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Sun–Jupiter–Saturn System May Exist: A Verified Computation of Quasiperiodic Solutions for the Planar Three-Body Problem

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Abstract
In this paper, we present evidence of the stability of a model of our Solar System when taking into account the two biggest planets, a planar (Newtonian) Sun–Jupiter–Saturn system with realistic data: masses of the Sun and the planets, their semiaxes, eccentricities and (apsidal) precessions of the planets close to the real ones. (We emphasize that our system is not in the perturbative regime but for fixed parameters.) The evidence is based on convincing numerics that a KAM theorem can be applied to the Hamiltonian equations of the model to produce quasiperiodic motion (on an invariant torus) with the appropriate frequencies. To do so, we first use KAM numerical schemes to compute translated tori to continue from the Kepler approximation (two uncoupled two-body problems) up to the actual Hamiltonian of the system, for which the translated torus is an invariant torus. Second, we use KAM numerical schemes for invariant tori to refine the solution giving the desired torus. Lastly, the convergence of the KAM scheme for the invariant torus is (numerically) checked by applying several times a KAM–iterative lemma, from which we obtain that the final torus (numerically) satisfies the existence conditions given by a KAM theorem.

Keywords Three-Body problem · Sun–Jupiter–Saturn · KAM theory

Mathematics Subject Classification Primary 70F07 · 70G60 · Secondary 70-08 · 37C55

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1 Introduction

In Newton (1687), Newton deduced the equations for the motion of planets and solved the two-body problem: Bounded orbits follow Kepler's motions (spin in ellipses with one focus on the center of mass), and unbounded ones are parabolae or hyperbolae. Then, Newton (Book 3, Proposition XIII, Theorem XIII) admits that observed planetary motion of Jupiter does not fit the equations, and explains it by noticing that Saturn's influence cannot be neglected. Since then, one of the most important problems in mathematics has been understanding the dynamics of the three (or higher)-body problem. Many researchers have pursued this question and realized in different temporal stages that there are two (among others) important questions: the stability of the solutions—do planets orbit around the Sun in a quasiperiodic motion ad perpetuum?, and the existence of chaos. This dichotomy was started by the pioneer work Poincaré (2017). In this paper, we are interested in the stability problem.

Several steps forward in time and we encounter a fundamental advance towards solving the stability problem. In 1954, Kolmogorov (1979) presented a methodology for proving the existence of Lagrangian invariant tori in Hamiltonian systems of n degrees of freedom close to integrable ones. Then, Arnold (1963a, b) and Moser (1962) further explored this and the KAM theory was officially born. Since then, a lot results have been produced, covering Lagrangian and lower dimensional tori, infinite dimensional systems, dissipative systems, etc. For the interested reader, we refer to the books Broer et al. (1996), de la Llave (2001), Chierchia (2003), and the popular book Dumas (2014).

It was clear from the very beginning that the three-body problem posed several obstacles that other Hamiltonians do not have. The integrable problem (Kepler's Hamiltonian) does not have all the frequencies that the full problem has: In a general four-degree- of-freedom Hamiltonian, invariant tori have four dimensions with four frequencies; while in Kepler's Hamiltonian they have two dimensions with two frequencies. In KAM terminology, it is said that the system is degenerate, and then, the full problem has several time-scale frequencies: the fast frequencies that correspond to the spinning of the planets around the Sun and the slow frequencies that correspond to the spinning of their orbital ellipses (precession motion) and, in the spatial problem, the changes in the inclination of the rotation planes (inclination motion).

A crucial advance was performed in Arnold (1962, 1963b) where he proved the persistence of quasiperiodic motion for the planar three-body problem for a ratio of the semimajor axis close to zero. The theory was later completed for the spatial N-body problem in remarkable works by Herman and Féjoz (2004), and Chierchia and Pinzari (2011), among others. In spite of the *fundamental* importance of all these theoretical results, they suffer the *practical* inconvenience that the ratio of the semimajor axis or the size of masses of the planets (used as the parameter measuring the distance to integrability) have to be ridiculously small. In fact, Hénon (1966) already took Arnold's paper and checked that this size, in the simpler and non-degenerate *restricted* three-body problem, is of order 10^{-333} (see the beautiful exposition of these facts in Laskar (2016)). After this result, Hénon asserted [1]: *"Thus, these theorems, although of a*

[1] From the English translation in Dumas (2014).

very great theoretical interest, do not seem applicable in their present state to practical problems, where the perturbations are always much larger than the thresholds [above].” This apparent lack of applicability of KAM theory to practical and physical problems led over time to some misunderstandings (and laughter) about KAM theory but, as emphasized in Dumas (2014), Hénon himself goes on to write: *“The numerical results we present here, and those obtained for other problems, indicate however that the [invariant] curves continue to exist for very strong perturbations, of the same order of magnitude as the leading term.”*

This last observation by Hénon is in fact what leads our research: the combination of qualitative KAM results with computers. The idea is the use of the computers for getting initial data that can be then checked to fulfill the conditions of the tailored KAM theorems. Following this line of thought in combination with all the previous (classical) KAM methodology, based on performing canonical transformations on the Hamiltonian, there has been important advancement toward the solution of the three-body problem for realistic masses Celletti and Chierchia (1995); Locatelli and Giorgilli (2005, 2007); Laskar and Robutel (1995); Robutel (1995). In particular, Locatelli and Giorgilli (2007) provides numerical evidence that Kolmogorov’s theorem applies to the Sun–Jupiter–Saturn system using computer-assisted techniques and computer algebra, implying the truncation of the Hamiltonian, but falls short from being a proof: The norms of the generating functions decrease geometrically with the normalization step up to values of order around 10^{-8} at the 17th step, and the reliability tests of the construction of Kolmogorov’s normal form provide discrepancies of order 10^{-3} between the numerical integration of an orbit and the semianalytical one. More recently, in Castan et al. (2017) a quantitative version of Arnold’s KAM theorem has been applied to the plane three-body problem to show, computer-assisted, the existence of quasiperiodic motion for a ratio of masses between the planets and the star that is close to 10^{-85}. (This estimate accounts for a mass of the planets smaller than 10^{-24} times the mass of the electron.) In this paper, however, we propose to use another approach, based on the so-called parameterization method (see the seminal works de la Llave (2001); de la Llave et al. (2005)), looking directly for the parameterizations of the invariant tori, mitigating the curse of dimensionality. In this approach, the KAM theorems are written in a posteriori format, so that the results are suitable for numerical verification and, finally, for computer-assisted proofs. This program has led to a rapid development of results (Haro et al. 2016; Figueras et al. 2017; Haro and Luque 2019; Figueras and Haro 2020; Calleja et al. 2013).

