac{6}{\kappa_5} \kappa. \) This value corresponds exactly to that of the matter content defined in Eq. (4.16). This implies that the presence of a positive cosmological constant on the brane requires a brane with tension greater than the one discussed above. This is called a supercritical brane.

Let us now look at the presence of gravity in the model. Another crucial aspect of Randal-Sundrum models is that gravity still appears as four-dimensional to observers confined to the hypersurface. This can be studied by using first order transverse and traceless gravitational perturbations (2.13) in five dimensions. Following [112, 113] and in a similar way as we did in section 2.2, one can find a differential equation of the form:

\[ -\frac{d^2 \zeta}{d\chi^2} + \left[ \frac{3}{4} \left( \frac{A'}{A} \right)^2 + \frac{3}{2} \frac{A''}{A} \right] \zeta = 0, \] (4.18)

where \( \zeta \) is the radial piece of the wave and a prime denotes a derivative respect to \( \chi \), which is a convenient conformal coordinate of the form:

\[ \chi = \int_0^r \frac{dr'}{A(r')} . \] (4.19)

The second derivative \( A'' \) is extremely important for identifying gravity on the braneworld. As the warp factor \( A(\chi) \) involves an absolute value, its second derivative will be a delta function \( \delta(r) \). Hence, the potential \( V(\chi) \) will peak at the location of the brane. Furthermore, when \( \Lambda_4 \geq 0 \), one can always find a normalisable solution for \( \zeta \) of the form:

\[ \zeta(\chi) \propto A^{3/2}(\chi). \] (4.20)

This implies that the wave peaks around the location of the brane and fades away when \( \chi \rightarrow \pm \infty \). This solution can be interpreted as the localisation of gravity and it means that gravity uniquely lives on the brane [110, 113].

In order to find how strong this four-dimensional gravity is, we can simply perform a dimensional reduction along the infinitely long fifth dimension \( r \), to find that [114]:

\[ \kappa_5 = \kappa_4 \int_{-\infty}^{\infty} dr e^{2[\epsilon, k, \theta(r) r + \epsilon, k, \theta(-r) r]} = \frac{\kappa_4^4}{k^4} . \] (4.21)

\[ \text{From action (4.11), perform a dimensional reduction along } r \text{ and identify the Planck mass in five and four dimensions.} \]
where we have imposed the choices (4.14). Although the extra dimension $r$ is not compact, it will still give a finite value for the dimensional reduction of the Newton's constant, as the volume along this direction is finite, due to the warping and the presence of the $\mathbb{Z}_2$ symmetry on the bubble wall.

Although the Randall-Sundrum construction uses elements that can easily be provided from string theory, it is not yet clear whether a phenomenological model can be obtained at all. The main point is that the type of brane required in order to induce a positive cosmological constant $\Lambda_4$, i.e. supercritical branes, are not those that can be found in string theory [115]. Moreover, the proposal discussed above seems to be in contradiction with the Maldacena-Nuñez no-go theorem [116]. Similarly, it was found in [117, 118] that the spacetime transverse to a domain wall interpolating between two different vacuum values always approaches the boundary of AdS. This implies that Randall-Sundrum four-dimensional de Sitter braneworld constructions are forbidden in string theory.

In any case, several iterations and variations of this model with interesting cosmological consequences have recently been studied in [119, 120]. The curious reader is welcome to get a better grasp of the proposals by reading these papers. In addition, we will extensively revisit braneworld constructions in stage III of this work. For now, we will take a quick look to the most famous mechanism for hiding extra dimensions: Compactifications.

### 4.2 Hills and valleys of the compactification

The second mechanism for "hiding" extra dimensions that we will explore in this chapter is that of compactifications. This technology is not something that has been proposed \textit{ad hoc} with the developments of string theory and its classical interpretation at the low energy limit, supergravity. In fact, compactification scenarios have already been studied by Kaluza and Klein in their seminal works [4, 10], where they described a way to unify classical electrodynamics and Einstein's theory by adding an extra space dimension to our four-dimensional cosmos. When this extra dimension is compact, for example, a circle, then dimensional reduction of the five-dimensional theory yields the usual four-dimensional Einstein-Maxwell equations, which involves a unification of gravity and electromagnetism. The main issue with this type of construction is that the lower-dimensional description also carries information about the size of the extra dimension in the form of a scalar.

\footnote{The Maldacena-Nuñez no-go theorem states that de Sitter solutions are not allowed in regular warped compactifications at tree level, i.e. de Sitter solutions would require from perturbative and/or non-perturbative corrections and/or the presence of $O_3$ planes, non-dynamical objects similar to branes, with mirror properties and negative tension.}
field.\textsuperscript{16} If this field turns out to be massless, its Yukawa interaction \cite{121} will be of infinite range, which would imply that this particle mediates a yet unobserved fifth force. Massless scalars have not been observed so far in the real world, so there must be a mechanism to develop a potential, with hills and valleys, that fixes the expected value of this scalar to something other than zero at the minimum of that potential. This technology is called \textit{stabilisation} and we will discuss it along this section.

Similarly to Kaluza and Klein, six out of the ten dimensions required by superstring theory must be compactified so that they are inaccesible to low-energy four-dimensional observers, i.e. so small that they are undetectable with the current technology at hand. The choice of the six-dimensional manifold to represent these compact dimensions will determine the properties of the four-dimensional world. However, there is a catch here: based on a statistical analysis of string compactifications, the number of possible solutions (vacua) is around $10^{500}$ \cite{122}. This huge variety is called the \textbf{Landscape}. This enormous collection should already raise a natural question: Which vacuum is the chosen one to represent the universe we live in? At the time of writing these lines, this is not well-known, and it is normally discussed under the umbrella of the anthropic mechanism \cite{123–125}.

It is not our aim to scrutinise the intricacies of such a vast landscape and the different combinations of stringy elements that can give rise to four-dimensional descriptions, but to present the basic concepts of the compactification mechanism in string theory. The best pedagogical approach to this is that of Freund-Rubin compactifications. Following along the lines of \cite{126}, we will present this topic in detail now.

Let us start with a $(D + d)$-dimensional spacetime $\mathcal{M}_{D+d}$, such that it can be decomposed into a product of the form $\mathcal{M}_{D+d} = \mathcal{M}_D \times \mathcal{M}_d$. While the $D$-dimensional space will so far be restricted to be maximally symmetric, the requirements for the $d$-dimensional one will be more severe. Inspired by the work of Kaluza and Klein \cite{4, 10}, we will require that this space is compact, i.e. closed and bounded. Compact directions will be identified as $(y \sim y + 1)$. The coordinate splitting will be such that:

$$X^M = \{x^1, \ldots, x^D, y^1, \ldots, y^d\}. \quad (4.22)$$

This splitting allows us to describe the space with the most general block-diagonal line invariant as

$$ds^2_{D+d} = g_{MN} dX^M dX^N = e^{2A(\phi(x) - \phi_0)} ds^2_D + e^{2B\phi(x)} ds^2_d, \quad (4.23)$$

\textsuperscript{16}This is the case for the dilaton field $\phi$ described at the introduction of this chapter and its connection to M-theory.
where $ds^2_D$ and $ds^2_d$ are respectively the metrics on $M_D$ and $M_d$ given the coordinates $\{x, y\}$ stated above. Note the presence of $\varphi(x)$ in the argument of the exponential factors before each line invariant. This object, which we will call *modulus*\(^{17}\), carries information about the size of the *internal* dimensions, hence their volume.

The action governing the dynamics of gravity in this space will be the $(D + d)$-dimensional version of the Einstein-Hilbert action discussed in section 2.1 without a cosmological constant. This reads:

$$S[g_{MN}] = \frac{1}{2\kappa_{D+d}} \int d^{D+d} X \sqrt{|g_{(D+d)}|} R_{(D+d)}. \quad (4.24)$$

If we now insert the Ansatz metric (4.23) into the action (4.24), we will see there is a linear combination of terms, such as the modulus $\varphi$, its derivatives and the Ricci scalar contributions from both *external* and *internal* space will arise. As the computational process can be frustrating with so many factors floating around, let us summarise the main steps to be taken to arrive at more familiar results:

• First, we notice that a metric of the form (4.23) yields a composite Ricci scalar (B.24), as discussed in appendix B.

• We then notice that the $\nabla^2$-terms do not contribute to the action, as they can be thought of total derivatives. The remaining terms in the Ricci expression will contribute with exponentials of $A, B$. Similarly, the determinant $|g_{D+d}|$, will leave a set of terms before the Ricci scalar $R_{(D)}$ of the form:

$$S = \frac{1}{2\kappa_{D+d}} \int d^D x \ d^d y \sqrt{|g_{(D)}| \sqrt{|g_{(d)}|}} \ e^{DA(\varphi(x))} e^{dB(\varphi(x))} e^{-2A(\varphi(x))} e^{2A(\varphi_0)} R_{(D)} + \cdots \quad (4.25)$$

We now impose the *Einstein* frame in such a way that there is no parametric dependence, but only the Newton’s constant (2.3) sits in front of the Ricci scalar. For this, we find:

$$B = -\frac{D-2}{d} A, \quad (4.26)$$

which allows us to make the identification:

$$\kappa_D = \frac{\kappa_{D+d}}{\text{Vol}_d} e^{(D-2)A\varphi_0}. \quad (4.27)$$

The previous equation shows that the gravitational coupling constant in the lower-dimensional theory is determined by the coupling in the

\(^{17}\)Moduli, in plural. These will be functions carrying parametric information about the *internal* directions.
higher-dimensional theory and by the volume of the \textit{internal} dimensions. This is the price we pay for being able to measure distances in the external space without the influence of the modulus $\varphi$, by choosing $\varphi_0$ as the vacuum expectation value of $\varphi$.

- In a similar way, it can be seen that when

\[
A^2 = \frac{d}{2(D + d - 2)(D - 2)},
\]

(4.28)

the factor in front of the kinetic term in expression (B.24) gets reduced to the canonical convention, i.e. 1/2.

- Finally, if we assume that the \textit{internal} Ricci scalar $R_{(d)}$ has a constant value, we can integrate the whole expression over the compact space directions, which will yield the Vol$_d$ factor shown earlier.

Carefully following the previous steps, we should arrive at the lower dimensional result:

\[
S[g_{\mu\nu}] = \frac{1}{2\kappa_D} \int d^D x \sqrt{|g(D)|} \left[ R(D) - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi(x) \partial_\nu \varphi(x) + e^{-2A\varphi_0 + 2(A-B)\varphi(x)} R(d) \right].
\]

(4.29)

The rich structure of this result contains very valuable information about the outcomes of a compactification of gravity. Let us look at each of the hidden features of the previous action.

Firstly, we should note the new factor before the integral. As discussed above, we are forcing the lower-dimensional theory to be in the \textit{Einstein} frame, which requires us to have the factor $\frac{1}{2\kappa_D}$ in front of the $D$ dimensional action. This leads us to the already discussed identification (4.27).

The second important thing to note in the action (4.29) is the presence of the function $\varphi(x)$ acting as a scalar field. This function, which was responsible for carrying information about the size of the compact dimensions has now been promoted to a scalar field in the $D$-dimensional theory. This is another relevant aspect of compactifications: Parametric functions which bear information about the properties of the \textit{internal} space get promoted to fields which decorate the lower-dimensional theory. Note the presence of the kinetic term for this volume field $\varphi(x)$. This entails that, although we have assumed the curvature of the \textit{internal} space to be constant, we can still see that its size (and in more general circumstances also the shape) are not fixed parameters. But the action (4.29) contains an additional term related to the volume modulus (third term in the expression). In fact, this term can be thought of as the potential that controls the dynamics of the volume modulus. This is subject to

\[
V(\varphi) = -e^{-2A\varphi_0 + 2(A-B)\varphi(x)} R(d).
\]

(4.30)
The vacuum expectation value of $\varphi$ can then be extracted from the potential above. If $\varphi(x)$ were massless, any fluctuation value of this field would be possible. This could have unwanted effects, as pathologies in the lower-dimensional theory that it decorates. Unfortunately, this would be the case for our volume modulus here as the potential (4.30) exhibits a runaway behaviour as $\varphi \to \infty$, assuming that the internal curvature does not vanish.

One might then be tempted to invoke quantum corrections to the action (4.29) to stabilise the moduli values. For example, if we were thinking of string theory, this could be achieved by taking into account curvature $\alpha'$ and string coupling $g_s$ corrections. However, Dine and Seiberg proved that using these techniques would lead us into regimes where we lose all perturbative control over the theory [127]. Perhaps, there is no need to resort to any quantum help. In fact, field strengths associated with gauge fields could provide some help with this task. We will call them fluxes.$^{18}$ For example, we can consider a $(D + d)$-dimensional flux which fills all directions of the internal space, as:$^{19}$

$$F_{MN\ldots W} = \begin{cases} q \epsilon_{ij\ldots k} & \text{if } \{M, N, \ldots, W\} = \{i, j, \ldots, k\}, \\ 0 & \text{else,} \end{cases}$$

(4.32)

where $\epsilon$ represents the Levi-Civita tensor (i.e. the volume form) on the compact manifold and $q$ is an arbitrary constant. The action (4.24) will now have an additional contribution representing the presence of the flux $F_{MN\ldots W}$ in the compact dimensions. This new term will also propagate down through the dimensional reduction, yielding a new contribution to the potential $V(\varphi)$ of the form:$^{20}$

$$S_{F_d} = -\frac{1}{2\kappa_{D+d}} \int d^{D+d} X \sqrt{|g_{(D+d)}|} \frac{1}{2d!} F_{MN\ldots W} F^{MN\ldots W}$$

$$= -\frac{1}{2\kappa_D} \int d^{D} x \sqrt{|g_{(D)}|} q^2 \frac{1}{2} e^{-2A\varphi_0 + 2(A-d)B\varphi(x)}.$$  \hspace{1cm} (4.33)

Finally, the total potential, including the flux term, reads:

$$V(\varphi) = e^{-2A\varphi_0} \left( e^{2(A-B)\varphi(x)} R_{(d)} + \frac{q^2}{2} e^{2(A-d)B\varphi(x)} \right).$$  \hspace{1cm} (4.34)

$^{18}$We are abusing the notation here. The fluxes are the integrals of these field strengths over the non-trivial cycles of the compact dimensions around which they wrap.

$^{19}$This choice of flux is convenient because it automatically satisfies the equation of motion and the Bianchi identity,

$$d \star F = dF = 0,$$

(4.31)

where $\star$ is the Hodge star operation described in appendix A.

$^{20}$Note that the factors $e^{-2B\varphi(x)}$ will come from the $d$ metric entries to lower the indices of $F^{MN\ldots}$. The contraction of the $\epsilon_{mnl\ldots} \epsilon^{mnl\ldots}$ will yield $n!$. 

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We now see that the flux contribution $q^2$ gives an extra term which can create a minimum. In fact, this is given by:

$$e^{2(d-1)B\phi_{\text{min}}^2} = q^2 \frac{A - dB}{2(A - B) R_{(d)}} = q^2 \frac{d(D - 1)}{2(D + d - 2) R_{(d)}},$$

(4.35)

where we have used relations (4.26) and (4.28) to explicitly show the dimensionality dependence. Note that the left hand side is definite positive. So are the dimensional factors on the right hand side of expression (4.35). Hence, the curvature of the internal space must be positive. Furthermore, if we now identify $\phi_{\text{min}}$, i.e. the vacuum expectation value of $\phi(x)$ with $\phi_0$, we obtain the potential energy at the minimum to be:

$$V(\phi_{\text{min}}) = -\frac{(d-1)(D-2)}{d(D-1)} e^{-2B\phi_0} R_{(d)}.$$

(4.36)

As previously commented, the curvature of the internal directions is positive, which means that the vacuum energy is negative. The good thing is that we have stabilised the modulus field, avoiding the aforementioned pathologies. When the field rests at this minimum, the potential will contribute as a constant in the dimensionally reduced theory. This can then be identified with the cosmological constant in the Einstein-Hilbert action (2.2). However, this identification means that we cannot obtain a positive cosmological constant $\Lambda$. The negative cosmological constant $\Lambda$ will then fix the external dimensions to be anti-de-Sitter-like. Not exactly what we observe in our cosmos. And this is not the only problem we can find in this construction.

If we take a closer look at the internal size scale evaluated at the minimum value of $\phi_{\text{min}}$, controlled by the factor $e^{2B\phi}$, we see that it scales as $R^2 \sim e^{2B\phi_{\text{min}}}$, where $R$ represents the volume scale of the internal manifold. If we apply exactly the same analysis to the cosmological constant in the lower-dimensional theory, we will see that its size, which corresponds to the length scale squared of the external space, scales in the same fashion.\textsuperscript{21}

This shows that the scales of both internal and external spaces are the same. This contradicts the very essence of compactifications, where compact dimensions are expected to be stabilised at unobservable length scales.

In principle, one could argue that these complications and unsatisfactory results are artifacts caused by our toy’s model choice. However, the construction discussed above is not a toy model, but a generalisation of concrete examples of stringy constructions studied over the years. If we look at dimensional reductions of ten or eleven-dimensional string theory on compact spaces in the form of $d$-dimensional spheres, we will find that the best known examples are: AdS$_5 \times S^5$ in type IIB \cite{128, 129} and AdS$_4 \times S^7$\textsuperscript{21}.

\textsuperscript{21}Observe that $\Lambda_{\text{AdS}}^2 \sim \Lambda^{-1} \sim \nu^{-1} \sim e^{2B\phi_{\text{min}}}$.  

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or AdS$_7 \times S^4$ from M-theory [130, 131]. However, one can already see from
the notation we used for the *external* space that the result was discovered to
be on similar grounds as our simple example [127].

In summary, Freund-Rubin solutions seem to suffer from two undesir-
able results that will not help us to configure a lower-dimensional descrip-
tion capable of reproducing our observable cosmos. These are:

- The vacuum energy is non-positive, $V(\phi_0) \leq 0$, leading to anti-de Sit-
ter or Minkowski solutions in the external space. This shows how chal-
lenging to obtain low-dimensional de Sitter solutions from string the-
ory is.

- There is no *scale separation*. The sizes of the *external* and *internal*
spaces are of the same order. This means that compactified extra di-
mensions are not small compared to the cosmological scales of our
four-dimensional universe.

One could then think of using more convoluted *internal* spaces to per-
form the compactification. Calabi-Yau three-folds (CY$_3$) are a much richer
alternative to $d$-spheres. This type of spaces are three-complex dimensional
manifolds that can be thought of six-real-dimensional spaces. This already
sounds promising, as if we were to start from type IIB supergravity (4.2), the
resulting external space will have four dimensions, ideal for making contact
with observable phenomenology. Although the literature about this type of
compactifications is extense, given the subtleties and intricacies discovered
over the years, we will only need to know very basic concepts about these
spaces to understand the lower-dimensional construction on them.\footnote{We would recommend this sweet set of notes by Van Riet and Zoccarato [126].}

First, Calabi-Yau spaces are Ricci-flat and are equipped with a complex
structure form $J$ and a globally defined non-vanishing complex three-form
$\Omega$. Similarly to the $d$-dimensional spheres discussed above, these spaces
have complex and Kähler moduli, which characterise the *shape* and *size*
of the manifold respectively. These fields are universal for any Calabi-Yau
compactification and carry information about *internal* sections of Calabi-
Yau spaces, which are called *cycles*. This more complicated structure of the
space under study will allow us to "wrap" fluxes along some of these cycles,
yielding more moduli that can decorate the lower-dimensional theory. In
the same way as in Freund-Rubin compactifications discussed above, these
fields should then be sitting at a minimum of the potential to be stabilised.

One of the most famous type IIB flux compactifications on Calabi-Yau
manifolds is that of Giddings, Kachru and Polchisnki (GKP) [68, 132]. If we
keep $\{F_3, H_3, F_5\}$ as the only switched on fluxes and redefine some of the
fields present in Eq. (4.2) as:

\[
\tau = C_0 + i e^{-\phi}, \quad G_3 = F_3 - \tau H_3, \quad \text{with } \star G_3 = i G_3, \quad (4.37)
\]

we see that expression (4.2) can be written as:

\[
S_{\text{IIIB}} = \frac{1}{2\kappa_{10} G_3^2} \int \sqrt{|g_{(10)}|} \left( R_{(10)} + \frac{\partial_M \tau \partial^M \bar{\tau}}{2(\text{Im } \tau)^2} - \frac{|G_3|^2}{2(\text{Im } \tau)} - \frac{|F_3|^2}{4} \right) + \frac{1}{2\kappa_{10}} \int \frac{1}{4\text{Im } \tau} C_4 \wedge G_3 \wedge \bar{G}_3, \quad (4.38)
\]

where a bar defines the complex conjugate of the quantity. The field \( \tau \) in Eq. (4.37) is called \textit{axio-dilaton}. This action can be compactified on a familiar metric Ansatz as:

\[
ds_{10}^2 = e^{2a(y)} ds_4^2 + e^{-2a(y)} ds_6^2. \quad (4.39)
\]

Note that we now require the exponential factors to depend on the compact dimensions \{y\}. Without going into the details arising from the compactification mechanism, it can be shown that the compactification of the action (4.38) using the aforementioned Ansatz (4.39) will yield a \( \mathcal{N} = 1 \) four-dimensional action of the familiar form:

\[
S = \frac{1}{2\kappa_4} \int d^4x \sqrt{|g_{(4)}|} \left( R_4 - \mathcal{K}_{ij} \partial_\mu \Phi^i \partial^\mu \bar{\Phi}^j - V(\Phi) \right) \quad (4.40)
\]

where the potential is

\[
V(\Phi) = \frac{2}{\kappa_4} e^\mathcal{K} \left( \mathcal{K}^{ij} (D_i \mathcal{W}) (D_j \bar{\mathcal{W}}) - 3|\mathcal{W}|^2 \right), \quad (4.41)
\]

with \( \mathcal{K} \) as the Kähler potential\(^{24}\) given by

\[
\mathcal{K} = -3 \log \left( -i (\rho - \bar{\rho}) \right) - \log \left( -i (\tau - \bar{\tau}) \right) - \log \left( i \int_{M_6} \Omega \wedge \bar{\Omega} \right) \quad (4.43)
\]

The first main difference to be noticed between the dimensional reduction of the action (4.29) and Eq. (4.40) is that the kinetic term has the presence of a metric-like object \( \mathcal{K}_{ij} \). This metric represents the moduli-space metric, i.e. the space of configurations that all moduli fields, represented by

\(^{23}\)Further details about the conditions on the fluxes, tadpole cancellations and more can be found in [104].

\(^{24}\)In fact, the Kähler potential gives rise to the metric of the moduli space such that

\[
\mathcal{K}_{ij} = \frac{\partial^2}{\partial \Phi^i \partial \Phi^j} \mathcal{K}. \quad (4.42)
\]
The potential controlling their dynamics is given by Eq. (4.41),

\[ \mathcal{W} = \int_{\mathcal{M}_6} G_3 \wedge \Omega_3. \]  

This is also known as the Gukov-Vafa-Witten (GVW) superpotential \[133\] and it carries information about the presence of the flux \( G_3 \) along the compact dimensions. Its covariant derivative \( D_i \mathcal{W} \) is called the \( F \)-term and it protects supersymmetry when its value is 0.

If we require supersymmetry to be preserved,

\[ D_i \mathcal{W} = \partial_i \mathcal{W} + (\partial_i \mathcal{K}) \mathcal{W} = 0, \]  

we can already see the result of the potential (4.41). A negative value. But in fact, this is not completely true. As it was shown in \[132\], the superpotential \( \mathcal{W} \) is independent of the volume modulus \( \rho \sim e^{2B\phi_0} \), which allows us to write

\[ \mathcal{K}^\rho \bar{\rho} D_\rho \mathcal{W} D_{\bar{\rho}} \mathcal{W} = 3|\mathcal{W}|^2, \]  

so we can remove the superpotential \( \mathcal{W} \) piece in Eq. (4.41) in favour of:

\[ V(\Phi) = \frac{2}{\kappa_4} \sum_{i,j \neq \rho} e^{\mathcal{K}} \left[ \mathcal{K}^{ij} (D_i \mathcal{W})(D_j \mathcal{W}) \right]. \]  

Note that now the indices do not run over the volume modulus \( \rho \). In this way, the potential remains flat along \( \rho \), the volume direction, which translates into a massless volume modulus. Hence, the scalar potential is of the no-scale type.

One can then use non-perturbative corrections to create a minimum for the volume modulus. This can be done by adding an extra piece to the tree-level superpotential of the form:\[25\]

\[ \mathcal{W} = \mathcal{W}_{\text{GVW}} + \mathcal{W}_{\text{non-pert}} = \mathcal{W}_0 + \mathcal{A} e^{-ia\rho}, \]  

where \( \mathcal{W}_{\text{GVW}} \) is the superpotential (4.44), \( \mathcal{A} \) is a factor independent of \( \rho \) and "\( a \)", the warp factor, is now a constant number. For simplicity, we will assume that \( \mathcal{W}_0, a \) and \( \mathcal{A} \) are real and the axionic piece of the volume modulus \( \rho \) is zero, i.e. \( \rho = i\sigma \). Again imposing supersymmetry, we find that:

\[ D_\rho \mathcal{W} = 0 \quad \rightarrow \quad \mathcal{W}_0 = -\mathcal{A} \left( \frac{2}{3} a\sigma_0 + 1 \right) e^{-a\sigma_0}. \]  

\[25\]The source of the non-perturbative effects may be Euclidean \( D_3 \)-branes wrapping four-cycles of the compact space. Another possible source is that of gluino condensation on \( D_7 \)-branes wrapping internal four-cycles.
This value of the superpotential yields a minimum of the potential 

\[ V_0 = -\frac{a^2 \mathcal{A}^2}{3\kappa_4 \sigma_0} e^{-2a\sigma}. \]  

(4.50)

As we can see, this additional term will contribute to stabilise the volume modulus and achieve the desired moduli stabilisation. However, the minimum is negative (see figure 4.2), resulting into an anti-de Sitter vacuum solution. This vacuum value can be uplifted by adding anti-D_3 branes\(^{26}\) to the construction. As observed in [134], the presence of these anti-branes will have two main effects: they will break supersymmetry and they will also add an extra energy contribution to the potential:\(^{27}\)

\[ \delta V(\sigma) = \frac{c(N)}{\sigma^3}, \]  

(4.51)

where \(c(N)\) is a coefficient which depends on the number of anti D_3-branes we set in the construction. Including the anti-D_3-brane contribution gives the scalar potential:

\[ V(\sigma) = \frac{a \mathcal{A}}{\kappa_4 \sigma^2} \left( \frac{1}{3} a \sigma \mathcal{A} e^{-a\sigma} + W_0 + \mathcal{A} e^{-a\sigma} \right) + \frac{c(N)}{\sigma^3}. \]  

(4.52)

For suitable choices of \(c(N)\), the Anti-de Sitter minimum becomes a de Sitter one, but the rest of the potential does not change too much, as can be seen in figure 4.2. The most important result here is the presence of the

![Figure 4.2. Four-dimensional potential without the presence of anti-D_3-branes and the uplifted version due to the brane contribution. Parametric choices: \(\{a = 0.15, \mathcal{A} = 1, W_0 = -0.5\}\) and \(c(N) = \{0, 0.29\}\), respectively.

\(^{26}\)D_3 branes with opposite charge to the usual ones.

\(^{27}\)However, both changes will be by a very small amount. The reason for this is that the anti-branes will try to minimise their energy, by moving to regions of the compact space where the warp factor is exponentially suppressed.
positive minimum, which gives a positive and small four-dimensional cosmological constant $\Lambda_4$ from string theory. This construction is called KKLT, after Kachru, Kallosh, Linde and Trivedi [135]. It is one of the most studied constructions of a small and positive four-dimensional constant from string theory. However, several of its aspects (i.e. stability of the Anti-de Sitter vacuum or the uplifted de Sitter one, for example) are still under active discussion and a consensus on its validity has not yet been reached [136–143].

Of course, this is not the only construction that the physics community has considered over the years. Constructions from type IIA theory easily allow for the stabilisation of all moduli, yielding supersymmetric AdS$_4$ solutions [144]. A promotion to dS$_4$ can be achieved by introducing other ingredients from string theory [145, 146] or using negatively curved manifolds with $R_6 < 0$. Nevertheless, it has been shown that all these dS solutions contain tachyons [147]. Alternatively, one can make use of non-geometric fluxes, i.e. fluxes without a geometric source in the theory, which can be obtained by using string dualities. Although this type of solutions seems to give the desired dS vacua [148, 149], one would need to find interpretations of the backgrounds from which those fluxes arise.

When it comes to studying compactifications from type IIB, we can find the aforementioned KKLT construction and the Large volume scenario (LVS) [150]. This proposal breaks supersymmetry in the GKP construction by using perturbative corrections to the superpotential $\mathcal{W}$. The advantage of this vacuum, over the KKLT AdS vacuum, is that the volume is typically much larger pushing the solution into a regime where the corrections are under better control. To uplift this solution to the desired small and positive vacuum, anti-branes can again be used. More constructions based on the GKP model, but using other corrections such as string loops, instantons, gaugino condensation or $\alpha'$-corrections can be found in [151–156].

However, it seems that part of the string phenomenology community feels that most of the above constructions may have their flaws, such as ingredients that are not well understood or the parametric control of supersymmetry breaking may not be completely controlled [156–159], etc. Extensive reviews of several other aspects of de Sitter constructions from string theory can be found in [160, 161]. What seems clear is that, although string theory easily accommodates a quantum description of gravity, together with perturbative renormalisation and no free parameters, when it comes to providing a lower-dimensional cosmological constant similar to the one we observe, it requires complicated constructions where not all the components are completely understood and where de Sitter solutions seems to be rare [147]. Even in the worst scenario, string theory may fail to provide a single de Sitter vacuum, as recent developments seem to indicate [162]. Given all evidence gathered over the years about the scarcity of low-
dimensional solutions with a positive cosmological constant, the physics community has changed their way of thinking about effective field theories and their relation to UV-complete theories. In order to understand all the intricacies and details of the aforementioned connection, the question that needs to be addressed is the following: What properties should an effective field theory have such that they can be obtained from a theory of quantum gravity?

This question will lead us to explore the lands beyond the String Kingdom domain. Unexplored territories where the map says: *Hic sunt dracones.*

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28In Latin, here are dragons. It makes reference to the medieval practice of putting illustrations of dragons, sea monsters and other mythological creatures in uncharted areas of maps where potential dangers were thought to exist.
5. The Swampland, a Bog in the Shadows

If everything in the Gurmag swamp hungers for (UV-complete theories seeking) human flesh, what bait could be more effective?

Adapted flavor text from Gurmag Angler’s Magic, The Gathering card

Chapter 4 was dedicated to providing a (quick) bird’s eye view of the Stringy Kingdom’s jurisdiction. We visited the prairies of braneworld constructions and learnt that some models lacked gravity, or did not take into account brane backreaction effects in the higher-dimensional picture or that the type of branes needed to induce a four-dimensional cosmological constant had to be supercritical. We then set off to the vast territory of the peaks and valleys of string flux compactifications. And here we learnt that this mechanism faces several fundamental problems. To build models where all the moduli are stabilised, equipped with scale separation and a small and positive cosmological constant $\Lambda_4$ seem to be unsolved issues [160, 161]. Even if some flux compactification constructions could avoid the above problems, any attempt to improve the model (i.e. to be able to reproduce some sectors of the standard model of particles) would require finding new solutions free from the problems discussed above.

These difficulties in string compactification have inspired researchers to go beyond the lands of the stringy kingdom, with the aim of charting its borders, if there are any. In order to do so, they proposed to take a different approach to the problem at hand; instead of aiming to obtain four-dimensional effective field theories with a positive cosmological constant $\Lambda_4$ from string theory, they thought that it could be more illuminating to identify, among all the existing lower-dimensional EFTs, those that can actually arise from quantum gravity. In order to avoid examining one by one the incalculable number of existing effective field theories, the idea is to find universal patterns that can be used as practical criteria to filter "good" EFTs from "bad" ones based on general characteristics. These screening criteria, which have not been proved yet,\(^1\) are formulated in the form of conjectures that represent the very core of the Swampland programme [164], i.e. to identify the borders between the landscape of good EFTs and the

\(^{1}\)See [163] for the first steps in proving one of these criteria.
swamp, those EFTs that do not meet the criteria to be obtainable from UV-complete theories.

The set of Swampland conjectures has grown since the programme was initiated in [164]. Interestingly, it has recently been shown that many of these conjectures are related, suggesting that they are no more than different aspects of some more fundamental principles of quantum gravity [162, 165–169]. In this chapter we will not deal with the ever-growing number of swampland conjectures, but we will select and present the basic concepts of some of them, which are relevant for the problem of finding de Sitter vacua in string theory and which are also the basis of the proposal to be presented in stage III of this work. The curious reader about the current state of the art of the Swampland is more than welcome to visit references presented above and the subsequent references therein.

This chapter is structured as follows: Section 5.1 discusses the difficulty of obtaining low-dimensional vacua with a positive cosmological constant. This is investigated by means of de Sitter conjecture(s), which set bounds on several features of de Sitter vacua from string theory. Although not related (at first sight) to the issue of positive vacua, we will examine in section 5.2 how gravity must always be the weakest force in any low energy description of quantum gravity with gauge forces. The consequences of the weakness of gravity for the fate of non-supersymmetric Anti-de Sitter spaces will be finally examined in section 5.3.

5.1 A watchtower to outlaw de Sitter theories

We have seen in chapter 4 how difficult it can be to construct de Sitter vacua in string theory. Furthermore, the few solutions that can be found are under scrutiny and are the source of debate about the components and/or mechanisms to build them [147, 160, 161]. One can then say that there is no full-fledged top-down de Sitter construction in a controlable regime of string theory. This suggests two different ways of thinking about this concern; on the one hand, it could be that these are just technical difficulties that will be overcome in the future as our understanding of the mathematical machinery used improves. On the other hand, this absence of de Sitter constructions points to more fundamental obstacles to the construction of de Sitter vacua in string theory. This is the route taken within the swampland programme.

We will now introduce the first swampland conjecture relevant to the present work, motivated by the difficulties of obtaining de Sitter vacua from string theory [170–172]:

78
in quantum gravity. This de Sitter Conjecture (dSC) can also be read as: de Sitter belongs to the swampland. This very strong conjecture has been formulated as different bounds on the scalar potential of an effective field theory which is coupled to Einstein gravity. These read:

\[
\min \left( \nabla_i \nabla_j V \right) \leq -\frac{c'}{M_{Pl}^2} V, \quad \text{and} \quad |\nabla V| \geq \frac{c}{M_{Pl}} V, \quad (5.1)
\]

with \( c \) and \( c' \) positive constants of \( O(1) \) in Planck units. The \( \min (\nabla_i \nabla_j V) \) represents the minimum possible eigenvalue of the mass matrix, i.e. the Hessian and \( |\nabla V| \) is the norm of the vector derivatives with respect to all scalar fields in the theory. This condition provides a lower bound on the slope of the potential and states that if the potential is positive, it must be step enough not to allow for extrema points \[170, 173\]. This last bound only needs to be imposed if the Hessian bound is violated, as proposed in the refined version of the de Sitter conjecture \[174\].

But these are not the only constraints that restrain the shape of the scalar potential. The Transplanckian Censorship Conjecture (TCC) states that any sub-Planckian quantum fluctuation should remain quantum. This implies that the expansion of the universe must slow down before any fluctuation become stretched beyond a Hubble size \[175\]. This restriction can be expressed in terms of the initial and final scale factors as follows:

\[
\frac{a_f}{a_i} \times \ell_{(D)} < \frac{1}{H_f}, \quad (5.2)
\]

where \( \ell_{(D)} \) is the \( D \)-dimensional Planck length and \( H_f \) is the final Hubble parameter (i.e. the inverse Hubble radius at which the expansion should end to prevent quantum fluctuations from becoming larger than the Hubble radius). Using both the classical Friedmann equations and the equations of motion for a scalar field \( \phi \) coupled to gravity controlled by a potential \( V(\phi) \), one can find a bound to the potential given by:

\[
|\nabla V| \geq \frac{2}{M_{Pl} \sqrt{(D-1)(D-2)}} V, \quad (5.3)
\]

where the potential \( V(\phi) \) is assumed to remain positive along the path connecting \( \phi_f \) and \( \phi_i \) in the moduli space. The bound (5.3) appears at asymptotic regimes, i.e. at the asymptotic boundaries of the moduli space, when \( \phi_i, \phi_f \rightarrow \infty \).

\[2\text{At least in the asymptotic regions of the moduli space. See the next footnote for a definition of such a space.}\]

\[3\text{The moduli space is the space of solutions to the equation of motion of all the fields } \Phi = \{\phi_1, \ldots, \phi_n\} \text{ in the theory. The asymptotic regions of this space contain the weak coupling limit theories. The values } \phi_f \text{ and } \phi_i \text{ are the coordinates of this space, and represent the final value of the scalar field when the scale factor is } a_f \text{ and } a_i, \text{ respectively.}\]
The similarities between the TCC bound (5.3) with the dSC bound (5.1) are not a coincidence. In fact, it can be shown that the undetermined constant $c$ in the de Sitter bound (5.1) becomes fixed to $2/\sqrt{(D-1)(D-2)}$ when one takes the values of the field $\phi$ to be in the asymptotic regimes of the moduli space. Using the Swampland Distance Conjecture (SDC), one can then show that the asymptotic behaviour of the scalar potential $V$ is the same as that predicted by the TCC.

5.2 Memorial to the weakness of gravity

Let us consider a $D$-dimensional effective field theory with a $U(1)$ gauge symmetry (i.e. a Maxwell theory) coupled to gravity, described by

$$S = \int d^D X \sqrt{|g(D)|} \left[ \frac{1}{2\kappa_D} \mathcal{R}^{(D)} - \frac{1}{4g^2} \mathcal{F}_{MN} \mathcal{F}^{MN} \right], \quad (5.4)$$

where $g$ is the gauge coupling of the theory. The (electric) version states that there must exist a particle in the theory with mass $m$ and charge $q$ that satisfies the inequality [177]:

$$m \leq \sqrt{\frac{D-2}{D-3}} g q \left( \frac{M_{\text{pl}}^{(D)}}{r} \right)^{\frac{D-2}{r}}, \quad (5.5)$$

where $M_{\text{pl}}^{(D)}$ is the $D$-dimensional Planck mass (2.3). This inequality becomes a strict inequality in the absence of supersymmetry. Another interpretation of the inequality (5.5) is as follows:

*Gravity is the weakest force of all in a theory of quantum gravity with gauge forces.*

This conjecture may seem *ad hoc* at first sight, but it has deep foundations in black hole physics. Consider a four-dimensional charged (i.e. Reisner-Nordström) black hole with line invariant:

$$ds_{\text{RN}}^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2 + r^2 \, d\Omega_2^2, \quad (5.6)$$

where

$$f(r) = 1 - \frac{2M_{\text{ADM}}}{r} + \frac{2g^2Q^2}{r^2}, \quad (5.7)$$

4The SDC states that an infinite tower of states of an effective field theory coupled to gravity becomes exponentially light at any infinite field distance limit in the moduli space [171, 176]. As no tower of light states has been observed in our universe, we can conclude that we may not be living in such asymptotic regimes.

