

JOIN OPERATION FOR THE BRUHAT ORDER AND VERMA MODULES

BY

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ABSTRACT

We observe that the join operation for the Bruhat order on a Weyl group agrees with the intersections of Verma modules in type A . The statement is not true in other types, and we propose a weaker correspondence. Namely, we introduce distinguished subsets of the Weyl group on which the join operation conjecturally agrees with the intersections of Verma modules. We also relate our conjecture with a statement about the socles of the cokernels of inclusions between Verma modules. The latter determines the first Ext space between a simple module and a Verma module. We give a conjectural complete description of such socles which we verify in a number of cases. Along the way, we determine the poset structure of the join-irreducible elements in Weyl groups and obtain closed formulae for certain families of Kazhdan–Lusztig polynomials.

1. Introduction

Let \mathfrak{g} be a finite dimensional complex semisimple Lie algebra with a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

Let W be the associated Weyl group, which we view as a Coxeter system (W, S) . Consider the associated BGG category \mathcal{O} and its principal block \mathcal{O}_0 (the indecomposable summand of \mathcal{O} containing the trivial \mathfrak{g} -module). The Verma modules in \mathcal{O}_0 are indexed by the Weyl group elements. For $w \in W$, we denote

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by Δ_w the Verma module of highest weight $w \cdot 0 = w\rho - \rho$, where ρ is the half of the sum of all positive roots. The set W has a poset structure with respect to the Bruhat order \leq . By a result of Bernstein, Gelfand and Gelfand, see [Di, Chapter 7], for $w, x \in W$, the following assertions are equivalent:

- there is a (necessarily unique up to scalar and injective) non-zero homomorphism from Δ_w to Δ_x .
- $w \geq x$.

This allows us to unambiguously write $\Delta_w \subseteq \Delta_x$ provided that $w \geq x$. In particular, we can view each Δ_w as a canonical submodule of the dominant Verma module Δ_e . In other words, we have an isomorphism of posets as follows:

$$(1) \quad (W, \leq) \cong (\{\Delta_w \subseteq \Delta_e \mid w \in W\}, \supseteq).$$

Being a poset, W has the join operation

$$\vee : 2^W \rightarrow W \amalg \{\text{'does not exist'}\},$$

where 2^W denotes the power set of W . The main purpose of the current paper is to understand the module-theoretic interpretation of this structure on the right-hand side of (1). The obvious candidate is the intersection:

Question A: Given $U \subseteq W$, do we have $w = \vee U$ if and only if $\bigcap_{x \in U} \Delta_x = \Delta_w$?

Using the results of [KMM], we observe that the answer to Question A is always positive in type A (see Corollary 4.12). A similar observation is made in [Ko].

THEOREM B: *Suppose $\mathfrak{g} = \mathfrak{sl}_n$. Then for any $U \subseteq W$, we have $\vee U = w \in W$ if and only if $\bigcap_{x \in U} \Delta_x = \Delta_w$.*

In general, the answer to Question A can be negative (see Example 5.42 in type E_6). In this paper, we explore a number of other special cases and examples which suggest that full understanding of Question A is very likely to be highly non-trivial. Our main idea for relating the join operation on W and the intersection of Verma submodules is to restrict \vee from 2^W to certain subsets consisting of join-irreducible elements, which we introduce now.

An element $w \in W$ is called **join-irreducible** if there is no ‘join expression’

$$w = \vee U$$

with $w \notin U$. We denote by \mathbf{JI} the set of join-irreducible elements in W . Then every element $w \in W$ has a distinguished join expression, as the join of join-irreducibles, given by the subset $\mathbf{JM}(w) \subset \mathbf{JI}$ defined as follows. First, recall that $s \in S$ is a left (resp., right) descent of $x \in W$ if $sx < x$ (resp., $xs < x$). Consider the set

$$\mathbf{JI}(w) = \{x \in \mathbf{JI} \mid x \leq w \text{ and each left (resp., right) decent of } x \text{ is a left (resp., right) descent of } w\}.$$

We will see in Lemma 3.1 that the elements in $\mathbf{JI}(w)$ in fact have unique left and right descent. Now we let

$$\mathbf{JM}(w) = \max \mathbf{JI}(w) = \text{the set of Bruhat maximal elements in } \mathbf{JI}(w).$$

Then $\mathbf{JM}(w)$ is, in fact, a join expression of w , that is, we have (see Lemma 3.4)

$$w = \bigvee \mathbf{JM}(w).$$

What we conjecture to hold, in all types, is that each Verma module has an expression given by $\mathbf{JM}(w)$.

CONJECTURE C: *For each $w \in W$, we have*

$$\Delta_w = \bigcap_{x \in \mathbf{JM}(w)} \Delta_x.$$

When W is dissective (see Subsection 2.4), Conjecture C answers Question A completely and positively (see Corollary 4.11), and, in fact, this is how we prove Theorem B in type A .

In our proof of Conjecture C in type A , the following property plays a crucial role and provides a connection to the results in [KMM]. Recall that, by definition and (1), for each $x \in \mathbf{JM}(w)$, we have a canonical map

$$\psi_x : \Delta_e / \Delta_w \twoheadrightarrow \Delta_e / \Delta_x.$$

Definition D: An element $w \in W$ is said to have the **socle-sum property** if

- (i) each ψ_x restricts to

$$\phi_x : \text{soc } \Delta_e / \Delta_w \rightarrow \text{soc } \Delta_e / \Delta_x;$$

- (ii) we have $\bigcap_{x \in \mathbf{JM}(w)} \ker \phi_x = 0$;
- (iii) the maps ϕ_x , where $x \in \mathbf{JM}(w)$, are surjective.

Note that the property (i) says that ψ_x maps $\text{soc } \Delta_e/\Delta_w$ to $\text{soc } \Delta_e/\Delta_x \subseteq \Delta_e/\Delta_x$. The properties (i) and (iii) make precise the statement that ‘ $\text{soc } \Delta_e/\Delta_w$ contains the sum of $\text{soc } \Delta_e/\Delta_x$ taken over all $x \in \mathbf{JM}(w)$ inside Δ_e ’. The properties (i) and (ii) together say that ‘ $\text{soc } \Delta_e/\Delta_w$ is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ ’. Often, and always in type A , this sum is direct. In such a case, the socle-sum property is equivalent to a simpler condition

$$\text{soc } \Delta_e/\Delta_w \cong \bigoplus_{x \in \mathbf{JM}(w)} \text{soc } \Delta_e/\Delta_x.$$

The main result of [KMM] can be interpreted as the socle-sum property in type A . The socle-sum property does not hold in general as we show in Remark 5.25 in type B and in Example 5.50 in type F . However, one direction of the socle-sum property is true in general.

THEOREM E: *For each $w \in W$, the properties (i) and (ii) hold. In other words, the socle of Δ_e/Δ_w is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ taken over all $x \in \mathbf{JM}(w)$.*

Motivated by the socle-sum property, we determine the socles of Δ_e/Δ_x , for $x \in \mathbf{JI}$, in almost all cases (we know the answer but do not have complete proofs in the remaining cases). We also establish the socle-sum property for many special cases and examples. In order to obtain a description of the socles, we explicitly give a set of generating relations for the poset of join-irreducible elements with fixed descent sets, for each W . There we encounter several interesting examples which illustrate the complexity of the combinatorics and representation theory outside type A . In exceptional types, this relies heavily on computer assisted computations.

Determining $\text{soc } \Delta_e/\Delta_w$, for $w \in W$, is an important problem independently of Question A or Conjecture C. As pointed out in [KMM], it has interesting applications to understanding various homological invariants of category \mathcal{O} . In particular, knowing $\text{soc } \Delta_e/\Delta_w$ would completely determine the dimension of the extension spaces

$$\text{Ext}^1(L_x, \Delta_w) \cong \text{Ext}^1(\nabla_w, L_x).$$

We use the notation

$${}_s\mathbf{JM}_t(w) = \{z \in \mathbf{JM}(w) \mid \text{the left (resp., right) descent of } z \text{ is } s \text{ (resp., } t)\},$$

for $w \in W$ and $s, t \in S$. Note that we have $\mathbf{JM}(w) = \coprod_{s, t \in S} {}_s\mathbf{JM}_t(w)$ by Lemma 3.1. Let $w_0 \in W$ be the longest element in (W, S) and let \mathcal{J} be the

two-sided Kazhdan–Lusztig cell containing the elements w_0S and consider the decomposition

$$\begin{aligned} \mathcal{J} &= \coprod_{s,t \in S} {}^s\mathcal{H}^t \\ &= \coprod_{s,t \in S} \{u \in \mathcal{J} \mid \text{the left (resp., right) descent of } z \text{ is } S \setminus s \text{ (resp., } S \setminus t)\} \end{aligned}$$

into the Kazhdan–Lusztig \mathbb{H} -cells (see Subsection 2.2). Then Theorem E, together with Propositions 2.3 and 2.6 and [Ma, Theorem 32], gives the following statement.

THEOREM F: *Let $w \in W$.*

- (a) *If $x \notin \mathcal{J} \coprod \{w_0\}$, then $\text{Ext}^1(L_x, \Delta_w) = 0$.*
- (b) *If $x = w_0$, then $\dim \text{Ext}^1(L_{w_0}, \Delta_w)$ equals the rank of the minimal parabolic subgroup containing w_0w .*
- (c) *If $x \in {}^s\mathcal{H}^t$, then*

$$(2) \quad \dim \text{Ext}^1(L_x, \Delta_w) \leq [\text{soc } \Delta_e/\Delta_b : L_x] \leq |{}_s\mathbf{JM}_t(w)|$$

where

$$b = \bigvee {}_s\mathbf{JM}_t(w).$$

In particular, $\text{Ext}^1(L_x, \Delta_w) = 0$ if ${}_s\mathbf{JM}_t(w) = \emptyset$.

For information on Theorem F(c) depending on the type of (W, S) , see Subsection 5. The number $[\text{soc } \Delta_e/\Delta_b : L_x]$ is bounded

- by 1 in type AB (see [KMM, Corollary 2] for type A where both inequalities in (c) are, in fact, the equalities),
- by 2 in type DF ,
- by 3, 4, and 6 in types E_6 , E_7 and E_8 , respectively.

Each of these bounds is achieved by $\dim \text{Ext}^1(L_x, \Delta_w)$, for some $x, w \in W$. The number $|{}_s\mathbf{JM}_t|$ can be larger, e.g., up to 21 in type E_8 , although we include it to give a purely combinatorial bound.

The paper is organized as follows: the next section contains all necessary preliminaries. Section 3 studies the join operation for Bruhat orders and then Section 4 discusses the module-theoretic interpretation of the join operation for Verma modules. Section 5 contains a case-by-case combinatorial analysis of the posets of join irreducible elements for all types and, in particular, determines the socles of Δ_e/Δ_x . Here we also provide figures illustrating the poset structure

of join irreducible elements and closed formulae for some families of Kazhdan–Lusztig polynomials in type BD . We finish the paper with an Appendix which provides tables of some families of Kazhdan–Lusztig polynomials in types E_7 and E_8 .

2. Preliminaries

2.1. HECKE ALGEBRA AND KAZHDAN–LUSZTIG BASIS. We refer to [KL, So] for more details on this subsection. A more expository reference is the book [EMTW]. Note that we use the normalization of the Hecke algebra as in [So] and [EMTW].

Let (W, S) be a finite Coxeter system. We denote by \leq the Bruhat order on W , and by $\ell : W \rightarrow \mathbb{Z}$ the associated length function. As usual, the identity element in W is denoted by e and the longest one by w_0 .

The Hecke algebra $H(W, S)$ associated to (W, S) is the $\mathbb{Z}[v, v^{-1}]$ -algebra generated by H_s , for $s \in S$, which satisfy the (Coxeter) braid relations and the quadratic relation

$$(H_s + v)(H_s - v^{-1}) = 0,$$

for all $s \in S$. Given a reduced expression $w = st \cdots u$ of $w \in W$, we let $H_w = H_s H_t \cdots H_u$. The element H_w is, in fact, independent of the choice of the reduced expression, and $\{H_w\}_{w \in W}$ is a $(\mathbb{Z}[v, v^{-1}])$ -basis of $H(W, S)$ called the **standard basis**. Now consider the $(\mathbb{Z}$ -algebra-)involution

$$\bar{} : H(W, S) \rightarrow H(W, S)$$

uniquely determined by $\bar{v} = v^{-1}$ and $\overline{H_s} = H_s^{-1}$. Then there is a unique element \underline{H}_w in $H(W, S)$ such that $\overline{\underline{H}_w} = \underline{H}_w$ and

$$(3) \quad \underline{H}_w = H_w + \sum_{y \in W} p_{y,w} H_y,$$

for some $p_{y,w} \in v\mathbb{Z}[v]$. The elements \underline{H}_w , where $w \in W$, form a basis of $H(W, S)$ called the **Kazhdan–Lusztig (KL) basis**. Given $x, y \in W$, the coefficient of v in $p_{x,y} + p_{y,x}$ is denoted by $\mu(x, y) = \mu(y, x)$, defining the **(Kazhdan–Lusztig) μ -function**. If $s \in S$, then $\underline{H}_s = H_s + v$, and

$$(4) \quad \underline{H}_s \underline{H}_y = \begin{cases} (v + v^{-1}) \underline{H}_y, & sy < y, \\ \underline{H}_{sy} + \sum_{sx < x, x < y} \mu(x, y) \underline{H}_x, & sy > y. \end{cases}$$

Another basic fact is that $p_{x,w_0} = v^{\ell(w_0) - \ell(x)}$.

2.2. THE SMALL AND PENULTIMATE KL CELLS. We call an element $s \in S$ a **left descent** of y if $sy < y$ and a **left ascent** of y if $sy > y$. **Right descent** and **right ascent** are defined similarly. We denote by $LD(y)$ and $RD(y)$, respectively, the sets of left and right descents of y , and by $LA(y)$ and $RA(y)$, respectively, the sets of left and right ascents of y . An element $w \in W$ is called **bigrassmanian** if it has a unique left and a unique right descent.

We consider the Kazhdan–Lusztig left, right, and two-sided orders on W , denoted, respectively, by \leq_L, \leq_R, \leq_J , and the associated equivalence relations \sim_L, \sim_R , and \sim_J . The corresponding equivalence classes in W are called left, right, and two-sided cells. We have an explicit description of the two smallest two-sided cells with respect to the two-sided order. The first one is $\mathcal{J}_0 = \{e\}$. The next one \mathcal{J}_1 , the so-called **small two-sided cell**, consists of all elements in W having a unique reduced expression. The left and the right cells in \mathcal{J}_1 are both indexed by S : for each $s \in S$, the elements in \mathcal{J}_1 with the right (resp., left) descent set $\{s\}$ form a left (resp., right) cell in \mathcal{J}_1 .

The μ -function is easy on \mathcal{J}_1 , see [KMMZ, KM]:

PROPOSITION 2.1: *Let $x, y \in \mathcal{J}_1$. Then $\mu(x, y) \neq 0$ if and only if x, y are adjacent in the Bruhat graph, in which case $\mu(x, y) = 1$.*

Multiplication by w_0 induces an involution on the set of two-sided cells, under which the small cell \mathcal{J}_1 corresponds to the **penultimate two-sided cell** denoted by \mathcal{J} . That is, $\mathcal{J} = w_0\mathcal{J}_1 = \mathcal{J}_1w_0$.

The left and the right cells in \mathcal{J} have unique ascents, so we index them by S . For example, \mathcal{L}^s , for $s \in S$, is the left cell that has the right ascent set $\{s\}$ (and hence the right descent set $S \setminus \{s\}$). Similarly, ${}^t\mathcal{R}$, for $t \in S$, is the right cell that has the left ascent set $\{t\}$. We index the intersections of left and right cells (called **H-cells**) by $S \times S$. An **H-cell** in \mathcal{J} is thus written as ${}^t\mathcal{H}^s = \mathcal{L}^s \cap {}^t\mathcal{R}$, and $y \in {}^t\mathcal{H}^s$ satisfies $RA(y) = \{s\}$ and $LA(y) = \{t\}$. In (Weyl) types A, D and E , all **H-cells** in \mathcal{J} are singletons. The latter property fails in types B, F and G .

PROPOSITION 2.2: *Let $x, y \in \mathcal{J}$. Then $\mu(x, y) \neq 0$ if and only if x, y are adjacent in the Bruhat graph, in which case $\mu(x, y) = 1$.*

Proof. By the Kazhdan–Lusztig inversion formula, see [KL, Section 3] (or, equivalently, by Koszul duality, see [BGS]), we have $\mu(x, y) = \mu(w_0x^{-1}, w_0y^{-1})$. Now, the claim follows from Proposition 2.1. ■

2.3. KAZHDAN–LUSZTIG COMBINATORICS, DESCENT SETS, AND VERMA MODULES. The statements in this subsection are proved in type A in [KMM, Subsection 2.2]. The general case has exactly the same proof.

