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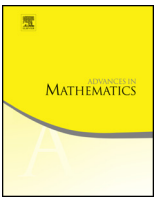

Backward dynamics of non-expanding maps in Gromov hyperbolic metric spaces

Leandro Arosio [a,1], Matteo Fiacchi [b,2], Lorenzo Guerini [c], Anders Karlsson [d,e,*,3]

[a] *Department of Mathematics of Università di Roma "Tor Vergata", Italy*
[b] *Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics, and Mechanics, Slovenia*
[c] *Korteweg de Vries Institute for Mathematics, University of Amsterdam, The Netherlands*
[d] *Section de mathématiques, Université de Genève, Switzerland*
[e] *Matematiska institutionen, Uppsala University, Sweden*

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ABSTRACT

We study the interplay between the backward dynamics of a non-expanding self-map f of a proper geodesic Gromov hyperbolic metric space X and the boundary regular fixed points of f in the Gromov boundary as defined in [8]. To do so, we introduce the notion of *stable dilation* at a boundary regular fixed point of the Gromov boundary, whose value is related to the dynamical behavior of the fixed point. This theory applies in particular to holomorphic self-maps of bounded domains $\Omega \subset\subset \mathbb{C}^q$, where Ω is either strongly pseudoconvex, convex finite type, or pseudoconvex finite type with $q = 2$, and solves several open problems from the literature. We extend results of holomorphic self-maps of

* Corresponding author.
E-mail addresses: arosio@mat.uniroma2.it (L. Arosio), matteo.fiacchi@fmf.uni-lj.si (M. Fiacchi), lorenzo.guerini92@gmail.com (L. Guerini), anders.karlsson@unige.ch (A. Karlsson).

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the disc $\mathbb{D} \subset \mathbb{C}$ obtained by Bracci and Poggi-Corradini in [14,27,28]. In particular, with our geometric approach we are able to answer a question, open even for the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$ (see [5,26]), namely that for holomorphic parabolic self-maps any escaping backward orbit with bounded step always converges to a point in the boundary.

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1. Introduction

The study of dynamics of a holomorphic self-map of the unit disk $f\colon \mathbb{D} \to \mathbb{D}$ goes back to Julia [22] who remarked that the case where f has a fixed point p could be easily treated using the Schwarz lemma: either every forward orbit converges to p or f is an elliptic automorphism. The situation when f has no fixed point is more interesting and was described by Wolff and Denjoy in [18,31]. Their celebrated theorem shows that if f has no fixed point, then there exists a point $\zeta \in \partial\mathbb{D}$, called the *Denjoy–Wolff point* of f, such that every forward orbit converges to ζ. Such result was later generalized to various bounded domains in several complex variables: in particular, Hervé [21] proved it for the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$, and Abate proved it for bounded strongly pseudoconvex domains:

Theorem 1.1 *([2]). Let $f\colon \Omega \to \Omega$ be a holomorphic self-map of a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^q$. If f admits an escaping orbit, then there exists a point $\zeta \in \partial\Omega$ such that every forward orbit converges to ζ.*

We call an orbit *escaping* if it eventually exits any given compact set in X. To prove Theorem 1.1, Abate exploited the non-expansivity of holomorphic maps w.r.t. the Kobayashi distance k_Ω and proved a "Wolff lemma", showing that a forward orbit is contained in a subset (a *big horosphere*) whose closure intersects the boundary in only one point. Notice that once one proves that a forward orbit converges to a point in the

boundary, it is immediate to show that every orbit has to converge to the same point, since the Kobayashi distance between $f^n(z)$ and $f^n(w)$ is bounded by $k_\Omega(z,w)$.

The metric nature of Denjoy–Wolff theorems became even more apparent with the work of Beardon [11], who proved (with a technical assumption) a Denjoy–Wolff theorem for non-expanding self-maps of a proper metric space X with a Hausdorff compactification $\overline{X}$ which satisfies a hyperbolicity condition at the boundary called Axiom II. The full Denjoy–Wolff theorem in this setting was later proved by the fourth-named author [23]. The proof is again based on a Wolff lemma,[4] and on a result by Calka [17] which asserts that the orbits of a non-expanding self-map of a proper metric space are either all relatively compact, or all escaping.

If the metric space is Gromov hyperbolic, then its Gromov compactification satisfies Axiom II, and thus one has the following result.

Theorem 1.2 *([23]). Let $f\colon X \to X$ be a non-expanding self-map of a proper Gromov hyperbolic metric space X. Then either*

i) *every forward orbit of f is relatively compact, or*
ii) *there exists a point ζ in the Gromov boundary $\partial_G X$ such that every forward orbit converges to ζ.*

Since by Balogh-Bonk [10] strongly pseudoconvex domains of $\mathbb{C}^q$ endowed with the Kobayashi distance are Gromov hyperbolic, and since the Gromov compactification is equivalent to the Euclidean compactification, this generalizes Abate's result. It also, together with all the results below of the present paper, applies to a very different setting, namely to homogeneous, order-preserving maps of proper convex cones in nonlinear Perron-Frobenius theory [25]. Such maps induce non-expanding maps in Hilbert's metric on a cross-section of the cone. This cross-section is Gromov hyperbolic for example when it is strictly convex and C^2-smooth [24,13]. For context let us also remark that the literature on non-expanding maps is vast, see e.g. [20].

Going back to the case of a holomorphic self-map of the unit disk $f\colon \mathbb{D} \to \mathbb{D}$, Bracci [14] and Poggi-Corradini [27,28] independently started the study of the backward dynamics of f and its interplay with boundary regular fixed points. The map f is not necessarily invertible, however it makes sense to study its backward dynamics by looking at its *backward orbits*, that is sequences $(z_n)_{n\in\mathbb{N}}$ such that $f(z_n) = z_{n-1}$ for all $n \in \mathbb{N}$. The *step* of a backward orbit (z_n) is defined as

$$\sigma_1 := \sup_n k_{\mathbb{D}}(z_n, z_{n+1}) \in (0, +\infty],$$

where k denotes the Kobayashi distance.

[4] In the same paper [23] a different proof is also given, not based on a Wolff lemma.

The map f does not necessarily extend holomorphically (or even continuously) to the boundary of the disc, yet there is a notion of fixed point ζ at the boundary which is natural in this context, and comes with a positive real number which is a sort of derivative of f at ζ. A point $\zeta \in \partial\mathbb{D}$ is a *boundary regular fixed point* (BRFP for short) if the non-tangential limit of f at ζ is ζ, and if the *dilation* λ_ζ defined by

$$\log \lambda_\zeta := \liminf_{z\to\zeta} k_{\mathbb{D}}(0,z) - k_{\mathbb{D}}(0,f(z)) \tag{1.1}$$

is finite. The dilation λ_ζ at a BRFP has an interesting interpretation: by the Julia–Wolff–Carathéodory theorem (see e.g. [1]) λ_ζ is equal to the non-tangential limit as $z \to \zeta$ of both the derivative $f'(z)$ and the incremental ratio $(f(z)-1)/(z-1)$.

A BRFP is *attracting* if $\lambda_\zeta < 1$, *indifferent* if $\lambda_\zeta = 1$, and *repelling* if $\lambda_\zeta > 1$. This allows to classify holomorphic self-maps of $\mathbb{D}$ as follows: f is *elliptic* if it has a fixed point in $\mathbb{D}$. If f is not elliptic, then one can show that its Denjoy–Wolff point ζ is a BRFP, which cannot be repelling. We then say that f is *parabolic* if ζ is indifferent, and that f is *hyperbolic* if ζ is attracting.

The backward dynamics of f is described by the following two results. The first result shows the existence of backward orbits with bounded step converging to a given repelling BRFP.

Theorem 1.3 *([27]). Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map, and let η be a repelling BRFP. Then there exists a backward orbit (z_n) converging radially to η with step $\log \lambda_\eta$.*

Poggi-Corradini used Theorem 1.3 to construct canonical pre-models (or Poincaré maps) associated with repelling BRFPs. The proof of Poggi-Corradini of Theorem 1.3 was generalized to the ball by Ostapyuk [26], and to strongly convex domains by Abate–Raissy [4,5], in both cases with the additional assumption that the BRFP is isolated. A proof without such assumption was later given in the ball in [9], and in strongly convex domains in [6]. In [9] and [6] this result was then used to develop a theory of canonical pre-models in several complex variables.

The second result can be thought of as a backward version of the Denjoy–Wolff theorem. Notice that relatively compact backward orbits are trivial and can only exist if the map is elliptic (see Proposition 6.2).

Theorem 1.4 *([14,28]). Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map, and let (z_n) be a backward orbit with bounded step. If (z_n) is not relatively compact, then it converges to a BRFP $\eta \in \partial\mathbb{D}$. Moreover, we have the following dichotomy: either*

i) *η is repelling with dilation satisfying $\log \lambda_\zeta \le \sigma_1$, and (z_n) converges to η non-tangentially, or*
ii) *η is indifferent, f is parabolic and η is its Denjoy–Wolff point, and (z_n) converges to η tangentially.*

Theorem 1.4 was applied by Bracci [14] to study boundary fixed points of commuting self-map of the disk. Theorem 1.4 had partial generalizations in several variables: Ostapyuk [26] treated the case of the ball $\mathbb{B}^q$ and Abate–Raissy considered strongly convex domains [4,5]. In both cases convergence to a BRFP was established for hyperbolic maps and for elliptic maps which admit a point p such that every forward orbit converges to p, also called *strongly elliptic maps*. The question whether for a parabolic self-map every non-relatively compact backward orbit with bounded step converges to a point of the boundary remained open even in the ball $\mathbb{B}^q$, see [26, Question 6.2.3], and [5, Remark 2.4]: "*[...] Thus the behavior of backward orbits for parabolic self-maps is still not understood, even (as far as we know) in the unit ball of* $\mathbb{C}^n$. Theorem 1.6 below gives a positive answer to this question.

In this paper we show that, as in the forward dynamics case, the holomorphic structure does not play a relevant role in Theorems 1.3 and 1.4. Indeed, we generalize both results to the case of a non-expanding self-map $f\colon X \to X$ of a proper geodesic Gromov hyperbolic metric space. To do so, one first needs to define the concepts of dilation and BRFP in this setting. This has been done in [8], under the additional assumption that the Gromov compactification of the metric space X is equivalent to the horofunction compactification. Under this assumption the authors prove in [8] a generalization of the classical Julia Lemma to the setting of non-expanding maps (see Theorem 2.19 below). As a consequence it is showed that the dilation introduced by Abate for holomorphic maps of strongly convex domains is the right notion also in this context: the dilation $\lambda_{\zeta,p}$ at a BRFP ζ is defined as

$$\log \lambda_{\zeta,p}(f) := \liminf_{x\to\zeta} d(x,p) - d(f(x),p),$$

where p is a given base-point. The dilation does not depend on the chosen base-point.

However, if the metric space X does not satisfy the assumption of equivalence of the two compactifications, then this definition of dilation fails to detect the attracting/indifferent/repelling character of the BRFP, as one can construct an example of a space X with a hyperbolic isometry whose dilation is strictly less than 1 at both the fixed points at the boundary (see Example 4.2). We show that this issue disappears if one considers instead the *stable dilation* Λ_ζ defined as

$$\log \Lambda_\zeta := \lim_{n\to+\infty} \frac{\log \lambda_{\zeta,p}(f^n)}{n}.$$

The limit in the above definition exists and is finite thanks to the results in Section 3 studying the behavior of the dilation $\lambda_{\zeta,p}$ under iteration of f. In Section 4 we prove that, even if the Gromov and horofunction compactifications are not equivalent, one can still prove an approximate Julia lemma (see Lemma 3.1), with an error depending only on the Gromov constant of the space X. As consequence we will show that the stable dilation enjoys all the expected properties, for instance if the map f is non-elliptic, then its

Denjoy–Wolff point is the only BRFP with stable dilation ≤ 1. Thus the stable dilation can be used to define attracting/indifferent/repelling BRFPs. Moreover we show that the stable dilation equals the dilation if the Gromov and horofunction compactifications are equivalent.

Before stating our two main results, we need a last definition. If (x_n) is a backward orbit with bounded step, then for all $m \geq 1$ set $\sigma_m(x_n) := \lim_{n\to+\infty} d(x_{n+m}, x_n)$. The *backward step rate* of (x_n) is then defined as

$$b(x_n) := \lim_{m\to+\infty} \sigma_m(x_n)/m.$$

This number plays an important role in understanding the dynamics of (x_n). Section 5 is devoted to the proof of the first main result, generalizing Theorem 1.3.

Theorem 1.5. *Let (X, d) be a proper geodesic Gromov hyperbolic space, and let $f\colon X \to X$ be a non-expanding map. Assume that $\eta \in \partial_G X$ is a repelling BRFP with stable dilation $\Lambda_\eta > 1$. Then there exists a backward orbit (x_n) converging to ζ with backward step rate $b(x_n) = \log \Lambda_\eta$.*

It is easy to construct an example where no backward orbit (x_n) converging to ζ has step $\sigma_1(x_n) = \log \Lambda_\zeta$, (for instance the backward orbits of Example 4.2). Similarly as in [9] and [6] one constructs the backward orbit (x_n) as the limit of an iterative process with stopping time prescribed by a horosphere centered at the BRFP ζ. The main novelty with respect to the previous proofs is the use of Gromov's four-point condition to show that this iterative process converges.