Applying KAM theory to the planar three-body problem (for realistic parameters and observations) is a very demanding problem, both mathematically and computationally. Hence, further steps must be performed to attack the problem. We have split this enterprise in three stages, of which the present paper is the centerpiece. Each stage deals with different questions and methods, so they could be of independent interest for different publics. Moreover, even though we have been thinking in the application to the three-body problem, and specifically to the Sun–Jupiter–Saturn system, the pieces can be applied to other problems. The first stage, appearing in Figueras and Haro (2024), is a KAM theorem based on a (modified) parameterization method for Hamiltonian systems, with sharp control on the bounds and the Diophantine frequencies (with precedents in Haro and Luque (2019); Villanueva (2017)). This first paper has two results that we use in this paper: the *KAM theorem* for verifying the existence of the invariant torus, and the *iterative lemma* used for giving an initial approximation with bounds on it; we perform several steps of the convergence scheme. This allows to refine the constants to be used later on the KAM theorem. The second stage, this paper, is a methodology to compute invariant tori in (close to degenerate) Hamiltonian systems with fast and slow time-scales, applied to numerically verify the existence of quasiperiodic solutions of the Sun–Jupiter–Saturn in the planar model with realistic masses and observations. We provide numerical evidence that Newton's method converges and that we can apply iteratively the iterative lemma to the approximately invariant torus to obtain an invariant torus, and we estimate the size of the analyticity strips of the real-analytic parameterization of the torus. The estimated error of invariance is of order 10^{-54}. We emphasize that the method does not involve a truncation of the Hamiltonian, as it is customary in normal form methods. The last stage is work in progress, and we plan to present how the numerics from this paper and the KAM theorem from Figueras and Haro (2024) are combined along with rigorous numerics for validating the results.

1.1 Our Results and Their Organization in the Paper

We start presenting the model we work on. It is a Hamiltonian with 3 degrees of freedom (the total angular momentum has been reduced) depending on a parameter μ that accounts for the masses, so that $\mu = 0$ corresponds to two uncoupled Kepler problems, and $\mu = \mu_0$ corresponds to the actual values of the masses of the planets (in our case $\mu_0 = 10^{-3}$). Then, we discuss the numerical methods used. The goal is computing a 3 dimensional invariant torus for the observed values of the frequencies (for $\mu = \mu_0$). Two of the frequencies are fast, and the other is slow and of the order of μ_0. A fundamental obstacle we encounter is that the torus does not come by continuation from a three-dimensional invariant torus for $\mu = 0$, because Kepler motions correspond to two-dimensional tori, and the slow frequency collapses to zero. Hence, we cannot apply a direct continuation technique of the *invariant tori* from the integrable problem since the problem is singular at $\mu = 0$. Following the lines of thought of González et al. (2014, 2022), we perform a continuation of *translated tori*, which are invariant tori for a modified Hamiltonian system to which we have added an extra term (a translation) that compensates the degeneracies of the actual problem. At $\mu = \mu_0$, the translation term should be zero. At this stage, the torus is no longer degenerate, allowing the use of KAM numerical schemes on the actual problem for its refinement, see de la Llave et al. (2005). This leads to, after iterating several times the KAM numerical scheme, obtaining a very accurate approximation for the invariant torus. Finally, with this approximation, we can run the iterative lemma in Figueras and Haro (2024) several times and, lastly, the KAM theorem so that all the bounds are satisfied and gives us the existence of a nearby invariant torus, hence giving a numerical verification of the existence of quasiperiodic solutions close to the observations of Sun–Jupiter–Saturn configuration. Paraphrasing Henón, the numerical results we present here indicate that the invariant torus exists for $\mu = \mu_0$.

2 The Planetary Model and the Problem

The planar $(1+n)$-body problem (the Sun plus n planets) in Poincaré heliocentric Cartesian coordinates has Hamiltonian (Laskar and Robutel 1995; Chierchia and Pinzari 2011) $H_C : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$\begin{aligned} H_C(x, y) &= \sum_{i=1}^{n} \left(\frac{\|y_i\|^2}{2m_i} - \frac{m_i}{\|x_i\|} \right) \\ &\quad + \mu \left(\sum_{i=1}^{n} \frac{\|y_i\|^2}{2} + \sum_{1 \le i < j \le n} \left(y_i \cdot y_j - \frac{m_i m_j}{\|x_i - x_j\|} \right) \right) \\ &= H_C^0(x, y) + \mu\, H_C^1(x, y), \end{aligned} \tag{2.1}$$

where the 0-th body (the Sun) has mass 1 and is fixed at the origin and the i-th body has mass μm_i and position–momentum coordinates $(x_i, y_i) = (x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2})$. Also, the length and time units are chosen so that the gravitational constant is 1 and the period of an elliptical orbit of semimajor axis 1 is 2π. (So its frequency is 1, and this is the case of the Earth in the Solar System.)

The $\mu = 0$ case corresponds to the integrable Keplerian motion of the planets around the Sun (no interaction *between* planets). Well-known angle-action coordinates for the Keplerian motion are Delaunay coordinates. See Appendix A for the sake of completeness. The Hamiltonian (2.1) is then written in Delaunay coordinates $(\ell, g, L, G) \in \mathbb{T}^n \times \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n$ as a function $H_D : \mathbb{T}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$

$$H_D(\ell, g, L, G) = \sum_{i=1}^{n} \frac{-m_i^3}{2L_i^2} + \mu\, H_C^1 \circ D(\ell, g, L, G) = H_D^0(L) + \mu\, H_D^1(\ell, g, L, G),$$

where D denotes the *Delaunay map* from Delaunay coordinates (ℓ, g, L, G) to Cartesian coordinates (x, y).

Let us denote by $\hat{G}_i = \sum_{1 \le k \le i} G_k$ the angular momentum of the i first planets. It is well known that the total angular momentum, $\hat{G}_n$, is a first integral of the Hamiltonian system, so that we can reduce by one the number of degrees of freedom by fixing the value $\hat{G}_n = \hat{G}_{n,0}$. By extending the angular momentum map above to a canonical transformation, taking $\hat{g}_i = g_i - g_{i+1}$ for $i = 1, \ldots, n-1$, and $\hat{g}_n = g_n$, one gets that $\hat{g}_n$ is a cyclic coordinate in the transformed Hamiltonian in the new coordinates. Hence, following the standard practice, by fixing the total angular momentum $\hat{G}_n$ to a given value $\hat{G}_{n,0}$, one gets a reduced Hamiltonian $H_{\hat{G}_{n,0}} : \mathbb{T}^{2n-1} \times \mathbb{R}^{2n-1} \to \mathbb{R}$ given by

$$H_{\hat{G}_{n,0}}(\ell, \hat{g}, L, \hat{G}) = H_D^0(L) + \mu H^1_{\hat{G}_{n,0}}(\ell, \hat{g}, L, \hat{G}), \tag{2.2}$$

with $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_{n-1})$ and $\hat{G} = (\hat{G}_1, \ldots, \hat{G}_{n-1})$. From now on, we will omit the dependence on $\hat{G}_{n,0}$ from the notation.

A Lagrangian invariant torus of H, with a $(2n-1)$-dimensional vector of frequencies $(\omega^\ell, \omega^{\hat{g}}) \in \mathbb{R}^n \times \mathbb{R}^{n-1}$ and total angular momentum $\hat{G}_{n,0}$, gives raise to a Lagrangian invariant torus of H_{D}, with a $2n$-dimensional vector of frequencies $(\omega^{\ell}, \omega^{g}) \in \mathbb{R}^n \times \mathbb{R}^n$. The frequencies are related by $\omega_i^{\hat{g}} = \omega_i^{g} - \omega_{i+1}^{g}$ for $i = 1, \ldots n-1$, and $\omega_n^{\hat{g}} = \omega_n^{g}$ is the average of $\frac{\partial H}{\partial \hat{G}_n}$ over the $(2n-1)$-dimensional invariant torus. We emphasize that ω^{ℓ} contains the fast frequencies (the ones coming from the Keplerian motion), and that $\omega^{\hat{g}}$ (or ω^{g}) contains the slow frequencies (that in our case are proportional to μ). This smallness is a main difficulty when facing the $(1+n)$-body problem with realistic data (big masses and no big axes).

