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MPCC: strong stability of weakly nondegenerate S-stationary points

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ABSTRACT

In this paper, we consider the class of mathematical programmes with complementarity constraints (MPCC). Specifically, we focus on *strong stability* of M- and S-stationary points for MPCC. Kojima introduced this concept for standard nonlinear optimization problems. It refers to several well-posedness properties of the underlying problem. Besides its topological definition, the challenge is to state an algebraic characterization of strong stability. We obtain such a description for S-stationary points whose components of Lagrange vectors corresponding to bi-active constraints do not mutually vanish. We call these points *weakly nondegenerate*. Moreover, we show that a particular constraint qualification is necessary for strong stability.

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1. Introduction

In this paper, we consider the following mathematical programme with complementarity constraints (MPCC):

$$\min_{x \in M[r,s]} f(x) \tag{1}$$

with

$$M[r,s] = \left\{x \in \mathbb{R}^n \mid \min\{r_l(x), s_l(x)\} = 0,\ l \in L\right\},$$

where L is a finite set and all describing functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $r_l, s_l\colon \mathbb{R}^n \to \mathbb{R}, l \in L$, are assumed to be twice continuously differentiable. We skip additional equality and inequality constraints to focus on the structure raised by complementarity constraints. If $s_l(x) = 1$ for some $l \in L$, then $r_l(x)$ can be treated as an equality constraint. Generalizing our results by including inequality constraints is laborious but technically not challenging.

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There are many applications for MPCC. we refer e.g. to the classical references [1–3]. Furthermore, we refer to applications in engineering, bilevel optimization, dynamic optimization, structure design, optimal control and stochastic programmes [4–8].

In this paper, we centre our attention on *strong stability* for MPCC. Strong stability is a desirable quality feature of some stationary points, ensuring that they are not affected much if the problem slightly changes. This guarantee is relevant in parametric optimization and sensitivity analysis. The property was introduced by Kojima [9] for standard nonlinear optimization problems. It holds at a certain stationary point if each sufficiently small perturbed problem has a stationary point that is locally unique and approaches that of the unperturbed problem as the perturbation becomes negligible. This concept applies to situations where the values of perturbations, along with their first and second derivatives, are considered without requiring them to depend on real parameters. Notably, if we only allow small linear and quadratic perturbations, we can readily use results on strong stability. The topological description given by Kojima can be used analogously for MPCC. However, the challenge lies in finding an algebraic characterization of this property.

Unlike the uniquely defined notion of a stationary point of a standard nonlinear optimization problem, there are several stationarity concepts for MPCC; among them are C-, M- and S-stationarity, see e.g. [10–12]. Next, we refer to some related results. In [13], an algebraic characterization of a strongly stable C-stationary point of MPCC is presented under the assumption that MPCC-LICQ is fulfilled. Recall that MPCC-LICQ holds if the gradients of the active constraints at the point under consideration are linearly independent. Moreover, we refer to [14–16] for analogous results when MPCC-LICQ is not satisfied. In [17], an algebraic characterization of strongly stable M- and S-stationary points under MPCC-LICQ is presented.

The area of MPCC has been actively studied over the recent three decades. Next, we mention some related references in addition to those cited above. In [18], MPCC is numerically solved as a standard nonlinear programme. Establishing global convergence to S-stationary points, an elastic mode approach and an interior-penalty method are presented in [19, 20], respectively. In [21], MPCC is smoothed by a lift onto a smooth manifold. The convergence properties of relaxation methods for MPCCs are revisited in [22]. We refer to [23] where sequential optimality conditions for MPCC are discussed together with some algorithmic consequences. Moreover, in [24] the authors discuss the convergence of regularization methods for MPCCs with approximate sequences of stationary points.

We now highlight the generic property of *nondegeneracy* introduced in [25] for so-called critical points. Nondegeneracy is fulfilled at a critical point if MPCC-LICQ holds, the components of the Lagrange vector corresponding to bi-active constraints are nonzero, and the Hessian of the Lagrange function is nonsingular on a certain tangent space. It is well known that nondegeneracy is a sufficient condition for strong stability, see e.g. [13, Corollary 3.2] and [12, Theorem 11]. However, we will introduce a weaker property to characterize strong stability: *weak nondegeneracy*. Although this concept is given for S-stationary points, it is mainly motivated by [17, Theorem 3.1] and [13, Theorem 4.1] where it plays an essential role in the description of strong stability of C-, M- and S-stationary points. There, this notion is implicitly used without giving it a particular name. Roughly speaking, an S-stationary point is weakly nondegenerate if at least one of the components of each Lagrange vector corresponding to bi-active constraints does not vanish. Obviously, nondegeneracy implies weak nondegeneracy. However, in general, MPCC-LICQ does not hold at weakly nondegenerate points.

Altogether, the goal of this paper is threefold:

- To characterize the interplay between stability properties of weakly nondegenerate S-stationary points when considered as M-stationary points.
- To show that a constraint qualification of Mangasarian-Fromovitz type, called M-MFCQ, is necessary for the strong stability of weakly nondegenerate S-stationary points.
- To provide an algebraic characterization of strongly stable weakly nondegenerate S-stationary points.

The remainder of this paper is divided into four sections. The next section contains basic notations, preliminary results and some properties of standard nonlinear optimization problems. Section 3 recollects the concepts of M- and S-stationarity for MPCC and discusses several relations between these two concepts. In particular, it introduces the notions of weak nondegeneracy and basic Lagrange vector. In Section 4, we show that M-MFCQ is a necessary condition for the strong stability of a weakly nondegenerate point. Section 5 contains the main contribution of this work: a topological and algebraic characterization of a strongly stable weakly nondegenerate S-stationary point.

2. Preliminaries

In this section, we present several notations used later. A substantial part of it is mainly taken from [26]. For $w \in \mathbb{R}^n$, always considered as a column vector, let $w_i \in \mathbb{R}$, $i = 1, \dots, n$ denote its components and define the index sets

$$I^0(w) = \{i \in \{1, \dots, n\} |\, w_i = 0\},$$
$$I^*(w) = \{i \in \{1, \dots, n\} |\, w_i \neq 0\}.$$

Given $\bar{x}, x \in \mathbb{R}^n$, let $\langle \bar{x}, x \rangle$ denote the scalar product of the vectors $\bar{x}$ and x; moreover, $\|x\|$ stands for the Euclidean norm of x, that is, $\|x\| = \sqrt{\langle x, x \rangle}$. Furthermore, for $\delta > 0$ let

$$B^n(\bar{x},\delta) = \left\{x \in \mathbb{R}^n \mid \|x - \bar{x}\| \le \delta\right\}.$$

We abbreviate the sentence 'V is a neighbourhood of $\bar{x}$' by letting $\mathcal{V}(\bar{x})$ be the *set of all neighbourhoods* of $\bar{x}$ and then simply stating '$V \in \mathcal{V}(\bar{x})$ '.

Let $\mathcal{C}^k(\mathbb{R}^n,\mathbb{R}^m)$ be the space of k−times continuously differentiable mappings with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$. For $f \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ denote the partial derivative of f at $\bar{x} \in \mathbb{R}^n$ with respect to x_i by $\frac{\partial f(\bar{x})}{\partial x_i}$, $i = 1,\dots,n$. In addition, $D_x f(\bar{x})$ stands for its gradient taken as a row vector and $D_x^2 f(\bar{x})$ for its Hessian at $\bar{x}$. Moreover, for $F \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^m)$ let $D_x F(\bar{x}) \in \mathbb{R}^{m\times n}$ be its Jacobian at $\bar{x}$. By $\mathbb{R}^n_+$, we denote the n-dimensional nonnegative orthant.

To apply the concept of strong stability, we need a seminorm for mappings. Let $V \in \mathcal{V}(\bar{x})$ and $G \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^m)$. Following [9], denote

$$\|G\|^V = \max\left\{\sup_{x\in V}\max_i\{|G_i(x)|\},\ \sup_{x\in V}\max_{i,j}\left\{\left|\frac{\partial G_i(x)}{\partial x_j}\right|\right\},\ \sup_{x\in V}\max_{i,j,k}\left\{\left|\frac{\partial^2 G_i(x)}{\partial x_j \partial x_k}\right|\right\}\right\} \tag{2}$$

where the indices i and j, k are varying in the sets $\{1,\dots,m\}$ and $\{1,\dots,n\}$, respectively.

For $O \subset \mathbb{R}^n$, let $\operatorname{conv} O$, $\operatorname{int} O$ and $O^\perp$ denote its *convex hull*, its *interior* and its *orthogonal complement*, respectively. If O is convex, then $\operatorname{ext} O$ is the set of its *extreme points*. If O is finite, let $|O|$ denote its cardinality.

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices and $A \in \mathbb{R}^{m\times n}$. The transpose of A is denoted by A^{t} and the linear subspace

$$\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

is the *kernel* of A. The following lemma is an obvious adaptation of well-known results from convex analysis.

Lemma 2.1: *Let $A \in \mathbb{R}^{m\times n}$, $a^1,\dots,a^n \in \mathbb{R}^m$ be the columns of A, $b \in \mathbb{R}^m$, $I^1 \subset \{1,\dots,n\}$, $I^2 = \{1,\dots,n\} \setminus I^1$ and*

$$\mathcal{A} = \left\{v \in \mathbb{R}^n \mid Av = b, v_i \ge 0, i \in I^1\right\}.$$

Then, the following conditions hold:

(i) $v^0 \in \operatorname{ext}\mathcal{A}$ *if and only if the vectors*

$$a^i,\ i \in [I^1 \cap I^*(v^0)] \cup I^2 \tag{3}$$

are linearly independent.

