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Defect groups of class $\mathcal{S}$ theories from the Coulomb branch

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Abstract: We study the global forms of class $\mathcal{S}[A_{N-1}]$ 4d $\mathcal{N}=2$ theories by deriving their defect groups (charges of line operators up to screening by local operators) from Coulomb branch data. Specifically, we employ an explicit construction of the BPS quiver for the case of full regular punctures to show that the defect group is $(\mathbb{Z}_N)^{2g}$, where g is the genus of the associated Riemann surface. This determines a sector of surface operators in the 5d symmetry TFT. We show how these can also be identified from dimensional reduction of M-theory. We discuss connections to the theory of cluster algebras.

Keywords: Global Symmetries, Topological Field Theories, M-Theory, Wilson, 't Hooft and Polyakov loops

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1 Introduction and overview

In exploring properties of quantum field theories that cannot be accessed through perturbative methods, symmetry is one of the precious few footholds available. In particular, anomalies of global symmetries provide quantities that, by the classic anomaly matching argument of [1], are invariant under renormalisation group flow.

A modern viewpoint on anomalies is in terms of invertible theories in one dimension higher, so that an anomalous theory lives on the boundary of an anomaly theory. Upon a background gauge transformation, the anomalous phases from the boundary and the bulk cancel, rendering the combined system anomaly-free; this is *anomaly inflow* [2–4]. In this framework, anomaly matching is the statement that anomaly theories are topological and therefore invariant under RG flow.

The concept of a symmetry, traditionally seen as a transformation on the fields that leaves the action and (up to an anomalous phase) partition function invariant, has in recent years been reexamined and generalised. The key observation is that the presence of a traditional symmetry is equivalent to the existence of operators/defects, supported on closed codimension-1 submanifolds of spacetime, and invariant under continuous deformations of those submanifolds. The group law of the symmetry is expressed in the fusion algebra of the corresponding defects. This perspective suggests the generalisation to p-form symmetries [5, 6], whose charged objects are p-dimensional extended operators acted on by codimension-$(p+1)$ symmetry operators. These may in addition mix with each other in higher group-like structures [7]. Another natural generalisation is to non-invertible symmetries [8–14].

A similar picture to that of anomaly inflow applies when one has several field theories with identical local dynamics, but different spectra of extended operators, that is, different

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global structures [15, 16]. The theories can then be viewed together as a *relative* field theory [17, 18] living on the boundary of a *non-invertible* TQFT, called the *symmetry TFT* (SymTFT) [5, 19–21], in one dimension higher. The set of all global forms is a property of the SymTFT itself, while a topological boundary condition picks out a boundary theory with a particular global structure (see figure 1 below). Indeed, the SymTFT and its topological boundary conditions can be studied abstractly, quite apart from any dynamical boundary theory, just as a group can be studied independently of its representations [20].

In the remainder of this section, we outline the structure of the paper, introduce the central concepts, summarise the main points and give some directions for future work.

[22] discussed a concrete example of the relation between global structures and the symmetry TFT, namely the case of a 4d QFT that has a Coulomb phase. We review it in some detail in section 2.

In this work, we apply the story of [22] to the case of the class $\mathcal{S}$ construction [23, 24]. We start from the 6d $\mathcal{N} = (2, 0)$ SCFT of type $\mathfrak{g}$ — a relative theory — and get a 4d $\mathcal{N} = 2$ supersymmetric relative theory by compactifying on a Riemann surface $\Sigma_{g,p}$ of genus g and with p punctures. We study the most straightforward case, where the punctures are all *regular* and *full* (see [25, chap. 12] for an elaboration on this) and $\mathfrak{g} = A_{N-1} = \mathfrak{su}(N)$. We call the 4d relative theory $\mathcal{S}[\mathfrak{su}(N), \Sigma_{g,p}]$. Concretely, we claim that the *defect group* [26] (the group $\mathbb{D}^{(1)}$ of 1-form symmetry charges of lines that appear in some global form) of this theory is

$$\mathbb{D}^{(1)}(\mathcal{S}[\mathfrak{su}(N), \Sigma_{g,p}]) \cong (\mathbb{Z}_N)^{2g}. \tag{1.1}$$

The defect group corresponds to the group of surface operators in the 5d SymTFT; it therefore allows us to determine the sector of the SymTFT that couples to the 1-form symmetry, as reviewed in section 2.

There are a priori good reasons to expect this defect group, and indeed special cases of the claim have appeared in earlier literature. [27] considered the case without punctures and (though the term *defect group* was not established) argued that

$$\mathbb{D}^{(1)}(\mathcal{S}[\mathfrak{g}, \Sigma_{g,0}]) \cong H^1(\Sigma_{g,0}; Z(G)) \tag{1.2}$$

where $Z(G)$ is the centre of the simply connected group G with algebra $\mathfrak{g}$; this agrees with (1.1). [28] proposed that regular untwisted punctures do not affect the defect group in general, so that in particular (1.2) holds also for $p > 0$. They verified this expectation in the cases $\mathcal{S}[\mathfrak{su}(N), \Sigma_{0,4}]$, $\mathcal{S}[\mathfrak{su}(2), \Sigma_{g,p}]$ and $\mathcal{S}[\mathfrak{su}(N), \Sigma_{1,\tilde{p}}]$, where $\Sigma_{1,\tilde{p}}$ denotes the torus with p *simple* punctures (see section 4.2), as well as many theories outside the scope of this paper. Our confirmation of (1.1) adds another class of examples to this list.

When $\mathfrak{g} = \mathfrak{su}(N)$ and there are no punctures, one can derive the defect group via the SymTFT that descends from the M-theory Chern-Simons term when realising the 6d theory as the worldvolume theory of a stack of M5-branes; we review this in section 3 following [29] and [10]. Furthermore, we add punctures using a geometric construction [30, 31] and argue that the defect group is unaffected as expected.

The main result of this work is presented in section 4. We make use of the known charge lattices of class $\mathcal{S}$ theories to verify (1.1) for small N, g and p by explicitly calculating the BPS quiver and defect group. This serves as a check of the paradigm from [22] that the

SymTFT can be accessed from Coulomb branch data. A similar analysis for 5d and 6d theories was carried out in [32]. In section 4.2, we do the same calculation for a class of theories where the punctures are not full. In section 4.3, we make some observations on the structure of the BPS quivers to motivate why (1.1) should hold.

There are several interesting avenues for generalisation and application of our methods in future work. Ideally, one would like a systematic, algorithmic construction of BPS quivers for class $\mathcal{S}$ theories with arbitrary simply laced gauge algebra $\mathfrak{g}$ and any configuration of regular punctures. Several proposals in this direction have been made [33, 34], but the constructions are complex and it is not clear to what extent they can be generalised. Here the work of [35] provides a promising starting point. Once one knows the BPS quiver of a theory, the natural application is to compute its BPS spectrum using the mutation method of [36]. In particular, this enables one to calculate the Schur index according to the conjecture of [37]; comparing this to the derivation of the index as a TQFT correlator on $\Sigma_{g,p}$ [38, 39] would be a good cross-check. The argument in section 3 on adding regular punctures is somewhat schematic; it would be useful to reproduce it at a higher level of rigour. Finally, filling in the proof sketch of section 4.3 would establish a result of general interest in the mathematics of cluster algebras.

Appendix A describes the details of the computer calculation using BPS quivers that confirms (1.1). Appendix B reviews some of the standard mathematical theorems used throughout the paper.

2 Defect group and SymTFT in the Coulomb phase

2.1 Wilson and 't Hooft lines in Maxwell theory

As preparation for the general case, let us consider four-dimensional Maxwell theory with a U(1) 1-form gauge field A. The field strength $F = \mathrm{d}A$ is a closed but not necessarily exact 2-form. We wish the Wilson loop $W_{\alpha_\mathrm{e}}(\gamma) = \mathrm{e}^{\mathrm{i}\alpha_\mathrm{e} \oint_\gamma A}$ of electric charge α_e to be well-defined. Consider deforming the loop from γ to γ' along a surface Σ with $\partial\Sigma = \gamma' - \gamma$; we find $W_{\alpha_\mathrm{e}}(\gamma') = \mathrm{e}^{\mathrm{i}\alpha_\mathrm{e} \int_\Sigma F} W_{\alpha_\mathrm{e}}(\gamma)$. In particular, from $\gamma' = \gamma$ we must have $\alpha_\mathrm{e} \oint_\Sigma F \in 2\pi\mathbb{Z}$ for all closed surfaces Σ. Next, we insert an 't Hooft loop $H_{\alpha_\mathrm{m}}(\ell)$, defined by sourcing a magnetic flux: $\oint_\Sigma \frac{F}{2\pi} = \alpha_\mathrm{m}$ whenever Σ links ℓ. In the presence of W_{α_e}, this is consistent with the previous condition when $\alpha_\mathrm{e}\alpha_\mathrm{m} \in \mathbb{Z}$; this is the Dirac quantisation condition. Thus the set of allowed Wilson lines constrain the allowed 't Hooft lines, and vice versa.