The second main result, generalizing Theorem 1.4, is proved in Section 6. An elliptic map is *strongly elliptic* if the union of the ω-limits of its forward orbits (the *limit retract* of f) is relatively compact in X, otherwise it is *weakly elliptic*. Notice that this definition agrees with the one previously given for holomorphic maps.

Theorem 1.6. *Let (X, d) be a proper geodesic Gromov hyperbolic space, and let $f\colon X \to X$ be a non-expanding map, not weakly elliptic. Let (x_n) be a backward orbit with bounded step. If (x_n) is not relatively compact, then it converges to a BRFP $\eta \in \partial_G X$. Moreover, we have the following dichotomy: either*

i) *$b(x_n) > 0$, η is repelling with stable dilation satisfying $\log \Lambda_\zeta = b(x_n)$, and (z_n) is a discrete quasigeodesic converging to η inside a geodesic region, or*
ii) *$b(x_n) = 0$, f is parabolic, η is the Denjoy–Wolff point of f, and (z_n) converges to η avoiding an horosphere $\{h_{a,p} \leq c\}$ centered in a point a of the horofunction boundary associated with η.*

This answers[5] positively [7, Question 9.6]. It is interesting to notice that the available proofs of the Denjoy–Wolff theorem do not appear to work when applied to backward orbits. In particular, the classical proof based the Wolff lemma cannot work since the Wolff lemma does not hold for backward orbits with bounded step, as shown by the following counterexample given by Poggi-Corradini in [28]: the parabolic holomorphic self-map $f(z) = \sqrt{z^2 - 1}$ of the upper half-plane has the backward orbit with bounded step $(\sqrt{n + i})$ converging to the Denjoy–Wolff point ∞ and eventually leaving any horosphere centered at ∞. On the other side, the proof of Theorem 1.6 may be easily adapted to forward dynamics to give an alternative proof of the Denjoy–Wolff theorem in proper geodesic Gromov hyperbolic metric spaces, different from the two proofs in [23]. The crucial point of the proof of Theorem 1.6 is that the backward step rate of (x_n) is strictly positive if and only if (x_n) is a discrete quasigeodesic. Hence, if $b(x_n) > 0$, we can exploit Gromov's shadowing lemma and obtain convergence of (x_n) to a repelling BRFP, inside a geodesic region. If $b(x_n) = 0$, we show that f cannot be strongly elliptic or hyperbolic, hence it has to be parabolic. The proof is then complete, since we show in Section 4 that the limit points of a backward orbit with bounded step are BRFPs with stable dilation ≤ 1. Hence the only limit point is the Denjoy–Wolff point of f. If the map f is weakly elliptic, then we cannot exclude that a backward orbit with bounded step could have limit set larger than a point and contained in the intersection of the Gromov boundary of X with the closure of the limit retract of f.

Notice that Theorems 1.5 and 1.6 can be applied to holomorphic self-maps of bounded strongly pseudoconvex domains in $\mathbb{C}^q$, to smoothly bounded convex domains of finite D'Angelo type in $\mathbb{C}^q$, and to smoothly bounded pseudoconvex domains of finite D'Angelo type in $\mathbb{C}^2$. The Gromov compactification is equivalent to the Euclidean compactifications in all those cases (see respectively [10], [32] and [19]).

2. Preliminaries

We start reviewing some basic definitions and results which we will need in the following sections.

2.1. Gromov hyperbolicity

Definition 2.1. Let $\delta > 0$. A proper geodesic metric space (X, d) is *δ-hyperbolic* if for every geodesic triangle, any side is contained in the δ-neighborhood of the union of the two other sides. The space (X, d) is *Gromov hyperbolic* if it is δ-hyperbolic for some δ.

Definition 2.2 *(Gromov compactification).* Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $\mathscr{R}(X)$ denote the set of geodesic rays in X. On $\mathscr{R}(X)$, the

[5] Given a holomorphic map $f\colon \mathbb{B}^q \to \mathbb{B}^q$, if a backward orbit with bounded step converges to the Denjoy–Wolff point, then its associated canonical model is parabolic.

relation

$$\gamma \sim_r \sigma \iff \gamma \text{ and } \sigma \text{ are asymptotic}$$

is an equivalence relation. The *Gromov boundary* of X is defined as $\partial_G X := \mathscr{R}(X)/_{\sim_r}$. The *Gromov compactification* of X is the set $\overline{X}^G := X \sqcup \partial_G X$ endowed with a suitable compact metrizable topology (see e.g. [15, Chapter III.H]).

Remark 2.3. There is an alternative definition of Gromov hyperbolic spaces. If (X, d) is a metric space, the *Gromov product* of x, y w.r.t. z is defined as

$$(x, y)_z := \frac{1}{2}(d(x, z) + d(y, z) - d(x, y)).$$

We say that the space (X, d) satisfies the *Gromov four-point condition* with constant $C \geq 0$ if

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - C, \quad \forall\, w, x, y, z \in X.$$

Let X be a proper geodesic metric space. If X is δ-hyperbolic then it satisfies the Gromov four-point condition with $C = 8\delta$. Conversely, if X satisfies the Gromov four-point condition with constant $C \geq 0$, then it is $4C$-hyperbolic (see e.g. [16, Proposition 3.6]).

Definition 2.4. Let (X, d) be a proper geodesic metric space. Fix $A \geq 1$, $B \geq 0$. Let $I \subset \mathbb{R}$ a closed (possibly unbounded) interval. A map $\gamma\colon I \to X$ is a (A, B)-*quasi-geodesic* if for every $s, t \geq 0$

$$A^{-1}|t - s| - B \leq d(\gamma(s), \gamma(t)) \leq A|t - s| + B.$$

If $I = \mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}$) we say that γ is a (A, B)-*quasi-geodesic ray* (resp. *line*).

A sequence $(x_n)_{n\geq 0}$ is a *discrete* (A, B)-*quasi-geodesic ray* if for every $n, m \geq 0$

$$A^{-1}|n - m| - B \leq d(x_n, x_m) \leq A|n - m| + B.$$

Similarly one can define discrete quasi-geodesics lines $(x_n)_{n\in\mathbb{Z}}$. By [8, Remark 6.22] a discrete (A, B)-quasi-geodesic ray can be interpolated with a $(A, A + B)$-quasi-geodesic ray.

If the metric space (X, d) is Gromov hyperbolic, then (A, B)-quasi-geodesics are "shadowed" by actual geodesics as the following fundamental result shows (for a proof, see e.g. [16, Théorème 3.1, p. 41]). Denote by $d_{\mathcal{H}}$ the Hausdorff distance.

Theorem 2.5 *(Gromov's shadowing lemma). Let (X, d) be a proper geodesic δ-hyperbolic metric space. Let $\sigma\colon [0, +\infty) \to X$ be a (A, B)-quasi-geodesic ray. Then there exist a geodesic ray $\gamma\colon [0, +\infty) \to X$ such that $\gamma(0) = \sigma(0)$ and such that*

$$d_{\mathcal{H}}(\gamma, \sigma) \leq C(\delta, A, B).$$

As a consequence, if α and β are two quasi-geodesic rays with the same endpoint in $\partial_G X$, then their Hausdorff distance $d_{\mathcal{H}}(\alpha, \beta)$ is bounded.

Let $f\colon X \to X$ be a non-expanding self-map of a proper geodesic Gromov hyperbolic metric space. The concepts of geodesic regions/geodesic limits were introduced in [8], generalizing classical concepts in complex analysis: Stolz regions/non-tangential limits in the disk $\mathbb{D} \subset \mathbb{C}$, and Koranyi regions/$K$-limits in the ball $\mathbb{B}^q \subset \mathbb{C}^q$ and in strongly convex domains (see e.g. [1]). The same is true for the concepts of dilation and boundary regular fixed point that will be introduced later on.

Definition 2.6 *(Geodesic region).* Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Given $R > 0$ and a geodesic ray $\gamma \in \mathscr{R}(X)$, the *geodesic region* $A(\gamma, R)$ is the open subset of X of the form

$$A(\gamma, R) := \{x \in X\colon d(x, \gamma) < R\}.$$

The point $[\gamma] \in \partial_G X$ is called the *vertex* of the geodesic region.

Definition 2.7 *(Geodesic limit).* Let (X, d) be a proper geodesic Gromov hyperbolic metric space and let Y be a Hausdorff topological space. Let $f\colon X \to Y$ a map, and let $\eta \in \partial_G X$, $\xi \in Y$. We say that f has *geodesic limit* ξ at η if for every sequence (x_n) converging to η contained in a geodesic region with vertex η, the sequence $(f(x_n))$ converges to ξ.

2.2. Horofunctions

Definition 2.8 *(Horofunction compactification).* Let (X, d) be a proper metric space. Let $C_*(X)$ be the quotient of $C(X)$ by the subspace of constant functions. Given $f \in C(X)$, we denote its equivalence class by $\overline{f} \in C_*(X)$.

Consider the embedding

$$i_H\colon X \longrightarrow C_*(X)$$

which sends a point $x \in X$ to the equivalence class of the function $d_x\colon y \mapsto d(x, y)$. The *horofunction compactification* $\overline{X}^H$ of X is the closure of $i_H(X)$ in $C_*(X)$. The *horofunction boundary* of X is the (compact) set

$$\partial_H X := \overline{X}^H \setminus i_H(X).$$

Let $a \in \partial_H X$. An *horofunction* centered at $a \in \partial_H X$ is an element $h \in C(X)$ satisfying $\bar{h} = a$. For every $p \in X$, the unique horofunction centered at a and vanishing at p is denoted by $h_{a,p}$. Let $c \in \mathbb{R}$. The level set $\{h_{a,p} < c\}$ (or $\{h_{a,p} \leq c\}$) is called a *horosphere* (or *horoball*) centered at a.

Let γ be a geodesic ray. The *Busemann function* $B_\gamma \colon X \times X \to \mathbb{R}$ associated with γ is defined as

$$B_\gamma(x,y) := \lim_{t\to+\infty} d(x,\gamma(t)) - d(\gamma(t),y).$$

For all $y \in X$, the function $x \mapsto B_\gamma(x,y)$ is a horofunction, and its class $\overline{B}_\gamma \in \partial_H X$ does not depend on $y \in X$.

The following results show how the Gromov and horofunction compactifications are related on a proper geodesic Gromov hyperbolic metric space. Given $A \subset X$, we denote by $\overline{A}^G$ the closure of A in the Gromov compactification.

Proposition 2.9 *([30, Proposition 4.6]). Let (X,d) be a proper geodesic Gromov hyperbolic metric space. There exists a continuous map $\Phi : \overline{X}^H \to \overline{X}^G$ such that $\mathrm{id}_X = \Phi \circ i_H$, where $i_H \colon X \to \overline{X}^H$ denotes the embedding of the space X into the horofunction compactification.*

Proposition 2.10 *([30, Proposition 4.4]). Let (X,d) be a proper geodesic Gromov hyperbolic metric space, and let $p \in X$. Given $a,b \in \partial_H X$, we have $\Phi(a) = \Phi(b)$ if and only if*

$$\sup_{x\in X} |h_{a,p}(x) - h_{b,p}(x)| \le M, \qquad \text{for some } M > 0.$$

Furthermore if the space is δ-hyperbolic and $\Phi(a) = \Phi(b)$, then we can choose $M = 2\delta$.

Proposition 2.11. *[8] Let (X,d) be a proper geodesic Gromov hyperbolic metric space, and let $p \in X, a \in \partial_H X, \eta = \Phi(a) \in \partial_G X$. Then*

i) *for all $c \in \mathbb{R}$, we have $\overline{\{h_{a,p} \le c\}}^G \cap \partial_G X = \{\eta\}$;*
ii) *$\bigcap_{c\in\mathbb{R}} \overline{\{h_{a,p} \le c\}}^G = \{\eta\}$, and thus if (x_n) is a sequence in X such that $h_{a,p}(x_n) \to -\infty$, then $x_n \to \eta$;*
iii) *if (x_n) is a sequence converging to η inside a geodesic region, then $h_{a,p}(x_n) \to -\infty$.*

The main tool to generalize the classical Julia Lemma to non-expanding maps is the following lemma.

Lemma 2.12. *[8, Lemma 6.14] Let (X,d) be a proper metric space and $f \colon X \to X$ a non-expanding map. Let $p \in X$. Assume that there exists a sequence (w_n) in X such that*

(1) $w_n \underset{n\to\infty}{\to} a \in \partial_H X$,
(2) $f(w_n) \underset{n\to\infty}{\to} b \in \partial_H X$,
(3) $d(p,w_n) - d(p,f(w_n)) \underset{n\to\infty}{\to} A < +\infty$.