(ii) *If the vectors a^i, $i \in I^2$ are linearly independent, then for each $v \in \mathcal{A}$ there exists $v^0 \in \operatorname{ext}\mathcal{A}$ with $I^*(v^0) \cap I^1 \subset I^*(v)$.*

Proof: The proof of (*i*) is analogous to that of [27, Proposition 3.3.3]. For proving (*ii*), let $v \in \mathcal{A}$. Following the argument in the proof of Carathéodory's theorem, we obtain $v^0 \in \mathcal{A}$ with $I^*(v^0) \cap I^1 \subset I^*(v)$ such that the vectors in (3) are linearly independent. From (*i*) we get $v^0 \in \operatorname{ext} \mathcal{A}$. ■

If $E \subset \mathbb{R}^n$ is a linear subspace and $A \in \mathbb{R}^{n\times n}$ a symmetric matrix, then A is called *positive definite on E* if

$$v^{\mathrm{t}} A v > 0$$

for all $v \in E \setminus \{0\}$, which is denoted by $A|_E \succ 0$. When $E = \mathbb{R}^n$, we simply write $A \succ 0$. We use the convention $A|_{\{0\}} \succ 0$.

Let $\Psi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The sets $\operatorname{dom}\Psi = \{z \in \mathbb{R}^n \,|\, \Psi(z) \neq \emptyset\}$ and

$$\operatorname{gr}\Psi = \{(z, \Psi(z)) \,|\, z \in \operatorname{dom}\Psi\}$$

are the *domain* and the *graph* of Ψ, respectively. If the set $\operatorname{gr}\Psi$ is closed, then Φ is said to be a *closed* set-valued mapping.

In this paper, we consider M- and S-stationary points of MPCC as well as their relation to stationary points of a certain standard nonlinear programme. For the sake of generality and, particularly, for using the results in [26], we consider the following generalized equation:

$$\text{Find } x \in \mathbb{R}^n \text{ such that } D_x f(x) = \theta^{\mathrm{t}} D_x F(x) \text{ for some } \theta \in \Theta(F(x)), \tag{4}$$

where $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $F \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$ and $\Theta\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a cone-valued mapping. The set $\Theta(x)$ can be empty or nonconvex for some $x \in \mathbb{R}^n$. To Problem (4), we associate the mapping $\mathcal{P}^\Theta\colon \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \to \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \times \{\Theta\}$ given by

$$\mathcal{P}^\Theta(f, F) = (f, F, \Theta),$$

which fixes Θ, and allows f and F to vary freely. Note that $\mathcal{P}^\Theta$ is a bijection and that elements in its image identify instances of Problem (4). Due to the latter, and for simplicity, we abuse language and refer to $P = \mathcal{P}^\Theta(f, F)$ as the problem under consideration.

Definition 2.2: Let $P = \mathcal{P}^\Theta(f, F)$.

(i) A point $x \in \mathbb{R}^n$ is called a *Fritz John* point for P if

$$\theta_0 D_x f(x) = \theta^{\mathrm{t}} D_x F(x)$$

for some $(\theta_0, \theta) \in [\mathbb{R}_+ \times \Theta(F(x))] \setminus \{0\}$. The set of Fritz John points for P is denoted by $\Sigma^{\mathrm{f}}(P)$.

(ii) A point $x \in \mathbb{R}^n$ that solves Problem (4) is called a *stationary* point for P. The set of stationary points for P is denoted by $\Sigma(P)$.

The three stationarity concepts mentioned above can be characterized by Definition 2.2 choosing Θ accordingly. Furthermore, for $x \in \mathbb{R}^n$ and $P = \mathcal{P}^\Theta(f, F)$ define the *set of Lagrange vectors* as

$$\mathcal{L}(P, x) = \left\{\theta \in \Theta(F(x)) \,|\, D_x f(x) = \theta^t D_x F(x)\right\}.$$

In particular, it holds that $x \in \Sigma(P)$ if and only if $\mathcal{L}(P, x) \neq \emptyset$. Next, we recall two generalized constraint qualifications, see [26].

Definition 2.3: We say that the Generalized Linear Independence constraint qualification (GLICQ) holds at x for F if

$$\ker D_x F(x)^t \cap \operatorname{span} \Theta(F(x)) = \{0\}.$$

Definition 2.4: We say that the Generalized Mangasarian-Fromovitz constraint qualification (GMFCQ) holds at x for F if

$$\ker D_x F(x)^t \cap \Theta(F(x)) \subset \{0\}.$$

Note that Definitions 2.3 and 2.4 are given for any $x \in \mathbb{R}^n$, and that $\operatorname{span} \emptyset = \{0\}$. In particular, both GLICQ and GMFCQ hold at x for F whenever $\Theta(F(x)) = \emptyset$. Moreover, if no confusion is possible, we say that a constraint qualification holds at x for $P = \mathcal{P}^\Theta(f, F)$ if it holds at x for F.

Since the mapping $\mathcal{P}^\Theta$ is a bijection, it naturally provides the set $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \times \{\Theta\}$ with the same structure as that of $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1})$. In particular, given $\bar{x} \in \mathbb{R}^n$, $V \in \mathcal{V}(\bar{x})$ and $P = \mathcal{P}^\Theta(f, F)$, we can define

$$\|P\|^V = \|(f, F)\|^V,$$

where the right-hand-side is obtained from (2) by choosing $G = (f, F)$. Furthermore, let $\bar{f} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $\bar{F} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$. For $\bar{P} = \mathcal{P}^\Theta(\bar{f}, \bar{F})$ and $\delta > 0$ define

$$B^V(\bar{P}, \delta) = \{P \,|\, \|P - \bar{P}\|^V \leq \delta\},$$

and let $\mathcal{W}^V(\bar{P})$ denote the *set of all neighbourhoods* of $\bar{P}$.

Throughout this paper, we use an overline as a notation when considering a particular item. For instance, $\bar{x}$ and $\bar{P}$ are sometimes the point and the problem under consideration, respectively. In those cases, x and P denote elements sufficiently close to $\bar{x}$ and $\bar{P}$, respectively.

Note that if Θ is closed, GMFCQ is related to the boundedness of Lagrange vectors after arbitrary sufficiently small perturbations.

Theorem 2.5 ([26, Theorem 4.3]): *Assume that Θ is closed. Then, the following two conditions are equivalent.*

(i) *GMFCQ holds at* $\bar{x}$ *for* $\bar{P}$.
(ii) *There exist* $V \in \mathcal{V}(\bar{x})$, $W \in \mathcal{W}^V(\bar{P})$ *and a compact set* $Q \subset \mathbb{R}^m$ *such that*

$$\mathcal{L}(P, x) \subset Q$$

for all $x \in V$ *and all* $P \in W$.

Next, we present two stability concepts given in [26] for generalized equations. Note that the first one is a generalization of that introduced by Kojima [9].

Definition 2.6: Let $\bar{P} = \mathcal{P}^{\Theta}(\bar{f}, \bar{F})$. A point $\bar{x} \in \Sigma(\bar{P})$ is called *strongly stable* if there exists a real number $\bar{\delta} > 0$ such that for each $\delta \in (0, \bar{\delta}]$ there exists a real number $\varepsilon > 0$ such that for every $P \in B^{B^n(\bar{x},\bar{\delta})}(\bar{P}, \varepsilon)$ it holds that

$$|\Sigma(P) \cap B^n(\bar{x}, \bar{\delta})| = |\Sigma(P) \cap B^n(\bar{x}, \delta)| = 1.$$

The set of strongly stable stationary points for $\bar{P}$ is denoted by $\Sigma^{\mathrm{s}}(\bar{P})$.

Definition 2.7: Let $\bar{P} = \mathcal{P}^{\Theta}(\bar{f}, \bar{F})$. A point $\bar{x} \in \Sigma(\bar{P})$ is called *weakly stable* if there exist real numbers $\bar{\delta} > 0$ and $\bar{\varepsilon} > 0$ such that for every $P \in B^{B^n(\bar{x},\bar{\delta})}(\bar{P}, \bar{\varepsilon})$ it holds that

$$|\Sigma(P) \cap B^n(\bar{x}, \bar{\delta})| = 1.$$

The set of weakly stable stationary points for $\bar{P}$ is denoted by $\Sigma^{\mathrm{w}}(\bar{P})$.

Note that in Definitions 2.6 and 2.7, the problem under consideration is $\bar{P}$, and the problem resulting from a sufficiently small perturbation is denoted by P. Obviously, it holds that

$$\Sigma^{\mathrm{s}}(\bar{P}) \subset \Sigma^{\mathrm{w}}(\bar{P}),$$

that is, the property defined in Definition 2.7 is weaker than that defined in Definition 2.6. The following result presents a necessary condition for weak stability.

Theorem 2.8 ([26, Theorem 3.19]): *If* $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P})$, *then there exist* $V \in \mathcal{V}(\bar{x})$ *and* $W \in \mathcal{W}^V(\bar{P})$ *such that*

$$\Sigma(P) \cap V = \Sigma^{\mathrm{f}}(P) \cap V$$

for all $P \in W$.