Let us now add massive dynamical states to the theory: a particle with electric charge q_e and a monopole with charge q_m. Integrating them out is equivalent to inserting Wilson and 't Hooft operators W_{q_e} and H_{q_m} supported on their worldlines in the IR path integral; thus these must be allowed line operators. In particular we have

$$q_\mathrm{e} q_\mathrm{m} \in \mathbb{Z}, \quad q_\mathrm{e}\alpha_\mathrm{m} \in \mathbb{Z}, \quad \alpha_\mathrm{e} q_\mathrm{m} \in \mathbb{Z}, \tag{2.1}$$

where α_e and α_m are the charges of any other allowed lines in the theory. Given the dynamical charges q_e and q_m, the set of solutions (α_e mod $q_\mathrm{e}, \alpha_\mathrm{m}$ mod q_m) to (2.1) up to screening by the dynamical states is called the *defect group*, in this case $\mathbb{Z}_{q_\mathrm{e}q_\mathrm{m}} \times \mathbb{Z}_{q_\mathrm{e}q_\mathrm{m}}$. Here the most general line is dyonic, with simultaneous electric and magnetic charges $(\alpha_\mathrm{e}, \alpha_\mathrm{m})$. The condition for two such lines to be well-defined in each others' presence is the Dirac-Schwinger-Zwanziger

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quantisation condition [40, 41] $\alpha_\mathrm{e}\alpha'_\mathrm{m} - \alpha_\mathrm{m}\alpha'_\mathrm{e} \in \mathbb{Z}$. A maximal subgroup of lines satisfying this will be (the Pontryagin dual of) the *1-form symmetry*. We will elaborate on this in the next section.

The reasoning so far holds regardless of the normalisation of the gauge field A. If the gauge transformations are $A \to A - \mathrm{d}\theta$ with the gauge parameter θ valued in $\mathbb{R}/2\pi\mathbb{Z}$, invariance of the Wilson line W_{α_e} under large gauge transformations requires $\alpha_\mathrm{e} \in \mathbb{Z}$. For convenience, however, we may rescale electric charges and the gauge field by $\frac{1}{q_\mathrm{e}}$ and magnetic charges by q_e without affecting the above discussion, ensuring that electric and magnetic charges of dynamical states are all integers, while lines may carry fractional electric charge. We will take this convention in the remainder of the paper.

2.2 Lines in the general case

In this section we review the main points of [22], in particular the notion of a defect group and its relation to the SymTFT. This generalises the arguments above in three respects: we allow for $r \geq 1$ gauge fields, we allow dyonic dynamical states as well as lines, and we also allow for flavour charge. For in-depth discussions of the various charge lattices, see [15, 16, 42].

We are interested in the 1-form symmetry of a four-dimensional $\mathrm{U}(1)^r$ gauge theory with charged massive dynamical states. This theory has generalised dyonic line operators L, carrying electric and magnetic charges $\boldsymbol{\alpha} = (\alpha_\mathrm{e}^1, \alpha_\mathrm{m}^1, \ldots, \alpha_\mathrm{e}^r, \alpha_\mathrm{m}^r)$ forming a lattice $\Gamma_L \subset \mathbb{R}^{2r}$. To be well-defined in the presence of charged states, they need to satisfy the quantisation condition $\langle \Gamma_L, \Gamma \rangle \subset \mathbb{Z}$, where

$$\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}' \rangle = \sum_{i=1}^{r} \left(\alpha_\mathrm{e}^i \alpha'^i_\mathrm{m} - \alpha_\mathrm{m}^i \alpha'^i_\mathrm{e} \right) \tag{2.2}$$

is the Dirac pairing and $\Gamma \subset \mathbb{Z}^{2r}$ is the lattice of dynamical charges, assumed to be of full rank. We denote this as $\Gamma_L \subset \Gamma^*$, where $\Gamma^* = \{\boldsymbol{\alpha} \in \mathbb{R}^{2r} \mid \langle \boldsymbol{\alpha}, \Gamma \rangle \subset \mathbb{Z}\}$ is the dual of Γ with respect to the Dirac pairing.

The worldlines of dynamical states form a subgroup $\Gamma \subset \Gamma_L$. As these lines can end on a local operator insertion, they cannot be charged under the one-form symmetry (the local operator is said to *screen* the line). The 1-form symmetry charges therefore take values in the *defect group* $\mathbb{D}^{(1)} = \Gamma^*/\Gamma$. Because the Dirac pairing is perfect, it induces an isomorphism $\Gamma^* \cong \mathrm{Hom}_\mathbb{Z}(\Gamma, \mathbb{Z})$ via $\boldsymbol{\alpha} \mapsto \langle \cdot, \boldsymbol{\alpha} \rangle$, and we can express the defect group in terms of its restriction $\mathcal{Q} \colon \Gamma \to \mathrm{Hom}_\mathbb{Z}(\Gamma, \mathbb{Z})$ as $\mathbb{D}^{(1)} \cong \operatorname{coker} \mathcal{Q}$.

A genuine line operator needs to be well-defined not only in the presence of dynamical states, but also in the presence of other lines. A configuration of two lines with charges $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ is subject to a phase ambiguity $\mathrm{e}^{2\pi \mathrm{i} \langle \boldsymbol{\alpha}, \boldsymbol{\alpha}' \rangle}$, so this requires the stronger condition $\langle \Gamma_L, \Gamma_L \rangle \subset \mathbb{Z}$. There are multiple maximal solutions Γ_L (maximal isotropic, or Lagrangian, sublattices of Γ^*) to this constraint, each defining a global structure of the theory.

Let us make this reasoning more concrete. Since the Dirac pairing is skew-symmetric, there is a basis $\{\gamma^{\mathrm{e},i}, \gamma^{\mathrm{m},i}\}_{i=1}^r$ for Γ where it takes the simple form

$$\left\langle \gamma^{\mathrm{e},i}, \gamma^{\mathrm{m},j} \right\rangle = n_i \delta_{ij} \tag{2.3}$$

$$\left\langle \gamma^{\mathrm{e},i}, \gamma^{\mathrm{e},j} \right\rangle = \left\langle \gamma^{\mathrm{m},i}, \gamma^{\mathrm{m},j} \right\rangle = 0 \tag{2.4}$$

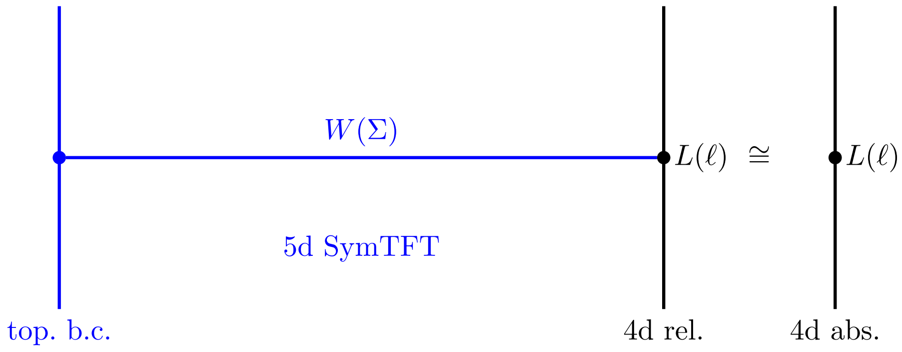

Figure 1. Line operators $L(\ell)$ in the relative 4d theory bound surface operators $W(\Sigma)$ in the SymTFT. A line operator is genuine (that is, survives in the absolute theory) if the surface operator it bounds can end on the topological boundary. See [18] for a detailed account.

with $n_i \in \mathbb{N}$. Then, since we assume Γ has full rank, it is clear that $\{\frac{1}{n_i}\gamma^{\mathrm{e},i}, \frac{1}{n_i}\gamma^{\mathrm{m},i}\}_{i=1}^r$ is a basis for Γ^*, and the defect group is

$$\mathbb{D}^{(1)} = \Gamma^*/\Gamma \cong \bigoplus_{i=1}^r (\mathbb{Z}/n_i\mathbb{Z})^2. \tag{2.5}$$

Indeed, the Dirac pairing expressed as a matrix $\mathcal{Q}^{\alpha\beta} = \left\langle \gamma^\alpha, \gamma^\beta \right\rangle$ becomes in this basis

$$\mathcal{Q} = \bigoplus_{i=1}^r \begin{pmatrix} 0 & n_i \\ -n_i & 0 \end{pmatrix}, \tag{2.6}$$

and we see that coker $\mathcal{Q}$ agrees with (2.5). The integers n_i are the non-zero invariant factors of $\mathcal{Q}$ — the diagonal entries in its Smith normal form.