PROPOSITION 2.3: *Let $x, y, z \in W$ be such that $x \geq y$ and L_z is in the socle of Δ_y/Δ_x . Then $z \in \mathcal{J}$.*

By the Kazhdan–Lusztig conjecture (proved in [BeBe, BK, EW]), the multiplicities of the graded composition factors in Δ_y isomorphic to each L_z , for $z \in \mathcal{J}$, are determined by the Kazhdan–Lusztig polynomials as

$$(5) \quad \sum_d [\Delta_y : L_z\langle -d \rangle] v^d = p_{y,z},$$

where $p_{y,z}$ is defined in (3). Since each Verma module Δ_y is canonically a submodule of Δ_e , we restrict our attention to the case $y = e$.

We denote by $\mathbf{a}: W \rightarrow \mathbb{Z}_{\geq 0}$ Lusztig’s \mathbf{a} -function, see [Lu]. The value $\mathbf{a}(w)$ does not change when w varies over a two-sided cell \mathcal{J}' . We write $\mathbf{a}(\mathcal{J}') = \mathbf{a}(w)$, for $w \in \mathcal{J}'$. By definition, the value $\mathbf{a}(w)$, for $w \in \mathcal{J}'$, describes the minimal possible degree shift d , for which there exists $u \in \mathcal{J}'$ such that Δ_e has $L_u\langle -d \rangle$ as a simple subquotient. The minimal shift is achieved exactly when u is a Duflo element (called distinguished involution in [Lu, Section 1]). It follows that

$$(6) \quad p_{e,w} \in \mathbb{Z}\{v^a \mid \mathbf{a}(w) \leq a \leq \ell(w)\},$$

where the first equality holds if and only if w is a Duflo element. Moreover, all exponents appearing in $p_{x,y}$ are of the same parity. Every Duflo element is an involution, and hence belongs to a diagonal H-cell.

PROPOSITION 2.4: *Let $x \in W$ and $s \in S$.*

- (i) *If $sx < x$, then the socle of Δ_e/Δ_x contains some L_y such that $sy > y$.*
- (ii) *If $xs < x$, then the socle of Δ_e/Δ_x contains some L_y such that $ys > y$.*

COROLLARY 2.5: *Let $w \in W$ be such that the socle of Δ_e/Δ_w is simple. Then w is bigrassmannian.*

PROPOSITION 2.6: *Assume that $x \in W$ and $y \in {}^s\mathcal{H}^t$, for $s, t \in S$, are such that L_y is in the socle of Δ_e/Δ_x . Then $s \in LD(x)$ and $t \in RD(x)$.*

The following statement is implicit in [KMM], but we provide a proof.

LEMMA 2.7: *Let $x, y \in W$, with $x \leq y$, and suppose that the socle of Δ_e/Δ_x contains a graded component X (of Δ_e) isomorphic to $L_w\langle d \rangle$, for $w \in {}^s\mathcal{H}^t$. If there exist $z \in W$ with $x \leq z \leq y$, such that $s \notin LD(z)$ or $t \notin RD(z)$ (equivalently, if $x \leq sy$ or $x \leq yt$), then $\text{soc } \Delta_e/\Delta_y$ does not contain X .*

Proof. Consider the quotients maps $\Delta_e/\Delta_y \twoheadrightarrow \Delta_e/\Delta_z \twoheadrightarrow \Delta_e/\Delta_x$. Since $s \notin LD(z)$, by Proposition 2.6, the socle of Δ_e/Δ_z does not contain any $L_w\langle d \rangle$. So X is not in the socle of Δ_e/Δ_z and thus is not in the socle of Δ_e/Δ_y either. ■

2.4. JOIN-IRREDUCIBLE ELEMENTS AND DISSECTORS. Let (P, \leq) be a partially ordered set, and let \vee denote the join operation (i.e., supremum) in P . Of course, not every subset of P needs to have a join. An element $x \in P$ is called **join-irreducible** if $x \neq \vee Y$ for any $Y \subseteq P \setminus \{x\}$. Equivalently (see [LS, Lemme 2.3]), there exists $y \in P$ such that x is minimal in $P \setminus \{z \in P : z \leq y\}$. Note that the smallest element e in P (if it exists) is not join-irreducible, since then, by convention, we have $\vee \emptyset = e$.

Denote by **JI** the set of all join-irreducible elements in P . This set forms a *base* of P in the sense of [GK, Page 284]. In more detail, denote by $\mathcal{P}(\mathbf{JI})$ the poset of all subsets of **JI** ordered by inclusion. The map $P \rightarrow \mathcal{P}(\mathbf{JI})$,

$$w \mapsto \{x \in \mathbf{JI} : x \leq w\}$$

is an isomorphism of posets onto the image, and, moreover, any subset of P with this property must contain **JI**.

For $w \in P$, denote

$$\mathbf{JM}'(w) := \text{the set of maximal elements in } \{x \in \mathbf{JI} : x \leq w\}.$$

LEMMA 2.8 ([GK, Page 284] or [Re, Proposition 9]): *For any $w \in P$, we have $w = \vee \mathbf{JM}'(w)$.*

An element $x \in P$ is called a **dissector** if P can be written as a disjoint union

$$P = \{z : z \geq x\} \coprod \{z : z \leq y\},$$

for some $y \in P$ (such y is called a **codissector** associated to x).

LEMMA 2.9 ([Re, Proposition 12]): *Every dissector is join-irreducible.*

The poset P is called **dissective** (one also says that it admits **clivage**, see [GK]) if every join-irreducible element is also a dissector.

PROPOSITION 2.10: *Suppose P is dissective, and fix $w \in P$. Then $\mathbf{JM}'(w)$ is the unique minimal set of join-irreducible elements with the property that $w = \bigvee \mathbf{JM}'(w)$.*

Proof. Assume $w = \bigvee Y$ for $Y \subseteq \mathbf{JI}$. We will prove that $\mathbf{JM}'(w) \subseteq Y$. Suppose $x \in \mathbf{JM}'(w) \setminus Y$, and let y be the codissector associated to the dissector x . Since, for $z \in Y \subseteq \{u \in \mathbf{JI} : u \leq w\}$, we cannot have $z \geq x$ by the maximality of x in the latter set $\{u \in \mathbf{JI} : u \leq w\}$, we must have $z \leq y$. Also, since $w \geq x$, we cannot have $w \leq y$. This contradicts $w = \bigvee Y$. ■

3. Join operation on Bruhat orders

In this section we collect and prove some properties of the join operation on a Coxeter group (W, S) , partially ordered with respect to the strong Bruhat order \leq .

LEMMA 3.1 ([GK, Theorems 2.5, 3.4, 4.6, and Table I]): *Every join-irreducible element in W is bigrassmanian. For finite Coxeter groups, the converse is true if and only if (W, S) is of type A, B_3 or of rank 2.*

LEMMA 3.2 ([LS] and [GK, Table I]): *A finite Coxeter group W is dissective if and only if it is of type A, B, H , or of rank 2.*

Given $w \in W$, we can find $x, \dots, y \in \mathbf{JI}$ such that $w = x \vee \dots \vee y$. We call this a **join expression** of w . One way to find a join expression is to consider $\mathbf{JM}'(w)$. We can find a potentially smaller join expression by restricting the descent sets. More precisely, for $T, U \subseteq S$, denote

$${}_T W_U := \{x \in W : LD(x) \subseteq T, RD(x) \subseteq U\}.$$

This is a poset with the order induced from the Bruhat order on W . Note that ${}_T W_U$ is the set of Bruhat minimal representatives of the two-sided cosets $W_{\tilde{T}} \backslash W / W_{\tilde{U}}$, where $W_{\tilde{T}}, W_{\tilde{U}}$ are the parabolic subgroups of W generated by $\tilde{T} = S \setminus T$ and $\tilde{U} = S \setminus U$, respectively. We would like to point out that our notation ${}_T W_U$ is different from, in particular, the notation ${}^{\tilde{T}} W^{\tilde{U}}$ used in [Re]. Also note that, for $s, t \in S$, the set ${}_{\{s\}} W_{\{t\}} \setminus \{e\}$ is exactly the set of all bigrassmannians with left descent s and right descent t . We will simplify the

notation in such a case and write

$${}_s\mathbf{BG}_t = \{{}_s\}W_{\{t\}} \setminus \{e\}.$$

For fixed simple reflections $s, t \in S$, we let

$${}_s\mathbf{JI}_t := \mathbf{JI} \cap {}_s\mathbf{BG}_t,$$

the (po)set of all join-irreducible elements with fixed descents.

For $w \in W$, define

$$\begin{aligned} \mathbf{JM}(w) := & \text{the set of Bruhat maximal elements} \\ & \text{in } \{x \in \mathbf{JI} \cap ({}_{LD(w)}W_{RD(w)}): x \leq w\}. \end{aligned}$$

LEMMA 3.3: *Let $X \subseteq {}_TW_U$ and $w \in W$ such that w is a minimal upper bound of X in W . Then $w \in {}_TW_U$. Consequently, the following diagram commutes:*

$$\begin{array}{ccc} 2^W & \xrightarrow{\vee} & W \coprod \{\text{'does not exist'}\} \\ \cap & & \cap \\ 2^{{}_TW_U} & \xrightarrow{\vee} & {}_TW_U \coprod \{\text{'does not exist'}.\} \end{array}$$

Proof. Suppose $s \in LD(w) \setminus T$. Then $sw < w$ and $sx > x$, for all $x \in X$. From $w > x$ and the lifting property of Coxeter groups, it follows that $sw > x$, for all $x \in X$, which is in contradiction with the minimality of w . This shows $LD(w) \subseteq T$. An analogous argument applies for the right descents of w . ■

By Lemma 3.3, the join operation for the poset (W, \leq) can be computed in the subposets $({}_s\mathbf{BG}_t, \leq)$. It is worth noting that, according to [Re, Proposition 32] (see also [GK, Corollary 2.8]), an element $x \in {}_sW_t$ is join-irreducible (resp. a dissector) in W if and only if it is join-irreducible (resp. a dissector) in the poset ${}_sW_t$.

LEMMA 3.4: *For any $w \in W$, we have $w = \vee \mathbf{JM}(w)$.*

Proof. According to [Re, Proposition 27 and Proposition 31], the join-irreducible elements in the poset ${}_TW_U$ (with $T = LD(w)$ and $U = RD(w)$) are exactly $\mathbf{JI} \cap {}_TW_U$. From Lemma 2.8 we have that $w = \vee \mathbf{JM}(w)$, where the join is taken inside the poset ${}_TW_U$. From Lemma 3.3 we see that the same equality also holds in the poset W . ■

LEMMA 3.5: Given $w \in W$, if $\mathbf{JM}'(w) \subseteq \mathbf{JM}(w)$ then $\mathbf{JM}'(w) = \mathbf{JM}(w)$.

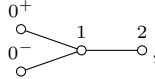
Proof. Take $x \in \mathbf{JM}(w)$ and assume that $x \notin \mathbf{JM}'(w)$. Then there is $y \in \mathbf{JI}$ such that $x < y \leq w$. Without loss of generality we can take y to be maximal with this property. This gives $y \in \mathbf{JM}'(w)$. But then $y \in \mathbf{JM}(w)$ which contradicts $x \in \mathbf{JM}(w)$. This proves $\mathbf{JM}'(w) = \mathbf{JM}(w)$. ■

COROLLARY 3.6: If W is dissective, then $\mathbf{JM}'(w) = \mathbf{JM}(w)$, for all $w \in W$.

Proof. This follows from Proposition 2.10, Lemma 3.4 and Lemma 3.5.

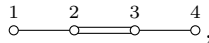
Alternatively, this also follows from the essential set description of $\mathbf{JM}'(w)$ from [KMM, Subsection 4.3] (originally introduced in [Fu] for type A , and [An, Theorem 2.2] for type B). ■

Example 3.7: Corollary 3.6 is not true in non-dissective Weyl groups. For example, in type D_4 with the following labeling of simple roots:



take $w = 0^-10^+210^-10^+$. Then we have $\mathbf{JM}(w) = \{0^-10^+210^-, 10^-210^+\}$, but $\mathbf{JM}'(w) = \mathbf{JM}(w) \cup \{10^-0^+1\}$.

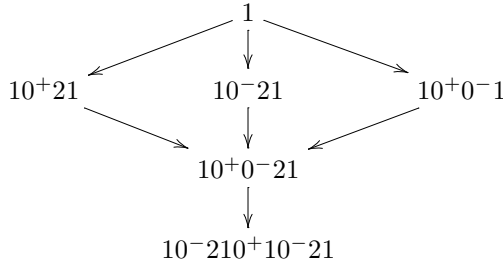
In type F_4 , with the following labeling of simple roots:



take $w = 3423123432$. In this case, we have $\mathbf{JM}(w) = \{34231234, 32341232\}$, but $\mathbf{JM}'(w) = \mathbf{JM}(w) \cup \{3234323\}$.

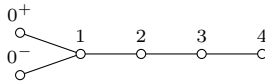
Remark 3.8: Note that $w = \bigvee \mathbf{JM}(w)$ is the unique minimal join expression of w in the dissective types (see Proposition 2.10, Lemma 3.2 and Corollary 3.6). This is not true in general. The smallest type where one can find a counterexample is D_4 . In this type, we have a unique counterexample, namely $w = 10^+0^-21$. It has $\mathbf{JM}(w) = \mathbf{JM}'(w) = \{10^+21, 10^-21, 10^+0^-1\}$. None of the latter three elements is a dissector. However, w can be written as the join of any two of them. In particular, w is a bigrassmannian which is not join-irreducible. We

give the Bruhat graph of ${}^1\mathbf{BG}_1$, which contains all the above mentioned elements:



In Subsection 5.4.2 below, we show that these three elements correspond to three distinct simple submodules of the isotypic subquotient $2L_x\langle -9 \rangle$ in Δ_e , and the sum of any two is necessarily the entire $2L_x\langle -9 \rangle$. This is the place (unique in type D_4) where the polynomial $p_{e,x}$, with $x \in \mathcal{J}$, has a coefficient bigger than one. Namely, $x = 0^+10^-210^+210^-12 \in {}^1\mathcal{H}^1$ and $p_{e,x} = v^{11} + 2v^9 + v^7$.

Example 3.9: The smallest W where $\mathbf{JM}(w) \not\subseteq \mathbf{JM}'(w)$, for some w , is of type D_6 . We denote the simple roots by



and take $w = 10^+210^-3210^+43210^-3210^+12$. Then we have

$$\begin{aligned} \mathbf{JM}(w) &= \{43210^-0^+12, 10^+210^-3210^+43210^-, \\ &\quad 10^-210^+3210^-43210^+, 10^+210^-10^+2132\}, \\ \mathbf{JM}'(w) &= \{43210^-0^+12, 10^+210^-3210^+43210^-, \\ &\quad 10^-210^+3210^-43210^+, 10^-210^+3210^-210^+321\}, \end{aligned}$$

while $\mathbf{JM}(w) \cap \mathbf{JM}'(w)$ contains only 3 elements.

Put

$$(7) \quad \mathbf{JM}''(w) := \{x \in \mathbf{JM}'(w) \mid x \in {}_{LD(w)}W_{RD(w)}\}$$

and note that $\mathbf{JM}''(w) = \mathbf{JM}(w) \cap \mathbf{JM}'(w)$. We do not know if $w = \bigvee \mathbf{JM}''(w)$ holds in general. In type D_6 (and all types of rank smaller than 6) we do have $w = \bigvee \mathbf{JM}''(w)$.

4. Join operation and Verma modules

4.1. INTERSECTION CONJECTURE. Our main conjecture is that the join expression $w = \vee \mathbf{JM}(w)$ gives rise to a join expression for the Verma module Δ_w .

CONJECTURE 4.1: For $w \in W$, we have

$$(8) \quad \Delta_w = \bigcap_{x \in \mathbf{JM}(w)} \Delta_x,$$

where the intersection is taken inside (the ungraded module) Δ_e .

Each Verma module Δ_w is uniquely determined by the socle of Δ_e/Δ_w . By Proposition 2.3, each simple subquotient of the latter socle is isomorphic to a shift of L_u , for some $u \in \mathcal{J}$. Thus, if each $L_u\langle d \rangle$ (for $u \in \mathcal{J}$ and $d \in \mathbb{Z}$) is multiplicity-free in Δ_e , then (8) is equivalent to

$$(9) \quad [\Delta_w\langle -\ell(w) \rangle : L_u\langle d \rangle] = \min_{x \in \mathbf{JM}(w)} [\Delta_x\langle -\ell(x) \rangle : L_u\langle d \rangle],$$

for all $u \in \mathcal{J}$ and $d \in \mathbb{Z}$. This reduces the module-theoretic Conjecture 4.1 to Kazhdan–Lusztig combinatorics. The multiplicity-free assumption holds in types A and B . In type A , Conjecture 4.1 follows from the results of [KMM], see Corollary 4.5. In type B , we checked (9) for small ranks on a computer.

