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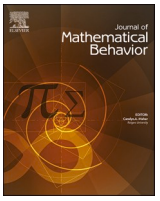

# Different types of talk in mixed-attainment problem-solving groups: Contributions to individual students' combinatorial thinking☆

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ABSTRACT

This study contributes to previous research on collaborative approaches to instruction in mathematics. The study focuses on the relationships between the type of talk in groups and individual students' combinatorial thinking. Four case studies of mixed-attainment groups of middle-school students working with mathematical problem solving in combinatorics were conducted. Video-recordings of dyad and group work, as well as interviews with four students (one per group), were analyzed. The results reveal how affordances and constraints in different types of talk (exploratory, cumulative, disputational talk) in mixed-attainment groups can contribute to individual students' combinatorial thinking. The results highlight the interconnectedness of collective and individual reasoning in combinatorics, emphasizing the role of quality of group talk.

## 1. Introduction

All students, regardless of their current level of achievement in mathematics, need opportunities to engage in mathematical problem-solving activities and see mathematical relationships (Bottge, 2001; Schoenfeld 1994). Moreover, teaching mathematical problem solving "provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding" (Lester & Cai, 2016, p. 120). Students can clarify their thinking when engaged in explaining and justifying their ideas among peers and challenging each other's ideas and explanations (e.g. Lampert, 1990; Lester & Cai, 2016).

However, collaborative approaches to teaching mathematics that are based on students' discussions of different solutions to mathematical problems with peers need deeper investigation as to under what conditions they promote achievement among all students (Boaler, 2002; Jackson & Cobb, 2010; Staples, 2008). In other words, we need to know more about "the nature of collaborative work and the interactional forces that shape the mathematics that students construct through solving problems jointly" (Langer-Osuna, 2016, p. 122). Researchers concur that there is a need for a deeper understanding of the processes and conditions by

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which students learn mathematics in peer groups, in particular regarding students who struggle with mathematics (Barclay, 2021; Webb & Mastergeorge, 2003).

Previous research has emphasized the importance of constructive and collaborative talk for students' mathematical learning in groups (Esmonde, 2009; Mercer & Littleton, 2007). The current study focuses on the subject area of combinatorics, which is a rich domain that suggests multiple interpretations of and solutions to a problem and can therefore create fruitful opportunities for engaging in collaborative problem solving in groups (Batanero et al., 1997; Fransisco, 2013; Lockwood, 2013). Combinatorics requires few mathematical prerequisites (e.g., Lockwood et al., 2020). At the same time, many students find combinatorial problems difficult (e.g., Batanero et al., 1997; English, 1991; 1996; Lockwood & Gibson, 2016; Lockwood et al., 2020; Mashiach Eizenberg & Zaslavsky, 2003) since they often cannot be solved by applying a specific, prescribed procedure (Lockwood et al., 2020). In short, what makes combinatorics both so challenging and rich is that combinatorial problems require thinking (Lockwood et al., 2020).

This study aims at extending the knowledge on collaborative approaches in mathematics by studying the potential relationships between the type of talk in mixed-attainment groups and individual students' combinatorial thinking. We study four mixed-attainment groups of middle school students (11–12 years old) who solve problems jointly. In particular, we focus on the reasoning of one student per group who, according to their teacher, had previously experienced difficulties in mathematics; that is, we focus on the combinatorial thinking of four individual students whom we denote "focal students". The following research question guides our study:

- In what ways can different types of talk in mixed-attainment mathematical problem-solving groups contribute to individual students' combinatorial thinking?

## 2. Literature background

After outlining relevant research on mathematical problem solving and combinatorics, we focus specifically on collaborative problem solving in mixed-attainment groups. Finally, we make explicit the theoretical assumptions that underlie our view on the relationships between group talk and individual students' reasoning.

### *2.1. Mathematical problem solving and combinatorics*

In many countries, problem solving is considered a central aspect of mathematics teaching (Cai & Lester, 2010; Lester & Cai, 2016). In this study, problem solving is seen as "engaging in a task for which the solution method is not known in advance" (National Council of Teachers of Mathematics, 2000, p. 52). Schoenfeld (1994) emphasizes that we should create opportunities for all students to solve problems and develop a mathematical point of view, regardless of their current level of achievement in mathematics. A widely endorsed view among scholars within the field of mathematics education is that problem solving is a fundamental part of students' mathematics learning (Lester & Cai, 2016). Students who are challenged with problems and activities that require that they struggle with important mathematical ideas can develop not only their problem-solving ability but also other mathematical competencies such as conceptual understanding (Hiebert & Grouws, 2007; Lester & Cai, 2016). As Lester & Cai (2016) point out, problem solving is an integral part of students' mathematics learning across different content areas.

One area of mathematics that offers a basis for group collaboration in problem solving is combinatorics (e.g., English, 1996). English (2005) suggests that "[c]ombinatorics may be defined as a principle of calculation involving the selection and arrangement of objects in a finite set" (p. 121). Combinatorics is a rich domain that suggests multiple interpretations of a problem and does not offer a ready-made solution, and thus serves as a good ground for collaborative problem solving (Batanero et al., 1997; Fransisco, 2013; Lockwood, 2013; Lockwood et al., 2020). Further, combinatorial problems offer a diversity in kinds of representations and reasoning (see Ayalon & Rubel, 2022). Lockwood et al. (2020) emphasize the accessibility of combinatorics to all students since it requires few mathematical prerequisites; therefore combinatorial problems are suitable for exploration. That is, students "can begin a solution path by symbolizing outcomes, and, in the process of symbolizing these outcomes, they can engage with and discuss key combinatorial ideas" (Lockwood et al., 2020, p. 5). As Lockwood et al. (2020) point out, combinatorial problems offer rich, challenging mathematical thinking for all students with opportunities for deep, critical thinking and novel problem-solving approaches since there is not always a clear path towards a solution. In particular, Goldin (2004) emphasizes that discrete mathematics, including combinatorics, can create opportunities for problem solving, especially for "students who have been 'turned off' by traditional school mathematics" (p. 56). Taken together, this makes combinatorics a suitable choice for our study on collaborative problem-solving activity, and how different types of talk in the mixed-attainment groups can contribute to individual students' mathematical thinking.

Research on students' solving of combinatorial counting problems emphasizes the relationship between three components of a model of students' combinatorial thinking: 1) counting processes, 2) sets of outcomes that are generated by counting processes, and 3) producing a formula (Lockwood, 2013). These three components represent important components of the model of combinatorial thinking and are closely linked to each other. Lockwood (2013) defines the set of outcomes as the "collection of objects being counted – those sets of elements that one can imagine being generated or enumerated by a counting process" (p. 253). Students develop from a random listing of sets of outcomes to a more systematic listing (English, 1991). Importantly, students who begin their process by listing elements from the set of outcomes are more successful in solving problems than those who use a formula immediately (Lockwood, 2013). Thus, it is important that middle school teaching provides opportunities for students to generate and organize sets of outcomes.

In combinatorics, organizing sets of outcomes systematically is a crucial aspect; that is, students need to develop from unsystematic to systematic listing of sets of outcomes (e.g., English, 1991; 2005; Lockwood & Gibson, 2016; Pourazima et al., 2023). Listing outcomes has shown to be positively correlated with solving counting problems correctly and even partial listing can be beneficial in order

to identify patterns (Lockwood & Gibson, 2016). Further, Batanero et al. (1997) found that the main difficulty prior to instruction in combinatorics was for students (aged 14–15 years) to list objects in a systematic manner. At the same time, research has shown (e.g., English, 1991) that even young children can combine objects systematically, given suitable materials and meaningful problem contexts. English (1991); (1993); (1996) has in her extensive work on K-7 students' combinatorial strategies focused on "the children's methods of item selection and combination" (English, 1991, p. 457). English (1996) identified three stages of student solutions among 9-year-olds to a problem similar to the Clothes problem in our study: a non-planning, a transitional, and an odometer stage. In Tillema's (2013) words: "When students reached the odometer stage, they had a systematic way of selecting items to produce all possible outcomes: They held one item from one quantity constant (e.g., a yellow shirt) while using a cyclic pattern for the items of the other quantity (e.g., a yellow, blue, and red pants), and then they repeated this process" (p. 334). As Lockwood & Gibson (2016) emphasize, the odometer strategy "convincingly provides a rationale for why no outcome is missed" (p. 251). In their study on combinatorial strategies that third-grade students use when solving permutation problems with three or four objects, Pourazima et al.'s (2023) results show three different subtypes of the odometer strategy: incomplete-odometer strategy, irregular-odometer strategy, and regular-odometer strategy. These all include holding one item constant while varying the other(s), but the incomplete-odometer strategy does not lead to all possible outcomes (Pourazima et al. 2023). The incomplete-odometer strategy would, however, not be considered an odometer strategy according to English (1996), Lockwood & Gibson (2016), and Tillema (2013), since all possible outcomes should be produced in order to constitute an odometer strategy. Both the irregular-odometer and the regular-odometer strategy lead to all possible outcomes but the most systematic and efficient strategy is the regular-odometer strategy in that the outcomes have a conventional order that depends on the order of the items in the first listed outcome (Pourazima et al., 2023).

Interestingly, English (1996) shows that level of achievement in mathematics does not predict which students will use the odometer strategy. The odometer stage was also a critical step for the focal student Stephanie in Maher and Martino's (1996) 5-year case study, following her development from 1st to 5th grade regarding her combinatorial thinking, in particular regarding classifying, organizing, and reorganizing data. Given extensive opportunities to build multiple representations and engage in explaining and justifying her ideas for the Clothes problem and Towers problem,[4] Stephanie began looking for patterns, started monitoring for duplicates, and eventually moved from local to global arrangement of ideas and varied items systematically (Maher & Martino, 1996).