5As the magnetic version is not relevant for this work, we refer the reader to [177].

6According to the Coleman-Mandula theorem, the only symmetry that can relate the mass (related to Poincaré symmetry) and the charge (related to an internal symmetry) is supersymmetry [178].
where $M_{\text{ADM}}$ is the mass of the black hole\(^7\) and $Q$ is its charge. The quadratic form of the function $f(r)$ gives two horizons for a Reisner-Nordström black hole as:

$$r_{\pm} = M_{\text{ADM}} \pm \sqrt{M_{\text{ADM}}^2 - 2g^2 Q^2}. \quad (5.8)$$

The horizon is **degenerate** when $r_+ = r_-$. Furthermore, the horizon(s) require that the black hole mass satisfies an extremality bound in order to avoid the presence of naked singularities [179]. This reads:

$$M_{\text{ADM}} \geq \sqrt{2} g Q M_{\text{Pl}}, \quad (5.9)$$

where $M_{\text{Pl}}$ has been restored to match dimensionality. A black hole that saturates the previous bound is called **extremal**. One can then see that the outer horizon shrinks towards the inner horizon as one approaches the extremality bound. The violation of this bound is associated with a naked singularity. Note that when $D = 4$, expression (5.5) and the lower bound (5.9) are the same with **opposite** orientation of the inequality. This coincidence is at the heart of black hole decay [177]. This can be understood by studying the evaporation process of a black hole.

![Figure 5.1. Decay of an extremal black hole.](image)

Black holes which are charged under a gauge $U(1)$ symmetry will induce an electric field around their horizon. This field allows for a discharge process analogous to Hawking radiation [180, 181]. While there are two different discharge channels, depending on the temperature-to-mass ratio of the

\(^7\)A definition of the ADM formalism can be found in section 3.1.
black hole, the one we are interested in is that of a cold black hole. This happens when \( T_H \ll m \). In this case, the black hole discharges electrically through Schwinger pair production.

Now consider an extremal black hole such that \( M_{\text{BH}}/Q_{\text{BH}} = 1 \) as in figure 5.1. A requirement for the discharge of this black hole is that the theory must contain a charged particle that can be emitted by the Schwinger process. If such particle exists, the black hole can decay into a pair of products such that one satisfies the extremality bound for black holes, i.e. \( M_1 \geq Q_1 \). By kinematic conservation (i.e. some of the total mass at rest of the initial black hole must be converted into kinetic energy), the second product cannot have a charge less than the mass, but greater than! This implies that the second object is not a smaller black hole, but a particle. This kinematic restriction can be derived from the conservation of energy and charge as follows. The mass \( M_{\text{BH}} \) of the initial black hole state must be greater than the sum of the decay product masses \( M_i \). However, the charge \( Q_{\text{BH}} \) of the initial black hole should be equal to the sum of the products \( Q_i \). This is

\[
M_{\text{BH}} \geq \sum_i M_i, \quad Q_{\text{BH}} = \sum_i q_i. \tag{5.10}
\]

This entails

\[
\frac{M_{\text{BH}}}{Q_{\text{BH}}} \geq \frac{1}{Q_{\text{BH}}} \sum_i M_i = \frac{1}{Q_{\text{BH}}} \sum_i \frac{M_i}{q_i} q_i \geq \frac{1}{Q_{\text{BH}}} \left( \frac{M}{Q} \right)_{\min} \sum_i q_i = \left( \frac{M}{Q} \right)_{\min} \tag{5.11}
\]

Therefore, the existence of a particle with a charge-to-mass ratio greater than that of the black hole is need for the latter to decay.

The previous discussion has been focused on a field theory with a \( U(1) \) gauge symmetry. More generally, one can consider antisymmetric tensors or rank \( p \), as the ones that charge objects such as the \( (p - 1) \)-dimensional branes that we saw in the introduction of chapter 4. It has been proposed that the WGC can be extended to these (black) branes. Hence, given an abelian \( p \)-form with gauge coupling \( g \), the WGC requires the existence of a charged \( (p - 1) \)-brane of tension \( T \) and integer charge \( Q \) such that \[177,184\]:

\[
\frac{p(D - p - 2)}{D - 2} T^2 \leq \frac{Q^2}{\kappa^{2-D}}. \tag{5.12}
\]

---

8The Hawking temperature of a black hole is proportional to the difference between the outer and inner horizons. It is intrinsically related to the extremality of the black hole [182]. At \( T_H \gg m \), the black hole thermally discharges itself.

9A Schwinger effect is a predicted physical phenomenon that states that pairs of matter-antimatter particles can be created as a vacuum decay process in the presence of an electric field [183].

10The adjective black is added to create an analogy with black holes, as these objects are also surrounded by a horizon.
We will see that this has interesting consequences for the when applied to the stability of non-supersymmetric Anti-de Sitter vacua in string theory.

5.3 The necropolis of the fallen non-susy AdS vacua

In order to understand the implications of the weak gravity conjecture described above, we will consider a $D$-dimensional Anti-de Sitter vacuum supported by fluxes. As it was discussed in section 4.2, the presence of fluxes (let us denote them by the letter $f$) will generate a potential that stabilises some of the moduli resulting from the compactification. Let us assume that the minimum corresponds to an AdS vacuum. These fluxes are Hodge dual to the top form gauge field strengths $\tilde{F}_D$, which are related to the gauge fields $\mathcal{C}_{D-1}$. According to the WGC discussed above, this form will charge a $(D-2)$-dimensional brane, which will play the role of an hypersurface of co-dimension one (see appendix C for further details). This object will have a tension $T$, charge $Q$ and will obey the rules defined in Eq. (5.12).

This hypersurface, located somewhere in the previously discussed $D$-dimensional AdS space, will interpolate between two different vacua with different values for the fluxes. Due to charge conservation the value of the flux$^{11}$ on one side of the brane will be $f$, while on the other side of the wall it will be equal to $f + Q$. If the vacuum hosting the brane is supersymmetric, then the brane will saturate the inequality (5.12) and remain steady at its position, separating two different supersymmetric configurations. However, if the vacuum is non-supersymmetric, then the weak gravity conjecture requires the strict inequality of equation (5.12). But such a co-dimensional one brane with tension less than the charge will correspond to an instability in the Anti-de Sitter space. Intuitively, we can see this process as simple conservation of energy: As the tension must be less than the charge, the energy cost of expanding the bubble will be less than the energy gain from the electric repulsion between different points on the wall. Therefore the brane will expand, mediating the decay of the vacuum with $f + Q$ to that inside the brane (i.e. the bubble) with vacuum $f$. We can then say that:

_**any non-supersymmetric AdS geometry supported by flux is unstable.**_

In other words, supersymmetry is the only mechanism to protect a vacuum decay from decaying into quantum gravity. This conjecture first appeared in [185, 186], but its foundations can be found in an older observation called _AdS fragmentation_ in [187]. In this work, it was shown that in a $D$-dimensional Anti-de Sitter space containing a spacetime filling flux of the same rank, there will be $(D-2)$-branes charged with respect to the flux,

---

$^{11}$Understand this value as the integral of $F_p$ over some internal non-trivial $p$-cycle of the compact space.
which can nucleate and expand towards the boundary of the AdS space, leaving behind an AdS space with one less unit of flux. The viability of this process depends on the charge-to-tension ratio of the brane. In order to understand this process, let us construct the instanton solution\textsuperscript{12} that mediates the vacuum decay.

Let us consider the following parametrisation of an AdS\textsubscript{D} metric,
\begin{equation}
\text{d}s^2 = L^2 \left( \cosh^2 (r) \text{d}r^2 + \text{d}r^2 + \sinh^2 (r) \text{d}\Omega_{D-2}^2 \right),
\end{equation}
where $L$ is the AdS radius. A spherical $(D-2)$-brane with radius $r(\tau)$ wrapping around $S^{D-2}$ and evolving in the Euclidean time $\tau$ is described by the action [185]:
\begin{equation}
S = L^{D-1} \Omega_{D-2} \int \text{d}\tau \left( T \sinh^{D-2} (r) \sqrt{\cosh^2 (r) + \left( \frac{\text{d}r}{\text{d}\tau} \right)^2} - Q \sinh^{D-1} (r) \right),
\end{equation}
where $\Omega_{D-2}$ is the volume of the unit $(D-2)$-sphere. One can then solve the Euclidean Einstein's equations, to find the radius for the spherical brane solution that extremises the action to be:
\begin{equation}
r_0 = \text{tanh}^{-1} \left( \frac{T}{Q} \right),
\end{equation}
in Planck units. In the Lorentzian signature, this is associated with the nucleation at rest of a bubble of radius $r_0$, which begins to expand, and mediates the decay of the vacua. Imposing the strict inequality of the WGC condition, we see that
\begin{equation}
r_0 < \text{tanh}^{-1} (1).
\end{equation}
This implies that a finite value of the Euclidean action can only be achieved with the strict inequality version of expression 5.12. On the contrary, if the vacuum is supersymmetric, we could satisfy the WGC by a BPS brane saturating the inequality (5.12), which implies an infinite radius solution, i.e. a non-expanding straight brane interpolating between two different supersymmetric vacua.

With this final conjecture we conclude our brief summary of the swamp-land program. Perhaps our dear reader is now wondering if a brane like the one mediating the aforementioned decay could be used to realise similar braneworld constructions to those discussed in section 4.1. In contrast to de Sitter solutions, Anti-de Sitter ones are quite abundant in supergravity and have been studied in extension. If some of these spaces have broken supersymmetry, they should decay via nucleation of a co-dimension one

\textsuperscript{12}The decay via brane nucleation will be a non-perturbative process, governed by the instanton (i.e. the brane) solution of the nucleated brane. More information about these solutions can be found in appendix D.
brane as discussed in 5.3. Inspired by the Randal-Sundrum construction discussed in 4.1, one could then find a way to induce a positive cosmological constant on the brane. However, this may not be an easy task, as the Randal-Sundrum construction requires a supercritical brane, i.e. $\sigma > Q$, while the aforementioned decay is mediated by subcritical ones $T < Q$.

At this point we would like to invite the reader to continue reading stage III of this work. There, we will jump into one of the nucleated branes discussed above and explore its induced cosmology.
Part III:
The (Dark) Bubble Universe
6. Dark Bubble Cosmology: A First Taste

Remember! Reality's an illusion, the universe is a *holo*gram bubble, buy gold!

Adapted cite by Bill Cipher, Gravity Falls

In this chapter you will learn...

- what the dark bubble model is and its stringy/swampy foundations
- how to induce four-dimensional positive dark energy on an expanding $D_3$-brane mediating a vacua decay

We have modestly explored the state of art of (string) cosmology in previous chapters. Chapters 2 and 3 were devoted to explaining our understanding of the universe and its dynamics from four-dimensional classical and quantum perspectives, respectively. These dynamics are strongly influenced by the presence of an unknown type of energy called *dark energy*.

We also discussed the physics community’s hope of gaining an understanding of dark energy by deriving four-dimensional cosmologies from the higher-dimensional theories with ultraviolet completion, such as string theory. Chapter 4 narrates how, infused with determination, they departed to the plains of the *Braneworlds* and mountains and valleys of the *Landscape*, promised lands of effective low-dimensional vacua derived from string theory.

Focusing only on the Landscape, this plethora of vacua had the potential to host solutions equipped with a small, yet positive four dimensional cosmological constant. But some sort of stringy complot seemed to make such achievement almost impossible. This fact motivated the community to explore the *Swampland* surrounding the Stringy Kingdom. Perhaps here, we could gain some understanding of why their expectations regarding the behaviour of UV-complete theories at low energies may have been wrong. Some of the highlights of this yet unfinished journey were elaborated in chapter 5.

At this point, one fact has to be addressed: Almost all our attention has been devoted to the mechanism of compactifications throughout stage II in this thesis. However, we have also presented an alternative to string compactifications, *Braneworld* scenarios, in section 4.1. Can a tweaked version...
of this mechanism be used to realise four-dimensional de Sitter cosmologies from string theory and at the same time avoid some of the swamp-land conjectures discussed in chapter 5? Yes, and we would like to raise the stakes: There is a braneworld scenario which can realise four-dimensional accelerated expanding cosmos making use of some of the aforementioned conjectures in chapter 5. This is model is known as the **Dark Bubble** model.

The inspirational bedrock of this proposal has already been presented in sections 5.2 and 5.3. In summary, the clues are:

- Unlike to de Sitter vacuum solutions, Anti-de Sitter ones are more common in string theory. For example, AdS$_5 \times S^5$ is one of the most studied and well-understood backgrounds for any flux value [188]. However, AdS vacua generally do not exhibit scale separation [189].

- If the AdS vacuum is non-supersymmetric and supported by fluxes, the conjecture described in section 5.3 states that such AdS$_5$ space is unstable and will decay non-perturbately, via brane nucleation [185, 186]. This brane is a spherical co-dimension one hypersurface that mediates the decay between the initial configuration of the vacuum and that enclosed by the brane, with one unit of flux less.

- The nucleated brane, which can be described as a Brown-Teitelboim instanton [190], will obey the rules of the weak gravity conjecture 5.2, i.e. its electric (gauge) repulsion will be greater than its gravitational attraction. This is known as a **subcritical** tension brane. This will be the fundamental feature driving its expansion within the vacua.

The Dark Bubble model makes use of the previous facts, starting from a non-supersymmetric five-dimensional Anti-de Sitter time independent flux compactification in string theory.\(^1\) This vacuum solution is supported, by at least, one $\mathbb{F}_5$ flux. The presence of this field strength will be responsible of fixing the compact dimension moduli potential to a minimum value $\Lambda_5$. According to the non-supersymmetric AdS conjecture, this configuration is unstable and will eventually decay. The decay channel will be the nucleation of a $D_3$-brane, mediating between the new vacuum with "deeper" minimum $\Lambda_5$ (which we will call _True vacuum_) and the old vacuum (False vacuum) that has not yet undergone the decay. The brane, which is a non-saturated object ($Q > \sigma$) will experience a gauge repulsion greater than its gravitational attraction.\(^2\) This repulsion will force it to expand, "eating" up the old vacuum and "replacing" it with the one enclosed in its ever-growing inside.

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\(^1\)Chapter 7 describes an explicit embedding of this model in supergravity.

\(^2\)We will denote the tension with the letter $\sigma$. We will explain its relation to the tension of a fundamental brane in chapter 7.
The setup discussed above grants us the playground to develop the Dark Bubble proposal. This model can be summarised as:

*The Dark Bubble model proposes that a four-dimensional accelerated expanding cosmology can be realised on the co-dimensional one boundary of a five-dimensional true vacuum bubble, mediating the decay of a false AdS one.*

In the remainder of this work, we will present and elaborate on this novel proposal suggested by Banerjee, Danielsson et al. in [37] and developed in subsequent papers [38–40]. This first chapter of stage III will introduce how to induce a small and positive dark energy density $\rho_A$ on the boundary of the aforementioned expanding bubble. The procedure will be extremely easy: First, we will dispose and prepare the ingredients to use; The clues commented on above are part of the constituents, but the star of the course will be the AdS$_5$ geometry hosting the bubble. This delicate procedure will be discuss in section 6.1. With our first ingredients ready, section 6.2 will guide us through the creation of the desired four-dimensional de Sitter cosmology riding the bubble of true vacua. Section 6.3 will tease out the details on the Dark Bubble recipe to be discussed, leading the hungry minds towards the different culinary aspects of such creation.

### 6.1 Chopping the geometry

Let us start with the usual Einstein-Hilbert action for a $D$-dimensional geometry,

$$S[g_{MN}] = \frac{1}{2\kappa_D} \int d^D X \sqrt{|g(D)|} (R - 2\Lambda_D). \quad (6.1)$$

Note that we are not (yet) accounting for any matter content in this geometry, hence the absence of Lagrangian representing it.

Anti-de Sitter spaces are maximally symmetric Lorentzian spaces characterised by a negative Ricci scalar. For convenience, we will express the $D$-dimensional cosmological constant as:

$$\Lambda_D = -\frac{1}{2} (D-1)(D-2) \frac{L^3_{\text{AdS}}}{D}. \quad (6.2)$$

The $L_{\text{AdS}}$ parameter has dimensions of length and sets the overall scale of the space-time geometry. From now on, we will drop the subscript AdS in order not to oversaturate our notation. A general Ansatz for the metric of

---

3This is a generalisation of Eq. (2.6) for $D$ dimensions and a negatively curved space-time. Note that one can easily see the dimensionality of the cosmological constant in this way.
such a space can be given as:

\[ ds_D^2 = -e^{2A(r)} dt^2 + e^{-2A(r)} dr^2 + r^2 d\Omega_{D-2}^2, \]  

(6.3)

where \( d\Omega_{D-2}^2 \) represents the line-invariant of a \( (D-2) \)-sphere. As usual, varying Eq. (6.1) with respect to \( g_{MN} \) will provide the already well-known Einstein equation (2.4). Introducing the previous Ansatz (6.3) will result in a set of differential equations with solution

\[ e^{2A(r)} = 1 + \frac{r^2}{L^2}. \]  

(6.4)

This solution represents the global set of coordinates for a \( D \)-dimensional anti-de Sitter space. Note that the whole space is covered, as the radial coordinate \( r \) varies in the range of \( 0 \leq r < \infty \). The angular variables cover the \( (D-2) \)-sphere and time ranges between \( -\infty < t < \infty \).

Although the previous set of global coordinates covers the whole anti-de Sitter space, one can think of approximations for specific regimes. A very useful regime is that of large radius \( r \). This is when \( (r/L)^2 \gg 1 \). In this sense, one can think of really large spatial sections of \( S_{D-2} \) along the radial direction \( r \), such that the curvature of the sphere is negligible. The "unhygienic" approximation \( \{\alpha, \beta, \gamma, \cdots\} \approx \frac{1}{L} \{x, y, z, \cdots\} \) would allow us to rewrite the line invariant as:

\[ ds_D^2 = k^2 r^2 \eta_{MN} dX^M dX^N + \frac{1}{k^2 r^2} dr^2, \]  

(6.5)

where \( \eta_{MN} \) is the usual Minkowski metric for representing patches of a really large \( S_{D-2} \), so that they look flat. Note the presence of the new variable \( k \) we have written; This is the AdS\(_D\) vacuum scale, i.e. the inverse of the AdS-length \( k = L^{-1} \). This new coordinate system is named after Henri Poincaré. Poincaré coordinates are simpler than global ones but they do not cover all AdS\(_D\) geometry.

We are just one step away from having our geometry ready; the "gluing" process. Recall the presence of the nucleated brane mediating the decay of the old vacuum into the new one. This \( D_p \)-brane, which is an \( O(D-1) \) Coleman-de Luccia instanton [191], will expand radially. The position of this hypersurface with co-dimension one along the throat direction can be then parametrised by \( r = a(t) \), generating the two aforementioned regions: the inside, located at \( r \in (0, a(t)) \) and the outside, located at \( r \in (a(t), \infty) \).

---

4Actually, a more general geometry would be with \( A(r) \) and \( B(r) \) as the argument of each exponent, but these can be fixed to \( A(r) = -B(r) \) by imposing maximal symmetry in the space-time geometry.

5From a five-dimensional point of view, this would be the \( S^3 \), i.e. our usual spatial directions with a closed geometry.

6Such dirty tricks should not be used when cooking. Dear reader, please keep some cleanliness when preparing food!
It may be convenient to introduce new notation at this point: From now on, we will refer to the vacuum living inside the hyper-volume encoded by the $D_p$-brane, i.e. the *inside* vacuum, with a minus (-) subscript sign. The whole higher-dimensional space receives the name of *Bulk* and the tandem brane-inside will be colloquially called *Bubble*. On the other hand, the *outside* of the bubble that has not yet decayed will be denoted by a positive (+) subscript sign.

The notation presented above allows us to write an overall bulk geometry taking into account both regions ($\pm$). The complete metric is given by:

$$\text{ds}^2 = g_{MN} \text{d}X^M \text{d}X^N = -f(r) \text{d}t^2 + f^{-1}(r) \text{d}r^2 + r^2 d\Omega^2_{D-2}, \quad (6.6)$$

where

$$f(r) = 1 + k^2 r^2 + \Theta(r - a(t)) k^2. \quad (6.7)$$

Here $\Theta(x)$ is the Heaviside *step* function. It is important to emphasise on the jump accross the bubble wall. The change in the AdS$_D$ vacuum scale $k_{\pm}$ will cause a jump in the extrinsic curvature $K_{ab}$. We will see in the next section that this jump is responsible for the geometry and contents that will be induced on the hypersurface, helping us identify an expanding cosmology which decorates the $D-1$-boundary of the bubble.

### 6.2 Recipe to induce dark energy on a bubble surface

The aim of this thesis is to discuss cosmological and phenomenological aspects in four dimensions, and in view of the discussion in appendix C about co-dimension one hypersurfaces, we will fix $D = 5$. This implies that the boundary $\partial B$ of the nucleated five-dimensional bubble will have $d = D-1 = 4$ dimensions.

From now on, **Greek** indices will represent the five-dimensional bulk coordinates, while **Latin** ones will be used to describe quantities associated with the *induced* geometry on the four-dimensional boundary. In this dimensionality, the line invariant (6.6) can be cast in the ($\pm$)-form:

$$\text{ds}^2_{\pm} = g_{\mu\nu}^{\pm} \text{d}x^\mu \text{d}x^\nu = -f_{\pm}(r) \text{d}t_{\pm}^2 + f^{-1}_{\pm}(r) \text{d}r^2 + r^2 d\Omega^2_3, \quad (6.8)$$

where

$$f_{\pm}(r) = e^{2A_{\pm}(r)} = 1 + k^2_{\pm} r^2 + \chi(r, t, k_{\pm}, q_1, \ldots, q_m). \quad (6.9)$$

---

7This will contain explicit information on how the brane is embedded from both points of view; inside and outside. See appendix C for further details.

8Furthermore, first letters of the latin alphabet {$a, b, c$} will be used to describe all four dimensions, while {$i, j, k$} will represent just the spatial sections. See appendices A and C for further information.
The extra piece $\chi(r, t, k_\pm, q_1, \ldots, q_m)$ in the function $f_\pm(r)$ has been added to encode additional properties of the bulk, i.e. a mass $M$, any charge $q_i$ or any other parametric dependence. However, these will only be relevant in the following chapters. For now on, let us fix $\chi(r, t, k_\pm, q_1, \ldots, q_m) = 0$.

Before we start computing the Israel’s junction conditions described in appendix C, it is important to realise the parametrical dependence of the expanding bubble. As described in appendix D, the nucleated bubble is a Coleman-de Luccia instanton [191], which has an $O(4)$-symmetry. The two relevant coordinates for this derivation are the global time $t$ and the radial coordinate $r$, which also depends on $t$, as the bubble expands, hence evolves in time. Both will be related to the proper time $\tau$ experienced by any observer living on the boundary $\partial B$. The choice of coordinates to describe the bulk geometry and the induced one are:

\[
x^\mu = \{t(\tau), r(\tau), \alpha, \beta, \gamma\}, \quad y^a = \{\tau, \alpha, \beta, \gamma\}.
\] (6.10)

Let us now calculate the first junction condition described in Eq. (C.9). This is no more than:

\[
ds^2_{\text{ind}} = h_{ab}dy^ady^b = -N(\tau)^2d\tau^2 + a^2(\tau)d\Omega_3^2,
\] (6.11)

where

\[
N(\tau)^2 = f'(a(\tau))i^2 - \frac{\dot{a}(\tau)^2}{f(a(\tau))}.
\] (6.12)

The metric $h_{ab}$ represents the four-dimensional induced geometry on the hypersurface, i.e. the $D_3$-brane, given by the parametrisation of $\{y^a\}$, the set of coordinates on the brane. The lapse function $N(\tau)$ has been introduced to manifest the invariance of the reparametrisation of time. One can always choose the right parametrical relation between $f(a(\tau)), i(\tau)$ and $a(\tau)$ such
that \( N(\tau) = 1 \). In that case, a dot will represent a proper-time derivative (i.e. the time an observer living on the hypersurface will experience). Thus, Eq. (6.11) will describe an expanding four-dimensional cosmology, characterised by a Friedmann-Robertson-Lemaître-Walker metric with closed spatial sections.

Let us now compute the second Israel junction condition. As discussed above, the presence of the expanding \( D_3 \)-brane divides the bulk space into an inside and an outside. This forces the presence of a non-zero induced energy-stress tensor on the brane, so that the whole configuration remains a solution to the bulk Einstein equation. This is what the second Israel junction condition (C.35) accounts for. To simplify this example, let us assume we have an empty brane. This implies that its energy-momentum tensor \( S_{ab} \) only contains information about its tension \( \sigma \).\(^9\) This can be expressed as:

\[
S_{ab} = -\sigma h_{ab}. \tag{6.13}
\]

Israel’s second condition refers to the discontinuity in the extrinsic curvature between the inside and the outside geometries separated by the brane. To be as pedagogical as possible in this introductory chapter, we will also compute the most relevant steps to obtain the induced energy-momentum tensor.\(^10\)

First, we need to identify the normal and tangent vectors \( \{ n^\mu, e^\mu_a \} \) in relation to the embedding. These vectors can be found using Eqs. (C.6) and (C.7):

\[
n^\mu = \left( \frac{-\dot{a}/\dot{t}}{f(a)\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, \frac{f(a)}{\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, 0, 0, 0 \right) \tag{6.14}
\]

and

\[
e^\mu_a = \begin{pmatrix} i(\tau) & 0 & 0 & 0 \\ \dot{f}(a) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.15}
\]

Note that the parametric dependence on the proper time \( \tau \) is not explicitly stated, but is still implicit in \( a(\tau) \) and \( t(\tau) \). Following the guidelines of appendix C, the next computation is that of the extrinsic curvature \( K_{ab} \) (C.15).

---

\(^9\)Recall that the tension of a \( D_3 \)-brane is given by \( T_{D_3} \propto \frac{1}{\ell_s^4} \), with \( \ell_s \) the string length. So its dimensionality is \([\text{Length}]^{-4}\).

\(^10\)The whole derivation is left as an exercise for the reader.
It is explicitly stated by

\[
K_{\tau\tau} = \frac{N(\tau)^2}{\dot{a}^2} \frac{d}{d\tau} \left[ \frac{1}{N(\tau)} \sqrt{f(a)N(\tau)^2 + \dot{a}^2} \right],
\]

\[
K_{ij} = -\frac{a}{N(\tau)} \gamma_{ij} \sqrt{f(a)N(\tau)^2 + \dot{a}^2},
\]

(6.16)

with \(\gamma_{ij}\) as the metric on the sphere \(S^3\). It is then straightforward to compute the second junction condition (C.35). Assuming a pure constant tension brane (6.13), the following expression can be derived:

\[
\sigma = \frac{3}{\kappa_5} \left( \sqrt{\frac{f_-(a)}{a^2} + \frac{\dot{a}^2}{N(\tau)^2 a^2}} - \sqrt{\frac{f_+(a)}{a^2} + \frac{\dot{a}^2}{N(\tau)^2 a^2}} \right),
\]

(6.17)

where \(\kappa_5 = 8\pi G_5\) and \(\sigma\) corresponds to the tension of the bubble wall. Note how the tension of the shell is governed by the four-dimensional Hubble parameter \(H = \dot{a}/(N(\tau)a)\) and the difference in the AdS_5 scales of the five-dimensional bulk.

A little bit of algebra and patience reveals a Friedmann-like equation in the form:

\[
H^2 = -\frac{1}{a^2} + \frac{1}{3} \left( \frac{\kappa_5^2 \sigma^2}{12} - \frac{3}{2} \left( k_+^2 + k_-^2 \right) + \frac{27}{4} \left( \frac{k_- - k_+}{\kappa_5 \sigma} \right)^2 \right),
\]

(6.18)

where we have identified the second term with a four-dimensional cosmological constant \(\Lambda_4\). Dimensional analysis can easily demonstrate that: \([k_\pm] = [\text{Length}]^{-1}\), while \(\kappa_5 = 8\pi G_5 = \ell_5^3\), where \(\ell_5\) is the five-dimensional Planck length. So \([\kappa_5] = [\text{Length}]^3\). From Eq. (4.9), we also know that \([\sigma] = [\text{Length}]^{-4}\). It is then easy to conclude that the dimensionality of that piece is that of a four-dimensional cosmological constant. What values can it take?

Note that the non-linear expression for \(\Lambda_4\) in Eq. (6.18) has two zeros\(^{11}\) at:

\[
\sigma_{cr} = \frac{3}{\kappa_5} (k_- \pm k_+).
\]

(6.19)

These correspond to the value of the critical tension.\(^{12}\) Based only in observational evidence, our objective is to find solutions with a small and positive \(\Lambda_4\), then the lower half of the plot can be omitted. In addition, recall

\(^{11}\)Negative values of the tension \(\sigma\) have no physical meaning for this construction.

\(^{12}\)Note that the choice of the solution with the plus sign corresponds to that of the Randall-Sundrum construction discussed in section 4.1.
Figure 6.2. Left: Parametric dependence of $\Lambda_4$ on the tension $\sigma$. Since only sub-critical branes can nucleate, only the solid lines are valid. The red solid line are the values for the Dark bubble construction, while the blue solid line are the supercritical values required by the Randall-Sundrum model discussed in section 4.1. Right: A change of coordinates $r \rightarrow \frac{1}{k} \log(kr)$ in the Poincaré patch (6.5) reveals the change in the transverse direction warping, which will be reflected in the extrinsic curvature $K_{ab}$. The brane is located at $r = 0$, and the spatial sections grow monotonically throughout the AdS throat.

that any nucleated brane according to the non-susy AdS conjecture is a non-saturated object, so its tension must be subcritical. This restricts the set of valid solutions with $\Lambda_4 > 0$ to be in the regime:

$$\sigma < \sigma_{cr} = \frac{3}{\kappa_5} (k_- - k_+) .$$ (6.20)

The Friedmann Eq. (6.18) can be further simplified, such that the cosmological constant $\Lambda_4$ becomes a linear expression of $k_\pm$ and $\sigma$. In fact, our main phenomenological target in this chapter is to provide late-time cosmological descriptions, where a small dark energy density $\rho_\Lambda$ dominates the dynamics of the induced cosmos. This phenomenological regime can be reproduced when:

$$k_\pm a \gg 1, \quad \sigma = (1 - \varepsilon) \sigma_{cr} .$$ (6.21)

The first requirement implies that the scaled scale factor $a$ is large enough, so that observers living on the hypersurface do not infer any spatial curvature. The second condition adjusts the brane to a tension which is nearly subcritical. Expanding Eq. (6.18) for small $\varepsilon$, we can find:

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = -\frac{1}{a^2} + \frac{8\pi}{3} \frac{2k_- k_+}{k_+ - k_-} G_5 (\sigma_{cr} - \sigma) + \mathcal{O}(\varepsilon^2) .$$ (6.22)

We can then perform a similar dimensional analysis as above to conclude that $[G_5 \kappa^2] = \text{[Length]}^2$. This is matches four-dimensional Newton’s constant $G_4$ dimensionality.\(^{13}\) This will allow us to make the following iden-

\(^{13}\)Recall, as one can see in Eq. (2.3), that $\kappa_D = \ell_D^{D-2}$. 
fifications:
\[ \kappa_4 = 2 \frac{k_- k_+}{k_- - k_+} \kappa_5, \quad \rho_\Lambda = \sigma_{cr} - \sigma. \]  

We can see that this regime allows us to relate the induced dark energy \( \rho_\Lambda \) to the "subcriticality" of the nucleated bubble. This requires a tuning of \( \varepsilon \sim 10^{-120} \) (in Planck units) to make contact with its observed value. We will see that this somewhat \textit{ad hoc} parameter \( \varepsilon \) can find a stringy interpretation in section 7.7. It is also important to note the unique relationship between four and five-dimensional Newton’s constant in this model. The difference between the AdS vacuum scales in the denominator of Eq. (6.23) already points to an inversion of the common scheme of scales discussed in chapter 4. In fact, when \( k_+ \sim k_- \sim k \), one finds that gravity will be a stronger force in four dimensions than five. Although concerns about divergence may make our legs tremble, we will see in section 7.7 how this hierarchy is under control when the bubble is embedded in a ten-dimensional description.

\textbf{Figure 6.3.} Pictorial representation of the inside/outside distribution of AdS-vacua and the bubble with tension \( \sigma \) mediating between them. Note the orientation of the normals \( n^\pm_\mu \), which will always point in the direction of increasing volume of spatial sections, contrary to the Randall-Sundrum case in figure 4.1.

It could be convenient to point out some of the main differences between the Dark Bubble model [37] and the Randall-Sundrum model [110] discussed in section 4.1:

- The first notable absence is that of the \( Z_2 \) symmetry, which identifies the vacua scales \( k \) to both sides of the brane. Even more important is the natural choice of the normal orientation for the dark bubble, where both point in the direction of growing spatial sections. This can be seen from the behaviour of the warp factors for both dark bubbles (6.2) and the Randall-Sundrum model (4.1).

- This brings us to the next consequence, that of the criticality of the brane tension \( \sigma \). While a subcritical tension is required to have a dark bubble nucleate and induce a positive cosmological constant at its boundary, the opposite, i.e. a supercritical tension is necessary to
achieve the same result for the Randall-Sundrum scenario. However, it is clear from the swampland conjectures discussed in chapter 5, that only branes with $\sigma < Q$ are accepted.

- Finally, it is important to notice the difference between the Newton’s constant in Randall-Sundrum and the dark bubble models. While the former scenario has a four-dimensional constant controlled by the vacuum scale $k$ of the mirrored AdS$_5$, the latter depends on the difference. This is crucial, as if this difference is extremely small, it can yield a situation where gravity results stronger in four dimensions than five, i.e. $G_4 \gg G_5$, as we will see in section 7.7.

In summary: The Dark Bubble model proposes the realisation of a four-dimensional accelerated expanding cosmology on the boundary of a nucleated five-dimensional bubble of a decaying AdS$_5$ vacuum. In this model, the low-dimensional dark energy density $\rho_\Lambda$ finds a dynamical interpretation in the higher-dimensional picture, where the difference between the critical and subcritical tension of the nucleate brane controls the motion of this in the extra dimensions.

6.3 Towards a gourmet cosmology

We would like to conclude this first chapter of stage III with a discussion on points that will be relevant and hence discussed in the following chapters. In this way, our dear reader can easily find those topics that align with their sophisticated palate.

- This chapters omits the presence of extra dimensions required by supergravity models. This will be the main discussion in chapter 7, where we will embed the dark bubble model in string theory. Its connections to the swampland will be also discussed in that chapter.

- We have talked about expanding branes in the bulk, but, how probable is such bubble nucleation? Can this event in higher dimensions be related to the standard cosmology in four dimensions? This will be reviewed and connected to the quantum cosmology discussion of chapter 3 in chapter 8.

- How can we decorate the four-dimensional universe in this proposal? Where is gravity? Does it remain localised on the induced geometry or does it propagate across extra dimensions? This will be examined in chapter 9, where we will be study how strings associated with the expanding brane are required to gain a good understanding of gravity in this model.

- We have only discussed how to induce dark energy $\rho_\Lambda$ on the brane, but the observable universe encompasses much more than that. This
will be the topic of, where the induced cosmology will be decorated with matter, radiation and waves coming from the extra dimensions. There we will explore the true power of the second Israel condition, its relation to the Gauss-Codazzi equation as described in appendix C and also how to interpret these concepts correctly to obtain a meaningful induced energy-momentum tensor.
7. The Dark Bubble Rises (from String Theory)

Deshi Basara! (Rise!)
Prisioners in the pit, The Dark Knight Rises

In this chapter you will learn...

- the first stringy embedding of the dark bubble and its dual field theory description
- which the main ten-dimensional characters are and their lower-dimensional interpretation
- how to induce four-dimensional $\Lambda_4 > 0$ on the bubble boundary from ten-dimensional supergravity and the associated energy hierarchy to this construction

If our dear reader has read this work from the very beginning and/or is a seasoned explorer of the bewitched lands described in stage II of our journey, they should already be aware of the difficulty in finding metastable vacua solutions in string theory that are equipped with a positive cosmological $\Lambda$. The mechanism of flux compactification has always led the string phenomenology community to challenging (yet illuminating) situations, giving them a hard time when aiming to stabilise all moduli arising from the compactification [160, 161, 192] or unleashing intractable instabilities when breaking supersymmetry [156–159].

Previous chapter 6 introduced a new braneworld technology for obtaining four-dimensional cosmologies equipped with a positive cosmological constant $\Lambda$ from higher dimensions. We did use a language based on string theory and related constituents, such as the WGC and the branes that mediate the decay of non-supersymmetric AdS$_5$-vacua. This chapter is a subsequent continuation of the preceding discussion with one single objective:

To embed the dark bubble model in string theory.

In this chapter, we will elaborate the first stringy embedding of the dark bubble cosmology proposed by Danielsson, Henriksson and Panizo and its phenomenological consequences for the associated energy hierarchy to the
embedding [45, 47]. In order to easily navigate through such a voluminous chapter, let us introduce the content of its sections:

Section 7.1 will present a small introduction to the dual field theory (DFT) description of the setup used and explains its two faces (DFT and gravitational description) relate to each other. We will then visit the near horizon geometry (a.k.a. the pit) which is sourced by a stack of $D_3$ branes in section 7.2 and then follow all the path of the brane that (almost) escaped from it in section 7.3. Its dynamics will not be unfamiliar to us, as they were previously introduced in chapter 6. Differences and similarities between the two approaches (Junction conditions vs Hamiltonian) in describing that motion will be commented in section 7.4. We will discover that the brane will eventually fall down in section 7.5; This will cause the induced cosmology to bounce back, failing on reproducing a cosmological era of accelerated expansion. This will force us to explore stringy corrections (a.k.a the hero) that can support the escaping brane providing the final boost to make the right jump in 7.6. Out of the well in section 7.7, we will realise how the pit’s shadow tendrils extend beyond and control all phenomenological energy scales, providing the construction with a novel scale hierarchy.