Then

$$h_{b,p} \circ f \leq h_{a,p} + A. \tag{2.1}$$

Definition 2.13 *(Dilation).* Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f\colon X \to X$ be a non-expanding self-map. Given $\eta \in \partial_G X$, the *dilation* of f at η with respect to the base point $p \in X$ as the number $\lambda_{\eta,p} > 0$ defined by

$$\log \lambda_{\eta,p} = \liminf_{z \to \eta} d(z, p) - d(f(z), p).$$

Remark 2.14. It is easy to see that $\log \lambda_{\eta,p} > -\infty$, and that the condition $\lambda_{\eta,p} < +\infty$ is independent on the choice of the base point $p \in X$. Indeed If, $p, q \in X$,

$$\log \lambda_{\eta,q} \leq \log \lambda_{\eta,p} + 2d(p, q).$$

Proposition 2.15. *[8, Proposition 6.15 and 6.16] Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f\colon X \to X$ be a non-expanding self-map. Let $p \in X$ and $\eta \in \partial_G X$ such that $\lambda_{\eta,p} < +\infty$. Then there exists $\xi \in \partial_G X$ such that f has geodesic limit ξ at η. Moreover, if (x_n) is a sequence in X converging to η such that the sequence $d(x_n, p) - d(f(x_n), p)$ is bounded from above, then $f(x_n)$ converges to ξ.*

Proof. We recall the proof since it will be relevant in what follows. Let (x_n) be a sequence converging to η such that $d(x_n, p) - d(f(x_n), p) \leq A$, with $A \in \mathbb{R}$. Up to extracting subsequences we may assume that $x_n \to a \in \partial_H X$ and $f(x_n) \to b \in \partial_H X$, where $\Phi(a) = \eta$. Then by Lemma 2.12 we have that

$$h_{b,p} \circ f \leq h_{a,p} + A.$$

Let now (w_n) be a sequence converging to η inside a geodesic region. Then by Proposition 2.11 iii) we have that $h_{a,p}(w_n) \to -\infty$, and thus $h_{b,p}(f(w_n)) \to -\infty$. This means that $(f(w_n))$ is eventually contained in a horosphere centered in b, and thus by Proposition 2.11 we have that $f(w_n) \to \Phi(b) \in \partial_G X$. □

Definition 2.16 *(Boundary regular fixed points).* Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f\colon X \to X$ be a non-expanding map. We say that a point $\eta \in \partial_G X$ is a *boundary regular fixed point* (BRFP for short) if $\lambda_{\eta,p} < +\infty$ and if f has geodesic limit η at η.

Lemma 2.17. *Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f : X \to X$ be a non-expanding map, let $p \in X$ and let $\eta \in \partial_G X$ be a BRFP. Suppose that (x_n) is a sequence in X converging to η and such that*

$$\limsup_{n \to \infty} d(p, x_n) - d(p, f(x_n)) = A < +\infty.$$

Then for any other $q \in X$ it holds

$$\limsup_{n\to\infty} d(q, x_n) - d(q, f(x_n)) \leq A + 2\delta.$$

If moreover $\overline{X}^H$ is equivalent to $\overline{X}^G$, then

$$\limsup_{n\to\infty} d(q, x_n) - d(q, f(x_n)) \leq A.$$

Proof. Since η is a BRFP, by Proposition 2.15 the sequence $(f(x_n))$ converges to η.

Let (w_k) be a subsequence of (x_n) such that

$$d(q, w_k) - d(q, f(w_k)) \overset{k\to\infty}{\longrightarrow} \limsup_{n\to\infty} d(q, x_n) - d(q, f(x_n)).$$

Up to extracting further we can assume that there exist $a, b \in \Phi^{-1}(\eta)$ such that $w_k \to a$, $f(w_k) \to b$. We have

$$\begin{aligned}&\Big(d(q, w_k) - d(q, f(w_k))\Big) - \Big(d(p, w_k) - d(p, f(w_k))\Big)\\ &= \Big(d(q, w_k) - d(p, w_k)\Big) - \Big(d(q, f(w_k)) - d(p, f(w_k))\Big) \overset{k\to\infty}{\longrightarrow} h_{a,p}(q) - h_{b,p}(q) \leq 2\delta,\end{aligned}$$

where in the last inequality we used Proposition 2.10. The proof of the last statement is similar. □

Remark 2.18. Hence if η is a BRFP, then for all $p, q \in X$

$$|\log \lambda_{\eta,p} - \log \lambda_{\eta,q}| \leq 2\delta.$$

If $\overline{X}^H \simeq \overline{X}^G$ the previous lemma shows that the dilation $\lambda_{\eta,p}$ is independent of the base-point p, and will thus be denoted λ_η in what follows (cf. [8, Proposition 6.30])).

A direct generalization of the Julia lemma is obtained in [8, Theorem 6.28] on proper geodesic Gromov hyperbolic metric spaces such that the Gromov compactification of X is equivalent to the horofunction compactification of X.

Theorem 2.19 *(Metric Julia lemma). Let (X, d) be a proper geodesic Gromov hyperbolic metric space such that $\overline{X}^H$ is equivalent to $\overline{X}^G$. Let $f\colon X \to X$ be a non-expanding self-map. Let $\eta \in \partial_G X$ and $p \in X$ be such that $\lambda_{\eta,p} < +\infty$. Let $\xi \in \partial_G X$ be the geodesic limit of f at η. Then*

$$h_{\eta,p} \circ f \leq h_{\xi,p} + \log \lambda_{\eta,p}. \tag{2.2}$$

2.3. Forward dynamics of elliptic maps

A result of Calka shows that an interesting dynamical dichotomy holds for non-expanding self-maps of proper metric spaces: orbits are either bounded or *escaping* (that is, there are no bounded subsequences).

Theorem 2.20. *[17] Let (X, d) a proper metric space and let $f\colon X \to X$ be a non-expanding map. If a forward orbit $(f^n(x))$ is unbounded, then $d(f^n(x), x) \to +\infty$.*

Definition 2.21. Let $f\colon X \to X$ a non-expanding self-map of a proper metric space. We say that f is *elliptic* if it has a bounded orbit, or equivalently, if every orbit is bounded.

Definition 2.22. Let $f\colon X \to X$ a non-expanding self-map of a proper metric space, and let $x \in X$. The *ω-limit* of f at x, denoted $\omega_f(x)$, is the limit set of the forward orbit $(f^n(x))$, that is the set of limits points of the sequence $(f^n(x))$. We denote

$$\omega_f := \bigcup_{x\in X} \omega_f(x).$$

The following result is classical in the context of holomorphic self-maps of taut manifolds, see Abate [1] (see also Bedford [12]). Abate's proof adapts immediately to the case of non-expanding maps of a proper metric space. A proof in the non-expanding case of points (i) and (ii) in the theorem below is given in Lemmens-Nussbaum [25].

Theorem 2.23. *Let (X, d) a proper metric space and let $f\colon X \to X$ be an elliptic non-expanding map.*

i) *There exists a subsequence of iterates (f^{n_k}) converging uniformly on compact subsets to a non-expanding retraction $r\colon X \to X$ whose image is ω_f.*
ii) *$f(\omega_f) = \omega_f$ and $f|_{\omega_f}$ is an isometry of ω_f.*
iii) *Every limit point $h\colon X \to X$ of the sequence (f^n) is of the form*

$$h = \gamma \circ r,$$

where γ is an isometry of ω_f.

Remark 2.24. Being a retract of a Hausdorff space, ω_f is a closed subset of X.

Definition 2.25. We call ω_f the *limit retract* of the map f. An elliptic non-expanding self-map is *strongly elliptic* if the limit retract is compact, and is *weakly elliptic* otherwise.

Remark 2.26. When X is a complete hyperbolic complex domain in $\mathbb{C}^q$ endowed with the Kobayashi distance and the map f is holomorphic, the limit retract is a holomorphic

retract. Since every holomorphic retract is a complex submanifold of X, it is compact if and only if it is a point. Hence Definition 2.25 agrees with the one given by Abate–Raissy in [4].

We end this section recalling an interesting property of images of non-expanding retracts.

Definition 2.27. Let (X, d) be a metric space. A subset $Z \subset X$ is a *non-expanding retract* if it is the image of a non-expanding retraction $\rho\colon X \to X$.

Remark 2.28. Endow a non-expanding retract Z of X with the metric d' induced from d. Then the metric space (Z, d') is geodesic since for all geodesic γ in X connecting points $x, y \in Z$ the composition $\rho \circ \gamma$ is a geodesic in Z connecting x and y. Moreover Z is a closed subset of X and thus (Z, d') is proper. If X is Gromov hyperbolic, then also Z is Gromov hyperbolic. The inclusion $i\colon (Z, d') \to (X, d)$ is an isometric embedding of proper geodesic Gromov spaces and thus by [15, Theorem 3.9, Chapter III.H], it extends to a topological embedding $i\colon \overline{Z}^G \to \overline{X}^G$.

3. Behavior of the dilation at a BRFP under iterates

In this section we study how the dilation at a BRFP changes when we iterate the map f. This will be relevant in Section 4 when we introduce the concept of *stable dilation*.

We start showing that an approximate Julia Lemma holds even if the Gromov and horofunction compactifications are not equivalent. Moreover the "error" depends only on the Gromov constant δ.

Lemma 3.1 *(δ-Julia Lemma). Let (X, d) be a proper geodesic δ-hyperbolic metric space and let $f : X \to X$ be a non-expanding map. Let $\eta \in \partial_G X$ and $p \in X$ be such that $\lambda_{\eta,p} < +\infty$. Let $\xi \in \partial_G X$ be the geodesic limit of f at η. Then, if $a \in \Phi^{-1}(\eta)$ and $b \in \Phi^{-1}(\xi)$, we have*

$$h_{b,p} \circ f \leq h_{a,p} + \log \lambda_{\eta,p} + 4\delta.$$

Proof. Let (x_n) be a sequence in X converging to η and such that $d(x_n, p) - d(f(x_n), p) \to \log \lambda_{\eta,p}$. By Proposition 2.15 we have that $(f(x_n))$ converges to ξ. Up to extracting subsequences, we may assume that $x_n \to a_0 \in \Phi^{-1}(\eta)$ and $f(x_n) \to b_0 \in \Phi^{-1}(\xi)$. By Lemma 2.12 we have

$$h_{b_0,p} \circ f \leq h_{a_0,p} + \log \lambda_{\eta,p}$$

Now if $a \in \Phi^{-1}(\eta)$ and $b \in \Phi^{-1}(\xi)$, by Proposition 2.10 we have, for all $x \in X$,

$$h_{a_0,p}(x) \leq h_{a,p}(x) + 2\delta$$

and

$$h_{b,p}(f(x)) \leq h_{b_0,p}(f(x)) + 2\delta$$

which implies

$$h_{b,p}(f(x)) \leq h_{a,p}(x) + \log \lambda_{\eta,p} + 4\delta. \quad \square$$

With this tool in hand, we can prove the main result of this section.

Proposition 3.2. *Let (X, d) be a proper geodesic Gromov hyperbolic metric space and let $f\colon X \to X$ be a non-expanding self-map. Let $p \in X$ and let η be a BRFP. Then for all $n \geq 2$ the point η is also a BRFP for the map f^n. Moreover, the sequence $(\frac{1}{n} \log \lambda_{\eta,p}(f^n))$ is bounded.*

The proof requires several preliminary results.

Lemma 3.3. *Let (X, d) be a proper geodesic δ-hyperbolic metric space and let $f : X \to X$ be a non-expanding map. Let $\eta \in \partial_G X$ be a BRFP and let $\gamma : [0, +\infty) \to X$ be a geodesic ray such that $\gamma(+\infty) = \eta$. Then*

$$\lim_{t \to +\infty} d(\gamma(t), \gamma(0)) - d(f(\gamma(t)), \gamma(0)) \leq \log \lambda_{\eta,\gamma(0)} + 4\delta,$$

and for all $p \in X$,

$$\limsup_{t \to +\infty} d(\gamma(t), p) - d(f(\gamma(t)), p) \leq \log \lambda_{\eta,p} + 8\delta.$$

Proof. Notice that $d(\gamma(t), \gamma(0))) - d(f(\gamma(t)), \gamma(0)))$ is non-decreasing in t, hence its limit exists. Let $a \in \partial_H X$ the Busemann point of γ. By the δ-Julia Lemma 3.1 we have, for all $t \geq 0$,

$$h_{a,\gamma(0)}(f(\gamma(t))) - h_{a,\gamma(0)}(\gamma(t)) \leq \log \lambda_{\eta,\gamma(0)} + 4\delta.$$

Since $h_{a,\gamma(0)}(f(\gamma(t))) \geq -d(f(\gamma(t)), \gamma(0))$ and $h_{a,\gamma(0)}(\gamma(t)) = -t$, it follows that

$$t - d(f(\gamma(t)), \gamma(0)) \leq \log \lambda_{\eta,\gamma(0)} + 4\delta.$$

The result now follows from Lemma 2.17. $\square$

Lemma 3.4. *Let (X, d) be a proper geodesic δ-hyperbolic metric space and let $f : X \to X$ be a non-expanding map. Let $\eta \in \partial_G X$ be a BRFP and let $\gamma : [0, +\infty) \to X$ be a geodesic ray such that $\gamma(+\infty) = \eta$. Then there exists $T \geq 0$ such that $f \circ \gamma : [T, +\infty) \to X$ is a $(1, 4\delta + 2)$-quasi-geodesic with endpoint η.*

Proof. First of all, since f is non-expanding we have for each $t_1, t_2 \geq 0$

$$d(f(\gamma(t_1)), f(\gamma(t_2))) \leq |t_1 - t_2|$$

Let $T \geq 0$ such that for each $t \geq T$ we have

$$\log \lambda_{\eta,\gamma(0)} - 1 \leq t - d(f(\gamma(t)), \gamma(0))) \leq \log \lambda_{\eta,\gamma(0)} + 4\delta + 1.$$