### 2.2. *Problem solving in mixed-attainment groups*

As students engage in discussing and solving combinatorial problems in groups, they have the opportunity to provide arguments for their ideas, and thus enhance their own understanding of the problem at hand (e.g., Yackel et al., 1991). Studies indicate that students show responsibility for their learning in productive collaborative group settings with peers (Langer-Osuna, 2016; Yackel et al., 1991). Collaboration involves a coordinated, joint attempt to solve a problem and continual (re)negotiation of meaning (Mercer & Howe, 2012). Activities that encourage students to share their ideas and strategies, and to strive at jointly explaining and justifying their solutions to reach a shared conclusion, can support students in formulating mathematical arguments on a general level (e.g., Fredriksdotter et al., 2022). Furthermore, as Bottge (2001) emphasizes, peer discussions are important for the development of problem-solving proficiency for both lower-attaining students and higher-attaining students due to several reasons. Among those reasons are that students discuss several plausible options with each other instead of settling for only one option, which is common when they work individually. Students who are lower attaining can benefit more from mixed-attainment groups as they can get necessary explanations, support, and guidance (Lou et al., 1996). Actually, the act of explaining is beneficial not only for students receiving the explanation but also for those giving it (Yackel et al., 1991), as they then have the opportunity to reorganize and clarify their thinking (Lampert, 1990; Lester & Cai, 2016; Webb & Mastergeorge, 2003). Importantly, as Barclay (2021) emphasizes, lower-attaining students not only benefit from support from their peers, but also make valuable contributions to mixed-attainment group discussions in mathematics. Therefore, it is essential to explore the potential relationship between problem solving in groups and students' individual reasoning about problems.

Webb et al. (2023) challenged the assumption that students' gains in collaborative groups may be dependent on preexisting differences in students' proficiency in mathematics; i.e., students considered to have extensive knowledge in mathematics would be more engaged and participate more actively in discussions, which might contribute to higher gains in mathematics learning. Rather, in their study of problem-solving in mixed-attainment groups, Webb et al. (2023) found that all students enhanced their understanding of mathematics when they were engaged with each other's ideas through activities of explaining, questioning, and making connections among group members' solutions. Important prerequisites for productive discussions in mixed-attainment groups include the following: a safe collaborative environment, where students are given time to explore and engage in each other's ideas; open-ended tasks allowing for multiple solutions; and cultivation of "a culture of reasoning" (Mueller, 2009). In the area of combinatorics in particular, previous research suggests that since combinatorics often cannot be solved by applying a specific, prescribed procedure, having students work together on solving problems in dyads or small groups "affords students opportunities to think and reason about their own arguments and the arguments of other students" (Lockwood et al., 2020, p. 8). Further, since combinatorial problems lack well-established verification strategies (Mashiach Eizenberg & Zaslavsky, 2003), there is a need for interaction with peers for discussing the different suggestions in groups. Students have to engage in each others' reasoning and try to understand why different solutions can be correct. Considering the ample possibilities for group discussions in the rich domain of combinatorics and the

[4] Having two colors of cubes to work with, the tasks were to find all the different towers of height 3, 4, 5, or 6 cubes.

accessibility of combinatorics to all students, this study sets out to investigate how different types of talk during discussions in dyads and groups can contribute to individual students' combinatorial thinking.

### 2.3. Relationships between group talk and individual students' reasoning

Recognizing the potential of collaborative learning in groups, this study takes interest in how group talk can be related to students' individual reasoning in combinatorics. In line with Vygotsky (1962, 2012), we view students' individual reasoning as embedded in social and cultural practices in the classroom. The difficulties that students may face in mathematics can thus be explained and conditioned upon the type of talk that emerges in small-group interaction rather than students' individual prerequisites (Mercer, 2004). Though not regarding individual reasoning and group talk as separate entities, we are interested, in line with Mercer (2004) and Mercer and Howe (2012), in the relationships between the social practices of group talk and students' individual reasoning. Mercer and Howe (2012) state that "'intermental' (social, interactional) activity forges some of the most important 'intramental' (individual, cognitive) capabilities, with children's involvement in joint activities generating new understandings and ways of thinking – not only for them, but also sometimes for those with whom they are interacting" (p. 13). With the central notion of language in mind, we are interested in characterizing talk in collaborative activity, in which shared meanings and interpretations of phenomena are created. We analyze how the individual students can appropriate new ways of solving combinatorial problems through social interaction in groups (Säljö, 2009).

## 3. Analytical approach

In line with the importance of combining theoretical approaches in searching for answers on complex issues in mathematics instruction (Prediger et al., 2008) and "to get a multi-faceted insight into the empirical phenomenon in view" (Prediger et al., 2008, p. 173), two conceptual frameworks were used to analyze the data: (1) types of group talk (Mercer & Littleton, 2007), and (2) students' combinatorial thinking (Jones et al., 1997; Melusova & Vidermanova, 2015).

Firstly, patterns of group talk were categorized with regard to type of talk in accordance with Mercer & Littleton (2007). Acknowledging the fundamental role of language as a tool for learning, Mercer & Littleton (2007) created what they call a typology of three archetypal forms of talk: *disputational*, *cumulative*, and *exploratory* talk. Disputational talk is characterized by individual decision-making and disagreements that are not resolved constructively. Participants may criticize each other's ideas, but not in a constructive way; there are few opportunities for sharing ideas, as the members of a group are competing rather than collaborating with each other. The talk has specific linguistic features, such as short assertions and counter-assertions, which are not justified with mathematical reasons. In contrast, both cumulative and exploratory types of talk involve constructive knowledge generation. In cumulative talk, a common understanding of ideas is created through accumulation. The ideas are not challenged, however, thus creating little space for constructive argument. The linguistic features of this type of talk are repetitions, confirmations, and elaborations. Exploratory talk, on the other hand, is characterized by collaborative reasoning and constructive conflict: knowledge is made publicly available and accountable as suggestions are proposed for joint consideration, decisions are made jointly once all the opinions in the group are analyzed and evaluated, reasoning is visible, and the participants challenge each other's ideas and provide justifications for their propositions. The linguistic features of exploratory talk are challenges, explanations, justifications, and requests for clarification. In this study, the type of talk in the mixed-attainment groups was characterized as either disputational, cumulative, or exploratory.

Secondly, we used a framework with levels of combinatorial thinking, originating from Jones et al.'s (1997) framework for assessing probabilistic thinking, to analyze shifts in combinatorial thinking during problem solving for four focal students. Melusova & Vidermanova (2015) have concretized the framework by Jones et al. (1997) with particular focus on combinatorial thinking, resulting in the following five levels, from lower to higher level of combinatorial thinking: (1) Subjective thinking reflects the students listing outcomes in random order or listing an incomplete set of outcomes, without looking for a systematic strategy; (2) Transitional level between subjective thinking and naïve quantitative thinking reflects the use of organizational principles suitable for two-dimensional problems when solving combinatorial problems, such as listing outcomes in a systematic way, listing a complete set of outcomes, or using a table of elements, vertex graph, or tree diagram; (3) Informal quantitative thinking level involves consistently listing outcomes and using a generative strategy for more-dimensional combinatorial situations, such as using expressions involving informal algebra, expressions using numbers, or expressions using variables; (4) Numerical reasoning level involves using combinatorial formulas for solving combinatorial problems, as well as verbal reasoning; (5) Extended abstract thinking involves the generalization of formulas/expressions, such as making mental calculations without a written procedure. In addition to Melusova & Vidermanova's (2015) levels, we have added level 0 which reflects that the student has not yet shown an understanding of the required set of outcomes, or has not yet grasped what was being asked. We added level 0 since these are well-established difficulties in combinatorics (e.g., Lockwood, 2012) which could also be seen in our analysis of the empirical material. As shown in Section 4.5, *Method of Analysis of Focal Students' Combinatorial Thinking*, only the first three levels of combinatorial thinking are present in the results of the study, which is not surprising since few students in grade 6 use and generalize combinatorial formulas. In Table 3, our application of the framework is exemplified in terms of levels of combinatorial thinking.

In summary, we first analyze the predominant type of talk (Mercer & Littleton, 2007) in four groups. Further, we relate the type of group talk to the acts of combinatorial thinking (Melusova & Vidermanova, 2015) that the four focal students in our study shared in their respective group and expressed in post-lesson interviews, based on their solution notes in the beginning and in the end of the group talk.

## 4. Method

This study is based on the investigation of students' work in dyads and groups in two grade-6 classrooms (i.e., students who are 11–12 years old) at a mainstream school situated in a large municipality of Sweden. Below, we present the educational setting of the study, the combinatorial problems that were assigned to the students during the observed lessons, the procedures of the data collection, and our method of analyzing students' group discussions and combinatorial thinking.

### 4.1. The educational setting of the study

Prior to this study, the school had participated in the large-scale project *Inclusion through group learning*, which sought to investigate how inclusion can be promoted through cooperative learning (see Klang et al., 2021). Ethical approval for the *Inclusion through group learning* project was obtained from the Regional Ethical Board of Uppsala (Dnr. 2017/372). Informed consent was obtained from the students' parents, and accessible information about the study was given to the students.

At the selected school, two classes in grade 6, denoted "6 A" and "6B", participated in the current study (22 students in 6 A and 24 students in 6B). Each class was managed by a classroom teacher. The classroom teachers, who had taught their classes for several terms, often applied collaborative working methods during ordinary classroom activities. In addition, the teachers had received training in cooperative learning as part of the *Inclusion through group learning* project. Based on prior continuous assessments of their students' academic performance in mathematics, the teachers created mixed-attainment groups of four students. That is, the teacher placed students with various levels of knowledge in mathematics together during classroom group work. Thus, each dyad and group whose work we analyse in this study consisted of students with various levels of knowledge in mathematics.