7.1 Alter ego field theory

We will start by reviewing the dual language of the string background used in [45] to embed the dark bubble model. The statement is based on one of the most extensively studied examples of the AdS/CFT\footnote{We recommend reading [193] for an exquisite introduction to the topic.} correspondence [128, 194, 195]. Condensing the duality in some lines, it proposes that type IIB supergravity with AdS$_5 \times S^5$ geometry sourced in the near horizon limit of a stack of $N$ $D_3$-branes can also be described by a strongly coupled $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory when $N \to \infty$.

The construction used in the dark bubble embedding is a modification\footnote{This family of solutions was originally introduced in [196] as a consistent truncation to five-dimensions of $\mathcal{N} = 2$ supergravity. The whole ten-dimensional uplift can be found in [197].} of the original work. The gravitational side of the duality comprises of a stack of $N$ black branes, with angular momentum in some of their transverse directions. The near horizon geometry sourced by such a stack will resemble that of a ten-dimensional Kerr black hole; This is because the rotation along some of the compact directions will mix the time and angular components of the metric. As in the original case, the whole ten-dimensional geometry can be decomposed asymptotically into AdS$_5 \times S^5$. If we decided to reduce on the compact dimensions $S^5$, the resulting five-dimensional geometry would resemble that of a Reissner-Nordström black hole.
In order to understand how such effective charge appears in the five-dimensional space, we are required to examine the dual field theory description. This is the low energy effective description of the stack of \( N D_3 \)-branes. These rotating branes in the compact dimensions \( S^5 \) exhibit an \( SO(6) \) rotational symmetry. This set of isometries can be "isomorphically" translated to an \( SU(4) \) global R-symmetry acting on the fields of the \( \mathcal{N} = 4 \) super Yang-Mills theory.

Among the fields in the \( \mathcal{N} = 4 \) super Yang-Mills description, we will find a gauge field \( A_\mu \), some fermions and six scalars, all in the adjoint representation of the \( SU(N) \) group. The eigenvalue scalars, which correspond to directions in a \( \mathbb{R}^6 \) space, will span a moduli space. The position of the eigenvalues in this moduli space will describe what the configuration of the gravitational description is. If the distance between each of the eigenvalues is small, one can determine the ten-dimensional geometry. But one single isolated eigenvalue points to the existence of a probe brane which has been embedded into the geometry sourced by the stack of branes.

These isolated eigenvalue points can appear when the \( \mathcal{N} = 4 \) super Yang-Mills description is decorated with additional background fields. These represent chemical potentials \( \mu_i \) and a non-zero temperature \( T \neq 0 \). The presence of these fields will cause two relevant changes to the system:

- The configuration of eigenvalues in the \( \mathbb{R}^6 \) moduli space will start to rotate. This will represent states with non-zero \( R \)-charge. The rotational velocity is determined by the chemical potentials \( \mu_i \).
- The background fields will introduce an effective negative mass contribution in the Lagrangian describing the interaction of the scalar fields. This will signal to an instability of the system, i.e. the rotational configuration of the previous point may eventually change to the "intruder" one if the rotational velocity reaches a threshold, i.e. the chemical potentials reach such threshold.

The presence of the chemical potentials \( \mu_i \) and the non-zero temperature \( T \neq 0 \) indicates to a Higgsing instability of the gauge group \( SU(N) \to SU(N-\Delta N) \times U(\Delta N) \). This results in a spontaneous color symmetry breaking.

What does all this mean from the gravitational perspective? Remember that the ten-dimensional geometry is the near-horizon limit sourced by the stack of \( N D_3 \)-branes. The angular velocities in the compact directions of \( S^5 \) correspond to those of the rotating moduli in the \( \mathbb{R}^6 \) space described above which is also related to the chemical potentials \( \mu_i \). When these potentials \( \mu_i \) cause a change in the rotational configuration of the moduli and the higgsing process occurs, this is replicated by the emission of \( \Delta N \) branes on the

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3This field will be relevant to describe the coordinate-mixing in the gravitational description later on.
gravity side, which will escape from the stack that is the source of the Kerr geometry. This brane nucleation\footnote{This decay process is exponentially suppressed $\sim e^{-N}$ [198].} implies that the membrane(s) will tunnel through the outer horizon of the black hole to minimise its energy. The new born brane(s) will then start to expand, in order to reach the minimum, carrying with them a portion of the total angular momentum $J$ associated with the rotational motion of the background in the compact directions. If we "zoom-out" to only see the AdS$_5$-space, we will observe a similar configuration, i.e. expanding brane(s) nucleated from the outer horizon of the Reisser-Nordström black hole which carry an R-charge $Q$, related to angular momentum $J$ in the compact directions.

This is, \textit{grosso modo},\footnote{In Latin, "roughly speaking".} the description of the two different aspects of the same coin. It can be inferred that this set-up represents a suitable supergravity configuration to embed the dark bubble model discussed in chapter \ref{chap:dark-bubbles}. In order to simplify our computations, we will from now restrict ourselves to an isotropic case of the aforementioned chemical potentials. This means that $\mu_1 = \mu_2 = \mu_3 = \mu$.

\section{The pit}

Let us start by defining the parametric coordinates\footnote{Remember that Latin indices will represent the usual four-dimensional coordinates. Greek indices denote bulk coordinates and in this chapter, we will introduce capital Latin indices $\{M,N\}$ for the ten-dimensional coordinates. For more information, see appendix A.} that will be used to describe the ten dimensional rotating background:

\begin{equation}
X^M = \{t, \alpha, \beta, \gamma, z, \theta, \psi, \phi_1, \phi_2, \phi_3\},
\end{equation}

where $t$ represents the ten-dimensional ambient space time while $\{\alpha, \beta, \gamma\}$ represent the three non-compact usual dimensions. The coordinate $z$ refers to the AdS$_5$-throat. Directions in the compact $S^5$ are $\{\theta, \psi, \phi_1, \phi_2, \phi_3\}$. The line invariant of this ten-dimensional geometry is given by following expression:

\begin{equation}
ds_{10}^2 = ds_5^2 + L^2 \sum_{i=1}^3 \left\{d\sigma_i^2 + \sigma_i^2 \left(d\phi_i + \frac{1}{L} A(z)\right)^2\right\},
\end{equation}

where $A(z)$ is a ten-dimensional one-form that will be later discussed and the $\sigma_i$-functions are combinations of trigonometric functions of two angles of the $S^5$ as:

\begin{equation}
\sigma_1 = \sin \theta, \quad \sigma_2 = \cos \theta \sin \psi, \quad \sigma_3 = \cos \theta \cos \psi.
\end{equation}

As stated in chapter \ref{chap:dark-bubbles}, $L$ represents the AdS$_5$ radius in Eq. (7.2). It is important to note that this radius also determines the scale of the extra compact
dimensions of $S^5$. This could be seen as a problematic feature, as it points to lack of scale separation, as discussed in 4. However, this will be a key characteristic of the dark bubble embedding into supergravity, as this model achieves the aforementioned scale separation via tension-to-charge ratio instead [45, 47]. We will further examine this feature in section 7.7.

Refocusing our attention on the line invariant (7.2), we can define the five-dimensional asymptotically-AdS metric $ds_5$ as follows:

$$ds_5^2 = -j(z)^{-2} s(z) dt^2 + j(z) [s(z)^{-1} dz^2 + z^2 d\Omega_3^2] ,$$  \hspace{1cm} (7.4)

where $d\Omega_3^2$ is the usual unit metric of the 3-sphere $(2.19)$. The radial functions $j(z)$ and $s(z)$ are expressed as

$$j(z) = 1 + \frac{q^2}{z^2} ,$$  \hspace{1cm} (7.5)

$$s(z) = 1 - \frac{m}{z^2} + \frac{z^2}{L^2} j(z)^3 ,$$

where $m$ and $q$ are representing respectively the mass and charge of the five-dimensional Reissner-Nordström black hole discussed in section 7.1. Let us then perform a "massage" to Eqs. (7.4, 7.5) to provide a more familiar Reissner-Nordström black hole line invariant [199]. To begin this task, one can perform a straightforward change of coordinates by defining a new radial coordinate $r$ as:

$$r^2 = z^2 j(z) = q^2 + z^2 .$$  \hspace{1cm} (7.6)

This change of coordinates will substantially transform Eq. (7.4), resulting in:

$$ds_5^2 = -g(r) dt^2 + g(r)^{-1} dr^2 + r^2 d\Omega_3^2 ,$$

$$g(r) = \frac{1 + \frac{k^2 r^2}{r^2} - \frac{2\kappa_5 M}{r^2} + \frac{\kappa_5^2 Q^2}{r^4}}{f(r)} ,$$  \hspace{1cm} (7.7)

with

$$M = \frac{1}{2\kappa_5} (m + 2q^2) , \hspace{1cm} Q^2 = \frac{1}{\kappa_5} q^2 (m + q^2) ,$$  \hspace{1cm} (7.8)

where $\kappa_5 = 8\pi G_5$. It is now easy to see that the change of variables as described in Eq. (7.6) transforms the line invariant (7.4) into the familiar form\(^7\) of a five dimensional Reissner-Nordström black hole living in an AdS vacuum. It should be noted that, for a small black hole, with $r_H \ll L$, we recover a flat space description and we need $Q < M$ to get a horizon and no

\(^7\)The patch described by the metric (7.4) corresponds to the requirement $r \geq q$. This is actually not an issue since the horizon $r_H$ will be covered by the patch.
naked singularity. However, if we want to study a horizon larger that the AdS₅ radius, this immediately implies $Q \ll M$.

Now let’s review the interpretation of both ten and-five dimensional geometries. In the ten-dimensional approach, we will observe a Kerr black hole rotating in three of the compact directions of $S^5$ (i.e. $\{\phi_i\}$). When we zoom out and the five-dimensional sphere is no longer resolved, the black hole no longer rotates. It becomes "static" from the point of view of a five-dimensional observer, but it acquires a charge $q$ due to its motion in the compact directions (further explanation will be provide later). This charge becomes effective (i.e. charge $Q$) when the change of coordinates (7.6) is performed to recover the more familiar description of a Reissner-Nordström black hole living in AdS [199].

As it is well known, this type of black hole has two different horizons [59]: The outer horizon (i.e. event horizon) and the inner one (i.e. the Cauchy horizon). When these two horizons become degenerate (i.e. $r_H = r_h$), this corresponds to a description of an extremal black hole. We will not further discuss charged black holes and their connection to the weak gravity conjecture here (see section 5.2 for more details), but, for computational purposes in the incoming sections, we find more convenient to express (7.7) in terms of the two aforementioned horizons: $\{r_h, r_H\}$ as the inner and outer horizons [45]. In this way, we can express $g(r)$ in Eq. (7.7) with the following Ansatz:

$$g(r) = \frac{k^2}{r^4} \left( r^2 + c \right) \left( r^2 - r_h^2 \right) \left( r^2 - r_H^2 \right),$$

with $c \in \mathbb{R}$. A simple match between Eqs. (7.7, 7.9) shows that:

$$M = \frac{1}{2\kappa_5} \left[ r_h^2 + r_H^2 + k^2 \left( r_h^4 + r_H^4 + r_h^2 r_H^2 r_H^4 \right) \right]$$

$$Q^2 = \frac{r_h^2 r_H^2}{\kappa_5^2} \left[ 1 + k^2 \left( r_h^2 + r_H^2 \right) \right],$$

$$c = r_h^2 + r_H^2 + \frac{1}{k^2}.$$  

which simplifies when the black hole is extremal. Let us keep in mind these identifications for future sections and return our attention back to the one-form potential $A(z)$ in Eq. (7.2). This form acts as a gauge field from the five-dimensional point of view. In term of the old coordinates (i.e. $\{t, \alpha, \beta, \gamma, z\}$) it can be written as:

$$A(z) = \frac{q}{z_H^2 + q^2} \sqrt{ \frac{q^2 + z_H^2}{L^2} } \int(z_H) \left( 1 - \frac{z_H^2 + q^2}{z^2 + q^2} \right) dt.$$  

Note that gauge freedom has been used to add a constant that sets $A(z_H) = 0$. This is required as a $A(z)$ is a temporal gauge potential, and must vanish
at the horizon. This potential can be expressed in a more handable way in two steps:

1. Part of the argument of the root in Eq. (7.11) is the mass \( m \) of the black hole.\(^8\) This is:

\[
m = z_H^2 \left( 1 + \frac{z_H^2}{L^2} j(z_H)^3 \right).
\] (7.12)

2. Furthermore, performing the change of variables (7.6) and making use of the identifications (7.8) to further simplify in the new coordinate system, one obtains:

\[
A(r) = \kappa_5 Q \left( \frac{1}{r_H^2} - \frac{1}{r^2} \right).
\] (7.13)

The self-dual \( \mathcal{F}_5 \) Ramond-Ramond (RR) field strength is sourced by the stack of rotating \( D_3 \)-branes in the ten-dimensional background,\(^9\) which can be expressed in terms of the radial function \( j(z) \) and the one-form \( A(z) \). However, what we will need for future computations is the corresponding four-form potential \( \mathcal{C}_4 \), with \( d\mathcal{C}_4 = \mathcal{F}_5 \). This is:

\[
\mathcal{C}_4\left(z, z_H, q\right) = \frac{1}{L} \left[ (z^2 + q^2)^2 - (z_H^2 + q^2)^2 \right] dt \wedge \epsilon_3
\]
\[
+ L^2 q \sqrt{z_H^2 + q^2 + \frac{z_H^4}{L^2} j(z_H)^3} \sum_i \sigma_i^2 d\phi_i \wedge \epsilon_3,
\] (7.14)

where \( \epsilon_3 \) corresponds to the volume-form associated with \( d\Omega_3 \) in Eq. (7.4).

Following the recipe describe above, one cast the four-form \( \mathcal{C}_4 \) in terms of the new variables as:

\[
\mathcal{C}_4\left(r, r_H, Q\right) = \frac{1}{L} \left[ r^4 - r_H^4 \right] dt \wedge \epsilon_3 + L^2 Q \kappa_5 \sum_i \sigma_i^2 d\phi_i \wedge \epsilon_3.
\] (7.15)

Note that we have split the four-form \( \mathcal{C}_4 \) into its time-dependent and \( \phi_i \)-dependent pieces. This notation will simplify further computations.

With this we have completed the description of the black hole in ten and five-dimensional perspectives. Let us now explore how the \( D_3 \)-brane is explicitly embedded in the ten-dimensional ambient space.

---

\(^8\)This can be easily obtained by computing the outer horizon in Eq. (7.4).

\(^9\)Ten-dimensional fields will be represented with a gothic calligraphy. See appendix A for more notation conventions.
7.3 The brane that (almost) escaped from the pit

Let us now move on and study the motion of a nucleated $D_3$-brane. The starting point of such enterprise is the action governing its dynamics (4.7):

$$S_{D_3} = S_{DBI} + S_{WZ} = -T_3 \int d^4 \xi \sqrt{|(P[G + 2\pi \alpha' \mathfrak{R}] + T_3 \int P[\mathfrak{C}_4]. (7.16)$$

The Dirac-Born-Infeld (DBI) term describes both the geometrical features of the embedding and the coupling of the brane to Neveu-Schwarz (NS) gauge fields living in the bulk, while the Wess-Zumino (WZ) term captures the non-gravitational forces on the brane due to fields external to it. $P[\ldots]$ denotes the pullback of fields to the worldvolume of the brane. This is no more than the projection of ten-dimensional coordinates onto the three-dimensional geometry of the $D_3$-brane, with tension $T_3$. These, the embedding functions of the brane will be denoted by capital and gothic versions of the coordinates: of spacetime $X^M(\xi)$. For simplicity, we will set the dilaton $\phi$ constant.

The dynamical description of the branes is as follows: The stack of $D_3$-branes, and hence the nucleated one, wrap the three-sphere $d\Omega_3$ inside the $\text{AdS}_5$ piece of the metric, with the radial $r$-coordinate (hence $\mathfrak{R}$) depending only on the time component. The black hole in the background has ten dimensions and rotates in the azimuthal $\phi_i$-directions on $S^5$. This implies that a $D_3$-brane escaping from the stack will preserve a portion of the angular momentum of this. Therefore, it will also move in those directions. For the sake of simplicity in the construction of this model, set the three angular momenta associated with each $\phi_i$-directions equal in the background. This implies that the brane will rotate with the same angular velocity in all $\phi_i$ directions. Finally, we will assume a constant position in the remaining coordinates of the compact $S^5$, i.e. $\{\Psi = \psi_0, \Theta = \theta_0\}$. The embedding functions of the brane are then given by:

$$\mathfrak{X} = \mathfrak{X}(\tau), \quad \mathfrak{R} = \mathfrak{R}(\tau), \quad \Theta = \theta_0, \quad \Psi = \psi_0, \quad \Phi_1 = \Phi_2 = \Phi_3 = \Phi(\tau). \quad (7.17)$$

Here, although $\tau$ is used as a time parameter on the induced geometry, we do not yet specify whether it denotes its proper time. We will later see that this is the case. From the point of view of an observer living in $\text{AdS}_5$, the rotation translates into an effective charge $Q$ under the gauge field $A$. When a brane nucleates, carrying a portion of the charge, it will also effectively reduce the charge of the remaining black hole.

---

10Note that we will denote the tension of the brane with $T_3$ and not $T_{D_3}$ as in Eq. (4.9). This will be explained later in the chapter.

11The construction described in this chapter does not include NS gauge fields, meaning that $\mathfrak{F} = \mathfrak{F}_{MN} = 0$ in the bulk. Further discussion on this topic can be found in section 10.3.
Figure 7.1. Schematic table to describe the position of the nucleated D3-brane. The brane wraps around the three standard spatial directions, given by coordinates \( \alpha, \beta, \gamma \). It will move radially along the \( \mathcal{R} \) direction of the throat. Green coordinates represent the compact directions of \( S^5 \). The brane is assumed to be fixed at \( \{ \Theta, \Psi \} \), while moving with the same angular momentum \( J_i \) along the three other directions \( \{ \Phi_i \} \).

Before analysing the action (7.16), we need some more practical notation. Derivatives with respect to \( \tau \) will be denoted by a dot. The components of the ten-dimensional metric (7.1) will be denoted by \( G_{MN} \). It will also be useful to define the combination of the azimuthal-time and azimuthal-azimuthal components in the compact piece of Eq. (7.1) as
\[
P_{[G]} = G_{t\tau} \dot{\tau}^2 + G_{r\tau} \dot{\mathcal{R}}^2 + 2 G_{t\phi} \dot{\Phi} \dot{\mathcal{R}} + G_{\phi\phi} \dot{\Phi}^2.
\]
This is also the square of the velocity that a comoving observer living on the brane will experience,
\[
\dot{X}^2 = \dot{X}_M \dot{X}^M = G_{tt} \dot{\tau}^2 + G_{rr} \dot{\mathcal{R}}^2 + 2 G_{t\phi} \dot{\Phi} \dot{\mathcal{R}} + G_{\phi\phi} \dot{\Phi}^2,
\]
with
\[
\dot{X} \equiv \frac{\partial X^M}{\partial \tau} \partial_M = \dot{\mathcal{R}} \partial_\tau + \dot{\Phi} \partial_r + \dot{\Phi} \sum_{i=1}^{\mathcal{3}} \partial_\phi_i.
\]
The induced line element on the brane worldvolume is:
\[
ds_4^2 = \dot{X}^2 d\tau^2 + \mathcal{R}(\tau)^2 d\Omega_3^2.
\]
Now one can see that picking \( \tau \) to be the proper time of an observer living on the brane would set \( \dot{X}^2 = -1 \), and the induced metric takes the usual FLRW-form 6.11, with \( \mathcal{R}(\tau) \) acting as the scale factor.

Similarly, the pullback of the four-form \( \mathcal{C}_4 \) is:
\[
P[\mathcal{C}_4] = \left( (\mathcal{C}_4)_t \frac{\partial \mathcal{X}}{\partial \tau} + \sum_i (\mathcal{C}_4)_\phi \frac{\partial \Phi_i}{\partial \tau} \right) d\tau \wedge \epsilon_3.
\]
Finally, the action (7.16) can be rewritten as:
\[
S_{D_3} \equiv \int d\tau L_{D_3} = -2\pi^2 T_3 \int d\tau \left\{ \mathcal{R}^3 \sqrt{-\dot{X}^2} - (\mathcal{C}_4)_t \dot{\mathcal{R}} - (\mathcal{C}_4)_\phi \dot{\Phi} \right\}.
\]
where $\mathcal{L}_{D3}$ represents the Lagrangian of the $D_3$-brane. Note that integration over the $S^3$ has already been taken into account. As can be observed, the Lagrangian does not depend on the time component $\Sigma$ or the angular coordinates $\Phi_i$, only their derivatives. This points to conserved quantities within the system. To make our computations as simple as possible, let us define the conserved angular momentum

$$J \equiv \frac{1}{2\pi^2 T_3} \frac{d\mathcal{L}_{D3}}{d\Phi} = \frac{\mathcal{G}^3}{\sqrt{-\dot{X}^2}} \left( G_{t\phi} \dot{X} + G_{\phi\phi} \dot{\Phi} \right) + (\mathcal{C}_4) \dot{\phi}. \quad (7.24)$$

Using Legendre transformation, we can substitute $\dot{\Phi}$ for $J$, which automatically satisfies $\Phi$'s EoM. Additionally, let's define $J_C = J - (\mathcal{C}_4) \dot{\phi}$, which will be given a physical interpretation later. By selecting the positive root, so that the gauge force can compensate the tension, we then obtain:

$$\Phi = - \frac{G_{t\phi}}{G_{\phi\phi}} \dot{X} \pm J_C \sqrt{\left( \frac{G^2_{t\phi}}{G^2_{\phi\phi}} - \frac{G^2_{tt}}{G_{\phi\phi}} \right) \dot{X}^2 - \frac{G_{rr} \mathcal{Y}^2}{G_{\phi\phi}}}.$$

and the Legendre transformation becomes:

$$\mathcal{L}_{D3}^J = \mathcal{L}_{D3} - \Phi J \left\{ - (\mathcal{C}_4)_t - \frac{G_{t\phi}}{G_{\phi\phi}} J_C + \sqrt{\left( \mathcal{G}^6 \frac{J_C^2}{G_{\phi\phi}} + G_{tt} \mathcal{R}^2 \right) \left( \frac{G_{tt}}{G_{\phi\phi}} + G_{rr} \mathcal{R}^2 \right)} \right\}.

Note that the derivative is now represented by a prime instead of a dot. For computational convenience, one may choose to write $\frac{\partial R}{\partial T} = \frac{\dot{X}}{\Sigma}$, which implies that prime is a derivative with respect to the time-coordinate of spacetime. Introducing the components of the ten-dimensional line invariant (7.2), Eq.(7.26) can be expressed as

$$\mathcal{L}_{D3}^J = -2\pi^2 T_3 \dot{X} \left\{ - (\mathcal{C}_4)_t - \frac{J_C}{L} A_t + \sqrt{\left( \mathcal{G}^6 \frac{J_C^2}{L^2} \right) \left( g_{tt} + g_{rr} \mathcal{R}^2 \right)} \right\}, \quad (7.27)$$

where $B = g_{tt}$ results after simplification of the metric entries. The linear term $J_C$ in (7.27) shows that the brane carries charge and couples to the bulk gauge field. Note that $J_C$ rather than $J$ measures the charge [45]. This relative shift can be explained by frame dragging caused by the rotation in the extra dimensions.\footnote{The right angular momentum of the expanding bubble is obtained when its rotation in the compact direction is measured with respect to a "static" reference system of the background. Hence the relative shift.}
Let us elaborate on the conceptual picture that the square root term in Eq. (7.27) represents: In the absence of $J_C$, this is simply an effective five-dimensional DBI-term. The presence of $J_C$ can be understood as an additional flux representing an uniform density of charged particles over the D3-brane. This dissolved particle density spread over the brane can be interpreted as the charge $Q$ from a five-dimensional point of view. \[ \text{eq:13} \]

Lastly, the $(\mathcal{C}_4)_t$-term can be studied as a five-dimensional WZ-term; One can always define an effective five-dimensional flux $F_{\text{eff}}^5 = d\left[(\mathcal{C}_4)_t dt \wedge \epsilon_3\right]$ with $F_{\text{eff}}^5$ proportional to the volume form of the line invariant $ds_5^2$.

Our $D_3$-dynamics equation (7.27) can be written in a simpler expression in terms of the Hamiltonian

\[
H = 2\pi^2 T_3 \frac{\dot{\mathcal{X}}}{\sqrt{-\left(g_{tt} + g_{rr} R'^2\right)}} \left[-g_{tt}\sqrt{\mathcal{R}^6 + \frac{J_C^2}{L^2}} - (\mathcal{C}_4)_t - \frac{J_C}{L} A_t\right], \tag{7.29}
\]

given in terms of the proper time of the brane. Careful here! It is of extreme importance to realize that this is not the time variable relevant for a four-dimensional observer like our dear reader [45]. Such a witness moves along with the nucleated brane in AdS$_5$, but does not move together with the brane in its compact dimensions motion. This is a similar argument to that Kaluza-Klein (KK) compactifications to four dimensions. Any observer, made out of KK particles, is governed by a time component which does not involve the internal momenta or velocities of their particles. We will refer to the relevant four-dimensional proper time as $\tilde{\tau}$, which is computed as

\[
d\tilde{\tau} = dt \sqrt{-g_{tt} + g_{rr} R'^2}. \tag{7.30}
\]

This allows us to rewrite the Hamiltonian (7.29) in terms of the proper time of the four-dimensional observer as:

\[
H = 2\pi^2 T_3 \frac{d\Sigma}{d\tilde{\tau}} \left[\sqrt{\mathcal{R}^6} \frac{J_C^2}{L^2} \sqrt{g(\mathcal{R}) + \mathcal{R}^2} - \frac{1}{L} (\mathcal{R}^4 - r_H^4) - \frac{\kappa_5 J_C Q}{L} \left(\frac{1}{r_H^2} - \frac{1}{\mathcal{R}^2}\right)\right], \tag{7.31}
\]

\[\text{eq:111}\]

One can formally collapse the brane to a point (i.e. set $\mathcal{R} = 0$), to see Eq. (7.27) become:

\[
-\frac{2\pi^2 T_3 J_C}{L} \frac{\dot{\mathcal{X}}}{\sqrt{-g_{tt} + g_{rr} R'^2}}. \tag{7.28}
\]

In the absence of the dissolved particles, the total mass of the $D_3$ approaches zero in this limit. In this case the mass converges to $\frac{2\pi^2 T_3 J_C}{L}$. This can be interpreted as the total mass of the dissolved particles.
where Eq. (7.30) has been used to rewrite the first term in the Hamiltonian (7.29), while its second and third pieces correspond to expressions (7.15) and (7.13) respectively. Recall that $g(R)$ is the function to describe the AdS$_5$-Reissner-Nordström geometry (7.7).

This Hamiltonian can be equated to a conserved energy $E$,\textsuperscript{14} while the constant in $(C_4)_T$ can be considered a gauge choice that also shifts $E$\textsuperscript{45}. In order to fix such conserved energy, it is important to know that the bubble, acting as a Coleman-de Lucia instanton, will nucleate at rest. This results into zero total energy in the approximately extremal background.\textsuperscript{15} Imposing $E = 0$, one can then solve Eq. (7.31) to find the equation of motion controlling the dynamics of the brane as:

$$
\dot{R}^2 = -f(R) - k^2 R^2 + \frac{k^2 \left( R^4 - r_H^4 + \frac{Q \kappa_5 J_c}{r_H^2} \left( 1 - \frac{r_H^2}{R^2} \right) \right)^2}{k^2 J_c^2 + R^6},
$$

(7.32)

where $f(R) = g(R) - k^2 R^2$ and terms in green and blue will be discussed in the next section. Eq. (7.32) still contains both geometrical information, through its first DBI term (i.e. $g(R)$) and WZ information (i.e the last term). This equation of motion is the furthest we can go simplifying the action of the $D_3$-brane (7.16) into the ten-dimensional background. In order to understand why the embedding of the dark bubble into string works, we will compare Eq. (7.32) to the equation governing a four-dimensional expanding cosmology derived from the junction conditions, as shown in Eq. (6.22). This will be the aim of the next section.

7.4 We are not so different after all, said Israel to Hamilton

As stated above, the five-dimensional picture is given by an AdS$_5$-Reissner-Nordström geometry. The black hole (i.e. the stack of $D_3$-branes), is unstable due to the presence of the large chemical potential $\mu$ and one or several can escape (i.e. nucleate) from the outer horizon $r_H$. Let’s begin our comparison by considering one single nucleated brane. It has tunnelled from the outer horizon $r_H$ at radius $r > r_H$ and starts to expand. Similarly to the simplest example described in section 6, this shell will mediate the decay between two different vacua: The outside, which has no idea yet that neither vacuum scale $k$, nor the black hole charge $Q$ and $M$ have changed and

\textsuperscript{14}Although we did not derive this conserved quantity, we saw that the Lagrangian density (7.23) does not depend on the coordinate $\zeta$. This implies $E$ to be conserved.

\textsuperscript{15}As described in the introduction of this chapter, the ten-dimensional ambient space has broken supersymmetry, due to the presence of close-to-zero temperature $T$ and chemical potential $\mu$. 

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the inside, where this information has already been taken into account, re-
sulting in a change in the geometry. Therefore, the inner and outer metric 
argument \( g(r) \) can be written as:

\[
g_{\pm}(r) = k_{\pm}^2 r^2 + 1 - \frac{2\kappa_5 M_{\pm}}{r^2} + \frac{\kappa_5^2 Q_{\pm}^2}{r^4},
\]

(7.33)

In the same spirit as in section 6, the junction condition (6.17) becomes:

\[
T_3 = \frac{3}{\kappa_5} \left( \sqrt{g_-(a)} \frac{\dot{a}^2}{a^2} + \sqrt{g_+(a)} \frac{\dot{a}^2}{a^2} \right).
\]

(7.34)

Here \( T_3 \) represents the tension of the brane (we will later see why \( T_3 \) and not \( \sigma \)). The location of the brane along the AdS5-throat is \( r = a(\tau) \).

In order to obtain a similar equation to (7.32) to compare with, one can 
solve exactly for \( \dot{a}^2 \) to obtain a Friedmann-like equation:

\[
\dot{a}^2 = \frac{1}{4\sigma^2 a^2} \left[ (k_-^2 - k_+^2) a^2 + (f_-(a) - f_+(a))^2 \right.
\]

\[
+ \frac{\sigma^2}{4} a^2 - \frac{1}{2} \left[ (f_-(a) + f_+(a)) + (k_-^2 + k_+^2) a^2 \right],
\]

(7.35)

As a handy notation, let us refer to the critical tension of this brane (6.19) as \( \sigma = T_3 \frac{\kappa_5}{3} = (k_- - k_+) = \Delta k \). In order to compare with (7.32), it is important 
to ensure that we are in the correct regime, i.e. that of the probe approxi-
mation. This means that we must work at linear order in the perturbations 
when making expansions of the form:

\[
k_{\pm} = k \mp \frac{1}{2} \Delta k, \quad Q_{\pm} = Q \mp \frac{1}{2} \Delta Q, \quad M_{\pm} = M \mp \frac{1}{2} \Delta M.
\]

(7.36)

Note that \( \Delta Q = Q_- - Q_+ < 0 \). This means that when the brane leaves the 
stack, it carries mass (i.e. tension) and a little bit of charge (the angular 
momentum \( J_c \) from the ten-dimensional perspective) with it. In this way, 
one ensures that the physical picture is respected by the approximation.

With approximations (7.36) at hand, let us substitute them in Eq. (7.35) 
to read:

\[
\dot{a}^2 = - f(a) - k_2 a^2 + \frac{k_2^2}{a^6} \left( a^4 + \frac{a^2}{2k\Delta k} (f_-(a) - f_+(a)) \right)^2,
\]

(7.37)

where only linear order terms have been considered. Here \( f(a) \) condenses 
zeroth order information about the mass \( M \) and charge \( Q \), while \( f_-(a) -

\[\text{This } \tau \text{ is the real proper time discussed in Eq. (7.39). The tilde was dropped to obtain a cleaner notation.}\]
\( f_4(a) \) encodes linear terms in \( \Delta Q \) and \( \Delta M \). The structure of Eq. (7.37) starts to resemble that of the embedding potential (7.32). But we aim for a closer match. Notice that this expression can be simplified even further. If one evaluates order by order the junction condition (C.25) and \( g(r) \) at the horizon \( r_H \), accounting that \( g_\pm(r) \) must vanish at the horizon \( r_H \), it results in the following constraints at the zeroth and first order:

0\textsuperscript{th}: \[ k^2 a^2 + g(a) = 0, \]

1\textsuperscript{st}: \[ k \Delta k a_H^2 - \frac{\kappa_5 \Delta M}{a_H^2} + \frac{\kappa_5 Q \Delta Q}{a_H^4} = 0. \] (7.38)

Substitute (7.38) in Eq. (7.37) results in:

\[ \dot{a}^2 = -f(a) - k^2 a^2 + \frac{k^2}{a^6} \left( a^4 - a_H^4 - \frac{\kappa_5 Q \Delta Q}{a_H^4} k \Delta k \left( 1 - \frac{a_H^2}{a^2} \right) \right)^2. \] (7.39)

Overall, the whole structure of Eq. (7.39) is identical to that of expression (7.32). Recall that, although we had chosen capital letters \( X \) to denote the embedding functions of the brane, the five-dimensional picture uses lower-case letters. In any case, both represent the same coordinates. Let us discuss the terms highlighted in colours:

- **fraction**: The comparison of both terms easily yields:

\[ J_C = -\frac{\kappa_5 \Delta Q}{k \Delta k}. \] (7.40)

This has already been discussed; It is the realisation that the brane’s effective angular momentum in the compact directions is equivalent to an effective charge \( \Delta Q \) carried by the brane when observing the system from a five-dimensional perspective.

- **denominator**: At first glance, one can observe the absence of a \( J_C^2 \) term in Eq. (7.39). This is because one important character, present in the string theory brane action, but not in the junction conditions, was not taken into account; The flux related to the dissolved charge density discussed above. Opposite to the DBI + WZ action, the junction condition (7.34) did not account for the presence of this dissolved flux. In fact, this can be used to enhance the junction conditions. Replacing \( T_3 \) in expression (7.34) by \( T_3 \sqrt{1 + k^2 J_C^2 a^{-6}} \) the result is the same as in Eq. (7.32).

After having related the two problematic terms, the match is complete. One can easily see that, the equation of motion for the \( D_3 \)-brane given by Eq. (7.16) and the junction conditions across the brane (7.34) are the same.\footnote{This will be a recurrent scheme in this part of the thesis. We will find that the equations of motion of the bubble obtained from the action are also reproduced by the junction conditions.}
expression. This just tells us that the construction from the DBI+WZ action (7.16) in the ten-dimensional stringy ambient space description produces the same dynamics as the junction condition for the co-dimension one shell in the five-dimensional AdS$_5$ bulk. This can just point to one simple fact:

*The dark bubble model has been embedded into string theory.*

### 7.5 Why do we fall?  
A collapsing cosmology

There is one little detail we have not commented on yet: This embedding of the dark bubble model within string theory is not able to reproduce a late time cosmology dominated by a cosmological constant (yet). This implies that the curvature of the universe will eventually become dominant, reaching a maximum value for the scale factor $a(\tau)$ and collapsing back, in the form of a bouncing cosmology. To realise this, one simply needs to check the behaviour of the Friedmann-like expression (7.32) for late-time cosmologies (i.e. $k\mathcal{R} \to \infty$). In this regime, the $k^2 \mathcal{R}(\tau)^2$ cancels, indicating that the brane's tension is equal to the critical one, as discussed in section 6. But let us examine this issue from the perspective of the brane tension $T_3$.

Junction conditions are formulated from a general relativity perspective; If both vacua (inside and outside) are standing-alone solutions to Einstein equation, what induced energy momentum tensor (6.13) should the brane be equipped with such that the whole is a solution $t$ of spacetimeo the Einstein equation? This implies that Newton's constant $G_5$ is constant from this perspective. This does not imply that any other combination of variables to describe this constant cannot change, as long as the resulting Newton's variable $G_5$ remains constant.

Given the AdS/CFT dictionary [128], one finds that:

$$G_5 = \frac{\pi L^3}{2N^2}, \quad (7.41)$$

where again $L$ is the curvature radius of the compact direction and the AdS-throat, while $N$ is the number of branes sitting in the stack of $D_3$-branes sourcing the geometry of the ambient space. Imposing the Newton's constant to be constant across the brane (i.e. $\Delta G_5 = 0$), we get:

$$\frac{\Delta L}{L} = \frac{2}{3} \frac{\Delta N}{N}, \quad (7.42)$$

where $\Delta L$ represent the difference in the curvature radius across the brane and $\Delta N$ encodes how many $D_3$-branes have nucleated together, mediating

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$^{18}$So we can learn to pick our bubble up.
the decay. This relation (7.42) can be used to rewrite the critical tension of the brane as:

\[ T_3 = \frac{3\Delta k}{8\pi G_5} \to -\frac{3}{8\pi G_5} \frac{\Delta L}{L^2} = -\frac{1}{4\pi G_5 L} \frac{\Delta N}{N} = -\frac{1}{2\pi^2 L^4} N \Delta N \]

(7.43)

Note that relation (7.42) has been used to reach the third step. The fourth equality is obtained using Eq. (7.41), and the final expression just requires us to use Eq. (4.9) and another AdS/CFT dictionary key:

\[ L^4 = 4\pi g_s N \ell_s^4. \]  

(7.44)

When \( \Delta N = N_{\text{final}} - N_{\text{initial}} = -1 \) (i.e. just one nucleated brane), Eq. (7.43) states that the critical tension is exactly that of a \( D_3 \)-brane. This means that the induced cosmological constant will be zero [45]. Before we decide to delve into the intricacies of inducing an effective positive, yet small four-dimensional dark energy density \( \rho_\Lambda \) in the model, let us first analyse how the induced four-dimensional cosmology, driven to the failure by the absence of a late-time cosmological constant dominance, behaves under the parameters controlling its dynamics.

For starters, let us express Eq. (7.39) in terms of the horizons \{a_h, a_H\} (recall that from a four-dimensional perspective \( R(\tau) = r(\tau) = a(\tau) \)) by substituting Eq. (7.10). This gives:

\[
\dot{a}^2 = \frac{k^2(a_h^2 - a^2)(a_H^2 - a^2)(a_h^2 + a_H^2 + \frac{1}{k^2} + a^2)}{a^4}
+ \frac{k^2}{a^4} \left( a^4 - a_H^4 + \frac{a_h J_c}{a_H} \sqrt{1 + k^2 a_h^2 + a_h^2} \left( 1 - \frac{a_h^2}{a_H^2} \right)^2 \right)
+ \frac{k^2 J_c^2 + a^6}{k^2 J_c^2 + a^6}.
\]

(7.45)

The first property to note is that \( \dot{a} = 0 \) when \( a = a_H \). As mentioned above, the brane nucleates at rest, which not only the junction condition, but also the DBI action from string theory remark in this construction. When both horizons coincide (i.e. \( r_h = r_H \)), one recovers the extremal description of the Reissner-Nordström black hole living in the AdS_5 bulk. Figure 7.2 will allow us to further explore other critical points of this construction.