In the next subsections, we discuss another approach to Conjecture 4.1.

4.2. SOCLE-SUM PROPERTY AND THE INTERSECTION CONJECTURE. The following proposition explains a relation between the intersection of Verma modules and the socles of the corresponding quotients.

PROPOSITION 4.2: Let $w \in W$ and $U \subset W$. Assume that the following holds:

- $x \leq w$, for any $x \in U$;
- the natural surjection $\Delta_e/\Delta_w \rightarrow \Delta_e/\Delta_x$ restricts to a surjection $\phi_x : \text{soc } \Delta_e/\Delta_w \rightarrow \text{soc } \Delta_e/\Delta_x$, for any $x \in U$;
- $\bigcap_{x \in U} \ker(\phi_x) = 0$.

Then $\bigcap_{x \in U} \Delta_x = \Delta_w$.

Proof. That $\Delta_w \subseteq \bigcap_{x \in U} \Delta_x$ follows directly from the first assumption. For the opposite inclusion, consider the quotient map $\pi : \Delta_e \twoheadrightarrow \Delta_e/\Delta_w$. We claim that the kernel of π contains $\bigcap_{x \in U} \Delta_x =: M$. Suppose not. Then $\pi(M)$ intersects the socle of Δ_e/Δ_w . If the nonzero intersection $\pi(M) \cap \text{soc } \Delta_e/\Delta_w \subseteq \ker(\phi_x)$ for all $x \in U$, then our assumption $\bigcap_{x \in U} \ker(\phi_x) = 0$ is not satisfied. Thus

there exist $x \in U$ such that $\phi_x(\pi(M) \cap \text{soc } \Delta_e/\Delta_w) \neq 0$. But this contradicts $M \subseteq \Delta_x$. Therefore, the map π factors through $\Delta_e/M \twoheadrightarrow \Delta_e/\Delta_w$, which implies that $M \subseteq \Delta_w$. ■

Note that the converse of Proposition 4.2 is not true. For example, if $x < y = w$, then $\Delta_x \cap \Delta_y = \Delta_w$. However, the map $\text{soc } \Delta_e/\Delta_y \rightarrow \text{soc } \Delta_e/\Delta_x$ is often not surjective.

Proposition 4.2 and Conjecture 4.1 motivate Definition D.

COROLLARY 4.3: *If every $w \in W$ has the socle-sum property, then Conjecture 4.1 is true for (W, S) .*

Proof. This follows from Proposition 4.2. ■

The result in [KMM] readily proves the socle-sum property and Conjecture 4.1 in type A.

THEOREM 4.4: *Let (W, S) be of type A. Then every $w \in W$ has the socle-sum property.*

Proof. Since the bigrassmanians are exactly the join-irreducibles in type A, the statement follows from the main theorem in [KMM]. ■

COROLLARY 4.5: *Conjecture 4.1 is true in type A.*

Proof. This follows from Corollary 4.3 and Theorem 4.4. ■

Note that it is crucial to take the intersection over $\mathbf{JM}(w)$ in Definition D, as we see in the following easy example.

Example 4.6: We have $w = w \vee x$, for any $x < w$. In this case, $\text{soc } \Delta_e/\Delta_x$ does not contribute to $\text{soc } \Delta_e/\Delta_w$. For example, if $w = st$ for $s, t \in S$ with $st \neq ts$, then $\text{soc } \Delta_e/\Delta_w$ is of the form L_x (up to shift) with $x \in {}^s\mathcal{H}^t$ (see Proposition 2.3 and Proposition 2.6). Writing $st = st \vee s$ we see that $\text{soc } \Delta_e/\Delta_s$, which is of the form L_x for $x \in {}^s\mathcal{H}^s$ (see Proposition 2.3 and Proposition 2.6), is not contained in $\text{soc } \Delta_e/\Delta_w$.

In contrast, Conjecture 4.1 seems to be less sensitive to the choice of a join expression. Here is a comparison with the situation in Example 4.6.

Example 4.7: We have $w = w \vee x$, for any $x < w$. In this case, we have $\Delta_w = \Delta_w \cap \Delta_x$ since $\Delta_x \supseteq \Delta_w$.

The following proposition also suggests that (8) may not require the join expression to be given by **JM**.

PROPOSITION 4.8: *If $\bigcap_{x \in U} \Delta_x = \Delta_w$, then $w = \bigvee U$.*

Proof. Suppose $\bigcap_{x \in U} \Delta_x = \Delta_w$. Then we have $x \leq w$, for each $x \in U$. If $z \in W$ is such that $x \leq z$, for all $x \in U$, then $\Delta_x \supseteq \Delta_z$, for all $x \in U$, so

$$\Delta_w = \bigcap_{x \in U} \Delta_x \supseteq \Delta_z.$$

The latter implies $w \leq z$ which proves the claim. ■

This motivates the main question of the paper (Question A).

Question 4.9: Given $U \subseteq W$, do we have $w = \bigvee U$ if and only if $\bigcap_{x \in U} \Delta_x = \Delta_w$?

Note that the join $\bigvee U$ does not exist in general, in which case $\bigcap_{x \in U} \Delta_x$ is not isomorphic to a Verma module by Proposition 4.8. Therefore, for such $U \subseteq W$, the answer to Question 4.9 is positive.

Under a combinatorial condition on the Bruhat order, the socle-sum property guarantees that the answer to Question 4.9 is positive.

PROPOSITION 4.10: *Let (W, S) be dissective and suppose Conjecture 4.1 holds for W . Then the answer to Question 4.9 is positive, for any $U \subseteq W$.*

Proof. The “if” claim follows from Proposition 4.8. To prove the “only if” part, let $U \subseteq W$ and $\bigvee U = w$. Consider the equality

$$(10) \quad \bigvee_{y \in U} \bigvee_{x \in \mathbf{JM}(y)} x = \bigvee_{z \in \mathbf{JM}(w)} z.$$

Since W is dissective, by Proposition 2.10 and Corollary 3.6, we have that $\mathbf{JM}'(w) = \mathbf{JM}(w)$ is the unique minimal set of join-irreducible elements whose join is w . It follows that the expression on the right-hand side of (10) is a subexpression of the left-hand side. Therefore, for any $y \in U$ and each $x \in \mathbf{JM}(y)$, either $x \in \mathbf{JM}(w)$ or $x \leq z$, for some $z \in \mathbf{JM}(w)$. Therefore, we have

$$(11) \quad \bigcap_{y \in U} \bigcap_{x \in \mathbf{JM}(y)} \Delta_x = \bigcap_{z \in \mathbf{JM}(w)} \Delta_z.$$

Now the validity of Conjecture 4.1 reduces (11) to $\bigcap_{y \in U} \Delta_y = \Delta_w$. ■

COROLLARY 4.11: *Let (W, S) be dissective and suppose all $w \in W$ have the socle-sum property. Then the answer to Question 4.9 is always positive.*

Proof. This follows from Corollary 4.3 and Proposition 4.10. ■

COROLLARY 4.12: *Let (W, S) be of type A. Then, for any $U \subseteq W$, we have $\bigvee U = w \in W$ if and only if $\bigcap_{x \in U} \Delta_x = \Delta_w$.*

Proof. This follows from Theorem 4.4, Lemma 3.2, and Proposition 4.10. ■

As we will see later in Example 5.42, in the situation when W is not dissective, the answer to Question 4.9 can be negative, for some $U \subseteq W$. Therefore, we ask a more precise question:

Question 4.13: Given $w \in W$, which $U \subseteq W$ satisfying $w = \bigvee U$ have the property $\bigcap_{x \in U} \Delta_x = \Delta_w$?

4.3. ANOTHER INTERPRETATION OF QUESTION 4.9. Let Sub be the poset of all submodules of Δ_e where the poset structure is given by inclusion. Then we have an embedding of posets

$$(12) \quad \Delta : (W, \leq) \rightarrow (\text{Sub}, \subseteq)^{\text{op}}$$

given by

$$w \mapsto \Delta_w \subseteq \Delta_e.$$

In $(\text{Sub}, \supseteq) = (\text{Sub}, \subseteq)^{\text{op}}$, arbitrary joins exist, since joins are given by intersections. If a join of some elements in the image $\Delta(W, \leq)$ lies in $\Delta(W, \leq)$ again, then the join exists already in (W, \leq) . While the above statement is true for any poset inclusion, the converse, that is, whether joins are mapped to joins under the inclusion, is not true in general.

Let CSub be the subset of Sub consisting of the submodules of Δ_e with simple head. The join operation on (Sub, \supseteq) restricts to that in (CSub, \supseteq) , that is, the meet in Sub restricts to CSub , in the following sense.

PROPOSITION 4.14: *For $P \subseteq \text{CSub}$, the meet $\bigwedge P$ exists if and only if $\bigcap_{M \in P} M$ has simple head, in which case we have $\bigwedge P = \bigcap_{M \in P} M$.*

Proof. The “if” statement is clear. Conversely, if the head of $\bigcap_{M \in P} M$ is not simple, then there are two incomparable submodules $X, Y \subset \bigcap_{M \in P} M$ with simple heads, none of which is properly contained in any other submodule of $\bigcap_{M \in P} M$ with simple head. If the meet exists, then it satisfies $\bigcap_{M \in P} M \supseteq \bigwedge P \supseteq X, Y$, contradicting the maximality of X, Y . ■

COROLLARY 4.15: *The answer to Question 4.9 is positive, for every $U \subseteq W$, if and only if the join operation on CSub^{op} restricts to the join operation on W .*

Proof. Consider the following diagram where the downward map

$$\text{Sub}^{\text{op}} \rightarrow \text{CSub}^{\text{op}} \coprod \{\text{'does not exist'}\}$$

is given by $M \mapsto M$, if $M \in \text{CSub}$, and $M \mapsto \text{'does not exist'}$, otherwise. The map below the latter is given similarly.

$$\begin{array}{ccc} 2^{\text{Sub}^{\text{op}}} & \xrightarrow{\vee} & \text{Sub}^{\text{op}} \\ \cap & & \uparrow \\ 2^{\text{CSub}^{\text{op}}} & \xrightarrow{\vee} & \text{CSub}^{\text{op}} \coprod \{\text{'does not exist'}\} \\ \cap & & \uparrow \\ 2^W & \xrightarrow{\vee} & W \coprod \{\text{'does not exist'}\} \end{array}$$

Conjecture 4.9 says that the big square commutes. Proposition 4.14 says that the upper square commutes. It follows that Conjecture 4.9 is equivalent to the lower square being commutative, which is our claim. ■

Intuitively, the poset CSub is isomorphic to the poset of simple subquotients of Δ_e where the partial order is given by (some kind of) generation. Let us make this statement more general and precise. For $M \in \mathcal{O}$, consider the **set of subquotients** of M given by

$$\text{SQ}(M) = \{(X, Y) \mid Y \subseteq \text{rad } X \subseteq M\},$$

where one thinks of (X, Y) as the subquotient X/Y of M . Then $\text{SQ}(M)$ is a poset with the partial order

$$(X, Y) \leq (Z, V) \iff X \subseteq Z \text{ and } Y \subseteq V,$$

which should be thought of as the induced map $X/Y \rightarrow Z/V$. (Here we are not considering all subquotients, for example, we only consider ‘quotients of submodules’ and not ‘submodules of quotients’. The zero subquotients are also not included in $\text{SQ}(M)$. But the above definition suffices for our purpose and allows us to avoid lengthy discussions.) The **poset of simple subquotients** of M is the subposet $\text{SSQ}(M)$ in $\text{SQ}(M)$ consisting of the pairs (X, Y) such that $\text{hd } X$ is simple and $Y = \text{rad } X$ (so that X/Y is simple). Then $\text{CSub}(M)$ (the submodules of M with simple head) is isomorphic to $\text{SSQ}(M)$, the isomorphism being given by $X \mapsto (X, \text{rad } X)$.

Note that in $\text{SSQ}(M)$ we have $(X, Y) \leq (Z, V)$ if and only if $X \subseteq Z$, since the second condition $Y \subseteq V$ is automatic. We thus see that the partial order on $\text{SSQ}(M)$ is alternatively given by $(X, Y) \leq (Z, V)$ if and only if there is a composition series of the form $0 \subsetneq \cdots \subsetneq X \subsetneq \cdots \subsetneq Z$, which is the most natural partial order that can be given to the simple subquotients of M .

Taking $M = \Delta_e$ and relating to the poset (W, \leq) , we have the diagram of posets

$$(13) \quad (W, \leq) \xrightarrow{\Delta} (\text{CSub}, \supseteq) \xrightarrow[\simeq]{\text{hd}} (\text{SSQ}, \geq) \xrightarrow{\text{deg}} (Z, \leq),$$

where the first map is induced from (12) (where (CSub, \supseteq) denotes the opposite poset of CSub and similarly for SSQ) and the last map takes the minimal degree of a (not necessarily graded) simple subquotient in the graded module Δ_e . The composition in (13) gives the length function on (W, S) .

The discussion above proves the following reformulation of Corollary 4.15.

COROLLARY 4.16: *The answer to Question 4.9 is positive, for every $U \subseteq W$, if and only if the join operation on SSQ restricts to the join operation on W , i.e., if and only if the following diagram commutes.*

$$\begin{array}{ccc} 2^{\text{SSQ}^{\text{op}}} & \xrightarrow{\vee} & \text{SSQ}^{\text{op}} \coprod \{\text{'does not exist'}\} \\ \cap & & \uparrow \\ 2^W & \xrightarrow{\vee} & W \coprod \{\text{'does not exist'}\} \end{array}$$

We note that Corollary 4.16 is non-trivial. Indeed, it follows from Corollary 4.16 and Example 5.42 below that the join operation on SSQ or CSub does not restrict to the join expression on W in type E .

4.4. HALF SOCLE-SUM PROPERTY VIA COMBINATORICS. Here we give a criterion for the socle-sum property to hold in one direction. Type by type analysis in the next section will prove that the criterion is always satisfied, hence establishing Theorem E.

LEMMA 4.17: *Let $s, t \in S$ and $m := \sum_{u \in {}^s \mathcal{H}^t} p_{e,u}(1)$. Suppose there exists a chain*

$$x_1 < \cdots < x_m$$

in ${}^s \mathbf{BG}_t$. Then every composition factor of Δ_e isomorphic to L_u with $u \in {}^s \mathcal{H}^t$ is contained in the sum of $\text{soc } \Delta_e / \Delta_y$, taken over all $y \in {}^s \mathbf{J}\mathbf{I}_t$.

By ‘ X is contained in the sum of socles of Δ_e/Δ_{y_i} ’ we mean the following: Given a maximal isotypic subquotient $X \cong cL_x\langle d \rangle$ in Δ_e , let M be the submodule of Δ_e generated by X , so that $\text{hd } M = X$. Then there are some $y_i \in {}_s\mathbf{J}\mathbf{I}_t$ so that $\Delta_{y_i} \subseteq \text{rad } M$ and the kernels of the induced maps

$$\psi_i : X \hookrightarrow \text{soc}(\Delta_e/\text{rad } M) \rightarrow \text{soc}(\Delta_e/\Delta_{y_i})$$

have trivial intersection in X .

It is more convenient to formulate the dual statement. Let K_x be the kernel of the canonical surjection $\nabla_e \rightarrow \nabla_x$, so that a statement about $\text{soc } \Delta_e/\Delta_x$ is dual to a statement about $\text{hd } K_x$. Let Y be a graded isotypic component in ∇_e corresponding to an element in $\mathcal{H}(a, b)$ and $N \subset \nabla_e$ be the submodule generated by Y . Then the statement in the previous paragraph is equivalent to

$$\bigoplus K_y \xrightarrow{\oplus \eta_y} N$$

being surjective. Here y runs over all elements in ${}_a\mathbf{J}\mathbf{I}_b$ for which the map $K_y \hookrightarrow \nabla_e$ restricts to $\eta_i : K_y \hookrightarrow N$.