2.1 Invariance Equation

In the light of the parameterization method, finding invariant tori for H with frequency vector $\omega = (\omega^{\ell}, \omega^{\hat{g}})$ reduces to finding a parameterization of the torus $K : \mathbb{T}^{2n-1} \to \mathbb{T}^{2n-1} \times \mathbb{R}^{2n-1}$ satisfying the *invariant torus equation*

$$\mathfrak{L}_{\omega} K(\theta) + X_H(K(\theta)) = 0, \tag{2.3}$$

where $\mathfrak{L}_{\omega}$ is the Lie operator acting on any smooth function f with domain $\mathbb{T}^{2n-1}$ by $\mathfrak{L}_{\omega} f(\theta) = -\mathrm{D} f(\theta)\omega$, and $X_H = \Omega^{-1}(\mathrm{D}H)^{\top}$ is the Hamiltonian vector field with respect to the standard symplectic form given by the matrix

$$\Omega = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}.$$

We write $J = \Omega$ when we think of such a matrix as a linear map instead of as a 2-form. In particular, we use J to define a normal bundle to the torus parameterized by K, framed by the columns of $N(\theta) = J\mathrm{D}K(\theta)(\mathrm{D}K(\theta)^{\top}\mathrm{D}K(\theta))^{-1}$, where the columns of $\mathrm{D}K(\theta)$ frame the tangent bundle. Moreover, the symmetric matrix

$$T(\theta) = N(\theta)^{\top}\Omega(\mathrm{D}X_H(K(\theta)) + J\mathrm{D}X_H(K(\theta))J)N(\theta)$$

measures how much the normal bundle is twisted. (The tangent bundle of an invariant torus is fixed.) The non-degeneracy of the average of T, the *torsion*, plays the role of the classical Kolmogorov non-degeneracy condition in KAM theory.

As it is also customary in KAM theory, we assume that ω is Diophantine, i.e., there exists $\gamma > 0$ and $\tau \geq 2n-2$ such that for any $k \in \mathbb{Z}^{2n-1}$ and $k \neq 0$, $|k \cdot \omega| \geq \gamma |k|_1^{-\tau}$. Given any real-analytic function $s : \mathbb{T}^{2n-1} \to \mathbb{R}^M$, we denote by $\mathfrak{R}_{\omega}(s)$ the only real-analytic function $f : \mathbb{T}^{2n-1} \to \mathbb{R}^M$, with average zero, that satisfies $\mathfrak{L}_{\omega} f = s - \langle s \rangle$, where $\langle s \rangle$ denotes the average of the function s. This operator is in the core of KAM theory.

3 Continuation from the Integrable Case with Translated Tori Methods

Notice that for $\mu = 0$ the reduced Hamiltonian (2.2) has the invariant tori

$$K_{\hat{G}_0}(\theta^\ell, \theta^{\hat{g}}) = (\theta^\ell, \theta^{\hat{g}}, L_0, \hat{G}_0) \tag{3.1}$$

where the components of L_0 are determined by the masses of the bodies and the fast frequencies ω^ℓ (by the third Kepler's law), but the secular frequency $\omega_0^{\hat{g}}$ is zero (not $\omega^{\hat{g}}$!) and $\hat{G}_0$ is free: There is an $(n-1)-$parameter family of $(2n-1)$-dimensional tori foliated by n-dimensional invariant tori. As a result, the torsion is noninvertible (since there is no twist in the $\hat{G}$ direction). In summary, the problem is degenerate.

3.1 A Translated Torus Algorithm

As mentioned above, the degeneracy of the problem imposes a first obstacle for applying any numerical KAM scheme for performing any continuation with respect to μ. In the spirit of González et al. (2014, 2022), we can overcome this degeneracy by introducing a counterterm $\lambda\, \Pi_{\hat{G}}$ to the Hamiltonian (2.2), where $\Pi_{\hat{G}} : \mathbb{T}^{2n-1} \times \mathbb{R}^{2n-1} \to \mathbb{R}$ is the projection onto the $\hat{G}$ coordinate. Hence, by denoting $X_{\hat{G}} = X_{\Pi_{\hat{G}}}$, instead of solving Equation (2.3), we solve the extended system

$$\begin{cases} \mathfrak{L}_\omega K(\theta) + X_H(K(\theta)) + X_{\hat{G}}(K(\theta))\lambda = 0, \\ \langle \Pi_{\hat{G}}(K(\theta)) \rangle - \hat{G}_0 = 0, \end{cases} \tag{3.2}$$

for a fixed constant $\hat{G}_0$. The *invariant tori* satisfying (3.2) are *translated tori* for the original Hamiltonian system. Since $\hat{G}_0$ is an extra parameter, under appropriate non-degeneracy conditions (that we will see later are very mild), we can find families of translated tori, labeled by $\hat{G}_0$. The use of counterterms in KAM theory goes back to the works of Moser (1967), Herman (1998), Féjoz (2004, 2017). We emphasize that, here, we introduce a minimum number of counterterms to unfold the singularity of the problem. Moreover, there is a correlation between the added counterterm and the extra equation and, a posteriori, λ is a gradient-like function of the moment $\hat{G}_0$ (González et al. 2014).

Notice that, for a given $\hat{G}_0$, for $\mu = 0$ the parameterization (3.1) satisfies (3.2) with frequency $\omega = (\omega^\ell, \omega^{\hat{g}})$, by selecting $\lambda = \omega^{\hat{g}}$. The idea is then performing a continuation method for solving the *translated torus equation* (3.2) for couples (K, λ) up to the value $\mu = \mu_0$. The rationale behind this method is that if there was an invariant torus with frequency $\omega = (\omega^\ell, \omega^{\hat{g}})$ and $\langle \Pi_{\hat{G}} \circ K \rangle = \hat{G}_0$ for $\mu = \mu_0$, then after the continuation procedure we would find (K, λ) with $\lambda = 0$. Using perturbation theory up to order one (expanding in Poincaré–Lindstedt series), we get an approximation of $\hat{G}_0$ by solving the equation $\langle \Pi_{\hat{g}} X_{H^1} \circ K_{\hat{G}} \rangle = \omega^{\hat{g}}/\mu_0$, where $\Pi_{\hat{g}}$ is the projection onto the $\hat{g}$ component. We then continue this solution from $\mu = 0$ to $\mu = \mu_0$ by solving the equations at each step, see below. Finally, since this value of $\hat{G}_0$ is not exact, we do not get $\lambda = 0$ at $\mu = \mu_0$. However, at μ_0 one can also tune $\hat{G}_0$ to get $\lambda = 0$ by a Newton method (again using perturbation methods for computing $\frac{\partial \lambda}{\partial \hat{G}_0}$). [2]