At the beginning of the problem-solving lessons, the teacher presented the assigned mathematical problem; however, she did not instruct her students to apply a particular strategy when solving the problem as one of the aims of the lessons was to give the students the opportunity to explore their own problem-solving strategies. After the teacher's presentation of the problem, each student received a worksheet with the wording of the problem. Time (around 2–3 minutes) was allotted for individual thinking and then the teacher instructed the students to share their ideas in dyads and groups, to try to find a joint solution. The purpose of instructing the students to start approaching the problem individually was to allow some time to grasp what the problem is about, and (possibly) to formulate an initial idea of how to solve it. After individual thinking, the majority of the lesson consisted of the students' discussions with peers while further working on the problem, first in dyads and then in groups of four, in order to attempt to reach a shared conclusion. In the groups of four, the two dyads presented their solutions to each other and also strived to find a joint way of explaining and justifying their shared conclusion.

### 4.2. Combinatorial problems

In the Swedish national syllabus for mathematics (National Agency for Education 2022), "simple combinatorics in concrete situations" (p. 60) is a core content in grades 4–6, placed under Probability and statistics. In grades 7–9, the core content is extended to "combinatorial principles and how they can be used in different situations" (p. 61). During the lessons analyzed in this study, the students worked on the combinatorial problems (both counting problems) shown in Fig. 1:

Combinatorics was a new content area in mathematics for the classes that participated in this study. All problems had a built-in increased level of difficulty to benefit students with various levels of understanding in line with Cohen (1994) and Lou et al. (1996), and to enable students to build on their existing knowledge in subsequent problems (English, 1996). The two combinatorial counting problems presented in Fig. 1 are commonly used in Swedish classrooms and are considered well suited for work on simple combinatorics in concrete situations. The teachers presented one problem per lesson. In total, observations from four lessons are presented (see Table 2 in Section 4.3).

The students had previously worked on linear-meaning multiplication problems, i.e., problems that involve a direct proportion (that is, an equivalence between two ratios) (Tillema, 2013). An example of such a problem is: "There are 5 cookies in a package. David has 3 packages of cookies. How many cookies does he have?" As Tillema (2013) emphasizes, "the ratios in the direct proportion involve

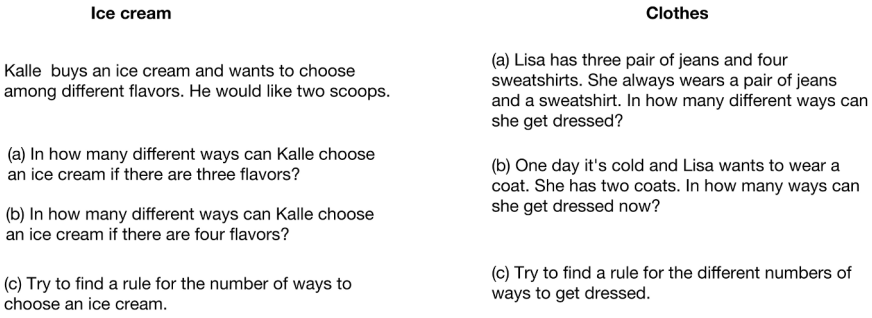



**Fig. 1.** The two combinatorial problems.

a multiplicative relationship between the two measure spaces" (p. 333), i.e., the number of cookies and the number of packages in this case. The students had not previously worked on non-linear-meaning multiplication problems which do not involve a direct proportional relationship (Tillema & Gatza, 2016). Combinatorial problems such as the Ice-cream problem and the Clothes problem both involve non-linear meanings of multiplication. As Maher & Yankelewitz (2011) emphasize (in the context of the Clothes problem), this type of multiplication is generally the most difficult for students. A difference between the Ice-cream problem and the Clothes problem, emphasized by Tillema (2013), is that for the Ice-cream problem the two proportions are dependent on each other; if we increase the number of flavors there are more flavors for both the first and the second scoop. In contrast, for the Clothes problem the two proportions are normally perceived as independent of each other, i.e., the number of jeans and the number of sweatshirts can be varied independently of each other.

For the Clothes problem, adding a coat entails going from a two-dimensional combinatorics problem in part a) to a three-dimensional combinatorics problem in part b) (English, 1996; Tillema & Gatza, 2016). The number of combinations of clothes can then be calculated by multiplying the number of jeans ($n_j$), the number of sweatshirts ($n_s$), and the number of coats ($n_c$), i.e., $n_j \bullet n_s \bullet n_c$. In the Ice-cream problem, we deliberately chose not to include a three-dimensional part (i.e., we hold the number of scoops constant to two scoops instead of increasing to three scoops). Instead, we increase the number of flavors from part a) to part b) of the problem (see Fig. 1). The reason for this is that the Ice-cream problem entails other challenges due to its openness. The Ice-cream problem can be interpreted as having four possible cases that yield four different answers, depending on whether the order of the scoops matters, and whether it is allowed to have two scoops with the same flavor, or not. The formulas for the calculations of combinations according to the four interpretations of the Ice-cream problem are presented in Table 1, in which the number of flavors is denoted *n* and the number of scoops is denoted *s*.

Thus, the Ice-cream problem can be interpreted in four ways, depending on the conditions presented above (see Table 1). We decided to not constrain this openness of the Ice-cream problem as we thought it would be interesting to analyze the group interactions concerning which conditions to apply. This decision is further discussed in Section 5, *Analysis and Results*, and in Section 6, *Discussion*.

As Maher and Martino (1996) point out, two-letter codes can be used as notation to represent the ice creams and outfits, with each single letter representing a flavor or a color. Combining ice creams or outfits systematically, holding one item constant while varying the other(s), is a strategy emphasized by Maher and Martino (1996) to simultaneously coordinate two variables. When students use this strategy, they have reached the odometer stage according to English's (1996) stages.

We expected that some of our students in grade 6 would combine items systematically and some unsystematically, the latter sometimes resulting in incomplete sets of outcomes, or sets of outcomes with duplicate combinations. For the solutions in which items were combined in systematic or unsystematic manners (or partly systematic manners), we expected to see representations of different kinds; drawn ice creams that resemble real ice-cream cones or drawn jeans and sweatshirts to make outfits, lists with two-letter codes, and lists with written names of flavors or colors to make combinations. Previous research has shown that middle-school students can symbolize the set of outcomes with written lists (Lockwood et al., 2020; Maher & Yankelewitz, 2011). Listing elements of the set of outcomes unsystematically is found to be more common (Melusova & Vidermanova, 2015) than listing items systematically, and we expected more student solutions that list items without any organizational principle.

We also expected pictures with lines that connect different ice-cream flavors or clothes. Solutions with number expressions (multiplication and/or repeated addition), that could be tied to a picture with drawn lines between the combined items (see Maher & Yankelewitz, 2011), to a tree diagram or even to a systematic table with all the combinations could be expected in some cases (although tables and graphs are found to be the least often used strategies by Melusova and Vidermanova (2015). Related to expected difficulties, we anticipated to see initial difficulties in understanding and interpreting the problem at hand. That is, since understanding the required set of outcomes is a common issue in combinatorics as well as understanding what is being asked (e.g., Lockwood, 2012), we expected some students to experience difficulties with this – i.e., to find *all* the possible combinations of ice creams or outfits and justify with arguments that all the desired outcomes have been counted exactly once (Lockwood & Gibson, 2016; Lockwood et al., 2020).

### 4.3. Data collection

The students' classroom work was video-recorded using two cameras on stands placed near the observed group. The cameras were arranged in such a way as to capture as much as possible of the students' verbal talk, gestures and use of artefacts (such as notes on their worksheets). After each lesson, the students' notes were collected and used as a resource during individual interviews with the students of the observed group. However, in this study, only the interviews with the focal students are part of the empirical data. In other words, the empirical material that is analyzed in this study comprises the video-recordings of discussions in the dyads and groups in which the

**Table 1**
The formulas for the four possible interpretations of the Ice-cream problem.

| Repetition of flavors<br>Order of the scoops | Flavors cannot be repeated, i.e. each flavor can be chosen only once | Flavors can be repeated, i.e. each flavor can be chosen more than once |
|---|---|---|
| The order of the flavors does not matter, i.e. "vanilla-chocolate" is the same as "chocolate-vanilla" | $\binom{n}{s}$ | $\binom{n+s-1}{s}$ |
| The order of the flavors matters, i.e. "vanilla-chocolate" is not the same as "chocolate-vanilla" | $\frac{n!}{(n-s)!}$ | $n^s$ |

**Table 2**
Empirical data in this study.

| Problem | Observed students (Class) | Length of lesson | Focal student | Length of interview |
|---|---|---|---|---|
| Ice cream | Berit, Betty, Bilal, Bosse (6B) | 70 min | Berit | 10 min |
| Ice cream | Arvid, Alice, Amelia, Alfred (6 A) | 50 min | Arvid | 10 min |
| Clothes | Anna, Ali, Alva, Axel (6 A) | 50 min | Anna | 15 min |
| Clothes | Billy, Benny, Bibbi, Bea (6B) | 55 min | Billy | 5 min |

focal students participated (for each problem, two groups were video-recorded), the collected student notes, and the researcher-conducted interviews with the focal students (see Table 2).

Based on the study's interest in accessible opportunities for mathematical learning for all students, we asked the teachers to choose focal students who had previously experienced difficulties in mathematics. Following the grading system in Sweden, formal grades are introduced at the end of the 6th school year. Therefore, the teachers suggested focal students based on their continuous informal classroom assessments during the preceding years.

### *4.4. Method of analysis of group talk*

The video-recordings of the four combinatorial lessons (total 225 minutes) and the interviews were transcribed verbatim. In categorizing the type of talk, each dyad and group talk was analyzed independently by two researchers. The unit of analysis was the dyad or group discussion for each part (a, b, c) of the combinatorial problems. Meetings between all three researchers were held continuously to resolve disagreements in the analyses and to reach consensus.