There are three (four) particular points to consider in figure 7.2:

- Both horizons \{a_h, a_H\} (first two roots in the non-extremal case): They correspond to the inner and outer horizons of the Reissner-Nordström black hole respectively. In the extremal case, both horizons coincide, which corresponds to the first root of the blue plot.
• The nucleation point \( a_{\text{nuc}} \) (2\(^{\text{nd}}\) or 3\(^{\text{rd}}\) roots, respectively): This is the value of \( a \) at which the \( D_3 \) brane (i.e. the four-dimensional universe) will nucleate. The region between \( a_H \) and \( a_{\text{nuc}} \) is the classical forbidden region through which the brane must tunnel. See chapter 8 for further information about the nucleation event.

• The maximum size \( a_b \) of the collapsing universe: This is given by the last root of the potential \( V(a) \). At this point, the size of the bubble is maximal, and the four-dimensional universe, without positive dark energy density, but equipped with positive curvature, will reach its maximum size and start contracting, in the form of a bouncing cosmology.

Which different features are controlled by which parameters? By construction, the potential (7.45) goes to 1 when the scale factor \( a \rightarrow \infty \). The AdS scale \( k^{-1} \) is in charge of controlling the overall value of the potential (i.e. basically scaling the \( y \)-axis). The angular momentum of the brane \( J_C \) will regulate the barrier of the potential.

\[ V(a) \]

\[ a \]

*Figure 7.2.* The potential controlling the dynamics of the four-dimensional cosmology riding on the \( D_3 \)-brane. Non-extremal and Extremal cases, with respective non-degenerate and degenerate horizons, will ultimately bounce back due to the absence of an effective positive cosmological constant (observe the asymptotic behaviour towards one when \( a \rightarrow \infty \)). When the brane has induced dark energy (see section 7.6), the (dashed) potential will dictate a never-ending expansion of the four-dimensional cosmos.

The potential described in Eq. (7.45) lacks a sustained cosmological constant and experiences only a fraction of an e-folding just after creation. The
expansion rate vanishes at nucleation, the universe then accelerates for a short time after which it enters a decelerated phase. As the curvature is positive, and there is no dark energy in the game, the universe will eventually stop expanding and start contracting, entering a bouncing phase [45].

Of course, this is not the desired fate for the dark bubble model in its stringy embedding. Our goal is to find a low-dimensional model capable of describing dark energy derived from string theory. Do previous computations indicate that the dark bubble model embedding should be ruled out? Absolutely not. All stringy computations so far have been performed assuming that the brane is a BPS one, although the ten-dimensional ambient space hosting it has broken supersymmetry. This will be taken into account in the incoming sections.

7.6 A hero can be anyone: Curvature corrections to obtain $\Lambda_4$

Looking carefully at Eqs. (7.43) and (7.45), it is easy to see the absence of an induced dark energy density $\rho_\Lambda$, at least, at leading order in the brane action. At the beginning of this section, we argued that the angular momentum of the stack of branes$^{19}$ breaks supersymmetry. We also discussed the implications that the WGC would have in non-supersymmetric backgrounds in section 5.3. Broken supersymmetry leads to a non-saturated inequality (i.e. $Q_{D_p} < T_{D_p}$) for extended objects in string theory. This would require the presence of corrections to the $D_p$-brane action, going beyond the rough sketch provided by the leading order, specially in the description of the brane’s tension, making it subcritical. In the dual field theory description of this embedding, the nucleation of a $D_3$-brane can be understood as if some of the $N = 4$ SYM scalar fields obtained a non-vanishing expectation value. This results in a Higgsing of the gauge group as $SU(N) \rightarrow SU(N - 1) \times U(1)$. Coming back to the gravity side, this process can be described as corrections of the order $1/N$, where $N$ is the number of $D_3$-branes in the stack.$^{20}$

If this is the case, What are the options for correcting the action? There are two types of corrections (up to a power of the string coupling $g_s$) that can contribute roughly at the same order of magnitude $1/N$.

- "Stringy" corrections: These corrections at $O(a'^2)$ can be obtained for both relevant parts of the $D_p$-brane action. On the one hand, cur-

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$^{19}$The high chemical potential $\mu$ and the non-zero temperature $T$ in the dual field theory.

$^{20}$One could argue that the expected corrections should be of order $1/N^2$, because all fields would belong to the adjoint representation. However, the relevant description of the group is after nucleation, hence the fundamental representation of $SU(N - 1)$, with the expected correction translation to $1/N$. 

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vature corrections for the DBI action were found in [200]. On simple
grounds, each term in the effective action is required to be consistent
with those terms of order $\mathcal{O}(\alpha'^2)$ of the corresponding disk-level scat-
tering amplitude [201]. On the other hand, the WZ curvature correc-
tions can be found by imposing that the chiral anomaly on the world
volume of intersecting D-branes cancels with the anomalous varia-
tion of the WZ action [202]. These additional terms would schemati-
cally represent a correction to the tension $\delta \sigma$ of the form:

$$\delta \sigma \sim T_D \frac{\alpha'^2}{L^4} \sim T_D \frac{1}{g_s N} \sim \frac{1}{g_s L^4 N^2} \sim \frac{1}{g_s L^4},$$

(7.46)

where we have used same set of relations as in Eq. (7.43). Note that,
since the length scale $\ell_s^2 = \alpha'$ is a classical scale set by the tension of
the string, these contributions can be interpreted as a tree level effect.

- **Quantum loop corrections**: Quantum loop corrections can be written
  as:

$$T_D \frac{\ell_{10}^4}{L^4} \sim \frac{1}{g_s \ell_s^4 L^4} \sim \frac{1}{L^4},$$

(7.47)

with $\ell_{10}$ as the ten-dimensional Planck length. The Planck length is
obtained from the classical Newton’s constant by introducing Planck’s
constant, showing that this correction comes from quantum loops
[47]. A renormalized one-loop calculation for a massive field of mass
$m$, leads to a contribution to the vacuum energy of order $m^4$. Hence,
we see that the loops above would be generated by fields with $m \sim 1/L$,
which are present in the theory. 21

So we have two possible candidates. It is not hard to see that the most
prominent correction is that of higher curvature corrections (7.46), since
we want to obtain results for vacua that are weakly coupled (i.e. $g_s \ll 1$). Let
us then dissect the individual contributions to the $D_p$-brane action.

### Corrections to the DBI term

As discussed before, string theory corrections to the DBI-action at order $\alpha'^2$
due to curvature were found in [200]:

$$S_{DBI}^{\alpha'^2} = - T_D \frac{\pi^2 \alpha'^2}{48} \int d^4 \xi \, e^{-\Phi} \sqrt{|\mathcal{G}|} \left( (R_T)_{abcd} (R_T)^{abcd} 
- 2(R_T)_{ab} (R_T)^{ab} - (R_N)_{abij} (R_N)^{abij} + 2(\bar{R})_{ij} (\bar{R})^{ij} \right).$$

(7.48)

21Note that this also is the mass-scale of the neutrino.
Note that $\left(R_T\right)_{abcd} = R^{(4)}_{abcd}$ is the induced curvature on the brane. The curvature of the normal bundle is given by $\left(R_N\right)_{abij}$. Both are related to the bulk and extrinsic curvatures through the Gauss-Codazzi equation \text{(C.18)}. However, there is one term that cannot be found in appendix C. This is $\bar{\mathcal{R}}_{ij}$, which can be defined as:

$$\bar{R}^{ij} = h^{ab}R^{(4+d)ij}_{ab} + h^{ab}h^{cd}\left(K^i_{ac}K^j_{bd} - \chi K^i_{ab}K^j_{cd}\right).$$

Contrary to other corrections to the action, Eq. \text{(7.49)} has no clear geometrical interpretation. Let us focus our attention on its two terms containing extrinsic curvature pieces. There is an important story here.

The first one, $K^i_{ac}K^{acij}$, was derived for the first time in [200]. The second term, $K^a_{ai}K^b_{bj}$, has an undetermined coefficient $\chi$, which could not be determined from the scattering amplitudes in [200]. How do we then fix its value? In fact, this term seems to be the responsible for contributing in a non-trivial way to the correction of the DBI piece of the dark bubble’s action \text{(7.16)}. In short: T and S-dualities of string theory can be used in order to fix $\chi = 1$, as was done in [203]. One has to be careful with terms that can be mixed up by the action of S-duality, but fortunately, none of them are mixed up in the computation. Therefore the result $\chi = 1$ from [203] can be trusted.

Let us now use this non-trivial correction to shift the tension of the brane in the DBI piece \text{(7.48)}. As the cosmological constant we would like to induce will only be relevant at late-time descriptions of the expanding cosmology, it will be enough to study a pure AdS$_5$-background. Any matter component induced by the Reissner-Nordström piece of the metric \text{(7.2)} can be ignored. Furthermore, as the cosmological constant is small compared to all other scales in the model, we can ignore the expansion of the dark bubble itself when calculating its value. With this in mind, let us assume that the brane sits at a fixed position along the AdS$_5$-throat, (i.e. $r = r_0$) and on the $S^5$. With $AdS_5 \times S^5$ as a direct product, and the brane positioned within the AdS$_5$, all contributions to the extrinsic curvature $K^i_{ab}$ with $i \neq r$ vanish. In this sense, we recover the co-dimension one shell extensively which is discussed in detail in appendix C. With such single extra dimension, one can start to simplify the terms. By definition, $\left(R_N\right)_{rr} = 0$, as it is anti-symmetric. $R_T$ will also vanish, as the brane is flat in the late-time cosmological approximation that has been taken. Hence, the only surviving term in \text{(7.48)} is $\bar{R}^{ij}$. Trusting $\chi = 1$ in Eq. \text{(7.49)}, as derived in [203],

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22 Note that \{$i, j\}$ indices represent directions that are co-dimensions of the embedding space. We will later drop these indices as we are dealing with a co-dimension one hypersurface.

23 We refer the reader to [47], in order to discover the subtle details of such derivation.

24 Correction coming from induced curvature itself will be sub-sub-leading and irrelevant for the computations to be shown.
it gives:

\[ \tilde{R}^{rr} = -16k^2 g^{rr}. \]  

(7.50)

Substituting this result back into Eq. (7.48), and simplifying factors, it is easy to see:

\[ T_{D_3} \rightarrow T_{D_3} \left( 1 - \frac{32}{3} \pi^2 a'^2 k^4 \right). \]  

(7.51)

As expected, there has been a shift in the tension, making it smaller than the original leading contribution. This is in line with WGC implications, as discussed above. Let us now turn our attention back to the Wess-Zumino piece of the action.

**Corrections to the WZ term**

To discuss the possible correction contributions to the Wess-Zumino piece of the \( D_3 \)-action (7.16), let us first examine its most general expression. It describes the coupling of the brane to Ramond-Ramond fields of different dimensions (represented by \( C \)). It is given by:

\[ S_{WZ} = T_{D_p} \int d^{p+1}x \mathcal{C} \sqrt{\frac{\mathcal{A}(4\pi^2 a'^2 R_T)}{\mathcal{A}(4\pi^2 a'^2 R_N)}}. \]  

(7.52)

Here \( \mathcal{A} \) represents the amplitude computation that contains tangential or normal terms. In our specific case, since \( R_T \) and \( R_N \) vanish, there are no relevant corrections from this expression. However, there is a potential contribution related to the above, as it was shown in [204]. This is:

\[ S_{WZ} \supset - \frac{\pi^2 a'^2}{12} \int d^4x e^{-\Phi} \sqrt{|g|} \epsilon^{a_0a_1a_2a_3} \frac{1}{(p+1)!} \partial_r \tilde{\mathcal{F}}_{(5)}^{(5)} a_{a_0a_1a_2a_3} \tilde{R}^{rr}, \]  

(7.53)

where \( \tilde{\mathcal{F}}_{(5)}^{(5)} \) is the flux sourced by the presence of the \( D_3 \)-brane along the non-compact directions and \( \epsilon \) is the Levi-Civita symbol. This is the only relevant term of all those derived in [204] for the case studied. Note the presence of \( \tilde{R}^{rr} \), which contains the trace of the extrinsic curvature discussed above. Unfortunately, it is not possible to fully trust the results derived in [204].

However, one can argue the following: Eq. (7.53) is equipped with a term of the form

\[ \partial_r \tilde{\mathcal{F}}_{(5)}^{(5)} a_{a_0a_1a_2a_3}. \]  

(7.54)

In the analysis of [204], assuming that the trace of the extrinsic curvature vanishes, there is no difference between \( \partial \) and the full covariant derivative \( \nabla = \partial + ... \). One could therefore argue that the correct expression should be (7.53) with a covariant derivative. Then the correction to the WZ will simply
vanish as a result of the equations of motion for \( F \) \cite{47}! However, an explicit scattering calculation is needed to confirm these results.

Surprisingly, the previous argument only leaves the correction as the simple term derived in Eq. (7.51). What consequences can this have for the embedding of the dark bubble? Let us explore them.

7.7 N, the parameter in the shadows controlling scales

As previously discussed in chapter 4, regular compactifications already impose severe restrictions on the size of compact dimensions, so that gravitational effects have no effect on the four-dimensional physics. But the dark bubble scenario presented above is not a regular compactification model. Could it be that its energy hierarchy holds surprises? Let us explore this in detail.

Although we may have enjoyed our short stay in the four-dimensional world, in order to obtain an explicit value for the induced dark energy, we must climb up back to the ten-dimensional ambient space described at the beginning of this chapter 7.

This ten-dimensional rotating background,\(^{25}\) sourced by the presence of the famous \( D_3 \)-brane stack, is equipped with a ten-dimensional Planck length \( \ell_{10} \) described as:

\[
\ell_{10}^8 = 2^6 \pi^7 g_s^2 \alpha'^4, \tag{7.55}
\]

where again, \( g_s \) is the string coupling and \( \alpha' \) is the Regge slope parameter. Although the determination of this parameter will be key to understanding the novelty provided by the embedding of the dark bubble, it will be more convenient to refer to its associated value, the length scale of the string\(^{26}\) (i.e. how long a string is), given by:

\[
\ell_s = \sqrt{\alpha'} \tag{7.56}
\]

Let us go down to five dimensions: Here our rotating black hole is no longer moving along some of the compact dimensions of \( S^5 \). It is a regular Reissner Nordström black hole and the secret information about its hidden angular momentum is encoded in its charge \( Q \). Following the discussion in chapter 4, the five-dimensional Planck length is given by

\[
\ell_5^3 = \frac{\ell_{10}^8}{\text{Vol}(S^5)} = 2^6 \pi^4 g_s^2 \frac{\alpha'^4}{L^5}. \tag{7.57}
\]

\(^{25}\)From now on and until we say otherwise, we will work in natural units \( \hbar = c = 1 \). We will restore them later to talk about the energy scales.

\(^{26}\)This section will be an interesting dance between lengths, masses and energies. It may be appropriate to have in mind the definition of Newton’s constant (2.3).
This can be rewritten in terms of the number of branes located in the $D_3$-stack. Using Eq. (7.44), Eq. (7.57) becomes:

$$L^3 = \frac{N^2 \ell_5^3}{4\pi^2}. \quad (7.58)$$

If you have a basic understanding of the correspondence discussed in [128], you may be concerned that this model has neither a low coupling nor a high number of branes $N$. We will see later that the number of branes in the stack is large enough for the correspondence to hold.\footnote{If every brane was worth a single $, we would be hobnobbing with Mr. Bezos and Mr. Musk.} For the moment, let us take a leap of faith and assume that $g_s N \gg 1$. This implies the following hierarchy of scales:

$$\frac{(N/2\pi)^{2/3}\ell_5}{L} \gg \frac{\pi^{3/8}(N/2\pi)^{5/12}\ell_5}{\ell_{10}} \gg \ell_5, \quad (7.59)$$

Down to this point, there is nothing new. A regular compactification of a ten-dimensional space down to a five-dimensional one. All the energy scales are the expected ones. But now we have a toll ahead: the junction conditions. As discussed in chapter 6, the presence of the brane which mediates the decay between the vacua imposes a jump in the extrinsic curvature. This leads to an induced energy-momentum tensor $S_{ab}$, so that the whole system remains a solution of the Einstein equation. In this section we have obtained an extremely important expression (6.23) which will be used in our next task: To find the four-dimensional Planck length in this construction.

Making use of Eq. (7.42) to rewrite Eq. (6.23), it yields:

$$\kappa_4 = 2 \frac{k_- k_+}{k_- - k_+} \kappa_5 = \frac{3k^2}{-\Delta N k} N \kappa_5 = 3 \frac{N}{L} \kappa_5, \quad (7.60)$$

where we have written $k_- \approx k_+$ (recall the relevant regime described in Eq. (7.36)) and $\Delta N = -1$. Again, we can play the game and rewrite the AdS-length (7.58) in terms of $\ell_4 = \sqrt{\kappa_4}$ as:

$$L = \frac{N^{1/2}}{2\sqrt{3}\pi} \ell_4, \quad (7.61)$$

which allow us to massage the hierarchy (7.59) in terms of $\ell_4$ instead:

$$\frac{N^{1/2}}{2\sqrt{3}\pi} \ell_4 \gg \frac{N^{1/4}}{2^{3/4}\sqrt{3}\pi^{3/8}} \ell_4 \gg \ell_4 \gg \frac{N^{-1/6}}{\sqrt{3}(2\pi)^{1/3}} \ell_4. \quad (7.62)$$
And it is at this point when surprises come in. It is of crucial importance to notice how unusual the hierarchy $\ell_5 \ll \ell_4$ is. In regular compactifications, $L \gg \ell_5$ immediately leads to $\ell_5 \gg \ell_4$. In this way, gravity is stronger in the compact dimensions and hence the upper bound on the volume of those closed directions to avoid conflict with observations. But this is not the case for our bubbly construction [47]. This can be already deduced from Eq. (7.60). The presence of the large factor $N$ is the term responsible for this inversion in the hierarchy. With a weaker gravity in the extra dimensions, there is no strict upper bound to be imposed. In this way, the dark bubble scenario acquires its own energy hierarchy not through the usual scale separation as in regular compactifications, but through charge-to-tension ratio (i.e. the sub-extremality of the bubble).

Let us now find the explicit value for the dark energy density and the energy scale associated with this embedding. From Eqs. (7.51) and (6.23), we see that:

$$\rho_\Lambda = \frac{32}{3} \frac{\pi^2}{L^4} T_{D_3} \rightarrow \rho_\Lambda = \frac{4}{3\pi g_s L^4}, \quad (7.63)$$

where Eq. (7.44) has been used. Rewriting for one final time expression (7.61), by using equations (2.3), (7.58) and (7.60) we find:

$$L^2 = \frac{2}{3\pi} NG_4. \quad (7.64)$$

Finally, Eq. (7.64) can be used in Eq. (7.63) to write:

$$N = \frac{\sqrt{3\pi}}{G_4 \sqrt{g_s \rho_\Lambda}}. \quad (7.65)$$

The total number of branes $N$ sitting in the stack which source the ten-dimensional geometry (7.2) can then be determined by the observed dark energy density in the low-dimensional geometry, its Newton’s constant and the string coupling $g_s$. In order to find the low-energy value of the string coupling, we take a result to be discussed\(^{28}\) in chapter 9, Eq. (9.8):

$$M_{\text{particle}} = T_s L, \quad (7.66)$$

which corresponds to the mass of a four-dimensional particle represented by the end point of a fundamental string. This string has charge $e$ and tension $T_s = 1/2\pi \alpha'$ and it is attached to the $D_3$-brane [38]. The end point mass will induce a four-dimensional Reisser-Nordström geometry with line invariant:

$$d_{\text{RN4D}}^2 = -g(r)_{\text{RN}} dr^2 + g(r)^{-1}_{\text{RN}} dr^2 + r^2 d\Omega^2_2, \quad (7.67)$$

\(^{28}\)This should be the only placeholder used in this thesis, i.e. the only result to be taken from a future rather than a past chapter.
where

$$g(r)_{RN} = 1 - \frac{2cM_{RN}}{8\pi \hbar r} \ell_4^2 + \frac{e^2}{4\pi \epsilon_0 c \hbar} \frac{\ell_4^2}{8\pi r^2},$$

(7.68)

with $\alpha_{EM} \approx \frac{1}{137}$, is the fine structure constant. Since the string is fundamental (i.e., a BPS object saturating the WGC, $q = m$), the induced geometry will be that of an entremal Reissner-Nordström geometry. With this in mind, one can obtain the following relationship between the string coupling $g_s$ and the fine-structure constant $\alpha_{EM}$:

$$M_{\text{particle}} = M_{RN} \rightarrow g_s = \frac{2}{3} \alpha_{EM}. \quad (7.69)$$

Electromagnetism is the only long-range force, apart from gravity, and it is therefore natural that the low energy value of the string coupling is determined by it.

With relation (7.69), we have all the relevant ingredients to obtain the explicit energy hierarchy of the dark bubble. Introducing this result in expression (7.65), together with $\rho_\Lambda \approx 6.8 \times 10^{-27} \text{kg/m}^3$ [55], we obtain (now with $\hbar$ and $c$ reinserted):

$$N = 1.2 \times 10^{63}, \quad (7.70)$$

which fixes the hierarchy of scales (7.62) of the dark bubble construction as described in table (7.3). All the scales previously discussed are unambiguously fixed. Throughout the calculations, we have used the reduced Planck scale for convenience. In the table we have chosen instead to use $G_d = \tilde{\ell}_d^{-2}$.  

<table>
<thead>
<tr>
<th>Scale</th>
<th>Length ($m$)</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$5.1 \times 10^{-5}$</td>
<td>3.8meV</td>
</tr>
<tr>
<td>$\sqrt{\alpha'}$</td>
<td>$1.8 \times 10^{-20}$</td>
<td>11.2TeV</td>
</tr>
<tr>
<td>$\tilde{\ell}_{10}$</td>
<td>$1.4 \times 10^{-20}$</td>
<td>13.7TeV</td>
</tr>
<tr>
<td>$\tilde{\ell}_5$</td>
<td>$3.9 \times 10^{-45}$</td>
<td>$5.1 \times 10^{28}$ TeV</td>
</tr>
</tbody>
</table>

*Figure 7.3. New hierarchy of scales associated to the embedding of the dark bubble in string theory.*

Let us look at some of the implications of these results:

- As shown above, the dark bubble model does not enjoy the usual scale separation required in regular compactifications. It is not the size $L$ of these extra dimensions (including the curvature of the AdS-throat), but the charge-to-tension ratio, controlled by $N$, the key feature to obtain the hierarchy presented in table (7.3). It is noteworthy that,
even though any value of $L$ could have been possible, the model predicts modifications of gravity slightly beyond our current observations [205]. Furthermore, although this order of magnitude is not new in the literature [206], it is obtained by computations based on first principles and from a completely different approach that the aforementioned reference.

- Also surprising are the resulting values for both the string and the ten-dimensional scales \{$\sqrt{\alpha'} \ell_1$, $\ell_{10}$\}. They are also marginally above our current observations. These new scales could be detected in the form of string excitations related to photons or any other light known particles.29

We finish this chapter with these impressive results. Let us summarise how we did get to this point:

We started with a well-defined ten-dimensional ambient space, with its geometry sourced as a near horizon limit to the rotating stack of $D_3$-branes. From a five-dimensional point of view, the geometry is that of an Reissner-Nordström black hole living in an AdS space. Due to the presence of a high chemical potential $\mu$ and non-zero temperature $T$ in the dual field theory of the model, supersymmetry is broken. This implies an unstable configuration of the stack, allowing branes to escape through the horizon $r_H$ of the black hole and start to expand. At leading order, these escape branes are BPS objects, which translate in the absence of an induced dark energy density in the expanding four-dimensional cosmology they accommodate. However, this fact was changed when considered for higher curvature corrections to the $D_3$-action. This lowered the tension of the exiled brane, causing it to be non-critical, hence decorating the construction with the sought-after cosmological constant. Furthermore, feeding in our observational dark energy density to the model, resulted in a novel energy hierarchy, with precise new scales marginally beyond our sharpest observations. Is it not monumental that such a simple string theory construction yields an explicit and novel hierarchy of scales, just around the corner from our current observations?

29It is important to emphasise that the conditions for such a test of the like may be more complicated than just a threshold of energy. The luminosity of each beam and the pair-production bounds may also affect to the detection of such excitations, pushing back the testability of the predictions in time.
8. Quantum Bubbles: The Higher-Dimensional Solution

Every puzzle has an answer.

Hersel Layton, Professor Layton games

In this chapter you will learn...

- to quantise the action of an expading bubble
- how likely the bubble nucleation event is
- the relationship between a higher-dimensional nucleation event and the choice of boundary conditions in four-dimensional quantum cosmology

We have discussed in detail the technicalities of how to induce dark energy $\rho_\Lambda$ on the bubble wall in chapter 6 and how that simple model can be easily embedded in string theory through chapter 7. In that same chapter, we made some comments on how likely this process is to occur, but we did not provide an explicit and detailed calculation of the nucleation probability $P$. In this chapter we will present the results of Danielsson, Panizo et al. in [43] on the bubble nucleation process and any possible implications for the low-dimensional cosmology.

As we have seen in chapter 3, there is still an open discussion about the right choice of boundary conditions for the wave function describing our universe. An out-of-the-box perspective to address this issue can be to analyse this problem from a higher-dimensional picture. In other words, we ask ultraviolet-complete models of cosmology within string theory what the right choice of boundary conditions should be in lower-dimensional quantum cosmology. Given the previously presented dark bubble scenario, capable of reproducing an expanding four-dimensional cosmology equipped with a positive cosmological constant $\Lambda_4$, one question to raise could be: Can a quantised description of the dark bubble provide insight into the four-dimensional quantum cosmological realm? Let us explore this further.

This is a short and insightful chapter divided into two sections. Section 8.1 will be devoted to presenting the classical bubble action to quantise and
how we can extract information from its components. This will allow us to identify similarities and differences between the bubble motion and that of those of the four-dimensional cosmos discussed in chapter 3. Following the aforementioned similarities, we will be able to identify the bubble nucleation probability for the right choice of boundary conditions for the four-dimensional quantum cosmological description in section 8.2.

8.1 Scene reconstruction of the quantum bubble

We will now study the quantum nucleation of such a bubble by closely following the treatment of [207] for bubble nucleation in five dimensions.\(^1\) The five-dimensional action describing the dynamics of the subcritical tension bubble is given by:

\[
S_{\text{bubble}} = S_{\text{bulk}} + S_{\text{matter}} + S_{\text{GHY}}. \tag{8.1}
\]

Let us discuss each contribution individually:

- **The bulk contribution**, written as
  \[
  S_{\text{bulk}} = \frac{1}{2\kappa_5} \int d^5x \sqrt{|g(5)|R(5)}, \tag{8.2}
  \]
  has the role of describing the five-dimensional curvature of the ambient space where the bubble will live in.

- **We should also consider any matter content in the bulk and on the brane.** This is the duty of the second term in expression (8.1), described as:
  \[
  S_{\text{matter}} = \int d^5x \sqrt{|g(5)|} \mathcal{L}_{\text{matter}}, \tag{8.3}
  \]
  where
  \[
  \mathcal{L}_{\text{matter}} = \begin{cases} 
  \Lambda^{(5)}(5)/\kappa_5 & \text{in } \mathcal{M}_-, \\
  \Lambda^{(5)}_+(5)/\kappa_5 & \text{in } \mathcal{M}_+, \\
  -\rho \delta(\Sigma) & \text{on } \Sigma.
  \end{cases} \tag{8.4}
  \]

  Here \(\mathcal{M}_i\) represents each piece of the bulk geometry on the outside and the inside of the mediating brane, while \(\rho\) encodes its content. The position of the brane in the bulk is represented by \(\delta(\Sigma)\).

- **Finally, one should also take into account any possible information about how the brane is embedded in the five-dimensional bulk.** This

\(^1\)Note that we will not include in our description the compact directions of \(S^5\). We assume a small compact dimension with respect to the size of the five-dimensional bubble.
is done by the contribution of the extrinsic curvature $K_{ab}$ of the bubble into the five-dimensional space. This a higher-dimensional extension of the Gibbons-Hawking-York term presented in chapter 3, which looks like:

$$S_{\text{GHY}} = \frac{1}{\kappa_5} \int d^4 y \sqrt{|h|} K.$$  \hspace{1cm} (8.5)

It should be clear by now that the whole system is made up of: An outside bulk, a brane mediating the decay and an inside bulk geometries. This implies that one should be careful when integrating along the radial direction $r$, as the integral will be split into two integration regimes.

As the aim of this section is to provide a clear and simple conceptual picture of the bubble nucleation problem, for the sake of simplicity in our computations, we will choose an empty\(^2\) AdS-bulk described by Eq. (6.8), with

$$f_{\pm}(r) = 1 - \frac{\Lambda^{(5)}_{\pm}}{6} r^2.$$  \hspace{1cm} (8.6)

The induced geometry on the bubble is again an FRLW expanding cosmology (6.11). For computational convenience, let us define a handy notation of the form:

$$\beta_{\pm}(\tau) = f_{\pm}(r) \frac{dt_{\pm}(\tau)}{d\tau} \bigg|_{r=a(\tau)} = \sqrt{f(a(\tau)) N(\tau)^2 + \dot{a}^2},$$  \hspace{1cm} (8.7)

where again, $N(\tau)$ is the lapse function (6.12) and a dot represent a derivative with respect to $\tau$.

Our aim now is to rewrite Eq. (8.1) to obtain a Lagrangian $L_{\text{total}}^{(5)}$. We will then extract a Hamiltonian $\mathcal{H}$, which we will later quantise, etc. Basically, we will follow the discussion in chapter 3 applied to this bubble cosmology. In this sense, we will move the cosmological constant $\Lambda^{(5)}$ to the bulk contribution (8.2) and leave the brane content alone. Furthermore, we will use Eq. (8.7) to write the action in terms of the proper time $\tau$. Finally, each term contribution is given by:

$$S_{\text{bulk}} = \frac{2\pi^2}{\kappa_5} \int d\tau \left[ 3\beta(\tau) a^2 + \Lambda^{(5)} a^4 + \frac{a^3}{\beta(\tau)} \left( \dot{a} + \frac{N(\tau)^2}{2} \frac{df(\tau)}{da} - \frac{\dot{a}N(\tau)}{N(\tau)} \right) \right]^-,$$

$$S_{\text{GHY}} = -\frac{2\pi^2}{\kappa_5} \int d\tau \left[ 3a^2 \dot{a} \text{tanh}^{-1} \left( \frac{\dot{a}}{\beta(\tau)} \right) + \frac{a^3}{\beta(\tau)} \left( \dot{a} - \frac{\dot{a}^2}{2a} \frac{df(\tau)}{da} - \frac{\dot{a}N(\tau)}{N(\tau)} \right) \right]^-,$$

$$S_{\text{brane}} = -2\pi^2 \sigma \int d\tau a^3 N(\tau).$$  \hspace{1cm} (8.8)

\(^2\)The presence of mass in the bulk will be dynamically irrelevant for the computations of the equation of motion. Moreover, the charge $Q$ associated with the five-dimensional Reissner-Nordström black hole in chapter 7 will also be irrelevant. Hence, these terms will not affect the resulting Hartle-Hawking equation. However, this need not be the case when computing the nucleation probability of the bubble. As our aim is to recapitulate the results of Danielsson, Panizo et al. [43], we will stick to an empty AdS computation and leave further modifications of the bulk space to future work.
were we have defined $[A]_+ = A_- - A_+$. Summing up all terms, and ignoring terms that do not affect the dynamics of the shell, the mini-superspace Lagrangian looks like:

$$L^{(5)}_{\text{total}} = \frac{6\pi^2}{\kappa_5} \left[ -a^2 \dot{a} \tanh^{-1}\left( \frac{\dot{a}}{\beta(\tau)} \right) + a^2 \beta(\tau) \right] + 2\pi^2 a^3 \sigma N(\tau). \quad (8.9)$$

Note that in the phenomenological "bubble" regime, i.e. when $ka \gg 1$ and $H/k \gg 1$, Eq. (8.9) becomes the simple four-dimensional Lagrangian described in Eq. 3.17. This already points out to a connection between the four-dimensional description of quantum cosmology discussed in chapter 3 and that of the higher-dimensional bubble. This shows that our five-dimensional ultraviolet completion gives full control over an infinite tower of corrections (coming from the argument of tanh) to the four-dimensional cosmological model. But this is not the only connection to be found.

Let us now compute the Hamiltonian. Following the discussion in chapter 3, the conjugate momenta are then given by:

$$\pi_{N(\tau)} = \frac{\partial L}{\partial \dot{N}(\tau)} = 0, \quad \pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{6\pi^2 a^2}{\kappa_5} \tanh^{-1}\left( \frac{\dot{a}}{\beta(\tau)} \right) \bigg|_-. \quad (8.10)$$

The conjugate momentum $\pi_{N(\tau)}$ is already pointing to an important feature of the model: The lapse-function $N(\tau)$ is not dynamical and will act as a Lagrange multiplier, enforcing the constraint $H = 0$. In fact, this allows us to write the Hamiltonian as:

$$\mathcal{H} = 2\pi^2 N(\tau) a^3 \left[ \sigma - \frac{3(\beta_-(\tau) - \beta_+(\tau))}{a \kappa_5} \right]. \quad (8.11)$$

Substituting the $\beta$-expression (8.7) and enforcing the constraint $H = 0$, it is easy to see that the Hamiltonian (8.11) is identical to the junction condition (6.17). This is nothing new to us, as we have already shown it in section 7.4. In any case, it is reassuring to find this exact relation from a different construction.

Although Eq. (8.11) shows the true power of the junction condition, it is not well prepared to be quantised and become the Wheeler-de-Witt equation discussed in Eq. (3.20). For this, the conjugate momentum $\pi_a$ should be given explicitly in the expression. By rewriting it as:

$$\cosh\left( \frac{\kappa_5 \pi_a}{6\pi^2 a^2} \right) = \frac{\beta_-(\tau) \beta_+(\tau) - \dot{a}^2}{N(\tau)^2 \sqrt{f_-(\tau) f_+(\tau)}}, \quad (8.12)$$

3This only affects the $\Lambda^{(5)}$ term, which results to be dynamically irrelevant, i.e. it will not appear in the equations of motion. This can be seen from the first junction condition (C.25) applied to the bulk geometry.

4i.e. the junction condition is the Hamiltonian as described in section 7.4.
one can substitute in the Hamiltonian $H$, to yield:

$$
\mathcal{H} = -\frac{6\pi^2}{\kappa_5} \left( f_-(\tau) + f_+(\tau) - 2\sqrt{f_-(\tau)f_+(\tau)} \cosh\left( \frac{\kappa_5 \pi a}{6\pi^2 a^2} \right) \right)^{1/2} + 2\pi^2 N(\tau) a^3 \sigma = 0.
$$

(8.13)

The presence of the momentum $\pi a$ in the argument of $\cosh$ implies that the Wheeler-de-Witt equation becomes one of infinite order in $\pi a$. Therefore, we will focus on the limit for small $\pi a$, up to quadratic order.\(^5\) If we now quantise the system by replacing the operator

$$
\pi a \rightarrow -\frac{i}{\alpha^{3/2}} \frac{d}{d\alpha},
$$

(8.14)

one obtains,

$$
\mathcal{H}\psi_{5D} = \left( -\frac{1}{24\pi^2 \alpha^{3/2}} \frac{d^2}{d\alpha^2}(\alpha^{3/2}) + 6\pi^2 V(a) \right) \psi_{5D} = 0.
$$

(8.15)

Note that this Wheeler-de-Witt equation is exactly Eq. (3.20). Here the wavefunction $\psi$ is supported in four spatial dimensions and it is related to that of the four-dimensional cosmology approach (3.22) by the identification $\psi_{4D} = \alpha^{3/2} \psi_{5D}$. All in all, it is remarkable to see that the dynamics used to describe the four-dimensional cosmology discussed in chapter 3 can be encoded by those of an expanding bubble in a higher-dimensional environment [43].

8.2 Whatever remains, however improbable, must be the truth

Although the results obtained in Eq. (8.15) are fine on their own, that was not the aim of this chapter. We wanted to use the higher-dimensional approach to determine the choice of boundary conditions in the description of four-dimensional quantum cosmology. A relevant question for this purpose might be: How likely is the nucleation of a dark bubble?

In the four-dimensional description discussed in chapter 3, we saw that the nucleation probability (3.26) differed by a sign in the exponential argument, depending on the choice of boundary conditions. If we now turn our attention to the five-dimensional bubble nucleation probability, we should recall that the dark bubble is a Brown-Teitelboim instanton [190]. Details of this type of solution can be found in appendix D. According to the Brown-Teitelboim computations presented there, the probability for such bubble nucleation to happen is given by:

$$
P = e^{-B},
$$

(8.16)

\(^5\)Recall that the bubble nucleates at rest $\dot{a} = 0$ and will expand from there.
where $B$ is the instanton bounce, as discussed in appendix D. The Euclidean instanton can be obtained by integrating $(8.1)$ over an $O(5)$ sphere of radius $r = a$ (i.e. the radius at which the bubble nucleates). Evaluating Eq. (D.14) for this case gives:

$$B = \sigma \Omega_4 a^4 + \frac{1}{\kappa_5} \left[ 4 k^2 \text{Vol}_5(a, k) - \frac{4}{R} \beta \Omega_4 \right], \quad (8.17)$$

with $\Omega_4 = 8\pi^2/3$ and $d\text{Vol}_5/da = \Omega_4/\beta$. Extremising using $dB/da = 0$ implies the junction condition $(8.11)$ and fixes $a$ to the critical value $a_*$. Expanding in the late big bubble regime, i.e. $ka_* \gg 1$, we have

$$\text{Vol}_5 = \frac{\Omega_4}{4k} \left( 1 - \frac{1}{k^2 a_*^2} \right). \quad (8.18)$$

This expression can be substituted back into Eq. (8.17). Furthermore, if we want to connect to four-dimensional quantities, i.e. $\kappa_5$ to $\kappa_4$, we must use of Eq. (6.23). Finally, this yields:

$$B = \frac{24\pi^2}{\kappa_4^2 \rho_{\Lambda_4}}. \quad (8.19)$$

Similarly, if one decides to use the WKB wall penetration probability given by:

$$P \sim e^{-B}, \quad \text{with} \quad B = 2 \int \pi_\alpha d\tau, \quad (8.20)$$

one then recovers the same result as in (8.19).