3.1.1 Solving Equations (3.2)

More concretely, from an approximate solution $(K(\theta), \lambda)$ of (3.2) we can perform a quasi-Newton correction of the form $(P(\theta)\xi(\theta), \Delta\lambda)$, where the matrix $P(\theta) = \big(\mathrm{D}K(\theta)\ N(\theta)\big)$, obtained by juxtaposing the tangent and normal frames given above, is approximately symplectic. Taking into account that the inverse of $P(\theta)$ is close to $-\Omega P(\theta)^T \Omega$, we end up with the linear system

$$\begin{cases} \mathfrak{L}_\omega \xi^{\mathrm{DK}}(\theta) + T(\theta)\xi^N(\theta) + b^{\mathrm{DK}}(\theta)\Delta\lambda = \eta^{\mathrm{DK}}(\theta), \\ \mathfrak{L}_\omega \xi^N(\theta) + \qquad\qquad\qquad b^N(\theta)\Delta\lambda = \eta^N(\theta), \\ \langle \Pi_{\hat{G}}(\mathrm{D}K(\theta)\xi^{\mathrm{DK}}(\theta) + N(\theta)\xi^N(\theta))\rangle = \eta^{\hat{G}_0}, \end{cases}$$

where $\begin{pmatrix} b^{\mathrm{DK}}(\theta) \\ b^N(\theta) \end{pmatrix} = \begin{pmatrix} N(\theta)^\top \\ -\mathrm{D}K(\theta)^\top \end{pmatrix} \Omega X_{\hat{G}}(K(\theta))$, $\begin{pmatrix} \eta^{\mathrm{DK}}(\theta) \\ \eta^N(\theta) \end{pmatrix} = \begin{pmatrix} -N(\theta)^\top \\ \mathrm{D}K(\theta)^\top \end{pmatrix} \Omega$ $(\mathfrak{L}_\omega K(\theta) + X_H(K(\theta)) + \lambda X_{\hat{G}}(K(\theta)))$, $\eta^{\hat{G}_0} = -\langle \Pi_{\hat{G}} \circ K\rangle + \hat{G}_0$.

As it is customary in these types of schemes, one solves them up to some a-priori unknowns: In this case, the average $\xi_0^N = \langle \xi^N \rangle$ and $\Delta\lambda$. These two satisfy the linear system

$$\begin{pmatrix} \langle T \rangle & \langle \tilde{b}^{\mathrm{DK}} \rangle \\ \langle \Pi_{\hat{G}}(N - DK\mathfrak{R}_\omega T)\rangle & -\langle \Pi_{\hat{G}}(DK\mathfrak{R}_\omega \tilde{b}^{\mathrm{DK}} + N\mathfrak{R}_\omega b^N)\rangle \end{pmatrix} \begin{pmatrix} \xi_0^N \\ \Delta\lambda \end{pmatrix} = \begin{pmatrix} \langle \tilde{\eta}^{\mathrm{DK}} \rangle \\ \eta^{\hat{G}_0} - \langle \Pi_{\hat{G}} \mathrm{D}K \mathfrak{R}_\omega \tilde{\eta}^{\mathrm{DK}} \rangle \end{pmatrix}, \tag{3.3}$$

where $\tilde{b}^{\mathrm{DK}} = b^{\mathrm{DK}} - T\mathfrak{R}_\omega b^N$ and $\tilde{\eta}^{\mathrm{DK}} = \eta^{\mathrm{DK}} - T\mathfrak{R}_\omega \eta^N$. If the matrix in (3.3), to which we will refer to as the *supertorsion* $\langle \hat{T} \rangle$, is regular, then the method can continue by computing

$$\begin{cases} \xi^N(\theta) = \xi_0^N + \mathfrak{R}_\omega \eta^N(\theta) - \mathfrak{R}_\omega b^N(\theta)\Delta\lambda, \\ \xi^L(\theta) = \mathfrak{R}_\omega \tilde{\eta}^{\mathrm{DK}}(\theta) - \mathfrak{R}_\omega T(\theta)\xi_0^N - \mathfrak{R}_\omega \tilde{b}^{\mathrm{DK}}(\theta)\Delta\lambda, \end{cases}$$

and at the next step we get a quadratically better estimate $(K + P\xi, \lambda + \Delta\lambda)$.

[2] In applications, estimates of $\hat{G}_0$ could also be obtained by methods such as averaging the $\hat{G}$ components of a quasiperiodic orbit obtained using frequency analysis (Laskar 2005; Gómez et al. 2010; Luque and Villanueva 2009; Das et al. 2017).

Remark 3.1 In particular, in the case $\mu = 0$, the torsion and the supertorsion of the torus (3.1) are

$$\langle T \rangle = \begin{pmatrix} \mathrm{D}^2 H^0(L_0) & O \\ O & O \end{pmatrix}, \quad \langle \hat{T} \rangle = \begin{pmatrix} \mathrm{D}^2 H^0(L_0) & O & O \\ O & O & I \\ O & I & O \end{pmatrix},$$

respectively. Notice that even though the torsion is degenerate, the supertorsion is not, permitting to start the continuation of *translated tori* from $\mu = 0$.

3.2 An Invariant Torus Algorithm

Once the continuation explained in Sect. 3.1 reaches the parameter value μ_0, one obtains a translated torus K with translation λ that, ideally, should be zero. We can then tune parameter $\hat{G}_0$ so that one gets $\lambda = 0$, by solving the equation $\lambda(\hat{G}_0) = 0$ by, say, a Newton method. See González et al. (2014, 2022). Further refinements can be done to solve directly Equation (2.3), in which the unknown is K. Following de la Llave et al. (2005), given an approximate solution K of (2.3), its correction is given by $P(\theta)\xi(\theta)$ where $\xi(\theta)$ solves the linear system

$$\begin{cases} \mathfrak{L}_\omega \xi^{\mathrm{DK}}(\theta) + T(\theta)\xi^N(\theta) = \eta^{\mathrm{DK}}(\theta), \\ \mathfrak{L}_\omega \xi^N(\theta) \qquad\qquad\qquad\quad = \eta^N(\theta), \end{cases}$$

where $\begin{pmatrix} \eta^{\mathrm{DK}}(\theta) \\ \eta^N(\theta) \end{pmatrix} = \begin{pmatrix} -N(\theta)^\top \\ \mathrm{D}K(\theta)^\top \end{pmatrix} \Omega(\mathfrak{L}_\omega K(\theta) + X_H(K(\theta)).$

Notice that in this case we assume that the torsion $\langle T \rangle$ is non-degenerate, so that this last system can be solved as

$$\begin{cases} \xi_0^N = \langle T(\theta)\rangle^{-1} \langle \eta^{\mathrm{DK}}(\theta) - T(\theta)\mathfrak{R}_\omega \eta^N(\theta)\rangle, \\ \xi^N(\theta) = \xi_0^N + \mathfrak{R}_\omega \eta^N(\theta), \\ \xi^{\mathrm{DK}}(\theta) = \mathfrak{R}_\omega \left(\eta^{\mathrm{DK}}(\theta) - T(\theta)\xi^N(\theta)\right). \end{cases}$$

Remark 3.2 A minor difference with the constructs presented in de la Llave et al. (2005) is that here we use explicitly the approximation

$$\left(\mathrm{D}K(\theta)\ N(\theta)\right)^{-1} \approx \begin{pmatrix} N(\theta)^\top \\ -\mathrm{D}K(\theta)^\top \end{pmatrix} \Omega.$$

4 Application to the Sun–Jupiter–Saturn Problem

We have implemented the algorithms discussed in Sect. 3 in C++ (see, e.g., Haro et al. (2016) for similar implementations). A key point of the implementation is to use FFT routines for fast evaluations of the vector field on the parameterization, and for evaluating the operator $\mathfrak{R}_\omega$. We have adapted the FFT routines from Press et al. (2002) to work with multiprecision arithmetics with mpfr (see Fousse et al. (2007)). Another key point is parallelization using `openmp` (see Chandra et al. (2001)).

Here, we present the specifics for the planar Sun–Jupiter–Saturn problem with realistic values of their parameters (masses, frequencies, ephemerides, etc.). The source for the values of the parameters we have used comes from astronomical observations from NASA: Planetary fact sheet (2023), Sun fact (2023). (Other important tools such as frequency analysis (Laskar 2005; Gómez et al. 2010; Luque and Villanueva 2009; Das et al. 2017) could have also been used for getting these data.)