Now for each $T \leq t_1 \leq t_2$ we have

$$d(f(\gamma(t_1)), f(\gamma(t_2))) \geq d(f(\gamma(t_2)), \gamma(0)) - d(f(\gamma(t_1)), \gamma(0)) \geq t_2 - t_1 - 4\delta - 2.$$

The endpoint of $f \circ \gamma$ is η since by assumption f has geodesic limit η at η. □

Proof of Proposition 3.2. First of all notice that by Remark 2.14, if we prove the result for a given base-point p, then it holds for all base-points. Let γ be a geodesic ray with endpoint η, and set $p := \gamma(0)$. Denote $a_n := \log \lambda_{\eta,p}(f^n)$. By Lemma 3.4 there exists $T \geq 0$ such that the curve $\sigma := f \circ \gamma$ is a $(1, 4\delta + 2)$-quasi-geodesic when $t \geq T$, and $\sigma(+\infty) = \eta$. It follows from [8, Lemma 5.8] that there exists $M \geq 0$ such that $d(\gamma(t), \sigma(t)) \leq M$ for all $t \geq 0$. We show by induction that for all $n \geq 1$ we have

$$a_n \leq n[a_1 + 8\delta + M + d(f(p), p)],$$

and that f^n has geodesic limit η at η. This is clear for $n = 1$. Assume it true for $n \geq 1$. We have

$$\begin{aligned}
&\liminf_{t\to+\infty} d(\sigma(t), p) - d(f^n(\sigma(t)), p) \\
\leq\ &\liminf_{t\to+\infty} d(\sigma(0), p) + t - d(p, f^n(\gamma(t))) + d(f^n(\sigma(t)), f^n(\gamma(t))) \\
\leq\ &a_n + 4\delta + M + d(f(p), p).
\end{aligned}$$

Now

$$\begin{aligned}
a_{n+1} &\leq \liminf_{t\to+\infty} d(\gamma(t), p) - d(f^{n+1}(\gamma(t)), p) \\
&= \liminf_{t\to+\infty} d(\gamma(t), p) - d(f(\gamma(t)), p) + d(f(\gamma(t)), p) - d(f^{n+1}(\gamma(t)), p) \\
&\leq a_1 + 4\delta + a_n + 4\delta + M + d(f(p), p) \\
&\leq (n+1)[a_1 + 8\delta + M + d(f(p), p)].
\end{aligned}$$

By Proposition 2.15 it follows that the map f^{n+1} has a geodesic limit as $x \to \eta$. This limit is η, since $f^{n+1} \circ \gamma = f^n \circ (f \circ \gamma)$, and f^n has geodesic limit η at η, while $f \circ \gamma$ is contained in a geodesic region. □

Remark 3.5. Let $f\colon D \to D$ be a holomorphic self-map of a bounded strongly convex domain $D \subset \mathbb{C}^q$. Let $\eta \in \partial D$ be a BRFP, and let $\varphi\colon \mathbb{D} \to D$ be a complex geodesic such that $\varphi(1) = \eta$. Then Abate–Raissy proved [4, Lemma 3.1] that

$$\lim_{t\to 1-} k_D(\varphi(t), f(\varphi(t))) = |\log \lambda_\eta|.$$

The following result shows that an approximate version of this result holds for non-expanding maps of proper geodesic Gromov hyperbolic metric spaces, where again the error depends only on the Gromov constant δ.

Proposition 3.6. *Let (X, d) be a proper geodesic Gromov hyperbolic metric space and let $f : X \to X$ be a non-expanding map. Let γ be a geodesic ray converging to a BRFP η. Let $p = \gamma(0)$. Then there exists a constant $C(\delta) \geq 0$, depending only on δ, such that*

$$|\log \lambda_{\eta,p}(f)| \leq \liminf_{t\to+\infty} d(\gamma(t), f(\gamma(t))) \leq \limsup_{t\to+\infty} d(\gamma(t), f(\gamma(t))) \leq |\log \lambda_{\eta,p}(f)| + C(\delta).$$

Proof. The first inequality follows immediately from the triangle inequality:

$$d(\gamma(t), f(\gamma(t))) \geq |d(\gamma(t), p) - d(f(\gamma(t)), p)|.$$

We now prove the third inequality. By Lemma 3.4 there exists $T \geq 0$ such that the curve $f \circ \gamma|_{[T,+\infty)}$ is a $(1, 4\delta + 2)$-quasi-geodesic with endpoint η. Let θ be a geodesic ray with $\theta(0) = f(\gamma(T))$ and endpoint η. By Gromov's shadowing lemma (Theorem 2.5) there exists a constant $C'(\delta) \geq 0$ such that the curve $f \circ \gamma|_{[T,+\infty)}$ is contained in a $C'(\delta)$-neighborhood of the geodesic θ. By [15, Lemma 3.3, Chapter III.H] there exist $T_1, T_2 \geq 0$ such that for all $t \geq 0$,

$$d(\gamma(t + T_1), \theta(t + T_2)) \leq 5\delta.$$

Set $C''(\delta) := C'(\delta) + 5\delta$. It immediately follows that there exists a constant $T_3 \geq 0$ such that

$$d(f(\gamma(t)), \gamma) \leq C''(\delta), \quad \forall\, t \geq T_3.$$

For all $t \geq T_3$ let $s_t \geq 0$ be such that $d(f(\gamma(t)), \gamma(s_t)) = d(f(\gamma(t)), \gamma)$. Then for all $t \geq T_3$,

$$d(\gamma(t), f(\gamma(t))) \leq d(\gamma(t), \gamma(s_t)) + d(\gamma(s_t), f(\gamma(t)) \leq d(\gamma(t), \gamma(s_t)) + C''(\delta).$$

Moreover

$$\begin{aligned} d(\gamma(t),\gamma(s_t)) &= |d(\gamma(t),p) - d(\gamma(s_t),p)| \\ &= |d(\gamma(t),p) - d(p,f(\gamma(t))) + d(p,f(\gamma(t))) - d(\gamma(s_t),p)| \\ &\leq |d(\gamma(t),p) - d(p,f(\gamma(t)))| + |d(p,f(\gamma(t))) - d(\gamma(s_t),p)| \\ &\leq |d(\gamma(t),p) - d(p,f(\gamma(t)))| + d(f(\gamma(t)),\gamma(s_t)) \\ &\leq |d(\gamma(t),p) - d(p,f(\gamma(t)))| + C''(\delta). \end{aligned}$$

Thus for all $t \geq T_3$,

$$d(\gamma(t),f(\gamma(t))) \leq |d(\gamma(t),p) - d(p,f(\gamma(t)))| + 2C''(\delta)$$

letting $t \to +\infty$ we obtain, using Corollary 3.3,

$$\limsup_{t\to+\infty} d(\gamma(t),f(\gamma(t))) \leq |\log \lambda_{\eta,p}| + 2C''(\delta) + 4\delta. \quad \square$$

Remark 3.7. In view of Remark 3.5, it is natural to ask whether, with notation from the previous proposition,

$$\lim_{t\to+\infty} d(\gamma(t),f(\gamma(t))) = |\log \lambda_{\eta,p}(f)|,$$

assuming that the Gromov compactification of X is equivalent to the horofunction compactification of X. This turns out to be false, as the following example shows. Consider the infinite cylinder $X := \{(x,y,z) \in \mathbb{R}^3 \colon y^2 + z^2 = 1\}$ with the Riemannian metric inherited from the euclidean metric on $\mathbb{R}^3$. If d is the associated distance, we have that $\overline{X}^H$ is equivalent to $\overline{X}^G$, and both boundaries consist of a point $-\infty$ and a point $+\infty$. Consider the isometry

$$f(x,y,z) = (x+1,-y,-z).$$

Then $d((x,y,z),f(x,y,z)) = \sqrt{1+\pi^2}$ for all $(x,y,z) \in X$, while $\log \lambda_{+\infty,p}(f) = -1$.

Next we prove some equivalent characterizations of BRFPs.

Proposition 3.8. *Let (X,d) be a proper geodesic Gromov hyperbolic metric space and let $f : X \to X$ be a non-expanding map. Let $\eta \in \partial_G X$. The following are equivalent:*

(1) *η is a BRFP;*
(2) *there exists a geodesic ray γ with endpoint η such that the curve $f \circ \gamma$ is a $(1,B)$-quasi-geodesic for some $B \geq 0$ with endpoint η;*
(3) *there exists a geodesic ray γ with endpoint η such that*

$$\limsup_{t\to+\infty} d(\gamma(t),f(\gamma(t))) < +\infty;$$

(4) *we have*

$$\liminf_{z\to\eta} d(z, f(z)) < +\infty.$$

Proof. [(1) ⇒ (2)] follows Proposition 3.4. [(2) ⇒ (3)] follows from [8, Lemma 5.8]. [(3) ⇒ (4)] is trivial. [(4) ⇒ (1)] for all $p \in X$ we have

$$\liminf_{z\to\eta} d(z,p) - d(f(z),p) \leq \liminf_{z\to\eta} d(z, f(z)) < +\infty. \quad \square$$

3.1. The case of equivalent compactifications

We end this section obtaining a refined version of Proposition 3.2 when the Gromov compactification of X is equivalent to the horofunction compactification of X. Recall that in this case the dilation at a BRFP η does not depend on the base-point p.

Proposition 3.9. *Let (X, d) be a proper geodesic Gromov hyperbolic metric space such that $\overline{X}^H$ is equivalent to $\overline{X}^G$. Let $f\colon X \to X$ be a non-expanding self-map and let $\eta \in \partial_G X$ be a BRFP. Then for all $n \geq 1$ we have*

$$\lambda_\eta(f^n) = \lambda_\eta(f)^n.$$

We need some preliminary result.

Definition 3.10. Let (X, d) be a geodesic metric space. We say that $\gamma : [0, +\infty) \to X$ is an *almost geodesic* if for each $\epsilon > 0$ there exists $t_\epsilon \geq 0$ such that for all $t_1, t_2 \geq t_\epsilon$

$$|t_1 - t_2| - \epsilon \leq d(\gamma(t_1), \gamma(t_2)) \leq |t_1 - t_2|.$$

Remark 3.11. Let $f\colon D \to D$ be a holomorphic self-map of a bounded strongly convex domain $D \subset \mathbb{C}^q$. Let $p \in D$, let $\eta \in \partial D$ be a BRFP, and let $\varphi\colon \mathbb{D} \to D$ be a complex geodesic such that $\varphi(1) = \eta$. Then Abate proved [1, Lemma 2.7.22] that

$$\lim_{t\to 1-} k_D(\varphi(t), p) - k_D(f(\varphi(t)), p) = \log \lambda_\eta.$$

This result can be generalized to our setting as follows.

Lemma 3.12. *Let (X, d) be a proper geodesic Gromov hyperbolic metric space such that $\overline{X}^H$ is equivalent to $\overline{X}^G$. Let $f\colon X \to X$ be a non-expanding self-map, let $\eta \in \partial_G X$ be a BRFP and let $\gamma : [0, +\infty) \to X$ be an almost geodesic with $\gamma(+\infty) = \eta$. Then*

$$\lim_{t\to+\infty} d(p, \gamma(t)) - d(p, f(\gamma(t))) = \log \lambda_\eta. \tag{3.1}$$

Proof. By Lemma 2.17, we may assume $p = \gamma(0)$. By definition of dilation

$$\liminf_{t\to+\infty} d(p,\gamma(t)) - d(p, f(\gamma(t))) \geq \log \lambda_\eta.$$

Since η is a BRFP, it follows that the curve $f(\gamma(t))$ converges to η. Hence for all $s \geq 0$ we have

$$\begin{aligned}\lim_{t\to+\infty} d(\gamma(s),\gamma(t)) - d(\gamma(0),\gamma(t)) &= h_{\eta,\gamma(0)}(\gamma(s)) \\ &= \lim_{t\to+\infty} d(\gamma(s), f(\gamma(t))) - d(\gamma(0), f(\gamma(t))),\end{aligned}$$

which implies

$$\limsup_{t\to+\infty} d(\gamma(0),\gamma(t)) - d(\gamma(0), f(\gamma(t))) = \limsup_{t\to+\infty} d(\gamma(s),\gamma(t)) - d(\gamma(s), f(\gamma(t))).$$

Now, for each $\epsilon > 0$ let $t_\epsilon \geq 0$ be given by the definition of almost geodesic. By Julia's Lemma (Theorem 2.19) we have, for all $t \geq t_\epsilon$,

$$h_{\eta,\gamma(t_\epsilon)}(f(\gamma(t))) - h_{\eta,\gamma(t_\epsilon)}(\gamma(t)) \leq \log \lambda_\eta.$$

On the other hand by the triangle inequality we have $-d(\gamma(t_\epsilon), f(\gamma(t))) \leq h_{\eta,\gamma(t_\epsilon)}(f(\gamma(t)))$. Moreover,

$$h_{\eta,\gamma(t_\epsilon)}(\gamma(t)) \leq -d(\gamma(t),\gamma(t_\epsilon)) + \epsilon,$$

hence

$$\begin{aligned}&\limsup_{t\to+\infty} d(\gamma(0),\gamma(t)) - d(\gamma(0), f(\gamma(t))) \\ =\ &\limsup_{t\to\infty} d(\gamma(t_\epsilon),\gamma(t)) - d(\gamma(t_\epsilon), f(\gamma(t))) \leq \log \lambda_\eta + \epsilon,\end{aligned}$$

for all $\epsilon > 0$. □

Lemma 3.13. *Let (X,d) be a proper geodesic Gromov hyperbolic metric space such that $\overline{X}^H$ is equivalent to $\overline{X}^G$. Let $f\colon X \to X$ be a non-expanding self-map, let $\eta \in \partial_G X$ be a BRFP and let $\gamma : [0,+\infty) \to X$ be an almost geodesic with $\gamma(+\infty) = \eta$. Then the curve $f \circ \gamma$ is an almost geodesic.*