Either way, the results are clear: this higher-dimensional approach to the quantum cosmology boundary choice problem leads us to select unambiguously the Vilenkin amplitude $(3.26)$ from the four-dimensional point of view. The bubble nucleation probability in five dimensions is equivalent to a "tunneling from nothing" probability in the four-dimensional realm. Furthermore, there is no Big Bang singularity to worry about. It does correspond to the fiducial zero size of the bubble (i.e. its absence before nucleation). Moreover, we have shown that the physics at all the length scales

---

6It could be tempting to replace the dark energy description $(7.65)$ from chapter 7. This would return the nucleation probability as $P \propto e^{-N^2}$, which is more highly suppressed than the one discussed in chapter 7, for a $D_3$-brane to escape from the $N$ stack. It is important to recall that we have not used the five-dimensional Reisser-Nordström black hole in this chapter, as we followed the results obtained by Danielsson, Panizo et al. in [43]. This suppressed difference can be caused by the absence of angular momentum in the description, which translates into a charge $Q$ in the five-dimensional AdS. The absence of extra fields in the background could be the reason for this difference in the arguments for both constructions. In any case, it might be interesting to revisit this quantum construction equipped with a more populated mini-superspace.
involved in the process of nucleation and expansion are essentially understood (see Eq. (8.9)): it is semi-classical gravity in five dimensions, that provides our ultraviolet completion of four-dimensional cosmology.

Finally, it is reassuring to see that this choice of boundary conditions follows naturally from such construction. This low-dimensional "tunneling into existence" as a higher-dimensional false vacuum decay interpretation has been suggested as the only way to realise bubble nucleation in string theory [157].
9. **Cejuelas y Selletas: Hanging Strings and Gravity**¹

All gods are homemade, and it is we who pull their strings, and so, give them the power to pull ours.

---

Aldous Huxley

---

In this chapter you will learn...

- how to induce massive point particles on the four-dimensional cosmos by using strings in the bulk
- the bending reaction of the brane to any changes in the bulk
- how gravity, which extends along the bulk, dictates the brane dynamics and how it behaves on it

In chapter 6, we focused on obtaining a positive and small cosmological constant $\Lambda_4$ for the induced cosmology living on the $D_3$-brane mediating the decay of two AdS$_5$-vacua. Furthermore, we showed how to embed this "bubbly" five-dimensional construction into string theory in chapter 7. We have also discussed the likelihood of the nucleation of such a bubble and how this higher-dimensional "pop" sheds light on the long standing problem of boundary choice in four-dimensional quantum cosmology reviewed in chapter 8. But nothing has been said so far about any other energy densities, such as matter or radiation, on the induced cosmology, other than that of dark energy $\rho_\Lambda$. In order to match the observational evidence of our cosmos, we need to extend this phenomenological bubbly-construction, so that it can predict the existence and host familiar energy densities $\rho_i$ on top of the brane.

Any impatient reader willing to decorate the induced cosmology as fast as possible is gently invited to take a shortcut and visit chapter 10. However, learning to run without having developed a steady and solid walk may not be the best idea. This chapter, inspired (in part) by [39, 40], is devoted to start the decoration of such an empty universe with the most basic element associated with branes: strings.

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¹In Spanish, nut and saddle, parts of a guitar. The nut can be thought of as the first fret while the saddle is where the strings end up, under the sound hole.
This chapter is organised as follows: In section 9.1, we will explore the simplest stringy configuration; A radially stretching string with one of its end points on the brane. This will induce a massive point particle on the bubble wall. Any deformations caused on the brane by the presence of the string and/or matter on the brane will be studied in section 9.2. We will learn there that the brane and the bulk geometries depend on each other’s changes. At first sight, some of the results obtained might appear to be problematic. Section 9.3 will be devoted to remove any concerns about these results, as it will be explicitly shown that gravity behaves in the familiar four-dimensional way when projected onto the induced cosmos.

9.1 Diddley bow string

Let us imagine just one string with one of its end points on our familiar $D_3$-brane. Furthermore, for simplicity in further computations and concept development along this section, we will assume the following points:

- In the same spirit as in chapter 8, we will leave the bulk "empty". In addition to this, we are interested in presenting results for late-time cosmologies, in such a way that the bubble’s patch of interest looks flat along the throat direction. This can be easily seen by changing the variables to Poincaré coordinates with conformally flat slices along the radial direction, giving:4

  \[ ds_5^2 = dr^2 + a(r) \eta_{ab} dx^a dx^b, \]  

  with the scale factor \( a(r) = e^{2kr} \).

- Let us freeze the expansion of the bubble and assume that it is located at \( r = r_0 \). Note that in this set of coordinates, the Poincaré horizon sits at \( r \to -\infty \), while the boundary to \( \text{AdS}_5 \) is at \( r \to \infty \).

- The string, which is assumed to stretch along the radial direction \( r \), will have one of its end points on the brane, while the other one will extend all the way up to the \( \text{AdS}_5 \)-horizon.5

---

2A diddley bow is a single-stringed American instrument that influenced the development of the blues sound. It consists of a single string of baling wire stretched between two nails on a board. It was traditionally considered a beginner’s or children’s instrument in the Deep South of the USA, especially in the African-American community. It may have been influenced to some extent by West African instruments.

3We will not consider extra dimensions in this chapter as our focus is on large length scales and small energy scales.

4Starting from a Poincaré line invariant (6.5), the required change of variables is \( r = \frac{1}{k} \log(\kappa \zeta) \). Notice that the throat direction is always represented by the letter \( r \), although the warping associated with this direction may vary from chapter to chapter.

5In case the title of this chapter is not clear, the brane represents the saddle of the guitar while the horizon is the nut.
Given the previous points, we are then ready to study how the presence of such string decorates the five-dimensional geometry. Its dynamics are controlled by the Polyakov action \[ S_{\text{string}} = -\frac{1}{2} T_s \int d^2 \xi \sqrt{|\omega_{\alpha\beta}|} \omega^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \] (9.2)

Note that \( \omega_{\alpha\beta} \) represents the world-sheet’s metric, with the indices \( \{\alpha, \beta\} \) representing its coordinates, while the usual indices \( \{\mu, \nu\} \) refer to the coordinates of the target space. \( T_s \) is the fundamental tension of the string, given by:

\[ T_s = \frac{1}{2\pi \alpha'}. \] (9.3)

The associated energy-momentum tensor of the string can be computed by varying the action (9.2) with respect to \( g_{\mu\nu} \). This gives:

\[ T^{\mu\nu} = - T_s \int d^2 \xi \frac{1}{\sqrt{|g_{\mu\nu}|}} \sqrt{|\omega_{\alpha\beta}|} \omega^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \delta^5 (X - x), \] (9.4)

where the \( \delta \)-function represents the source divergences due to the presence of the string, i.e. the position at which it is localised. In order to simplify our computations, we will use the \textit{static} gauge of the string. This is \( \xi^0 = t \) and \( \xi^1 = r \). It is then easy to see that the only non-vanishing components of the energy-momentum tensor are given by [40]:

\[ T^t_t = T^r_r = \frac{T_s a(r)^3}{r^3} \delta \left( x^a - x^a_0 \right), \] (9.5)

where \( x^a \) describes the three spatial directions tangent to the wall of the bubble and \( x^a_0 \) the exact position of the string on it. However, there is a catch here. This energy-momentum tensor corresponds to an equation of state \( p = \rho \) along the throat direction \( r \) and \( p = 0 \) in the transverse directions. This is consistent with what to expect from a string stretched \textit{infinitely} in the radial direction. But this is not what we assumed in the third point stated above. In this sense, we need to "cut" the string to end at the brane. This implies that the pressure component of the energy-momentum tensor should also be cut in the radial direction. This can be done by a Heaviside function \( \Theta(r) \), such as that:

\[ T^r_r = \frac{T_s}{a(r)^3} \delta \left( x^a - x^a_0 \right) \Theta(r - r_0). \] (9.6)

One can then obtain the new associated energy density \( \rho \) of a string ending on the brane by imposing the covariant conservation\(^6\) \[ \nabla_\alpha T^{\alpha}_{\beta} = 0, \] which

\(^6\)This is the analogous Bianchi identity to the right-hand side of Einstein’s equation (2.4) so that the whole construction remains consistent.
requires $T^t_t$ to be:

$$T^t_t = \frac{T_s}{a(r)^3} \delta \left( x^a - x_0^a \right) \Theta (r - r_0) + \frac{T_s}{k a(r)^3} \delta \left( x^a - x_0^a \right) \Theta (r - r_0).$$  \hspace{1cm} (9.7)

The first term of the previous expression is already known: It corresponds to the string stretching along the bulk, down to the position of the "frozen" brane. The second term is the interesting one; in fact, such scaling for the energy density localised on the brane $r = r_0$, is not new in this thesis. We discussed in section 2.1 that non-relativistic matter decays as $\rho_{\text{matter}} \sim a^{-3}$. This implies that the end point on the brane acts as a point particle in the induced expanding cosmology, with mass given by [40]:

$$M_{\text{particle}} = \frac{T_s}{k}. \hspace{1cm} (9.8)$$

This is our very first decoration of the induced cosmos. Radially stretching strings in the bulk will induce massive\(^7\) particles on the bubble’s wall. This can be thought as a point mass suspended by a string in the gravitational well of the AdS\(_5\) space. As the universe expands and the brane moves upwards, the string pulls the point mass so that it can move along the brane, ensuring the aforementioned covariant conservation.

Our dear reader, though full of joy knowing that the induced cosmology will no longer be a wasteland, may not be at ease. Any type of worry caused by the point-particle mass on the brane is probably based on a simple everyday experience; If one places an object on a stretched out sheet, the sheet will sag down in response to the presence of such an object. Should this higher dimensional situation not be similar in concept? Could this somehow affect to the sign of the induced energy density? A thought experiment of this kind can even raise more fundamental questions: Where is gravity? Is it localised on the brane or does it extend along the entire bulk? These are questions to be answered in the following sections of this chapter. Let us start with the everyday example of bending.

### 9.2 Tirando y Apoyando\(^8\)

The situation described above is at the same heart of in the concept of the **backreaction**. The line invariant (9.1) represents a five-dimensional empty

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\(^7\)Note that given the scales in table 7.3, the mass of the induced particles will be of order $M_{\text{particle}} \sim M_{\text{Planck}}$. These would not be normal massive particles. To induce ordinary ones would need to have non-fundamental strings with end point on the brane and stretching up to the AdS boundary. An interesting possibility could be to have $n$ coincidental $D_3$-branes along the radial direction $r$ but at different positions in the compact dimensions $S^5$ in the string theory embedding discussed in chapter 7. Strings stretching between different branes would have the potential to induce standard model-like particles [102, 103].

\(^8\)In Spanish, to pull and to rest. Makes reference to two different ways of plucking guitar strings: "Tirando" means that the finger does not touch the string that is next lowest in
space with conformally flat slices along the throat direction $r$. Any kind of source setting there, not only on the bubble, but in the whole ambient space will cause a deformation of the bulk. This new geometry, which contains information about the presence of the source is called backreacted geometry.\(^9\)

Let us now consider perturbations of the bulk metric in the presence of induced matter on the brane, as a result of the presence of the string in the bulk. These would be described by two perturbation parameters on the brane:

1. How much matter density $\rho$ you put on the brane.
2. How much the radial coordinate on the brane $\xi$ would be deformed by the presence of this mass.

However, these aforementioned changes only apply to the geometry on the brane. If we also bring back the stretched sheet image, the brane should also bend along the throat direction \([40,208]\). And this change is correlated with how "big" the adjustment is in the $\xi$ coordinate, i.e. the radial coordinate in the three-dimensional space. This can be expressed as $r = \tilde{r} + c(\xi)$. All these changes can be captured by following line invariant:

$$
\text{d}s_{\text{back}}^2 = \text{d}\tilde{r}^2 + a(\tilde{r})^2 \left[ -\left( 1 - \frac{4\rho}{\xi} + 2F(\xi) \right) \text{d}r^2 + \left( 1 + \frac{2\rho}{\xi} + 2F(\xi) \right) \text{d}\xi^2 + \xi^2 \text{d}\Omega^2 \right],
$$

(9.9)

where $d\Omega^2_2$ is the line invariant of the two-sphere in the usual spatial sections and $F(\xi)$ is a function on $\xi$ which can be written as

$$
F(\xi) = k f(\xi) = \frac{\rho}{2\xi},
$$

(9.10)

so that the metric (9.9) is a solution of Einstein’s equation, up to linear order in the perturbation expansion commented above.

The next step is to compute a good old friend: The Israel’s junction condition (C.35). However, one has to be careful in this case; The induced energy-momentum tensor will not only be that of a brane (with $\rho = -p$), but will also have contributions coming from the induced matter on top, i.e. the end point of the string. This implies $S_{ab} = -\sigma h_{ab} + T_{ab}$. This yields:

$$
\frac{1}{2\kappa_5} \left[ \partial_r h_{ab} \right]^+ + \frac{\sigma}{3} h_{ab} = - \left( T_{ab} - \frac{1}{3} T h_{ab} \right),
$$

(9.11)

\[\text{pitch (physically higher) on the guitar, as is the case does with apoyando. Although the note is exactly the same, the colour and robustness of the sound varies according to the technique used.}\]

\[\text{\(^9\)The derivation of the geometry to be discussed will not be presented in detail, as it is involved and full of technicalities. Explicit computations of backreaction geometries, more relevant to the results in this thesis can be found in chapter 10. Of course, a detailed discussion of the present one can be found in [39].}\]
where we have exploited the fact that $K_{ab} = \frac{1}{2} \partial_r h_{ab}$ in the coordinates (9.1) and $[K]_+ = \frac{\kappa}{3} (4\sigma - T)$, with $T = \hat{h}_{ab} T_{ab}$.

There is a problem in our previous line of reasoning; although Eq. (9.11) accounts for the presence of induced matter on the brane, the induced metric $h_{ab}$ does not completely reflect any deformations of the bulk geometry. In other words: Eq. (9.11) is aware of the matter density $\rho$ but not of the deformation induced on the brane. Any Gaussian normal perturbation on the bulk geometry of the form $g_{\alpha\beta} \rightarrow g_{\alpha\beta} + \delta g_{\alpha\beta}$ will induce a metric $a^2(r) \eta_{ab} \rightarrow a^2(r) \eta_{ab} + \hat{\gamma}_{ab}$, where $\hat{\gamma}_{ab}$ is the induced perturbation piece, caused by the matter density $\chi$ and its radial ($\xi$) deformation. This implies that the correct junction condition to consider is:

$$\frac{1}{2\kappa} \left[ \partial_r \hat{\gamma}_{ab} \right]_+ + \frac{\sigma}{3} \gamma_{ab} = - \left( T_{ab} - \frac{1}{3} T \eta_{ab} \right).$$

(9.12)

We are not done yet: due to the presence of matter $\rho$, in a similar way than the aforementioned sheet bending under the weight of the mass above, the bending of the brane in the radial direction will make the junction conditions hard to apply. The position of the brane will be given in terms of two different functions $r = f_{\pm}(x)$, depending on which side you look from. This requires us to apply two diffeomorphisms on the outside and inside to bring the brane back to $r = 0$, as it has been deformed due to the presence of the mass. This implies that $f(x) < 0$ represents the brane bending "down", towards the Poincaré horizon. Conversely, $f(x) > 0$ means that the brane bends "upwards", towards the AdS horizon. Requiring the brane to sit at $r = r_0 = 0$ in the new coordinates from both sides, one finds [208]:

$$r \rightarrow r - f(x), \quad x^a \rightarrow x^a + \frac{1}{2k} \left(1 - a^{-2}\right) \partial^a f(x) + w^a(x),$$

(9.13)

where $w^a(x)$ is an arbitrary function of the transverse coordinates. In these new coordinates, the induced metric perturbation piece $\gamma_{ab}$ becomes:

$$\hat{\gamma}_{ab} = \gamma_{ab} + \frac{1}{k} \left(1 - a^{-2}\right) \partial_a \partial_b f + 2kf \hat{h}_{ab} - 2a^2 w_{[a,b]},$$

(9.14)

where $w_{[a,b]} = \partial_a q_b + \partial_b q_a$. Introducing Eq. (9.14) into the junction condition (9.12) evaluated at $r = r_0$ and imposing continuity of the induced metric [39] one finally obtains:

$$\frac{1}{2\kappa} \left[ \partial_r \hat{\gamma}_{ab} \right]_+ + \frac{\sigma}{3} \gamma_{ab} = - \Sigma_{ab}$$

(9.15)

where the tensor $\Sigma_{ab}$ is given by:

$$\Sigma_{ab} = \left( T_{ab} - \frac{1}{3} T \eta_{ab} \right) - \frac{1}{\kappa} \left( \frac{1}{k^+} - \frac{1}{k^-} \right) \partial_a \partial_b F(\xi).$$

(9.16)
The expression above contains new information that needs to be discussed. While the first term on the right hand side of equation (9.16) has remained the same, the second term is a result of the deformation (9.14). This part contains the bending sample $F(\xi)$ of the backreaction (9.9). If we choose this bending gauge $F(\xi)$ such that $\Sigma_{ab}$ becomes traceless, Eq. (9.16) becomes [208]:

$$T = -\frac{3}{\kappa_5} \left( \frac{1}{k_+} - \frac{1}{k_-} \right) \partial^a \partial_a F(\xi),$$  \hspace{1cm} (9.17)

with $T = h_{ab} T^{ab}$ as the trace of the energy-momentum tensor on the brane. Recall that $F(\xi) \sim f(\xi)$, which was defined above as the "gauge" that identifies how much the brane bends in response to the presence of matter. In this sense, Eq. (9.17) can be used to determine how much and in which direction the brane bends with a non-zero energy-stress tensor $T_{ab}$. In fact, let us study three relevant cases:

- Any point mass on the brane has an energy density $\rho_{\text{Mass}}$ given by $T_{00} \sim \delta(\xi)$. To compute the trace $T$, we raise an index so that $T \sim -\delta(\xi)$. This implies that $\partial^a \partial_a F(\xi) \sim \delta(\xi)$. Hence $f(\xi) < 0$. This means that the brane sags down in response to it.

- Similarly, a string extending from the AdS boundary and with one of its end points on the brane (i.e. Eq.(9.7)), will induce a contribution $T_{00} \sim -\delta(\xi)$. Going up one index, to compute the trace, we get $T \sim \delta(\xi)$. This brings us to $\partial^a \partial_a F(\xi) \sim -\delta(\xi)$. Therefore $f(\xi) > 0$. This means that the brane is pulled up in response to the presence of the string.

- As we can see from the string case, any content in the bulk between the AdS boundary and the brane that contributes with a positive trace $T$ will pull the brane up.

Two of these cases are shown in figure 9.1 (see section 10.3 for further information on the third case). Let us recapitulate what we have done so far: We started with a brane that had conformally flat slices (9.1) along the radial direction. We then considered any deformations of the brane geometry that any mass in the bulk and on the brane could generate in Eq. (9.9). This is encoded in the gauge (9.10). In the case we study a situation like the one described in the second point above (i.e. a string with an end point on the brane), the gauge will tell us that the brane will be deformed upwards. If we wanted to recover the flat description, we would need to add a mass to the end point of the string, to pull the string down, as described in the first bullet point above. This mass has already been computed in Eq. (9.8), demanding energy conservation. This proves that a string stretching outwards, pulling from the brane will induce a positive mass, yielding the ex-

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10Conversely, the opposite is true for branes stretching from the inside.
pected density on the four-dimensional cosmos. On the other hand, matter sitting on the brane will produce negative masses.

Figure 9.1. Sketch representation of the bubble, whose boundary responds to the presence of sources. Strings with end point on the bran will deform the conformally flat geometry and pull up from the brane. The opposite case would be generated by masses on the brane, which will sag down the brane, yielding a negative contribution to the four-dimensional energy-momentum tensor.

The previous discussion about the end points of strings acting as four-dimensional positive masses can be extended to reproduce an homogeneous and isotropic distribution of "dust" in the induced expanding four-dimensional cosmos. This will be achieved in section 10.1 by using an homogeneous and isotropic distribution of such strings around the bubble.

In fact, the presence of this cloud of strings is crucial to realise Einstein's gravity on the four-dimensional wall. Contrary to the case discussed in section 4.1, gravity is not localised in the dark bubble scenario. The reason for this is that the presence of sources in the bulk (i.e. the strings) introduces non-normalisable modes of $\zeta$, contrary to what we saw in section 4.1. These non-normalisable solutions are required to have the right propagator of the graviton so that it can be interpreted as the right tensorial description of gravity in four dimensions. However, the scale of violation of localisation is controlled by the AdS length scale $L$ and it is microscopic. We will not discuss these aspects further here, but we refer the curious reader to [39, 40].

We will now continue delving into the interplay between the bulk behaviour due to the presence of any source in it and how this can affect the induced geometry on the brane. It is time to unmask the true power of Gauss-Codazzi equations (C.18).
9.3 Harmonious gravity and its dimensional transposition

We have previously studied how the behaviour of the brane depends on the position of the matter. The conformally flat direction \( r \) is bent upwards when the end point of a string is attached to it from the outside. The direction will be downward when matter is placed directly on top of it. Gravity is present in all bulk space and it is the force responsible for the bending behaviour described above. But what is the nature of the four-dimensional gravity on the braneworld? How does gravity behave on the bubble wall?

In order to gain a deeper understanding on the nature of gravity on and across the bulk space, our starting point will be the Gauss-Codazzi equation (C.18) with full tangential projection described in appendix C. This expression can be modified to help us read off any induced energy-momentum tensor \( T_{ab} \) due to the contents of the bulk and on the brane [38]. The aim is to rewrite this Gauss-Codazzi equation (C.18) so that on one side of the equation we end up with some sort of four-dimensional geometry description. On the other side, something reminiscent of an induced energy-stress tensor. We can start this task by tracing Eq. (C.18),

\[
J_{ab} = R_{\alpha\beta\gamma\delta}^{(5)} e^a_\alpha e^b_\beta e^\gamma_\delta e^\delta_\epsilon h^{\epsilon\delta} = R_{ab}^{(4)} + (K_{ac} K^c_b - K^c_c K_{ab}) ,
\]

where \( J_{ab} \) plays the role of the "pre-induced" energy-momentum tensor. One can continue to massage this term to write:

\[
J_{ab} = e^\beta_\alpha e^\delta_b \left( R_{\alpha\beta\gamma\delta}^{(5)} e^a_\gamma e^\epsilon_d h^{\epsilon\delta} \right) = e^\beta_\alpha e^\delta_b \left( R_{\beta\delta}^{(5)} - R_{\mu\nu\rho\sigma} n^\mu n^\nu \right) ,
\]

where in the second step we have split the five-dimensional metric \( g_{\mu\nu} \) into its normal and tangential pieces, with the help of the metric decomposition described in Eq. (C.10).

So far, we have only applied useful geometric relationships between tangential and normal coordinates. Let us now put physics to work for us. Although not specified in previous chapters, we have assumed \((k \gg H)\) that the extrinsic curvature was mainly dominated by the AdS5-vacua scale \( k \). This implies that the extrinsic curvature \( K_{ab} \) can be written as

\[
K_{ab} = kh_{ab} + \zeta_{ab} ,
\]

where \( h_{ab} \) is the induced FRLW metric on the brane (6.11) and \( \zeta_{ab} \) contains subleading contributions with respect to \( k \) [38]. Inserting this approximation into Eq. (9.18) and accounting for the jump in the extrinsic curvature (i.e. the jump in the AdS5-vacua scale \( k \)) yields:

\[
\left[ \frac{J_{ab}}{k} - \frac{R_{ab}^{(4)}}{k} \right]^- = \left[ 3kh_{ab} + (2\zeta_{ab} + \zeta h_{ab}) \right]^+ + \ldots ,
\]
where $\zeta$ is the trace of the subleading piece in $K_{ab}$ and the dots ... represent the suppressed\textsuperscript{11} contributions of $\zeta/k$. If we explicitly open the previous expression (9.21) we will see that:

$$\left(\frac{\mathcal{I}^-}{k_-} - \frac{\mathcal{I}^+}{k_+}\right) - R^{(4)}_{ab}\left(\frac{1}{k_-} - \frac{1}{k_+}\right) = 3(k_- - k_+) h_{ab} + 2\left(\zeta^+_{ab} - \zeta^-_{ab}\right) + \left(\zeta^+ - \zeta^-\right) h_{ab},$$

(9.22)

where we have dropped the suppressed contributions. Note that the right-hand side of Eq. (9.22) contains pieces that may vaguely resemble those inside the second Israel's junction condition (C.35). In fact, if one rewrites $\zeta_{ab} = K_{ab} - kh_{ab}$ and rearranges the terms to leave $R^{(4)}_{ab}$ alone, it yields:

$$R^{(4)}_{ab} = \left(\frac{k_+ k_-}{k_- - k_+}\right) \left[3(k_- - k_+) h_{ab} + \left(\frac{\mathcal{I}^+}{k_+} - \frac{\mathcal{I}^-}{k_-}\right) + \left(k^+ - K^-\right) h_{ab} + 2\left(K^+_{ab} - K^-_{ab}\right)\right].$$

(9.23)

On the left-hand side, we see the induced Ricci tensor. On the right, we find geometrical information about the bulk and how the brane is embedded. This can be trace-reversed to compute the four-dimensional Einstein tensor $G^{(4)}_{ab} = R^{(4)}_{ab} - \frac{R^{(4)}}{2} h_{ab}$, i.e.,

$$G^{(4)}_{ab} = \left(\frac{k_+ k_-}{k_- - k_+}\right) \left[\left(\frac{\mathcal{I}^+}{k_+} - \frac{\mathcal{I}^-}{k_-}\right) - \frac{1}{2} h_{ab} \left(\frac{\mathcal{I}^+}{k_+} - \frac{\mathcal{I}^-}{k_-}\right) - 3(k_- - k_+) h_{ab} - 2\left((K^+_{ab} - K^-_{ab}) - (K^+ - K^-) h_{ab}\right)\right].$$

(9.24)

We will call this equation the reversed Gauss-Codazzi equation. Although it already has a very rich structure, it may be useful to give it a final massage to appreciate its power in all its glory. In fact, this can be done based on two facts:

- The "pre-induced" energy-momentum tensor $\mathcal{I}_{ab}$ contains a piece which represents the AdS$_5$ vacuum $k$ as:

$$\frac{\mathcal{I}_{ab}}{k} = -3 k h_{ab} + \tilde{\mathcal{I}}_{ab},$$

(9.25)

where $\tilde{\mathcal{I}}_{ab}$ stands for any other extrinsic curvature contribution generated by $\chi(r, t, \ldots)$ in Eq. (6.9).

- The most general energy-momentum tensor $S_{ab}$ is given by:

$$S_{ab} = -\sigma h_{ab} + T_{ab},$$

(9.26)

with $T_{ab}$ as the energy-momentum tensor of any field living on the brane.

\textsuperscript{11}These would go as $\mathcal{O}\left(\frac{\zeta^2}{k^2}\right)$, we will comment on these terms later.
These two points allow us to rewrite Eq. (9.24) as:

\[
R_{ab}^{(4)} - \frac{1}{2} R^{(4)} h_{ab} + \Lambda_4 h_{ab} = T_{ab}^{\text{extrinsic}} - T_{ab}^{\text{brane}} = T_{ab}^{\text{effective}},
\]

where we have used Eqs. (6.20) and (6.23) to rewrite the cosmological constant \( \Lambda_4 = \kappa_4 \rho_\Lambda \). The extrinsic energy-momentum tensor is given by [37,46]:

\[
T_{ab}^{\text{extrinsic}} = \left( \frac{k_+ k_-}{k_- - k_+} \right) \left( \mathcal{J}_+ - \mathcal{J}_- \right) - \frac{1}{2} h_{ab} \left( \frac{\mathcal{J}_+}{k_+} - \frac{\mathcal{J}_-}{k_-} \right).
\]

Expression (9.27) has a very rich structure, so let us extract the information out of it. First, it is undeniable that one recovers the right four-dimensional Einstein gravity when projecting down the bulk information through the Gauss-Codazzi equation (C.18). Gravity, which extends all over the bulk space, will effectively behave as the usual Einstein gravity when restricted to the boundary of the bubble at \( r = a(\tau) \). Any corrections coming from higher dimensions are suppressed. This can be easily observed by analysing the Friedmann equations of the system while keeping the subleading contributions \( \varsigma_{ab} \) described in Eq. (9.21). These quadratic couplings will contribute with corrections of the form \( \kappa_4^2 \rho^2 k^{-2} \). Using the energy hierarchy of the embedding (7.62), this implies that corrections will only take place at "stringy" scales such as [46]:

\[
\Theta \left( \frac{\varsigma^2}{k^2} \right) = \frac{\varsigma_4^2 \rho^2}{k^2} \times \kappa_4 \rho \sim \ell_0^4 \rho \times \kappa_4 \rho \sim g_s \ell_s^4 \rho \times \kappa_4 \rho.
\]

The whole Gauss-Codazzi projection (9.24) shows the usual effective low-energy four-dimensional description, which is only disturbed when densities are close to the ten-dimensional Planck scale. This proves that gravity effectively behaves as four-dimensional Einstein gravity on the brane, although it extends to the bulk.

The final comments of this section are devoted to the preservation of harmony on the dark bubble. The right-hand side of expression (9.27) shows the presence of two different contributions to energy-momentum tensor. As we have seen throughout stage III of this work, the dark bubble construction is based on two main sets of equations: Einstein’s equations in the bulk (with the junction condition (C.35) playing the role of an equation of motion) together with the Gauss-Codazzi equations (C.18). Combining them, the four-dimensional Einstein equation (9.27) can be obtained. This equation can be thought of as the usual \( G_{ab}^{(4)} = T_{ab}^{(4)} \) Einstein equation, which is covariantly conserved. This means that it will be conserved under time evolution and any choice of initial conditions will hence be maintained. For example, if we impose the initial condition to be

\[
G_{ab}^{(4)} = T_{\text{effective}} = T_{\text{brane}},
\]

(9.30)
this already indicates that $T_{\text{extrinsic}} = 2T_{\text{brane}}$ initially. The reason why this remains true for the subsequent time evolution is twofold:

1. $T_{\text{brane}}$ is covariantly conserved on its own by its equations of motion derived from the brane action. Therefore, there cannot be any exchange of energy between $T_{\text{brane}}$ and anything else.

2. The total energy-momentum tensor, involving $T_{\text{extrinsic}}$ is also covariantly conserved since it is equal to the 4D Einstein tensor (2.4). Hence, the Einstein equation in the form (9.30) is conserved throughout.

From a physical perspective, this can be understood as follows: let us assume matter fields living on the brane, i.e. those of the standard model. We will call this the visible sector. Then we can set up initial conditions on the brane and in the bulk to make sure that the induced four-dimensional energy-momentum is exactly the expected one of the visible sector. By the covariant conservation cases discussed above, the previous time evolution will that of the visible sector coupled to four-dimensional gravity. However, in the case that there is an initial mismatch between the energy-momentum tensor of the visible sector and that of the induced four-dimensional energy-momentum tensor, we would interpret this as the presence of a dark sector. This implies that the dark sector will couple to ordinary gravity from a four-dimensional point of view, coupled to the usual gravity in the same way as the visible sector [46]. For example, the best known representative of this sector has been extensively discussed in this work; this is dark energy. But this would not be the only constituent. In fact, the rich structure of the dark bubble model suggests that other more dynamical components associated with the extra dimensions could also be switched on. This would be a very interesting aspect to explore in within this research line.

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12 We will explore in detail this with the electromagnetic field. Other fields, such as the weak and strong forces would require brane configurations a little bit more elaborated.

13 This sector makes reference to roughly $\sim 95\%$ of the content of the universe that does not interact with the visible one. Dark matter $\sim 25\%$ could be massive fields weakly coupled to gravity. However, we have focused all our attention on another components of the dark sector in this work: Dark energy $\sim 70\%$. An excellent review of some of the aspects of the dark sector can be found in [209].
In this chapter you will learn...

- to populate the four-dimensional cosmos with different types of energy density $\rho_i$ using bulk features
- how the previous process is encoded in the backreacted bulk geometry and projects down to four dimensions by the Gauss-Codazzi equation

In our attempt to enrich the empty cosmos obtained in the first chapters of stage III, chapter 9 was devoted to studying how the presence of strings stretching all the way up to the boundary of AdS$_5$ could induce massive point particles on the bubble wall. These were our first sketches to decorating the four-dimensional universe with more than just dark energy $\rho_\Lambda$. Moreover, such strings helped us to realise that their motion in the bulk also dictates the motion of the ending points in the form of geodesics on the four-dimensional cosmos. This encouraged us to delve in the interplay between five and four-dimensional gravity. Although not localised on the brane, the right nature of the four-dimensional Einstein equation arises as a projection of bulk features onto the brane.

In this chapter we will go one step further and try to "paint" the bubble wall with contents well known from our usual four-dimensional description: matter, radiation and waves. This will be achieved through the backreaction process studied in chapter 9. In this way, when the bulk features are accurately encoded within the bulk geometry, so that the extrinsic curvature accounts for the deformation produced by them, the reversed Gauss-Codazzi equation (9.24) will give the induced effective four-dimensional energy momentum tensor $T_{ab}^{\text{eff}}$.

The organisation of this chapter is as follows: Inspired by section 9.2, we will start by populating the bulk geometry with radially stretching strings and matter in the bulk, along the lines [38]. We will then see, as predicted in the previous section, that the distribution of strings induces "dust" on the four-dimensional cosmos, while matter in the bulk will contribute with
a radiation-like density. In section 10.2 we will review the main results obtained by Danielsson, Panizo and Tielemans in [44]. The aim will be to induce the same gravitational waves already discussed in section 2.2.1 from similar five-dimensional ones propagating in the bulk. Equivalently in section 10.3, following the work of Basile, Danielsson, Giri and Panizo [46], we will study how higher-dimensional gauge fields in the bulk will be responsible for decorating the induced expanding cosmos with the electromagnetic radiation introduced in 2.2.2.

10.1 Glaze and splatter

The embedding of the dark bubble into string theory discussed in chapter 7 required a five-dimensional bulk equipped with a non-extremal and unstable Reisser-Nordström hole, which would decay through the emission of $D_3$-branes. Although we presented the general form of the induced Friedmann-like equation (the potential $V(a)$ in Eq. (7.45)), we focused on the fate of such cosmos.

Let us now turn our attention to the contents of such an induced universe. For simplicity, we will assume the black hole charge $Q = 0$, so that we obtain an asymptotic AdS-Schwarzschild geometry in the five-dimensional bulk [199]. In addition, we will include subcritical tension $D_3$-branes as the ones discussed in sec 7.6, so that one naturally obtains an induced dark energy density $\rho_\Lambda$ on the bubble wall. The line invariant describing the geometry of such bulk is that of Eq. (6.8) with the metric argument

$$f_\pm(r) = 1 + k_\pm^2 r^2 - \frac{2\kappa_5 M_\pm}{r^2}.$$  

Inserting this geometry into Eq. (9.24), imposing a pure brane with $S_{ab} = -\sigma h_{ab}$ expanding in the radial direction ($r = a(\tau)$) and raising one index to the whole expression, we obtain:

$$G_{a}^{(4)} b = -\frac{6\kappa_5}{a(\tau)^4} \left( \frac{M_+ k_- - M_- k_+}{k_- - k_+} \right) \left( \delta_0^a \delta_0^b - \frac{1}{3} \sum_{i=1}^{3} \delta_i^a \delta_i^b \right) - 2 k_+ k_- \left( 3 - \frac{\kappa_5}{k_- - k_+} \sigma \right) \delta_a^b,$$

where the second line has been rewritten using of the dark bubble identifications (6.23). Note that one can directly identify the induced energy-momentum tensor on the brane on the right hand side of the expression.

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1Both make reference to painting techniques. To glaze consists of adding a glossy illumination effect to some layer of the paint, while to splatter is to soil with splashes of the desired color on a surface.
The second line corresponds to the presence of dark energy \( p = \rho \), while the first line represents a homogeneous distribution of radiation with equation of state \( p = \rho/3 \), with energy density

\[
\rho_{\text{rad}} \sim \frac{2M}{k a^4},
\]

(10.3)

where we have taken similar masses in and out, i.e. \( M_+ = M_- = M \) and imposed the approximation (7.36). Note that this is only an effective form of radiation; the expansion of the bubble will be slowed down by the black hole in the bulk in the same way as a normal four–dimensional cosmology would be slowed down by the presence of radiation. The presence of the black hole does not imply a true form of radiation on the bubble cosmology, such as regular gravitational or electromagnetic radiation. These will be discussed in sections 10.2 and 10.3 respectively.

![Figure 10.1.](image)

*Figure 10.1.* Left: A mass in the bulk, i.e. a black hole, will induce radiation on the bubble wall. Right: A homogeneous cloud of semi-infinite strings, extending radially towards the bulk’s boundary, with end points on the brane, will induce a uniform a non-relativistic matter density on the expanding cosmology.

Let us now investigate how to induce non relativistic matter on the brane. This task is slightly more subtle, but we have already seen an introduction in section 9.1. A string stretched along the throat direction \( r \), with its end point on the brane, will induce a positive mass \( M_{\text{particle}} = T L \) on the wall of the bubble. It is then natural to think that a homogeneous distribution of such strings ending on the brane will hence induce a uniform distribution of non-relativistic masses on the brane. If we take the action (9.2) for \( N_s \) strings stretching along the throat direction of an empty \( \text{AdS}_5 \) geometry, we can see that the associated energy-stress tensor is:

\[
T^{\mu\nu} = -\kappa_5 T_s \sum_i \frac{N_s}{4} \int d^2 \xi \frac{1}{\sqrt{|g_{\mu\nu}|}} \sqrt{|\omega_{\alpha\beta}|} \omega^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \delta_5^5 (X - x_i). \quad (10.4)
\]
Again, we can take the static gauge for all the strings (i.e. $\xi^0 = t$ and $\xi^1 = r$) and smear their energy-momentum tensor which would yield:

$$T^\mu_\nu = \kappa_5 \int T^\mu_\nu \, d^3 \alpha = -\kappa_5 \frac{T_s N_5}{V_3} \frac{1}{r^3} \delta^\mu_\nu,$$  \hspace{1cm} (10.5)

with non vanishing components $T^t_t$ and $T^r_r$. How do we encode this bulk information so that the reversed Gauss Codazzi equation (9.24) ends up projecting its features down to four dimensions? As we saw in section 9.3, it is through the bulk geometric information that we can gain insight into the induced four-dimensional geometry. This requires us to "cook up" a geometry that accounts for the presence of such a cloud of strings. In other words: We need to find out how the geometry reacts to the presence of these strings. This is the backreaction of the original metric to the changes of the energy-momentum tensor sourcing it. The stretching strings will now be the source of the new geometry that we need to describe.