The masses of Jupiter and Saturn are taken to be $0.9546 \cdot 10^{-3}$ and $0.2856 \cdot 10^{-3}$, respectively, so $m_1 = 0.9546$, $m_2 = 0.2856$ and $\mu_0 = 10^{-3}$. The corresponding semiaxis and eccentricities are taken to be

$$a_1 = 5.21723814923184120006, a_2 = 9.56220231823193419964$$

and

$$e_1 = 0.048901221343923, e_2 = 0.056520213912482,$$

from which we get the first approximation of the torus (and, in particular, its actions). Also, from the semiaxis and Kepler's law we fix the frequencies of the Keplerian motions, say

$$\omega^\ell = (8.39549288702546301204... \cdot 10^{-2}, 3.38240117059304358259... \cdot 10^{-2}), \tag{4.1}$$

(with machine precision). From the frequency analysis performed in Locatelli and Giorgilli (2007), we take the secular frequencies to be

$$\omega^g = (2.32008443270689 \cdot 10^{-5}, 4.17016431348284 \cdot 10^{-5})$$

from which we take

$$\omega^{\hat{g}_1} = \omega^{g_1} - \omega^{g_2} = -1.85007988077595 \cdot 10^{-5}. \tag{4.2}$$

The important fact is that the order of magnitude of the secular frequencies is the one of the fast frequencies times $\mu = 10^{-3}$. Finally, we take the total angular momentum to be

$$\hat{G}_{2,0} = 3.05839852910896096675.$$

While the total angular momentum $\hat{G}_2$ is preserved (and equal to the value $\hat{G}_{2,0}$), the one of Jupiter, $\hat{G}$, is not. An approximation of the average of the angular momentum of Jupiter comes from the Kepler approximation, and approximations of order μ_0 are obtained by solving the equation $\langle \Pi_{\hat{g}} X_{H^1} \circ K_{\hat{G}} \rangle = \omega^{\hat{g}}/\mu_0$. After several iterations of the secant method, we obtain the approximation $\hat{G}_0 = 2.17647359010273488684$.

These preliminary computations are performed with `long double` precision C arithmetic, and the parameterizations are given by grids of size 128^3.

The first thing we have done is finding the value of $\hat{G}_0$ such that at $\mu = \mu_0$, when performing the continuation of the translated torus algorithm, we obtain $\lambda = 0$. To do this, we performed the following 4 times: At $\mu = 0$, we have the invariant torus (3.1) with an approximation of the desired $\hat{G}_0$ value. Then, we perform a continuation using the translated torus algorithm from $\mu = 0$ to $\mu = 10^{-3}$ and get an approximately translated torus (that, unfortunately, does not satisfy $\lambda = 0$). Finally, by doing a Newton step for solving $\lambda(\hat{G}_0) = 0$ we obtain a better estimate of $\hat{G}_0$. Then, we repeat the process. By doing this, in the first run we obtain an approximately translated torus with invariance error $4.8 \cdot 10^{-9}$, moment error $\langle \Pi_{\hat{G}} K \rangle - \hat{G}_0 = 1.5 \cdot 10^{-14}$, and $\lambda = -8.67345532598763273085 \cdot 10^{-7}$. The Newton step gives us a better estimate $\hat{G}_0 = 2.17658253666877214401$. After the fourth time we do this, we obtain an approximately translated torus with $\hat{G}_0 = 2.17657425006565519231$, invariance error $5.3 \cdot 10^{-10}$, moment error $\langle \Pi_{\hat{G}} K \rangle - \hat{G}_0 = 6.5 \cdot 10^{-17}$ and, $\lambda = 3.14309229785154830993 \cdot 10^{-14}$. For this last torus, the supertorsion (3.3) is

$$\begin{pmatrix} -1.15830235 \cdot 10^{-1} & 1.43198239 \cdot 10^{-3} & 7.41752484 \cdot 10^{-4} & 2.11280898 \cdot 10^{-1} \\ 1.43198239 \cdot 10^{-3} & -1.23922829 \cdot 10^{-1} & -5.21182674 \cdot 10^{-3} & -3.45764287 \cdot 10^{-1} \\ 7.41752484 \cdot 10^{-4} & -5.21182674 \cdot 10^{-3} & -3.18265383 \cdot 10^{-3} & 5.20444218 \cdot 10^{-1} \\ 2.11280898 \cdot 10^{-1} & -3.45764287 \cdot 10^{-1} & 5.20444218 \cdot 10^{-1} & 8.38171177 \cdot 10^{0} \end{pmatrix}$$

The norm of the inverse of the supertorsion is $3.9 \cdot 10^1$ and that of the inverse of the torsion is $3.5 \cdot 10^2$. The whole computation takes less than one hour, with the first continuation taking around 17 min, and the last around 12 min. Different projections of the invariant torus are shown in Figs. 1 and 2.

Later on we refined the approximately invariant torus using the invariant torus algorithm by increasing the precision with `long double`, `__float128` and, finally `mpfr`. We gradually increased the accuracy and the size of the grids. In the last run, the input torus was given with a grid of size 512^3 with 57 digits, and the output with a grid of size 1024^3 with 76 digits. The input error was $9.1 \cdot 10^{-29}$, and the error saturated at the first step to $3.9 \cdot 10^{-54}$. (The error at the second step was $2.9 \cdot 10^{-54}$.) Although it is not used during the computations, we obtain that this torus has $\hat{G}_0 = 2.1765741888387268935208468527794 3$. In this case, one Newton step took around one week, and the top size of RAM memory used was 194 G.

From this last torus, we have estimated how fast the Fourier coefficients decrease and, so, its analyticity radius. To do so, we have fit the Fourier coefficients of the complexifications $K^{\ell_1} + \mathrm{i}K^{L_1}$, $K^{\ell_2} + \mathrm{i}K^{L_2}$ and $K^{\hat{g}} + \mathrm{i}K^{\hat{G}}$ with respect to each of the angles $\theta^{\ell_1}, \theta^{\ell_2}, \theta^{\hat{g}}$, thus obtaining estimates of

the analyticity strips $\rho_{\ell_1}, \rho_{\ell_2}, \rho_{\hat{g}}$. The results are shown in Fig. 3 where, for a Fourier expansion

$$f(\theta^{\ell_1}, \theta^{\ell_2}, \theta^{\hat{g}}) = \sum_{k_{\ell_1}, k_{\ell_2}, k_{\hat{g}}} f_{k_{\ell_1}, k_{\ell_2}, k_{\hat{g}}} e^{\mathrm{i}(k_{\ell_1}\theta^{\ell_1} + k_{\ell_2}\theta^{\ell_2} + k_{\hat{g}}\theta^{\hat{g}})},$$

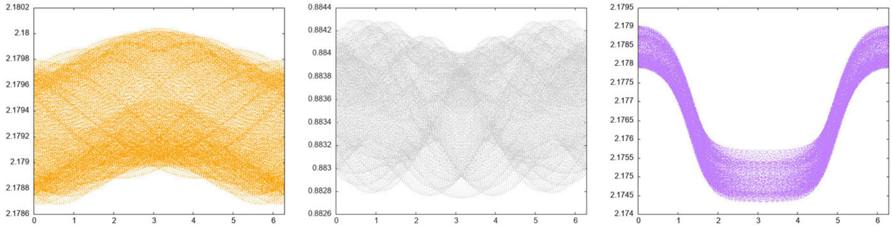


Fig. 1 Projections of the 3D invariant torus in Delaunay coordinates onto (ℓ_1, L_1), (ℓ_2, L_2) and $(\hat{g}, \hat{G})$ components