For later use, in the remainder of this section we recollect several concepts related to a standard nonlinear programme (NLP) with finitely many equality and inequality constraints given as

$$\min_{x \in M^{\mathrm{sn}}[h,g]} f(x),$$

where

$$M^{\mathrm{sn}}[h,g] = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{matrix} h_i(x) = 0, & i \in I, \\ g_j(x) \geq 0, & j \in J \end{matrix} \right\}$$

with finite index sets I and J as well as functions $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $h_i \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}), i \in I$ and $g_j \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}), j \in J$. The problem of finding a stationary point for NLP can be written as in (4) by letting

$$F(x) = \begin{pmatrix} h_i(x),\ i \in I \\ g_j(x),\ j \in J \end{pmatrix},$$

$$\Theta^{\mathrm{sn}}(y^h, y^g) = \begin{cases} \mathbb{R}^{|I|} \times (\mathbb{R}^{|J|}_+ \cap \{y^g\}^\perp), & \text{if } y^h_i = 0,\ i \in I,\ y^g_j \geq 0,\ j \in J \\ \emptyset, & \text{otherwise} \end{cases},$$

where $y^h \in \mathbb{R}^{|I|}$ and $y^g \in \mathbb{R}^{|J|}$. For simplicity of notation, denote the mapping $\mathcal{P}^{\Theta^{\mathrm{sn}}}$ by $\mathcal{P}^{\mathrm{sn}}$. Note that for $x \in M^{\mathrm{sn}}[h,g]$ and $P^{\mathrm{sn}} = \mathcal{P}^{\mathrm{sn}}(f,h,g)$, GLICQ and GMFCQ are simply the well-known LICQ and MFCQ for standard nonlinear programmes, respectively. Let $J^0_g(x) = \{j \in J | \, g_j(x) = 0\}$. We recall that:

- The *Linear Independence constraint qualification (LICQ)* holds at x for P^{sn} if the vectors

  $$D_x h_i(x),\ i \in I, \quad D_x g_j(x),\ j \in J^0_g(x)$$

  are linearly independent.
- The *Mangasarian-Fromovitz constraint qualification (MFCQ)* holds at x for P^{sn} if the vectors

  $$D_x h_i(x),\ i \in I$$

  are linearly independent and there exists $v \in \mathbb{R}^n$ such that

  $$D_x h_i(x) v = 0,\ i \in I, \quad D_x g_j(x) v > 0,\ j \in J^0_g(x).$$

In the remainder of this section, let $P^{\mathrm{sn}} = \mathcal{P}^{\mathrm{sn}}(f,h,g)$ and $\bar{P}^{\mathrm{sn}} = \mathcal{P}^{\mathrm{sn}}(\bar{f}, \bar{h}, \bar{g})$. For $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^{|I|}$ and $\mu \in \mathbb{R}^{|J|}$, let

$$\mathbf{L}^{\mathrm{sn}}(x, \lambda, \mu) = f(x) - \sum_{i \in I} \lambda_i h_i(x) - \sum_{j \in J} \mu_j g_j(x)$$

be the *Lagrange function* for P^{sn}. In particular, let $\bar{\mathbf{L}}^{\mathrm{sn}}$ denote the Lagrange function for $\bar{P}^{\mathrm{sn}}$. Note that if $x \in \Sigma(P^{\mathrm{sn}})$, then $D_x \mathbf{L}^{\mathrm{sn}}(x, \lambda, \mu) = 0$ for some $(\lambda, \mu) \in \Theta^{\mathrm{sn}}(h(x), g(x))$, that is, $(\lambda, \mu) \in \mathcal{L}(P^{\mathrm{sn}}, x)$.

Next, we present two auxiliary results under the assumption that MFCQ holds at the stationary point $\bar{x}$ for $\bar{P}^{\mathrm{sn}}$. The first one characterizes a property of Lagrange vectors after a sufficiently small perturbation. The second one states that, under MFCQ, a lack of strong stability implies locally a bifurcation of stationary points for a particular sequence of problems.

Lemma 2.9: *Let $\bar{x} \in \Sigma(\bar{P}^{\mathrm{sn}})$ and assume that MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{sn}}$. Then, there exists $\bar{V} \in \mathcal{V}(\bar{x})$ such that for every $V \in \mathcal{V}(\bar{x})$ with $V \subset \bar{V}$ there exists $W \in \mathcal{W}^V(\bar{P}^{\mathrm{sn}})$ such that for all $P^{\mathrm{sn}} \in W$, $x \in V \cap \Sigma(P^{\mathrm{sn}})$ and each $(\lambda, \mu) \in \mathcal{L}(P^{\mathrm{sn}}, x)$ it holds that*

$$I^*(\bar{\mu}) \subset I^*(\mu)$$

for some $(\bar{\lambda}, \bar{\mu}) \in \operatorname{ext} \mathcal{L}(\bar{P}^{\mathrm{sn}}, \bar{x})$.

Proof: By Theorem 2.5 and a continuity argument, there exists $\bar{V} \in \mathcal{V}(\bar{x})$ such that for every $V \in \mathcal{V}(\bar{x})$ with $V \subset \bar{V}$ there exists $W \in \mathcal{W}^V(\bar{P}^{\mathrm{sn}})$ such that for all $P^{\mathrm{sn}} \in W$, $x \in V \cap \Sigma(P^{\mathrm{sn}})$ and each $(\lambda, \mu) \in \mathcal{L}(P^{\mathrm{sn}}, x)$ it holds that

$$I^*(\bar{\mu}) \subset I^*(\mu)$$

for some $(\bar{\lambda}, \bar{\mu}) \in \mathcal{L}(\bar{P}^{\mathrm{sn}}, \bar{x})$. An application of Lemma 2.1 yields the desired result. ■

Lemma 2.10: *Let $\bar{x} \in \Sigma(\bar{P}^{\mathrm{sn}})$ and assume that MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{sn}}$. If $\bar{x} \notin \Sigma^{s}(\bar{P}^{\mathrm{sn}})$, then there exist sequences $x^{k,1}$, $x^{k,2}$ and P^k with $x^{k,1} \neq x^{k,2}$, $x^{k,1} \to \bar{x}$, $x^{k,2} \to \bar{x}$, $\{x^{k,1}, x^{k,2}\} \subset \Sigma(P^k)$, and $\|P^k - \bar{P}^{\mathrm{sn}}\|^V \to 0$ for every bounded $V \in \mathcal{V}(\bar{x})$.*

Proof: Analogously to the proofs of both [9, Corollary 4.3] and the 'only if' part of [9, Theorem 7.2], a sequence P^k is obtained by adding sufficiently small quadratic perturbations to the functions describing $\bar{P}^{\mathrm{sn}}$. Thus, it holds that $\|P^k - \bar{P}^{\mathrm{sn}}\|^V \to 0$ for every bounded $V \in \mathcal{V}(\bar{x})$. Moreover, by the aforementioned proofs, there exist sequences $x^{k,1}$, $x^{k,2}$ and P^k with $x^{k,1} \neq x^{k,2}$, $x^{k,1} \to \bar{x}$, $x^{k,2} \to \bar{x}$, $\{x^{k,1}, x^{k,2}\} \subset \Sigma(P^k)$, which completes the proof. ■

We end this section by recalling Kojima's well-known algebraic characterization of strong stability when MFCQ holds but LICQ does not hold. For $\bar{x} \in \Sigma(\bar{P}^{\mathrm{sn}})$ and $(\bar{\lambda}, \bar{\mu}) \in \mathcal{L}(\bar{P}^{\mathrm{sn}}, \bar{x})$ let

$$T_{\bar{x}}(\bar{h}, \bar{g}, \bar{\lambda}, \bar{\mu}) = \left\{ v \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} D_x \bar{h}_i(\bar{x}) v = 0, & i \in I, \\ D_x \bar{g}_j(\bar{x}) v = 0, & j \in I^*(\bar{\mu}) \end{array} \right\}$$

be the corresponding *tangent space*.

Theorem 2.11 ([9, Theorem 7.2]): *Let $\bar{x} \in \Sigma(\bar{P}^{\text{sn}})$ and assume that MFCQ holds and that LICQ does not hold at $\bar{x}$ for $\bar{P}^{\text{sn}}$. Then, the following two conditions are equivalent:*

(i) $\bar{x} \in \Sigma^s(\bar{P}^{\text{sn}})$.
(ii) *For all $(\bar{\lambda}, \bar{\mu}) \in \text{ext}\,\mathcal{L}(\bar{P}^{\text{sn}}, \bar{x})$ it holds that*

$$D_x^2 \bar{\mathbf{L}}^{\text{sn}}(\bar{x}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}}(\bar{h}, \bar{g}, \bar{\lambda}, \bar{\mu})} \succ 0.$$

3. Stationarity concepts and basic Lagrange vectors

In this section, we consider a mathematical programme with complementarity constraints as defined in (1). We give special attention to M- and S-stationarity, and present the notion of a *basic Lagrange vector*. For a feasible point $x \in M[r,s]$ define the following active index sets:

$$\begin{aligned}
\bar{I}_r(x) &= \{l \in L |\, r_l(x) = 0\}, \\
\bar{I}_s(x) &= \{l \in L |\, s_l(x) = 0\}, \\
I_r(x) &= \{l \in L |\, r_l(x) = 0, s_l(x) > 0\}, \\
I_s(x) &= \{l \in L |\, r_l(x) > 0, s_l(x) = 0\}, \\
I_{rs}(x) &= \{l \in L |\, r_l(x) = 0, s_l(x) = 0\}.
\end{aligned}$$

By definition, the Linear Independence constraint qualification for MPCC (MPCC-LICQ) is fulfilled at x for (r,s) if the vectors

$$D_x r_l(x),\ l \in \bar{I}_r(x), \quad D_x s_l(x),\ l \in \bar{I}_s(x)$$

are linearly independent. Next, we recall the concepts of M- and S-stationarity, see e.g. [11, 12].