10.1.1 Spanning encore: Backreaction

Let us start by parametrising a backreacted Ansatz for this new geometry of the form:

$$d^2_{\text{back}} = f_s(r) dt^2 + f_s(r)^{-1} dr^2 + r^2 d\Omega^2_3,$$  \hspace{1cm} (10.6)

where $f_s(r)$ is the metric argument account for the "cloud". This metric should yield an energy-momentum tensor such that its non-zero $T^\mu_\nu$ components are given by those discussed in Eq. (10.5) plus the five-dimensional dark energy density ones. Solving Einstein’s equation one then finds:

$$f_s(r) = 1 + k^2 r^2 - \frac{2B}{r^2} - \frac{2\kappa_5 \alpha}{3r},$$  \hspace{1cm} (10.7)

where $B$ a constant and $\alpha = T_s N_5 / V_3$. $B$ would be related to the ADM mass [81, 214] in the bulk space, if it existed. For the case in question, where only strings are present in the bulk, let $B = 0$.

One can now go back to the reversed Gauss-Codazzi equation (9.24) and insert this cloud of strings to obtain the induced energy-momentum tensor on the brane. This yields:

$$G^{(4)}_{ab} = -\frac{2\kappa_5}{a(\tau)^3} \left( \frac{\alpha_+ k_- - \alpha_- k_+}{k_- - k_+} \right) \delta^a_0 \delta^0_b - 2k_+ k_- \left( \frac{\kappa_5}{k_- - k_+} \right) \delta^a_b,$$  \hspace{1cm} (10.8)

where again, the second piece corresponds to the dark energy induced on the brane, while the first part represents an energy density $\rho$ that dilutes as

---

2This metric was already constructed a long time ago. We will present the main derivation steps as a pedagogical resource in this thesis. See [210–213] for further information.
\[ \rho_{\text{matter}} \sim \frac{T_s}{k} \frac{N_s}{V_3} \frac{1}{a^3}, \]  

which corresponds to the energy density distribution produced by \( N_s \) massive point particles distributed homogeneously and isotropically through the four-dimensional cosmos.

There are a few physical notes to be made before the end of this section. Our dear reader has surely noticed the minus sign in front of the four-dimensional contributions \( M_\text{part} \) and \( \alpha_\text{part} \) for both the bulk black hole and the string cloud. In fact, this is the same contribution with negative sign as any field placed on the brane. To understand why this is physically correct sign, let us start again with the AdS-Schwarzschild background example described above. If we assume the same mass parameter \( M \) on both sides of the brane, we have shown that the induced cosmology is decorated with radiation, with the energy density \( \rho_{\text{rad}} \) positive and proportional to \( M/k_+ - M/k_- > 0 \). Now, as a thought experiment, let us imagine that the black hole is replaced by a shell with the same mass \( M \). The metric outside of the spherically symmetric shell remains the same, as does the induced four-dimensional physics. If we increase the radius of the shell so that it touches the brane, then the bulk matter can be deposited on the braneworld with no change in the physics. Eventually, all the mass would have been placed on the brane. Therefore, the interior of the brane will be pure AdS. The term \( -M/k_- \) should then be removed and replaced by the extra energy density on the brane, i.e. \( \rho_{\text{brane}} \). As we have shown in Eq. (9.27), it will contribute with the negative sign noted earlier. This shows what is going on. It is a mistake to conclude that adding matter to the brane will result in a negative energy density in four dimensions without any other consequences. What happens is that the added matter necessarily back reacts on the five-dimensional spacetime, yielding a contribution to the extrinsic curvature that gives an overall positive four-dimensional energy density \( \rho_i \) [114].

### 10.2 Shake and stir

We have extensively discussed a uniform background of gravitational perturbations in section 2.2.1. How can we induce such waves from the higher-

\(^3\)To avoid the pigments of the painting to sit at the bottom of the container, so the applied layer is homogenous in color and bright.
dimensional embedding that the bubble represents? The most logical approach to this puzzle is to start by studying geometrical fluctuations in the AdS$_5$ bulk space. We will consider the most general possible perturbation: in addition to a time-dependent piece, one can also find variations along the throat-direction $r$ [44]. These will later be required to behave in such a way that they evolve with the bubble, i.e. one can recover the four-dimensional description (2.26) when "slicing" the five-dimensional ones to the top of brane, located at $r = a(\tau)$.

Let us start by studying the transverse and traceless perturbations that propagates in the AdS$_5$ bulk. The line invariant describing this geometry is given by:

$$d\gamma^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (\gamma_{ij} + \epsilon P_{ij}(t, r, \alpha)) d\Omega^2_3,$$  \hspace{1cm} (10.10)

where $d\Omega^2_3$ is as in Eq. (2.19). $\gamma_{ij}$ is the unit matrix for the three spatial directions and $P_{ij}(t, r, \alpha)$ is the transverse and traceless tensorial perturbation and $\epsilon$ is a small parameter. It is easy to check that the induced metric on the brane in conformal coordinates then corresponds to (2.17).

As we did in section 2.2.1, the transverse and traceless perturbations can be decomposed into $S^3$ harmonics (see appendix B). Inspired by the four-dimensional case, we will first solve for a single mode (i.e. a single wave with wave number $n$) $P_{ij} = \zeta_{5D}(\eta, r) Y_{ij}$. The resulting gravitational wave equation in this geometry is [44]:

$$\frac{\partial^2 \zeta_{5D}}{\partial t^2} - f^2 \frac{\partial^2 \zeta_{5D}}{\partial r^2} - \frac{f}{r} \left( 2 + 4 k^2 r^2 + f \right) \frac{\partial \zeta_{5D}}{\partial r} + \frac{n^2 - 1}{r^2} f \zeta_{5D} = 0,$$  \hspace{1cm} (10.11)

where we assume empty$^5$ AdS$_5$ with $f(r) = 1 + k^2 r^2$ and $\zeta_{5D}(\eta, r)$ represents the time and radial dependence of the wave. This determines the evolution of a gravitational wave throughout the AdS$_5$ bulk.

In order to make contact with the four-dimensional conformal time $\eta$ (2.18) in the low-dimensional case (2.17), it will be useful to work with coordinates that provide a simple five-dimensional uplift of Eq. (2.23). These are:

$$w = \cos(k t) = \frac{\cos \eta}{\sqrt{1 + \left( \frac{H}{k} \right)^2 \sin^2 \eta}} = \cos \eta + O \left( \left( \frac{H}{k} \right)^2 \sin^2 \eta \right)$$ \hspace{1cm} (10.12)

$^4$Note that in our conventions the coordinates $t$ and $r$ have the dimension of length and the coordinates $\alpha^i$ are dimensionless.

$^5$We have assumed this for simplicity. One could also consider a five-dimensional bulk black hole as in chapter 7. As we have seen in section 10.1, each contribution will come separately in the expansion. To avoid long equations, we stick to an empty AdS and its gravitational perturbations.
where the relation between bulk time and conformal time on the brane (6.12) has been used. If we want to connect back to the four-dimensional description, that is, \( t \to \eta \), the relevant limit approximation implies \( H/k \ll 1 \), where one has \( w \approx v \). In particular,

\[
kt = \eta + \Theta \left( \left( \frac{H}{k} \right)^2 \sin(2\eta) \right).
\]  

(10.13)

Note that this relation is only meaningful once the bubble has nucleated. This occurs at \( \eta = -\pi/2 \) as commented on the four-dimensional treatment (2.22). It corresponds to a bulk time \( t \approx -\pi/2k \). The bulk time has in principle the full range \(-\infty < t < +\infty \). However, one has to take into account the composite inside-outside geometry and the non-eternity of the bubble [44]. The time range of outside geometry is \(-\infty < t_+ < 0 \) where the limit \( t_+ \to 0 \) corresponds to the bubble having eaten all of \( \text{AdS}_5^+ \). The inside geometry is only present once a bubble has nucleated after which it persists forever. Therefore \(-\pi/2k < t_- < +\infty \).

Inserting Eq. (10.12) into the five-dimensional gravitational wave equation in the bulk (10.11) yields:

\[
k^2 \left[ (1 - w^2) \frac{\partial^2 \zeta_{SD}}{\partial w^2} - w \frac{\partial \zeta_{SD}}{\partial w} \right] - f^2 \frac{\partial^2 \zeta_{SD}}{\partial r^2} - \frac{f}{r} \left( 2 + 4k^2 r^2 + f \right) \frac{\partial \zeta_{SD}}{\partial r} + \frac{n^2 - 1}{r^2} f \zeta_{SD} = 0.
\]  

(10.14)

This previous equation will control the dynamics of the bulk gravitational perturbations both inside and outside the bubble. And also those of the induced gravitational waves on the brane. In fact, this equation needs to be complemented by suitable boundary conditions [44]. These are:

1. when \( \zeta_{SD} \) is restricted to the brane (i.e. the to-be "induced wave" \( \zeta_{\text{ind}} \)), we require that \( \zeta_{\text{ind}} \) coincides with the four-dimensional wave \( \zeta \) (2.26), up to leading order in \( H/k \). This requires to impose the boundary conditions at the location of the bubble \( r = a(w) \), assuming \( ka \gg 1 \), i.e.

\[
\zeta_{\text{ind}}(w) \equiv \zeta_{SD} \left( w, \frac{1}{H\sqrt{1-w^2}} \right) = \zeta(v) + \Theta \left( \left( \frac{H}{k} \right)^2 \right).
\]  

(10.15)

2. Furthermore, as we did in the four-dimensional case, we require the waves to be sourceless. This means that the wave should fade out when approaching the centre of the bubble, i.e.

\[
\lim_{r \to 0} \zeta_{SD}(w, r) = 0.
\]  

(10.16)

Note that, although we are talking about boundary conditions for a wave, there are two different waves: inside and outside the bubble. So it seems natural to insist that only the inside wave decays and only the outside wave
does not blow up as $r \to \infty$. Note however that their evolution is governed by the same wave equation (10.14), up to a difference in $k$. The first boundary condition at the location of the brane must be imposed for both of them. This boundary condition uniquely fixes the inside and the outside waves [44]. This means that if the outside wave were extrapolated in the supposed limit $r \to 0$, it would still vanish.

Figure 10.2. Gravitational waves in the bulk are sourceless, i.e. they vanish at the centre of the bubble. They "freeze out" at large $r$ and behave like the induced wave when $r = \left( \frac{H \sqrt{1 - \bar{w}^2}}{2} \right)$.

Solving for Eq. (10.14) and considering both boundary conditions (10.15, 10.16), re-inserting the global time $t$ through Eq. (10.12) and massaging the resulting hypergeometric functions in terms of Chebyshev polynomials (B.4, B.5), we can write:

$$\zeta_{5D}(t, r) = D(r, n) \left[ C(r, n) \cos((n+1)kt) + \sin(kt) \sin(nkt) \right],$$

$$\tilde{\zeta}_{5D}(t, r) = D(r, n) \left[ C(r, n) \sin((n+1)kt) - \sin(kt) \cos(nkt) \right],$$

where

$$D(r, n) = \frac{(kr)^{n-1}}{(1 + k^2 r^2)^{\frac{n-1}{2}}}, \quad C(r, n) = \frac{\frac{1}{2} (1 + n)(2 - n) + k^2 r^2}{(n + 1) (1 + k^2 r^2)}.$$

Just as in the case of four-dimensional gravitational waves, one might also be tempted to use the large $n$, late time ($t \to \infty$) limit to find an uplift of the gravitational waves in a flat universe (2.26). However, one must also take into account that, to reach this flat limit, the wave must be considered close to the bubble. This implies an additional condition: the limit in which $kr$ is large;

$$\zeta_{5D}(t, z) = -\frac{1}{2} n^2 + \frac{k^2 r^2}{nk^2 r^2} \cos(nkt) + kt \sin(knt),$$

$$\tilde{\zeta}_{5D}(t, z) = -\frac{1}{2} n^2 + \frac{k^2 r^2}{nk^2 r^2} \sin(nkt) - kt \cos(knt).$$
Note that, when \( n \) is large, there are corrections beyond the four-dimensional wave at high momentum (or energy carried by the wave) \( p = n/r \).

For example, at a fixed position in the throat, i.e. \( kr \), there is a competition between \( n \) and \( k \) in the first term. The induced four-dimensional wave receives a modification when \( kr \sim n \), which translates to a momentum \( p \) of the same order of magnitude than the AdS_5 scale, i.e. \( p \sim n/r \sim k \). In this way, one can therefore conclude that the AdS-scale \( k \) represents a UV-scale where new physics appears on the four-dimensional cosmology [44]. This is agreement with the results that we previously discussed in section 7.7.

In the same spirit as we did for four-dimensional gravitational waves in section 2.2.1, we can compose an uniform background of waves that fills in the five-dimensional bulk. Following the usual average-isotropy process, we find an isotropic stress tensor consisting of three pieces: curvature, radiation, and flux: \(^6\)

\[
\langle T^\mu_\nu \rangle_{\text{iso}} = \langle T^\mu_\nu \rangle_c + \langle T^\mu_\nu \rangle_r + \langle T^\mu_\nu \rangle_f, \tag{10.20}
\]

where each component is given by

\[
\langle T^\mu_\nu \rangle_c = \frac{1}{8\kappa_5 r^2} \left( 7 - \frac{n^4}{2k^4 r^2} \right) \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} + 1_{5 \times 5} \mathcal{O} \left( \frac{n^2}{k^2 r^2} \right),
\]

\[
\langle T^\mu_\nu \rangle_r = \frac{k^2 n^2 r^2}{4\kappa_5 r^2} \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \langle T^\mu_\nu \rangle_f = \frac{n^2}{8\kappa_5 r^2} \begin{pmatrix}
0 & 0 & 0 & -\frac{2f}{k^2 r^3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2k^2 tr & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{10.21}
\]

Note that the momentum of the wave is \( p \sim n/r \), appears as a correction in the curvature contribution, but not in the radiation piece. The flux contribution shows how these gravitational waves represent a net flow of energy in the positive \( r \) bulk’s direction.

10.2.1 Trembling encore: Backreaction

In section 2.2.1, we reviewed how the geometry can be modified when you go up to first order in the gravitational perturbation expansion (2.13). We had a geometry at leading order and on top of it, we put a small perturbation of this, representing a propagating wave (2.26). One can then ask

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\(^6\)This is discussed in section 2.2.1.

\(^7\)Note that the asymmetry in the indices is caused the choice of covariant-contravariant. The tensor is symmetric when both are of the same type.
how the whole geometry will be deformed by the presence of these waves, similar to what we did in section 10.1. Had we not thought of the possible changes in the geometry caused by the energy density of gravitational waves, we would have said that they were probe. However, let us explore how the geometry backreacts to its presence. For example, an isotropic and homogeneous background of gravitational waves (10.20) can be captured by a backreacted geometry with Ansatz [44]:

\[
\text{ds}^2_{\text{back}} = (g^{(0)}_{\mu\nu} + \varepsilon^2 g^{(2)}_{\mu\nu}) \text{d}x^\mu \text{d}x^\nu \approx - f_1(t, r) \text{d}t^2 + \frac{\text{d}r^2}{f_2(t, r)} + r^2 \text{d}Ω^2.
\]  

(10.22)

with

\[
f_1(t, r) = 1 + k^2 r^2 + \varepsilon^2 \left( q_1 - q_2 k^2 t^2 \right),
\]

\[
f_2(t, r) = 1 + k^2 r^2 + \varepsilon^2 \left( q_1 - q_3 k^2 t^2 \right).
\]  

(10.23)

The set of coefficients \( q_i \) \( \in \mathbb{Z} \) will be determined later. Note that when the expansion parameter \( \varepsilon \to 0 \), we recover the AdS$_5$ background (10.10). The backreaction piece \( g^{(2)}_{\mu\nu} \) is given by the \( \varepsilon^2 \) coefficient in the small \( \varepsilon \) expansion, in particular

\[
g^{(2)}_{tt} = q_1 - q_2 k^2 t^2, \quad g^{(2)}_{rr} = \frac{q_3 k^2 t^2 - q_1}{(1 + k^2 r^2)^2}.
\]  

(10.24)

How do you fix the \( q_i \)-coefficients? As we want the Ansatz (10.22) to capture the gravitational waves energy-momentum tensor (10.20), one then has to compute the second order Einstein tensor (2.16) and compare to Eq. (10.20). This yields

\[
q_1 = -\frac{7}{24}, \quad q_2 = -\frac{n^2}{6}, \quad q_3 = \frac{q_2}{2} = -\frac{n^2}{12}.
\]  

(10.25)

The change in the bulk geometry described by the Ansatz (10.22) will affect the dynamics of the expanding bubble. In fact, as this backreacted geometry captures the presence of the gravitational waves (10.19), the junction condition 6.13 will be sensitive to this and reflect these changes through the projection of the induced energy-momentum tensor \( T_{ab} \). By computing the extrinsic curvature (C.15), inserting this in the junction condition (6.13) and massaging the terms to the left and right, one then finds (in the large \( k \) limit) that the induced Friedmann-like equation on the brane is:

\[
\delta_b^a \left( 3H^2 + \frac{1}{a^2} \right) + \frac{2}{a^2} \delta_0^a \delta_0^b = \Lambda_4 \delta_b^a + \kappa_4 \sigma + \varepsilon^2 \left( \frac{7}{24\kappa_4 a^2} (3\delta_0^a \delta_0^i + \delta_i^a \delta_i^b) - \frac{n^2}{4\kappa_4 H^2 a^4} \left( \delta_0^a \delta_0^b - \frac{1}{3} \delta_i^a \delta_i^b \right) \right),
\]

\[
= T_b^a.
\]  

(10.26)

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where \( \sigma \) corresponds to the tension of the brane. Note that the second line is the induced four-dimensional energy-momentum tensor \( T^a_b \) which we found in Eq. (2.28). Alternatively, one can use the Gauss-Codazzi equation (9.24) to obtain the same result in the form \([44]\)

\[
G^{(4)}_{ab} = -\Lambda_4 \delta^a_b - \epsilon^2 \left( \frac{7}{24\kappa_4 a^2} (3\delta^a_0 \delta^0_b + \delta^a_i \delta^i_b) - \frac{n^2}{4\kappa_4 H^2 a^4} \left( \delta^a_0 \delta^0_b - \frac{1}{3} \delta^a_i \delta^i_b \right) \right) .
\] (10.27)

This already shows the power of the junction condition: First, it is aware of changes in the bulk geometry, i.e. the Ansatz to capture the backreaction of the five-dimensional spacetime to the presence of an uniform background of gravitational perturbations. This translates to changes in the extrinsic curvature of the brane \( K_{ab} \), which are reflected in the projection of bulk features onto the brane surface, i.e. the induced expanding cosmology. This implies that the junction conditions take care of gravitational perturbations in the bulk, providing a clear connection between the cosmological features of the bulk and and the boundary bubble. But as we will see in the next section, gravitational waves bulk-boundary dynamics are not the only feature controlled by the Gauss-Codazzi equation. Any feature, living on the brane or in the bulk, will pass through the Gauss-Codazzi toll, modifying the bubble dynamics and hence, the induced geometry on its boundary. Gravity, which sees *everything*, will be the conductor of such ballet.

### 10.3 Chiaroscuro\(^8\)

Let us now study how gauge fields in the AdS\(_5\) bulk, which also enjoy a ten-dimensional supergravity interpretation, can give rise to their four-dimensional counterparts (2.39) on the boundary of the bubble \([46]\).

While Ramond-Ramond fields may seem appealing candidates for generating gauge fields on the brane,\(^9\) there is an even more straightforward candidate for this task; the gauge field \( \mathfrak{F}_{\mu\nu} \) represented in the DBI-piece of the \( D_3 \)-action (7.16):

\[
S_{\text{DBI}} = -T_3 \int d^4\xi \sqrt{\left| (P|G + 2\pi\alpha'\mathfrak{F}) \right|} .
\] (10.28)

\(^8\)Painting technique, used to create strong contrast between light and dark tones.

\(^9\)We will not consider any possible contribution they may make in this section.
If we explicitly expand (10.28) up to second order for small $\alpha'$, we obtain:

$$S_{\text{DBI}} = -\int d^4x \sqrt{|g^{(4)}|} \left( T_3 + T_3 \frac{(2\pi \alpha')^2}{4} \tilde{\phi}^{\mu\nu} \tilde{\phi}^{\mu\nu} + O(\tilde{\phi}^2) \right),$$  

(10.29)

where $g^2 = 2\pi g_s$ is the gauge coupling. The second piece in Eq. (10.29) can then be identified with the action of electromagnetism in four dimensions (2.29). In turn, the gauge field on the brane can be written as:

$$2\pi \alpha' \tilde{F}^{\mu\nu} = 2\pi \alpha' F^{\mu\nu} + \mathcal{B}^{\mu\nu},$$  

(10.30)

where $F^{\mu\nu}$ is the usual Maxwell field strength and $\mathcal{B}^{\mu\nu}$ is the antisymmetric two-form under which extended objects in string theory can be charged. Note that any possible change in the gauge of this field $\mathcal{B}^{\mu\nu}$ should be accordingly compensated for a shift in $F^{\mu\nu}$. In this way, $\tilde{\phi}$ remains gauge invariant. The bulk field strength $\tilde{\phi} = d\mathcal{B}$ is of course also gauge invariant.

If we now perform a variation of the DBI-action (10.29) with respect to $\mathcal{B}$, we then obtain a source term for the field strength $H^{\mu\nu}$ given simply by $F$. This can be understood as follows: The $D_3$-expanding brane is equipped with the gauge invariant $\tilde{\phi}$ field strength. This will be the source of the field strength in the bulk $\tilde{\phi}^{\mu\nu}$. To make our computations easier, we can make two simplifications [46]:

1. There is no $\tilde{\phi}^{\mu\nu\rho}$ flux inside the space enclosed by the $D_3$-brane.

2. The outer $\tilde{\phi}^{\mu\nu\rho}$ flux is purely due to the presence of the electromagnetic field $\tilde{\phi}$ on the brane.

With this description in hand, the total action describing the dynamics of the tandem bulk-brane in the presence of the source field strength $\tilde{\phi}$ and the bulk flux $\tilde{\phi}$ can be given by:

$$S_5[g^{\mu\nu}, \tilde{\phi}^{\mu\nu}, \tilde{\phi}^{\mu\nu\rho}] = S_{\text{bulk}}[g^{\mu\nu}, \tilde{\phi}^{\mu\nu\rho}] + S_{\text{DBI}}[P[G^{\mu\nu}, \tilde{\phi}^{\mu\nu}]],$$  

(10.31)

with

$$S_{\text{bulk}}[g^{\mu\nu}, \tilde{\phi}^{\mu\nu\rho}] = \frac{1}{2\kappa_5} \int d^5x \sqrt{|g^{(5)}|} \left( R - \frac{1}{12g_s} \tilde{\phi}^2 \right),$$

$$S_{\text{DBI}}[P[G^{\mu\nu}, \tilde{\phi}]] = -T_3 \int d^5x \delta(r - a[\eta]) \sqrt{|g^{(4)} + 2\pi \alpha' \tilde{\phi}|}.$$  

(10.32)

Here we will work in the five-dimensional Einstein frame, with fixed dilaton coupling $e^{-\phi} = g_s^{-1}$. The compact dimension moduli are assumed to be stabilised. Furthermore, note that the $D_3$-brane will be localised at $r = a[\eta]$,  

Note that we will be using Greek indices even though we refer to the four usual coordinates to describe the expanding cosmos, normally represented by Latin indices. This is because we will study the $D_3$-brane and its contents from a five-dimensional point of view in this section.
where $\eta$ the conformal time on the brane. The line invariant describing the AdS$_5$ geometry is that of the Poincaré patch (6.5). This implies the following relationship of the metric determinants:

$$\sqrt{|g^{(5)}|} = \frac{1}{kr} \sqrt{|g^{(4)}|}$$

(10.33)

If we expand for small $\alpha'$, as we did in expression (10.29) and then vary the action with respect to $\mathcal{B}$, we find the equations of motion

$$\partial_r \mathcal{S}^{\nu\mu} = 2\kappa_5 \frac{k r}{\lambda' \pi^2} \mathcal{S}^{\nu\mu} \mathcal{A} (r - a(\eta)),$$

(10.34)

which integrated across the brane at $r = a(\eta)$ gives

$$\Delta \mathcal{S}^{\nu\mu} |_{r=a} = \mathcal{S}^{\nu\mu} |_{r=a} = \frac{2\kappa_5 k a}{\lambda' \pi^2} \mathcal{S}^{\nu\mu} |_{r=a}. \quad (10.35)$$

This equation describes that the $D_3$-brane is equipped with the invariant gauge field strength $\mathcal{S}^{\nu\mu}$ and mediates the decay of the two AdS$_5$ vacua. Furthermore, it is also responsible of sourcing the field strength $\mathcal{S}_{\mu\nu\rho}$ located outside the bubble. And how do we identify the field strength (10.30) with the usual Maxwell one described in (2.37)? They are just the same with different coordinate parametrisations, i.e. conformal in the four-dimensional case and global in the five-dimensional spacetime, related by expression (10.12).

We can then reproduce our computations from section 2.2.2. To be specific, we focus on electromagnetic waves propagating along $z$, with the electric field polarized along $x$, and the magnetic field polarized along $y$. Given a wave propagating along $z$ direction, and $E, B$ respectively polarised along $\{x, y\}$ directions, we have

$$\mathcal{S}^{tx} = \frac{\mathcal{E}(t, z)}{k^2 a^4}, \quad \mathcal{S}^{xz} = \frac{\mathcal{E}(t, z)}{k^2 a^4}, \quad (10.36)$$

where $\mathcal{E}$ is a dimensional function that determines the amplitude of the wave. These waves act as sources for the $\mathcal{S}_{\mu\nu\rho}$ field strength in the bulk. In fact, imposing the absence of this inside of the bubble, the equation of motion (10.35) yields [46]:

$$\mathcal{S}^{tx} |_{r=a} = h_1(t, a, z) = \frac{2\kappa_5 k a}{\lambda' \pi^2} \mathcal{S}^{tx} = \frac{2\kappa_5}{\lambda' \pi^2 k} \mathcal{E}(t, z), \quad \mathcal{S}^{xz} |_{r=a} = h_2(t, a, z) = \frac{2\kappa_5 k a}{\lambda' \pi^2} \mathcal{S}^{xz} = \frac{2\kappa_5}{\lambda' \pi^2 k} \mathcal{E}(t, z). \quad (10.37)$$

In figure 10.3 we can see how the electromagnetic waves (hence the $\mathcal{S}$ field strength) on the brane source the $\mathcal{S}$-field in the bulk above the brane. The

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11The computations in this section are performed for large $ka$-limit, which means late-time induced cosmologies. This implies that the curvature of the bubble, therefore that of the bulk space, is negligible.
Figure 10.3. A sketch of a portion of the dark bubble carrying an electromagnetic field strength $\mathcal{F}$, which sources the field strength $F$ in the bulk. The junction conditions imply a jump in the $\mathcal{F}$-field encoded in Eq. (10.35). The bending is caused by the backreaction of the $\mathcal{F}$-field (see section 9.2 for further details).

$\mathcal{F}$-field must solve the equation of motion (10.37) and the Bianchi identities associated with such a gauge field, away from the brane.\(^{12}\) The Bianchi identities imply:

$$\begin{align*}
\star d \star \mathcal{F} & = 0 \rightarrow \nabla_\alpha \mathcal{F}^{\alpha\beta\gamma} = 0, \\
\star d \mathcal{F} & = 0 \rightarrow \epsilon^{\mu\nu\lambda}_{\kappa} \partial_\alpha \mathcal{F}_{\mu\nu\lambda} = 0,
\end{align*}$$

(10.38)

where $\star$ is the five-dimensional Hodge star and $d$ the usual exterior derivative. In order to make the presentation of the results as smooth as possible to spare our dear reader pain and tears, we introduce a non-trivial time dependence\(^{13}\) in the bulk field strength $\mathcal{F}$. This can be done by adding an extra "leg" to this field, in the form of:

$$\mathcal{F}^{txz} \equiv h_3(t, r, z)$$

(10.39)

Inserting Eq. (10.39), together with Eq. (10.36) into the equation of motion (10.35) yields

$$\begin{align*}
\left( \partial_r + \frac{3}{r} \right) h_1 + \partial_z h_3 & = 0, \\
\left( \partial_r + \frac{3}{r} \right) h_2 + \partial_t h_3 & = 0, \\
\partial_z h_2 - \partial_t h_1 & = 0,
\end{align*}$$

(10.40)

while the Bianchi identity (10.38) becomes

$$r^2 (\partial_t h_2 - \partial_z h_1) + k^4 \partial_r (r^6 h_3) = 0$$

(10.41)

\(^{12}\)Again, for simplicity, we assume that the dilaton as well as any other moduli of the background are fixed.

\(^{13}\)Science is trial and error. In the early stages of this research, we observed that the absence of such a non-trivial dependence would cause the four-dimensional observer to detect an unphysical change in the energy density of the brane. As the aim of the dark bubble model is to make contact with reality, we rejected this possibility. See [46] for further information.
Let us now explicitly solve for the $H$-field strength. We make the ansatz:

$$h_1 = f_1(t, \Xi, z) \frac{1}{k^2 r^3}, \quad h_2 = f_2(t, \Xi, z) \frac{1}{k^2 r^3}, \quad h_3 = f_3(t, \Xi, z) \frac{1}{k^3 r^4},$$  \hspace{1em} (10.42)

with $f_i(t, \Xi, z)$ functions

$$f_i(t, \Xi, z) \equiv \alpha_i(t, \Xi) \sin (k n (t + z)) + \beta_i(t, \Xi) \cos (k n (t + z)),$$  \hspace{1em} (10.43)

where the functions $\alpha_i$ and $\beta_i$ are dimensionless. The parameter $\Xi$ is given by:

$$\Xi = \frac{n}{2k^3 tr^2}.$$  \hspace{1em} (10.44)

This whole massage of the coordinates and the Ansatz itself deserves a good explanation [46]:

1. First, note that an observer on the brane, with conformal time $\eta = kt = -(Hr)^{-1}$, will experience the following relation for the argument within the trigonometric pieces of Eq. (10.43):

$$\frac{1}{knt} \sim \frac{rH}{n} \sim \frac{H}{p},$$  \hspace{1em} (10.45)

where $p$ is the energy of the wave. Similar to gravitational waves (see section 10.2), the momentum of the wave is greater than the Hubble horizon $p \gg H$, so $(knt)^{-1} \ll 1$ becomes a natural variable for expansion.

2. Furthermore, the choice of the new variable $\Xi$ follows the same argument, as:

$$\Xi = \frac{n}{2k^3 tr^2} \sim \frac{nH}{k^2r} \sim \frac{pH}{k^2} \sim \frac{H}{k} \frac{p}{k} \sim \frac{p}{M_{\text{pl}}},$$  \hspace{1em} (10.46)

where in the last step, we have used the relations from section 7.7, exactly Eq. (7.61) and $H \sim \rho^{1/2}_\Lambda$ with $\rho_\Lambda$ as in Eq. (7.63). $M_{\text{pl}}$ is the Planck mass in four dimensions, restored to the correct dimensions in the previous expressions. This implies that $\Xi$ is a small parameter compared to the four-dimensional Planck mass, provided that $H \ll p \ll M_{\text{pl}}$. This is the regime that is relevant for our computations.

By imposing both restrictions discussed above, one can expand for small $\Xi$ and small momentum $p$ and solve the equations of motion (10.40) and Bianchi identities (10.38) order by order. In this way, we find the solution, up to order $(knt)^{-1}$ but in all orders in $\Xi$ to be given by:

$$h_1(t, \Xi, z) = \frac{\alpha}{k^2 r^3} \sin (\theta(t, z)) + \frac{2\beta}{k^2 r^3} \left[ \sin \left( \frac{\Xi}{2} + \theta(t, z) \right) - \sin (\theta(t, z)) \right],$$
$$h_2(t, \Xi, z) = h_1(t, \Xi, z) - \frac{\beta \Xi}{k^3 n tr^3} \cos (\frac{\Xi}{2}) \sin (\theta(t, z)),$$
$$h_3(t, \Xi, z) = \frac{2\beta \Xi}{nk^3 r^4} \sin (\frac{\Xi}{2} + \theta(t, z)).$$  \hspace{1em} (10.47)
where \( \theta(t, z) = kn(t + z) \) and \( \alpha \) and \( \beta \) are now constants.\(^{14}\) We note that only \( h_2 \) requires a correction at \((knt)^{-1}\) to the equations of motion up to leading order in \((knt)^{-1}\).

With solutions \((10.47)\) at hand, we only need to perform one extra step to find the energy-momentum tensor associated with the \( \delta_{\mu\nu\rho} \) field in the bulk. Varying the action \((10.31)\) with respect to \( g_{\mu\nu} \) we find:

\[
T_{\mu\nu} = \frac{1}{\kappa_5} \left( -\frac{1}{2} g_{\mu\nu} \delta^2 + 3 \delta_{\mu\rho\omega} \delta_{\nu}^{\rho\omega} \right). \tag{10.48}
\]

Similar to what we have already done for all previous homogeneous backgrounds in sections 2.2 and 10.2, we must consider for all possible waves travelling in all directions, incoming and outgoing ones with any wavenumber \( n \). Homogenising \((B.15)\) and isotropising \((B.15)\) yields:

\[
\langle T_{\mu\nu}^\mu \rangle_{\text{EM}} = \frac{3\alpha^2}{\kappa_5 k^2 r^4} \begin{pmatrix}
-1 & 0 & 0 & 0 & -\beta \\
0 & 1/3 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 0 \\
\beta_r & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{10.49}
\]

It is really easy to see the similarities and the differences respect to that of the gravitational waves \((10.20)\). Contrary to that case, here we do not observe any contribution in the form of curvature. We do identify two main pieces: That of the diagonal, corresponding to the electromagnetic radiation, with an energy density diluting as \( \rho_{\text{EM}} \sim a^{-4} \) and an off-diagonal piece showing an energy flux. This flux represents how the expanding bubble, equipped with the \( \delta_{\mu\nu} \) field strength that sources the \( \delta_{\mu\nu\rho} \) pushes the latter, carrying it out in the expansion in the AdS\(_5\)-bulk [46].

### 10.3.1 Glowing encore: Backreaction

In the eventuality that our dear reader has got lost conceptually in the previous paragraphs, we would like to remind them what the aim of this whole chapter is: We would like to decorate the induced expanding four dimensional cosmology by the action of bulk features "projected down" to the \( D_3 \)-brane geometry. In this particular case, the aim is to reproduce the energy-momentum tensor associated with the homogeneous background radiation described in Eq. \((2.41)\).

So far, we have understood how the correspondence between the source field-strength \( \delta_{\mu\nu} \) of the \( D_3 \)-brane and the bulk field \( \delta_{\mu\nu\rho} \) works, but we still need to induce the expected energy-momentum tensor \( T_{ab} \) \((2.41)\) on

\(^{14}\)This follows from the expansion of \( \{\alpha, \beta\} \)-function for small \( c \) and \( p \).
the brane. According to section 10.1, we need to find a backreacted metric that takes into account for the presence of the homogeneous and isotropic background of electromagnetic waves described in Eq. (10.49).

The backreacted geometry Ansatz that captures the presence of the background radiation can be written as [46]:

$$d s^2_{\text{back}} \approx -j_1(t, r) d t^2 + j_2(t, r)^{-1} d r^2 + r^2 d \Omega^2_3,$$  \hspace{1cm} (10.50)

where

$$j_i(t, r) \equiv k^2 r^2 - \frac{\epsilon^2 (\log(-\rho k^2 t r) + q^\text{EM}_i)}{k^2 r^2}$$  \hspace{1cm} (10.51)

with \( \{q^\text{EM}_i\} \) as undetermined numbers and \( \rho \) as a free parameter. This line invariant yields an energy-momentum tensor of the form:

$$\langle T^\mu_\nu \rangle_{\text{back}} = \frac{3 \epsilon^2}{2 \kappa_5 k^2 r^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{r}{k^2} & 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (10.52)

where \( q^\text{EM}_1 - q^\text{EM}_2 = 1/4 \) has already been "fixed" to resemble (10.49). Coefficients \( \{\alpha, \beta\} \) in Eq. (10.49) can also be related to the expansion parameter \( \epsilon \) and the amplitude \( \mathcal{E} \) of the four-dimensional electromagnetic wave in Eq. (10.35) as:

$$\alpha = \beta = \frac{\epsilon}{\sqrt{2}} = \frac{2 \kappa_5 k}{\alpha' \pi^2} \mathcal{E}. \hspace{1cm} (10.53)$$

Note that \( \rho \) does not appear in (10.52). However, this does not make it an irrelevant term; this parameter can activate an AdS-Schwarzschild background behaviour in the calculation of the induced four-dimensional cosmology, in the same way as in section 10.1.

Finally, let us study how the backreaction of the bulk geometry (10.50), due to the presence of the \( \mathcal{F}_{\mu\nu\rho} \) field-strength, changes the motion of the bubble wall, and hence the induced energy-momentum tensor \( T_{\alpha\beta} \). As in previous sections, one can equivalently arrive to the same result by applying Israel’s second junction condition (C.35) or the reversed Gauss-Codazzi equation (9.24). However, one must be careful with the following computation; Whenever we explored how the bulk’s content induces certain energy densities in the expanding cosmology, with the presence of strings and matter in section 10.1 and gravitational waves in section 10.2, we assumed a pure brane, i.e. with nothing on top apart except the pure tension of the brane \( \sigma \). This left the Israel’s junction condition (C.35) to be as simple as in Eq. (6.13).

\[\text{We will definitely settle them to a certain value later on.}\]
This is not the current case at hand, as we have initially equipped the brane with a gauge field strength $\tilde{F}_{\mu\nu}$. This requires us to account for an additional gauge field contribution living on the brane. This can be written as:

$$S_{ab} = -\sigma h_{ab} + T_{ab}^{EM}. \quad (10.54)$$

In this way, the resulting energy-momentum tensor $T_{ab}$ induced on the brane will have two main contributions:

1. A **positive** contribution coming from the extrinsic curvature.
2. A piece that comes from the gauge field on the brane, which adds to the tension and thus gives a **negative** contribution.