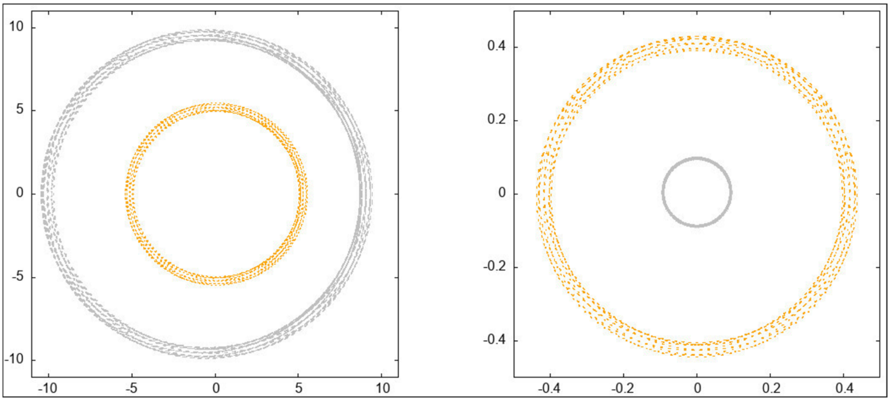


Fig. 2 Projections of the 3D invariant torus (generating a 4D torus) in Cartesian coordinates onto positions $x_1 = (x_{1,1}, x_{1,2}), x_2 = (x_{2,1}, x_{2,2})$ and momenta $y_1 = (y_{1,1}, y_{1,2}), y_2 = (y_{2,1}, y_{2,2})$. Coordinates x_1, y_1 correspond to Jupiter and x_2, y_2 correspond to Saturn and are plot in orange and gray, respectively

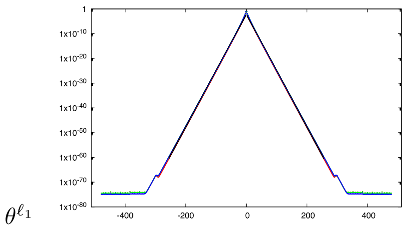
we fit the analyticity strips $\rho_{\ell_1}, \rho_{\ell_2}, \rho_{\hat{g}}$ of each of the angles by considering the univariate Fourier series

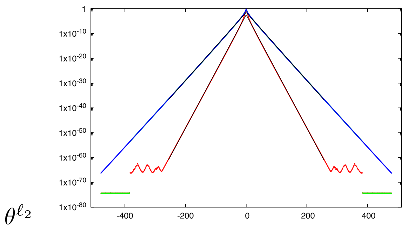
$$\tilde{f}_{\ell_1}(\theta^{\ell_1}) = \sum_{k_{\ell_1}} \left(\sum_{k_{\ell_2}, k_{\hat{g}}} |f_{k_{\ell_1}, k_{\ell_2}, k_{\hat{g}}}| \right) e^{\mathrm{i}(k_{\ell_1} \theta^{\ell_1})},$$

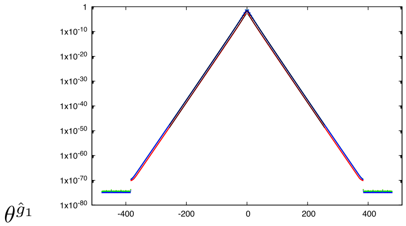
$$\tilde{f}_{\ell_2}(\theta^{\ell_2}) = \sum_{k_{\ell_2}} \left(\sum_{k_{\ell_1}, k_{\hat{g}}} |f_{k_{\ell_1}, k_{\ell_2}, k_{\hat{g}}}| \right) e^{\mathrm{i}(k_{\ell_2} \theta^{\ell_2})},$$

$$\tilde{f}_{\hat{g}}(\theta^{\hat{g}}) = \sum_{k_{\hat{g}}} \left(\sum_{k_{\ell_1}, k_{\ell_2}} |f_{k_{\ell_1}, k_{\ell_2}, k_{\hat{g}}}| \right) e^{\mathrm{i}(k_{\hat{g}} \theta^{\hat{g}})}$$

and doing a standard fit on their coefficients.







	$K^{\ell_1} + \mathrm{i}K^{L_1}$	$K^{\ell_2} + \mathrm{i}K^{L_2}$	$K^{\hat{g}} + \mathrm{i}K^{\hat{G}}$
ρ_{ℓ_1}	0.524453	0.521794	0.522239
ρ_{ℓ_2}	0.416514	0.417078	0.417001
$\rho_{\hat{g}}$	0.525773	0.316526	0.316557

Fig. 3 Fits of Fourier coefficients of the (complexified) components of the parameterization $\tilde{f}_{\ell_1}$, $\tilde{f}_{\ell_2}$ and $\tilde{f}_{\hat{g}}$, and estimates of analyticity strips

5 Numerical Verification of the KAM Constants

The numerical certification of the existence of the invariant torus is based on the KAM theorem and the Iterative Lemma appearing in Figueras and Haro (2024). For the sake of completeness, we include their tailored and simplified versions (with the most relevant hypotheses) in Appendix B, so it will guide us in all the data needed for doing the validation. For the specific expression of all the constants, we refer the reader to Figueras and Haro (2024), where they appear in the appendices.

Given the $\omega = (\omega^{\ell}, \omega^{\hat{g}_1})$ in (4.1), (4.2) we can certify (using the validation techniques in Figueras et al. (2017)) that at distance less than 10^{-80} there is a Diophantine vector with $\tau = 2.4$ and $\gamma = 1.69 \cdot 10^{-6}$. Moreover, we choose the radius of analyticity to be $\rho = 0.1$ and $\delta = \frac{\rho}{6}$.

The hypotheses in H_1 control the Hamiltonian and its associated vector field in a tubular neighborhood of the torus $K(\mathbb{T}^m_\rho)$. In our case, it is enough to take these constants to be

$$c_{X_h} = 0.09,\; c_{\mathrm{D}X_h} = 129,\; c_{(\mathrm{D}X_h)^\top} = 129,\; c_{\mathrm{D}^2 X_h} = 5 \cdot 10^{10}.$$

The hypotheses in H_2 control the parameterization K and all the geometric information it has (the bundles $\mathrm{D}K$, N, and so on). In our case, these constants are computed with the approximation and obtained

$$\begin{aligned} \|\mathrm{D}K\| &= 4.7811815833,\; \|(\mathrm{D}K)^\top\| = 6.8755882886,\; \|B\| = 7.35806411265,\\ \|N\| &= 3.5704498717,\; \|N^\top\| = 2.8621242724,\; |\langle T\rangle|^{-1} = 354.07743243. \end{aligned}$$

The corresponding σ constants are obtained by multiplying these norms by a factor $1 + 10^{-10}$.

With this information, we can run the Iterative Lemma several steps, say 10, and with different initial invariance errors and then apply the KAM theorem to see if it converges. (Inequality (B.1) is fulfilled.) We have obtained that with $\|\eta^{\mathrm{D}K}\| = 10^{-38}$, $\|\eta^N\| = 10^{-44}$. However, from our numerics we obtain that our torus satisfies $\|\eta^{\mathrm{D}K}\| = 3.6 \cdot 10^{-54}$ and $\|\eta^N\| = 1.7 \cdot 10^{-57}$, which are very much smaller than the thresholds!