Proof. By Proposition 2.3 and Proposition 2.6, the numbers

$$\sum_{u \in {}^s\mathcal{H}^t} [\Delta_e/\Delta_{x_i} : L_u]$$

strictly increase along $x_1 < \dots < x_m$, and moreover, each component of the form L_u with $u \in {}^s\mathcal{H}^t$ in $\Delta_{x_{i-1}}/\Delta_{x_i}$ belongs to $\text{soc } \Delta_e/\Delta_{x_i}$ under

$$\Delta_{x_{i-1}}/\Delta_{x_i} \hookrightarrow \Delta_e/\Delta_{x_i}.$$

Thus the length of the sum of $\text{soc } \Delta_e/\Delta_{x_1}, \dots, \text{soc } \Delta_e/\Delta_{x_i}$ strictly increases with i . In particular, the sum of $\text{soc } \Delta_e/\Delta_{x_1}, \dots, \text{soc } \Delta_e/\Delta_{x_m}$ has length at least m . But, by Proposition 2.6 and (5), the length is at most m . The claim follows. ■

PROPOSITION 4.18: *Suppose that, for each $s, t \in S$, every composition factor of Δ_e isomorphic to L_u , with $u \in {}^s\mathcal{H}^t$, is contained in the sum of $\text{soc } \Delta_e/\Delta_y$ taken over $y \in {}_s\mathbf{J}\mathbf{I}_t$. Then, for each $w \in W$, we have*

$$\bigcap_{y \in \mathbf{J}\mathbf{M}(w)} \ker\{\text{soc } \Delta_e/\Delta_w \xrightarrow{\phi_y} \text{soc } \Delta_e/\Delta_y\} = 0,$$

where the maps ϕ_y are induced by $\Delta_y \subseteq \Delta_w$. In other words, the socle of Δ_e/Δ_w is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ taken over $x \in \mathbf{J}\mathbf{M}(w)$.

Proof. Suppose not. Then we have a simple socle component $X \subseteq \text{soc } \Delta_e/\Delta_w$ such that $\phi_y(X) = 0$, for all $y \in \mathbf{JM}(w)$. By Proposition 2.3, it is isomorphic to (some shift of) $L_x \in \mathcal{J}$. Let a and b be the left and right ascents of x , respectively, i.e., $x \in {}^a\mathcal{H}^b$. Then the assumption provides $y_i \in {}_a\mathbf{J}\mathbf{I}_b$ such that the sum of $\text{soc } \Delta_e/\Delta_{y_i}$ contains X . By replacing $y_i \leq w$ by $y'_i \in \mathbf{JM}(w)$ with $y_i \leq y'_i$, we get a contradiction. ■

PROPOSITION 4.19: *Suppose, for each $s, t \in S$, there exists a chain in ${}_s\mathbf{BG}_t$ of length $\sum_{u \in {}^s\mathcal{H}^t} p_{e,u}(1)$. Then, for each $w \in W$, we have*

$$\bigcap_{y \in \mathbf{JM}(w)} \ker\{\text{soc } \Delta_e/\Delta_w \xrightarrow{\phi_y} \text{soc } \Delta_e/\Delta_y\} = 0,$$

where the maps ϕ_y are induced by $\Delta_y \subseteq \Delta_w$. In other words, the socle of Δ_e/Δ_w is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ taken over $x \in \mathbf{JM}(w)$.

Proof. This follows from Lemma 4.17 and Proposition 4.18. ■

The proof of Corollary 4.19 is also easier to make precise in the dual setting. The dual statement of Lemma 4.17 gives that $\bigoplus_{y_i} K_{y_i} \twoheadrightarrow M$, for some $y_i \in {}_a\mathbf{J}\mathbf{I}_b$, where M is the submodule of K_w generated by a head component isomorphic to L_x , for $x \in \mathcal{H}(a, b)$. Then, taking $y'_i \in \mathbf{JM}(w)$ with $y'_i > y_i$, we factor the above surjective map as $\bigoplus_{y_i} K_{y_i} \rightarrow \bigoplus_{y'_i} K_{y'_i} \xrightarrow{\eta} M$. This implies that η is surjective, as desired.

4.5. SOCLE-SUM PROPERTY AND \mathbf{JM}'' . Recall the set $\mathbf{JM}''(w)$, for $w \in W$, defined in (7). One could ask whether $\mathbf{JM}''(w)$ might be a better join expression than $\mathbf{JM}(w)$ for the socle-sum property. However, Example 5.50 gives an example of $w \in W$ with $\mathbf{JM}(w) \supsetneq \mathbf{JM}''(w)$ such that

$$\text{soc } \Delta_e/\Delta_w = \sum_{z \in \mathbf{JM}''(w)} \text{soc } \Delta_e/\Delta_z \subsetneq \sum_{z \in \mathbf{JM}(w)} \text{soc } \Delta_e/\Delta_z.$$

Also, Remark 5.25 shows that the socle-sum property does not hold, in general, if we replace \mathbf{JM} by \mathbf{JM}'' either. In fact, Remark 5.25 shows that there is no join expression such that the socle-sum property always holds, since it provides a non-join-irreducible element $f \in W$ such that

$$\text{soc } \Delta_e/\Delta_f = L\langle d \rangle \oplus L'\langle d' \rangle$$

while there is no $z \in W$ with $\text{soc } \Delta_e/\Delta_z \cong L\langle d \rangle$. We also do not know whether \mathbf{JM}'' provides a join expression, i.e., whether $\bigvee \mathbf{JM}''(w) = w$ is true in general.

5. Join-irreducibles and Verma modules

The goal of this section is to explicitly describe $\text{soc } \Delta_e / \Delta_y$, for $y \in \mathbf{JI} \subset W$, by closely studying combinatorial aspects of (W, S) . Our first ingredient is the Kazhdan–Lusztig combinatorics. We explicitly compute $p_{e,y}$, for $y \in \mathcal{J}$, which determines the multiplicities of the composition subquotients of Δ_e of the form $L_y \langle d \rangle$. The second ingredient is the Bruhat order on W . We determine the subposet ${}_s \mathbf{JI}_t \subset W$, for each $s, t \in S$, and match them with the Kazhdan–Lusztig polynomials.

We collect some general lemmas in Subsection 5.1 and proceed to Weyl type analysis in the following subsections.

5.1. STRATEGIES.

5.1.1. *Useful equations in KL polynomials.*

LEMMA 5.1: *Let $y \in \mathcal{J}$, $z \in W$ and $s \in S$. If $z < sz$ and $z \neq y, sy$, then we have*

$$p_{sz,y} = \begin{cases} v^{-1} \cdot p_{z,y}, & sy < y, \\ p_{z,sy} + \sum_{\substack{t \in S \\ ty \sim_{\mathbf{L}} y \\ ty < y}} p_{z,ty} - v \cdot p_{z,y}, & sy > y, \end{cases}$$

if $z < zs$ and $z \neq y, ys$, then we have

$$p_{zs,y} = \begin{cases} v^{-1} \cdot p_{z,y}, & ys < y, \\ p_{z,ys} + \sum_{\substack{t \in S \\ yt \sim_{\mathbf{R}} y \\ yt < y}} p_{z,yt} - v \cdot p_{z,y}, & ys > y. \end{cases}$$

Proof. We prove the first case. The second case has the same proof. In Equation (4) we take the coefficients next to H_z on both sides. Note that all x appearing in the sum there satisfy $x \sim_{\mathbf{L}} y$, by the definition of the order $\leq_{\mathbf{L}}$ and the penultimate cell \mathcal{J} . Proposition 2.2 finishes the proof. ■

We record separately the case $z = e$.

LEMMA 5.2: *Suppose $y \in \mathcal{J}$ and $s \in S$ are such that $sy > y$. Then we have*

$$(14) \quad v \cdot p_{e,y} + p_{s,y} = p_{e,sy} + \sum_{\substack{u \in S \\ uy \sim_{\mathbf{L}} y \\ uy < y}} p_{e,uy}.$$

If $y \in \mathcal{J}$ and $s \in S$ are such that $ys > y$, then we have

$$(15) \quad v \cdot p_{e,y} + p_{s,y} = p_{e,ys} + \sum_{\substack{u \in S \\ yu \sim_R y \\ yu < y}} p_{e,yu}.$$

Moreover, if $y \in {}^i\mathcal{H}^j$ with $i \neq j$ in either case, then both sides of (14) or (15) are equal to $(v + v^{-1}) \cdot p_{e,y}$.

5.1.2. *Socle-killing combinatorics.* By Lemma 2.7, the following combinatorial property plays an important role in determining socles.

Definition 5.3: A relation $x < y$, with $y \in {}_s\mathbf{J}\mathbf{I}_t$, is called **socle-killing** if either $x \leq sy$ or $x \leq yt$ holds. A chain $x_0 < x_1 < \dots < x_m$ in ${}_s\mathbf{J}\mathbf{I}_t$ is called **socle-killing** if the pair $x_i < x_{i+1}$ is socle-killing, for each $i = 0, 1, \dots, m - 1$.

To bound the degrees in which the socle of Δ_e/Δ_x can appear, we start with the following lemma. Here we denote by $\max.\text{deg. } M$, for a graded module $M = \bigoplus_{d \in \mathbb{Z}} M_d$, the maximal value of $d \in \mathbb{Z}$ such that $M_d \neq 0$.

LEMMA 5.4: *Let $x, y \in W$.*

- (i) *If $x \leq y$, then $\max.\text{deg. } \Delta_e/\Delta_x \leq \max.\text{deg. } \Delta_e/\Delta_y$. If, moreover, $\max.\text{deg. } \Delta_e/\Delta_x = \max.\text{deg. } \Delta_e/\Delta_y$, then the maximal degree component of Δ_e/Δ_x is contained in the maximal degree component of Δ_e/Δ_y .*
- (ii) *If y is join-irreducible, and the relation $x < y$ is socle-killing, then we have*

$$\max.\text{deg. } \Delta_e/\Delta_x < \max.\text{deg. } \Delta_e/\Delta_y.$$

- (iii) *If all the composition factors of $\text{soc } \Delta_e/\Delta_x$ appear in Δ_e/Δ_y , then $x \leq y$.*

Proof. The first two claims follow directly from the definitions. To prove the last claim, assume that $x \not\leq y$. Then the composition factor L_y that comes from the top of Δ_y , appears in Δ_e/Δ_x . So, the image of Δ_y in Δ_e/Δ_x has a non-trivial intersection with the socle of Δ_e/Δ_x . This is a contradiction to our assumption. ■

Let us introduce some auxiliary notation. For a fixed penultimate H-cell ${}^s\mathcal{H}^t$, write

$$(16) \quad p_{st} := \sum_{w \in {}^s\mathcal{H}^t} p_{e,w} = c_0 v^{d_0} + c_1 v^{d_1} + \dots + c_r v^{d_r}.$$

Here $c_i = c_i(s, t) \neq 0$ and, furthermore, $d_i = d_i(s, t) \in \mathbb{N}$ form a strictly increasing sequence. In particular, the number $r + 1 = r(s, t) + 1$ is the number of homogeneous terms in p_{st} . For $y \in {}_s\mathbf{J}\mathbf{I}_t$, define $\text{skal}(y)$ (here “skal” stands for **socle killing above length**) to be the maximal length of all socle-killing chains in ${}_s\mathbf{J}\mathbf{I}_t$ ending at y , i.e., the maximal m such that there exists a socle-killing chain $y_0 < \dots < y_m = y$ with $y_i \in {}_s\mathbf{J}\mathbf{I}_t$. Similarly, define $\text{skbl}(y)$ (**socle killing below length**) to be the maximal length of the socle-killing chains in ${}_s\mathbf{J}\mathbf{I}_t$ starting from y .

LEMMA 5.5: *Let $y \in {}_s\mathbf{J}\mathbf{I}_t$. Then we have*

$$d_{\text{skal}(y)} \leq \max. \text{deg. soc } \Delta_e / \Delta_y \leq d_{(r - \text{skbl}(y))}.$$

Proof. This follows from Lemma 5.4. ■

The minimal degree is not bounded by socle-killing chains on the nose but can be determined inductively by the following lemma.

LEMMA 5.6: *Let $y \in {}_s\mathbf{J}\mathbf{I}_t$ and suppose that the sum of $\text{soc } \Delta_e / \Delta_x$, taken over all $x \in {}_s\mathbf{J}\mathbf{I}_t$ such that $x < y$ is socle-killing, contains all subquotients in Δ_e isomorphic to $L_w \langle -d_i \rangle$ with $w \in {}^s\mathcal{H}^t$ and $i \leq m$. Then we have*

$$d_m < \min. \text{deg. soc } \Delta_e / \Delta_y.$$

Proof. This follows from Proposition 2.6 and Lemma 2.7. ■

In particular, if $c_i = 1$, for all $i < \text{skal}(y)$, then we have

$$d_{\text{skal}(y)} \leq \min. \text{deg. soc } \Delta_e / \Delta_y \leq \max. \text{deg. soc } \Delta_e / \Delta_y \leq d_{(r - \text{skbl}(y))},$$

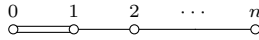
where the first inequality comes from Lemma 5.6 and the last one from Lemma 5.5. If, moreover, y belongs to a socle-killing chain of length r , i.e., if

$$\text{skal}(y) + \text{skbl}(y) = r,$$

then $\text{soc } \Delta_e / \Delta_y$ is homogeneous.

5.2. TYPE A. In type A, the join-irreducibles in (W, S) are exactly the bigrassmannians in (W, S) , see Lemma 3.1. The relevant Kazhdan–Lusztig polynomials are computed in [KMM, Proposition 12] and a complete description of the socles of the cokernels of inclusions of Verma modules is given in [KMM].

5.3. TYPE *B*. Let (W, S) be of type B_{n+1} . We use the following labeling of S :



We have $\ell(w_0) = (n + 1)^2$. The element w_0 is central, and the multiplication by w_0 gives rise to a natural bijection between the H -cells in the small two-sided cell and the H -cells in the penultimate two-sided cell \mathcal{J} , which preserves diagonal H -cells. Denote by $s_{ij} \in W$ the product $i \cdots j$ of simple reflections along the unique shortest path starting in i and ending in j in the Dynkin diagram. Put also

$$\begin{aligned} t_{ij} &:= s_{i1} \cdot 0 \cdot s_{1j} = i(i - 1) \cdots 101 \cdots (j - 1)j, & \text{if } i \neq 0 \text{ or } j \neq 0, \\ t_{00} &:= 010, \\ w_{ij} &:= s_{ij} \cdot w_0, \\ u_{ij} &:= t_{ij} \cdot w_0. \end{aligned} \tag{17}$$

The H -cell ${}^i\mathcal{H}^j$ is equal to the two-element set $\{u_{ij}, w_{ij}\}$, if both $i, j \neq 0$ or $i = j = 0$, and to the singleton $\{u_{ij} = w_{ij}\}$, otherwise. We present the Bruhat graph of the penultimate cell \mathcal{J} in Figure 1. There, the cell ${}^i\mathcal{H}^j$ is the gray square placed in the i -th row and j -th column. The left cell \mathcal{L}^j consists of all gray squares in the j -th column, and the right cell ${}^i\mathcal{R}$ consists of all gray squares in the i -th row.

5.3.1. *Some Kazhdan–Lusztig computation in type B.* Note that $u_{nn} \in {}^n\mathcal{H}^n$ is the maximal element in the parabolic subgroup generated by

$$I = \{0, 1, \dots, n - 1\} \subset S.$$

From this, we have

$$\mathbf{a}(\mathcal{J}) = \mathbf{a}(u_{nn}) = \ell(u_{nn}) = \ell(w_0) - 2n - 1 = n^2.$$

PROPOSITION 5.7: *Let $y \in {}^i\mathcal{H}^j \subset \mathcal{J}$. If $i = n$ or $j = n$, then $p_{e,y} = v^{\ell(y)}$.*

Proof. The equality $p_{e,u_{nn}} = v^{n^2}$ follows from the fact that u_{nn} is the maximal element in a parabolic subgroup. Moreover, we have $p_{e,u_{n-1,n}} = v^{n^2+1}$, because of the formulae $\ell(u_{n-1,n}) = n^2 + 1$ and $\mathbf{a}(u_{n-1,n}) = n^2$, Equation (6) and the parity condition. From Lemma 5.2, we get a recursion $(v + v^{-1})p_{e,b} = p_{e,a} + p_{e,c}$, for any consecutive $a \rightarrow b \rightarrow c$ in the chain

$$u_{nn} \rightarrow u_{n-1,n} \rightarrow \cdots \rightarrow u_{0n} \rightarrow w_{1n} \rightarrow w_{2n} \rightarrow \cdots \rightarrow w_{nn}$$

(see Figure 1).