This can be read off from the Friedmann-like equation governing the dynamics of the induced cosmology which the second Israel's condition (C.35) spits out as:

$$\left(\frac{\dot{a}}{a}\right)^2 \delta^a_b = \frac{\Lambda_4}{3} \delta^a_b + \frac{\kappa_4}{3} \left(\ddot{\delta}_{bc} \delta^{ac} \delta^{ij} \delta_{ij} \right) - \frac{1}{4} \delta^a_b \delta_{ij} \delta^{ij}$$

$$+ \epsilon^2 \frac{\kappa_4}{6\kappa_5 k_+^3 a^4} \left(3\delta_{ij} \delta^0_b - \delta^i_j \delta^0_b\right) \left[ q_{2}^{EM} + \log \left[ -\rho k / H \right] \right]. \quad (10.55)$$

The left-hand side represents the usual geometrical evolution of the scale factor for late time cosmologies (i.e. there is no contribution from the curvature term) in the Friedmann equations. The two contributions on the right-hand side correspond to the energy-momentum tensor associated with the induced cosmology. Note that by setting $a = b = 0$, we recover the first Friedmann equation. It is crucial to note that the gauge field on the brane, in the first line, gives a **negative** contribution to the energy density, as does the brane tension. This needs to be corrected to a positive contribution by the second term due to the extrinsic curvature. In other words, this is the gravitational backreaction of the bulk.

We can do this by properly adjusting the free parameters $\rho$ and $q_{2}^{EM}$, which are degenerate and can be shifted into each other. First, the parameter $\rho$ can be chosen such that there is no logarithmic contribution in Eq. (10.55), i.e. $-\rho k / H = 1$. Furthermore, if we impose that the aforementioned combination of contributions to the energy-momentum tensor must be that of the electromagnetic tensor (2.41), we would find [46]:

$$\epsilon^2 \frac{q_{2}^{EM}}{2a^4 k_+^3 \kappa_5} \left(3\delta_{ij} \delta^0_b - \delta^i_j \delta^0_b\right) \left(\ddot{\delta}_{bc} \delta^{ac} \delta^{ij} \delta_{ij} \right) \equiv -\langle T_{b}^{a} \rangle_{iso}. \quad (10.56)$$

One can then solve for $q_{2}^{EM}$ imposing that the right-hand side corresponds to the usual **positive** electromagnetic energy density (recall that in our con-
ventions $T_0^0 = -\rho$). Using relations (10.53) and (7.60) we can fix $q^{EM}_2$ to be:

$$q^{EM}_2 = k^2 M_{pl}^2 N \pi,$$

where $N$ is the number of $D_3$-branes in the ten-dimensional background (7.65) and $T_{D_3}$ is the fundamental tension of one of them.

The sharp mind of our dear reader may have noticed that the imposition on the right-hand side of Eq. (10.56) to be rather ad-hoc. What would happen if one does not impose it? What if the backreaction is not exact enough to compensate for the negative contribution to the energy density from the fields on the brane? That would correspond to a mismatch between the induced energy-momentum tensor and the observed one, Would it not? Would it not be seen as a failure of the model to predict the right energy-momentum tensor? Dear Padawan, these are questions that only the dark side sector can answer.

As discussed in section 9.3, any initial mismatch between the energy-momentum tensor of the visible sector, i.e. Eq. (2.41) and the effective induced four-dimensional one (10.55) can be interpreted as the presence of a dark sector. This layer would also couple to four-dimensional gravity like the visible one, but it will not interact with visible fields. This tempting proposal should be further explored in future works about the dark bubble scenario.
Part IV:
To Infinity and Beyond
11. And Go on till You Come to the End; Then Stop

*What is the question that we should be asking that we are not?*

Lyra Silvertongue, His Dark Materials

It has been a long journey up to this point. It is quite likely that our dear reader’s mind is exhausted and their eyes dry after reading all these pages. Although answers may have been found across this work, it is quite that possible they have broken down into in smaller and more subtle questions the answers to which may lie in unexplored regions of the cosmos. In this chapter, we will collect some of the anecdotes we have lived during our pilgrimage and offer clues and/or guides that could lead us to interesting discoveries in the future.

We started the expedition in our four-dimensional cosmological shire. In chapter 2, we briefly reviewed our current best tool for describing our cosmos: the Λ-CDM model. However, we also learnt that such elegant framework has its flaws. Specially when it comes to explaining the universe at energetical regimes where the classical description of spacetime breaks, i.e. distances in the Planck length $\ell_{Pl}$ range. This helped us to see the need for acquiring a more fundamental theory well defined at arbitrarily high energies; a theory of quantum gravity. Our first wandering steps beyond our classical courtyard led us to understand the foundations of quantum cosmology in chapter 3. In this new point of view of the universe, we learnt to describe the cosmos in terms of its quantum state $|\psi\rangle$ and its evolution governed by the Wheeler-De Witt equation (3.14). However, we also found out that one needs to impose a good choice of boundary conditions to determine which solution of the previous equation corresponds to our universe. At this point, we met three gentlemen who were discussing which boundary condition was the suitable one for our cosmos: Hartle, Hawking and Vilenkin [25,26]. All their arguments in favour of their own choices sounded good enough, though they seemed to have been stuck in the same discussion for years with very little or no consensus. Could this boundary puzzle be solved by a ultraviolet completion of general relativity?

In the hope of finding solutions to this choice of boundaries and in the search for a theory of quantum gravity that could describe dark energy from
a microscopic point of view, we left the realm of quantum cosmology and headed towards the jurisdiction of string theory in chapter 4. There we learnt that the sovereign of these lands, string theory, was a promising theory of quantum gravity, offering a framework to unify all the fundamental forces under the same ten-dimensional spacetime formalism. Nevertheless, two different mechanisms had been engineered to hide the extra dimensions required by supergravity, the low-energy description of string theory: Braneworlds and string compactifications. The former proposed that our universe was confined to a three-dimensional hypersurface, called brane, that could move in the extra dimensions. The latter, (flux) compactifications, claimed that extra dimensions are tiny and rolled up on themselves. Main features of the non-compact directions, such as the observable four-dimensional cosmological constant $\Lambda_4$, would be controlled by the internal configuration of compact spaces.

However, these two territories were not free from controversy and problems: The calm meadows of the braneworld mechanism required supercritical branes to induce an observable positive cosmological constant in their four-dimensional volume. Moreover, although they made use of stringy ingredients, they did not seem to have an explicit embedding in string theory yet. The seek for the right positive valley between the cliffs of the compactification landscape showed every sign of being an arduous task. Not to mention the lack of scale separation between compact and non-compact dimensions in each of the compactifications. In summary: obtaining lower-dimensional effective field theories from string theory where all components of the construction are understood seems to remain as a challenging task [160].

Following a trail of evidences left by brave researchers in the lead, we made our way out of the stringy kingdom’s domain, and into the swampland in chapter 5. This muddy terrain contained all the apparently consistent low-energy descriptions of quantum gravity that do not meet certain criteria to be obtained from UV-complete theories. All de Sitter cosmologies lay in one of the swamps [170,172,185]. Decaying non-supersymmetric Anti-de Sitter vacua spawned a broad cementery [185] because of the weakness of gravity [177, 184]. What kind of fundamental principle of quantum gravity might lie in the deepest part of the swampland, where all conjectures seem to converge [162, 166–169]?

We noticed that, every now and then, a brane could nucleate from the bottom of the non-supersymmetric anti-de Sitter vacua cementery [185]. As the swampland did not seem to be the right place to fulfil our search for four-dimensional de Sitter solutions from UV-complete theories, we decided to jump on one of these nucleated branes, to escape from such horrible place in chapter 6. These co-dimensional one branes mediated the
decay between two different five-dimensional anti-de Sitter vacua, with the vacuum enclosed by the brane equipped with a deeper value of the five-dimensional cosmological constant than the outer region. With a little bit of dimensional analysis and the junction conditions at hand [215], it was not hard to see that this difference, together with the subcritical tension of the brane imposed by the weak gravity conjecture [177,184], was the reason why the four-dimensional cosmos induced on the boundary of the bubble was decorated with a small and positive cosmological constant [37].

The exploration of this dark bubble universe continued yielding more impressive findings, such as the heroic story of the brane we rode. In chapter 7 we discovered that the brane we had jumped into had risen from a "pit", a near horizon geometry similar to a ten-dimensional Kerr black hole sourced by a stack of \( N \) black \( D_3 \) branes. The presence of non-zero temperature \( T \) and chemical potentials \( \mu_i \) in the dual field theory describing such stringy model were responsible for breaking supersymmetry and making the whole construction unstable. In this way, the system could decay via brane nucleation. Any brane escaping from the stack will have tunneled through the outer horizon of the black hole, and will start to expand to minimise its energy. Both equations of motion resulting from the \( D_3 \) DBI and WZ action (7.32) and the junction condition (7.39) of the dark bubble definitely point to a clear fact: this dark bubble comes from string theory [45]. Curvature corrections to the action (7.16) were responsible for lowering its tension beyond criticality [47], in line with the expectations of the weak gravity conjecture, resulting in the small but positive dark energy that decorated the induced four-dimensional cosmos of the brane. Furthermore, we saw that the scale separation was achieved thanks to the tension-to-charge ratio of the expanding brane and that the energy hierarchy associated with this bubble was not the usual one, with gravity being stronger in four dimensions than five. The presence of large extra dimensions with size \( L \sim 10^{-3}\text{m} \) in such construction suggests corrections to Newton's gravity at this scale [47].

After we studied the constructions of the bubble universe embedded in string theory, we decided to analyse the rate of nucleation rate of such bubbles. Indeed we saw that nucleation probability of bubbles in five dimensions is exactly the same as a tunneling from nothing in the four dimensional interpretation of quantum cosmology [43]. This higher-dimensional approach unambiguously chooses Vilenkin boundary conditions, providing an explicit solution to the four-dimensional quantum puzzle discussed in chapter 3.

Then in chapter 9 we told how we examined the bubble wall in more detail. We realised that four-dimensional point particles of matter were no more than the end points of stretching strings extending into the bulk.
These strings will also pull up from the brane, contributing with an unexpected energy density sign \[40\]. However, it turned out to be the right one. Gravity, which is not localised on the boundary of the bubble, but which extends along the bulk, was aware of this. Through the reversed Gauss-Codazzi equation (9.24) we saw that gravity also accounts for the extrinsic deformations in the embedding geometry, providing a harmonious balance between five and four-dimensional geometrical features, resulting in the expected energy densities of the four-dimensional cosmos \[38, 46\].

Finally, we decided to explore the colourful contents of the induced cosmos in chapter 10. The reversed Gauss-Codazzi equation (9.24) proved to be useful in this task, as the only way to contemplate this was when the bulk backreaction effects were taken into account. Dust matter and both gravitational and electromagnetic waves were painting the four-dimensional cosmos, as projection of five-dimensional bulk features onto the bubble wall \[44, 46\].

And here we are, recapitulating our experiences during this work. Although these adventures have provided us with a new way of realising dark energy from string theory in the form of a four-dimensional universe riding on a dark bubble, our original questions may have matured and for sure, new ones may have appeared. As the dark bubble construction seems a promising but still unexplored territory to accommodate four-dimensional effective field theories equipped with a positive cosmological constant, we will then sketch some of our future lines that could lead us to new and exciting revelations.

First, it will be interesting to see the results of future experiments testing deviations from Newton's gravity on the scales of microns \[216, 217\] and of forthcoming tests at higher energies and luminosity at the LHC. In this way, the predictions arising from the stringy embedding discussed in chapter 7 could be falsified. It remains to be seen what kind of fascinating results future experiments in both fields will yield.

The landscape of string theory compactifications has been extensively explored, so that there is now a good understanding on how the geometry of the internal manifold and its corresponding fluxes give rise to external manifolds with different properties depending on the compact spaces \[126, 160, 218\]. However, there is no analogue of this "catalogue" for braneworld scenarios, and more specifically, for dark bubble constructions. The first steps in this direction could be taken by revisiting the compactification proposed in \[37\]. This would first require us to prove that the construction cannot perturbatively decay, which could be done by adapting the Kaluza-Klein spectrometry method developed in \[219\] to check that all modes satisfy the Breitenlohner-Freedman bound. Once this is confirmed, the construction will only decay via non-perturbative mechanisms, such as brane
nucleation. At this point, it will be necessary to engineer a process that favours the nucleation of such branes. Alternatively, one can also explore other type of constructions. For example, using a type 0’B supergravity model \[220\] as starting point. Supersymmetry is already broken in this construction, so a stack of $D_3$ branes placed in this geometry will be unstable, provoking the emission of branes as in \[45\]. To study the induced four-dimensional cosmology on the nucleated brane, one can then compute its potential from the worldvolume action controlling its dynamics. Following a similar derivation as in \[45\], one will gather information on how the dark bubble embedding works for different backgrounds, opening up a new study of the dark bubble landscape.

The dark bubble construction cannot only provide a higher-dimensional interpretation of the microscopic nature of dark energy but, as shown in section 10.3, it also derives the induced effective field theory of electromagnetism from the dynamics of supergravity fields in higher dimensions and their interaction with the expanding bubble. This suggests that other four-dimensional forces, such as the weak interaction, may also have a higher dimensional interpretation in this construction. A similar conclusion can be drawn by noting that the sum of the neutrino masses is bounded by the inverse of the scale of the extra dimensions $L$ of the dark bubble model. To develop this further, one can investigate whether the neutrino masses have an intrinsically higher-dimensional interpretation, specifically, to the Kaluza-Klein modes of the internal dimensions of the compactification \[221\]. This can be done by computing the extra dimensional volume dependence of the coupling between left-handed neutrinos living on the brane and massless bulk right-handed neutrinos. This coupling can then be used to construct an Ansatz for the lepton interaction action and read off the exact value of the left-handed neutrinos masses from the generated Kaluza-Klein tower of modes in the extra dimensions. The numerical bounds could again be tested against the most recent experimental values \[222\].

The rich construction of the dark bubble embedding discussed in chapter 7 also presents new opportunities for understanding the $SU(2)$ interaction, i.e. the electroweak force and the Higgs process as the embedding of these in a higher-dimensional construction. In particular, one can compute the Higgs field as the zero mode of an open string stretching between the two branes, coincidental in the radial direction, but separated by a distance $\ell$ in the compact space of $S^5$. The mass of this mode $\sim \ell T_s$ could then be identified with the vacuum expectation value of the Higgs boson. This computation could be done both at early and late cosmologies to test whether the vacuum expectation value is modified by the dynamics of the universe. This might shed light on the current Hubble tension \[62\] discussed in section 2.3, as a higher value of the Higgs vev has been proposed to alleviate such problem \[223\].
These could be two concrete examples of planned trips to explore uncharted territories where lower-dimensional effective field theories may lie. However, we believe that the dark bubble model also has the potential to accommodate explanations for matter beyond the visible sector of particle physics, i.e. dark matter, as we saw in 9.3 and in similar way as discussed in \[170, 224, 225\]. It can also explain non-trivial gravitational phenomena such as black holes \[41\]. Moreover, it seems to be equipped to offer alternative mechanisms to inflation in cosmology, more along the lines of string gas cosmology \[226, 227\].

Could this model be the right alternative to realise reliable and realistic four-dimensional effective field theories equipped with a small and positive dark energy from string theory? So far, it seems so. However, further research is needed, so that we can test its limits. Above we have just sketched some of the directions in which to go to assess the capabilities of the dark bubble model. Although the directions in our compass may seem clear, the path to the (desired) destination may be full of muddy monsters, obstructions, swamp puddles and abrupt abysses that would require us to recalculate our route, go back and start again. Nothing that we physicists are not used to. We are well aware that paths are not written. Paths are made, as Don Antonio Machado said:

\[
\begin{align*}
\text{Caminante, son tus huellas} \\
\text{el camino y nada más;} \\
\text{caminante, no hay camino,} \\
\text{se hace camino al andar.}
\end{align*}
\]

\[
\begin{align*}
\text{Al andar se hace camino} \\
\text{y al volver la vista atrás} \\
\text{se ve la senda que nunca} \\
\text{se ha de volver a pisar.}
\end{align*}
\]

\[
\begin{align*}
\text{Caminante no hay camino,} \\
\text{¡sino estelas en la mar!}
\end{align*}
\]

---

\[1\]

Wanderer, it is your footprints winding down, and nothing more; wanderer, no roads lie waiting, roads you make as you explore.

Step by step your road is charted, and behind your turning head lies the path that you have trodden, not again for you to tread.

Wanderer, there are no roadways, only wakes upon the sea!
Appendix A.
Notation

This appendix will outline the basic notation and conventions used in this work.

The metric and its coordinates
The metric can be written in a $X^M$-coordinate chart as:

$$g = g_{MN}(X) \, dX^M \otimes dX^N,$$

(A.1)

with a Lorentzian/Euclidean signature of mostly plus entries

$$g = \text{diag}(-, +, \cdots, +), \quad g^E = \text{diag}(+, +, \cdots, +).$$

(A.2)

The determinant of the metric will be represented by $|g| = \mp \det g$. As we are going to deal with Lorentzian metrics of different dimensionalities, it could be adequate to specify the index notation used to describe these coordinates. This can be summarised as:

- Four-dimensional spacetime coordinates: Latin lower case letters

$${a, b, c, \cdots}.$$  

(A.3)

- Spatial coordinates in three dimensions or more directions:

$${i, j, k}.$$  

(A.4)

- Five-dimensional spacetime and external coordinates of higher dimensional spaces: Greek lower case letters

$${\mu, \nu, \gamma}.$$  

(A.5)

- Ten or $D$-dimensional spacetime coordinates: Latin capital letters

$${M, N, P}.$$  

(A.6)

In addition, it could be useful to define how coordinates and/or tensors are represented in different dimensionalities.
• Generic coordinates in four or five dimensions are defined as $x$.
• Generic coordinates that make reference to induced or internal geometries are indicated with $y$.
• Generic coordinates from ten or $D$-dimensions are given by $X$.
• Ten or $D$-dimensional non-geometrical tensors (like field strengths) are written with upper and lower case Gothic letters.

Finally, it could be convenient to specify the form and signs of the energy-momentum tensor $T_{MN}$. A perfect fluid is described by:

$$T_{MN} = (\rho + p) u_M u_N + p g_{MN},$$  \hspace{1cm} \text{(A.7)}$$

where $u_M = (-N(t),0,0,0)$ is the fluid four-velocity (assuming no relativistic motion of this), and $(\rho, p)$ its energy density and pressure. To easily read the right entries of the energy-momentum tensor, one needs the tensor to have one contravariant index and a covariant one as:

$$T^a_b = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$  \hspace{1cm} \text{(A.8)}$$

where we have a four-dimensional case as an example.

Differential forms

A $p$-form in a coordinate chart $X^M$ is expressed in component as:

$$\omega^{(p)} = \frac{1}{p!} \omega^{(p)}_{M_1...M_p} dX^{M_1} \land ... \land dX^{M_p},$$  \hspace{1cm} \text{(A.9)}$$

where $\land$ is the wedge product. This product gives a $(p+q)$-form of the factors $p$-form and a $q$-form, with $p+q \leq D$, with $D$ the dimensionality of the space on which these forms are defined on. It is defined to be totally anti-symmetric and satisfies

$$\omega^{(p)} \land \omega^{(q)} = (-)^{pq} \omega^{(q)} \land \omega^{(p)}. \hspace{1cm} \text{(A.10)}$$

Continuing with operations on forms, when two $p$-forms of the same rank are contracted, we will use the notation:

$$|\omega^{(p)}| = \frac{1}{p!} \omega_{M_1...M_p} \omega^{M_1...M_p},$$
$$|\omega^{(p)}|_{MN} = \frac{1}{(p-1)!} \omega_{M_1...M_p} \omega^{M_2...M_p}. \hspace{1cm} \text{(A.11)}$$
Among the forms, one of the most important ones is the canonical volume form of a manifold $\mathcal{M}$. This tensor is given by the combination of the Levi-Civita symbol $\epsilon^{(LC)}_{M_1\ldots M_p}$ and the Jacobian $\sqrt{|g|}$ as:

$$\epsilon_{M_1\ldots M_p} = \sqrt{|g|} \epsilon^{(LC)}_{M_1\ldots M_p},$$

(A.12)

where

$$
\epsilon^{(LC)}_{M_1 M_2 \ldots M_D} = \begin{cases} +1 & \text{if } M_1 M_2 \ldots M_D \text{ is even permutation of } 01 \ldots D, \\ -1 & \text{if } M_1 M_2 \ldots M_D \text{ is odd permutation of } 01 \ldots D, \\ 0 & \text{else}. \end{cases}
$$

(A.13)

This tensor allows us to define the Hodge operator as:

$$\star (dX^{M_1} \wedge \ldots \wedge dX^{M_p}) = \frac{1}{(D - p)!} \epsilon^{N_1 \ldots N_{D-p}}_{M_1 \ldots M_p} dX^{N_1} \wedge \ldots \wedge dX^{N_{D-p}}.$$

(A.14)

Finally, we define the exterior derivative $d$ as a linear map from $p$-forms to $(p + 1)$-forms of the form:

$$d\omega^{(p)} = \frac{1}{p!} \partial_{[N} \omega^{(p)}_{M_1 \ldots M_p]} dX^N \wedge dX^{M_1} \wedge \ldots \wedge dX^{M_p}.$$ 

(A.15)
Hypergeometric functions and Chebyshev polynomials

Hypergeometric differential equations are of the form:
\[ z(1-z)\frac{d^2 w}{dz^2} + [c - (a + b + 1)z] \frac{dw}{dz} - abw = 0, \tag{B.1} \]
with three singular points at \( z = \{0,1,\infty\} \). Solutions are given by hypergeometric functions as:
\[ _2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots \tag{B.2} \]
When \( b = -a \), with \( a \in \mathbb{Z} \) and \( c = 1/2 \), Eq. (B.2) can be expressed as:
\[ _2F_1(a, -a, 1/2; z) = T_n(1-2z). \tag{B.3} \]
Here \( T_n(x) \) represents the Chebyshev polynomial of first kind, given by:
\[ T_n(x) = \cos(n \arccos(x)), \tag{B.4} \]
and second kind \( U_n(x) \):
\[ U_n(x)^2 = \frac{T_{n+1}(x)^2 - 1}{x^2 - 1}. \tag{B.5} \]

Vector and tensor spherical harmonic decomposition

Our dear reader is probably aware of the joy and relief that scalar spherical harmonics provide. This celebration can be continued with the decomposition of vectors and tensors in such a delightful basis. We will not explicitly describe each of its components in this section (see [228]). Here we will only present its basic properties.

The spatial geometry will be that of \( S^3 \) as described in Eq. (2.19). Any vector can be decomposed as:
\[ V^a = g^{ab} \sum_{k=k_{\text{min}}}^{\infty} \sum_{\ell=\ell_{\text{min}}}^{k} \sum_{m=-\ell}^{\ell} V^{k\ell m} Y_{(1)b}^{k\ell m}, \tag{B.6} \]
where \( Y^{k\ell m} \) is a Fourier coefficients with time dependence. \( Y^{k\ell m}_b \) is the spherical harmonic part, that can be composed in three different ways. For convenience, we will only show here the first way in Lindblom's paper [228].

\[
Y^{k\ell m}_{(1)a} = \frac{1}{\sqrt{\ell(\ell+1)}} \epsilon_{bc}^{a} \nabla_{b} Y^{k\ell m} \nabla_{c} \cos \gamma, \tag{B.7}
\]

where \( \epsilon_{bc}^{a} \) is the full antisymmetric tensor volume and \( \nabla \) the covariant derivative with respect to \( S^3 \) geometry. This harmonic decomposition satisfies the following eigenvalue relation and divergence identity:

\[
\nabla^{b} \nabla_{b} Y^{k\ell m}_{(1)a} = 1 - k(k+2) Y^{k\ell m}_{(1)a}, \tag{B.8}
\]

\[
\nabla^{a} Y^{k\ell m}_{(1)a} = 0.
\]

**Tensors** allow for richer and more involved compositions (up to five different ways). In this thesis we will only present the fourth one from Lindblom:

\[
T_{ab} = g_{ab} \sum_{k=k_{\text{min}}}^{\infty} \sum_{\ell=\ell_{\text{min}}}^{k} \sum_{m=-\ell}^{\ell} T^{k\ell m}_{(4)ab} Y^{k\ell m}_{(4)ab}, \tag{B.9}
\]

where

\[
Y^{k\ell m}_{(4)ab} = \sqrt{(\ell-1)(\ell+2)} \left\{ \frac{1}{2} E^{k\ell} \left( \nabla_{a} F_{b}^{\ell m} + \nabla_{b} F_{a}^{\ell m} \right) \right. \\
+ \csc^{2} \gamma \left( \frac{1}{2} (\ell-1) \cos \gamma E^{k\ell} + C^{k\ell} \right) \left( F_{a}^{\ell m} \nabla_{b} \cos \gamma + F_{b}^{\ell m} \nabla_{a} \cos \gamma \right) \right\}. \tag{B.10}
\]

The quantity \( E^{k\ell} \) is given by:

\[
E^{\ell\ell}(\gamma) = -\frac{2 \cos \gamma}{\ell-1} C^{\ell\ell}(\gamma), \tag{B.11}
\]

if \( k = \ell \). For \( k > \ell \):

\[
E^{k\ell}(\gamma) = -\frac{2(k+2) \cos \gamma}{(\ell-1)(\ell+2)} C^{k\ell}(\gamma) + \frac{2 \sqrt{(k+1)(k-\ell)(k+\ell+1)}}{(\ell-1)(\ell+2)\sqrt{k}} C^{k-1\ell}(\gamma), \tag{B.12}
\]

where \( C^{k\ell} \) are Gegenbauer polynomials [228]. Finally, \( F_{a}^{\ell m} \) is:

\[
F_{a}^{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \epsilon_{bc}^{a} \nabla_{b} \left( \sin^{\ell} \gamma Y^{\ell m} \right) \nabla_{c} \cos \gamma, \tag{B.13}
\]

As harmonic functions, they satisfy the following relationships (the eigenvalue, divergence and traceless equations):

\[
\nabla^{c} \nabla_{c} Y^{k\ell m}_{(4)ab} = 2 - k(k+2) Y^{k\ell m}_{(4)ab}, \tag{B.14}
\]

\[
\nabla^{a} Y^{k\ell m}_{(4)ab} = 0, \tag{B.15}
\]

\[
g^{ab} Y^{k\ell m}_{(4)ab} = 0.
\]
Isotropy and homogeneity
To integrate over all possible phases of a wave (chapters 2 and 10).

\[
\langle T^i_j \rangle = \lim_{L \to \infty} \left( \frac{1}{L} \int_1^L \theta \, d\theta \, T^i_j \right). \tag{B.15}
\]

To isotropise:

\[
\langle T^i_j \rangle_{\text{Total}} = \frac{1}{3} \left( \langle T^i_j \rangle_{\theta_1} + \langle T^i_j \rangle_{\theta_2} + \langle T^i_j \rangle_{\theta_3} \right) \tag{B.16}
\]

Some Riemannian geometry relations

\[
[\nabla_a, \nabla_b] X^{cd} = R^c_{iab} X^i d + R^d_{jab} X^j d \tag{B.17}
\]

\[
\nabla_a v^a = \frac{1}{\sqrt{|g|}} \partial_a \left( \sqrt{|g|} v^a \right) \tag{B.18}
\]

Weyl rescaling and warped products

A Weyl rescaling of the form:

\[
\tilde{g}_{MN} = e^{2\omega} g_{MN}, \tag{B.19}
\]

Implies that

\[
\sqrt{|\tilde{g}|} = e^{n\omega} \sqrt{|g|} \tag{B.20}
\]

and

\[
\tilde{R} = e^{-2\omega} \left( R - 2(n - 1) \nabla^2 \omega - (n - 2)(n - 1) \partial^K \omega \partial^K \omega \right), \tag{B.21}
\]

where \( n \) is the dimension of the space of study.

Any \((D + d)\)-dimensional metric of the block diagonal form:

\[
d^2 s_{(D+d)} = g_{MN} dX^M X^N = e^{2A(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{2B(x)} j_{mn} dy^m dy^n, \tag{B.22}
\]

with coordinate splitting as:

\[
X^M = \{ x^1, \ldots, x^D, y^1, \ldots, y^d \}, \tag{B.23}
\]

will have a composite Ricci scalar described as:

\[
e^{2B(x)} R_{(D+d)} = R_{(D)} + e^{2B(x) - A(x)} \left[ R_{(D)} - 2(D-1)(D-2)\nabla^2 (A(x) - B(x)) - (D-1)(D-2)\partial_a (A(x) - B(x))\partial^a (A(x) - B(x)) - (D + d - 1)(D + d - 2)\partial_\gamma B(x)\partial^\gamma B(x) - 2(D + d - 1) \left[ \nabla^2 B(x) + (D - 2)\partial_\rho B(x)\partial^\rho (A(x) - B(x)) \right] \right] \tag{B.24}
\]
Appendix C.
Hypersurfaces, Gauss-Codazzi Equation and Junction Conditions

The aim of this appendix is to provide the reader with a quick overview of the wonderful world of hypersurfaces, i.e. submanifolds of dimension \( \text{dim} = D - d \) which are "slices" of \( D \)-dimensional manifolds. This appendix will follow along the lines of [229] and [230].

The mathematical definition of hypersurface can be cast in the following form:

\[
S = \{ x^\alpha \in \mathcal{M} | \Phi(x^\alpha) = 0 \} \subset \mathcal{M}. \quad (C.1)
\]

Perhaps mathematical definitions will cause our dear reader to have a rash, so let us simplify the previous description in more "peasant" language. In addition, as we also care about our reader’s sanity, we will restrict this study to hypersurfaces of co-dimension one. This is \( d = 1 \). More brave mathematical warriors, willing to fight through co-dimension \( d \) sub-manifolds, are welcome to read the exquisite selected literature on the topic [231–233].

As we said before, we define a hypersurface \( \Sigma \) as a "slice" of a higher-dimensional space with metric \( g_{\mu\nu} \). This is something that has been known since the good old high school days. For example, one can define a two-dimensional sphere in a three-dimensional flat Euclidean space by

\[
\Phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0, \quad (C.2)
\]

with \( R \) as its radius. The embedding map \( \Phi \) tells us how to "insert" the hypersurface \( \Sigma \) (the sphere) in the manifold \( \mathcal{M} \), i.e. the three-dimensional Euclidean space. Note that this description of the sphere respects the definition given in expression (C.1). Its coordinates \( \{x, y, z\} \) are contained in \( \mathcal{M} \), the three-dimensional Euclidean space and Eq. (C.2) corresponds to the restriction in the second part of the definition. Eventually, one can also choose a new set of coordinates that are intrinsic to the sphere itself, such as \( y^a = \{\phi, \theta\} \), so that we can relate the extrinsic coordinates \( \{x, y, z\} \) to those of the sphere by the well-known parametric relations:

\[
x = R \cos \phi \sin \theta, \\
y = R \sin \phi \sin \theta, \\
z = R \cos \theta. \quad (C.3)
\]
This parametric equation can be written in a more general way as:

\[ x^\alpha = x^\alpha (y^\alpha). \]  

(C.4)

Although it has been already specified in appendix A, the significance of the following notation demands to emphasise again that:

- **Extrinsic** coordinates will be denoted by **Greek** letters \{\(\alpha, \beta, \gamma, \cdots\}\).
- **Intrinsic** coordinates will be denoted by **Latin** letters \{\(a, b, c, \cdots\}\).

It is extremely important that this notation is crystal clear, as the aim of this appendix, and by extension this thesis, is to relate extrinsic and intrinsic properties of manifolds one another, so that we can get as much information as possible from both coordinate systems.

Returning to our simple spherical example, let us continue with more definitions. As a surface, it can be equipped with vectors. The ones of our interest are of two different types: The normal and the tangent vectors.

![Figure C.1](image)

*Figure C.1.* Two of the most basic elements to describe the embedding of a codimension one hypersurface \(\Sigma\) in a \(D\)-dimensional manifold are the normal vector \(n^\mu\) and the tangent vectors \(e^a_\alpha\).

### Normal vector

It is easy to think of a normal vector \(n^\alpha\) in the spherical case described above: A stingy arrow pointing orthogonally (outside or inside) with respect to the surface \(\Sigma\). The problem arises when dealing with dimensions greater than three or signatures beyond the Euclidean one. How do you define the normal vector?
We can define a unit normal unit vector \( n^\alpha \) imposing unitarity:

\[
n^\alpha n_\alpha = \epsilon = \pm 1,
\]

where (+) corresponds to a timelike hypersurface and (−) represents spacelike ones.\(^1\) Furthermore, we require that \( n^\alpha \) points in the direction of increasing \( \Phi \). In the case we that we are looking at a spacelike surface, the normal vector will point in the direction of growing spatial sections, i.e. \( n^\alpha \partial_\alpha \Phi > 0 \). This implies that the normal vector can be defined as:

\[
n_\alpha = \frac{\epsilon \partial_\alpha \Phi}{\sqrt{g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi}}. \tag{C.6}
\]

Note that for the two-dimensional sphere described above, as the metric \( g_\mu^\nu = \mathbb{1}_{3\times3} \), one recovers its usual euclidean definition.

**Tangent vectors**

Unlike normal vectors, tangent vectors live on the hypersurface. They are defined by:

\[
e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}, \tag{C.7}
\]

where again, \( x^\alpha \) coordinates belong to the manifold \( \mathcal{M} \) and \( y^a \) are coordinates of \( \Sigma \), related by the parametric relation (C.4). Note that \( e^\alpha_a \) will be a matrix, i.e. the Jacobian of the parametric transformation, with \( D \) rows and \( D - 1 \) columns, as we are dealing with hypersurfaces of co-dimension one.

As expected, tangent and normal vectors are orthogonal to each other, which means that their scalar product is null,

\[
n_\alpha e^\alpha_a = 0. \tag{C.8}
\]

Furthermore, tangent vectors, acting as "projectors" of \( D \)-dimensional coordinates \( x^\alpha \) onto the hypersurface \( \Sigma \), can be used to describe the intrinsic or induced line invariant on the surface. This is done by restricting the line element of the \( D \)-dimensional space to displacements confined to \( \Sigma \), as:

\[
ds_\Sigma = g_{\alpha\beta} dx^\alpha dx^\beta \bigg|_\Sigma
\]

\[
= g_{\alpha\beta} \left( \frac{\partial x^\alpha}{\partial y^a} dy^a \right) \left( \frac{\partial x^\beta}{\partial y^b} dy^b \right)
\]

\[
= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} dy^a dy^b h_{ab} \tag{C.9}
\]

\(^1\)Null hypersurfaces are trickier. Check [229] for further details.
where $h_{ab}$ is the **induced** metric or first fundamental form. This will allow us to define the following completeness relation for the metric $g_{a\beta}$ as:

$$g^{a\beta} = \epsilon n^a n^\beta + h^{ab} e_a^\alpha e_b^\beta. \quad (C.10)$$

This previous expression will be very useful in the following pages, as it relates the tangential and normal parts of the embedding in the line invariant hosting the hypersurface.

Let us now imagine a full tangential tensor $A^{a\beta}$ defined on $\Sigma$, with no components in the normal directions (i.e. $A^{a\beta}n_a = 0$). Such tensor admits the decomposition

$$A^{a\beta} \cdots = A^{ab} \cdots e_a^\alpha e_b^\beta \cdots = h^{ai} h^{bj} \cdots A_{ij} \cdots e_a^\alpha e_b^\beta \cdots \quad (C.11)$$

To understand how these tensors differentiate, we can simply apply the usual covariant derivative. However, the resulting information will depend on the chosen set of coordinates. If we choose the hypersurface coordinates $y^a$, it is easy to prove that:

$$\nabla_b A_a = \nabla_\beta A_\alpha e_\beta^b = \cdots = \partial_b A_a - \Gamma^i_{ab} A_i, \quad (C.12)$$

where $\cdots$ are intermediate steps of the computation. The expression (C.12) corresponds to the well-known **intrinsic** covariant differentiation. But this is not the end of the story. We can reproduce the same computation, but splitting the components of the metric $g_{a\beta}$ into its normal and tangential pieces. For indices convenience, let us consider the vector $\nabla_\beta A_\alpha e_\beta^b$, whose tangential components are given by Eq. (C.12). This is:

$$\nabla_\beta A_\alpha e_\beta^b = g_{\gamma\gamma} \nabla_\beta A_\gamma e_\gamma^b \nabla_\beta A_\gamma e_\gamma^b = \epsilon (n_\gamma n_\gamma + h_{ij} n_i e_\gamma^i e_\gamma^j) \nabla_\beta A_\gamma e_\gamma^b = \epsilon \left( n_\gamma \nabla_\beta A_\gamma e_\gamma^b \right) n_\alpha + h_{ij} \left( \nabla_\beta A_\gamma e_\gamma^b e_\alpha^i \right), \quad (C.13)$$

Note that the first term can be rewritten using the fact that $A_\gamma n_\gamma = 0$, as we assume that the tensor $A_\alpha$ is completely tangential. This allows us to rewrite the previous expression as:

$$\cdots = \nabla_b A_i e_a^i - \epsilon A_i \left( \nabla_\beta n_\gamma e_\gamma^i e_\gamma^b \right) n_\alpha, \quad (C.14)$$

where we have defined the symmetric **extrinsic** curvature of the hypersurface $\Sigma$ or second fundamental form of the hypersurface as:

$$K_{ab} = \nabla_\beta n_a e_a^\alpha e_b^\beta. \quad (C.15)$$
with trace computed after contraction against the induced metric $h_{ab}$

$$K = K^{ab} h_{ab} = \nabla_a n^a. \quad (C.16)$$

Note that the starting point in Eqs. (C.12) and (C.13) is the same; the covariant derivative of the tangent form $A^a$ lives on $\Sigma$. Eq. (C.14) shows a pure tangential piece of the vector field (the first term) and its normal component (the second term). This piece carries geometrical information about how the hypersurface $\Sigma$ is embedded in the hosting space $\mathcal{M}$ and hence, what kind of curvature it acquires. This term can only be zero if and only if the extrinsic curvature vanishes.

**Gauss-Codazzi Equations**

The next logical step in this discussion is to explore if the intrinsic Riemann tensor of the hypersurface $\Sigma$ can also be expressed in terms of extrinsic information. Let us first recall the definition of a purely intrinsic curvature tensor as:

$$[\nabla_a, \nabla_b] A^c = R^c_{dab} A^d. \quad (C.17)$$

In the same spirit as in the previous computations, one can use the identities relating normal and tangent tensors in order to relate both extrinsic and intrinsic curvature tensors. This requires a modest amount of algebra and we refer the curious reader to [229]. Here we will only show the final result of the computation,

$$R = R_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} + \epsilon (K_{ad} K_{bc} - K_{ac} K_{bd}),$$

$$R_{\mu\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c n^\mu = \nabla_c K_{ab} - \nabla_c K_{ac}. \quad (C.18)$$

These are known as the **Gauss-Codazzi** equations. They show that some components of the curvature tensor of any geometry in $\mathcal{M}$ can be decomposed into terms of the intrinsic and extrinsic curvature pieces of the hypersurface it may be hosting.