In accordance with recommendations (Mercer & Littleton, 2007), both an overall characterization and linguistic features of conversation sequences were used to draw conclusions regarding the predominant type of talk involving the focal students. The overarching characterization of each type of talk constituted the qualities for determining the type of talk as a whole for each dyad and group talk. In addition, the linguistic features of each type of talk were marked with particular colors in the transcripts. The overarching characterizations corresponded well to the linguistic features of each type of talk. The excerpts below serve as an illustration of the method of analysis for the type of talk. In the excerpts, the characteristic linguistic features of the exemplified type of talk are highlighted with capital letters in bold text.

#### *4.4.1. Disputational talk*

As stated in Section 3, (*Analytical Approach)*, disputational talk is characterized by individual decision making, which also involves assertions and counter-assertions that are not justified (Mercer & Littleton, 2007). This type of talk is exemplified by an excerpt from the discussion about the Ice-cream problem, involving the focal student Arvid working with Alice, Alfred and Amelia. The two dyads (Arvid with Alice and Alfred with Amelia) have interpreted the conditions of the problem in different ways and quarrel whether or not the order of the scoops with different flavors matters. After a while, Alice makes an attempt to explain her suggestion but is interrupted by Alfred, who is supported by Amelia:

> Alice: Quiet now! If this is the cone…
>
> Alfred: But my question is, will it be a different flavor just because you put the scoops in different places?
>
> **CHALLENGE**
>
> Amelia: No.
>
> Alfred: No.
>
> Arvid: It's still different ways.
>
> **ASSERTION**
>
> Alice: Hello, hello, hello, hello!
>
> Amelia: The answer to part a is three.
>
> **COUNTER-ASSERTION THAT IS NOT JUSTIFIED**

This excerpt is categorized as Disputational talk, as neither of the dyads seems to listen to the others' explanations. Instead the

students interrupt one another, which (together with Alice's calling "hello") indicates that the group members are competing for the opportunity to speak.

#### 4.4.2. Cumulative talk

In contrast to Disputational talk, Cumulative talk is characterized by the non-critical acceptance of others' suggestions. The linguistic features of this type of talk are repetitions and confirmations, as well as elaborations (Mercer & Littleton, 2007). The Cumulative type of talk is exemplified by an excerpt from a discussion about part b) of the Clothes problem, involving the focal student Billy, together with his peers Bibbi, Benny and Bea:

Bibbi: It's twenty four.

Benny: It's twenty four, yes.

**REPETITION**

Bibbi: Mm.

Bea: Mm.

**CONFIRMATION**

Benny: The answer is twenty four with the jackets.

**REPETITION**

Bibbi: 'Cause you can put this jacket on twelve times in diff- since they are different, right?

**ELABORATION**

Benny: Yeah.

Billy: [nods]

**CONFIRMATION**

Bibbi: And this jacket, you can take that one twelve times, 'cause if one is black and the other one is white, you can put it on twelve ways since there are twelve different kinds, and then it's twelve with the black one too.

**CONTINUED ELABORATION**

Benny: Yeah.

**CONFIRMATION**

Bibbi: Do you get it?

Bea: Mm.

**CONFIRMATION**

This excerpt is categorized as an example of Cumulative talk as Bibbi's suggestion "twenty four" is repeated; in addition, Bibbi's solution, as well as her elaboration, is confirmed by short responses from her peers (i.e., nodding, and saying "yeah" and "mm"). Bibbi's explicit question "do you get it" is also responded to by Bea's short and confirming "mm".

#### 4.4.3. Exploratory talk

The exploratory type of talk is also exemplified by an excerpt from a discussion about part b) of the Clothes problem. Here the focal student Anna is involved in a discussion with her peers Alva, Axel and Ali:

Alva: Should it be twelve times four then? Cause you can't have a coat with a pair of pants.

**SUGGESTION OFFERED FOR JOINT CONSIDERATION**

Axel: No, she should have the coat, the sweatshirt and…

**CHALLENGE & JUSTIFICATION**

Ali: She should have a coat, she should wear everything every time.

**JUSTIFICATION**

Axel: But then it's just double, it'll be just double, 'cause there's a coat…

**ALTERNATIVE HYPOTHESIS OFFERED**

Anna: Hello, I still don't understand.

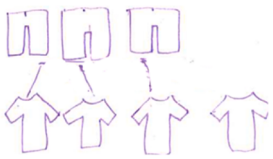
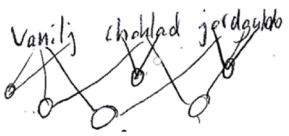
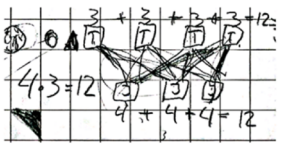
Axel: If it had been one coat, it would have still been twelve.

**JUSTIFICATION**

Ali: Anna doesn't get it.

Anna: I don't get how you come to twenty-four.

**Table 3**
Levels of combinatorial thinking – examples of our application of the framework.

| Level | Definition | Example |
|---|---|---|
| Level 0<br>Emergent understanding of sets of outcomes | The student has not yet shown an understanding of the required set of outcomes, or has not yet grasped what is being asked. |  <br>Figure 4.1. Anna's first drawing for part a of Clothes problem<br>Each of the three pairs of jeans is combined with one sweatshirt. No more combinations can be created since "all clothes have been used". |
| Level 1<br>Subjective thinking | Listing outcomes in random order or listing an incomplete set of outcomes, without looking for a systematic strategy. |  <br>Figure 2.4. Bilal's drawing for part b of the Ice-cream problem.<br>The ice-cream combinations are listed unsystematically, leading to duplicating one combination (VC and CV). |
| Level 2<br>Transitional level between subjective thinking and naïve quantitative thinking | Use of organizational principles suitable for two-dimensional problems, such as listing outcomes in a systematic way, e.g. using a table, a vertex graph or a tree diagram. |  <br>Figure 2.2. Betty's drawing for part a of the Ice-cream problem.<br>The ice-cream flavors are combined systematically, using a graph. |
| Level 3<br>Informal quantitative thinking | Consistently listing outcomes and using a generative strategy suitable for more-dimensional combinatorial situations, such as using expressions involving numbers, e.g. multiplicative or additive expressions. |  <br>Figure 5.2. Bibbi's drawing for part a of Clothes problem<br>Generative strategies using multiplicative and additive expressions are used, with the following explanation from Bibbi:<br>"So we thought like that with different sweatshirts, we say this is black, this is grey, this is white, and this is green [points to her notes; see Figure 5.2] and then we have three, blue jeans, black jeans, and white jeans. How many combinations all in all with a black sweatshirt, but I tested and counted for control by drawing the combinations." |
| Level 4<br>Numerical reasoning | Using combinatorial formulas as well as verbal reasoning. | (not identified in our data) |
| Level 5<br>Extended abstract thinking | Generalization of formulas/expressions, such as making mental calculations without a written procedure. | (not identified in our data) |

Axel: Think like this, although there are two coats, but if there was only one coat it would still be twelve. But since you have to do the whole thing one more time, it'll be double.

**JUSTIFICATION**

Anna: Okay, so twelve times two.

Axel: Yes, it'll just be twelve times two.

Alva: You take twelve times two. It's because both pants and sweatshirts together are twelve combinations, times two, because it's two coats. Then it'll be twenty-four.

**JUSTIFICATION & JOINTLY MADE DECISION**

The students' talk is characterized as exploratory since the group members engage both critically and constructively with each other's ideas, suggestions and justifications that are made visible for joint consideration. Moreover, all group members actively participate and make decisions jointly (Mercer & Littleton, 2007).

Alva's suggestion of multiplying the twelve combinations of pants and sweatshirts by four is challenged by Axel, who provides a justification, which is further elaborated by Ali. An alternative hypothesis is then offered by Axel, which is justified. Furthermore, knowledge – as well as lack of knowledge – is made publicly available in the group. When Anna repeatedly notifies her group members that she does not understand the solution and requests clarification, the other group members contribute by offering justifications before a decision is jointly made in the group.

### *4.5. Method of analysis of focal students' combinatorial thinking*

Analyses of the four focal students' utterances and solution notes were conducted by all three researchers in recurring data analysis sessions in relation to the framework of combinatorial thinking (Melusova & Vidermanova, 2015). To be categorized as level 0, the student has not yet shown an understanding of the required set of outcomes, or has not yet grasped what was being asked, which is a common issue in combinatorics (e.g., Lockwood, 2012). For example, a common student conception is that when the three colors of jeans have been combined with one color of sweatshirt each, there can be no more combinations since all clothes have been used (see Figure 4.1 and 5.1). To be categorized as level 1, the student lists outcomes unsystematically, i.e., in random order, which often yields an incomplete set of outcomes. An example is shown in Figure 2.4 in which the student lists combinations of ice-cream flavors unsystematically, in this case leading to duplicating one combination. To be categorized as level 2, the student uses organizational principles suitable for two-dimensional problems, for example listing outcomes systematically by drawing lines between the different ice-cream flavors (see Figure 2.2 and 3.3), drawing a table, a vertex graph, or a tree diagram (Melusova & Vidermanova, 2015). To be categorized as level 3, the student uses expressions with numbers, such as multiplicative and/or additive expressions (see Figure 5.2 and Figure 5.3). These generative strategies are suitable for more-dimensional combinatorial situations (Melusova & Vidermanova, 2015). Our application of the framework of combinatorial thinking to our empirical data is exemplified in Table 3.