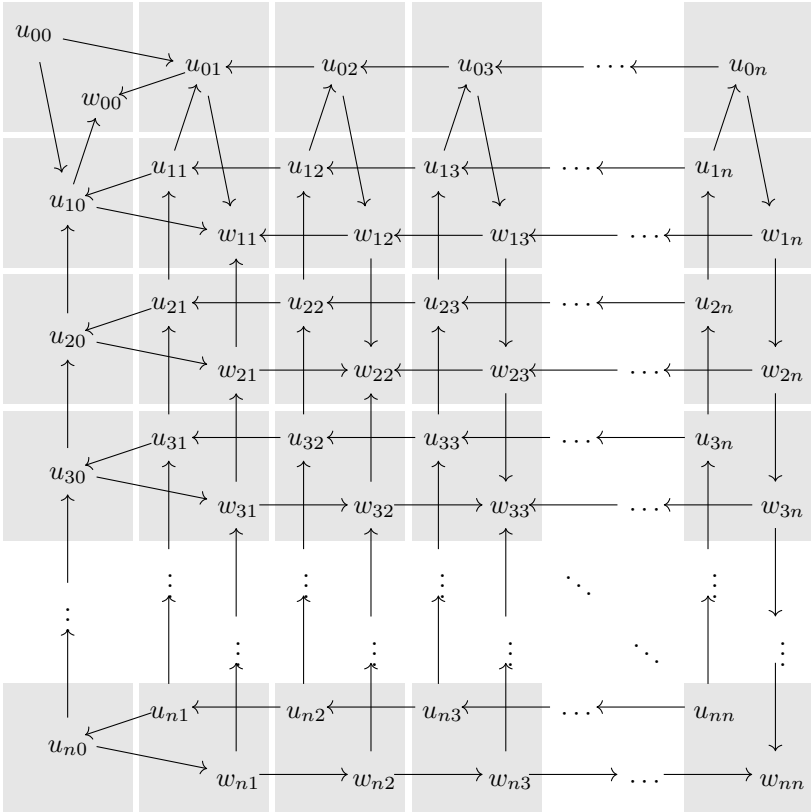


Figure 1. Bruhat graph of the penultimate two-sided cell in type B_{n+1} .

By a two-step induction, we get the claim of the proposition in the case $j = n$. In the case $i = n$, the claim follows by recalling that $(\mathcal{L}^n)^{-1} = {}^n\mathcal{R}$, and that $p_{e,w} = p_{e,w^{-1}}$. ■

A similar computation to Proposition 5.7 (as well as a part of Proposition 5.8 below) is made in [Bo] which is given a geometric interpretation in [Pe].

PROPOSITION 5.8: *Let $y \in {}^i\mathcal{H}^j \subseteq \mathcal{J}$ be such that $(i, j) \neq (0, 0)$. Then we have*

$$(18) \quad p_{e,y} = v^{\ell(y)} + v^{\ell(y)-2} + \dots + v^{\ell(y)-2d(y)}, \quad \text{where } d(y) := \min(n-i, n-j).$$

Proof. The case $d(y) = 0$ is considered in Proposition 5.7, which gives us the basis of the induction.

Take $y \in {}^i\mathcal{H}^j$ with $d := d(y) \in \{1, \dots, n-1\}$, and assume that the assertion of the proposition is true for all $y' \in \mathcal{J}$, for which $d(y') < d$. Assume that $i = n-d$ and $j \leq n-d$ (the opposite case will follow by symmetry). Note that

$$y' := (i+1) \cdot y \in {}^{i+1}\mathcal{H}^j$$

(see Equation (17) and Figure 1). Applying Lemma 5.2 to y' (observe that $i+1 \neq j$), we get

$$(v + v^{-1}) \cdot p_{e,y'} = \begin{cases} p_{e,y}, & d = 1, \\ p_{e,y} + p_{e,y''}, & d \geq 2, \end{cases}$$

for

$$y'' := (i+2) \cdot y' \in {}^{i+2}\mathcal{H}^j.$$

Applying the inductive assumption to $p_{e,y'}$ and $p_{e,y''}$, we get that $p_{e,y}$ has the desired form. ■

We now consider the remaining cases, namely the two-sided cell ${}^0\mathcal{H}^0$, which has the following two elements: $w_{00} = 0w_0$ and $u_{00} = 010w_0$ (note that here the symbols 0 and 1 denote the simple reflections corresponding to the respective vertices of the Dynkin diagram).

PROPOSITION 5.9: *We have*

$$(19) \quad p_{e,w_{00}} + p_{e,u_{00}} = v^{\ell(w_0)-1} + v^{\ell(w_0)-3} + \dots + v^{\ell(w_0)-2n-1}.$$

More specifically, if $n+1$ is odd, then w_{00} is the Duflo element in ${}^0\mathcal{H}^0$ and we have

$$(20) \quad \begin{aligned} p_{e,w_{00}} &= v^{\ell(w_0)-1} + v^{\ell(w_0)-5} + \dots + v^{\ell(w_0)-2n-1}, \\ p_{e,u_{00}} &= v^{\ell(w_0)-3} + v^{\ell(w_0)-7} + \dots + v^{\ell(w_0)-2n+1}; \end{aligned}$$

if $n+1$ is even, then u_{00} is the Duflo element in ${}^0\mathcal{H}^0$ and we have

$$(21) \quad \begin{aligned} p_{e,w_{00}} &= v^{\ell(w_0)-1} + v^{\ell(w_0)-5} + \dots + v^{\ell(w_0)-2n+1}, \\ p_{e,u_{00}} &= v^{\ell(w_0)-3} + v^{\ell(w_0)-7} + \dots + v^{\ell(w_0)-2n-1}. \end{aligned}$$

Proof. Applying Lemma 5.2 to $y = u_{10}$ and $sy = w_{00}$ (see also Equation (17) and Figure 1), we get the equation

$$(v + v^{-1}) \cdot p_{e,u_{10}} = p_{e,w_{00}} + p_{e,u_{00}} + p_{e,u_{20}}.$$

Plugging in $p_{e,u_{10}}$ and $p_{e,u_{20}}$ from Proposition 5.8, we obtain (19).

Note that $x \in \mathcal{J}$ is a Duflo element if and only if $p_{e,x}$ contains the term $v^{\ell(w_0)-2n-1}$, so it remains to prove Formulae (20) and (21).

Applying Lemma 5.2 to $y = w_{00}$, $s = 0$, and, respectively, to $y = u_{00}$, $s = 0$, we get

$$(22) \quad v \cdot p_{e,w_{00}} + p_{0,w_{00}} = v^{\ell(w_0)} + v^{\ell(w_0)-2} + v^{\ell(w_0)-4} + \dots + v^{\ell(w_0)-2n},$$

$$(23) \quad v \cdot p_{e,u_{00}} + p_{0,u_{00}} = v^{\ell(w_0)-2} + v^{\ell(w_0)-4} + \dots + v^{\ell(w_0)-2n}.$$

From (5) and the fact that $\Delta_s \langle -1 \rangle \subset \Delta_e$, it follows that, for $s \in S$, $x \in W$, the polynomial $p_{e,x} - v \cdot p_{s,x}$ has non-negative coefficients. In particular, we have

$$(24) \quad p_{s,x}(1) \leq p_{e,x}(1).$$

Assume that $n = 2k$, the other case being similar and therefore omitted. By evaluating $v = 1$ in (22) and (23), and using (24), it follows that $p_{e,w_{00}}(1) \geq k+1$, and that $p_{e,u_{00}}(1) \geq k$. Combining this with (19), we see that $p_{e,w_{00}}(1) = k+1$, and that $p_{e,u_{00}}(1) = k$. Going back to (23), we see that $p_{0,u_{00}}(1) = k$. So the polynomial

$$p_{e,u_{00}} - v \cdot p_{0,u_{00}},$$

with non-negative coefficients, vanishes at $v = 1$. It follows that

$$p_{e,u_{00}} = v \cdot p_{0,u_{00}}.$$

This makes (23) into

$$(v + v^{-1}) \cdot p_{e,u_{00}} = v^{\ell(w_0)-2} + v^{\ell(w_0)-4} + \dots + v^{\ell(w_0)-4k},$$

which, in turn, gives (20). ■

We visualize the results in Figure 2 by depicting each simple subquotient $L_w \langle -k \rangle$ of Δ_e , $w \in {}^i\mathcal{H}^j$, as $(i, j, k) \in \mathbb{Z}^3$. If $|{}^i\mathcal{H}^j| = 1$ or if w is the shorter one of the two elements there, we denote this point by \circ ; otherwise we denote it by \star . Note that both \circ and \star can appear at the same coordinate. The bottom of the picture consists of the points that come from the elements $w_{ii} = i \cdot w_0$, for $i \in S$, and the top of the picture comes from the Duflo elements.

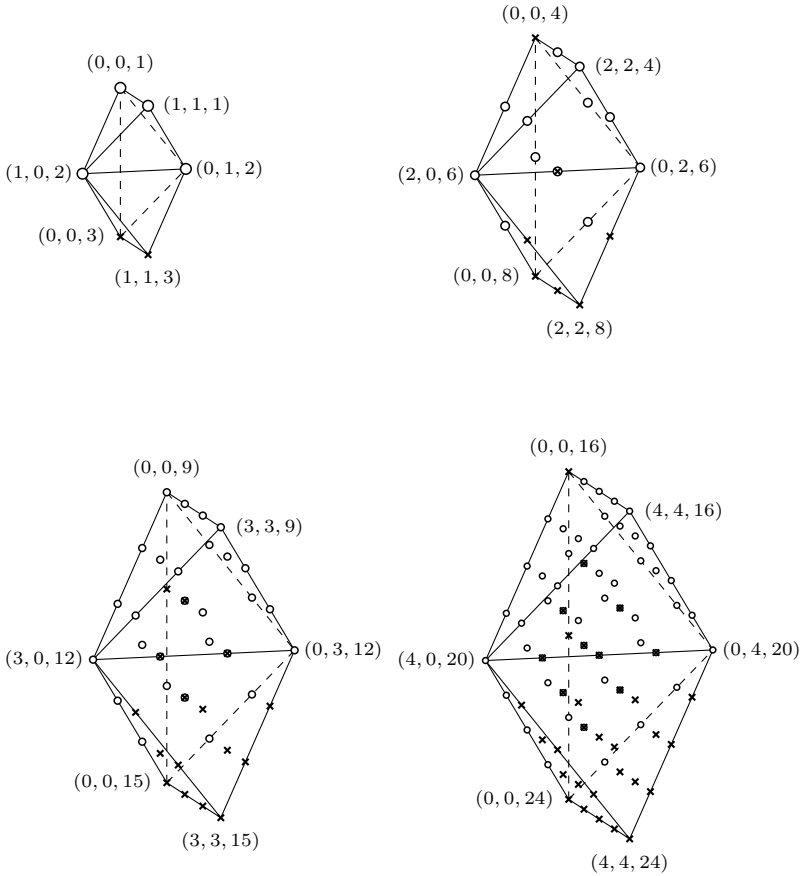


Figure 2. Composition factors of Δ_e from the penultimate cell, for $n = 1, 2, 3, 4$.

Remark 5.10: The number of graded composition factors in Δ_e isomorphic to L_w , for $w \in \mathcal{J}$, adds up to the octahedral number

$$\frac{(2n^2 + 4n + 3)(n + 1)}{3}.$$

If we depict $L_{w_{ij}}\langle -k \rangle$, for $i, j > 0$, as a point with the coordinates

$$(n + 1 - i, n + 1 - j, k),$$

we indeed get a “combinatorial” octahedron inscribed in the cuboid

$$[0, n] \times [0, n] \times [n^2, n^2 + 2n].$$

We present it in Figure 3 for small ranks.

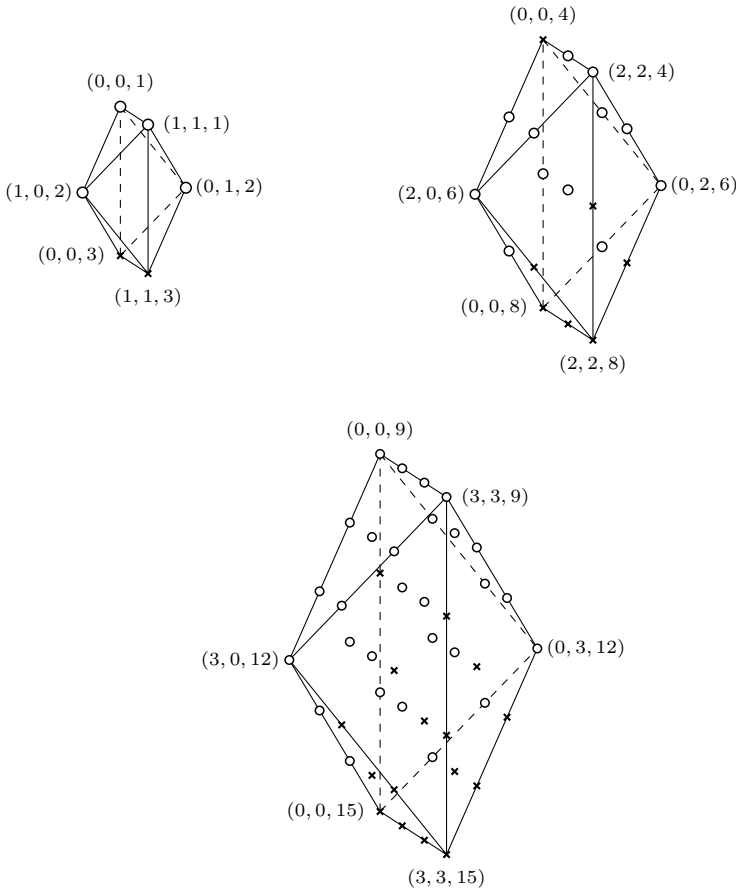
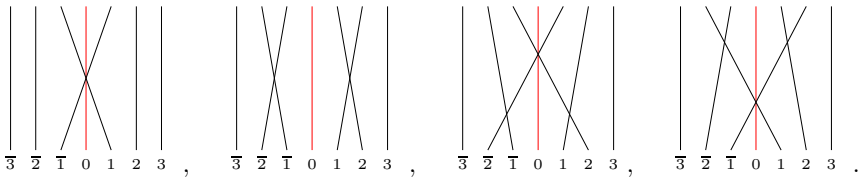


Figure 3. The octahedron for $n = 1, 2, 3$.

5.3.2. *Join-irreducibles in type B.* Join-irreducible elements in W of type B are explicitly determined in [GK, Section 4], and, in a different way, in [An, Section 2]. We describe these and determine the Bruhat order on each $i\mathbf{J}_j$. By symmetry, we only consider the case $i \leq j$.

The reduced expressions appearing here are rather lengthy. So, we also present elements in W as signed permutations and provide the corresponding pictures. Recall that a signed permutation is a permutation w of $(-n - 1, -n, \dots, -1, 0, 1, \dots, n, n + 1)$ with the property that $w(-k) = -w(k)$. We identify such w with the sequence $(w(1), w(2), \dots, w(n))$, which uniquely determines w . We write \bar{k} instead of $-k$ (only in this subsection).

For example, when $n = 2$, i.e., in type B_3 , we have $s_0 = (\bar{1}, 2, 3)$, $s_1 = (2, 1, 3)$, $s_0s_1 = (2, \bar{1}, 3)$ and $s_1s_0 = (\bar{2}, 1, 3)$ depicted, respectively, as follows:

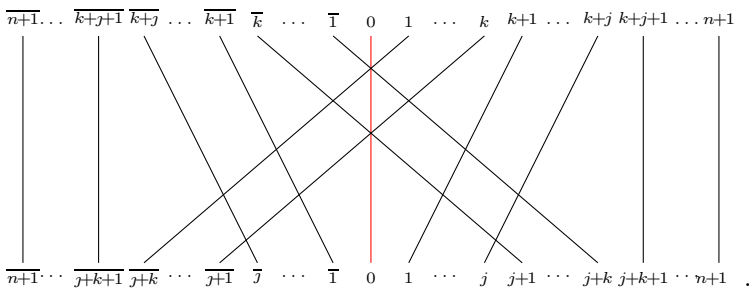


We read the pictures from bottom to top and read the expression from right to left.