The picture we have outlined is easily generalised to incorporate flavour. We allow for the possibility of f flavour charges (a rank f flavour group) so that Γ has rank $2r+f$. These have zero Dirac pairing with any other charge; they generate ker $\mathcal{Q}$. We quotient out by the flavour charges to define reduced charge lattices

$$\tilde{\Gamma} = \frac{\Gamma}{\ker \mathcal{Q}}, \qquad \tilde{\Gamma}^* = \frac{\Gamma^*}{\mathbb{R} \otimes_{\mathbb{Z}} \ker \mathcal{Q}} \tag{2.7}$$

and define the defect group by $\mathbb{D}^{(1)} = \tilde{\Gamma}^*/\tilde{\Gamma}$. As above, the Dirac pairing defines an isomorphism $\tilde{\Gamma}^* \cong \mathrm{Hom}_{\mathbb{Z}}(\tilde{\Gamma}, \mathbb{Z})$ with restriction $\tilde{\mathcal{Q}} \colon \tilde{\Gamma} \to \mathrm{Hom}_{\mathbb{Z}}(\tilde{\Gamma}, \mathbb{Z})$, such that

$$\mathbb{D}^{(1)} \cong \operatorname{coker} \tilde{\mathcal{Q}} \cong \operatorname{Tor} \operatorname{coker} \mathcal{Q} \tag{2.8}$$

and

$$\operatorname{coker} \mathcal{Q} \cong \mathbb{D}^{(1)} \oplus \mathbb{Z}^f. \tag{2.9}$$

The central claim of [22] is that the symmetry TFT capturing the choice of global structure explained above is described by the action

$$S = \frac{\mathrm{i}}{2\pi} \frac{\mathcal{Q}^{\alpha\beta}}{2} \int_{X_5} B_\alpha \wedge \mathrm{d}B_\beta \tag{2.10}$$

where the 4d theory lives on $\mathcal{M}_4 \subset \partial X_5$.[1] The fields B_α are 2-form higher U(1) gauge fields. (Properly, they are cocycles in differential cohomology [43, 44]; see also [21, 45, 46]. Roughly speaking, they are locally defined 2-forms such that $\frac{1}{2\pi}\,\mathrm{d}B_\alpha$ are globally defined 3-forms with integer periods.[2]) In the special basis introduced above, the action reduces to

$$S = \frac{\mathrm{i}}{2\pi}\sum_{i=1}^{r} n_i\left(\int_{X_5} B_{\mathrm{e},i}\wedge \mathrm{d}B_{\mathrm{m},i} - \frac{1}{2}\int_{\partial X_5} B_{\mathrm{e},i}\wedge B_{\mathrm{m},i}\right). \tag{2.11}$$

We disregard the boundary terms, which are local in the boundary gauge fields, and focus on the bulk terms, which describe a product of r BF theories [5, 29]. The terms with $n_i = 1$ describe trivial (invertible) field theories, but those with $n_i > 1$ give a non-invertible field theory containing two-dimensional surface operators ending on the 4d line operators. They form the defect group $\mathbb{D}^{(1)}$, and their linking relations

$$\left\langle \exp\left(\mathrm{i}\oint_\Sigma B_{\mathrm{e},i}\right)\exp\left(\mathrm{i}\oint_{\Sigma'} B_{\mathrm{m},j}\right)\right\rangle = \exp\left(\frac{2\pi \mathrm{i}}{n_i}\delta_{ij}\,\mathrm{link}(\Sigma,\Sigma')\right) \tag{2.12}$$

capture the phase ambiguity between 4d lines. A topological boundary condition determines the set of line operators in the absolute theory; see figure 1. From this perspective it is clear that the defect group is a property of the SymTFT bulk, while a maximal isotropic sublattice Γ_L/Γ is determined by a choice of topological boundary condition, as derived in the 3d case in [47].[3] In this work, we focus on the SymTFT itself.

3 Reduction from M-theory

One derivation of the symmetry TFT of the class $\mathcal{S}[\mathfrak{su}(N), \Sigma_{g,p}]$ theory proceeds by dimensional reduction of M-theory. We realise the $\mathcal{N} = (2, 0)$ theory on a six-dimensional spacetime $\mathcal{M}_6$ as the worldvolume theory on a stack of N coincident M5-branes. [29] showed that, in the limit $N \to \infty$, the near-horizon geometry is $X_7 \times S^4$ where the conformal boundary of X_7 is $\partial X_7 = \mathcal{M}_6$. He further argued that the low-energy theory close to the branes has a topological sector described by the Chern-Simons action

$$S = -\frac{\mathrm{i}N}{2\cdot 2\pi}\int_{X_7} C\wedge \mathrm{d}C\,, \tag{3.1}$$

where C is the M-theory U(1) 3-form gauge field. This can be justified as follows: upon reduction of the Chern-Simons term in the Euclidean 11d supergravity action,

$$S = -\frac{\mathrm{i}}{6(2\pi)^2}\int_{X_7\times S^4} C\wedge \mathrm{d}C\wedge \mathrm{d}C\,, \tag{3.2}$$

the single-derivative term is

$$S = -\frac{\mathrm{i}}{2\cdot 2\pi}\int_{X_7} C\wedge \mathrm{d}C\ \frac{1}{2\pi}\oint_{S^4}\mathrm{d}C\,. \tag{3.3}$$

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[1] Importantly, ∂X_5 can have components other than $\mathcal{M}_4$; in the simplest case $X_5 = \mathcal{M}_4 \times [0, 1]$ as in figure 1.

[2] Compared to the fields b_α of [22], our fields are $B_\alpha = 2\pi\, b_\alpha$. This is the conventional normalisation used in physics, although the b_α are mathematically more natural.

[3] The charge lattice Γ_L/Γ does not completely specify a topological boundary condition, but additional data is needed [48, 49].

Since each M5-brane sources one unit of flux for $\mathrm{d}C$, the S^4 integral evaluates to N and we recover (3.1). This reduction is well-known, and has been performed in greater generality in the framework of differential cohomology [21].

Next, we perform Kaluza-Klein reduction on $X_7 = X_5 \times \Sigma_{g,p}$, beginning with the case $p = 0$, detailed in [10]. Expand C in terms of eigen-1-forms of the Laplace-de Rham operator $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ on $\Sigma_{g,0}$:

$$C = \sum_i B_i \wedge \omega^i, \qquad \Delta\omega^i = \lambda_i \omega^i \tag{3.4}$$

with $\oint_{\Sigma_{g,0}} \omega^i \wedge \star\omega^j = 0$ if $\lambda_i \neq \lambda_j$. The coefficients B_i are 2-forms on X_5. Reducing the kinetic term $\frac{1}{2\kappa^2}\int_{X_7} \mathrm{d}C \wedge \star\mathrm{d}C$ produces a mass term for B_i unless $\mathrm{d}\omega^i = 0$; we truncate to these massless modes. Now reduce the topological term (3.1) to obtain

$$S = -\frac{\mathrm{i}N}{2\cdot 2\pi} \sum_{i,j} \int_{X_5} B_i \wedge \mathrm{d}B_j \oint_{\Sigma_{g,0}} \omega^i \wedge \omega^j. \tag{3.5}$$