With the application of the Iterative Lemma, we validate the existence of the 3D invariant torus for the reduced 3-degree-of-freedom Hamiltonian $H_{\hat{G}_{n,0}}$, in Delaunay coordinates. We then obtain a 4D invariant torus for the whole system, corresponding to a realistic planar motion of Sun–Jupiter–Saturn.

Remark 5.1 In order to emphasize the sensitivity of the computations (and their convergence) to the choice of coordinates, we mention that we have also computed a 3D invariant torus for a properly modified 4-degree-of-freedom Hamiltonian in Cartesian coordinates, that generates a 4D invariant torus for the original Hamiltonian (2.1). This is made thanks to a reduction lemma in Figueras and Haro (2024) that implies that, for our problem, since the total angular momentum induces a circle action on the phase space, then one can generate 4D invariant tori from 3D invariant tori of a modified Hamiltonian, *without reducing the number of degrees of freedom of the problem*. The algorithms in Haro and Luque (2019), Figueras and Haro (2024) are close to the ones used in the present paper. Hence, we have also applied this methodology to the particular Sun–Jupiter–Saturn system. The convergence of the algorithms is compromised by the bad condition numbers of B and $\langle T \rangle$. Even when taking the torus produced in Delaunay coordinates back to Cartesian coordinates and obtaining errors of the order 10^{-54}, the iterative lemma fails.

Remark 5.2 The choice of appropriate coordinates in Celestial Mechanics is a key point for solving the general N-body problem Chierchia and Pinzari (2011). It would be interesting to study the suitability of different systems of coordinates depending on the data (eccentricities, semiaxis, frequencies, etc.). Important results in this direction are in Pinzari (2018). We have modestly only considered a particular planetary system, but there are many exoplanetary systems (or even other problems) for which the methodology could be applied. In this respect, we mention the important applications of KAM theory to the study of exoplanetary systems Caracciolo et al. (2024); Mastroianni and Locatelli (2024, 2023a, b), using normal forms. It would be also interesting merging different methodologies.

6 Computation Details

For running the continuation method on the translated torus algorithm from the integrable system, we run the programs in an out of date MacBook Air laptop with one CPU 1.7 GHz Dual-Core Intel i7 and RAM memory 8 G, since for the approximation we work with `long double` C arithmetics and the tori are discretized in 128^3 nodes, accounting to 32 M of memory for each of the six components of the parameterization of the torus. We have also adapted and tested the programs to work with quadruple precision `__float128` C arithmetics. For the invariant torus algorithm, we have used an iMac Pro with one CPU 3,2 GHz Intel Xeon W with 8 cores and RAM Memory 256 G, working with several extended precision arithmetics with `mpfr` (up to 76 decimal digits, that correspond to 64 bytes, respectively) and the torus is discretized in 1024^3 nodes, accounting 64 G of memory for each to the components. This last computation has also been run in the UPPMAX supercomputer.

Finally, we give some numbers to provide an idea of the order of magnitude of the managed data structures at the final stages of the computations. The data structures are complex vectors that store couples of real grids. Moreover, handling of memory (both RAM and disk) by `mpfr` is anisotropic. For instance, for the computation of the torus with 76 digits the program uses up to 194 G of RAM memory for handling one single complex grid of size 1024^3, and 891 G of memory disk to store the objects being computed by the program. For files storing the same number of `mpfr` objects, $2 \cdot 1024^3/8$, the sizes range from 87 M to 8.8G.

Appendix A. Delaunay Coordinates

Delaunay coordinates are defined body-wise. For the i-th body, they are $(\ell_i, g_i, L_i, G_i) \in \mathbb{T}^2 \times \mathbb{R}^2$, with $G_i < L_i$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and which are mapped to the Cartesian coordinates $(x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}) \in \mathbb{R}^4$ through the following steps:

$$e_i = \sqrt{1 - \left(\frac{G_i}{L_i}\right)^2}, \; a_i = \frac{(L_i)^2}{m_i^2}, \; b_i = \frac{m_i^2}{L_i}, \; E_i = K(\ell_i, e_i),$$

$$\begin{pmatrix} q_{i,1} \\ q_{i,2} \end{pmatrix} = a_i \begin{pmatrix} \cos(E_i) - e_i \\ \dfrac{G_i}{L_i} \sin(E_i) \end{pmatrix}, \qquad \begin{pmatrix} x_{i,1} \\ x_{i,2} \end{pmatrix} = \begin{pmatrix} \cos(g_i) & -\sin(g_i) \\ \sin(g_i) & \cos(g_i) \end{pmatrix} \begin{pmatrix} q_{i,1} \\ q_{i,2} \end{pmatrix}$$

$$\begin{pmatrix} p_{i,1} \\ p_{i,2} \end{pmatrix} = \frac{b_i}{1 - e_i \cos(E_i)} \begin{pmatrix} -\sin(E_i) \\ \dfrac{G_i}{L_i} \cos(E_i) \end{pmatrix}, \qquad \begin{pmatrix} y_{i,1} \\ y_{i,2} \end{pmatrix} = \begin{pmatrix} \cos(g_i) & -\sin(g_i) \\ \sin(g_i) & \cos(g_i) \end{pmatrix} \begin{pmatrix} p_{i,1} \\ p_{i,2} \end{pmatrix}$$

where $E = K(\ell, e)$ denotes the solution of the Kepler equation $\ell = E - e \sin(E)$.

Appendix B. KAM Theorem and Iterative Lemma

Here, we gather both the KAM theorem and the iterative lemma in a tailored form. For a more detailed exposition of them have a look at Figueras and Haro (2024).

Theorem B.1 *Let $h : \mathcal{U} \to \mathbb{C}$ be a real-analytic Hamiltonian, defined in an open set $\mathcal{U} \subset \mathbb{T}^m_{\mathbb{C}} \times \mathbb{C}^m$. Let $K : \bar{\mathbb{T}}^m_\rho \to \mathcal{U}$ be a continuous map, real-analytic in $\mathbb{T}^m_\rho$, whose* *derivatives are also continuous in* $\bar{\mathbb{T}}^m_\rho$*, defining an homotopic to the zero-section embedding of* $\bar{\mathbb{T}}^m_\rho$ *into* $\mathbb{T}^m_{\mathbb{C}} \times \mathbb{C}^m$ *(in particular* $K(\theta) - (\theta, 0)$ *is* 2π*-periodic). Let* $\omega \in \mathcal{D}^m_{\gamma,\tau}$ *be a Diophantine vector, for some* $\gamma > 0$ *and* $\tau \geq m - 1$*. We also assume:*

H_1 *There exist constants* $c_{X_h}, c_{\mathrm{D}X_h}, c_{(\mathrm{D}X_h)^\top}, c_{\mathrm{D}^2 X_h}$ *such that*

$$\|X_h\|_{\mathcal{U}} \leq c_{X_h}, \quad \|\mathrm{D}X_h\|_{\mathcal{U}} \leq c_{\mathrm{D}X_h}, \quad \left\|(\mathrm{D}X_h)^\top\right\|_{\mathcal{U}} \leq c_{(\mathrm{D}X_h)^\top}, \quad \left\|\mathrm{D}^2 X_h\right\|_{\mathcal{U}} \leq c_{\mathrm{D}^2 X_h}.$$