Proof. By Proposition 3.12 for each $s \geq 0$ we have

$$\lim_{t\to+\infty} d(\gamma(s),\gamma(t)) - d(\gamma(s), f(\gamma(t))) = \log \lambda_\eta.$$

Therefore, given $\epsilon > 0$ we can choose $t_0 \geq 0$ so that for every $t, s \geq t_0$ it holds that

$$|d(\gamma(t_0),\gamma(t)) - d(\gamma(t_0), f(\gamma(t))) - \log\lambda_\eta| \le \epsilon/3,$$

and

$$|t-s| - \epsilon/3 \le d(\gamma(t),\gamma(s)) \le |t-s|.$$

Assume by contradiction that there exists $t_2 > t_1 \ge t_0$ so that

$$d(f(\gamma(t_1)), f(\gamma(t_2))) < |t_2 - t_1| - \varepsilon = t_2 - t_1 - \varepsilon.$$

Then

$$\begin{aligned}
\log\lambda_\eta &\ge d(\gamma(t_0),\gamma(t_2)) - d(\gamma(t_0), f(\gamma(t_2))) - \epsilon/3\\
&\ge d(\gamma(t_0),\gamma(t_1)) - d(\gamma(t_1),\gamma(t_2)) - d(\gamma(t_0), f(\gamma(t_1)))\\
&\quad - d(f(\gamma(t_1)), f(\gamma(t_2))) - \epsilon/3\\
&\ge \log\lambda_\eta - \epsilon/3 - d(\gamma(t_1),\gamma(t_2)) - d(f(\gamma(t_1)), f(\gamma(t_2))) - \epsilon/3\\
&> \log\lambda_\eta + \epsilon - 2\epsilon/3 = \log\lambda_\eta + \epsilon/3,
\end{aligned}$$

which is a contradiction. □

Proof of Proposition 3.9. Let $\gamma : [0,\infty) \to X$ be a geodesic ray such that $\gamma(+\infty) = \eta$. Then

$$\begin{aligned}
\lambda_\eta(f^n) &= \lim_{t\to+\infty} d(p,\gamma(t)) - d(p, f^n(\gamma(t)))\\
&= \lim_{t\to+\infty} [d(p,\gamma(t)) - d(p, f(\gamma(t))] + \cdots + [d(p, f^{n-1}(\gamma(t))) - d(p, f^n(\gamma(t))]\\
&= n\log\lambda_\eta(f). \quad \square
\end{aligned}$$

4. Stable dilation at a BRFP

When $\overline{X}^H$ is equivalent to $\overline{X}^G$ the dilation λ_η is deeply related to the dynamical behavior of the BRFP η, see [8, Theorem 6.32, Proposition 6.34]. For this reason the following definition was given in [8, Definition 6.31] (see [1,4] for the same definition in the case of holomorphic self-maps of the ball or, more generally, of a strongly convex domain in $\mathbb{C}^q$).

Definition 4.1. Let (X,d) be a proper geodesic Gromov hyperbolic metric space such that $\overline{X}^H$ is topologically equivalent to $\overline{X}^G$. Let $f\colon X \to X$ be a non-expanding self-map, and let η be a BRFP. We say that η is *attracting* if $\lambda_\eta < 1$, it is *indifferent* if $\lambda_\eta = 1$, and it is *repelling* if $\lambda_\eta > 1$.

However, the previous definition cannot be carried *verbatim* to the case when $\overline{X}^H$ is not equivalent to $\overline{X}^G$. Indeed, in this case the dilation at a BRFP may depend on the base-point. More surprisingly, even when the dilation does not depend on the base-point, it turns out not to be the right tool to distinguish between attracting, indifferent and repelling BRFPs, as the following example shows.

Example 4.2. Consider $(\mathbb{R}, d)$ and $(\mathbb{S}^1, d')$ where d is the euclidean distance and d' is the inner distance induced by the euclidean distance. Endow $X := \mathbb{R} \times \mathbb{S}^1$ with the distance $d''((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d'(y_1, y_2)$. Then X is a proper geodesic Gromov hyperbolic space. The Gromov boundary consists of two points $+\infty$ and $-\infty$, while the horofunction boundary is the disjoint union of two $\mathbb{S}^1$. Let $\vartheta \in [0, \pi]$ and let $R_\vartheta \colon \mathbb{S}^1 \to \mathbb{S}^1$ be the counterclockwise rotation by an angle ϑ. The hyperbolic isometry $f \colon X \to X$ defined by $f(x, y) = (x+1, R_\vartheta(y))$ has Denjoy–Wolff point $+\infty$, while $-\infty$ is the Denjoy–Wolff point of f^{-1}. It is easy to see that $\log \lambda_{-\infty,p}$ does not depend on the base point p and is equal to $1 - \vartheta$. Hence depending on ϑ the dilation $\lambda_{-\infty,p}$ can be strictly larger than 1, equal to 1, or strictly smaller than 1.

This motivates the following definition.

Definition 4.3 *(Stable dilation).* Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f \colon X \to X$ be a non-expanding self-map. Let $p \in X$ and let η be a BRFP. We define the *stable dilation* of f at η as

$$\log \Lambda_\eta := \lim_{n\to\infty} \frac{\log \lambda_{\eta,p}(f^n)}{n}.$$

Remark 4.4. If the Gromov and horofunction compactifications are equivalent, then by Proposition 3.9 it follows that $\Lambda_\eta = \lambda_\eta$.

Example 4.5. Let $f : X \to X$ be the isometry of Example 4.2. A simple computation shows that for every $x \in \mathbb{R}$, $t \geq 0$ and $y, y' \in \mathbb{S}^1$ we have

$$t \leq d''((x, y), (x + t, y')) \leq t + 2\pi,$$

hence for a fixed $n \geq 1$, for all $x \in \mathbb{R}$ close to $-\infty$ and for all $y \in \mathbb{S}^1$ we have

$$n - 2\pi \leq d''(p, (x, y)) - d''(p, f^n(x, y)) \leq n + 2\pi,$$

hence

$$n - 2\pi \leq \log \lambda_{-\infty,p}(f^n) \leq n + 2\pi$$

which implies that $\log \Lambda_{-\infty} = 1$.

We have to show that the limit in the previous definition exists. This is the content of the next result.

Theorem 4.6. *The sequence* $(\log \lambda_{\eta,p}(f^n))_{n\geq 1}$ *is superadditive.*

Remark 4.7. By the Fekete Lemma this implies that

$$\lim_{n\to\infty} \frac{\log \lambda_{\eta,p}(f^n)}{n} = \sup_{n\geq 1} \frac{\log \lambda_{\eta,p}(f^n)}{n} \in (-\infty, \infty].$$

By Proposition 3.2 we have that the limit of $\log \lambda_{\eta,p}(f^n)/n$ is actually finite.

Proof. Fix $n, m \geq 1$. Let (z_k) be a sequence in X converging to ζ such that

$$d(z_k, p) - d(f^{n+m}(z_k), p) \to \log \lambda_{\zeta,p}(f^{n+m}).$$

Denote $q := f^m(p)$. It follows that

$$d(z_k, p) - d(f^n(z_k), p) \leq d(z_k, p) - d(f^{n+m}(z_k), q) \leq d(z_k, p) - d(f^{n+m}(z_k), p) + d(p, q)$$

which is uniformly bounded from above. By Proposition 3.2 we know that the geodesic limit of f^n as $z \to \eta$ is η, hence Proposition 2.15 yields that $f^n(z_k) \to \eta$. We have

$$\begin{aligned}
\log \lambda_{\eta,p}(f^{n+m}) &= \lim_{k\to+\infty} d(z_k, p) - d(f^{n+m}(z_k), p) \\
&= \lim_{k\to+\infty} d(z_k, p) - d(f^n(z_k), p) + d(f^n(z_k), p) - d(f^{n+m}(z_k), p) \\
&\geq \liminf_{k\to+\infty} d(z_k, p) - d(f^n(z_k), p) + \liminf_{k\to+\infty} d(f^n(z_k), p) - d(f^{n+m}(z_k), p) \\
&\geq \log \lambda_{\eta,p}(f^n) + \log \lambda_{\eta,p}(f^m). \quad \square
\end{aligned}$$

It follows immediately from Remark 2.14 that the stable dilation Λ_η at a BRFP does not depend on the base-point p. Notice also that $\Lambda_\eta(f^n) = \Lambda_\eta(f)^n$.

Definition 4.8. Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f : X \to X$ be a non-expanding map and $\eta \in \partial_G X$ a BRFP. We say that η is *attracting* if $\Lambda_\eta < 1$, *parabolic* if $\Lambda_\eta = 1$, and *repelling* if $\Lambda_\eta > 1$.

Remark 4.9. Notice that by Remark 4.7 it is enough for a BRFP η to have one integer n such that $\lambda_{\eta,p}(f^n) > 1$ to conclude that η is repelling.

Definition 4.10 *(Divergence rate).* Let (X, d) be a metric space and $f\colon X \to X$ be a non-expanding self-map. Let $x \in X$, the *divergence rate* (or *translation length*, or *escape rate*) $c(f)$ of f is the limit

$$c(f) := \lim_{n\to\infty} \frac{d(x, f^n(x))}{n}. \tag{4.1}$$

Remark 4.11. The sequence $(d(x, f^n(x)))$ is subadditive, indeed if $n, m \geq 0$,

$$d(x, f^{n+m}(x)) \leq d(x, f^n(x)) + d(f^n(x), f^{n+m}(x)) \leq d(x, f^n(x)) + d(x, f^m(x)).$$

Hence by the Fekete Lemma the limit (4.1) exists and equals

$$\inf_{n\geq 1} \frac{d(x, f^n(x))}{n}.$$

Moreover, the limit (4.1) does not depend on $x \in X$, indeed for all $x, y \in X$ we have

$$|d(x, f^n(x)) - d(y, f^n(y))| \leq d(x, y) + d(f^n(x), f^n(y)) \leq 2d(x, y).$$

Proposition 4.12. *Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f\colon X \to X$ be a non-elliptic non-expanding map. Let $\zeta \in \partial_G X$ be its Denjoy–Wolff point. Then*

$$\log \Lambda_\zeta = -c(f).$$

Proof. Let $a \in \Phi^{-1}(\zeta)$. By the δ-Julia Lemma 3.1 we have

$$\log \lambda_{\zeta,p}(f^n) + 4\delta \geq h_{a,p}(f^n(x)) - h_{a,p}(x) \geq -d(f^n(x), x),$$

hence

$$-c(f) = \lim_{n\to+\infty} \frac{-d(f^n(x), x)}{n} \leq \lim_{n\to+\infty} \frac{\log \lambda_{\zeta,p}(f^n) + 4\delta}{n} = \log \Lambda_\zeta.$$

Moreover, by (1) of [8, Proposition 6.19],

$$\log \Lambda_\zeta = \lim_{n\to+\infty} \frac{\log \lambda_{\zeta,p}(f^n)}{n} \leq \lim_{n\to+\infty} \frac{-c(f^n)}{n} = -c(f). \quad \square$$

Definition 4.13. Let (X, d) be a proper geodesic Gromov hyperbolic metric space. Let $f\colon X \to X$ be a non-expanding map. Recall (Definition 2.21) that f is called elliptic if it admits a forward orbit which is not escaping (equivalently by Calka's theorem, every forward orbit is relatively compact). If f is non-elliptic, we say that it is

- *parabolic* if its Denjoy–Wolff point is indifferent, or equivalently if $c(f) = 0$;
- *hyperbolic* if its Denjoy–Wolff point is attracting, or equivalently if $c(f) > 0$.

Remark 4.14. This definition generalizes both the classification of holomorphic self-maps of a bounded strongly convex domain of $\mathbb{C}^n$ (see e.g. [4]), and the classification of isometries of a proper geodesic Gromov hyperbolic metric space (see for instance [16]).

We briefly describe the BRFPs of isometries. An isometry $f\colon X \to X$ of a proper geodesic Gromov hyperbolic space extends to a homeomorphism $\tilde{f}\colon \overline{X}^G \to \overline{X}^G$ (see [15, Theorem 3.9, Chapter III.H]). Clearly if $\eta \in \partial_G X$ is a BRFP, then it is a fixed point of $\tilde{f}$. Conversely, if $\eta \in \partial_G X$ is a fixed point of $\tilde{f}$, then given a geodesic ray γ with endpoint η, the curve $f \circ \gamma$ is also a geodesic ray with endpoint η, and thus by Proposition 3.8 the point η is a BRFP. By a classical result (see e.g. [16]), if an isometry f is not elliptic then either

(1) the unique fixed point of $\tilde{f}$ in $\partial_G X$ is the Denjoy–Wolff point of f, and in this case f is parabolic; or
(2) $\tilde{f}$ has exactly two fixed points in $\partial_G X$, which are the Denjoy–Wolff points of f and f^{-1}, and in this case f is hyperbolic.