PROPOSITION 5.11: *Let $0 \leq j \leq n$. Then the set ${}_0\mathbf{J}\mathbf{I}_j$ consists of the elements*

$$(25) \quad b_k^{0j} = b_k := s_{00} \cdot s_{10} \cdots s_{k-1,0} \cdot s_{k1} \cdot s_{k+1,2} \cdots s_{k+j-1,j}, \quad 1 \leq k \leq n + 1 - j,$$

with b_k having the following picture:



There is no cancellation in the above product, i.e., one obtains a reduced expressions for b_k by concatenating the reduced expressions for the factors. Moreover, as a subposet of (W, \leq) , ${}_0\mathbf{J}\mathbf{I}_j$ is a chain

$$(26) \quad \{b_1 < b_2 < \cdots < b_{n+1-j}\}.$$

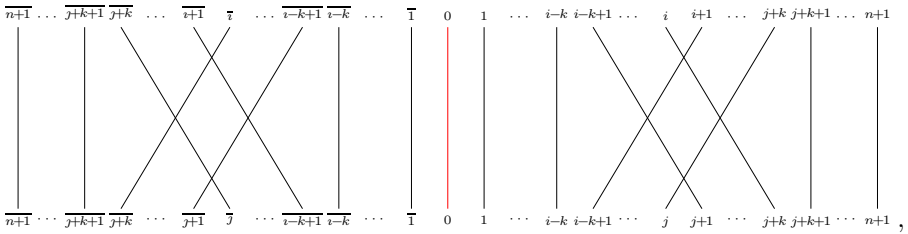
Proof. The first part follows directly from [GK, Theorem 4.6]. The Bruhat relations in (26) follow from the reduced expressions obtained from (25). ■

PROPOSITION 5.12: *Let $1 \leq i \leq j \leq n$. Then the set ${}_i\mathbf{J}\mathbf{J}_j$ consists of the following three types of elements:*

- (type O_A) for $1 \leq k \leq \min\{i, n + 1 - j\}$:

$$b_{o,k}^{ij} = b_{o,k} := s_{i,i-k+1} \cdot s_{i+1,i-k+2} \cdots s_{k+j-1,j},$$

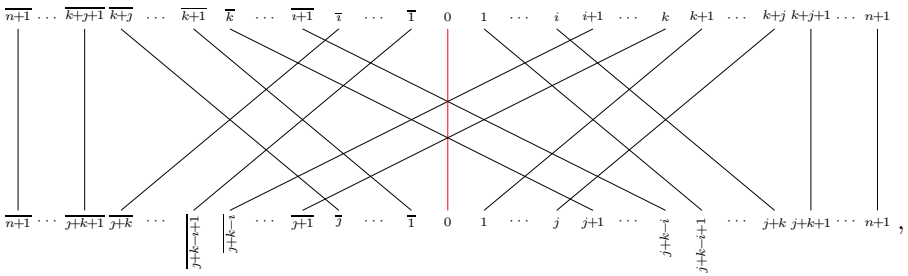
with the picture



- (type O_B) for $i < k \leq n + 1 - j$:

$$b_{o,k}^{ij} = b_{o,k} := s_{i0} \cdot s_{i+1,0} \cdots s_{k-1,0} \cdot s_{k,1} \cdot s_{k+1,2} \cdots s_{k+j-1,j},$$

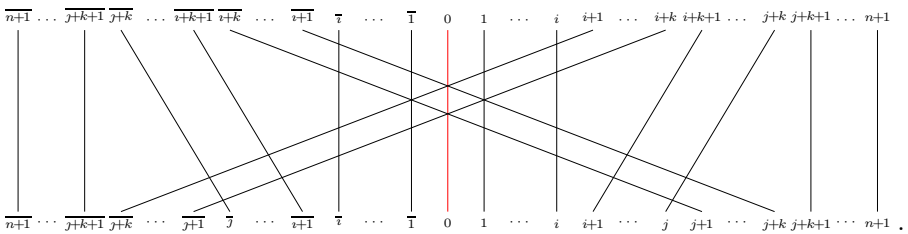
with the picture



- (type X) for $1 \leq k \leq n + 1 - j$:

$$b_{x,k}^{ij} = b_{x,k} := t_{ii} \cdot t_{i+1,i} \cdots t_{i+k-1,i} \cdot s_{i+k,i+1} \cdot s_{i+k+1,i+2} \cdots s_{j+k-1,j},$$

with the picture



There are no cancellations in the above products, i.e., one obtains reduced expressions for $b_{\circ,k}$ and $b_{\times,k}$ by concatenating the reduced expressions for the factors. Moreover, the partial order on ${}_i\mathbf{J}\mathbf{I}_j$ is generated by the following relations (see Figure 5):

- (27) $b_{\circ,k} < b_{\circ,k+1}, \quad 1 \leq k \leq n - j,$
- (28) $b_{\times,k} < b_{\times,k+1}, \quad 1 \leq k \leq n - j,$
- (29) $b_{\circ,k} < b_{\times,k}, \quad 1 \leq k \leq n + 1 - j,$
- (30) $b_{\times,k} < b_{\circ,k+i}, \quad 1 \leq k \leq n + 1 - j - i.$

Proof. The fact that ${}_i\mathbf{J}\mathbf{I}_j$ consists of the elements as above follows from [GK, Theorem 4.6]. The reduced expressions we choose to give agree with the expressions used in [GK]. The case $l = 0$ in [GK, Theorem 4.6] is our type O_A ; $l_1 = 0$ or $l = c$ is our type X (for $i = j$ or $i \neq j$), and $c = m + l$ is our type O_B . The Bruhat relations (27), (28) and (29) follow directly from the reduced expressions given above. For the relation (30), we see from the reduced expressions above that it is enough to prove it in the case $i = j$, which follows easily from, e.g., [BjBr, Theorem 8.1.8].

To prove that (27)–(30) are generating relations in ${}_i\mathbf{J}\mathbf{I}_j$, it is enough to observe that $b_{\circ,k} \not\leq b_{\times,k-1}$, since $b_{\circ,k}$ involves the simple reflection $(j + k - 1)$ in its reduced expressions while $b_{\times,k-1}$ does not, and, moreover, that $b_{\times,k} \not\leq b_{\circ,k'}$ for $k' < k + i$, by comparing the number of occurrences of s_0 in their reduced expressions. ■

COROLLARY 5.13: *Let $i, j \in S$. Then*

$$|{}_i\mathbf{J}\mathbf{I}_j| = \sum_{y \in {}_i\mathcal{H}^j} p_{e,y}(1) = \begin{cases} n + 1 - \max\{i, j\}, & i = 0 \text{ or } j = 0, \\ 2(n + 1 - \max\{i, j\}), & \text{otherwise.} \end{cases}$$

Proof. Follows from comparing Propositions 5.11 and 5.12 with Propositions 5.8 and 5.9. ■

We will need also to use some information on ${}_i\mathbf{B}\mathbf{G}_j$. However, neither a complete enumeration of ${}_i\mathbf{B}\mathbf{G}_j \setminus {}_i\mathbf{J}\mathbf{I}_j$ nor determination of the corresponding poset structure is needed. The following proposition is enough for our purposes.

PROPOSITION 5.14: *Let $1 \leq i \leq j \leq n$. The elements*

$$f_k^{ij} = b_{o,k+1}^{ij} \vee b_{\times,k}^{ij} \in {}_i\mathbf{BG}_j \setminus {}_i\mathbf{J}\mathbf{I}_j$$

are well-defined, for each $1 \leq k \leq n - j$, and

$$(31) \quad \mathbf{JM}(f_k^{ij}) = \{b_{o,k+1}^{ij}, b_{\times,k}^{ij}\}.$$

Moreover, we have

$$(32) \quad f_k^{ij} < b_{\times,k+1}^{ij},$$

for each $1 \leq k \leq n - j$.

Proof. Fix i, j and $1 \leq k \leq n - j$. We define the elements $f = f_k^{ij}$ as follows. Take the reduced expression for $b_{o,k+1}^{ij}$ given in Proposition 5.12. Note that the k -th factor is $s_{i+k-1, i-1}$, both for $k \leq i$ and $i < k$. Let f be given by the expression obtained from the reduced expression of $b_{o,k+1}^{ij}$ by replacing the first k factors of the form $s_{m,l}$ by $t_{m, i-1}$. The result is an expression of the form used in Section 4 in [GK]. In particular, it is reduced and the element belongs to ${}_i\mathbf{BG}_j$. More explicitly, we have

$$f = t_{i, i-1} \cdot t_{i+1, i-1} \cdots t_{i+k-1, i-1} \cdot s_{i+k, i} \cdot s_{i+k+1, i+1} \cdots s_{j+k, j}.$$

To show that $f = b_{o,k+1}^{ij} \vee b_{\times,k}^{ij}$, we use [GK, Proposition 4.5], which states that any $f \in {}_i\mathbf{BG}_j$ is written as $f = f_- \vee f_+$, whenever $f \geq f_+, f_- \in {}_i\mathbf{BG}_j$ are such that

- (i) f and f_- have the same number of s_0 appearing in their reduced expressions (note that the number is independent of reduced expressions);
- (ii) $f, f_+ \in W_m \setminus W_{m-1}$, for some $m \leq n$, where $W_m \subset W$ is the parabolic subgroup generated by $\{s_0, \dots, s_m\} \subset S$.

By construction, the pair $f_- = b_{\times,k}^{ij}$ and $f_+ = b_{o,k+1}^{ij}$ satisfies the conditions, with the numbers k and $m = j + k$, and establishes the first claim. Dissectivity, Proposition 2.10, and Corollary 3.6 give (31).

The relation (32) follows from the given reduced expressions since the two are comparable factor by factor. ■

LEMMA 5.15: *The Bruhat relations in (26), (27), (28) and (29), as well as (32), are socle-killing.*

Proof. This is a routine check of all the cases, e.g., using the given reduced expressions. ■

PROPOSITION 5.16: *For any $0 \leq i, j \leq n$, there exists a chain of length $\sum_{y \in {}^i\mathcal{H}^j} p_{e,y}(1)$ in ${}^i\mathbf{BG}_j$.*

Proof. When $i = 0$ or $j = 0$, the elements in ${}^i\mathbf{J}\mathbf{I}_j$ form a chain, as shown in Proposition 5.11. This chain is of the desired length.

When $i \neq 0 \neq j$, then Proposition 5.12 and Proposition 5.14 provide the chain

$$(33) \quad b_{0,1} < b_{\times,1} < f_1 < b_{\times,2} < f_2 < \dots < f_{n-\max\{i,j\}} < b_{\times,n-\max\{i,j\}+1}$$

which has the desired length by Corollary 5.13. ■

COROLLARY 5.17: *Let $w \in W$. The socle of Δ_e/Δ_w is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ taken over $x \in \mathbf{JM}(w)$.*

Proof. This follows from Proposition 5.16 and Proposition 4.19. ■

5.3.3. *The join-irreducible socles in type B.* Recall from Subsection 5.3.1 the structure of the penultimate cell \mathcal{J} .

PROPOSITION 5.18: *For $b_k \in {}_0\mathbf{J}\mathbf{I}_0$, where $1 \leq k \leq n + 1$, we have*

$$\text{soc } \Delta_e/\Delta_{b_k} \cong \begin{cases} L_{u_{00}}\langle -\ell(u_{00}) + 2(n - k) \rangle, & k \equiv n \pmod{2}, \\ L_{w_{00}}\langle -\ell(w_{00}) + 2(n + 1 - k) \rangle, & k \not\equiv n \pmod{2}. \end{cases}$$

Proof. From Proposition 5.11 and Lemma 5.15 we know that ${}_0\mathbf{J}\mathbf{I}_0$ is a socle-killing chain. By Proposition 2.3 and Proposition 2.6, the socles corresponding to the elements b_k are direct sums of shifts of $L_{u_{00}}$ and $L_{w_{00}}$. Proposition 5.9 gives the degrees in Δ_e in which these composition factors appear. The claim now follows from Lemma 2.7, Lemma 5.13 and the pigeonhole principle. ■

PROPOSITION 5.19: *For $1 \leq j \leq n$ and $b_k \in {}_0\mathbf{J}\mathbf{I}_j$, where $1 \leq k \leq n + 1 - j$, we have*

$$\text{soc } \Delta_e/\Delta_{b_k} \cong L_{u_{0j}}\langle -\ell(u_{0j}) + 2(n + 1 - j - k) \rangle.$$

Proof. Analogous to the proof of Proposition 5.18. ■

To determine the case $1 \leq i \leq j \leq n$, we need some further computation of KL polynomials. The following lemma, which is derivable from Lemma 5.1 directly, determines all $p_{x,y}$, for $x \in W$ and $y \in \mathcal{J}$ inductively. Note that the induction also works when we restrict to $y \in \mathcal{L} \subset \mathcal{J}$.

LEMMA 5.20: *Let $z \in W$ and $s \in S$ be such that $zs > s$.*

(i) *For $i > 0$ and $y_{i,j} \in {}^i\mathcal{H}^j$, we have*

$$(34) \quad p_{zs,y_{i,j}} = \begin{cases} v^{-1}p_{z,y_{i,j}}, & \text{if } s \neq s_j, \\ -vp_{z,y_{i,j}} + p_{z,y_{i,j+1}} + p_{z,y_{i,j-1}} + \delta_{y_{i,j},s_j,w_0}p_{z,w_0}, & \text{if } s = s_j. \end{cases}$$

Here, by convention, $u_{i,j} = w_{i,-j}$, if $j < 0$, and $y_{i,j} = 0$, if $j > n$. Note that $\delta_{y_{i,j},s_j,w_0} = \delta_{y_{i,j},w_{i,i}}$.

(ii) *For $y_{0,j} \in {}^0\mathcal{H}^j$, we have*

$$(35) \quad p_{zs,w_{0,j}} = \begin{cases} v^{-1}p_{z,w_{0,j}}, & \text{if } s \neq s_j, \\ -vp_{z,w_{0,j}} + p_{z,w_{0,j+1}} + p_{z,w_{0,j},s_j} + \delta_{1,j}p_{z,u_{0,0}}, & \text{if } s = s_j, \end{cases}$$

and

$$(36) \quad p_{z.s,u_{0,0}} = \begin{cases} v^{-1}p_{z,u_{0,0}}, & \text{if } s \neq s_0, \\ -vp_{z,u_{0,0}} + p_{z,u_{0,1}}, & \text{if } s = s_0. \end{cases}$$

LEMMA 5.21: *For $0 \leq i \leq j \leq n$ and $0 < j$, we have:*

$$(37) \quad p_{s_{ij},u_{ij}} = v^{\ell(w_0)-2(j+1)} + v^{\ell(w_0)-2(j+2)} + \dots + v^{\ell(w_0)-2n},$$

$$(38) \quad p_{s_{ij},w_{ij}} = v^{\ell(w_0)-2(j-i+1)} + v^{\ell(w_0)-2(j-i+2)} + \dots + v^{\ell(w_0)-2(n+1-i)} \\ = v^{2(i-1)}(v^{\ell(w_0)-2j} + p_{s_{ij},u_{ij}}),$$

$$(39) \quad p_{t_{ij},u_{ij}} = v^{-2i} \cdot p_{s_{ij},u_{ij}},$$

$$(40) \quad p_{t_{ij},w_{ij}} = p_{s_{ij},u_{ij}}.$$

Proof. Follows from Lemma 5.20 by various inductions. ■

COROLLARY 5.22: *For $0 \leq i \leq j \leq n$ and $0 < j$, we have*

$$[\Delta_e/\Delta_{b_{\times,1}} : L_{u_{ij}}] = 0 \quad \text{and} \quad [\Delta_e/\Delta_{b_{\times,1}} : L_{w_{ij}}] = 1,$$

and, moreover, the composition factor $L_{w_{ij}}$ appearing in $\Delta_e/\Delta_{b_{\times,1}}$ is in the minimal degree among the composition factors $L_{w_{ij}}$ in Δ_e .

Proof. Comparing Proposition 5.21 with Proposition 5.8 gives the claim. ■

PROPOSITION 5.23: *Let $1 \leq i \leq j \leq n$. For each $1 \leq k \leq n + 1 - j$, and for the corresponding join-irreducible elements $b_{\circ,k}$ and $b_{\times,k}$ in ${}^i\mathbf{JJ}_j$, we have*

$$(41) \quad \text{soc } \Delta_e / \Delta_{b_{\circ,k}} \cong \begin{cases} L_{u_{ij}} \langle -\ell(u_{ij}) + 2(n + 1 - j - k) \rangle, & b_{\circ,k} \text{ is of type } O_A, \\ (L_{u_{ij}} \oplus L_{w_{ij}}) \langle -\ell(u_{ij}) + 2(n + 1 - j - k) \rangle, & b_{\circ,k} \text{ is of type } O_B, \end{cases}$$

and

$$(42) \quad \text{soc } \Delta_e / \Delta_{b_{\times,k}} \cong L_k \langle -\ell(w_{ij}) + 2(n + 1 - j - k) \rangle$$

for either $L_k \cong L_{w_{ij}}$ or $L_k \cong L_{u_{ij}}$.

Remark 5.24: For $k = 1$ and $k > n + 1 - i - j$, we have $L_k \cong L_{w_{ij}}$ by Corollary 5.22 and for degree reasons. We conjecture that $L_k \cong L_{w_{ij}}$ is the case for all k , but to prove this in general, a computation of KL polynomials similar to Lemma 5.21, e.g., using Lemma 5.20, seems required.

Remark 5.25: In the proof, we also prove

$$(43) \quad \text{soc } \Delta_e / \Delta_{f_k} = L'_k \langle \ell(u_{ij}) + 2(n + 1 - j - k) \rangle \oplus L_k \langle -\ell(w_{ij}) + 2(n + 1 - j - k) \rangle$$

with $L'_k \cong L_{w_{ij}}$, if $L_{k-i} \cong L_{u_{ij}}$, and $L'_k \cong L_{u_{ij}}$, if $L_{k-i} \cong L_{w_{ij}}$, where f_k is from Proposition 5.14. Thus the socle-sum property does not hold for f_k when $k > i$.

Proof. We use Proposition 5.8 and Lemma 2.7 throughout the proof without referring to it.