In the terms with $\lambda_i \neq 0$, $\omega^i = \frac{1}{\lambda_i}\mathrm{d}\delta\omega^i$ is exact and the integral over $\Sigma_{g,0}$ is zero; similarly if $\lambda_j \neq 0$. Thus the contributing ω^i are harmonic forms which we can take to have integral periods. These are by the Hodge theorem [50, Theorem 1.45] and Poincaré duality in bijection with the generators c^i of $H_1(\Sigma_{g,0};\mathbb{Z}) \cong \mathbb{Z}^{2g}$. The intersection pairing $(c^i, c^j) = \int_{\Sigma_{g,0}} \omega^i \wedge \omega^j \in \mathbb{Z}$ is

$$(c^{\mathrm{e},i}, c^{\mathrm{m},j}) = -(c^{\mathrm{m},j}, c^{\mathrm{e},i}) = -\delta^{ij} \tag{3.6}$$

$$(c^{\mathrm{e},i}, c^{\mathrm{e},j}) = (c^{\mathrm{m},i}, c^{\mathrm{m},j}) = 0 \tag{3.7}$$

with the appropriate choices of generators $c^{\mathrm{e},1}, \ldots, c^{\mathrm{e},g}, c^{\mathrm{m},1}, \ldots, c^{\mathrm{m},g}$. Thus

$$S = \frac{\mathrm{i}N}{2\pi} \sum_{i=1}^{g} \int_{X_5} B_{\mathrm{e},i} \wedge \mathrm{d}B_{\mathrm{m},i}\,, \tag{3.8}$$

(again, up to boundary terms) which reproduces (2.11) with g of the n_i equal to N. By (2.5), this indeed matches with (1.1). While Witten's construction relied on the holographic limit $N \to \infty$, we expect the result to hold for any N, and indeed check it for small N in the next section.

The presence of punctures complicates the analysis since there are now boundary conditions for C. Naively, one could observe that since (3.5) couples the 5d gauge fields through the intersection pairing on $H_1(\Sigma_{g,0};\mathbb{Z})$, and since the elementary cycles surrounding punctures are in the kernel of the intersection pairing on $H_1(\Sigma_{g,p};\mathbb{Z})$, adding punctures should not affect (3.8). The problem is that by the Hodge decomposition with boundary [51, Theorem 2.6.1] and Lefschetz duality [52, Theorem 3.43], this pairing results from the KK reduction above only if C is taken to have Dirichlet boundary conditions at the punctures, and this is not generally the case. Indeed, *irregular* punctures do introduce 1-form symmetries "trapped" at the punctures [18].

From the M-theory perspective, a regular puncture can be described using a construction of [30, 31]: we modify the space $\Sigma_{g,0} \times S^4$ in the above construction in a neighbourhood

$D \subset \Sigma_{g,0}$ of the puncture, replacing $D \times S^4$ with a space X_6 whose $\mathrm{d}C$ flux along four-cycles define the puncture data. We thus obtain the 11-dimensional spacetime $\mathcal{M}_{11} = X_5 \times Y_6$. The space X_6 is a fibration of the form $S^2 \to X_6 \to X_4$; $S^1 \to X_4 \to \mathbb{R}^3$ where the S^2 and S^1 shrink at certain singular loci.

We outline an argument that this structure means that the puncture does not modify the defect group. If X_6 and X_4 were non-singular fibrations, we would have the long exact sequences

$$\cdots \longrightarrow \pi_1(S^2) \longrightarrow \pi_1(X_6) \longrightarrow \pi_1(X_4) \longrightarrow \pi_0(S^2) \longrightarrow \cdots \tag{3.9}$$

$$\cdots \longrightarrow \pi_2(\mathbb{R}^3) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(X_4) \longrightarrow \pi_1(\mathbb{R}^3) \longrightarrow \cdots \tag{3.10}$$

establishing that $\pi_1(X_6) \cong \pi_1(X_4) \cong \pi_1(S^1) \cong \mathbb{Z}$, generated by the loop winding along S^1. In the present case, however, the S^1 shrinks and this loop is contractible; we therefore expect $\pi_1(X_6) = 0$ instead.[4] Then $H_1(X_6) = 0$ by the Hurewicz theorem [52, Theorem 2A.1]. Now, expanding $C_{\mathcal{M}_{11}} = \sum_i B^{(2)}_{i,X_5} \wedge \omega^{i(1)}_{Y_6} + \phi^{(0)}_{X_5} c^{(3)}_{Y_6} + \cdots$, the topological term (3.1) reduces to

$$-\frac{\mathrm{i}}{2\cdot 2\pi}\int_{X_5} B_i \wedge \mathrm{d}B_j \oint_{Y_6} \omega^i \wedge \omega^j \wedge \frac{\mathrm{d}c}{2\pi} \tag{3.11}$$

where ω^i and $\mathrm{d}c\,/2\pi$ have integral periods; in particular ω^i is valued in $H^1(Y_6;\mathbb{Z})$. Consider for simplicity the case of a single puncture; then we have $Y_6 = (\Sigma_{g,1} \times S^4) \cup X_6$ with $(\Sigma_{g,1} \times S^4) \cap X_6 \simeq S^1 \times S^4$, and the reduced Mayer-Vietoris sequence [53, section 4.6] becomes

$$\cdots \longrightarrow \overbrace{H_1(S^1 \times S^4)}^{\mathbb{Z}} \xrightarrow{i_*} H_1(\Sigma_{g,1} \times S^4) \oplus \overbrace{H_1(X_6)}^{0} \longrightarrow H_1(Y_6) \longrightarrow 0 \tag{3.12}$$

and it follows that

$$H_1(Y_6) \cong \frac{H_1(\Sigma_{g,1} \times S^4)}{\mathrm{im}\, i_*} \cong H_1(\Sigma_{g,0} \times S^4) \cong H_1(\Sigma_{g,0}). \tag{3.13}$$

The middle isomorphism above comes from the fact that $i\colon S^1 \times S^4 \to \Sigma_{g,1} \times S^4$ is the inclusion identifying S^1 with the non-contractible[5] loop surrounding the puncture; trivialising it amounts to closing the puncture. The universal coefficient theorem [52, Theorem 3.2] now gives $H^1(Y_6;\mathbb{Z}) \cong H^1(\Sigma_{g,0};\mathbb{Z})$ and ω^i are cocycles on $\Sigma_{g,0}$. Then, the integral of $\frac{\mathrm{d}c}{2\pi}$ captures the total flux N sourced by the branes; we recover (3.5) and confirm that the puncture does not affect the defect group.

4 Defect group from BPS quivers

In this section, we find the defect group and hence the SymTFT using an explicit construction of the 4d BPS quiver.

4.1 Full punctures

A BPS quiver [36, 54] of a 4d $\mathcal{N} = 2$ supersymmetric theory has nodes corresponding to charges of certain BPS states, and arrows such that the signed adjacency matrix is the matrix

[4] This is analogous to the singular fibration $S^1 \to S^2 \to [0,1]$ of the 2-sphere over the unit interval. The S^1 fibre shrinks to a point at the endpoints of the interval, so $\pi_1(S^2) = 0$ rather than $\mathbb{Z}$.

[5] This loop is non-contractible whenever $g \geq 1$. The case $g = 0$ corresponds to a sphere with a single punctures, but we consider only spheres with at least three punctures.

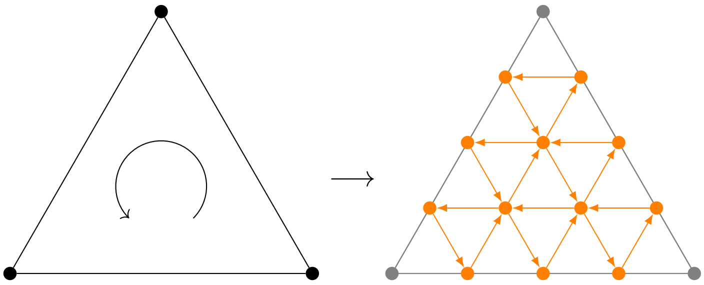
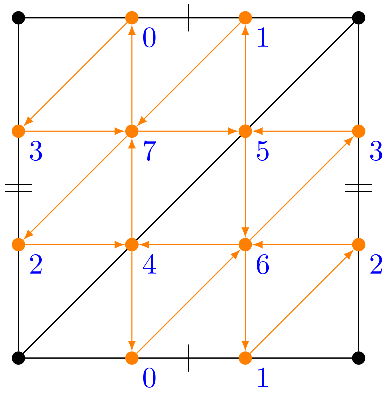


Figure 2. Constructing the class $\mathcal{S}[\mathfrak{su}(N)]$ BPS quiver from a triangulation ($N = 4$ is shown).