As an illustration of the method of analysis of the shifts in the focal students' combinatorial thinking, we present Anna's reasoning about the Clothes problem before and after the dyad and group discussion. We took departure in Anna's initial drawing and utterance when she starts drawing pants and sweatshirts with only three lines between them (see Figure 4.1) followed by her question "Can you use the same pants twice?" when she notes the multiple lines drawn between pants and sweatshirts in Ali's solution (see Figure 4.2). This reflects that Anna has not yet developed an understanding of the required set of outcomes which can be seen as level 0 that we added to Melusova & Vidermanova's (2015) framework. To capture how Anna expressed her reasoning after the group discussion, we analyzed Anna's utterances in the post-lesson interview in which she refers to her second drawing (see Figure 4.3):

**Anna**: Yeah exactly, or I just used the lines to get help, or like that, so first I counted one two three four (points to the lines between the first pair of pants and the four sweatshirts; see Figure 4.3) and then one two three four five six seven eight nine ten eleven twelve and so they said it would be three times four and then I thought yeah it would be more simple to count three times four instead of counting like that.

Since Anna also drew lines combining all the pants with all the sweatshirts (see Figure 4.3), we categorized her second drawing as showing transitional thinking (i.e. using organizational principles) which is on level 2 according to Melusova & Vidermanova's (2015) framework. In the above interview excerpt, Anna expresses that she first counted the four lines from the first pair of pants to the four different sweatshirts, and also uses expressions with numbers (3 • 4) which reflects level 3, i.e. informal quantitative thinking, according to Melusova & Vidermanova's (2015) framework. Moreover, for part b) of the problem, Anna provides the following explanation in the post-lesson interview which together with the multiplicative expression 2 • 12 in Figure 4.4 reflects informal quantitative thinking:

**Anna**: Eh, or first I didn't get anything of this first, eh, and then so so they said like it's two coats and you can use them with all the pairs with all the other clothes here (points to Figure 4.4). Yeah, so we took three times four times two and it was twenty-four.

In the above statement, informal quantitative thinking (i.e. using expressions with numbers) on level 3 (Melusova & Vidermanova, 2015) is reflected (3 • 4 • 2) in the post-lesson interview with Anna. Anna's combinatorial thinking had thus advanced from level 0, not

yet being sure of what was being asked in relation to the required set of outcomes, to level 3, using expressions with numbers. Interestingly, Anna says in the post-lesson interview that "it's two coats and you can use them with all the pairs with all the other clothes here (points to Figure 4.4). Yeah, so we took three times four times two and it was twenty-four". Anna's utterance that the two coats can be used with all the *pairs* of clothes, together with Anna's written solution to part b) of the Clothes problem in Figure 4.4., reflects that she treats the combinations of two clothes (jeans and sweatshirts) as a composite unit (a pair), to be combined with a unit of one (the coats) in the three-dimensional part b) of the Clothes problem (see Tillema, 2018; Tillema & Gatza, 2016).

## 5. Analysis and results

The analysis and results are presented as case studies depicting how the four focal students (Berit, Arvid, Anna, and Billy) participate in their respective groups (in dyads and in groups of four students) in problem-solving discussions on combinatorics. Table 4 summarizes the predominant type of talk that was identified in the focal students' discussion of each problem part (a, b, and c) in dyads and groups. The table also includes information about shifts in the focal students' combinatorial thinking.

Next, we exemplify the students' discussions by presenting analyses of excerpts from the video-recorded lessons and the interviews with the focal students.

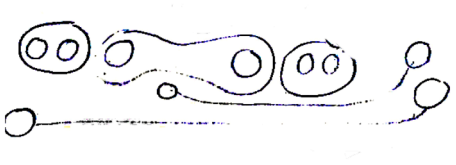
### 5.1. Case 1: Combining ice-cream scoops – Berit's appreciated contribution

In Case 1, the focus is on Berit. After individual work, Berit forms a dyad with Betty and later the dyad engages in group work together with Bosse and Bilal.

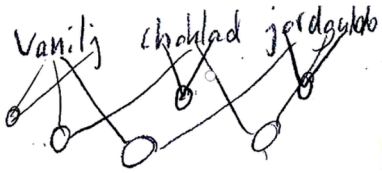
During individual work, Berit draws the ice-cream scoops in her notes (see Figure. 2.1), resulting in six combinations of three flavors. Berit and Betty then start comparing their solutions:

**Table 4**
Summary of type of talk (in dyads and groups) and shifts in focal student's combinatorial thinking.

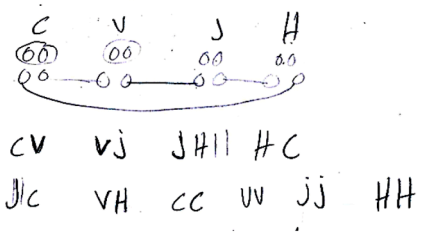
| Case<br>Problem | Focal student<br>Dyad | Parts discussed and type of talk in dyads | Parts discussedand type of talk in groups | Shifts in focal student's combinatorial thinking |
|---|---|---|---|---|
| 1<br>Ice cream | Berit<br>Berit-Betty | a) Exploratory<br>b) Exploratory<br>c) (not discussed) | a) Exploratory<br>b) Exploratory<br>c) (not discussed) | From level 1<br>to level 2 |
| 2<br>Ice cream | Arvid<br>Arvid-Alice | a) Disputational<br>b) (not discussed)<br>c) (not discussed) | a) Disputational<br>b) Disputational<br>c) (not discussed) | From level 1<br>to over-generalizing linear to non-linear meanings of multiplication |
| 3<br>Clothes | Anna<br>Anna-Ali | a) Exploratory<br>b) (not discussed)<br>c) (not discussed) | a) Exploratory<br>b) Exploratory<br>c) (not discussed) | From level 0<br>to level 3 |
| 4<br>Clothes | Billy<br>Billy-Benny | a) "Parallel talk"<br>b) (not discussed)<br>c) (not discussed) | a) "Parallel talk"/Disputational<br>b) Cumulative<br>c) "Parallel talk"/Cumulative | From level 0<br>to level 3 |



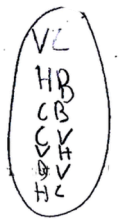
2.1. Berit's drawing for part a of Ice-cream problem.




2.2. Betty's drawing for part a of Ice-cream problem.




2.3. Berit's drawing for part b of Ice-cream problem.




2.4. Bilal's drawing for part b of Ice-cream problem.

**Fig. 2.** Student solutions to combining ice-cream scoops: Berit's appreciated contribution.

Betty: Yes, so that's ehm my "a" task.

**Berit**: Yes I got six.

Betty: I got three on the "a" task.

**Berit**: It's different. Yes but I wrote I could get two there, you can take two of them, two of them, two of them.

Betty: [looks at Berit's notes]

**Berit**: You can have two chocolate scoops, two vanilla scoops, two strawberry orchocolate vanilla.

Betty: Yes it's true. Yeeah, how smart! Then I'll do…

[draws three additional lines in her notes; see Figure 2.2]

The talk is characterized as exploratory. The knowledge is made publicly available, and Betty and Berit express true interest in each other's solutions. The conversation between Betty and Berit makes explicit that the two students have interpreted the task in different ways. Berit's interpretation is that the two scoops can have the same flavor, resulting in six combinations, whereas Betty's three combinations do not include the option of two scoops of the same flavor. Betty shows appreciation of Berit's solution, by exclaiming "how smart!" after Berit's justification of her solution, and by adding three more combinations to her original suggestion. Betty's way of drawing scoops, however, represents a more systematic way of looking at the sets of outcomes than Berit's drawing (see Figures 2.1 and 2.2). When Betty adds three more lines to her drawing after Berit's suggestion that the two scoops can have the same flavor, Berit looks carefully at Betty's notes, exclaiming "Now I get it!" which can be intrepreted as an expression of momentary understanding of Betty's systematic arrangement of outcomes.

The exploratory talk continues as Betty and Berit discuss four ice-cream flavors, for which they agree on ten combinations (see Figure 2.3). When the dyad comes together with Bilal and Bosse, they also show appreciation for Berit's contribution that the two scoops can have the same flavor. The following discussion unfolds, involving combining four flavors:

Bilal: Yes, and then we come up with seven ways.

No I think it's eleven for the "b" task.

**Berit**: We've got ten, 'cause look [inaudible speech]

Betty: Yes we've got ten.

**Berit**: [shows her notes; see Figure 2.3[5]] Here it's chocolate vanilla, vanilla strawberry, strawberry raspberry, eh, raspberry chocolate, eh, strawberry chocolate, vanilla raspberry, and then it's…

Bosse: [counts combinations in his own notes]

Bosse: It's four times.

**Berit**: …two chocolate, two vanilla, two strawberry, two raspberry.

Bosse: Four plus seven.

Betty: I counted to check here and I can't come up with more than ten.

Bosse: It feels like it's four plus seven.

Bilal: It's ten or seven first.

Bosse: It's four plus seven.

'Cause seven, if there's no more, as we said, then it must be …

Bilal: But look, we come up with seven ways

[shows his notes (see Figure 2.4) and counts them]

one two three four five six seven.

Betty: [leans forward to look at Bilal's notes] But wait, can I just look.

Bilal: Wait, I wrote CV twice. That explains it.

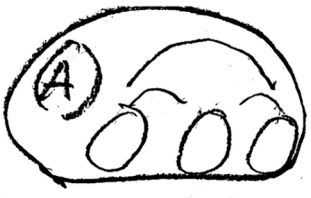
Betty: Okay, that's why.

In this discussion, which is characterized as exploratory, Bilal lists the sets of outcomes in an unsystematic way, which leads to duplicating one combination (writing both VC and CV which are not seen as two different combinations when the order does not

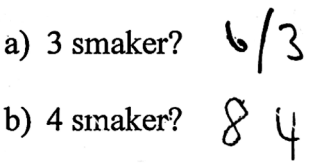
[5] In Figure 2.3, j stands for strawberry and h stands for raspberry due to the corresponding Swedish words.

3.1. Arvid's drawing for part a of Ice-cream problem.

3.2. Alice's drawing for part a of Ice-cream problem.




3.3. Amelia's drawing for part a of Ice-cream problem.




3.4. Arvid's notes for Ice-cream problem.

**Fig. 3.** Student solutions to combining ice-cream scoops: Arvid's group disagrees, resulting in a dispute.

matter). This reflects the common mistake of non-systematic listing when solving combinatorial problems (Batanero et al., 1997). As knowledge is publicly shared in the group and Bilal's solution is productively challenged by peers, it leads Bilal to reconsider his solution.