Definition 3.1 (M- and S-stationarity): A point $\bar{x} \in M[r,s]$ is called an *M-stationary point* for Problem (1) if there exists $(\rho, \sigma) \in \mathbb{R}^{2l}$ such that

$$D_x \mathbf{L}^{\text{cc}}(\bar{x}, \rho, \sigma) = 0, \tag{5}$$

$$\rho_l \cdot r_l(\bar{x}) = \sigma_l \cdot s_l(\bar{x}) = 0,\ l \in L,$$

$$\rho_l > 0,\ \sigma_l > 0 \text{ or } \rho_l \cdot \sigma_l = 0,\ l \in L, \tag{6}$$

where

$$\mathbf{L}^{\text{cc}}(x, \rho, \sigma) = f(x) - \sum_{l \in L} [\rho_l r_l(x) + \sigma_l s_l(x)]$$

is the so-called *MPCC-Lagrange function*. Moreover, a point $\bar{x} \in M[r,s]$ is called an *S-stationary point* for Problem (1) if there exists $(\rho, \sigma) \in \mathbb{R}^{2l}$ such that (5) and (6) hold, and that $\rho_l \geq 0,\ \sigma_l \geq 0, l \in I_{rs}(\bar{x})$.

Now, we characterize the relation between Definitions 2.2 and 3.1 by using appropriate describing functions and cone-valued mappings. Let $F = (r, s), \theta = (\rho, \sigma) \in \mathbb{R}^{2|L|}, y = (y^r, y^s) \in \mathbb{R}^{2|L|}$ and

$$\Theta^M(y^r, y^s) = \begin{cases} \Xi^M(y^r, y^s), & \text{if } \min\{y_l^r, y_l^s\} = 0,\ l \in L \\ \emptyset, & \text{otherwise} \end{cases},$$

where

$$\Xi^M(y^r, y^s) = \left\{ (\rho, \sigma) \in \mathbb{R}^{2|L|} \,\middle|\, \begin{array}{ll} y_l^r \cdot \rho_l = y_l^s \cdot \sigma_l = 0, & l \in L, \\ \rho_l > 0,\ \sigma_l > 0 \text{ or } \rho_l \cdot \sigma_l = 0, & l \in L \end{array} \right\}.$$

Moreover, let

$$\Theta^S(y^r, y^s) = \begin{cases} \Xi^S(y^r, y^s), & \text{if } \min\{y_l^r, y_l^s\} = 0,\ l \in L \\ \emptyset, & \text{otherwise} \end{cases},$$

where

$$\Xi^S(y^r, y^s) = \left\{ (\rho, \sigma) \in \mathbb{R}^{2|L|} \,\middle|\, \begin{array}{ll} y_l^r \cdot \rho_l = y_l^s \cdot \sigma_l = 0, & l \in L, \\ \rho_l \geq 0, \sigma_l \geq 0, & l \in I^0(y^r) \cap I^0(y^s) \end{array} \right\}.$$

Note that Θ^M is closed, whereas Θ^S is not.

Theorem 3.2: *Let $P^{\mathrm{cm}} = \mathcal{P}^{\Theta^M}(f, F)$ and $P^{\mathrm{cs}} = \mathcal{P}^{\Theta^S}(f, F)$.*

(i) *The point $\bar{x}$ is an M-stationary point for Problem (1) if and only if $\bar{x} \in \Sigma(P^{\mathrm{cm}})$.*
(ii) *The point $\bar{x}$ is an S-stationary point for Problem (1) if and only if $\bar{x} \in \Sigma(P^{\mathrm{cs}})$.*

The previous result is trivial but also crucial. Indeed, Theorem 3.2 allows us to focus on the strong stability of M- and S-stationary points given as in Definition 2.6. That is, there is no need to define independently strong stability for each stationarity concept. For simplicity of notation, we denote $\mathcal{P}^{\Theta^M}$ and $\mathcal{P}^{\Theta^S}$ by $\mathcal{P}^{\mathrm{cm}}$ and $\mathcal{P}^{\mathrm{cs}}$, respectively.

Note that GLICQ for both M- and S-stationary points is equivalent to the condition MPCC-LICQ, see [26, Section 5]. We say that M-MFCQ and S-MFCQ hold at $\bar{x}$ for P^{cm} and P^{cs} whenever GMFCQ holds at $\bar{x}$ for P^{cm} and P^{cs}, respectively. It is easy to see that

$$\mathcal{L}(P^{\mathrm{cs}}, x) \subset \mathcal{L}(P^{\mathrm{cm}}, x)$$

and that M-MFCQ implies S-MFCQ. As seen in the following two theorems, M-MFCQ and S-MFCQ are necessary conditions for the strong stability of M- and S-stationary points, respectively.

Theorem 3.3 ([26, Theorem 5.2]): *Let $\bar{P}^{\mathrm{cm}} = \mathcal{P}^{\mathrm{cm}}(\bar{f}, \bar{r}, \bar{s})$. If $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cm}})$, then M-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$.*

Theorem 3.4 ([26, Theorem 5.6]): *Let $\bar{P}^{\mathrm{cs}} = \mathcal{P}^{\mathrm{cs}}(\bar{f}, \bar{r}, \bar{s})$. If $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cs}})$, then S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$.*

As the paper title indicates, the forthcoming results focus mainly on strong stability and *weak nondegeneracy*. We formally define the latter next.

Definition 3.5 (Weak nondegeneracy): Let $P^{\mathrm{cs}} = \mathcal{P}^{\mathrm{cs}}(f, r, s)$ and $x \in \Sigma(P^{\mathrm{cs}})$. We say that x is a *weakly nondegenerate* point for P^{cs} if

$$I_{rs}(x) \subset I^*(\rho) \cup I^*(\sigma) \tag{7}$$

for all $(\rho, \sigma) \in \mathcal{L}(P^{\mathrm{cs}}, x)$.

According to the following theorem, weak nondegeneracy is sometimes necessary for weak stability. The proof of this statement is not entirely original but merely a recollection of different parts of the proofs of [16, Lemma 5.2], [13, Theorem 3.1] and [17, Theorem 4.1]. Nevertheless, for completeness, we prove it in detail after recalling an auxiliary result.

Lemma 3.6 ([17, Lemma 4.2]): *Assume that $\bar{f} \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$. If $\bar{f}(0) = 0$ and $D_x\bar{f}(0) = 0$, then there exist $v_{ij} \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$, $i, j = 1, \ldots, n$ such that*

$$\bar{f}(x) = \sum_{i,j=1}^{n} v_{ij}(x) x_i x_j.$$

Theorem 3.7: *Let $P^{\mathrm{cm}} = \mathcal{P}^{\mathrm{cm}}(f, r, s)$, $P^{\mathrm{cs}} = \mathcal{P}^{\mathrm{cs}}(f, r, s)$, $\bar{x} \in \Sigma(\bar{P}^{\mathrm{cs}})$ and assume that $|\bar{I}_{\bar{r}}(\bar{x})| + |\bar{I}_{\bar{s}}(\bar{x})| \leq n$. If $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}}) \cup \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cm}})$, then $\bar{x}$ is a weakly nondegenerate point for $\bar{P}^{\mathrm{cs}}$.*

Proof: Assume that $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$. Noting that $\Sigma(P^{\mathrm{cs}}) \subset \Sigma(P^{\mathrm{cm}})$, the proof of the case $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cm}})$ runs analogously. Fix $B^n(\bar{x}, \bar{\delta})$ and $B^{B^n(\bar{x},\bar{\delta})}(\bar{P}^{\mathrm{cs}}, \bar{\varepsilon})$ as in Definition 2.7 and suppose contrarily that $\bar{\rho}_{l^0} = \bar{\sigma}_{l^0} = 0$ for some $l^0 \in I_{\bar{r}\bar{s}}(\bar{x})$ and $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$. After possibly adding sufficiently small constants to the functions $\bar{r}_l$, $l \in [I^0(\bar{\rho}) \setminus \{l^0\}] \cap I_{\bar{r}\bar{s}}(\bar{x})$ and $\bar{s}_l$, $l \in [I^0(\bar{\sigma}) \setminus I^0(\bar{\rho})] \cap I_{\bar{r}\bar{s}}(\bar{x})$, assume that $[I^0(\bar{\rho}) \cup I^0(\bar{\sigma})] \cap I_{\bar{r}\bar{s}}(\bar{x}) = \{l^0\}$.

Next, we perform several successive perturbations onto $\bar{P}^{\mathrm{cs}}$, all remaining in the set $\operatorname{int} B^{B^n(\bar{x},\bar{\delta})}(\bar{P}^{\mathrm{cs}}, \bar{\varepsilon})$. The last perturbation will yield a contradiction to $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$. By $|\bar{I}_{\bar{r}}(\bar{x})| + |\bar{I}_{\bar{s}}(\bar{x})| \leq n$, assume, after perhaps interchanging constraints, that $\bar{I}_{\bar{r}}(\bar{x}) = L$ and that $|\bar{I}_{\bar{s}}(\bar{x})| \leq n - |L|$. For $\varepsilon^1 > 0$ sufficiently small define

$$
\begin{aligned}
r_l^1(x) &= \bar{r}_l(x) + \varepsilon^1(x_l - \bar{x}_l),\ l \in L,\\
s_l^1(x) &= \bar{s}_l(x) + \varepsilon^1(x_{l+|L|} - \bar{x}_{l+|L|}),\ l \in \bar{I}_{\bar{s}}(\bar{x}),\\
s_l^1(x) &= \bar{s}_l(x),\ l \in L \setminus \bar{I}_{\bar{s}}(\bar{x}),\\
f^1(x) &= \bar{f}(x) + \sum_{l \in L} \varepsilon^1 \bar{\rho}_l(x_l - \bar{x}_l) + \sum_{l \in \bar{I}_{\bar{s}}(\bar{x})} \varepsilon^1 \bar{\sigma}_l(x_{l+|L|} - \bar{x}_{l+|L|}),
\end{aligned}
$$

and let $P^1 = \mathcal{P}^{\mathrm{cs}}(f^1, r^1, s^1)$. By construction, MPCC-LICQ holds at $\bar{x}$ for P^1 and $\mathcal{L}(P^1, \bar{x}) = \{(\bar{\rho}, \bar{\sigma})\}$.