$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 3. BPS quiver and signed adjacency matrix $\mathcal{Q}$ for the theory $\mathcal{S}[\mathfrak{su}(3), \Sigma_{1,1}]$.

$\mathcal{Q}$ of Dirac pairings as in section 2. The nodes always correspond to physical BPS states; hence every charge in the lattice they generate is realised by a dynamical state (not in general a BPS state). We assume, as in [22], that this is the full charge lattice of the theory; in other words, every charge is a sum of BPS charges.

[54] described how to construct BPS quivers of class $\mathcal{S}[\mathfrak{su}(2)]$ theories from ideal triangulations [55] of $\Sigma_{g,p}$. The nodes of the quiver are the arcs[6] of the triangulation, and they are joined by arrows going counterclockwise around each triangle (with respect to the orientation of $\Sigma_{g,p}$). In the literature on cluster algebras, these are called quivers of surface type. The procedure generalises to other class $\mathcal{S}$ theories via work on Hitchin systems and spectral networks [15, 24, 56, 57] and their relation to the BPS quivers [33, 34]. We need only the fact that the BPS quiver for a class $\mathcal{S}[\mathfrak{su}(N)]$ theory with full punctures is a so-called $(N-1)$-*triangulation* [35, 58]: starting from an ideal triangulation, the quiver has $N-1$ nodes for each arc, as well as $\binom{N-1}{2}$ internal nodes in each triangle, connected as in figure 2. Once the quiver is known, the defect group (as well as the flavour rank f) can be extracted using (2.9).

For example, let us put $N = 3$, $g = 1$, $p = 1$ for the class $\mathcal{S}[\mathfrak{su}(3)]$ theory on a torus with one full puncture; see figure 3. The torus is triangulated with two triangles and three arcs that begin and end on the single puncture. The BPS quiver is the Fock-Goncharov

[6]While a triangulation and a BPS quiver are both a type of graph, we use distinct terminology for them in order to avoid confusion: a triangulation consists of *vertices* and *arcs*, while a BPS quiver consists of *nodes* and *arrows*.

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2-triangulation and has eight nodes. The corresponding signed adjacency matrix $\mathcal{Q}$ (figure 3) has the Smith normal form

$$\mathrm{diag}(1,\,1,\,1,\,1,\,3,\,3,\,0,\,0) \tag{4.1}$$

so that $\mathrm{coker}\,\mathcal{Q} \cong (\mathbb{Z}_3)^2 \oplus \mathbb{Z}^2$. This means that the theory has defect group $\mathbb{D}^{(1)} \cong (\mathbb{Z}_3)^2$ and two flavour charges.

Accompanying this work is a computer program to carry through this calculation for arbitrary N, g and p. Details on the computation are found in appendix A. We find that

$$(\mathbb{D}^{(1)} \oplus \mathbb{Z}^f)(\mathcal{S}[\mathfrak{su}(N), \Sigma_{g,p}]) \cong (\mathbb{Z}_N)^{2g} \oplus \mathbb{Z}^{(N-1)p} \tag{4.2}$$

in agreement with (1.1), for all values $2 \leq N \leq 7$, $0 \leq g \leq 8$ and $1 \leq p \leq 8$ ($3 \leq p \leq 8$ for $g = 0$).

In addition to the defect group, the calculation also determines the rank r and flavour rank f of the theory. Eq. (4.2) directly gives

$$f = (N-1)p. \tag{4.3}$$

To derive r, the dimension of the Coulomb branch of the 4d theory, note that the charge lattice has rank $2r+f$, equal to the number of nodes in the BPS quiver. As described above, the quiver has $2r + f = \binom{N-1}{2}t + (N-1)a$ nodes, where a is the number of arcs in the triangulation and t is the number of triangles. By Euler's formula $p - a + t = 2 - 2g$, as well as the fact that $2a = 3t$ in a triangulation, we find that $2r + f = (N^2-1)(2g+p-2)$. Together with the result (4.3) for f, this leads to the following formula for the rank:

$$r = (N^2-1)(g-1) + \binom{N}{2}p. \tag{4.4}$$

We can obtain some basic consistency checks on the above construction and results constructing $\mathcal{S}[\mathfrak{su}(N), \Sigma_{g,p}]$ by gluing together copies of the T_N theory [23, 59, 60], which is the compactification on a sphere with three full punctures: $T_N = \mathcal{S}[\mathfrak{su}(N), \Sigma_{0,3}]$. Each full puncture carries an $\mathrm{SU}(N)$ flavour symmetry [25] and connecting two punctures by a tube corresponds to gauging it. Thus we obtain $\Sigma_{g,p}$ from $2g+p-2$ T_N theories on thrice-punctured spheres after connecting $3g+p-3$ pairs of punctures by a tube (figure 4).

While it is not obvious how to extract the defect group from this picture, we can easily derive r and f. The $\mathrm{SU}(N)^p$ flavour symmetry evidently reproduces (4.3). As for the rank, the T_N theory on each sphere has $d-2$ Coulomb branch operators of scaling dimension d for each $d = 3, \ldots, N$ [60, Fact 5.7], so its rank is $\sum_{d=3}^{N}(d-2) = \binom{N-1}{2}$. Furthermore, each gauged $\mathrm{SU}(N)$ associated to a tube contributes $N-1$ Coulomb branch operators. The total rank is

$$r = \binom{N-1}{2}(2g+p-2) + (N-1)(3g+p-3) \tag{4.5}$$

which indeed simplifies to (4.4).

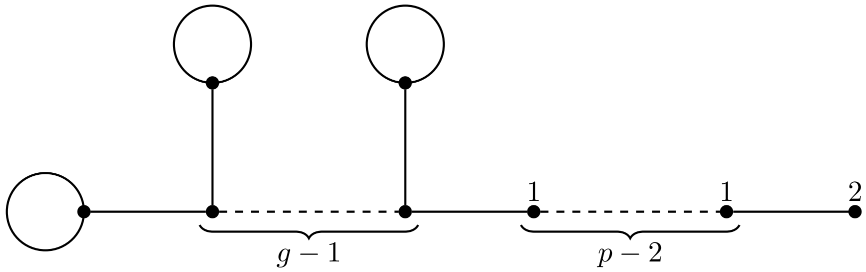

Figure 4. Decomposition of $\Sigma_{g,p}$ into $2g+p-2$ spheres (dots) with three full punctures, joined by $3g+p-3$ tubes (lines). Numbers denote punctures.

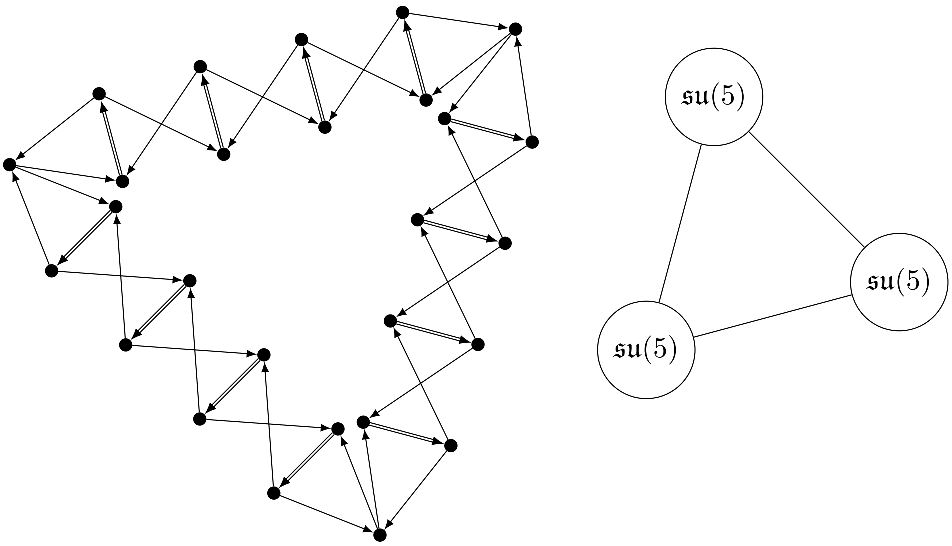

Figure 5. Left: BPS quiver for $\mathfrak{g} = \mathfrak{su}(N)$ compactified on a torus with p simple punctures ($N = 5$, $p = 3$ is shown).
Right: quiver gauge theory description. A node is a gauge algebra and an edge is a bifundamental hypermultiplet.