After the lesson, Berit expresses her experiences from working in the group:

> **Berit**: Today it was different 'cause usually I'm the one who experience difficulties, or how would you put it, at least I think so, but today I learned that I noticed that I know a little more than I used to know.

Overall, Berit was more of a help-giver than a help-receiver in this lesson, which she states is different from how it usually is. The help she gets from Betty concerns the systematic organization of the set of outcomes and Berit makes use of this help in her subsequent explanations to Bosse and Bilal.

Regarding the level of combinatorial thinking, Berit's first drawing of ice-cream scoops (see Figure 2.1) reflects that she uses no systematic strategy, corresponding to level 1, the level of subjective thinking (Melusova & Vidermanova, 2015). After discussing with Betty, Berit lists the set of outcomes in a more systematic way (see Figure 2.3), corresponding to level 2, the transitional level between subjective thinking and naïve quantitative thinking (see Section 3, *Analytical Approach*).

### *5.2. Case 2: Combining ice-cream scoops – Arvid's group disagrees, resulting in a dispute*

In the second case, we focus on Arvid. After individual work, and working together with Alice, his dyad forms a group with Alfred and Amelia.

During individual work, Arvid drew combinations of ice-cream scoops for part a) of the problem involving three flavors and two scoops, resulting in six combinations (see Figure 3.1). Although his partner Alice also arrived at six combinations (see Figure 3.2[6]), the dyad seems to have difficulties understanding each other's solutions and separating the number of flavors, the number of scoops, and the number of combinations:

> Alice: I think it's three different flavors.
>
> **Arvid**: Wait, he wants to have two scoops. He doesn't want to have three scoops. No, this is wrong wrong wrong [points to Alice's notes]

As illustrated in this short exchange, the talk consists of assertions and counter-assertions, and is characterized as disputational.

When the group members continue (with Amelia and Alfred included), they appear to be interested in the answers rather than a solution. Like Berit and Betty, Arvid and his peers have made different interpretations of the Ice-cream problem. However, while the discussion presented as Case 1 centers around the option of repeating the flavors, Case 2 instead focuses on whether or not the order of the flavors matters. While Amelia and Alfred think that the order of the scoops does not matter, and arrive at three combinations (see Figure 3.3), Alice and Arvid think that the order of the scoops does matter, and arrive at six combinations. The dyads disagree, and continue to argue:

> Amelia: Do you think it's six on "a"? What did you think?
>
> Alfred: If we draw it, then?

[6] In Figure 3.2, the scoops are numbered by Alice for each of the six ice-cream combinations as: (1, 2), (2, 1), (3, 2), (2, 3), (3, 1), (1, 3).

**Arvid**: I drew it like that [shows his notes; see Figure 3.1]

Amelia: We thought it was three.

Alice: Yes we think it's like that, 'cause you can do it like that [points to her notes

(see Figure 3.2), but is interrupted by Alfred and Amelia]

Alfred: But look, look [points at his notes] these two go, this one goes,

these two go.

Amelia: No but we draw it like that [refers to Figure 3.3].

Alfred: These two go, these two go, and these two go.

Alice: Well…

Alfred: There are no more.

Alice: Yes look here, these ones, this is vanilla-chocolate…

Alfred: Mm.

Alice: …and then chocolate-vanilla.

**Arvid**: Well…

Amelia: No, not like that. No.

Alfred: You can't do that.

Amelia: [laughs] It's flavors, it's not people.

Alice: But you don't get it, you can have them in different places.

**Arvid**: That's actually true.

The group talk is disputational, since the dyads oppose each other's solutions without listening to their justifications. Alice starts to provide a justification for her solution but is interrupted by Alfred and Amelia who focus on their own suggestion; they then reject Alice's interpretation of the conditions of the problem by stating "No, not like that", "You can't do that". The two dyads continue arguing whether the changed order of scoops can be regarded as a different combination of flavors. Here, the openness of the Ice-cream problem – which in Case 1 seemed to offer Berit the opportunity to contribute – instead leads to an unproductive dispute that is hard to resolve.

After the lesson, Arvid is interviewed about his solution:

> **Arvid**: I think I've done like this – one black one white one black one white and grey and black and grey and white, kind of got different combinations of that so I got six. And for four flavors, I didn't have much time left so I thought maybe it could be eight 'cause it'll be two four six two four six eight, so it'll be four times and then this one was three times so maybe it's like that (see Figure 3.4).

It appears that the group discussion did not contribute to Arvid's understanding of the problem. At the beginning of the problem-solving session, he draws ice-cream scoops in a partly systematic manner for part a) of the problem (see Figure 3.1), which can be seen as level 1 (see Section 3, *Analytical Approach*) but approaching level 2 since the use of organizational principles of listing outcomes systematically is beginning to emerge (Melusova & Vidermanova, 2015). However, Arvid's attempt to generalize his solution to more flavors is based on erroneous conclusions (2 4 6 for three flavors and 2 4 6 8 for four flavors which he explains in the interview by saying "so it'll be four times and this one was three times"). Arvid thus seems to have drawn an incorrect conclusion on part b) by overgeneralizing linear meanings of multiplication (Tillema & Gatza, 2016). This is hardly surprising, as he received no specific help or explanations to enhance his understanding during the group's disputational talk. Furthermore, the group's dispute over whether the changed order of flavors should be counted as combinations hindered them in part b) of the problem.

We move on to another combinatorial problem, further illustrating different types of talk in relation to different levels of combinatorial thinking.

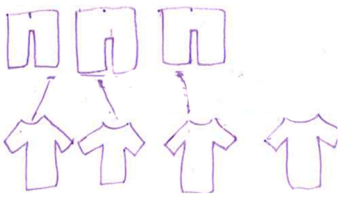
### *5.3. Case 3: Combining clothes – Anna's persistent quest for explanations*

Case 3 focuses on Anna's work. After individual work, she forms a dyad with Ali, and later a group together with Alva and Axel.

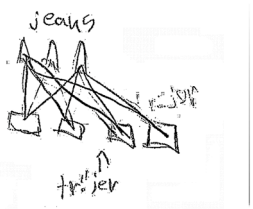
During individual work on the Clothes problem, having looked at Alva's solution, Anna starts drawing pants and sweatshirts with three lines between them (see Figure 4.1). She then joins Ali in the dyad work, and the following conversation unfolds:

Ali: What's your answer?

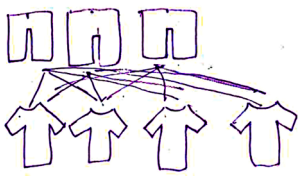
**Anna**: I have no idea 'cause I don't get it.



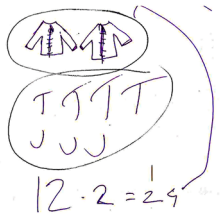
4.1. Anna's first drawing for part a of Clothes problem




4.2. Ali's first drawing for part a of Clothes problem



4.3. Anna's second drawing for part a of Clothes problem




4.4. Anna's drawing for part b of Clothes problem

**Fig. 4.** Student solutions to combining clothes: Anna's persistent quest for explanations.

Ali: This one there, this one there…

[draws lines between pants and sweatshirts in his notes; see Figure 4.2]

**Anna**: What?

Ali: This one there.

**Anna**: Can you use the same pants twice? [Anna refers to the multiple lines drawn between pants and sweatshirts in Ali's notes]

Ali: No, but you can do it more times.

**Anna**: Aaah or what?

Ali: Yeah, so, kind of, you can take these pants off and put on other pants, take the pants off and put on other ones and take the pants off and put on other ones [points to the lines between the pants and sweatshirts in his notes; see Figure 4.2].

This type of talk is characterized as exploratory. Ali shares his thoughts on the solution to the problem with Anna, who challenges Ali's solution by asking "Can you use the same pants twice?". Anna's questions elicit specific explanations from Ali. At first she appears to understand the problem as involving outfits that can only be used once, rather than finding all the possible different combinations of clothes. Ali's explanation of the possibility to take pants off and put on other pants several times gives Anna the idea that the clothes can be combined in different ways, and leads her to draw lines between all the pants and sweatshirts (see Figure 4.3). Anna then counts all the lines and exclaims "Tweeelve, oh yeah!":

Ali: Yes, three times four is twelve. This is simple, it's just three times four.

**Anna**: Tweeelve, oh yeah!

Anna and Ali join Axel and Alva to discuss their solutions to the problem. After agreeing on part a) of the problem, the group members start out to solve part b) with two coats to the already established combinations:

Alva: Should it be twelve times four then? Cause you can't have a coat with a pair of pants.

Axel: No, she should have the coat, the sweatshirt and…

Ali: She should have a coat, she should have everything on every time.

Axel: But then it's just double, it'll be just double, 'cause there's a coat…

**Anna**: Hello, I still don't understand.

Axel: If it had been one coat, it would have still been twelve.

Ali: Anna doesn't get it.

**Anna**: I don't get how you come to twenty-four.

Axel: Think like this, although it's two coats, but if it was only one coat it would still be twelve. But since you have to do the whole thing one more time, it'll be double.

**Anna**: Okay, so twelve times two.

Axel: Yes, it'll just be twelve times two.

Alva: You take twelve times two. It's because both pants and sweatshirts together are twelve combinations, times two, because it's two coats. Then it'll be twenty-four.

The group members' talk here is characterized as exploratory. They make propositions regarding the solutions to the problem, challenging each other's assertions and justifying their challenges. For example, Alva's suggestion of multiplying the twelve combinations of pants and sweatshirts by four is challenged by Axel, who provides a justification, which is further elaborated by Ali. Furthermore, knowledge and lack of knowledge are made publicly available in the group. When Anna repeatedly notifies her group members that she does not understand the solution, both Ali, Alva, and Axel help her and contribute by offering explanations.