Since $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is a dense subset of $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, there exists $f^2 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ sufficiently close to f^1 such that $f^2(\bar{x}) = f^1(\bar{x})$, $D_x f^2(\bar{x}) = D_x f^1(\bar{x})$, and $D_x^2 f^2(\bar{x}) = D_x^2 f^1(\bar{x})$, see [28, Theorem 2.4] and the proof of [29, Lemma 6.2.3, Case III]. Analogously, we obtain functions $r^2 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $s^2 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ sufficiently close to r^1 and s^1, respectively. Let $P^2 = \mathcal{P}^{\mathrm{cs}}(f^2, r^2, s^2)$ and note that $\mathcal{L}(P^2, \bar{x}) = \{(\bar{\rho}, \bar{\sigma})\}$.

For our local consideration, we perform a coordinate transformation by choosing as new coordinates a basis of $\mathbb{R}^n$ which contains the gradients of the active constraints at $\bar{x}$; see e.g. [13], where such a coordinate transformation is called *standard diffeomorphism*. As a consequence, we can restrict our analysis to the following case: $f^2 \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$, $\bar{x} = (0,0)^{\mathrm{t}}$, $f^2(0,0) = 0$, $r^2(x_1, x_2) = x_1$, and $s^2(x_1, x_2) = x_2$. That is, the problem becomes

$$
\begin{aligned}
&\min f^2(x_1, x_2)\\
&\text{s.t. } \min\{x_1, x_2\} = 0.
\end{aligned}
$$

It is well known that for problems in this form, the Lagrange vector is the derivative at the point under consideration. In particular for P^2 it holds that

$$
\mathcal{L}(P^2, \bar{x}) = \{D_x f^2(0,0)\} = \{(0,0)\}.
$$

By Lemma 3.6, we get

$$
f^2(x_1, x_2) = v_{11}(x_1, x_2)x_1^2 + v_{12}(x_1, x_2)x_1x_2 + v_{22}(x_1, x_2)x_2^2
$$

for some $v_{11}, v_{12}, v_{22} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$. Moreover, after possibly adding sufficiently small quadratic terms to f^2 we assume $v_{11}(x_1, x_2) \neq 0$ and $v_{22}(x_1, x_2) \neq 0$. Then, an appropriate local coordinate transformation leaves the feasible set unchanged, while the objective function becomes

$$
f^2(x_1, x_2) = c_1x_1^2 + v(x_1, x_2)x_1x_2 + c_2x_2^2,
$$

where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ and $v \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$. For $\varepsilon^3 > 0$ sufficiently small let

$$
f^3(x_1, x_2) = c_1(x_1 - \varepsilon^3)^2 + v(x_1, x_2)x_1x_2 + c_2(x_2 - \varepsilon^3)^2,
$$

and $P^3 = \mathcal{P}^{cs}(f^3, r^2, s^2)$. Note that

$$D_x f^3(\varepsilon^3, 0) = (0, \varepsilon^3 v(\varepsilon^3, 0) - 2c_2\varepsilon^3),$$
$$D_x f^3(0, \varepsilon^3) = (\varepsilon^3 v(0, \varepsilon^3) - 2c_1\varepsilon^3, 0).$$

Thus, $(\varepsilon^3, 0)^t, (0, \varepsilon^3)^t \in \Sigma(P^3)$, which contradicts that $\bar{x} \in \Sigma^w(\bar{P}^{cs})$. ■

Note that $\mathcal{L}(P^{cs}, x)$ is convex. Indeed, it holds that $\mathcal{L}(P^{cs}, x) = \mathcal{A}$, where the set $\mathcal{A}$ is given as in Lemma 2.1 as follows:

- $b = [D_x f(x)]^t$.
- A is a matrix whose columns are the transpose of the gradients $D_x r_l(x)$, $l \in \bar{I}_r(x)$ and $D_x s_l(x)$, $l \in \bar{I}_s(x)$.
- I^1 is the index set corresponding to the columns of the constraints $r_l(x), s_l(x)$, $l \in I_{rs}(x)$.

Observe that the concept of an extreme point plays an essential role in both Lemma 2.1 and Theorem 2.11. To obtain related results for the, in general, non-convex set $\mathcal{L}(P^{cm}, x)$, we introduce the notion of a basic Lagrange vector.

Definition 3.8 (cf. [16, Definition 4.1]): Let $x \in \Sigma(P^{cm})$. We say that $(\rho, \sigma) \in \mathcal{L}(P^{cm}, x)$ is a *basic Lagrange vector* if there does not exist $(\rho^0, \sigma^0) \in \mathcal{L}(P^{cm}, x)$ with $(\rho^0, \sigma^0) \neq (\rho, \sigma)$ and

$$I^*(\rho^0) \cap I_{rs}(x) \subset I^*(\rho), \quad I^*(\sigma^0) \cap I_{rs}(x) \subset I^*(\sigma).$$

The set of basic Lagrange vectors is denoted by $\mathcal{L}^b(P^{cm}, x)$.

The next result is analogous to Lemma 2.1 and characterizes basic Lagrange vectors.

Lemma 3.9: *The following two conditions are equivalent:*

(i) $(\rho, \sigma) \in \mathcal{L}^b(P^{cm}, x)$.
(ii) $(\rho, \sigma) \in \mathcal{L}(P^{cm}, x)$ *and the vectors*

$$D_x r_i(x), i \in I_r(x) \cup I^*(\rho), \quad D_x s_j(x), j \in I_s(x) \cup I^*(\sigma)$$

are linearly independent.

Proof: The proof runs analogously to that of [16, Lemma 4.1]. ■

The following corollaries relate the sets ext $\mathcal{L}(P^{cs}, x)$ and $\mathcal{L}^b(P^{cm}, x)$.

Corollary 3.10: *It holds that*

$$\operatorname{ext}\mathcal{L}(P^{\mathrm{cs}},x)=\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}},x)\cap\mathcal{L}(P^{\mathrm{cs}},x).$$

Proof: First, let $(\rho,\sigma)\in\operatorname{ext}\mathcal{L}(P^{\mathrm{cs}},x)$. By Lemma 2.1 (i), the vectors

$$D_x r_i(x), i\in I_r(x)\cup I^*(\rho),\quad D_x s_j(x), j\in I_s(x)\cup I^*(\sigma)$$

are linearly independent. Obviously, we have $(\rho,\sigma)\in\mathcal{L}(P^{\mathrm{cm}},x)$. Hence, by Lemma 3.9, we obtain $(\rho,\sigma)\in\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}},x)$. Now, let $(\rho,\sigma)\in\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}},x)\cap\mathcal{L}(P^{\mathrm{cs}},x)$. Note that (ii) in Lemma 3.9 holds and, therefore, Lemma 2.1 yields $(\rho,\sigma)\in\operatorname{ext}\mathcal{L}(P^{\mathrm{cs}},x)$. ∎

Corollary 3.11: *The following two conditions are equivalent:*

(i) $\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}})\subset\mathcal{L}(P^{\mathrm{cs}},x)$.
(ii) $\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}})=\operatorname{ext}\mathcal{L}(P^{\mathrm{cs}},x)$.

Proof: It readily follows from Corollary 3.10. ∎

Note that, however, the conditions in Corollary 3.11 do not necessarily imply that $\mathcal{L}(\bar{P}^{\mathrm{cm}},\bar{x})=\mathcal{L}(\bar{P}^{\mathrm{cs}},\bar{x})$, which is shown in the next example.

Example 3.12: Let $n=5,\bar{x}=(0,0,0,0,0)^{\mathrm{t}}$, and consider $\bar{P}^{\mathrm{cm}}$ and $\bar{P}^{\mathrm{cs}}$ associated to the problem

$$\begin{aligned}&\min x_1+3x_2+2x_3+4x_4\\ &\text{s.t.}\\ &\min\{x_1,x_2\}=0,\\ &\min\{x_3,x_4\}=0,\\ &\min\{x_5,x_1+x_2+x_3+x_4\}=0.\end{aligned}$$

We have

$$\mathcal{L}^{\mathrm{b}}(\bar{P}^{\mathrm{cm}},\bar{x})=\operatorname{ext}\mathcal{L}(\bar{P}^{\mathrm{cs}},\bar{x})=\left\{(1,2,0,3,4,0)^{\mathrm{t}},(0,1,0,2,3,1)^{\mathrm{t}}\right\}.$$

However, $\mathcal{L}(\bar{P}^{\mathrm{cs}},\bar{x})=\operatorname{conv}\operatorname{ext}\mathcal{L}(\bar{P}^{\mathrm{cs}},\bar{x})$, whereas

$$\mathcal{L}(\bar{P}^{\mathrm{cm}},\bar{x})=\left\{(1,2,0,3,4,0)^{\mathrm{t}}+\tau(1,1,0,1,1,-1)^{\mathrm{t}}\,|\,\tau\geq-1\right\}.$$

Therefore, $\mathcal{L}(\bar{P}^{\mathrm{cm}},\bar{x})\neq\mathcal{L}(\bar{P}^{\mathrm{cs}},\bar{x})$.

The following lemma characterizes the existence of basic Lagrange vectors.