H_2 *There are condition numbers* $\sigma_{\mathrm{D}K}, \sigma_{(\mathrm{D}K)^\top}, \sigma_B, \sigma_N, \sigma_{N^\top}$*, and* $\sigma_{\langle T\rangle^{-1}}$ *such that*

$$\|\mathrm{D}K\|_\rho < \sigma_{\mathrm{D}K}, \quad \left\|(\mathrm{D}K)^\top\right\|_\rho < \sigma_{(\mathrm{D}K)^\top}, \quad \|B\|_\rho < \sigma_B,$$
$$\|N\|_\rho < \sigma_N, \quad \left\|N^\top\right\|_\rho < \sigma_{N^\top}, \quad |\langle T\rangle^{-}1| < \sigma_{\langle T\rangle^{-1}};$$

Then, for each $\delta \in]0, \rho/6[$*, there exist constants* $\mathfrak{C}, \mathfrak{C}_{\Delta K}$ *depending on* ρ, δ *and the above constants and objects, such that if*

$$\frac{\mathfrak{C}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\} < 1, \tag{B.1}$$

where $\eta^{\mathrm{D}K} = -N^\top\Omega\,(\mathfrak{L}_\omega K + X_h \circ K)$*,* $\eta^N = (\mathrm{D}K)^\top\Omega\,(\mathfrak{L}_\omega K + X_h \circ K)$*, then, for* $\rho_\infty = \rho - 6\delta$*, there exists* $K_\infty : \bar{\mathbb{T}}^m_{\rho_\infty} \to \mathcal{U}$ *continuous, real-analytic in* $\mathbb{T}^m_{\rho_\infty}$*, whose derivatives are also continuous in* $\bar{\mathbb{T}}^m_{\rho_\infty}$*, defining an homotopic to the zero-section embedding of* $\bar{\mathbb{T}}^m_{\rho_\infty}$ *into* $\mathcal{U}$ *that is invariant under* X_h*, with frequency* ω*, so that*

$$\mathfrak{L}_\omega K_\infty + X_h \circ K_\infty = 0.$$

Moreover, K_∞ *satisfies hypothesis* H_2*, in* $\mathbb{T}^m_{\rho_\infty}$*, and it is close to* K*:*

$$\|K_\infty - K\|_{\rho_\infty} \leq \frac{\mathfrak{C}_{\Delta K}}{\gamma\delta^\tau} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\}.$$

The proof of the previous theorem consists of iteratively applying the following lemma.

Lemma B.2 [The Iterative Lemma] *Let us be under the same hypotheses as in Theorem B.1. For any* $\delta \in]0, \rho/3]$*, there exist constants* $C_{\mathrm{sym}}, C_{\xi^L}, C_{\Delta\bar{\bar{K}}}, C_{\Delta\mathrm{D}\bar{\bar{K}}}, C_{\Delta(\mathrm{D}\bar{\bar{K}})^\top}, C_{\Delta\bar{\bar{B}}}, C_{\Delta\bar{\bar{N}}}, C_{\Delta\bar{\bar{N}}^\top}, C_{\Delta\langle\bar{\bar{T}}\rangle^{-1}}, \hat{C}_\Delta$ *and* $Q_{\bar{\bar{\eta}}^L}, Q_{\bar{\bar{\eta}}^N}$*, such that if*

$$\frac{\hat{C}_\Delta}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\} < 1,$$

then we have a new real-analytic parameterization $\bar{\bar{K}} : \bar{\mathbb{T}}^m_{\rho-2\delta} \to \mathcal{U}$*, that defines new objects* $\mathrm{D}\bar{\bar{K}}, \bar{\bar{B}}, \bar{\bar{N}}$ *and* $\bar{\bar{T}}$ *(obtained replacing* K *by* $\bar{\bar{K}}$ *in the corresponding definitions) satisfying*

$$\left\|\mathrm{D}\bar{\bar{K}}\right\|_{\rho-3\delta} < \sigma_{\mathrm{D}K}, \left\|(\mathrm{D}\bar{\bar{K}})^\top\right\|_{\rho-3\delta} < \sigma_{(\mathrm{D}K)^\top}, \left\|\bar{\bar{B}}\right\|_{\rho-3\delta} < \sigma_B,$$
$$\left\|\bar{\bar{N}}\right\|_{\rho-3\delta} < \sigma_N, \left\|\bar{\bar{N}}^\top\right\|_{\rho-3\delta} < \sigma_{N^\top}, |\langle\bar{\bar{T}}\rangle^{-1}| < \sigma_{\langle T\rangle^{-1}},$$

and

$$\left\|\bar{\bar{K}} - K\right\|_{\rho-2\delta} \leq \frac{C_{\Delta\bar{\bar{K}}}}{\gamma\delta^\tau} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\},$$
$$\left\|\mathrm{D}\bar{\bar{K}} - \mathrm{D}K\right\|_{\rho-3\delta} \leq \frac{C_{\Delta\mathrm{D}\bar{\bar{K}}}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\},$$
$$\left\|(\mathrm{D}\bar{\bar{K}})^\top - (\mathrm{D}K)^\top\right\|_{\rho-3\delta} \leq \frac{C_{\Delta(\mathrm{D}\bar{\bar{K}})^\top}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\},$$
$$\left\|\bar{\bar{B}} - B\right\|_{\rho-3\delta} \leq \frac{C_{\Delta\bar{\bar{B}}}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\},$$
$$\left\|\bar{\bar{N}} - N\right\|_{\rho-3\delta} \leq \frac{C_{\Delta\bar{\bar{N}}}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\},$$
$$\left\|\bar{\bar{N}}^\top - N^\top\right\|_{\rho-3\delta} \leq \frac{C_{\Delta\bar{\bar{N}}^\top}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\},$$
$$|\langle\bar{\bar{T}}\rangle^{-1} - \langle T\rangle^{-1}| \leq \frac{C_{\Delta\langle\bar{\bar{T}}\rangle^{-1}}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\}.$$

Moreover, the tangent and normal components of the new error of invariance

$$\bar{\bar{E}} = X_h \circ \bar{\bar{K}} + \mathfrak{L}_\omega \bar{\bar{K}},$$

satisfy

$$\left\|\bar{\bar{\eta}}^N\right\|_{\rho-3\delta} \leq \frac{Q_{\bar{\bar{\eta}}^N}}{\delta} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\}^2,$$

and

$$\left\|\bar{\bar{\eta}}^L\right\|_{\rho-3\delta} \leq \frac{Q_{\bar{\bar{\eta}}^L}}{\gamma\delta^{\tau+1}} \max\left\{\left\|\eta^{\mathrm{D}K}\right\|_\rho, \frac{1}{\gamma\delta^\tau}\left\|\eta^N\right\|_\rho\right\}^2.$$

Remark B.3 We emphasize that this theorem obtains sharper KAM results than in de la Llave et al. (2005). Instead of obtaining at each step a new approximate solution as

$$K(\theta) + \mathrm{D}K(\theta)\xi^{\mathrm{D}K}(\theta) + N(\theta)\xi^N(\theta),$$

(as we made for the numerical algorithm in this paper), it is obtained as

$$K(\theta + \xi^{\mathrm{D}K}(\theta)) + N(\theta + \xi^{\mathrm{D}K}(\theta))\,\xi^N(\theta + \xi^{\mathrm{D}K}(\theta)).$$

See Villanueva (2017); Figueras and Haro (2024). However, from the numerical point of view, this choice seems to be much more difficult to implement efficiently because it involves compositions of functions.

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Data Availability No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest No datasets were generated or analyzed during the current study.

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