In the post-lesson interview, Anna explains the group's solution to part a) of the problem:

**Anna**: Yeah exactly, or I just used the lines to get help, or like that, so first I counted one two three four (points to the lines between the first pair of pants and the four sweatshirts; see Figure 4.3) and then one two three four five six seven eight nine ten eleven twelve and so they said it would be three times four and then I thought yeah it would be simpler to count three times four instead of counting like that.

For part b) of the problem, Anna provides the following explanation:

**Anna**: Eh, or first I didn't get anything of this first, eh, and then so so they said like it's two coats and you can use them with all the pairs with all the other clothes here (points to Figure 4.4). Yeah, so we took three times four times two and it was twenty-four.

It is evident that Anna has enhanced her understanding of how organizational principles can be used to list the complete set of outcomes (see Figure 4.3), corresponding to the transitional level of combinatorial thinking (Melusova & Vidermanova, 2015). In the interview, she expresses understanding of the number operation in her statement "then I thought yeah it would be simpler to count three times four instead of counting like that" and in her concluding statement "so we took three times four times two and it was twenty-four" (see also Figure 4.4). These statements reflect informal quantitative thinking since this level of thinking involves using a generative strategy such as using number expressions (see Section 3, *Analytical Approach*). Anna's momentary understanding has hence moved from level 0, not yet seeming to understand the required set of outcomes reflected in Figure 4.1, to level 3, informal quantitative thinking (Melusova & Vidermanova, 2015) reflected in the post-lesson interview together with Figure 4.3 and Figure 4.4. To summarize, Anna's persistent quest for explanations together with her group members' specific justifications and explanations may have served as enabling moments of insight into combinatorial reasoning for Anna.

### *5.4. Case 4: Combining clothes – Billy's parallel line of reasoning in group discussion*

In Case 4, Billy is our focal student. After individual work, he joins Benny in dyad work, and they later form a group with Bibbi and Bea.

During individual work, Billy draws lines between each pair of pants and sweatshirt, resulting in three outfits (see Figure 5.1); that is, he appears to understand the problem as involving outfits that can only be used once, rather than finding all the possible different combinations of clothes. When Billy and Benny talk, Billy states that he found three different ways, whereupon Benny answers "Maybe more ways?" They continue:

**Billy**: She has three pair of pants and three, four sweatshirts. So I thought that one pair of pants and a sweatshirt is a kind of outfit.

Benny: Yes.

**Billy**: And then one more, and one more. And so it'll be three, three times.

Benny: I thought twelve. It's twelve I think. I thought only three times four, but then I drew it so that I did it too. I think three times four is easy.

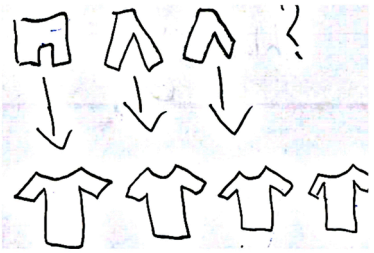
**Billy**: Yes.

Although Benny suggests a different solution than Billy, the talk between Billy and Benny could be characterized as cumulative (rather than disputational or exploratory) in the sense that they do not explicitly oppose each other's ideas. However, neither of them engages with or builds upon his peer's reasoning; instead, their dialogue shows two parallel lines of reasoning. After this exchange, the two of them do not engage in further discussion.

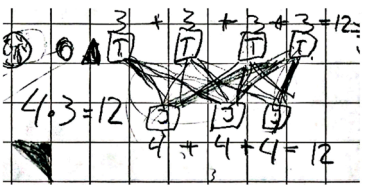
When Bea and Bibbi join the discussion, the following conversation unfolds:

Benny: What did you think?

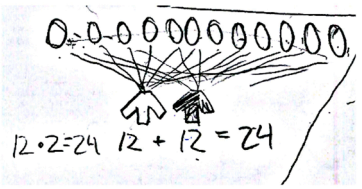
Bibbi: We thought four times three is twelve.



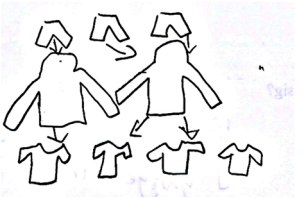
5.1. Billy's drawing for part a of Clothes problem




5.2. Bibbi's drawing for part a of Clothes problem




5.3. Bibbi's drawing for part b of Clothes problem



5.4. Billy's drawing for part b of Clothes problem

**Fig. 5.** Student solutions to combining clothes: Billy's parallel line of reasoning.

Bea: Mm

Benny: Yes.

Bibbi: So we thought like that with different sweatshirts, we say this is black, this is grey, this is white, and this is green [points to her notes; see Figure 5.2] and then we have three, blue jeans, black jeans, and white jeans. How many combinations all in all with a black sweatshirt, but I tested and counted for control by drawing the combinations.

**Billy**: She can have it like this, there are three pair of jeans and four sweatshirts, and then she can use three, three sweatshirts and three pair of jeans before she has to wash the jeans. So she can have it three times.

Bibbi: Okay, we do like this. Oh the time, we only have ten minutes. We can write it as one pair of jeans takes four sweatshirts, as she has four sweatshirts, three pair of jeans and four sweatshirts, then it's three times four.

The talk between Benny and Bibbi appears to be cumulative, with Bibbi taking a leading role in explaining her solution. In her explanations, a systematic pattern of organizing the set of outcomes appears, holding one item constant and systematically varying the other items (English, 1991). In further discussion, Billy insists on his understanding of a combination, proposing that the clothes need to be washed before being worn again and thus cannot be used several times. This indicates that real-world considerations (Maher & Yankelewitz, 2011) may stand in the way for Billy's thinking. He asks "Do you wear jeans and then use them the next day? That's not clean". However, Billy's idea about three outfits is consistently disregarded by his peers; it is neither challenged nor confirmed by his group members.

As the group moves on to part b) with two coats, the type of talk among the other three group members is characterized as cumulative, with Bibbi doing most of the talking:

Bibbi: It'll be twenty-four

Benny: Of course it'll be twenty-four

Bibbi: Mm, 'cause this coat, you can use it twelve times 'cause these are different, aren't they? [to Billy] Do you get it?

**Billy**: What?

Bibbi: And then this coat, you can use it twelve times as well, cause this one is black and that one is white, so you can, look, there, there, there, there, there, there, there, there [points to her notes; see Figure 5.3] and then it'll be twelve different, it's then twelve different outfits, and then the black coat, you can put it on with that one, that one, that one, that one, that one, that one…

Benny: Yes.

Over the course of the entire lesson, Billy leaves the group and returns several times. Nevertheless, when interviewed about the problem after the lesson, he surprisingly both talks about his own solution and acknowledges the group's solution as "smarter":

> **Billy**: So I thought that a pair of pants and a sweatshirt is an outfit, there's an outfit and then there and there. And then she has it three times. Benny's idea was smarter, that you can take, one pair of jeans can have four sweatshirts and then it'll be four times three. So it'll be twelve.
>
> […]
>
> **Billy**: And here it's the same. One coat, you can have it twelve times. And then there's one more coat, you can have it twelve times, so it's twenty-four (see Figure 5.4).

Although two parallel lines of reasoning are evident in the group discussions, the interview reveals that Billy appears to take note of the explanations of the group's solution. His understanding of the problem appears to move from level 0, not yet understanding the required set of outcomes (see Section 3, *Analytical Approach)* – reflected in Figure 5.1 as well as in statements from the classroom excerpts – to level 3, informal quantitative thinking using number expressions (Melusova & Vidermanova, 2015), describing how multiplication can be used to combine the clothes as reflected in the interview "one pair of jeans can have four sweatshirts and then it'll be four times three".

### 5.5. *Relationships Between Group Talk and Focal Students' Individual Reasoning*

In line with Vygotsky (1962, 2012) we view students' individual reasoning as embedded in the social practices of group talk. Although not attempting to separate individual students' reasoning from the social practices in which it occurs, we are interested, in line with Mercer (2004) and Mercer & Howe (2012), in the relationships between students' individual reasoning and the quality of group talk. The four different cases included in this study were characterized by three different types of relationships between students' individual reasoning and patterns of talk that arose in small groups and that appeared to contribute to the focal students' thinking.

Firstly, when the group talk is characterized as disputational (Case 2), it does not appear to enhance Arvid's understanding of combinatorial thinking. At the beginning of the problem-solving session, his individual solution is characterized by emerging use of organizational principles of listing outcomes in a systematic way. However, as can be seen from the interview after the group discussion, he draws erroneous conclusions about the generalization of his solution to more flavors. That Arvid's combinatorial reasoning is not enhanced is not a surprising finding due to the nature of disputational talk which is characterized by individual decision-making and disagreements that are not resolved constructively in the group.

Secondly, when the group talk is categorized as exploratory (Case 1 and Case 3), the focal students' understanding of combinatorial thinking appears to be enhanced through the group talk. In Case 1, the group mates' appreciation of Berit's additional combinations becomes integrated and seems to guide Berit into a more systematic representation of the different combinations. Likewise, in Case 3, Anna's initial difficulty to understand the required set of outcomes advances into the use of organizational principles to list the complete set of outcomes in the course of persistent explanations in the group talk. A specific connection that can be seen between students' combinatorial thinking and the group talk is evident in moments of enlightment during the group talk after a group mate's explanation, such as Anna exclaiming "Tweeelve, oh yeah!" after herself having drawn all twelve lines between the trousers and sweatshirts. Similarly, Berit looks carefully at Betty's notes, exclaiming "Now I get it!" which can be intrepreted as an expression of momentary understanding of Betty's systematic arrangement of outcomes of ice creams. It is also interesting to note that in Case 1, the exploratory talk emerges from Betty's response to the focal student Berit's suggestion, while the focal student Anna's persistence in raising questions contributed to the exploratory talk exemplified in Case 3.