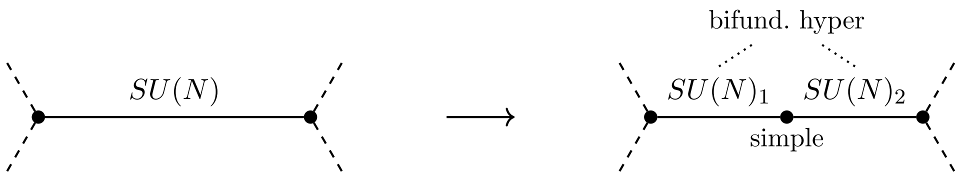

Figure 6. Adding a simple puncture.

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4.2 Non-full punctures

One limitation of our computation is that it deals with full punctures only. In general one can consider tame punctures labelled by any partition of N [25]. The general expectation is that the defect group should be the same for any choice of partitions of N at the punctures, as argued in [28] and in section 3. Our result confirms that changing a full puncture, which is labelled by $(1, \ldots, 1)$, into the absence of a puncture, which can be thought of as labelled by (N), does not affect the defect group. For theories with partial punctures, several constructions of the BPS quivers exist [33, 34], but here we will be content with checking a specific example. Namely, in the case of $\mathcal{S}[\mathfrak{su}(N), \Sigma_{1,\tilde{p}}]$, where $\Sigma_{1,\tilde{p}}$ denotes the torus with p *simple* punctures labelled by $(N-1, 1)$, the quiver is known [36] and displayed in figure 5. We have checked that for this quiver,

$$\operatorname{coker} \mathcal{Q} \cong (\mathbb{Z}_N)^2 \oplus \mathbb{Z}^p \tag{4.6}$$

for $2 \leq N \leq 7$ and $1 \leq p \leq 8$, so that the defect group is indeed $(\mathbb{Z}_N)^2$. This Coulomb branch result can also be confirmed from a quiver gauge theory description (figure 5) as in [28]. In a purely electric duality frame, the gauge group is $\mathrm{SU}(N)^p$ with Wilson lines charged under $Z(\mathrm{SU}(N)^p) \cong (\mathbb{Z}_N^p)^{(1)}$. The dynamical states in the bifundamental representations screen the 1-form symmetry to the diagonal $(\mathbb{Z}_N)^{(1)}$, identified with a maximal isotropic subgroup of $\mathbb{D}^{(1)} \cong (\mathbb{Z}_N)^2$.

Indeed, this can be slightly generalised to a field-theoretic argument for arbitrary simple punctures preserving the defect group: adding a simple puncture corresponds to replacing a gauged $\mathrm{SU}(N)$ by $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2$ and a hypermultiplet in the bifundamental (figure 6); see [25, figure 12.5]. The hypermultiplet has N-ality $(1, -1)$ and therefore the lines charged under the centre 1-form symmetries $(\mathbb{Z}_N)_1^{(1)}$ and $(\mathbb{Z}_N)_2^{(1)}$ are identified up to screening.

4.3 Structure of the BPS quiver

The result (4.2) is a purely combinatorial statement about the $(N-1)$-triangulations of [58], and it is interesting to consider it as such. It has been partially addressed in the mathematical literature; in particular, eq. (4.3) when $N = 2$ is a special case of Theorem 14.3 of [55]. Here, we present some observations on the structure of the quivers that make the result plausible in the form of a proof sketch. It would be interesting to see if this reasoning could be extended to a full elementary proof of (4.2).

Recall that we have the charge lattice $\Gamma = \mathbb{Z}^{2r+f}$ with standard basis given by the nodes of the quiver, and we conceptualise the Dirac pairing as a $\mathbb{Z}$-linear map $\mathcal{Q}\colon \Gamma \to \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$.

First, the free factor in $\operatorname{coker} \mathcal{Q}$ being $\mathbb{Z}^{(N-1)p}$ is equivalent to $\ker \mathcal{Q} \cong \mathbb{Z}^{(N-1)p}$. It is in fact easy to exhibit $(N-1)p$ null (flavour) vectors; $N-1$ for each elementary cycle wrapping a puncture in $\Sigma_{g,p}$, as in figure 7. Recall that a node in the quiver is a generator of Γ; in the figure, the marked nodes γ_i define a vector $\gamma = \sum_i \gamma_i \in \Gamma$. The covector $\mathcal{Q}\gamma = \langle \cdot, \gamma \rangle \in \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ is found by following the arrows adjacent to each marked point, adding 1 for outgoing arrows and -1 for incoming arrows. The construction of γ ensures that all such contributions (occurring at the nodes marked with blue circles) cancel, so that $\mathcal{Q}\gamma = 0$.

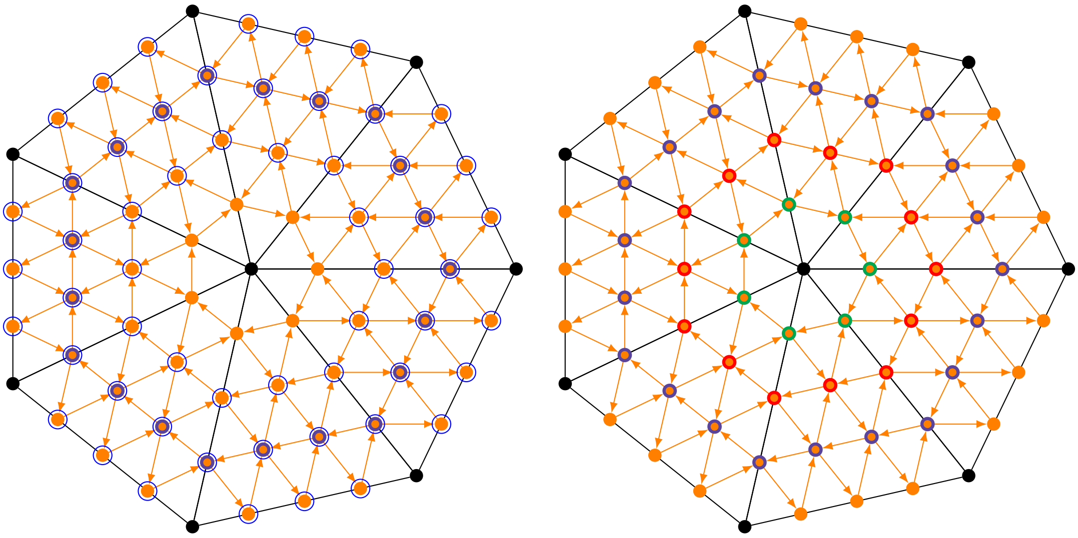

Figure 7. Left: construction of a flavour vector γ of $\mathcal{Q}$ around a generic puncture, with entries of 1 at marked nodes and 0 elsewhere. Blue circles mark where contributions from different nodes cancel in $\mathcal{Q}\gamma$.
Right: the $N-1$ such flavour vectors around each puncture. Here $N=4$ and the three flavour vectors are marked in green, red and blue respectively (with entries of 1 at marked nodes and 0 elsewhere).

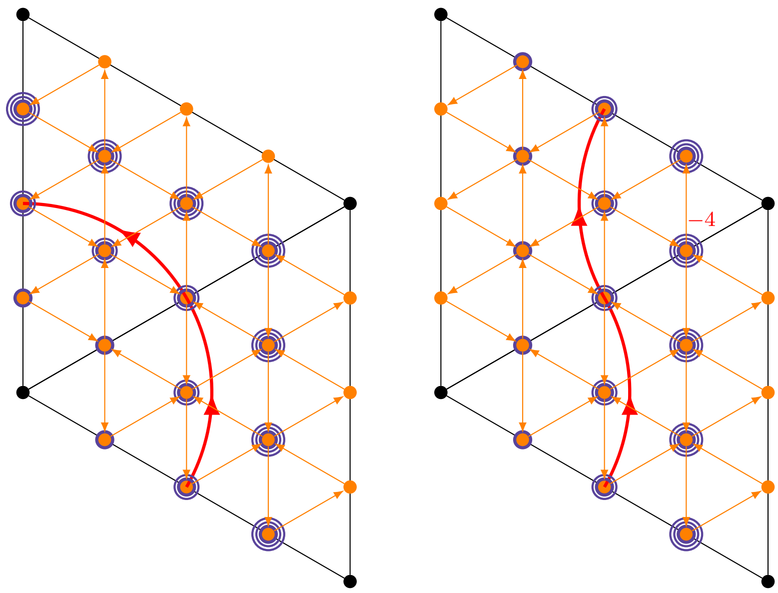

Figure 8. Charge vectors $v(c)$ mapping to torsion elements of coker $\mathcal{Q}$. Entries of 1, 2 and 3 are marked with the corresponding number of blue circles.
Left: a counterclockwise turning path segment, with all zeroes in the image.
Right: a direction change, with a single nonzero entry $\pm N$ in the image (here $N=4$).