It follows that if f is parabolic, then its indifferent Denjoy–Wolff point is the only BRFP. If f is hyperbolic, and ζ, η denote the Denjoy–Wolff points of f and f^{-1} respectively, then it follows from Proposition 4.12 and from $c(f) = c(f^{-1})$ that

$$\log \Lambda_\eta(f) = -\log \Lambda_\eta(f^{-1}) = c(f^{-1}) = c(f) = -\log \Lambda_\zeta(f),$$

hence ζ is a repelling BRFP for f with stable dilation

$$\Lambda_\eta(f) = \frac{1}{\Lambda_\zeta(f)}.$$

Lemma 4.15. *Let $f\colon X \to X$ be an elliptic isometry of a proper geodesic Gromov hyperbolic metric space. Then all BRFPs of f are indifferent.*

Proof. Let η be a BRFP. Fix a point $p \in X$. We have, for all $x \in X$, $n \geq 1$,

$$\begin{aligned} -d(x, f^n(x)) &\leq d(x,p) - d(f^n(x), p) \\ &\leq d(x,p) - d(f^n(x), f^n(p)) + d(p, f^n(p)) = d(p, f^n(p)). \end{aligned}$$

Since f is elliptic, the sequences $(d(p, f^n(p)))$ and $(d(x, f^n(x)))$ are bounded, and thus $\Lambda_\eta = 1$. □

Proposition 4.16. *Let $f\colon X \to X$ be a non-expanding self-map of a proper geodesic Gromov hyperbolic metric space.*

i) *If a BRFP η of f has stable dilation $\Lambda_\eta < 1$ then f is hyperbolic and η is its Denjoy–Wolff point.*
ii) *If f is non-elliptic then the only BRFP η with stable dilation $\Lambda_\eta \leq 1$ is the Denjoy–Wolff point.*

iii) *If f is elliptic and η is a BRFP then $\Lambda_\eta = 1$ if and only if η is contained in the Gromov closure of the limit retract ω_f (Definition 2.25).*

Proof. i) By Remark 4.7 we have that, for all $n \geq 1$,

$$\log \lambda_{\eta,p}(f^n) \leq n \log \Lambda_\eta.$$

Let $a \in \Phi^{-1}(\eta)$, $x_0 \in X$ and $n \geq 1$. By the δ-Julia Lemma 3.1 applied to f^n we have

$$h_{a,p}(f^n(x_0)) \leq h_{a,p}(x_0) + \log \lambda_{\eta,p}(f^n) + 4\delta \leq h_{a,p}(x_0) + n \log \Lambda_\eta + 4\delta.$$

Hence $h_{a,p}(f^n(x_0)) \overset{n\to\infty}{\longrightarrow} -\infty$, and by Proposition 2.11 ii) it follows that $f^n(x_0)$ converges to $\eta \in \partial_G X$. Thus f is non-elliptic, and η is the Denjoy-Wolff point of f.

ii) Assume f is non-elliptic and let η be a BRFP with stable dilation ≤ 1. Let $a \in \Phi^{-1}(\eta)$, $x_0 \in X$ and $n \geq 1$. Then by the δ-Julia Lemma 3.1 we have

$$h_{a,p}(f^n(x_0)) \leq h_{a,p}(x_0) + \log \lambda_{\eta,p}(f^n) + 4\delta \leq h_{a,p}(x_0) + 4\delta,$$

where we used Remark 4.7 to conclude that $\log \lambda_{\eta,p}(f^n) \leq 0$. Hence the forward orbit $(f^n(x_0))$ is contained in the horosphere $\{h_{a,p} \leq h_{a,p}(x_0) + 4\delta\}$. By Proposition 2.11 i) η is the Denjoy–Wolff point of f.

iii) Assume that f is elliptic and let η be a BRFP with stable dilation $\Lambda_\eta = 1$, not contained in $\overline{\omega_f}^G$. Let $a \in \Phi^{-1}(\eta)$. By Proposition 2.11 ii) there exists $c \in \mathbb{R}$ such that the horosphere $\{h_{a,p} \leq c\}$ is contained in the open neighborhood $\overline{X}^G \setminus \overline{\omega_f}^G$ of η, and thus

$$\{h_{a,p} \leq c\} \cap \omega_f = \varnothing.$$

Let $x_0 \in \{h_{a,p} \leq c - 4\delta\}$. Then as above the forward orbit $(f^n(x_0))$ is contained in the horosphere $\{h_{a,p} \leq c\}$ but this is a contradiction since every forward orbit of f admits a limit point in ω_f.

Assume conversely that η is a BRFP contained in $\overline{\omega_f}^G$. By Remark 2.28 η is also a point in the Gromov boundary of ω_f. By Theorem 2.23 the restriction $f|_{\omega_f} \colon \omega_f \to \omega_f$ is an elliptic isometry.

Let γ be a geodesic ray in ω_f with endpoint η. Then by Corollary 3.3 we have that

$$\lim_{t\to+\infty} d(\gamma(t), \gamma(0))) - d(f(\gamma(t)), \gamma(0))) \leq \log \lambda_{\eta,\gamma(0)} + 4\delta,$$

where $\lambda_{\eta,\gamma(0)}$ is the dilation at η as a BRFP of X. Hence η is a BRFP for $f|_{\omega_f}$ too. By Lemma 4.15 η is indifferent as a BRFP of ω_f. Then clearly the stable dilation of η as a BRFP of X satisfies $\log \Lambda_\eta \leq 0$. By point i) above it follows that $\log \Lambda_\eta = 0$. □

Proposition 4.17. *Let $f\colon X \to X$ be a non-elliptic non-expanding self-map of a proper geodesic Gromov hyperbolic metric space. Let ζ be the Denjoy–Wolff point of f and let η be a BRFP different from ζ. Then*

$$\log \Lambda_\eta \geq -\log \Lambda_\zeta.$$

Proof. Fix $p \in X$. By ii) of Proposition 4.16 η is a repelling BRFP. In what follows let $n \geq 0$ be large enough such that $\log \lambda_{\eta,p}(f^n) > 0$. By Proposition 3.6 there exists $x_n \in X$ such that

$$\log \lambda_{\eta,p}(f^n) \geq d(x_n, f^n(x_n)) - C(\delta) - 1.$$

Let $a \in \Phi^{-1}(\zeta)$, by the δ-Julia Lemma 3.1 we have

$$\log \lambda_{\zeta,p}(f^n) + 4\delta \geq h_{a,p}(f^n(x_n)) - h_{a,p}(x_n) \geq -d(x_n, f^n(x_n)).$$

Hence

$$\log \lambda_{\eta,p}(f^n) \geq -\log \lambda_{\zeta,p}(f^n) - 4\delta - C(\delta) - 1,$$

which implies

$$\log \Lambda_\eta = \lim_{n\to+\infty} \frac{\log \lambda_{\eta,p}(f^n)}{n} \geq \lim_{n\to+\infty} \frac{-\log \lambda_{\zeta,p}(f^n) - 4\delta - C(\delta) - 1}{n} = -\log \Lambda_\zeta. \quad \square$$

Remark 4.18. The results in this section generalize several results for holomorphic self-maps of bounded strongly convex domains in $\mathbb{C}^q$. For Proposition 4.12 see [6, Proposition 4.1]. See [4] for points i) and ii) of Proposition 4.16 and for Proposition 4.17. Finally see [3, Proposition 3.4] for point iii) of Proposition 4.16.

5. Backward orbits with bounded step converging to a repelling BRFP

We now introduce a number associated with every backward orbit with bounded step (x_n), which will turn out to detect most of its dynamical behavior (see Propositions 6.3, 6.4 and Corollary 6.5 below).

Definition 5.1 *(Backward step rate).* Let (X, d) be a proper geodesic Gromov hyperbolic space, and let $f\colon X \to X$ be a non-expanding map. Let (x_n) be a backward orbit with bounded step. If $m \geq 1$, the *m-step* of (x_n) is defined as

$$\sigma_m(x_n) := \lim_{n\to+\infty} d(x_n, x_{n+m}).$$

Notice that the limit exists because the sequence is not decreasing, and moreover the sequence $(\sigma_m(x_n))_m$ is subadditive. We define the *backward step rate* of (x_n) as

$$b(x_n) := \lim_{m\to+\infty} \frac{\sigma_m(x_n)}{m} = \inf_m \frac{\sigma_m(x_n)}{m}.$$

Remark 5.2. Clearly $b(x_n)$ is smaller than or equal to the step $\sigma_1(x_n)$ of the backward orbit. We will see that $b(x_n)$ carries far more dynamical information on the orbit than the step $\sigma_1(x_n)$. Also notice that for all $m \geq 1$ we have

$$\frac{d(x_0, x_m)}{m} \leq \frac{\sigma_m(x_m)}{m}$$

which implies

$$c(f) \leq b(x_n).$$

Proposition 5.3. *Let (X, d) be a proper geodesic Gromov hyperbolic space, and let $f\colon X \to X$ be a non-expanding map. If a point $\eta \in \partial_G X$ is a limit point of a backward orbit with bounded step (x_n), then η is a BRFP and*

$$0 \leq \log \Lambda_\eta \leq b(x_n).$$

Proof. Let (x_{n_k}) be a subsequence converging to η. For all $m \geq 1$,

$$\sigma_m(x_n) \geq \limsup_{k\to+\infty} d(x_{n_k}, p) - d(x_{n_k-m}, p) \geq \log \lambda_{\eta,p}(f^m).$$

This shows $\log \Lambda_\eta \leq b(x_n)$. Assume we know by contradiction that $\log \Lambda_\eta < 0$. Then η is attracting, and thus f is hyperbolic. Then by [8, Proposition 6.25] the backward orbit (x_n) has to converge to a BRFP different from the Denjoy–Wolff point η, contradiction. □

The goal of this section is to prove Theorem 1.5. We will actually prove the following.

Theorem 5.4. *Let (X, d) be a proper geodesic Gromov hyperbolic space, and let $f\colon X \to X$ be a non-expanding map. Assume that $\eta \in \partial_G X$ is a repelling BRFP with stable dilation $\Lambda_\eta > 1$. Then the following holds.*

i) *There exists a backward orbit (x_n) converging to η with backward step rate*

$$b(x_n) = \log \Lambda_\eta.$$

ii) *If (x_n) and (y_n) are two backward orbits with bounded step converging to η, then*

$$\sup_{n\geq 0} d(x_n, y_n) < +\infty.$$

iii) *Every backward orbit (y_n) with bounded step converging to η has backward step rate*

$$b(y_n) = \log \Lambda_\eta.$$

Lemma 5.5. *Let (X, d) be a proper geodesic Gromov hyperbolic space, let $f : X \to X$ be a non-expanding map and let $\eta \in \partial_G X$ be a repelling BRFP. Let $a \in \Phi^{-1}(\eta)$. Then there exists $c \in \mathbb{R}$ so that every forward orbit of f eventually avoids the horosphere $\{h_{a,p} \leq c\}$.*

Proof. If f is non-elliptic, then every forward orbit converges to the attracting or indifferent Denjoy–Wolff point, which is different from η. By Proposition 2.11 i) the result holds for all $c \in \mathbb{R}$.

Otherwise, assume that f is elliptic with limit retract ω_f. By Proposition 4.16 iii) the BRFP η is not contained in $\overline{\omega_f}^G$. It follows from Proposition 2.11 ii) that there exists $c \in \mathbb{R}$ such that

$$\{h_{a,p} \leq c\} \cap \omega_f = \varnothing.$$

The result follows since every forward orbit of f is eventually contained in any given neighborhood $\omega_f \subset U \subset X$. □

Let now γ be a geodesic ray with endpoint η, let $a \in \Phi^{-1}(\eta)$ be its Busemann point and let $p := \gamma(0)$. Choose $c \in \mathbb{R}$ as in Lemma 5.5. Every forward orbit starting in the horosphere $\{h_{a,p} \leq c\}$ eventually leaves the same set. Choose an increasing sequence (t_k) in $\mathbb{R}$ so that $t_k \geq -c$. Since $h_{a,p}(\gamma(t_k)) = -t_k \leq c$, it follows that for all $m \geq 1$ and $k \geq 0$ there exists $n_{m,k} \geq 0$ such that for all $0 \leq n \leq n_{m,k}$ we have

$$h_{a,p}(f^{mn}(\gamma(t_k))) \leq c, \quad \text{but} \quad h_{a,p}(f^{m(n_{m,k}+1)}(\gamma(t_k))) > c. \tag{5.1}$$

Set $x_{m,k} := f^{mn_{m,k}}(\gamma(t_k))$ and $y_{m,k} := f^{m(n_{m,k}+1)}(\gamma(t_k))$.

Proposition 5.6. *There exists $m \geq 1$ such that the sequences $(x_{m,k})_{k\geq 0}$ and $(y_{m,k})_{k\geq 0}$ are bounded.*

Proof. In what follows, let $m \geq 1$ be large enough such that $\log \lambda_{\eta,p}(f^m) > 0$. By Proposition 3.6 applied to f^m we have that

$$\limsup_{k\to+\infty} d(x_{m,k}, y_{m,k}) \leq \limsup_{k\to+\infty} d(\gamma(t_k), f^m(\gamma(t_k))) \leq \log \lambda_{\eta,p}(f^m) + C(\delta). \tag{5.2}$$

In particular the sequence $(d(x_{m,k}, y_{m,k}))_{k\geq 0}$ is bounded from above. Hence the sequence $(x_{m,k})_{k\geq 0}$ is bounded if and only if $(y_{m,k})_{k\geq 0}$ is bounded.