Thirdly, when the group talk is characterized as cumulative with parallel lines of reasoning between Billy (Case 4) and his group mates, Billy's reasoning about the problem develops from initial understanding of lists of outcomes as involving outfits that can only be used once, rather than finding all the possible different combinations of clothes using multiplication. In excerpts of group talk and in the interview it becomes evident how Billy, by participating in parallel lines of reasoning, takes note of the group members' explanations. Possible reasons for this are further discussed below (see Section 6, *Discussion*).

To conclude, in the course of constructive conversations in groups, characterized as both exploratory talk and cumulative talk with parallel lines of reasoning, the focal students appear to appropriate new ways of reasoning (Säljö, 2009) about combinatorial problems. Thus, the results highlight the interconnectedness of the collective and the individual thinking (Mercer, 2004; Mercer & Howe, 2012) and emphasize the role of language in mathematical problem solving as students give and use explanations and constructively challenge each others' ideas.

## 6. Discussion

This study aims at contributing to previous research on collaborative approaches in mathematics by studying the relationship between the type of talk in mixed-attainment groups and individual students' combinatorial thinking. Focusing on collaborative problem solving, the relationship between the type of talk and the combinatorial thinking of students who had previously experienced difficulties in mathematics was investigated through four cases of mixed-attainment groups. The results reveal how affordances and constraints through different types of group talk contribute to individual students' combinatorial thinking. Two aspects of the results are especially interesting to discuss.

The first aspect relates to the importance of identifying sets of outcomes as a crucial aspect of combinatorics for students to understand, stressed by Lockwood (2012) and Batanero et al. (1997). In Lockwood's (2013) study, students who started by listing sets of outcomes were more successful than those who focused solely on a formula. A progression is suggested from listing sets of outcomes to

more systematical listing, to counting, and finally to using formulas (English, 1991; Jones et al., 1997; Lockwood, 2012; 2013; Melusova & Vidermanova, 2015). Some of the focal students in this study appeared to have difficulties understanding what sets of outcomes were required. Anna (Case 3) and Billy (Case 4) understood combining clothes as creating outfits that can only be used once (cf., Pourazima et al., 2023), rather than finding all the possible different combinations of clothes, which reflects difficulties in understanding the required set of outcomes (Lockwood, 2012; 2013; Lockwood & Gibson, 2016). It was after requesting and receiving explanations from her group members in their exploratory talk that Anna grasped the required set of outcomes and also started listing sets of outcomes systematically. After the group discussions, Anna even demonstrated that she treated the combinations of two clothes (jeans and sweatshirts) as a composite unit (as she states in the post-lesson interview; as a pair), to be combined with a unit of one (the coats) in the three-dimensional part b) of the Clothes problem (Tillema, 2018; Tillema & Gatza, 2016). In collaborative reasoning the students can put their emergent understandings on trial and take advantage of their peers' explanations in systematically organizing sets of outcomes. This relation between the collective and individual thinking (cf. Mercer & Howe, 2012) leads us to the next interesting aspect to discuss.

The second aspect we discuss relates to the main focus of this study, i.e. how different types of talk in mixed-attainment groups can contribute to individual students' combinatorial thinking. In our study, exploratory talk was related to better learning outcomes while in the context of disputational talk the focal student did not show an increased understanding, which aligns with previous research by Mercer and colleagues (Mercer & Littleton, 2007; Mercer & Sams, 2006; Mercer, Wegerif & Dawes, 1999). These findings are not surprising since reasoning is visible in exploratory talk, in contrast to disputational talk in which the reasoning is not visible (Mercer et al., 1999). The question is how we can explain that focal students participating not only in exploratory talk as emphasized by Mercer and colleagues – but also in one instance of cumulative talk with parallel lines of reasoning – displayed an increased understanding of combinatorics in our study. On a general level, both cumulative and explorative talk involve constructive knowledge generation (Mercer & Littleton, 2007). In the particular example revealed in this study, elaborations made by Billy's group members in their cumulative talk seem to have enhanced Billy's understanding of different combinations of clothes (Case 4). More specifically, the reason may lie in the nature and use of explanations in the groups – how explanations are given and how they are used (Webb & Mastergeorge, 2003). Billy seems to listen – despite his parallel line of reasoning – to the elaborated explanations from Bibbi, which in turn are triggered by Billy's statements. A possible explanation as to why one of the focal students in this study showed an increased understanding of combinatorics not only in relation to explorative talk – as expected – but also in relation to cumulative talk with parallel lines of reasoning, might hence be students' activity in relation to seeking, receiving, giving and using explanations (Webb & Mastergeorge, 2003). It is essential that students' explanations are elaborated as well as adapted to and solicited by the one who receives the explanation, asking specific questions (Webb & Mastergeorge, 2003). That is, students need to engage with each other's propositions; Anna's persistent quest for more specific explanations that she also received from her group members, Berit's appreciated contribution with additional combinations leading to more elaborated explanations from both sides, Billy's persistent view of outfits that can only be used once (and then have to be washed) triggering further explanations from his group members – all these were made possible through the group's type of talk and also contributed to the type of talk. Thus, according to our findings, beneficial group talk appears to emerge when a group member's suggestions, persistent questions as well as challenges of the ideas under discussion, trigger other group members to persist in explaining and justifying their solutions.

Further examining the complexities involved in collaborative approaches with problem solving, the openness of the mathematical problem can also influence the interaction and reasoning. Tasks that are not only "characterized by multiple representations and solution paths" (Staples, 2008, p. 355) but also open to different interpretations depending on the conditions (such as the Ice-cream problem; see Section 4.2, *Combinatorial problems*) may stimulate interaction and reasoning in mixed-attainment groups. Interestingly, in Berit's case the openness of the Ice-cream problem seemed to create possibilities for her to make alternative, valuable contributions (cf. Barclay, 2021) that were appreciated by the group whose talk was characterized as exploratory. In Arvid's case this openness instead created a tension in the group whose talk was characterized as disputational. The students' disagreements about the possible conditions to the problem were not resolved and there were few attempts to pool their resources. Social norms focusing on the answer to the problem rather than the process (Esmonde, 2009) seemed to keep Arvid's group from trying different alternatives and to list the set of outcomes. This may have contributed to Arvid's overgeneralization of linear meanings of multiplication when trying to find the number of combinations of ice creams when the number of flavors were increased in part b). This is a prominent aspect of the well-established difficulty of establishing non-linear meanings of multiplication (Tillema & Gatza, 2016).

To conclude, this study contributes to further nuancing the teaching approach of collaborative problem solving by unpacking how group interactions in terms of students' type of talk (Mercer & Littleton, 2007) can contribute to individual students' combinatorial thinking. The study uncovers the supportive structure of collaborative reasoning and in particular exploratory talk, in line with previous research (Mercer & Sams, 2006). More specifically, in relation to students' combinatorial thinking, it reveals the affordances for reasoning about sets of outcomes in collaborative group work, which appeared to be challenging to the focal students in this study. In contrast to previous studies on the importance of open-ended tasks (Cohen, 1994), this study shows that, depending on the type of talk that unfolds in the group, open-ended tasks may create learning opportunities for some groups and unproductive interaction for others. This study has generated further insights into collaborative problem-solving interactions in mixed-attainment groups and under what circumstances they are beneficial for individual students' combinatorial thinking. Moreover, our results align with Barclay's (2021) findings in that lower-attaining students can both benefit from other's explanations and also make valid and valuable contributions to mixed-attainment group conversations. Still, an important prerequisite for this to happen is a group talk in which the students are engaged in and explore each other's ideas (Webb et al., 2023). Thus, in order to promote a "culture of reasoning" in mixed-attainment groups (Mueller, 2009) in which multiple ideas can be proposed, tested, put on trial, argued for and reflected upon, teachers need to create a safe collaborative environment.

In a wider sense, the findings of this study matter to the issue of equity in mathematics education in the context of group talk and have implications for teaching collaborative problem solving in schools. The issue of equity is raised by a number of scholars within the field (e.g. Boaler, 2002; Esmonde, 2009; Jackson & Cobb, 2010; Langer-Osuna, 2016; Lockwood et al., 2020; Staples, 2008). It is of crucial importance for equity in the teaching of mathematics in schools to create conditions for all students to participate substantially – but not necessarily in the same ways – in all phases of the lesson (cf. Jackson & Cobb, 2010). Esmonde (2009) states that "for a group's interactions to be equitable, they should explicitly attend to the meaning-making processes of each of the group members, not just the few who are considered "smart" or most competent" (p. 250–251). Regarding the domain of combinatorics, Lockwood et al. (2020) state that "combinatorics can broaden opportunities for students to feel like they are capable of doing mathematics" (p. 10) and that "questions of what mathematics is and who can do mathematics are fundamentally issues of access and equity" (p. 10).

As Boaler (2002) emphasizes, issues of equitable teaching lie *within* different teaching approaches. Further investigation is needed on the conditions that make the teaching approach with collaborative problem-solving discussions in groups beneficial for all students' mathematical sense making (Boaler, 2002; Jackson & Cobb, 2010; Langer-Osuna, 2016; Staples, 2008). Collaborative problem solving in groups entails its specific challenges related to accessible interactions, which need to be considered by teachers to create conditions that benefit all students. In order for students to face the challenging aspects of combinatorial problem solving so that they benefit from the rich opportunities arising from different solutions, teachers need to create a learning environment in which productive, collaborative group talk with elaborated explanations can contribute to individual students' thinking.

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## Ethical statement

The project was approved by Uppsala Ethical Regional Committee, Dnr. 2017/372.

## Declaration of Competing Interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Nina Klang reports financial support was provided by Swedish Research Council. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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