Consider the chain (33) which we rename $x_1 < \dots < x_{2n-2j+2}$. Since the total multiplicity of composition factors in Δ_e isomorphic to shifts of L_y with $y \in {}^i\mathcal{H}^j$ agrees with $2n - 2j + 2$ by Proposition 5.13, we have

$$[\Delta_e / \Delta_{x_m} : L_u] + [\Delta_e / \Delta_{x_m} : L_w] = m,$$

for each $1 \leq m \leq 2n - 2j + 2$ (see the proof of Proposition 4.17), where $u = u_{ij}$ and $w = w_{ij}$. Here we use that both subquotients L_u and L_w of Δ_e are graded multiplicity free. By Proposition 5.15, for each even number m , the relation $x_{m-1} < x_m$ is socle-killing. Thus, by Lemma 2.7, the socle of Δ_e / Δ_{x_m} is simple, for each even number m . That is, each $\text{soc } \Delta_e / \Delta_{b_{\times,k}}$ is simple. Now, Corollary 5.22 gives (42), for $k = 1$. Since $b_{\times,k} < b_{\times,k+1}$ is socle-killing, Lemma 5.4(ii) establishes (42) for all k .

Similarly, each subquotient $\Delta_{x_{m-2}} / \Delta_{x_m}$ contains exactly two penultimate composition factors (i.e., composition factors isomorphic, up to shift, either to L_u or to L_w). By the socle-killing relations and Lemma 2.7, this subquotient

Since \mathcal{J} is strongly regular, i.e., all \mathbb{H} -cells inside \mathcal{J} are singletons, we denote the unique element in ${}^i\mathcal{H}^j$ as w_{ij} . These elements can be described as follows. Denote by $\hat{\cdot}$ the identity on S , if n is even, and the unique automorphism of the Dynkin diagram that swaps $0^+ \leftrightarrow 0^-$, if n is odd. For simple reflections i, j , denote by $s_{ij} \in W$ the product $i \cdots j$ of simple reflections along the unique shortest path starting in i and ending in j in the Dynkin diagram. Then

$$w_{ij} = s_{i\hat{j}} \cdot w_0.$$

The Bruhat graph of \mathcal{J} , for n even, is given on Figure 4. The Bruhat graph of \mathcal{J} , for n odd, is obtained from Figure 4 by reversing all the arrows that start from or end in one of the following elements: w_{0-0^-} , w_{0-0^+} , w_{0+0^-} , w_{0+0^+} . Both for even and odd n , the rows are right cells, and the columns are left cells.

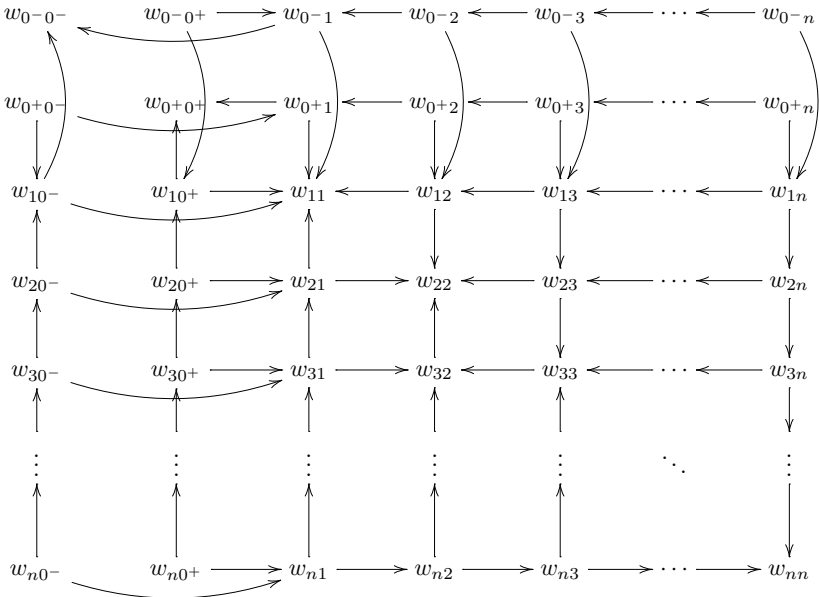


Figure 4. Bruhat graph of the penultimate two-sided cell in type D_{n+2} with n even.

5.4.1. *Some Kazhdan–Lusztig computation in type D.* Since the Coxeter number for D_{n+2} is $h = 2n + 2$, we have $\ell(w_0) = (n + 2)h/2 = (n + 2)(n + 1) = n^2 + 3n + 2$. We do not know any reference for an explicit formula for the value of the \mathbf{a} -function on \mathcal{J} , so we will compute this value below. We start with the following estimate.

LEMMA 5.26: We have $\mathbf{a}(\mathcal{J}) \geq \ell(w_0) - 2n - 1$.

Proof. Consider $I = \{0^-, 0^+, 1, \dots, n - 1\} \subset S$. We have

$$\ell(w_I) = (n + 1)2n/2 = n^2 + n = \ell(w_0) - 2n - 2.$$

We want to prove that w_I is not in \mathcal{J} . If it were, then it would have to be in ${}^n\mathcal{H}^n$. But ${}^n\mathcal{H}^n$ has only one element $w_{nn} = n \cdot w_0$, whose length is $\ell(w_0) - 1$, a contradiction. Now, $w_I <_J \mathcal{J}$ implies $\mathbf{a}(w_I) = \ell(w_I) = \ell(w_0) - 2n - 2 < \mathbf{a}(\mathcal{J})$, proving the claim. ■

We will simplify the notation and use $p_{ij} := p_{e, w_{ij}}$, for $i, j \in S$. The following is the base case of the inductive computation of the $p_{e, y}$, for $y \in \mathcal{J}$.

PROPOSITION 5.27: For $y = w_{i0^+}, w_{i0^-}, w_{0^+i},$ or w_{0^-i} , where $i \geq 1$, we have

$$(46) \quad p_{e, y} = v^{\ell(y)} + v^{\ell(y)-2} + \dots + v^{\ell(y)-2(n-i)}.$$

Proof. Applying Lemma 5.2 to $w_{i0^+} \rightarrow w_{i-1, 0^+}$, for, respectively, $i = 2, 3, \dots, n$ (see also Figure 4), we get a series of equalities:

$$(47) \quad \begin{aligned} (v + v^{-1}) \cdot p_{i0^+} &= p_{i-1, 0^+} + p_{i+1, 0^+}, & i = 2, 3, \dots, n - 1, \\ (v + v^{-1}) \cdot p_{n0^+} &= p_{n-1, 0^+}. \end{aligned}$$

Suppose that v^m is the smallest monomial that appears in p_{n0^+} , and note that $m \leq \ell(w_{n0^+}) = \ell(w_0) - n - 1$. By induction, it is easy to see, using (47), that p_{10^+} must contain v^{m-n+1} as a monomial. Since w_{10^+} is not a Duflo element, we have

$$m - n + 1 > \mathbf{a}(\mathcal{J}) \geq \ell(w_0) - 2n - 1$$

(see Lemma 5.26). This gives us $m = \ell(w_0) - n - 1$, and therefore $p_{n0^+} = v^{\ell(w_{n0^+})}$. The statement of the proposition, for $y = w_{i0^+}$, now follows from (47) by a two-step induction.

The statement for the remaining elements follows from the symmetry of the Dynkin diagram, and from the fact that $p_{ij} = p_{ji}$, for $i, j \in S$. ■

From the proof of Proposition 5.27 we also get:

COROLLARY 5.28: We have $\mathbf{a}(\mathcal{J}) = \ell(w_0) - 2n - 1 = n^2 + n + 1$.

PROPOSITION 5.29: For $1 \leq i \leq j \leq n$ and $y = w_{ij}$, we have

$$(48) \quad \begin{aligned} p_{ij} = p_{ji} = & v^{\ell(y)} + v^{\ell(y)-2} + \dots + v^{\ell(y)-2(n-j)} + \\ & + v^{\ell(y)-2i} + v^{\ell(y)-2i-2} + \dots + v^{\ell(y)-2(n-j+i)}. \end{aligned}$$

Note that higher multiplicities appear in (48) if and only if $i + j \leq n$. These multiplicities are at most 2.

Proof. We apply Lemma 5.2 to $w_{0+j} \rightarrow w_{1j}$ to get

$$(v + v^{-1}) \cdot p_{0+j} = p_{1j}.$$

Proposition 5.27 applied to p_{0+j} above proves our claim (48) for $i = 1$. Applying again Lemma 5.2 to $w_{1j} \rightarrow w_{2j}$ and $j \geq 2$, we get

$$(v + v^{-1}) \cdot p_{1j} = p_{2j} + p_{0+j} + p_{0-j}.$$

Here we apply Proposition 5.27 and (48) for $i = 1$, which proves (48) for $i = 2$. For $i > 2$ we apply Lemma 5.2 to $w_{i-1,j} \rightarrow w_{ij}$ and $j \geq i$ to get

$$(v + v^{-1}) \cdot p_{i-1,j} = p_{ij} + p_{i-2,j},$$

where the induction hypothesis establishes (48). The proof is complete. ■

Finally, we treat the remaining four cases where $i, j \in \{0^+, 0^-\}$. The result depends on the parity of the rank.

PROPOSITION 5.30: For n even, we have

$$(49) \quad p_{0+0+} = p_{0-0-} = v^{\ell(w_0)-1} + v^{\ell(w_0)-5} + \dots + v^{\ell(w_0)-2n-1},$$

$$(50) \quad p_{0+0-} = p_{0-0+} = v^{\ell(w_0)-3} + v^{\ell(w_0)-7} + \dots + v^{\ell(w_0)-2n+1}.$$

For n odd, we have

$$(51) \quad p_{0+0+} = p_{0-0-} = v^{\ell(w_0)-3} + v^{\ell(w_0)-7} + \dots + v^{\ell(w_0)-2n-1},$$

$$(52) \quad p_{0+0-} = p_{0-0+} = v^{\ell(w_0)-1} + v^{\ell(w_0)-5} + \dots + v^{\ell(w_0)-2n+1}.$$

Proof. The equalities $p_{0+0+} = p_{0-0-}$ and $p_{0+0-} = p_{0-0+}$ follow from the symmetry of the Dynkin diagram.

Assume first that n is even. Applying Lemma 5.2 to $w_{0-0+} \rightarrow w_{10+}$ yields

$$(v + v^{-1}) \cdot p_{0-0+} = p_{10+}$$

(see Figure 4), where the right-hand side is known from Proposition 5.27. This gives (50).

Applying Lemma 5.2 to $w_{10+} \rightarrow w_{0+0+}$ yields

$$(v + v^{-1}) \cdot p_{10+} = p_{0+0+} + p_{20+} + p_{0-0+}$$

(see Figure 4). Now we use Proposition 5.27 and (50) to get (49).

Assume now that n is odd. We apply Lemma 5.2 to $w_{0+0-} \rightarrow w_0 = 0^+ w_{0+0-}$, which gives

$$(v + v^{-1}) \cdot p_{0+0-} = v^{\ell(w_0)} + p_{10-}.$$

From this, one easily gets (52).

Finally, we apply Lemma 5.2 to $w_{10+} \rightarrow w_{0-0+}$, which gives

$$(v + v^{-1}) \cdot p_{10+} = p_{0-0+} + p_{20+} + p_{0+0+}.$$

From this, one easily gets (51). ■

5.4.2. *Join-irreducibles in type D.* In this subsection we study join-irreducibles in type D_{n+2} via relating the Weyl group with that of type B_{n+1} and using the results in Subsection 5.3.2. We note that the maps we use are not directly related to the ones in [GK] which relates type B_{n+1} with type D_{n+1} . Consider the Weyl groups $W(B_{n+1})$ and $W(D_{n+2})$ with Dynkin diagrams as in Subsection 5.3.1 and Subsection 5.4.1. Define a map

$$\phi: W(B_{n+1}) \rightarrow W(D_{n+2})$$

by $0 \mapsto 0^+0^- = 0^-0^+$ and $i \mapsto i$ for $1 \leq i \leq n$. Since

$$0^+0^-10^+0^-1 = 10^+0^-10^+0^-,$$

this defines a group homomorphism.

We denote by \mathbf{JI} the join-irreducibles in $W(D_{n+2})$, by \mathbf{JI}^B the join-irreducibles in $W(B_{n+1})$. From the braids of the elements in \mathbf{JI}^B given in Subsection 5.3.2, we see that no element in \mathbf{JI}^B has a reduced expression that contains consecutive simple reflections 0101, and so, in particular, no two reduced expressions of an element in \mathbf{JI}^B are related by the type B braid relation $0101 = 1010$. Therefore, we have two well-defined maps

$$\phi^\pm: \mathbf{JI}^B \rightarrow W(D_{n+2})$$

which are given by taking a reduced expression and replacing every other appearance of ‘0’ by 0^+ and 0^- alternatingly. The result is an expression of an element in $W(D_{n+2})$. We let $\phi^+(w)$ have 0^+ at the first (leftmost) appearance of 0 in a reduced expression of w , and $\phi^-(w)$ has 0^- . For example,

$$\begin{aligned} \phi(0102103210) &= 0^+0^-10^+0^-210^+0^-3210^+0^-, \\ \phi^+(0102103210) &= 0^+10^-210^+3210^-, \\ \phi^-(0102103210) &= 0^-10^+210^-3210^+. \end{aligned}$$

Note that ϕ^+ and ϕ^- differ by the automorphism of the Dynkin diagram which swaps $0^+ \leftrightarrow 0^-$ and preserves the other simple roots.

PROPOSITION 5.31: For $j \in S$, define the following elements in $W(D_{n+2})$:

$$d_k^{0^\pm j} := \phi^\pm(b_k^{0j}), \quad 1 \leq k \leq n + 1 - j.$$

(i) Then we have finite sets

$$(53) \quad {}_{0^+}\mathbf{J}\mathbf{I}_{0^+} = \{d_1^{0^+0} < d_3^{0^+0} < d_5^{0^+0} < \dots\},$$

$$(54) \quad {}_{0^+}\mathbf{J}\mathbf{I}_{0^-} = \{d_2^{0^+0} < d_4^{0^+0} < d_6^{0^+0} < \dots\},$$

where the last elements above are $d_n^{0^+0}$ and $d_{n+1}^{0^+0}$, depending on the parity of n .

(ii) For $j \geq 1$, we have

$$(55) \quad {}_{0^+}\mathbf{J}\mathbf{I}_j = \{d_1^{0^+j} < d_2^{0^+j} < d_3^{0^+j} < \dots < d_{n+1-j}^{0^+j}\}.$$

(iii) Analogous statements hold for ${}_{0^-}\mathbf{J}\mathbf{I}_j$.

Proof. Note that by applying ϕ^\pm to the reduced expressions of elements b_k^{0j} described in Proposition 5.11, we get exactly the reduced expressions of the elements from [GK, Theorem 5.7], from which the rest follows easily. ■

PROPOSITION 5.32: Let $1 \leq i \leq j \leq n$. The set ${}_i\mathbf{J}\mathbf{I}_j$ consists of the following three types of elements:

- (type O_A) for $1 \leq k \leq \min\{i, n + 1 - j\}$:

$$d_{\circ,k}^{ij} = d_{\circ,k} = \phi(b_{\circ,k}^{ij}) = \phi^+(b_{\circ,k}^{ij}) = \phi^-(b_{\circ,k}^{ij}),$$

- (type O_B) for $i < k \leq n + 1 - j$:

$$d_{\pm,k}^{ij} = d_{\pm,k} := \phi^\pm(b_{\circ,k}^{ij}),$$

- (type X) for $1 \leq k \leq n + 1 - j$:

$$d_{\times,k}^{ij} = d_{\times,k} := \phi(b_{\times,k}^{ij}).$$

Moreover, the partial order on ${}_i\mathbf{J}\mathbf{I}_j$ is generated by the following relations (see Figure 5):

- (56) $d_{\circ,k} < d_{\circ,k+1}, \quad 1 \leq k \leq \min\{i - 1, n - j\},$
- (57) $d_{\circ,i} < d_{\pm,i+1}, \quad \text{if } i \leq n - j,$
- (58) $d_{\pm,k} < d_{\pm,k+1}, \quad d_{\pm,k} < d_{\mp,k+1}, \quad i < k \leq n - j,$
- (59) $d_{\times,k} < d_{\times,k+1}, \quad 1 \leq k \leq n - j,$
- (60) $d_{\circ,k} < d_{\times,k}, \quad 1 \leq k \leq \min\{i, n + 1 - j\},$
- (61) $d_{\pm,k} < d_{\times,k} \quad i < k \leq n + 1 - j,$
- (62) $d_{\times,k} < d_{\pm,k+i+1}, \quad 1 \leq k \leq n - j - i.$

Proof. The fact that ${}_i\mathbf{J}\mathbf{I}_j$ consists of the elements as above follows from [GK, Theorem 5.7], as well as the fact that applying the maps ϕ and ϕ^\pm to the reduced expression of the corresponding element, we get a reduced expression. The Bruhat relations (56)–(61) follow directly the construction of the elements. For the relation (62), we see from the reduced expressions above that it is enough to prove it in the case $i = j$, which follows from e.g. [BjBr, Theorem 8.2.8].