Lemma 3.13: *Let $x \in \Sigma(P^{\mathrm{cm}})$. The following three conditions are equivalent:*

(i) $\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}}, x) \neq \emptyset$.

(iii) *The vectors*

$$D_x r_l(x),\ l \in I_r(x), \quad D_x s_l(x),\ l \in I_s(x)$$

are linearly independent.

(iii) *For all $(\rho, \sigma) \in \mathcal{L}(P^{\mathrm{cm}}, x)$ there exists $(\rho^0, \sigma^0) \in \mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}}, x)$ with*

$$I^*(\rho^0) \cap I_{rs}(x) \subset I^*(\rho), \quad I^*(\sigma^0) \cap I_{rs}(x) \subset I^*(\sigma)$$

and

$$\rho_l^0 \cdot \rho_l > 0,\ l \in I^*(\rho^0) \cap I_{rs}(x),$$
$$\sigma_l^0 \cdot \sigma_l > 0,\ l \in I^*(\sigma^0) \cap I_{rs}(x).$$

Proof: $(i) \implies (ii)$. It follows from Lemma 3.9.

$(ii) \implies (iii)$. For simplicity of notation, assume that $I_{rs}(x) = L$. The proof runs analogously when $I_{rs}(x) \neq L$. Choose $(\rho, \sigma) \in \mathcal{L}(P^{\mathrm{cm}}, x)$. For $\tau \in \mathbb{R}$, let $\mathrm{sgn}(\tau)$ denote its sign function, that is, $\mathrm{sgn}(\tau) = 0$ if $\tau = 0$, and $\mathrm{sgn}(\tau) = \tau \cdot |\tau|^{-1}$ otherwise. Let $I^1 = \{1, \ldots, 2|L|\}$ and $b = [D_x f(x)]^{\mathrm{t}}$. Moreover, let $v \in \mathbb{R}^{2|L|}$ be given as

$$v_l = \mathrm{sgn}(\rho_l)\rho_l,\ l \in L, \quad v_{l+|L|} = \mathrm{sgn}(\sigma_l)\sigma_l,\ l \in L$$

and let $A \in \mathbb{R}^{n \times 2|L|}$ be the matrix whose columns are of the vectors

$$\mathrm{sgn}(\rho_l)[D_x r_l(x)]^{\mathrm{t}}, \quad \mathrm{sgn}(\sigma_l)[D_x s_l(x)]^{\mathrm{t}}, l \in L.$$

Define $\mathcal{A}$ as in Lemma 2.1 and note that $v \in \mathcal{A}$. By Lemma 2.1, there exists $v^0 \in \operatorname{ext} \mathcal{A}$ with $I^*(v^0) \subset I^*(v)$. Let $(\rho^0, \sigma^0) \in \mathbb{R}^{2|L|}$ be a vector whose components are

$$\rho_l^0 = \mathrm{sgn}(\rho_l) v_l^0, \quad \sigma_l^0 = \mathrm{sgn}(\sigma_l) v_{l+|L|}^0,\ l \in L.$$

Using Lemma 3.9, it is easy to verify that (ρ^0, σ^0) has the desired properties.

$(iii) \implies (i)$. It is obvious. ∎

The final result of this section strengthens Corollary 3.11 whenever the point under consideration is weakly nondegenerate.

Lemma 3.14: *Assume that $x \in \Sigma(P^{\mathrm{cs}})$ is a weakly nondegenerate point for P^{cs}. Then, the following two conditions are equivalent:*

(i) $\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}}, x) \subset \mathcal{L}(P^{\mathrm{cs}}, x)$.

(ii) $\mathcal{L}(P^{\mathrm{cm}}, x) = \mathcal{L}(P^{\mathrm{cs}}, x)$.

Proof: $(ii) \implies (i)$. Obviously, it holds that

$$\mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}}, x) \subset \mathcal{L}(P^{\mathrm{cm}}, x) = \mathcal{L}(P^{\mathrm{cs}}, x).$$

$(i) \implies (ii)$. Suppose contrarily that there exists $(\rho, \sigma) \in \mathcal{L}(P^{\mathrm{cm}}, x) \setminus \mathcal{L}(P^{\mathrm{cs}}, x)$. After perhaps interchanging constraints, assume that

$$I^0(\rho) \cap I_{rs}(x) \subset I^0(\sigma),$$

and note that $\rho_{l^0} < 0$ for some $l^0 \in I^0(\sigma) \cap I_{rs}(x)$. By Lemma 3.13, there exists $(\rho^0, \sigma^0) \in \mathcal{L}^{\mathrm{b}}(P^{\mathrm{cm}}, x)$ with $\rho^0_{l_0} \le 0$. However, by (i) it holds that $\rho^0_{l_0} = \sigma^0_{l_0} = 0$, which contradicts that x is a weakly nondegenerate point for P^{cs}. ∎

Note that the point under consideration in Example 3.12 is not weakly nondegenerate.

4. M-MFCQ: a necessary condition for strong stability

In the remainder of this paper, we assume that:

- The point $\bar{x}$ is an S-stationary point for Problem (1), that is, $\bar{x} \in \Sigma(\bar{P}^{\mathrm{cs}})$.
- The point $\bar{x}$ is weakly nondegenerate.

In the following, we recall the relaxed problem, see [12], which we denote here by $\bar{P}^{\mathrm{rel}}$. Particularly, we focus on the perturbation P^{rel} of $\bar{P}^{\mathrm{rel}}$ which is a standard nonlinear programme given as

$$P^{\mathrm{rel}}: \quad \min_{x \in M^{\mathrm{rel}}[r,s]} f(x)$$

with

$$M^{\mathrm{rel}}[r,s] = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{lll} r_l(x) = 0, & s_l(x) \ge 0, & l \in I_{\bar{r}}(\bar{x}) \\ r_l(x) \ge 0, & s_l(x) = 0, & l \in I_{\bar{s}}(\bar{x}) \\ r_l(x) \ge 0, & s_l(x) \ge 0, & l \in I_{\bar{r}\bar{s}}(\bar{x}) \end{array} \right\},$$

where the mappings f, r and s are assumed to be near $\bar{f}$, $\bar{r}$ and $\bar{s}$, respectively. Note that P^{rel} does not depend only on the functions that describe P^{cs}, but also on those describing $\bar{P}^{\mathrm{cs}}$ as well as on the point $\bar{x}$. Obviously, we have $\mathcal{L}(\bar{P}^{\mathrm{rel}}, \bar{x}) = \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$.

Lemma 4.1: *Assume that S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$. Then, there exists $\bar{V} \in \mathcal{V}(\bar{x})$ such that for every $V \in \mathcal{V}(\bar{x})$ with $V \subset \bar{V}$ there exists $W \in \mathcal{W}^V(\bar{P}^{\mathrm{cs}})$ such that*

$$\Sigma(P^{\mathrm{rel}}) \cap V \subset \Sigma(P^{\mathrm{cs}})$$

for all $P^{\mathrm{cs}} \in W$.

Proof: After perhaps interchanging $\bar{r}_l$ and $\bar{s}_l$ for $l \in I_{\bar{s}}(\bar{x})$, assume for simplicity of notation that $I_{\bar{s}}(\bar{x}) = \emptyset$. Since S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$, MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{rel}}$. Hence, applying Lemma 2.9 to $\bar{P}^{\mathrm{rel}}$ we get $\bar{V} \in \mathcal{V}(\bar{x})$ such that for every $V \in \mathcal{V}(\bar{x})$ with $V \subset \bar{V}$ there exists $W \in \mathcal{W}^V(\bar{P}^{\mathrm{cs}})$ such that for all $P^{\mathrm{cs}} \in W$, $x \in V \cap \Sigma(P^{\mathrm{rel}})$ and each $(\rho, \sigma) \in \mathcal{L}(P^{\mathrm{rel}}, x)$ it holds that

$$I^*(\bar{\rho}) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\rho),$$
$$I^*(\bar{\sigma}) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\sigma),$$

for some $(\bar{\rho}, \bar{\sigma}) \in \operatorname{ext} \mathcal{L}(\bar{P}^{\mathrm{rel}}, \bar{x})$. This, together with (7), yields

$$I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\rho) \cup I^*(\sigma)$$

and, therefore,

$$\min\{r_l(x), s_l(x)\} = 0, \; l \in I_{\bar{r}\bar{s}}(\bar{x}).$$

Moreover, by a continuity argument we have

$$r_l(x) = 0, \quad s_l(x) > 0, \; l \in I_{\bar{r}}(\bar{x}).$$

Thus, it holds that $x \in M[r, s] \cap \Sigma(P^{\mathrm{rel}})$ and, by definition of P^{rel}, that

$$\mathcal{L}(P^{\mathrm{rel}}, x) \subset \mathcal{L}(P^{\mathrm{cs}}, x).$$

Hence, $\mathcal{L}(P^{\mathrm{cs}}, x) \neq \emptyset$ and, consequently, $x \in \Sigma(P^{\mathrm{cs}})$. ∎

Among other facts, the next example illustrates that the inclusion in Lemma 4.1 can be strict. Therefore, characterizing the strong stability of $\bar{x}$ for $\bar{P}^{\mathrm{rel}}$ is not enough to characterize that of $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$.