Suppose by contradiction that, for every $m \geq 1$, the sequence $(x_{m,k})_{k\geq 0}$ is not bounded. By taking a subsequence if necessary, we may then assume that for each m the

sequence $(x_{m,k})$ is escaping. By Proposition 2.11 i), since $h_{a,p}(x_{m,k}) \leq c$ we have that $(x_{m,k})$ converges to η. Therefore $(y_{m,k})$ also converges to η.

By (5.2), we have that

$$\begin{aligned}\frac{\log \lambda_{\eta,p}(f^m)}{m} &\leq \liminf_{k\to+\infty} \frac{1}{m}(d(p,x_{m,k}) - d(p,y_{m,k})) \leq \liminf_{k\to+\infty} \frac{1}{m} d(x_{m,k}, y_{m,k}) \\ &\leq \limsup_{k\to+\infty} \frac{1}{m} d(x_{m,k}, y_{m,k}) \leq \frac{\log \lambda_{\eta,p}(f^m) + C(\delta)}{m},\end{aligned}$$

and that

$$\begin{aligned}\frac{\log \lambda_{\eta,p}(f^m)}{m} &\leq \liminf_{k\to+\infty} \frac{1}{m}(d(p,x_{m,k}) - d(p,y_{m,k})) \leq \limsup_{k\to+\infty} \frac{1}{m}(d(p,x_{m,k}) - d_m(p,y_{m,k})) \\ &\leq \limsup_{k\to+\infty} \frac{1}{m} d(x_{m,k}, y_{m,k}) \leq \frac{\log \lambda_{\eta,p}(f^m) + C(\delta)}{m}.\end{aligned}$$

The inequalities above imply the existence of a divergent sequence (k_m) in $\mathbb{N}$ such that, if we write $\hat{x}_m := x_{m,k_m}$ and $\hat{y}_m := y_{m,k_m}$, we have

$$\lim_{m\to+\infty} \frac{d(\hat{x}_m, \hat{y}_m)}{m} = \lim_{m\to+\infty} \frac{d(p,\hat{x}_m) - d(p,\hat{y}_m)}{m} = \log \Lambda_\eta. \tag{5.3}$$

Let $\gamma_m : [0, T_m] \to X$ be a geodesic segment connecting p to $\hat{x}_m$, where $T_m := d(p, \hat{x}_m)$. Since we are assuming that for every m the sequence $(x_{m,k})_{k\geq 0}$ converges to the point η, the sequence (k_m) in the previous step can be chosen so that

$$d(p, \hat{x}_m) \geq c + 2\log \lambda_{\eta,p}(f^m). \tag{5.4}$$

In particular for every m the point $\hat{y}'_m := \gamma_m(T_m - \log \lambda_{\eta,p}(f^m))$ is well defined. Given any $w \in X$, by Remark 2.3 we have that

$$(p, \hat{x}_m)_w \geq \min((p, \hat{y}'_m)_w, (\hat{y}'_m, \hat{x}_m)_w) - 2\delta. \tag{5.5}$$

Suppose first that the minimum in the inequality above is realized by $(p, \hat{y}'_m)_w$. Then

$$2(p, \hat{x}_m)_w = d(\hat{x}_m, w) + d(p, w) - d(p, \hat{x}_m) \geq d(\hat{y}'_m, w) + d(p, w) - d(p, \hat{y}'_m) - 4\delta,$$

hence

$$d(\hat{y}'_m, w) \leq d(\hat{x}_m, w) + d(p, \hat{y}'_m) - d(p, \hat{x}_m) + 4\delta = d(\hat{x}_m, w) - \log \lambda_{\eta,p}(f^m) + 4\delta, \tag{5.6}$$

where we used the fact that $d(p, \hat{y}'_m) = d(p, \hat{x}_m) - \log \lambda_{p,\eta}(f^m)$, directly from the definition of the point $\hat{y}'_m$.

On the other hand, suppose that the minimum in (5.5) is realized by $(\hat{y}'_m, \hat{x}_m)_w$. Then we have that

$$2(p, \hat{x}_m)_w = d(\hat{x}_m, w) + d(p, w) - d(\hat{x}_m, p) \geq d(\hat{y}'_m, w) + d(\hat{x}_m, w) - d(\hat{x}_m, \hat{y}'_m) - 4\delta,$$

hence

$$d(\hat{y}'_m, w) - d(p, w) \leq d(\hat{x}_m, \hat{y}'_m) - d(\hat{x}_m, p) + 2\delta \leq -d(\hat{x}_m, p) + \log \lambda_{\eta,p}(f^m) + 4\delta. \quad (5.7)$$

Let (w_n) be a sequence in X converging to $a \in \partial^H X$. Then by taking a subsequence if necessary, we may assume that either the minimum in equation (5.5) is always realized by $(\hat{y}'_m, p)_{w_n}$ or that it is always realized by $(\hat{y}'_m, \hat{x}_m)_{w_n}$. In the first case, for every n, it follows from equation (5.6) that

$$d(\hat{y}'_m, w_n) - d(p, w_n) \leq d(\hat{x}_m, w_n) - d(p, w_n) - \log \lambda_{\eta,p}(f^m) + 4\delta,$$

and therefore that

$$h_{a,p}(\hat{y}'_m) \leq h_{a,p}(\hat{x}_m) - \log \lambda_{p,\eta}(f^m) + 4\delta \leq c - \log \lambda_{\eta,p}(f^m) + 4\delta.$$

In the second case we have instead, using equation (5.7),

$$h_{a,p}(\hat{y}'_m) \leq -d(\hat{x}_m, p) + \log \lambda_{\eta,p}(f^m) + 4\delta.$$

Since the sequence (k_m) was chosen so that equation (5.4) holds, we conclude that in both cases the following holds:

$$h_{a,p}(\hat{y}'_m) \leq c - \log \lambda_{\eta,p}(f^m) + 4\delta.$$

We claim that $\frac{1}{m} d(\hat{y}_m, \hat{y}'_m) \to 0$. This can be proved by considering again the inequality (5.5) in the case $w = \hat{y}_m$. Again, we may assume that the minimum in (5.5) is realized either by $(p, \hat{y}'_m)_{\hat{y}_m}$ or by $(\hat{y}'_m, \hat{x}_m)_{\hat{y}_m}$. In the first case, by equation (5.6) we have that

$$d(\hat{y}'_m, \hat{y}_m) \leq d(\hat{x}_m, \hat{y}_m) - \log \lambda_{\eta,p}(f^m) + 4\delta.$$

In the second case, by equation (5.7), we have instead that

$$d(\hat{y}'_m, \hat{y}_m) \leq d(p, \hat{y}_m) - d(\hat{x}_m, p) + \log \lambda_{\eta,p}(f^m) + 4\delta.$$

By equation (5.3) we conclude that in both cases $\frac{1}{m} d(\hat{y}_m, \hat{y}'_m) \to 0$.

In conclusion, we obtain that

$$\frac{h_{a,p}(\hat{y}_m)}{m} \leq \frac{h_{a,p}(\hat{y}'_m) + d(\hat{y}'_m, \hat{y}_m)}{m} \leq \frac{c + 4\delta}{m} - \frac{\log \lambda_{\eta,p}(f^m)}{m} + \frac{d(\hat{y}'_m, \hat{y}_m)}{m}.$$

The right hand side of the inequality above converges to $-\log \Lambda_\eta < 0$, which implies that whenever m is sufficiently large, $h_{a,p}(\hat{y}_m) \leq c$, contradicting equation (5.1). □

Proof of Theorem 5.4. [*Proof of* i)] The proof is similar to [9, Theorem 2], but we include it for the convenience of the reader. By Proposition 5.6 there exists $m \geq 1$ such that the sequence $(f^{mn_{m,k}}(\gamma(t_k)))$ is bounded. Denote for simplicity $n_{m,k} = n_k$. Then there exist $z_0 \in X$ and a subsequence $(n_{k_{0,h}})$ such that

$$f^{mn_{k_{0,h}}}(\gamma(t_{k_{0,h}})) \overset{h\to+\infty}{\longrightarrow} z_0.$$

Since f is not-expanding, by Proposition 3.6 it holds that

$$d(f^{mn_{k_{0,h}}}(\gamma(t_{k_{0,h}})), f^{mn_{k_{0,h}}-1}(\gamma(t_{k_{0,h}}))) \leq d(f(\gamma(t_{k_{0,h}})), \gamma(t_{k_{0,h}})) \leq |\log \lambda_{\eta,p}(f)| + C(\delta),$$

in particular we can find a subsequence $(k_{1,h})$ of $(k_{0,h})$ and $z_1 \in X$ such that

$$f^{mn_{k_{1,h}}-1}(\gamma(t_{k_{1,h}})) \overset{h\to+\infty}{\longrightarrow} z_1.$$

Notice that by continuity of f we have that $f(z_1) = z_0$. This procedure can be iterated, giving for every $\nu \geq 1$ a subsequence $(k_{\nu+1,h})$ of $(k_{\nu,h})$ such that the sequence $(f^{mn_{k_{\nu,h}}-\nu}(\gamma(t_{k_{\nu,h}})))$ converges to a point $z_\nu \in X$ such that $f(z_\nu) = z_{\nu-1}$. Furthermore, again by Proposition 3.6, we have for all $\mu \geq 1$,

$$\begin{aligned} d(z_\nu, z_{\nu-\mu}) &= \lim_{h\to+\infty} d(f^{mn_{k_{\nu,h}}-\nu}(\gamma(t_{k_{\nu,h}})), f^{mn_{k_{\nu,h}}-\nu-\mu}(\gamma(t_{k_{\nu,h}}))) \\ &\leq \lim_{h\to+\infty} d(f^{\mu}(\gamma(t_{k_{\nu,h}})), \gamma(t_{k_{\nu,h}})) \leq |\log \lambda_{\eta,p}(f^\mu)| + C(\delta), \end{aligned}$$

and therefore $b(z_n) \leq \log \Lambda_\eta$.

It remains to show that the backward orbit (z_ν) converges to η. It is enough to show that the subsequence $(z_{m\nu})$ converges to η. Notice that by construction for all $\nu \geq 0$ we have that the point $z_{m\nu}$ belongs to the horosphere $\{h_{a,p} \leq c\}$. By Proposition 2.11 i), either $z_{m\nu} \to \eta$ or there exists a subsequence $z_{m\nu_k} \to z' \in \{h_{a,p} \leq c\}$. In the second case for every $i \in \mathbb{N}$ we have that

$$f^{mi}(z') = \lim_{k\to\infty} f^{mi}(z_{m\nu_k}) = \lim_{k\to\infty} z_{m(\nu_k - i)} \in \{h_{a,p} \leq c\}.$$

We conclude that there exists a subsequence of the forward orbit of z' contained in the horosphere $\{h_{a,p} \leq c\}$, which is not possible thanks to the choice of c (see Lemma 5.5).

By Proposition 5.3 it follows that the backward step rate $b(z_\nu)$ is bounded from below by $\log \Lambda_\eta$, and therefore it must be equal to $\log \Lambda_\eta$.

[*Proof of* ii)] Let (x_n) and (y_n) be backward orbits with bounded step converging to η. The backward orbits are discrete quasi-geodesics, and thus can be interpolated with

quasi-geodesic rays. Hence by Gromov's shadowing lemma (Theorem 2.5) we have that there exists $M \geq 0$ such that

$$\sup_{n\geq 0} \inf_{m\geq 0} d(x_n, y_m) \leq M.$$

Consider the complete orbits of (x_n) and (z_n), setting $x_{-n} := f^n(x_0)$ and $y_{-n} := f^n(y_0)$ for all $n > 0$. The sequences (x_n) and (y_n) converge to η as $n \to +\infty$. When $n \to -\infty$, then if f is non-elliptic they converge to the Denjoy–Wolff point of f, which is different from η, while if f is elliptic they accumulate on the limit retract ω_f, which by Proposition 4.16 iii) does not contain η in its Gromov closure. Hence in both cases there exists $N \geq 0$ such that $d(x_N, y_m) > M$ for all $m < 0$. Moreover, there exists $L \geq 0$ such that $d(x_N, y_m) > M$ for all $m > L$.

For all $n \geq N$ let $m_n \in \mathbb{N}$ be such that $d(x_n, y_{m_n}) \leq M$. By the non-expansivity of f,

$$d(x_N, y_{m_n+N-n}) \leq M,$$

which implies

$$0 \leq m_n + N - n \leq L.$$

In particular

$$|n - m_n| \leq L + N.$$

Finally, for all $n \geq N$,

$$d(x_n, y_n) \leq d(x_n, y_{m_n}) + d(y_{m_n}, y_n) \leq M + \sigma_1 |n - m_n| \leq M + \sigma_1(L + N).$$

[*Proof of* iii)] Let (y_n) be a backward orbit with bounded step converging to η, and let (x_n) be the backward orbit given by point i). By ii) there exists $M \geq 0$ such that $d(x_n, y_n) \leq M$ for all $n \geq 0$. Hence for all $m \geq 0$,

$$|\sigma_m(x_n) - \sigma_m(y_n)| \leq 2M,$$

which implies $b(y_n) = b(x_n) = \log \Lambda_\eta$. □

We conclude this section showing that point i) and ii) of Theorem 5.4 are not true in general if the point η is indifferent.