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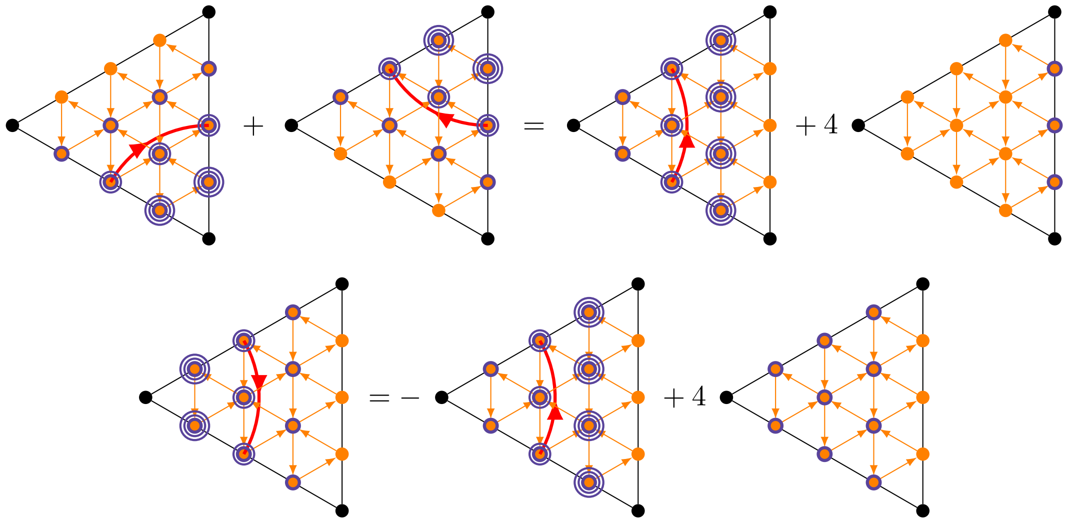

Figure 9. The assignment v of charges to cycles is additive modulo N (here $N = 4$). Entries of 1, 2 and 3 are marked with the corresponding number of blue circles.

This works no matter the number of arcs incident to the puncture. This shows that the rank of $\ker \mathcal{Q}$ is at least $(N-1)p$, but does not rule out hypothetical further flavour vectors.[7]

Next we search for the torsional part of $\operatorname{coker} \mathcal{Q}$; the defect group. There is an explicit map $v\colon H_1(\Sigma_{g,p};\mathbb{Z}) \to \Gamma$, defined in figure 8, such that $\operatorname{im}(\mathcal{Q}\circ v) \subset N\operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$. Figure 9 shows that it defines a homomorphism $v_N\colon H_1(\Sigma_{g,p};\mathbb{Z}_N) \to \Gamma\otimes\mathbb{Z}_N$. Each generator $c \in H_1(\Sigma_{g,p};\mathbb{Z})$ passes through a sequence of triangles, and in each triangle, turns either clockwise or counterclockwise. A cycle that turns purely clockwise or counterclockwise surrounds a single puncture; then $v(c)$ is a sum of flavour vectors as in figure 7. Thus v_N descends to a homomorphism $\tilde{v}_N\colon H_1(\Sigma_{g,0};\mathbb{Z}_N) \to \tilde{\Gamma}\otimes\mathbb{Z}_N$ such that $(\tilde{\mathcal{Q}}\otimes 1_{\mathbb{Z}_N})\circ\tilde{v}_N = 0$ (recall that $\tilde{\Gamma} = \Gamma/\ker\mathcal{Q}$, $\tilde{\Gamma}^* \cong \operatorname{Hom}_{\mathbb{Z}}(\tilde{\Gamma},\mathbb{Z})$ and $\mathbb{D}^{(1)} = \operatorname{coker}(\tilde{\mathcal{Q}}\colon \tilde{\Gamma}\hookrightarrow\tilde{\Gamma}^*)$).

Here we conjecture that in fact $\ker(\tilde{\mathcal{Q}}\otimes 1_{\mathbb{Z}_N}) = \operatorname{im}\tilde{v}_N$, so that we have an exact sequence

$$0 \longrightarrow H_1(\Sigma_{g,0};\mathbb{Z}_N) \xrightarrow{\tilde{v}_N} \tilde{\Gamma}\otimes\mathbb{Z}_N \xrightarrow{\tilde{\mathcal{Q}}\otimes 1_{\mathbb{Z}_N}} \tilde{\Gamma}^*\otimes\mathbb{Z}_N \longrightarrow \mathbb{D}^{(1)}\otimes\mathbb{Z}_N \longrightarrow 0. \tag{4.7}$$

In other words, $\operatorname{Tor}(\mathbb{D}^{(1)},\mathbb{Z}_N) \cong H_1(\Sigma_{g,0};\mathbb{Z}_N) \cong (\mathbb{Z}_N)^{2g}$. From this, if one can show that $\mathbb{D}^{(1)}$ is N-periodic ($N\mathbb{D}^{(1)} = 0$, that is, $N\tilde{\Gamma}^* \subset \operatorname{im}\tilde{\mathcal{Q}}$); then $\mathbb{D}^{(1)} \cong \operatorname{Tor}(\mathbb{D}^{(1)},\mathbb{Z}_N)$ and (4.2) follows.

Acknowledgments

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[7]The story is somewhat complicated by torsion. The cycles form a maximal linearly independent subset of $\ker\mathcal{Q}$, but not a generating set. For example, consider the case where the vertices of each triangle are all distinct. Summing all cycles gives a multiple of the null vector $\gamma = (1,\ldots,1)$, namely 3γ, but γ itself cannot in general be attained this way.

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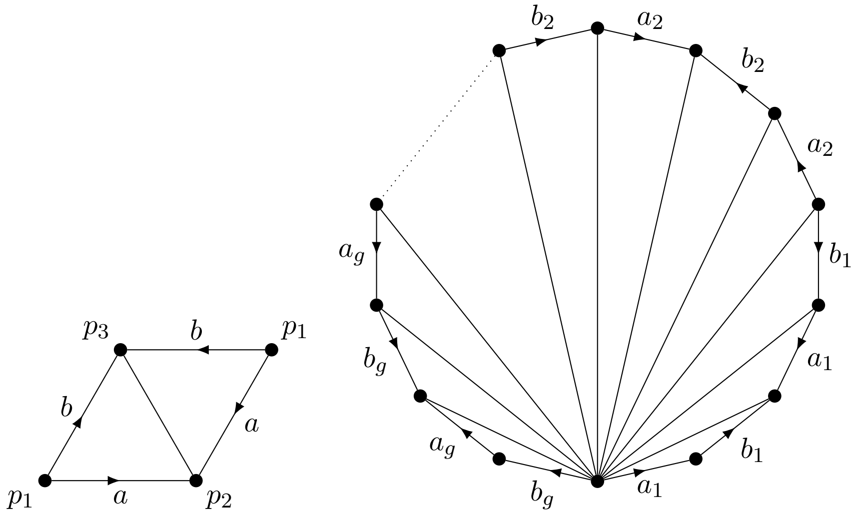

Figure 10. Ideal triangulations of $\Sigma_{0,3}$ and $\Sigma_{g,1}$ for $g > 0$.

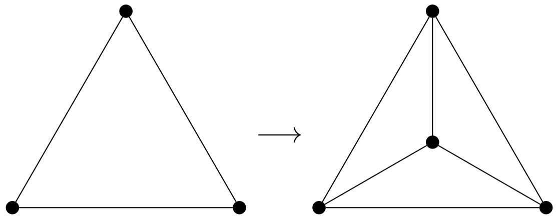

Figure 11. Adding a puncture in the interior of a triangle.