Example 4.2: Let $n = 3$, $\bar{x} = (0, 0, 0)^{\mathrm{t}}$ and consider $\bar{P}^{\mathrm{cm}}$ and $\bar{P}^{\mathrm{cs}}$ associated to the problem

$$\begin{aligned} &\min x_1 + 2x_2 + x_3 \\ &\text{s.t.} \\ &\min\{x_1, x_2\} = 0, \\ &\min\{x_3, x_1 + x_2 - x_3\} = 0. \end{aligned}$$

Note that M-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$. Hence, so does S-MFCQ at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$. Moreover,

$$\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x}) = \operatorname{conv}\{(1, 1, 2, 0)^{\mathrm{t}}, \; (0, 2, 1, 1)^{\mathrm{t}}\},$$

and $(2, 0, 3, -1)^{\mathrm{t}} \in \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$. For $\varepsilon > 0$, consider $\bar{P}^{\mathrm{cm},\varepsilon}$ and $\bar{P}^{\mathrm{cs},\varepsilon}$ associated to the problem with the objective function above and constraints

$$\min\{x_1, x_2\} = 0,$$
$$\min\{x_3 + \varepsilon, x_1 + x_2 - x_3\} = 0.$$

It is easy to see that

$$\{(0,0,0)^{\mathrm{t}}, (0,0,-\varepsilon)^{\mathrm{t}}\} \subset \Sigma(P^{\mathrm{cs},\varepsilon}),$$

and that

$$(0,0,0)^{\mathrm{t}} \notin \Sigma(P^{\mathrm{rel},\varepsilon}).$$

Moreover, since $\Sigma(P^{\mathrm{cs},\varepsilon}) \subset \Sigma(P^{\mathrm{cm},\varepsilon})$, we also have $\bar{x} \notin \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cm}}) \cup \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$.

In Example 4.2 we use the property $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x}) \neq \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$ for obtaining a perturbed problem whose S-stationary points are sufficiently close to $\bar{x}$. Next, we will apply a similar technique to prove a necessary condition for weak stability.

Theorem 4.3: *Assume that S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$. If $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}}) \cup \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cm}})$, then $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{rel}})$ and $\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x}) = \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$.*

Proof: Assume that $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$. If $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cm}})$, then the proof runs analogously noting that $\Sigma(P^{\mathrm{cs}}) \subset \Sigma(P^{\mathrm{cm}})$. Firstly, suppose contrarily that $\bar{x} \notin \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{rel}})$. Since S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$, it follows that MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{rel}}$. By Lemma 2.10, there exist sequences $x^{k,1}$, $x^{k,2}$ and $P^{\mathrm{rel},k}$ with $x^{k,1} \neq x^{k,2}$, $x^{k,1} \to \bar{x}$, $x^{k,2} \to \bar{x}$, $\{x^{k,1}, x^{k,2}\} \subset \Sigma(P^{\mathrm{rel},k})$, and $\|P^{\mathrm{rel},k} - \bar{P}^{\mathrm{rel}}\|^{V} \to 0$ for every bounded $V \in \mathcal{V}(\bar{x})$. Hence, by Lemma 4.1, for k sufficiently large, it holds that

$$\{x^{k,1}, x^{k,2}\} \subset \Sigma(P^{\mathrm{cs},k}),$$

which contradicts that $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$. Secondly, let $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{rel}})$, suppose contrarily that

$$\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x}) \neq \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$$

and, by Lemma 3.14, choose $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^{\mathrm{b}}(\bar{P}^{\mathrm{cm}}, \bar{x}) \setminus \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$. For simplicity of notation, assume that $I_{\bar{r}\bar{s}}(\bar{x}) = L$ and, after perhaps interchanging $\bar{r}_l$ and $\bar{s}_l$ for some $l \in L$, that $\bar{\sigma}_l \geq 0$, $l \in L$. By Lemma 3.9, it holds that the vectors

$$D_x\bar{r}_l(\bar{x}),\ l \in L, \quad D_x\bar{s}_l(\bar{x}),\ l \in I^*(\bar{\sigma})$$

are linearly independent. Since $(\bar{\rho}, \bar{\sigma}) \notin \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$, we have $\bar{\rho}_{l^0} < 0$ for some $l^0 \in L \setminus I^*(\bar{\sigma})$. For $\varepsilon > 0$ sufficiently small let

$$s_l^{\varepsilon}(x) = \bar{s}_l(x),\ l \in I^*(\bar{\sigma}),$$
$$s_l^{\varepsilon}(x) = \bar{s}_l(x) + \varepsilon,\ l \in L \setminus I^*(\bar{\sigma}),$$

and $P^{\varepsilon} = \mathcal{P}^{\mathrm{cs}}(\bar{f}, \bar{r}, s^{\varepsilon})$. Note that $\bar{x} \in \Sigma(P^{\varepsilon})$. Moreover, since $\bar{\rho}_{l^0} < 0$ and LICQ holds at $\bar{x}$ for $P^{\mathrm{rel},\varepsilon}$, it follows that $\bar{x} \notin \Sigma(P^{\mathrm{rel},\varepsilon})$ and, hence,

$$\bar{x} \in \Sigma(P^{\varepsilon}) \setminus \Sigma(P^{\mathrm{rel},\varepsilon}).$$

Since $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{rel}})$, there exists $x^{\varepsilon} \in \Sigma(P^{\mathrm{rel},\varepsilon})$. By Lemma 4.1, we have $x^{\varepsilon} \in \Sigma(P^{\varepsilon})$. Consequently, we obtain

$$\{\bar{x}, x^{\varepsilon}\} \subset \Sigma(P^{\varepsilon}),$$

which again contradicts that $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$. ■

We now refer to whether the previous theorem already indicates the necessity of M-MFCQ for the strong stability of an S-stationary point. Obviously, strong stability implies weak stability and, by Theorem 3.4, S-MFCQ holds. Hence, $\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$ is bounded and, by Theorem 4.3, it holds that $\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x}) = \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$. Thus, $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ is bounded. However, the following example shows that, in general, the boundedness of $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ does not imply M-MFCQ.

Example 4.4: Let $n = 3$, $\bar{x} = (0, 0, 0)^{\mathrm{t}}$ and consider $\bar{P}^{\mathrm{cm}}$ associated to the problem

$$\begin{aligned} &\min x_1 + x_2 + x_3 \\ &\text{s.t.} \\ &\min\{x_1, x_2\} = 0, \\ &\min\{x_1, x_3\} = 0. \end{aligned}$$

Note that M-MFCQ does not hold at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$. However, the set

$$\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x}) = \mathrm{conv}\{(1, 0, 1, 1)^{\mathrm{t}}, (0, 1, 1, 1)^{\mathrm{t}}\}$$

is bounded.

Nonetheless, as shown in the following lemma, the conjunction of weak stability and the boundedness of $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ does imply M-MFCQ.

Lemma 4.5: *Assume that $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$ and that $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ is bounded. Then, M-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$.*

Proof: For simplicity of notation assume that $I_{\overline{rs}}(\bar{x}) = L$. Suppose contrarily that M-MFCQ does not hold at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$. Therefore, there exists $(\bar{\alpha}, \bar{\beta}) \in$

$\Theta^M(\bar{r}(\bar{x}), \bar{s}(\bar{x})) \setminus \{0\}$ such that

$$\sum_{l \in L} \left[\bar{\alpha}_l D_x \bar{r}_l(\bar{x}) + \bar{\beta}_l D_x \bar{s}_l(\bar{x})\right] = 0.$$

Assume without loss of generality that $I^0(\bar{\alpha}) \subset I^0(\bar{\beta})$. Next, we show that the boundedness of $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ implies

$$I^0(\bar{\beta}) \cap I^*(\bar{\sigma}) \neq \emptyset \tag{8}$$

for all $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$. Suppose contrarily that

$$I^0(\bar{\beta}) \subset I^0(\bar{\sigma})$$

for some $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$. Note that for $t \in \mathbb{R}$ sufficiently large it holds that

$$\begin{aligned} &\bar{\alpha}_l t + \bar{\rho}_l > 0, \quad \bar{\beta}_l t + \bar{\sigma}_l > 0, \ l \in I^*(\bar{\beta}), \\ &\bar{\beta}_l t + \bar{\sigma}_l = 0, \ l \in I^0(\bar{\beta}). \end{aligned}$$

Hence,

$$(\bar{\alpha} t + \bar{\rho}, \bar{\beta} t + \bar{\sigma}) \in \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x}),$$

which contradicts the boundedness of $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ and, thus, we get (8).

Now, for $\varepsilon > 0$ sufficiently small let

$$\begin{aligned} s_l^{\varepsilon}(x) &= \bar{s}_l(x), \ l \in I^*(\bar{\beta}), \\ s_l^{\varepsilon}(x) &= \bar{s}_l(x) + \varepsilon, \ l \in I^0(\bar{\beta}), \end{aligned}$$

and $P^{\varepsilon} = \mathcal{P}^{\mathrm{cs}}(\bar{f}, \bar{r}, s^{\varepsilon})$. By (8), we obtain $\mathcal{L}(P^{\varepsilon}, \bar{x}) = \emptyset$. Therefore,

$$\bar{x} \in \Sigma^{\mathrm{f}}(P^{\varepsilon}) \setminus \Sigma(P^{\varepsilon}).$$

Thus, by Theorem 2.8, we get a contradiction to $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}})$. ■

Finally, we present the main result of this section.

Theorem 4.6: *If $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cs}})$, then M-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$.*

Proof: By Theorem 3.4, it follows that S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$. Since $\Theta^S(\bar{r}(\bar{x}), \bar{s}(\bar{x}))$ is a closed cone, we obtain that $\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$ is bounded. Moreover, by Theorem 4.3, we obtain $\mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x}) = \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ and, therefore, $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ is bounded. Hence, an application of Lemma 4.5 yields the desired result. ■

5. Characterizations of strong stability

In this section, we present a topological as well as an algebraic characterization of strongly stable weakly nondegenerate S-stationary points. To this end, we first refine Lemma 4.1.