Example 5.7. Consider the metric space $(\mathbb{R}_{>0}, d)$ with $d(x, y) = |\ln \frac{x}{y}|$, and let $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be the non-expanding map $f(t) = t + 1$. Then $\partial_G \mathbb{R}_{>0} = \{0, +\infty\}$, the indifferent BRFP $+\infty$ is the Denjoy–Wolff point of f but there are not backward orbits converging

to $+\infty$. A similar example in the holomorphic setting is given by the self-map $z \to z+1$ in the right half-plane $\mathbb{H}$ endowed with the Poincaré distance.

Example 5.8. Let $X = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ endowed with the Poincaré distance, and let $f\colon X \to X$ be defined by $f(z) = z+1$. Then the Denjoy–Wolff point ∞ is indifferent. The two backward orbits with bounded step $x_n := (-n, 1)$ and $y_n := (-n, -1)$ converge to the Denjoy–Wolff point, but $d(x_n, y_n) \to +\infty$.

Point iii) of Theorem 5.4 actually holds also if η is indifferent, as will be shown in the next section.

6. Backward Denjoy–Wolff theorem

In this section we prove Theorem 1.6. The condition that backward orbit has bounded step is crucial, as the following example shows.

Example 6.1 *(Backward orbits with unbounded step).* Let $X = \mathbb{S}^1 \times (0, +\infty)$ with the Riemannian metric

$$g(\alpha, t) = \frac{(d\alpha)^2 + (dt)^2}{t^2},$$

and d the corresponding distance. The metric space (X, d) is proper geodesic Gromov hyperbolic because is isometric to the punctured disk $\mathbb{D}^*$ with the hyperbolic distance (for the hyperbolicity of $\mathbb{D}^*$ see for example Lemma 5.4 in [29]) and its Gromov boundary is canonically homeomorphic to disjoint union between $\mathbb{S}^1$ and $+\infty$. Let $\vartheta \in [0, 2\pi) \setminus \pi\mathbb{Q}$ and let $R_\vartheta\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the counterclockwise rotation by an angle ϑ. Then the map

$$f((\alpha, t)) = (R_\vartheta(\alpha), 2t)$$

is a non-expanding hyperbolic map (with Denjoy-Wolff point $+\infty$). Fix $x_0 := (\alpha, t) \in X$, then the sequence $x_n := (R_{n\vartheta}^{-1}(\alpha), 2^{-n}t)$ is a backward orbit with $\mathbb{S}^1$ as limit set.

We start describing the only backward orbits which are not escaping.

Proposition 6.2. *Let X be a proper metric space and let $f\colon X \to X$ be a non-expanding self-map. If a backward orbit (x_n) is not escaping, then the map f is elliptic, and (x_n) is a relatively compact orbit of the form $(x_n) = (f|_{\omega_f}^{-n}(x_0))$.*

Proof. Assume that the backward orbit (x_n) is not escaping. Then there exists a subsequence (x_{n_k}) converging to a point w_0 in X. Then we have that

$$d(x_0, f^{n_k}(w_0)) = d(f^{n_k}(x_{n_k}), f^{n_k}(w_0)) \leq d(x_{n_k}, w_0) \to 0.$$

Hence f is elliptic and the point x_0 belongs to the limit set of the forward orbit of w_0. Hence $x_0 \in \omega_f$. Similarly we obtain $x_n \in \omega_f$ for all $n \geq 1$. □

The proof of Theorem 1.6 is split in the two following results.

Proposition 6.3. *Let X be a proper geodesic Gromov hyperbolic metric space and let $f\colon X \to X$ be a non-expanding self-map. Let (x_n) be an escaping backward orbit with bounded step. Then the following are equivalent:*

(1) $b(x_n) > 0$;
(2) (x_n) *is a discrete quasigeodesic;*
(3) (x_n) *converges to a BRFP η inside a geodesic region with vertex η;*
(4) (x_n) *converges to a BRFP η and*

$$\lim_{n\to+\infty} h_{a,p}(x_n) = -\infty, \quad \forall a \in \Phi^{-1}(\eta),$$

that is, (x_n) is eventually contained in every horosphere centered in a point of $\Phi^{-1}(\eta)$;
(5) (x_n) *converges to a BRFP η and*

$$\liminf_{n\to+\infty} h_{a,p}(x_n) = -\infty, \quad \forall a \in \Phi^{-1}(\eta);$$

(6) (x_n) *converges to a repelling BRFP;*
(7) (x_n) *converges to a repelling BRFP η and $b(x_n) = \log \Lambda_\eta$.*

Proof. (1) ⇒ (2) Interpolate (x_n) with a curve $\gamma : [0, +\infty) \to X$ defined as

$$\gamma(t) = \gamma_{\lfloor t \rfloor}(t - \lfloor t \rfloor), \quad t > 0,$$

where $\gamma_n : [0, 1] \to X$ is a curve from x_n to x_{n+1} with length equal to $d(x_n, x_{n+1})$. Since $b(x_n) = \inf_n \frac{\sigma_n(x_n)}{n}$, it follows that $\sigma_n \geq b(x_n)n$ for all $n > 0$. Let $A \geq 1$ be such that $b(x_n) \geq A^{-1}$, and let $N \geq 1$. Then there exists $M_N \geq 0$ such that for all $m \geq M_N$ and for all $n = 1, \dots, N + 1$ we have

$$d(x_m, x_{m+n}) \geq A^{-1}n.$$

We want to prove that for all $N \geq 1$ the curve $\gamma|_{[M_N,+\infty)}$ is a N-local $(A\sigma_1, 2A\sigma_1^2)$-quasigeodesic in the sense of [16, Definition 1.1] i.e. for all $M_N \leq s \leq t$ with $|t - s| \leq N$ we have

$$\ell(\gamma|_{[s,t]}) \leq A\sigma_1 d(\gamma(s), \gamma(t)) + 2A\sigma_1^2.$$

Indeed,

$$\ell(\gamma|_{[s,t]}) \leq \sum_{k=\lfloor s\rfloor}^{\lfloor t\rfloor} \ell(\gamma_k) = \sum_{k=\lfloor s\rfloor}^{\lfloor t\rfloor} d(x_k, x_{k+1}) \leq \sigma_1(\lfloor t\rfloor + 1 - \lfloor s\rfloor)$$
$$\leq A\sigma_1 d(x_{\lfloor s\rfloor}, x_{\lfloor t\rfloor+1}) \leq A\sigma_1 d(\gamma(s), \gamma(t)) + 2A\sigma_1^2.$$

Now by [16, Theorem 1.4] there exist $\hat{A} \geq 1, \hat{B} \geq 0$ and $N = N(A, \sigma_1, \delta) \geq 1$ such that $\gamma|_{[M_N, +\infty)}$ is a (global) $(\hat{A}, \hat{B})$-quasi-geodesic in the sense of [16], that is for all $M_N \leq s < t$

$$\ell(\gamma|_{[s,t]}) \leq \hat{A} d(\gamma(s), \gamma(t)) + \hat{B}.$$

In particular, for all $n \geq M_N$ and $m \geq 1$ we have

$$d(x_0, x_1)m \leq \sum_{k=n}^{n+m-1} d(x_k, x_{k+1}) = \sum_{k=n}^{n+m-1} \ell(\gamma|_{[k,k+1]}) = \ell(\gamma|_{[n,n+m]}) \leq \hat{A} d(x_n, x_{n+m}) + \hat{B}$$

so

$$d(x_n, x_{n+m}) \geq \hat{A}^{-1} d(x_0, x_1)m - \hat{A}^{-1}\hat{B}.$$

Finally,

$$d(x_n, x_{n+m}) \leq \sum_{k=n}^{n+m-1} d(x_k, x_{k+1}) \leq \sigma_1 m$$

so $(x_n)_{n \geq M_N}$ is a discrete quasi-geodesic.

(2) $\Rightarrow$ (3) The discrete quasi-geodesic (x_n) can be interpolated with a quasi-geodesic ray. Then the result follows from Gromov's shadowing lemma (Theorem 2.5).

(3) $\Rightarrow$ (4) Follows from Proposition 2.11 iii).

(4) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (6) by Proposition 5.3 the BRFP η cannot be attracting. Assume by contradiction that η is indifferent. Let $a \in \Phi^{-1}(\eta)$, and let (x_{n_k}) be a subsequence such that

$$\lim_{k \to +\infty} h_{a,p}(x_{n_k}) = -\infty.$$

Then by the δ-Julia Lemma 3.1 we have

$$h_{a,p}(x_0) \leq h_{a,p}(x_{n_k}) + 4\delta \overset{n \to +\infty}{\longrightarrow} -\infty,$$

which is a contradiction.

(6) $\Rightarrow$ (7) Follows from Theorem 5.4 iii).

(7) $\Rightarrow$ (1) Trivial. $\square$

Proposition 6.4. *Let X be a proper geodesic Gromov hyperbolic metric space and let $f\colon X \to X$ be a non-expanding map. Let (x_n) be an escaping backward orbit with bounded step and assume that $b(x_n) = 0$. Then f is either weakly elliptic or parabolic. In the parabolic case, the orbit (x_n) converges to the indifferent Denjoy–Wolff point ζ of f, and*

$$\liminf_{n\to+\infty} h_{a,p}(x_n) > -\infty, \quad \forall a \in \Phi^{-1}(\eta),$$

that is, there exists a horosphere centered in a point of $\Phi^{-1}(\zeta)$ which does not contain any point of (x_n).

Proof. By Proposition 5.3 the limits points of (x_n) are indifferent. Then the map f cannot be hyperbolic. The map f cannot be strongly elliptic either, indeed from (3) of Proposition 4.16 all the BRFP are repelling. Hence f is either weakly elliptic or parabolic. If f is parabolic, then the backward orbit (x_n) has to converge to the Denjoy–Wolff point of f. The last statement follows from Proposition 6.3. □

Corollary 6.5. *Let X be a proper geodesic Gromov hyperbolic metric space. Let $f\colon X \to X$ be a non-expanding map. Let (x_n) be an escaping backward orbit with bounded step. Then the limit*

$$c(x_n) := \lim_{n\to+\infty} \frac{d(x_0, x_n)}{n}$$

exists and equals the backward step rate $b(x_n)$.

Proof. Clearly for all $n \geq 0$

$$d(x_0, x_n) \leq \sigma_n,$$

so

$$\limsup_{n\to\infty} \frac{d(x_0, x_n)}{n} \leq b(x_n).$$

It follows that if $b(x_n) = 0$ the limit exists and is equal to 0. If instead $b(x_n) > 0$, then Theorem 6.3 (x_n) converges to a repelling point $\eta \in \partial_G X$ with stable dilation $\log \Lambda_\eta = b(x_n) > 0$. Let $0 < a < \log \Lambda_\eta$. Then there exists $m \geq 1$ such that

$$\frac{\log \lambda_{\eta,p}(f^m)}{m} > a.$$

It follows that there exists $N \geq 0$ such that for all $n \geq N$ we have

$$d(p, x_{m(n+1)}) - d(p, x_{mn}) \geq am.$$

Hence, for all $n, k \geq N$,

$$d(x_{mn}, x_{mk}) \geq am|n-k|,$$

which gives

$$\liminf_{n\to\infty} \frac{d(x_0, x_{mn})}{n} \geq am.$$

Now for all $n \geq 0$ there exists $k \geq 0$ such that $mk \leq n < m(k+1)$, so

$$d(x_0, x_n) \geq d(x_0, x_{mk}) - d(x_{mk}, x_n) \geq d(x_0, x_{mk}) - (n-mk)\sigma_1 \geq d(x_0, x_{mk}) - m\sigma_1,$$

which implies

$$\liminf_{n\to+\infty} \frac{d(x_0, x_n)}{n} \geq \liminf_{k\to+\infty} \left[\frac{d(x_0, x_{mk})}{mk} - \frac{m\sigma_1}{mk}\right] \geq a$$

for all $0 < a < \log \Lambda_\eta$, hence

$$\liminf_{n\to+\infty} \frac{d(x_0, x_n)}{n} \geq \log \Lambda_\eta = b(x_n). \quad \square$$

We leave the following open question.

Question 6.6. *Let (X, d) a Gromov hyperbolic metric space and let $f\colon X \to X$ be a weakly elliptic non-expanding map. Can there exist an escaping backward orbit with bounded step (x_n) not converging to a point of the Gromov boundary? Clearly such an orbit would satisfy $b(x_n) = 0$ and thus its limit set would be contained in the Gromov closure of the limit retract ω_f.*

We conclude giving an example of a weakly elliptic non-expanding map with a backward orbit with bounded step converging to a point in the Gromov closure of the limit retract.

Example 6.7. Let $\mathbb{H}_u \subset \mathbb{C}$ be the upper half-plane endowed with the Poincaré distance. Let $f : \mathbb{H}_u \longrightarrow \mathbb{H}_u$ be defined by $f(x+iy) = [x-1]_+ - [-x-1]_+ + iy$ where $[\cdot]_+$ is the positive part. The map f is weakly elliptic non-expanding with limit retract the geodesic line between 0 and ∞. The associated retraction is $g(x+iy) = iy$. Finally, the sequence $x_n = n + i$ is a backward orbit with bounded step converging to $+\infty$.

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