2023 school and the GCS2023 conference and school, respectively. Finally, I am grateful to the anonymous referee, whose comments greatly helped improve the paper's clarity.

A Computation

The computations in section 4 were done using SageMath [61]. The source code and results are available at https://gitlab.com/eliasrg/class-s-defect-groups.

The first step is to construct an ideal triangulation of the punctured Riemann surface $\Sigma_{g,p}$. We achieve this by starting from a triangulation of $\Sigma_{g,1}$, or of $\Sigma_{0,3}$ if $g = 0$. These are shown in figure 10. We then add additional punctures one by one, as shown in figure 11, until there are p punctures in total. In preparation for the next step, we also keep track of the orientation of each triangle.

Having obtained the triangulation, we next construct the class $\mathcal{S}[\mathfrak{su}(N)]$ BPS quiver. We put $N - 1$ nodes on each arc and $\binom{N-1}{2}$ internal nodes in each triangle, and connect them with arrows according to figure 2, counterclockwise with respect to the orientation of the surface, as in [35, figure 23].

Finally, we compute the defect group and flavour rank from the quiver's signed adjacency matrix $\mathcal{Q}$ using (2.9) and

$$\operatorname{coker} \mathcal{Q} \cong \operatorname{coker} S \tag{A.1}$$

where S is the Smith normal form of $\mathcal{Q}$. The result of this computation is eq. (4.2).

B Mathematical preliminaries

In this appendix, we collect the most important standard mathematical theorems used in the main text for the reader's convenience.

The skew normal form. The existence of the special basis of the charge lattice Γ follows from [62, Theorem IV.1]: if $\mathcal{Q}$ is a skew-symmetric $n \times n$ matrix with integer entries, then $\mathcal{Q}$ is congruent to a block-diagonal matrix $\mathcal{Q}'$ (meaning that there exists an integer matrix B, invertible over the integers, such that $\mathcal{Q} = B^T \mathcal{Q}' B$) of the form

$$\mathcal{Q}' = \bigoplus_{i=1}^{r} \begin{pmatrix} 0 & n_i \\ -n_i & 0 \end{pmatrix} \oplus 0_{f \times f} \tag{B.1}$$

where $n = 2r + f$ for some r and f (in particular, Ω has even rank $2r$, and the kernel is isomorphic to $\mathbb{Z}^f$).

Hodge isomorphism for manifolds with boundary. For a manifold X without boundary, the Hodge theorem [50, Theorem 1.45] states that there is an isomorphism between the group of harmonic p-forms (the kernel of the Laplacian $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ on p-forms) and the p-th de Rham cohomology of X. In other words, every class in $H^p(X;\mathbb{R}) \cong H^p_{\mathrm{dR}}(X)$ has a unique harmonic representative. There is a version of this for manifolds with boundary [51, Theorem 2.6.1, Corollary 2.6.2], which asserts isomorphisms

$$H^p(X;\mathbb{R}) \cong \mathcal{H}^p_N(X) \quad \text{and} \quad H^p(X, \partial X;\mathbb{R}) \cong \mathcal{H}^p_D(X) \tag{B.2}$$

where $\mathcal{H}^p_N(X)$ and $\mathcal{H}^p_D(X)$ are the harmonic p-forms with Neumann and Dirichlet boundary conditions on ∂X, respectively. This holds when X is a compact so-called ∂-manifold (orientable smooth manifold with boundary that is complete as a metric space).

The remaining facts are about algebraic topology. Most of them are found, with more detailed statements, in [52].

Fibrations. A *fibration* in the usual sense is a map $p\colon E \to B$ such that any homotopy $h\colon X \times [0,1] \to B$ can be lifted along any map $g\colon X \to E$ such that $p \circ g = h(-,0)$; that is, there exists a map $\tilde{h}\colon X \times [0,1] \to E$ with $p \circ \tilde{h} = h$. The fibres $p^{-1}(b)$ are all homotopy equivalent to some F [52, Theorem 4.61], and one writes $F \to E \to B$. The homotopy groups fit into a long exact sequence [52, Theorem 4.41]:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots . \tag{B.3}$$

The singular fibrations discussed in section 3 are not of this type; in particular, above certain singular points, the fibre shrinks to a point and not all fibres are homotopy equivalent.

The Hurewicz theorem for H_1. The Hurewicz theorem [52, Theorems 2A.1, 4.32] describes the relationship between the first nontrivial homotopy and homology groups of a space. For the argument in section 3, we have made use of the connection between π_1 and H_1, namely that for a path-connected space X, the first homology group $H_1(X)$ is isomorphic to the abelianisation of the fundamental group $\pi_1(X)$. In particular, if $\pi_1(X) = 0$, then $H_1(X) = 0$.

The reduced Mayer-Vietoris sequence. The *reduced* homology groups [53, section 4.3] $\tilde{H}_n(X)$ of a space (or chain complex) X are the same as $H_n(X)$ for $n \geq 1$, but $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. In particular, they have the convenient property that $\tilde{H}_0(X) = 0$ if X is path-connected.

For a space expressed as the union $A \cup B$ of two subspaces, the *reduced Mayer-Vietoris sequence* [53, section 4.6] is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A\cap B) \xrightarrow{i_*} \tilde{H}_n(A)\oplus\tilde{H}_n(B) \xrightarrow{j_*} \tilde{H}_n(A\cup B) \xrightarrow{\partial_*} \tilde{H}_{n-1}(A\cap B) \longrightarrow \cdots . \tag{B.4}$$

Here $i_* = i_{A*} \oplus (-i_{B*})$ and $j_* = j_{A*} + j_{B*}$, where $A \cap B \xhookrightarrow{i_A} A \xhookrightarrow{j_A} A \cup B$ and $A \cap B \xhookrightarrow{i_B} B \xhookrightarrow{j_B} A \cup B$ are the inclusions.

The universal coefficient theorem for cohomology. For a space X, the universal coefficient theorem for cohomology [52, Theorem 3.2] asserts the existence of split short exact sequences

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow \operatorname{Hom}(H_n(X), G) \longrightarrow 0 \tag{B.5}$$

for any n and any coefficient group G. Here $\operatorname{Ext}(H, G)$ is functorial in H and G and in particular $\operatorname{Ext}(H, G) = 0$ if H is a free abelian group. Therefore, since $H_0(X)$ is free, there is an isomorphism $H^1(X; G) \cong \operatorname{Hom}(H_1(X), G)$.

Lefschetz duality. Poincaré duality [52, Theorem 3.30] states that, for a closed orientable d-dimensional manifold X, there is an isomorphism $H^n(X) \cong H_{d-n}(X)$. Lefschetz duality [52, Theorem 3.43] is the analogous statement for manifolds with boundary: if X is a compact orientable d-dimensional manifold with boundary, then there are isomorphisms $H^n(X, \partial X) \cong H_{d-n}(X)$ and $H^n(X) \cong H_{d-n}(X, \partial X)$.

The Tor functor. The Tor functor was used to interpret the exact sequence (4.7). Given an abelian group G, the functor $G \otimes -\colon \mathbf{Ab} \to \mathbf{Ab}$ is right exact but not in general left exact. Its first left derived functor is called Tor (see e.g. [63, section 2.4]). Concretely, this means the following. Every abelian group H is a quotient of a free abelian group, that is, there is an exact sequence $0 \to F_1 \to F_0 \to H \to 0$ with F_0, F_1 free (a free resolution of H). Then the sequence $G \otimes F_1 \to G \otimes F_0 \to G \otimes H \to 0$ is exact, but the first morphism is not in general injective. Its kernel is known as $\operatorname{Tor}(G, H)$, so that the extended sequence

$$0 \longrightarrow \operatorname{Tor}(G, H) \longrightarrow G \otimes F_1 \longrightarrow G \otimes F_0 \longrightarrow G \otimes H \longrightarrow 0 \tag{B.6}$$

is exact. $\operatorname{Tor}(G, H)$ is functorial in G and H and satisfies in particular [52, Proposition 3A.5] that $\operatorname{Tor}(\mathbb{Z}_N, H) \cong \{h \in H \mid Nh = 0\}$, the subgroup of N-torsion elements of H.

Data Availability Statement. This article has associated data in a data repository.

Code Availability Statement. This article has associated code in a code repository.

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