Lemma 5.1: *Assume that M-MFCQ holds at $\bar{x}$ for $\bar{P}^{cm}$ and that $\mathcal{L}(\bar{P}^{cs},\bar{x}) = \mathcal{L}(\bar{P}^{cm},\bar{x})$. Then, there exists $\bar{V} \in \mathcal{V}(\bar{x})$ such that for every $V \in \mathcal{V}(\bar{x})$ with $V \subset \bar{V}$ there exists $W \in \mathcal{W}^V(\bar{P})$ such that*

$$\mathcal{L}(P^{rel},x) = \mathcal{L}(P^{cs},x) = \mathcal{L}(P^{cm},x),\ x \in V \tag{9}$$

and that

$$\Sigma(P^{rel}) \cap V = \Sigma(P^{cs}) \cap V = \Sigma(P^{cm}) \cap V \tag{10}$$

for all $P \in W$.

Proof: Firstly, we prove that there exists $\bar{V} \in \mathcal{V}(\bar{x})$ such that

$$\mathcal{L}(\bar{P}^{cm},x) \subset \mathcal{L}(\bar{P}^{rel},x)$$

for all $x \in \bar{V}$. Suppose contrarily that there exist sequences $x^k \to \bar{x}$ and $(\rho^k,\sigma^k) \in \mathcal{L}(\bar{P}^{cm},x^k)$ with $\rho^k_{l^0} < 0$ and $\sigma^k_{l^0} = 0$ for some $l^0 \in I_{\bar{r}\bar{s}}(\bar{x})$. Since M-MFCQ holds at $\bar{x}$ for $\bar{P}^{cm}$, by Theorem 2.5, the sequence (ρ^k,σ^k) is bounded. Hence, assume without loss of generality that $(\rho^k,\sigma^k) \to (\bar{\rho},\bar{\sigma})$. By a continuity argument, we get $(\bar{\rho},\bar{\sigma}) \in \mathcal{L}(\bar{P}^{cm},\bar{x})$. Since $\mathcal{L}(\bar{P}^{cs},\bar{x}) = \mathcal{L}(\bar{P}^{cm},\bar{x})$ and $l^0 \in I_{\bar{r}\bar{s}}(\bar{x})$, it follows that $\bar{\rho}_{l^0} = \bar{\sigma}_{l^0} = 0$. However, this contradicts that $\bar{x}$ is weakly nondegenerate.

Now, assume without loss of generality that $\bar{V}$ is compact. Analogously to the previous lines, it follows that for all $V \in \mathcal{V}(\bar{x})$ with $V \subset \bar{V}$ there exists $W \in \mathcal{W}^V(\bar{P})$ such that

$$\mathcal{L}(P^{cm},x) \subset \mathcal{L}(P^{rel},x),$$

for all $x \in V$ and all $P \in W$. Fix $x \in V$ and $P \in W$. Obviously, $\mathcal{L}(P^{cs},x) \subset \mathcal{L}(P^{cm},x)$. Moreover, by Lemma 4.1 we have $\mathcal{L}(P^{rel},x) \subset \mathcal{L}(P^{cs},x)$. Thus, (9) holds. Obviously, (9) implies (10), which completes the proof. ∎

The first main result of this section is a topological characterization of strong stability.

Theorem 5.2: *The following three conditions are equivalent:*

(i) $\bar{x} \in \Sigma^s(\bar{P}^{cs})$.
(ii) $\bar{x} \in \Sigma^s(\bar{P}^{cm})$.
(iii) $\bar{x} \in \Sigma^s(\bar{P}^{rel})$, $\mathcal{L}(\bar{P}^{cs},\bar{x}) = \mathcal{L}(\bar{P}^{cm},\bar{x})$ *and M-MFCQ holds at $\bar{x}$ for $\bar{P}^{cm}$.*

Proof: (*i*) $\Longrightarrow$ (*iii*). It follows from Theorems 4.3 and 4.6.

(*ii*) $\Longrightarrow$ (*iii*). By Theorem 3.3, it follows that M-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$. Therefore, S-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cs}}$. An application of Theorem 4.3 yields the desired result.

(*iii*) $\Longrightarrow$ (*i*) and (*iii*) $\Longrightarrow$ (*ii*). Since $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{rel}})$, after perhaps shrinking the balls $B^n(\bar{x}, \bar{\delta})$ and $B^{B^n(\bar{x},\bar{\delta})}(\bar{P}^{\mathrm{rel}}, \varepsilon)$ in Definition 2.6 and applying Lemma 5.1, we get

$$\Sigma(P^{\mathrm{rel}}) \cap B^n(\bar{x}, \bar{\delta}) = \Sigma(P^{\mathrm{cs}}) \cap B^n(\bar{x}, \bar{\delta}) = \Sigma(P^{\mathrm{cm}}) \cap B^n(\bar{x}, \bar{\delta})$$

and

$$\Sigma(P^{\mathrm{rel}}) \cap B^n(\bar{x}, \delta) = \Sigma(P^{\mathrm{cs}}) \cap B^n(\bar{x}, \delta) = \Sigma(P^{\mathrm{cm}}) \cap B^n(\bar{x}, \delta)$$

for all $\delta \in (0, \bar{\delta}]$ and all $P^{\mathrm{rel}} \in B^{B^n(\bar{x},\bar{\delta})}(\bar{P}^{\mathrm{rel}}, \varepsilon)$, which completes the proof. ■

Corollary 5.3: *Assume that M-MFCQ holds at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$. Then, the following two conditions are equivalent:*

(i) $\bar{x} \in \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cs}}) \cap \Sigma^{\mathrm{w}}(\bar{P}^{\mathrm{cm}})$.
(ii) $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cs}}) \cup \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cm}})$.

Proof: (*i*) $\Longrightarrow$ (*ii*). It follows from Theorems 4.3 and 5.2. The other implication is obvious. ■

For $\bar{x} \in \Sigma(\bar{P}^{\mathrm{cm}})$ and $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x})$ let

$$T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma}) = \left\{ v \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} D_x\bar{r}_l(\bar{x})v = 0, & l \in I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho}), \\ D_x\bar{s}_l(\bar{x})v = 0, & l \in I_{\bar{s}}(\bar{x}) \cup I^*(\bar{\sigma}) \end{array} \right\}$$

be the corresponding *tangent space*. The second main result of this section is an algebraic characterization of strong stability. Here, the concepts of extreme points and basic Lagrange vectors play an essential role.

Theorem 5.4: *Assume that M-MFCQ holds and that MPCC-LICQ does not hold at $\bar{x}$ for $\bar{P}^{\mathrm{cm}}$. Then, the following five conditions are equivalent:*

(i) $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cs}})$.
(ii) $\bar{x} \in \Sigma^{\mathrm{s}}(\bar{P}^{\mathrm{cm}})$.
(iii) $\mathcal{L}(\bar{P}^{\mathrm{cm}}, \bar{x}) = \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$ *and for every* $(\bar{\rho}, \bar{\sigma}) \in \mathrm{ext}\, \mathcal{L}(\bar{P}^{\mathrm{rel}}, \bar{x})$ *it holds that*

$$D_x^2 \bar{\mathbf{L}}^{\mathrm{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma})|_{T_{\bar{x}}(\bar{r},\bar{s},\bar{\rho},\bar{\sigma})} \succ 0. \tag{11}$$

(iv) $\mathcal{L}^{\mathrm{b}}(\bar{P}^{\mathrm{cm}}, \bar{x}) \subset \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$ *and for every* $(\bar{\rho}, \bar{\sigma}) \in \mathrm{ext}\, \mathcal{L}(\bar{P}^{\mathrm{cs}}, \bar{x})$ *it holds (11).*
(v) *For every* $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^{\mathrm{b}}(\bar{P}^{\mathrm{cm}}, \bar{x})$ *it holds that* $\bar{\rho}_l \geq 0, \bar{\sigma}_l \geq 0, l \in I_{\bar{r}\bar{s}}(\bar{x})$ *and (11).*

Proof: $(i) \iff (ii)$. It follows from Theorem 5.2.

$(ii) \iff (iii)$. It follows from Theorems 5.2 and 2.11.

$(iii) \iff (iv)$. By Lemma 3.14, we have $\mathcal{L}(\bar{P}^{\text{cm}}, \bar{x}) = \mathcal{L}(\bar{P}^{\text{cs}}, \bar{x})$ if and only if $\mathcal{L}^{\text{b}}(\bar{P}^{\text{cm}}, \bar{x}) \subset \mathcal{L}(\bar{P}^{\text{cs}}, \bar{x})$. Moreover, recall that $\mathcal{L}(\bar{P}^{\text{cs}}, \bar{x}) = \mathcal{L}(\bar{P}^{\text{rel}}, \bar{x})$ and, therefore, that $\text{ext}\, \mathcal{L}(\bar{P}^{\text{cs}}, \bar{x}) = \text{ext}\, \mathcal{L}(\bar{P}^{\text{rel}}, \bar{x})$.

$(iv) \iff (v)$. Obviously, $\mathcal{L}^{\text{b}}(\bar{P}^{\text{cm}}, \bar{x}) \subset \mathcal{L}(\bar{P}^{\text{cs}}, \bar{x})$ if and only if for every $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^{\text{b}}(\bar{P}^{\text{cm}}, \bar{x})$ it holds that $\bar{\rho}_l \geq 0$, $\bar{\sigma}_l \geq 0$, $l \in I_{\overline{rs}}(\bar{x})$. By Corollary 3.11, we get $\mathcal{L}^{\text{b}}(\bar{P}^{\text{cm}}, \bar{x}) = \text{ext}\, \mathcal{L}(\bar{P}^{\text{cs}}, \bar{x})$, which completes the proof. ■

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