Although the Riemann tensor, in any of its forms, contains valuable information about the geometry it represents, more practical tensorial objects will be found in everyday physic computations. Let us then find expressions for both the Ricci tensor and scalar given the metric decomposition described in Eq. (C.10). For the Ricci tensor we find:

$$R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta}$$

$$= \left( \epsilon n^\mu n^\nu + h^{mn} e^\mu_m e^\nu_n \right) R_{\mu\alpha\nu\beta}$$

$$= \epsilon R_{\mu\alpha\nu\beta} n^\mu n^\nu + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\nu_n. \quad (C.19)$$
while the Ricci scalar gives:

\[
R = g^{\alpha\beta} R_{\alpha\beta} \\
= (\epsilon n^\alpha n^\beta + h^{ab} e^a_\alpha e^\beta_b) \left( \epsilon R_{\mu\nu\alpha\beta} n^\mu n^\nu + h^{mn} R_{\mu\nu\alpha\beta} e^\mu_m e^\nu_n \right) \\
= 2\epsilon h^{ab} R_{\mu\nu\alpha\beta} n^\mu e^a_\alpha e^\beta_b + h^{mn} h^{\mu\nu} R_{\alpha\beta} e^\mu_m e^\nu_n e^a_\alpha e^\beta_b.
\]

(C.20)

It can be useful to make good use of relations (C.15) and (C.17) to further simplify the Ricci scalar expression (C.20). Some minutes of patience and algebra yield:

\[
R = (D-1) R + \epsilon \left( K^2 - K^{ab} K_{ab} \right) + 2\epsilon \nabla_\alpha \left( n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta \right).
\]

(C.21)

This expression is the evaluation of the \(D\)-dimensional Ricci scalar on the \((D-1)\)-dimensional hypersurface \(\Sigma\). This result is extremely practical in the context of branes and hypersurfaces, especially when the action governing their dynamics requires to be split, as discussed in chapter 3.

Junction conditions

The previous pages have been devoted to studying how the embedding of a co-dimension one hypersurface \(\Sigma\) can be used to provide a splitting of the hosting manifold \(\mathcal{M}\) into its tangential and normal components. However, the physical concepts of such picture have not yet received our attention. For example, one can find the following situation in physics: Suppose that such a hypersurface \(\Sigma\) divides a spacetime geometry into two regions \(V^+\) and \(V^-\). Both regions have different metrics \(g^\pm_{\alpha\beta}\). Furthermore, they are both solutions to the Einstein field equations. What conditions should be put on the metrics to ensure that both spaces join smoothly at \(\Sigma\), so that the whole union of spaces becomes a solution to the Einstein equation? This set of requirements demanded on the geometric features of the spaces \(\mathcal{M}^\pm\) and \(\Sigma\) are called Junction conditions. They were originally discussed in papers like [215, 234, 235].

Let us first imagine two \(D\)-dimensional manifold \(\mathcal{M}^\pm\), described by different sets of coordinates \(\{x^a_{\pm}\}\) and equipped with metrics \(g^\pm_{\alpha\beta}\). Furthermore, both spaces share a boundary \(\partial\mathcal{M}\), which is a hypersurface \(\Sigma\) of co-dimension one described by a set of coordinates \(\{y^a\}\). On top of all this, we assume that this composition space satisfies the \(D\)-dimensional Einstein equation. We can then try to define a general metric \(g_{\alpha\beta}\) that interpolates between two \(D\)-dimensional spaces. This can be written as:

\[
g_{\alpha\beta} = \Theta(\lambda) g^+_{\alpha\beta} + \Theta(-\lambda) g^-_{\alpha\beta},
\]

(C.22)
where $\Theta(\pm \lambda)$ is the Heaviside distribution function and $\lambda$ is an affine parameter describing geodesics connecting both regions, piercing through the hypersurface $\Sigma$. In this sense, when $\lambda > 0$, one could say to be placed in $\mathcal{M}^+$. Similarly, the minus sign represents a point of the geodesic in $\mathcal{M}^-$ and its null value is on the hypersurface $\Sigma$. Note that more complicated objects like the affine connection $\Gamma$ or the Riemann tensor depend on derivatives of the metric. One has to be careful then, as we are dealing with distribution functions and combinations of them can give rise to non-distribution terms, which makes it difficult to find a physical interpretation. In fact, one finds oneself in such a situation by simply the deriving expression (C.22) with respect to any coordinate $x^\gamma$. This yields:

$$\partial_\gamma g_{\alpha\beta} = \Theta(\lambda) \partial_\gamma g_{\alpha\beta}^+ + \Theta(-\lambda) \partial_\gamma g_{\alpha\beta}^- + \epsilon \delta(\lambda) n_\gamma \left[ g_{\alpha\beta}^+ - g_{\alpha\beta}^- \right],$$

(C.23)

where the last term comes from the derivative $\partial_\lambda \Theta(\lambda) = \delta(\lambda)$.\(^2\) Note that this term will yield contributions of the form $\Theta(\pm \lambda) \delta(\pm \lambda)$ when computing the Christoffel symbols. But such a combination of distributions is not one!\(^3\) It is therefore necessary to get rid of such a term. In order to achieve this task, let us impose continuity of the metric across the hypersurface $\Sigma$,

$$g_{\alpha\beta}^+|_\Sigma = g_{\alpha\beta}^-|_\Sigma.$$

(C.24)

This constraint can be further refined: The completeness relation (C.10) allows us to split between tangential and normal components of the continu-

\(^2\)The required change of variables in the derivative follows from the fact that any displacement away from the hypersurface along one of the geodesics described above is given by $dx^\mu = n^\mu d\lambda$.

\(^3\)Note that $\Theta(0) = \text{indeterminate}$, while $\delta(0) = 1$. What is this?
ity relation. Here we note that \([n^a]_+ = n^+ - n^- = 0\).\(^4\) Furthermore, coordinates \([y^a]\) are the same on both sides of the hypersurface \(\Sigma\). This implies that tangent vectors are uniquely defined on it. With these two facts, one can rewrite Eq. (C.24) as:

\[
h_{ab}^+ = h_{ab}^-.
\] (C.25)

This is called the **first junction condition**, which forces the induced metric \(h_{ab}\) to be the same on both sides of \(\Sigma\). This is an essential requirement for a well-defined geometry.

The derivation of the second junction can be done in different ways, with the previous mathematical approach requiring us to elaborate further along the lines discussed above.\(^5\) While this approach can be formal and elegant, we would need to introduce new concepts and complicated computations. Inspired by [230], a more physical approach will be presented in this section of the appendix, in line with the essence of this thesis.

We will first try to understand the geometry of a portion of the composition space presented above; A single manifold \(\mathcal{M}\) and its boundary \(\partial \mathcal{M}\) equipped with the metrics \(g_{\mu\nu}\) and \(h_{ab}\), respectively. The action describing the geometry and content of this space can be given as:

\[
S_{\text{Total}} = S_{\mathcal{M}} + S_{\partial \mathcal{M}} + S_{\Sigma},
\] (C.26)

where \(S_{\mathcal{M}}\) is the usual Einstein-Hilbert action as:

\[
S_{\mathcal{M}} = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left( \frac{1}{2\kappa_D} (D) R + \mathcal{L}_{\mathcal{M}} \right),
\] (C.27)

where \((D) R\) the \(D\)-dimensional Ricci scalar and \(\mathcal{L}_{\mathcal{M}}\) is the Lagrangian density for any kind of matter content in such space.

The action term \(S_{\partial \mathcal{M}}\) in Eq. (C.26) is the Gibbons-Hawking-York term \([82, 83]\) and is necessary for the proper definition of the variational principle since the Ricci scalar \(R\) is constructed from second derivatives of the metric. It describes how the submanifold is embedded as a boundary of \(\mathcal{M}\). Hence, it can be expressed in terms of the extrinsic geometric pieces as:

\[
S_{\partial \mathcal{M}} = \varepsilon \int_{\partial \mathcal{M}} d^{D-1} y \sqrt{|h|} K,
\] (C.28)

with the induced metric \(h_{ab}\) and the trace of the extrinsic curvature \(K_{ab}\) as described in (C.9) and (C.15). Note the presence of the chosen normalisation \(\varepsilon\) of the normal vector \(n^\alpha\). Finally, the term \(S_{\Sigma}\) represents any type of matter content living on the boundary.

\(^4\)This requirement follows from footnote 2 plus the continuity of \(\lambda\) and \(x^\alpha\) across the hypersurface.

\(^5\)These lines can be found in [229].
But this discussion is so far only valid for one manifold and its boundary. As described above, we will usually face situations where we find that the hypersurface $\Sigma$ is the boundary of two different manifolds $\mathcal{M}^\pm$. Hence, we have to "duplicate" the previous action (C.26) and glue them together, along the boundary $\partial \mathcal{M}$ mediating between $\mathcal{M}^-$ and $\mathcal{M}^+$. This is assumed to be as the only boundary present in the construction.

Let us now derive the junction condition by applying the variational principle with respect to $g_{\mu\nu}$. In order to simplify this task and present the results in the tidiest way, we will carry out this computation piece by piece in the action. For each manifold $\mathcal{M}^\pm$ we find:

$$\delta S_{\mathcal{M}^\pm} = \frac{1}{2\kappa_D} \int_{\mathcal{M}^\pm} d^D x \sqrt{|g|} \left[ \left( g_{\mu\nu}^{\pm} - \kappa_D T_{\mu\nu}^{\pm} \right) \delta g^{\mu\nu} + \nabla_{\mu} \left( g_{\alpha\beta} \nabla^{\mu} \delta g^{\alpha\beta} - \nabla^\alpha \delta g^{\alpha\mu} \right) \right],$$

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to any matter content in the $D$-dimensional spaces

$$T_{\mu\nu} = \mathcal{L}_{\mathcal{M}} g_{\mu\nu} - 2 \frac{\delta \mathcal{L}_{\mathcal{M}}}{\delta g^{\mu\nu}}.$$  

The boundary term yields a variation of the form:

$$\delta \partial \mathcal{M}^\pm = \frac{\epsilon}{2\kappa_D} \int_{\partial \mathcal{M}^\pm} d^{D-1} y \sqrt{|h|} \left[ \left( K_{\mu\nu}^{\pm} - K_{\mu\nu}^{\pm} g_{\mu\nu} \right) \delta g^{\mu\nu} + n_{\mu} \left( \nabla_{\delta} g_{\alpha\mu} - g_{\alpha\beta} \nabla^{\mu} \delta g^{\alpha\beta} \right) \right].$$

Note that the second line of each expression cancel each other out by the Gauss-Stokes theorem [229].

With the expressions (C.29) and (C.31) at hand and the action piece $S_{\mathcal{L}_\Sigma}$ in Eq. (C.26) representing matter content on the wall, one can then compute the dynamics of the whole composite space. However, one must be careful, as the normal $n^\mu$ is chosen to point in the direction of increasing volume in the transverse directions, i.e. from $\mathcal{M}^-$ to $\mathcal{M}^+$. This implies a change of the sign\(^6\) for $n^\mu_\pm$. This will affect the definition of the extrinsic curvature $K_{\mu\nu}$, as it contains the normal vector $n^\mu$ inside (see Eq. (C.15) and figure C.2). Consequently, we have:

$$n^\mu = -n^\mu_+ = n^\mu_-.$$  

\(^6\)but not in the value of the norm $\epsilon$.  

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The whole composition space (C.26) then reads:

\[
\delta S_{\text{Total}} = \delta S_{\mathcal{M}+} + \delta S_{\mathcal{M}-} - \delta S_{\partial \mathcal{M}+} + \delta S_{\partial \mathcal{M}-} + \delta S_{\Sigma} = \\
= \frac{1}{2\kappa_D} \int_{\mathcal{M}+} d^D x \sqrt{|g|} (G_{\mu\nu}^+ - \kappa_D T_{\mu\nu}^+) \delta g^{\mu\nu} \\
+ \frac{1}{2\kappa_D} \int_{\mathcal{M}-} d^D x \sqrt{|g|} (G_{\mu\nu}^- - \kappa_D T_{\mu\nu}^-) \delta g^{\mu\nu} \\
- \frac{\epsilon}{2\kappa_D} \int_{\Sigma} d^{D-1} y \sqrt{|h|} \left( K^+ g_{\mu\nu}^- - K^- g_{\mu\nu}^+ \right) \delta g^{\mu\nu} \\
+ \frac{\epsilon}{2\kappa_D} \int_{\Sigma} d^{D-1} y \sqrt{|h|} \left( K^- g_{\mu\nu}^+ - K^+ g_{\mu\nu}^- \right) \delta g^{\mu\nu} \\
- \frac{1}{2\kappa_D} \int_{\Sigma} d^{D-1} y \sqrt{|h|} \kappa_D S_{\mu\nu} \delta g^{\mu\nu},
\]

where \( S_{\mu\nu} \) represents the energy-momentum tensor (C.30) for all matter content living on the hypersurface \( \Sigma \). We can conveniently identify both Einstein equations for each manifold \( \mathcal{M}_\pm \) in the first two lines of expression (C.33). Both spaces are independent solutions of the Einstein equation, so they will not contribute to the equation of motion. The last three lines of previous expression correspond to the boundary geometric contributions \( \partial M_\pm \) (where the orientation of the normal has already been taken into account) and any possible matter fields living on such hypersurface. For the whole expression (C.33) to be zero, it is required that these last three terms cancel, so

\[
\kappa_D S_{\mu\nu} = \epsilon \left[ \left( K^- g_{\mu\nu}^- - K^- g_{\mu\nu}^+ \right) - \left( K^+ g_{\mu\nu}^+ - K^+ g_{\mu\nu}^- \right) \right]
\]

(C.34)

A small massage and projection down to tangential components, as these are tangential tensors (C.11), we finally find:

\[
\kappa_D S_{ab} = \epsilon \left( [K_{ab}]_+^+ - h_{ab} [K]_+^- \right),
\]

(C.35)

with \([A]^+_- = A^+ - A^-\). This is the (second) junction condition. Equation (C.35) states that the presence of a localised energy-momentum tensor \( S_{ab} \) on the hypersurface will source a jump discontinuity in the extrinsic curvature. Alternatively, one can read it backwards: a \( D-1 \)-dimensional hypersurface \( \Sigma \) acting as the boundary of two different \( D \)-dimensional manifolds \( \mathcal{M}_\pm \), which are independent solutions to the Einstein equation, must be equipped with an energy-momentum tensor \( S_{ab} \), proportional to the jump in the extrinsic curvature of the embedding, such that the whole composite space is also solution to the Einstein equation.
Appendix D.
Instantons

We will review two main types of instantons\(^1\) in this appendix: Coleman-de Lucia [191] and Brown-Teitelboim [190] ones. These types of solutions are constitutional foundations of the dark bubble model proposed in part III of this work.

Coleman-De Lucia instantons

Let us consider a theory with a single scalar field \(\phi(t, \bar{x})\) which is controlled by a potential \(V(\phi)\) in a \(D\)-dimensional spacetime. We further assume that this potential has \(\text{two} \) non-degenerate minima \(V_\pm\), such that \(V_+(\phi) > V_-(\phi)\). The shape of this potential is sketched in figure D.1. The action governing the dynamics of such configuration is:

\[
S = \int d^Dx \left( \frac{1}{2} \partial_M \phi \partial_M \phi - V(\phi) \right),
\]

where \(M\) goes from 0 to \(D - 1\). From a classical point of view, if the scalar field \(\phi\) is at rest in the local minimum \(V_+\), it will not have enough energy to climb over the potential barrier and reach the global minimum \(V_-\). However, if quantum mechanics is taken into account, there is a probability that the field will "tunnel" through the barrier and end up in the global minimum \(V_-\). In this framework, we will say that the local minimum \(V_+\) is metastable and a decay can take place, so that the field configuration "traverses" from the false vacuum \(V_+\) to the true vacuum \(V_-\).

The process described above was studied by Sidney Coleman and Curtis Callan in [236, 237]. The underlying physics of these decays is a first order transition\(^2\) and it can be easily understood by a thermodynamic analogy.

Consider a fluid, that is heated homogenously up to the point where it can start boiling. In its initial liquid phase, as much as you can try to have an even distribution of the temperature, there may be thermodynamic fluctuations at some given points where the temperature is slightly different from

\(^1\)Instantons are solutions to the classical Euclidean equations of motion which interpolate between real classical motions of the system, and thus provide a semiclassical "path" by which the system tunnels from one classical regime to the other.

\(^2\)i.e. the amount of energy absorbed or released by the system is fixed.
the surrounding volume. If this variation is favourable towards the phase transition temperature, a bubble of vapour phase will appear. This bubble can have two different fates: If its size is smaller that a certain threshold, so that the gain in energy density (i.e. the energy stored inside the bubble volume) is overcompensated by the loss of surface energy, then the bubble will collapse to nothing. On the contrary, a large enough bubble will have a favourable energy balance, which will cause the bubble to expand until all the liquid has undergone the phase transition to vapour.

This is a similar situation to the vacuum decay described some paragraphs above. We are now faced with a field decorating an empty space-time. The configuration of the field is such that it starts in its false vacuum. In this case, not thermodynamical, but quantum fluctuations of the vacuum can occur, triggering the phase transition towards the true configuration in some specific region. This will happen through the nucleation of a spherically symmetric bubble of true vacuum.

![Figure D.1](image)

*Figure D.1.* (Left): The potential $V(\phi)$ with its local and global minima. (Right): The inverted potential $U(\phi)$. This allows us to think in terms of the motion of a particle. Starting from $\phi_+$, we then see that it can roll up to $\phi_I$ and bounce back. Hence the name for this type of solutions.

The probability per unit of volume $\Gamma$ that such an event will occur can be determined in the semi-classical approximation $\hbar \ll 1$ and it is given by:

$$\Gamma \sim e^{-B} \left( 1 + \mathcal{O}(\hbar) \right),$$

with $B = S_E(\phi_I) - S_E(\phi_-)$. The term $S_E(\phi_-)$ is the Euclidean action of the system evaluated in the false vacuum while $S_E(\phi_I)$ is the Euclidean action for the bounce solution. The bounce is the instanton solution corresponding to the nucleated bubble, i.e. solutions that extremise and give a finite value to the Euclidean version of the action that governs the dynamics of the system. Let us discuss this quantity in more detail.

---

3This can be deduced from the transition matrix element between the ground state $|0\rangle$ evolving in time. When $\hbar$-corrections are taken into account, the associated eigen-energy of the process is $E = \frac{1}{2} \omega \hbar + hK \omega^{-B/\hbar}$, where $K \in \mathbb{C}$. Identifying the corrections with the non-Hermitian piece of the evolution operator $\mathcal{H}$ of the system, one can then relate the imaginary part of the energy $E$ with the decay channel $\Gamma$. 

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The bounce

In order to easily understand the bounce, we are going to work on three specific regimes that are the implicit ones in this work:

- The first one, by analytic continuation, we will perform all our computations in the Euclidean realm ($\tau = it$).

- The second one will affect the shape of the potential $V(\phi)$, as we will be working in the thin wall approximation. This is the case when the difference between the energy density of the two vacua is really small, so that one writes $|\Delta V(\phi)| = \varepsilon$ with $|\varepsilon| \ll 1$.

- The last regime will help us to think in terms of particle dynamics. If we invert the potential, i.e. $U(\phi) = -V(\phi)$, such as shown in figure D.1, we can then use a motion analogy. In this case, a particle sitting on the "lower" hill could start rolling down, up to the point of the potential where its energy will be equal to that of the starting position. As we know from high school, this is the point that the particle will reach with zero velocity and would then bounce back down the valley. Restoring the sign, we then see the meaning of that bouncing point: it would be the final position at which the tunneling process ends. We will call this field position as $\phi_I$, which is a solution of the equation of motion and extremise the value of the Euclidean action.

Given the previous assumptions and exploiting the implicit $O(D)$ symmetry of the Euclidean space of study, so that we can rewrite all coordinates $(\tau, \vec{x})$ in terms of a "radial" coordinate $r = \sqrt{\tau^2 + x_i x^i}$, it can be shown that the Euclidean version of action (D.1) is

$$S_{E}^{\text{total}} = \int d^D x \left( \frac{1}{2} \partial_M \phi \partial^M \phi + U(\phi) \right) = \Omega_{D-1} \int_{R}^{\infty} dr r^{D-1} \left[ \left( \frac{d\phi_+}{dr} \right)^2 + U_+(\phi) \right] + \Omega_{D-1} R^{D-1} \int dr \left[ \left( \frac{d\phi_I}{dr} \right)^2 + U_+(\phi_I) \right] + \Omega_{D-1} \int_{0}^{R} dr r^{D-1} \left[ \left( \frac{d\phi_-}{dr} \right)^2 + U_-(\phi) \right],$$

where $\Omega_{D-1}$ is the area of a unit-radius $(D-1)$-sphere given by:

$$\Omega_{D-1} = \frac{2 \pi^{D/2}}{\Gamma(D/2)}. \quad \text{(D.4)}$$

Note that we have divided the integration regime into three different intervals:

- That corresponding to the outside of the bubble (first line in Eq. (D.3)). The phase transition has not yet reach to this region, so the field and associated potential $U(\phi)$ correspond to the false configuration.
• The instanton solution, which corresponds to the location of the bubble (second line). We will call this piece \( S_1 \), and it represents the Euclidean action for the bounce solution.

• The inside volume enclosed by it (last line). The decay has already taken place here, so we write \( \phi_- \) and \( U(\phi) \).

Let us now specify our computations for a potential such that \( U(\phi_+) = 0 \) and \( U(\phi_-) \approx -\epsilon \). Plugging in these considerations into the action (D.3), we will obtain:

\[
S_{E}^{\text{total}} = S_{E}(\phi_I) - S_{E}(\phi_-) = \Omega_{D-1} \left( R^{D-1} S_1 - \frac{R^D}{D} \epsilon \right),
\]

which corresponds to the expression in the argument of Eq. (D.2). If we now derive the action with respect to \( R \), we will find the radius \( R \) that extremises the value of the action. This yields:

\[
B = \frac{\Omega_{D-1}}{D} \left( \frac{D-1}{\epsilon} \right)^{D-1} S_1^D. \tag{D.6}
\]

In this way we have found the closed-form expression for the coefficient \( B \) in the thin-wall approximation. The explicit value of this term will depend on the potential shape integrand in \( S_1 \). In any case, this value ensures that the bubble nucleation probability is maximal.

When gravity wants to blow bubbles

The previous discussion was made without considering the role of gravity. When gravity is not present, any false vacuum in quantum field theory can decay. However, this is not the case when the theory is coupled to gravity. This was later studied by Coleman and de Luccia in [191], who showed that gravity can stabilise some false vacua, making them persistent.\(^4\) Let us elaborate on this point:

The bubble, being an instanton solution, is formed at a radius \( R \) such that it minimises the Euclidean action. The formation of such a bubble has an energy cost, which is proportional to the bubble tension \( \sigma \) times the area of the bubble. But where does this energy come from? By conservation of energy, it must have come from reducing the vacuum energy inside its interior. This amount of energy is equal to the vacuum energy density times the

\(^4\)It can be shown that the bounce solution gets modified to by gravity as:

\[
B_{\text{gravity}} = \frac{B}{(1 + (R_0/2 \Lambda_D))^2}, \tag{D.7}
\]

where \( R_0 \) is the radius that minimises the action (D.5) and \( \Lambda_D \) is the vacuum energy of the \( D \)-dimensional spacetime, i.e. a cosmological constant.
volume enclosed by the bubble. This relationship between the energy contributions can be seen in Eq. (D.5). If the balance is precise so that one term compensates for the other, the bubble will be nucleated with the radius that minimises the action and remain at rest. In the case the bubble gains more energy than is needed to create it by reducing the vacuum energy inside it, the extra energy will provide the kinetics to expand the bubble. The bubble, which will nucleate at rest, is then accelerated outwards, asymptotically approaching the speed of light. This whole process can be understood by the expression

$$|V_{after} - V_{before}| = E_{wall} + E_{kinetic},$$

where $E_{kinetic}$ is the kinetic energy of the bubble if more energy was obtained by reducing the energy vacuum density inside. Note that when this term is zero, the tension $\sigma$, i.e. the "mass" of the bubble at rest is proportional to difference in vacuum energy discussed above.

**Brown-Teitelboim instantons**

The previous discussion only applies if the decay process occurs once. One can then wonder whether a tower of decays can be achieved within the precedent formalism. This was done by Brown and Teitelboim in [190], where they proved that one can have a tower of vacuum decays when antisymmetric tensor fields are considered in the bubble action.

Let us now suppose a vacuum supported by a cosmological constant $\Lambda_D$ which can be lowered in the presence of antisymmetric tensor field strength $F = dA$, where $A$ is the corresponding tensor field. In this case, the nucleated membrane will be charged under this field $A$. The Euclidean action to

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**Figure D.2.** From left to right: Bubble nucleation requires a delicate balance between the surface energy (represented by the dashed line) and the vacuum energy difference (the bar under each rectangle). When the values are exactly equal to compensate each other, a bubble will nucleate at rest and remain there. However, if more energy is extracted from the decay than is required to form the bubble, the extra energy will be kinetic and used to expand the bubble.
describing this type of instantons is given by:

\[
S_E = -\frac{1}{2\kappa_D}\int d^Dx\sqrt{|g|}(^{(D)}R - 2\Lambda_D) + \sigma\int d^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int d^{D-1}x\sqrt{|g|}\nabla M 
\]

\[
+ \frac{1}{(D-1)!}\int d^Dx\sqrt{|g|}\nabla M(F_{M\ldots A\ldots}) - \frac{1}{2D!}\int d^Dx\sqrt{|g|}|F|^2
\]

\[
+ \frac{q}{(D-1)!}\int d^{D-1}yA_{M_1\ldots M_{D-1}}e_{m_1}^{M_1}\cdots e_{m_{D-1}}^{M_{D-1}}e_{m_1}^{m_1}\cdots e_{m_{D-1}}^{m_{D-1}},
\]

with \(\epsilon\) is the Levi-Civitta symbol. As already discussed in appendix C, Eq. (C.26), the first line represents the gravitational contribution of the whole space. In this case, the energy-momentum tensor of all matter living on the hypersurface (i.e. the membrane), is its tension \(\sigma\). The second line \((D.9)\) represents the presence of the field strength in the \(D\)-dimensional space and a boundary term for it, while the third line denotes the coupling of the brane to the tensor field \(A\).

As \(F\) is a form of top degree, this means that it is proportional to the volume form:

\[
F_{M_1\ldots M_D} = E(x)e_{M_1\ldots M_D},
\]

for some scalar function \(E(x)\), i.e. the electric field. This can be shown to be constant away from the brane sources by examining the equation of motion for the tensor field \(A\):

\[
\partial_L E(x)e^{LM_1\ldots M_{D-1}} = -q\int d^{D-1}y\delta(x^a - x^a(y))e_{m_1}^{M_1}\cdots e_{m_{D-1}}^{M_{D-1}}e_{m_1}^{m_1}\cdots e_{m_{D-1}}^{m_{D-1}}.
\]

This implies that the electric field \(E(x)\) will jump one charge unit \(q\) across the brane.\(^5\) This jump will be an additional contribution to lowering the vacuum energy via brane nucleation. Indeed, if one inserts Eq. \((D.10)\) in the action \((D.9)\) one can read off an effective cosmological constant:

\[
\Lambda_{\text{eff}} = \Lambda_D + \frac{1}{2\kappa_D}E^2.
\]

Thus, when the equation of motion \((D.11)\) holds, we can rewrite the action \((D.9)\) as:

\[
S_E = -\frac{1}{2\kappa_D}\int d^Dx\sqrt{|g|}(^{(D)}R - 2\Lambda_D) + \frac{1}{2D!}\int d^Dx\sqrt{|g|}|F|^2
\]

\[
+ \sigma\int d^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int d^{D-1}x\sqrt{|g|}K,
\]

Let us now discuss the decay channel \((D.2)\) associated with the nucleation of a single bubble of true vacuum\(^6\) when the tensor field \(A\) permeates

\(^5\)Due to the conservation of charge, this charge difference will be carried by on the bubble’s boundary, i.e. the (mem)brane.

\(^6\)As the decay can occur repeatedly, we should speak of a less false vacuum. Let us stick to true to avoid confusion.
the vacuum. Here we will just highlight the most important results of [190]. The curious reader interested in the detailed steps of the derivation is referred to that paper.

It can be shown that the bounce of the action (D.13) is given by:

\[
B = \sigma \Omega_{D-1} R^{D-1} + \frac{1}{\kappa_D} \left[ \left( \frac{2\Lambda_i}{(D-2)} \right) \text{Vol}_D(R, \epsilon_i, \Lambda_i) + (D-1)\epsilon_i \sqrt{\frac{1}{R^2} - \frac{2\Lambda_i}{(D-2)(D-1)} \Omega_{D-1} R^{D-1}} \right],
\]

where we have dropped the "eff" superscript of the effective cosmological constant \( \Lambda \) and \( \epsilon_i \) represents the orientation choice for the normal \( n_\mu \) of the hypersurface \( \Sigma \), i.e. the membrane, as discussed in appendix C. The area covered by a unit-radius \((D-1)\)-dimensional Euclidean membrane is \( \Omega_{D-1} \) and is given by:

\[
\Omega_{D-1} = \int d^{D-1} \xi \sqrt{(D-1) g} = \frac{2\pi^{D/2}}{\Gamma(D/2)},
\]

where \( \Gamma(x) \) is the usual gamma function. The volume of the inside and the complement of the outside can be computed by:

\[
V_D(R, \epsilon_i, \Lambda_i) = \int_{\pm} d^D x \sqrt{g} = \left( \frac{(D-1)(D-2)}{2|\Lambda_i|} \right)^{D/2} \Omega_{D-1} \times \left| \int_{1} \left[ \sigma_i[1-2\Lambda_i R^2/(D-2)(D-1)]^{1/2} \right] \right| \quad \text{d}(\cos x) \sin^{D-2} x.
\]

In the case that the effective cosmological constant is \( \Lambda < 0 \), the trigonometric functions in Eq. (D.16) should be replaced by hyperbolic trigonometric functions.

In the same spirit as in the case of Coleman-De Lucia instantons, one can find the value of \( R \) that extremises the action \( B \) by deriving it with respect to \( R \). Substituting the resulting \( R_0 \) back into \( B \) gives the extremised nucleation probability of a Brown-Teitelboim bubble in \( D \) dimensions.

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7 This is the volume fraction of the background that is converted into the inside region when the bubble is created, i.e. the dashed volume in figure D.2.
12. Svensk Sammanfattning


Det finns två olika tekniker för att ”gömma” extra dimensionerna: branvärldar eller kompaktifieringar.


Den mörka bubblan är en modell som har goda förutsättningar att förklara den mörka energin med hjälp av strängteori. Dessutom verkar det som att man också kan förstå hur partikelfysikens alla krafter och partiklar kan ses ur ett högredimensionellt perspektiv. Vill du komma och rida en mörk bubbla tillsammans med oss?
13. So Long, and Thanks for All the Fish

There may be other methods for finding oneself, […] but I know of none better than that of reflecting upon one's educators and cultivators.

Friedrich Nietzsche, Schopenhauer as Educator

I sometimes prefer to ask for directions when I have to find some address in a city, rather than checking the maps app. In this way, non-exempt of imprecision, one can get to know subtle details that a map will never tell you beyond the desired location. There are things that can only be learnt from other people. After directions, one must always be grateful to the person who helped. Double grateful when one can safely stand in front of the desired destination, thanks to the directions of that person. One is "lost", go to someone, ask for directions, appreciate them, continue their way and reach destination, where new "paths" to choose from will reveal. Is this not the way we grow up? Following Nietzsche's recommendations, I would like to thank those ones that guided and helped me in my path in this city, almost a maze, that life is.

Sin duda alguna diré que Ana¹ y Pilar² son las que más, entre todos los "profes" del Rayuela, me ayudaron a poner mis primeros pilares de pensamiento crítico, orden y clarividencia a la hora aprender.³ He intentado interiorizar, mejorar y llevar por bandera sus recomendaciones sobre las bondades de una narrativa buena y ordenada a la hora de resolver problemas, bien sazonada con la necesaria "panchanguera" para entender que es lo que quiero resolver y como. Aunque claro, es difícil investigar sólo con esos trucos si uno no tiene ni pajolera de idea de Inglés. Tuve la suerte de contar con Susana y Sole, quienes con paciencia y tesón pudieron enseñarme a manejarme con tal idioma. ¿Dónde habría acabado sin ellas? Y para aprender, repetitio est mater studiorum,⁴ vamos, que uno necesita repetir y repetir para interiorizar. Y repetir require de diligencia y disciplina. Ahí entra mi buen Shifu⁵ Bruno, quién me enseñó que el kung fu es mucho

¹a.k.a. la "Picot".
²a.k.a. la Tanque.
³Bueno, sin contar a la señorita Charo, Maribel, Apro, Miguel Ángel, Vicky, Amelia, Antolín, Pulido, Rafa, etc. pero leñe tengo que poner ciertas cotas, que si no esto se hace eterno.
⁴La repetición es la madre del aprendizaje.
⁵Maestro en Chino.
más que tener equilibrio y golpear fuerte. Es entrega. Disciplina. Practicar siempre, sin excusas. Repetir, hasta alcanzar la perfección, que nunca llega. Es el compromiso para realizar el trabajo con la mejor voluntad posible y total concentración en la tarea a realizar.

Gracias al bueno de Rafa, quien nunca perdió la fe en mí como físico, aunque deje mucho que desear, pude seguir creciendo como teórico. ¿Qué quieres hacer un Máster en física teórica? Tira pa’ Uppsala, me dijo. Allí hace mucho frío y mal tiempo, lo que te "motivará" a pasar más tiempo estudiando en la biblioteca. Y para Suecia que fui. A new country and society to discover. I will always be grateful to Giussepe, who guided me in my first steps in the string cosmology field. Although I always messed up with my computations, he always had the patience to lead me in the right direction and to hang in there while facing hard computations. His perfect explanations and handwriting motivated me to enhance mine, so the process of learning could become easier and more satisfactory.

I would not be writing these lines if Magdalena had not pointed me to the CIM Ph.D. position I am about to complete. Neither would I be if Jordi had not allowed me to present my own Ph.D. proposal, nor Tobias or Giorgios had not supported me. Thanks to you all I could start doing research in what I always wanted: String cosmology.


But Ulf was not the only one that guided me and that I learnt from. Dear Thomas, you are the embodiment of what I consider the perfect balance between concept and computations. Thanks for your patience, your support and I hope we can do research together again soon. And Rob, although you left academia, it was really nice to share miseries (a.k.a. to compute together while both our supervisors were having fun together), as Thomas said, with you. I wish you the very best within the finance world. Oscar, it was amazing to have you by our side to embed the dark bubble into string

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6No fue tanto el frío como el sistema educativo para el aprendizaje "a tu ritmo" que me encontré en Uppsala lo que me motivó a estudiar, la verdad.
theory. And thanks also for inviting me to Helsinki! It is a city that is always nice to come back! I must be grateful to have done research together with you too, Ivano. It was an honor to join forces with such promising researcher. I hope we can continue enhancing the dark bubble together! And this also applies to you, Suvendu! I will always have problems to follow your amazing speed when reasoning, but I also know you will wait for my slow mind to catch up. Gentlemen, the dark bubble does not wait and requires some Standard model of particles to embed in. Shall we?

This five year Ph.D. would have been way more difficult sans mon bon ami, Ramón. Tu as été le meilleur compagnion de bureau et de fika au Café Liné. Tu m’as aussi enseigné beaucoup de mathématiques, à propos des frisbees, et ensemble nous avons vénéré notre Saint favori: Eddy. Je veux que tu continues cette tradition quand je serai plus là. Et tu n’échapperas pas à ma question "Qu’est-ce que tu vas faire ce weekend". Tu n’échapperas jamais à cette tradition. Je te souhaite une très bonne dernière année de Ph.D. et la meilleur recherche de postdoc possible. And remember that those good lunch discussions you and I had were enhanced by the good quality food we enjoyed. Our friend and chef Reda was always ready to help us start our weekend with the best possible food and service. Keep it being the way you are, man, without caring about what they may think of you!

Although close to half of the time has been spent inside Ångström during these years, there have always been good people guiding me through different paths, non-related to physics, outside this building. Vassilis, I am thankful to you for teaching me that Bach is just metronome and the three p’s: practice, practice and practice. You have been the best guitar teacher one could dream of. Thanks for your hospitality, your patience, our long discussions on how to play and interpret music. I am grateful you taught me to love the best possible string theory that exists; that of playing the six ones a guitar has. Cecilia, jag kan svenska tack vare dig. Du är den bästa sven-skälararen, inte bara för att du lär ut språket, men också svordomar (tyvärr är spanska fortfarande bäst på svordomar) och du lärde mig om den svenska kulturen och samhället. Till sist, mens sana in corpore sano! Martin. Amanda. Lennart. UHFS. Det har varit jättekul att lära mig att fäktas med er, och jag älskade våra diskussioner om hur man kan ta de bästa stegen för att hålla balans eller att tolka tekniker och använda dem på bästa sätt. Jag

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vet inte om jag kommer lämna Sverige, men räkna med mitt svärd och några snabba zwerchhau\textsuperscript{8} om jag stannar här.

Pero al final, Como el turrón, uno ha de volver a casa, porque es ahí donde se encuentran los mejores guías. No encuentro las palabras adecuadas para poder expresar la gratitud hacia todo lo que mis señores padres nos han brindado a mi hermana y a mí. Desde el gusto por la lectura, procurando que nunca faltase un buen libro que leer hasta no dejar que cayeramos en antojos banales, enseñándonos que un NO ante los caprichos es la mejor manera de mirar de tú a tú a la responsabilidad. Todo ello sin olvidar que la libertad y apoyo para elegir nuestros propios caminos, acompañados de la paciencia y fe infinitas que solo una Madre y un Padre pueden tener ante un hijo tan terco y cabezota como servidor, son aptitudes y actitudes que han de ser valoradas por encima de cualquier riqueza material. Y por último, mas no menos importante, quisiera reconocer y agradecer todo el apoyo de mi tía Paloma, artífice y razón de que pudiera continuar mis estudios en estas tierras. Gracias a tu sustento he podido crecer y cultivarme como físico teórico. Gracias por estar siempre ahí, atenta para que no me faltase nada y por todas esas fotos de \textit{Castropodame}, que a modo de pequeños recordatorios, han sido la manera más grande para no olvidar el mejor lugar al que siempre volver.

\textsuperscript{8}Esa velocidad, fuerza y destreza no la podría haber alcanzado sin la guía de mi entrenador personal, también primoh, Don Antonio de Lucas Benito. Gracias por haberme guiado para estar atlético y sano.
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