To prove that (56)–(62) are essentially all relations in ${}_i\mathbf{J}\mathbf{I}_j$, it is enough to observe the following:

- $d_{\circ,k} \not\leq d_{\times,k-1}$, for $1 \leq k \leq \min\{i, n + 1 - j\}$, since the former one involves the simple reflection $(j + k - 1)$ in its reduced expressions while the latter does not,
- $d_{\pm,k} \not\leq d_{\times,k-1}$, for $i \leq k \leq n + 1 - j$, for the same reason as above,
- $d_{\times,k} \not\leq d_{\pm,k+i}$, for $1 \leq k \leq n + 1 - j - i$, which can be checked using [BjBr, Theorem 8.2.8]. ■

LEMMA 5.33: *The Bruhat relations in (53)–(55) and (56)–(61) are socle-killing.*

Proof. This is a routine check of all the cases. ■

We note that the formulation of Lemma 5.33 lists all the relations from Proposition 5.32 except (62).

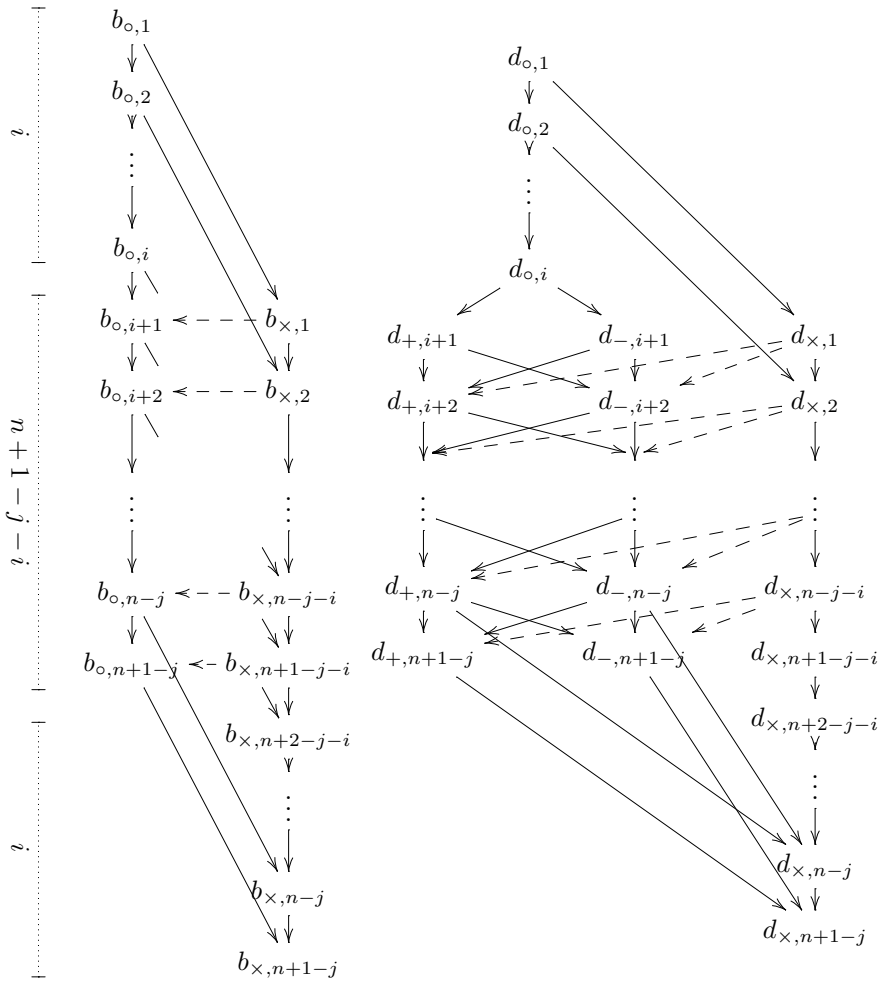


Figure 5. The Bruhat graph of $i\mathbf{J}\mathbf{I}_j$ in types B_{n+1} (left) and D_{n+2} (right), for $1 \leq i \leq j \leq n$ and $i \leq n - j$, with solid arrows socle-killing. In the right-hand side diagram, all but the first two arrows from (60) are omitted, as well as all but the last four from (61). The case $i > n - j$ is similar, but without type O_B elements (i.e., no dashed arrows).

As in type B , it helps to consider some of ${}_i\mathbf{BG}_j$ in addition to ${}_i\mathbf{J}_j$:

PROPOSITION 5.34: *Let $1 \leq i \leq j$. Then the elements*

$$g_k^{ij} = \phi(f_k^{ij}) \in W(D_{n+2})$$

belong to ${}_i\mathbf{BG}_j$ and we have in $W(D_{n+2})$

$$d_{\times,k}^{ij} < g_k^{ij} < d_{\times,k+1}^{ij}.$$

Proof. The left and right descents of $\phi(f_k^{ij})$ are $\{i\}$ and $\{j\}$, respectively, by construction of ϕ . The second claim follows from $b_{\times,k}^{ij} < f_k^{ij} < b_{\times,k}^{ij}$ in $W(B_{n+1})$ since the map ϕ preserves the order relations. ■

LEMMA 5.35: *Let $1 \leq i \leq j$. The relation $g_k^{ij} < d_{\times,k+1}^{ij}$ is socle-killing.*

Proof. This follows from the fact that $f = f_k^{ij} < b_{\times,k+1}^{ij} = b$ is socle-killing. In fact, the former implies either $f < s_i b$ or $f < b s_j$. If $f < s_i b$, then

$$g_k^{ij} = \phi(f) < \phi(s_i b) = \phi(s_i)\phi(b) = s_i d_{\times,k+1}^{ij},$$

and the second case is similar. ■

COROLLARY 5.36: *Let $i, j \in S$. Then*

$$|{}_i\mathbf{J}_j| = \begin{cases} p_{e,w_{ij}}(1), & i = 0^\pm \text{ or } j = 0^\pm, \\ p_{e,w_{ij}}(1) + c_{ij}, & 1 \leq i, j \leq n, \end{cases}$$

where $c_{ij} := \max\{0, n + 1 - j - i\}$.

Proof. The claim follows from comparing Proposition 5.31 and 5.32 with Proposition 5.27, 5.30, and 5.29. Note that c_{ij} is equal to one half of the number of type O_B elements, and also to the number of monomials in p_{ij} with coefficient 2. ■

PROPOSITION 5.37: *For any $i, j \in S$, there exists a chain of length $p_{e,w_{ij}}(1)$ in ${}_i\mathbf{BG}_j$.*

Proof. When $i = 0^\pm$ or $j = 0^\pm$, the elements in ${}_i\mathbf{J}_j$ form a chain, as shown in Proposition 5.31, of the desired length.

Otherwise, Proposition 5.32 and Proposition 5.34 provide the chain

$$(63) \quad d_{\circ,1} < d_{\times,1} < g_1 < d_{\times,2} < g_2 < \cdots < g_{n-\max\{i,j\}} < d_{\times,n-\max\{i,j\}+1}$$

which has the desired length by Corollary 5.36. ■

COROLLARY 5.38: *Let $w \in W$. The socle of Δ_e/Δ_w is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ taken over $x \in \mathbf{JM}(w)$.*

Proof. This follows from Proposition 5.37 and Proposition 4.19. ■

5.4.3. *The join-irreducible socles.*

PROPOSITION 5.39: *Let $i = 0^\pm$ and $j \in S$. Then, for $d_k \in {}_{0^\pm}\mathbf{J}\mathbf{I}_j$, we have*

$$\text{soc } \Delta_e/\Delta_{d_k} \cong \begin{cases} L_{w_{ij}}\langle -\ell(w_0) + 1 + 2(n + 1 - k) \rangle, & j = 0^\pm, \\ L_{w_{ij}}\langle -\ell(w_{ij}) + 2(n + 1 - j - k) \rangle, & j \geq 1. \end{cases}$$

Proof. Since ${}_i\mathbf{J}\mathbf{I}_j$ is a chain with socle-killing relations by Proposition 5.31 and Lemma 5.33, and since $|{}_i\mathbf{J}\mathbf{I}_j| = p_{ij}(1)$ by Proposition 5.30 and Propostion 5.27, the claim follows from Lemma 2.7 and the pigeonhole principle. ■

PROPOSITION 5.40: *Fix $1 \leq i \leq j \leq n$. For $d \in {}_i\mathbf{J}\mathbf{I}_j$, the socle of Δ_e/Δ_d is isomorphic to some shift of $L_{w_{ij}}$. To be more precise,*

(i) *for $1 \leq k \leq \min\{i, n + 1 - j\}$, we have*

$$\text{soc } \Delta_e/\Delta_{d_{o,k}} \cong L_{w_{ij}}\langle -\ell(w_{ij}) + 2(n + 1 - j + i - k) \rangle,$$

(ii) *for $i < k \leq n + 1 - j$, the socles $\text{soc } \Delta_e/\Delta_{d_{+,k}}$, $\text{soc } \Delta_e/\Delta_{d_{-,k}}$ and $\text{soc } \Delta_e/\Delta_{d_{\times,k-i}}$ are three distinct simple summands of the isotypic component $2L_{w_{ij}}\langle -\ell(w_{ij}) + 2(n + 1 - j + i - k) \rangle$ in Δ_e ,*

(iii) *for $n + 1 - j - i < k \leq n + 1 - j$, we have*

$$\text{soc } \Delta_e/\Delta_{d_{\times,k}} \cong L_{w_{ij}}\langle -\ell(w_{ij}) + 2(n + 1 - j - k) \rangle.$$

Proof. Using Lemma 5.1 and induction one obtains the type D analogue of Lemma 5.21 which shows that the penultimate simple multiplicity of maximal degree in $\Delta_e/\Delta_{d_{\times,1}}$ equals 1. This implies (ii) for $d_{\times,1}$. This allows us to follow the steps in the proof of Proposition 5.23.

We use Proposition 5.29 and Lemma 2.7 throughout the proof without referring to it and we set $w = w_{ij}$.

Consider the chain (63) which we rename $x_1 < \dots < x_{2n-2j+2}$. Since the total multiplicity of penultimate composition factors (i.e., composition factors isomorphic, up to shift, to L_w) in Δ_e agrees with $2n - 2j + 2$ by Proposition 5.29, we have

$$[\Delta_e/\Delta_{x_m} : L_w] = m,$$

for each $1 \leq m \leq 2n - 2j + 2$ (see the proof of Proposition 4.17). By Proposition 5.35, for each even number m , the relation $x_{m-1} < x_m$ is socle-killing, and

thus by Lemma 2.7 the socle of Δ_e/Δ_{x_m} , for each even number m , is simple, that is, each $\text{soc } \Delta_e/\Delta_{d_{\times,k}}$ is simple. As proved in the first paragraph, we have $\text{soc } \Delta_e/\Delta_{d_{\times,1}}$ as stated in the formulation, and, in particular, it is in the desired degree. Since $d_{\times,k} < d_{\times,k+1}$ is socle-killing, Lemma 5.4(ii) shows that, for all k , the socles $\text{soc } \Delta_e/\Delta_{d_{\times,k}}$ are as in (ii) and in (iii).

We prove, by induction on $l = n + 1 - j - k = 1, \dots, n - j$, that

$$(64) \quad \max \deg(\text{soc } \Delta_e/\Delta_{d_{\pm,k+1}}) = \ell(w) - 2i - 2(l - 1)$$

Note that Lemma 5.5 gives “ \geq ” in (64). Let $l = 1$. Then “ \leq ” in (64) follows from by combining the following facts:

- that there is at most one (graded) penultimate composition factor of Δ_e in each grade $> \ell(w) - 2i - 2(l - 1)$,
- that each such composition factor is the socle of $\Delta_e/\Delta_{d_{\times,k'}}$, for some $k' \geq n + 2 - i - j$, by the previous paragraph,
- that $d_{\pm,k+1} > d_{\times,k'}$, if true, is socle-killing (see Proposition 5.33), for each $k' \geq n + 2 - i - j$.

Now let $l > 1$. Then the socle-killing relation $d_{\pm,k+1} < d_{\pm,k+2}$ and induction provides the desired bound.

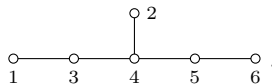
The socle-killing relation $d_{\pm,k-1} < d_{\pm,k}$ and Lemma 5.6 upgrades (64) to

$$\deg(\text{soc } \Delta_e/\Delta_{d_{\pm,k+1}}) = \ell(w) - 2i - 2(l - 1).$$

This implies (i) immediately. For (ii), it remains to observe that either of $\text{soc } \Delta_e/\Delta_{d_{\pm,k}}$ having the maximal possible multiplicity, namely 2, with respect to the composition factor L_w in the maximal degree contradicts that $d_{+,k}$ and $d_{-,k}$ are Bruhat incomparable. This proves the claims for $d_{\pm,k}$ and $d_{\circ,k}$.

The proposition is proved. ■

5.5. TYPE E_6 . Before we discuss type E in general, it is useful to first look at type E_6 in detail. Let (W, S) be of type E_6 . We denote the simple reflections by



Since \mathcal{J} is strongly regular, i.e., all \mathbb{H} -cells inside \mathcal{J} are singletons, we denote the unique element ${}^i\mathcal{H}^j$ as w_{ij} . These elements can be given as follows. Denote by $\hat{\cdot}$ the unique non-trivial automorphism of the Dynkin diagram. For simple reflections i, j , denote by $s_{ij} \in W$ the product $i \cdots j$ of simple reflections along

the unique shortest path starting in i and ending in j in the Dynkin diagram. Then we have $w_{ij} = s_{i\hat{j}} \cdot w_0$.

The Kazhdan–Lusztig polynomials $p_{e,w}$, for $w \in \mathcal{J}$, are collected in Table 1. We put $p_{e,w}$ in the position (i, j) of the table, where i is the left, and j the right ascent for w . The computation was performed in SageMath v.9.0.

Table 1. Kazhdan–Lusztig polynomials $p_{e,w}$, for $w \in \mathcal{J}$, in E_6 .

	1	2	3
1	$v^{31} + v^{25}$	$v^{32} + v^{28}$	$v^{32} + v^{30} + v^{26}$
2	$v^{32} + v^{28}$	$v^{35} + v^{31} + v^{29} + v^{25}$	$v^{33} + v^{31} + v^{29} + v^{27}$
3	$v^{32} + v^{30} + v^{26}$	$v^{33} + v^{31} + v^{29} + v^{27}$	$v^{33} + 2v^{31} + v^{29} + v^{27} + v^{25}$
4	$v^{33} + v^{31} + v^{29} + v^{27}$	$v^{34} + v^{32} + 2v^{30} + v^{28} + v^{26}$	$v^{34} + 2v^{32} + 2v^{30} + 2v^{28} + v^{26}$
5	$v^{34} + v^{30} + v^{28}$	$v^{33} + v^{31} + v^{29} + v^{27}$	$v^{35} + v^{33} + v^{31} + 2v^{29} + v^{27}$
6	$v^{35} + v^{29}$	$v^{32} + v^{28}$	$v^{34} + v^{30} + v^{28}$

	4	5	6
1	$v^{33} + v^{31} + v^{29} + v^{27}$	$v^{34} + v^{30} + v^{28}$	$v^{35} + v^{29}$
2	$v^{34} + v^{32} + 2v^{30} + v^{28} + v^{26}$	$v^{33} + v^{31} + v^{29} + v^{27}$	$v^{32} + v^{28}$
3	$v^{34} + 2v^{32} + 2v^{30} + 2v^{28} + v^{26}$	$v^{35} + v^{33} + v^{31} + 2v^{29} + v^{27}$	$v^{34} + v^{30} + v^{28}$
4	$v^{35} + 2v^{33} + 3v^{31} + 3v^{29} + 2v^{27} + v^{25}$	$v^{34} + 2v^{32} + 2v^{30} + 2v^{28} + v^{26}$	$v^{33} + v^{31} + v^{29} + v^{27}$
5	$v^{34} + 2v^{32} + 2v^{30} + 2v^{28} + v^{26}$	$v^{33} + 2v^{31} + v^{29} + v^{27} + v^{25}$	$v^{32} + v^{30} + v^{26}$
6	$v^{33} + v^{31} + v^{29} + v^{27}$	$v^{32} + v^{30} + v^{26}$	$v^{31} + v^{25}$

The posets ${}_1\mathbf{J}_j$, for $j = 1, 2, \dots, 6$, are given in Figure 6, with all the relations socle-killing.

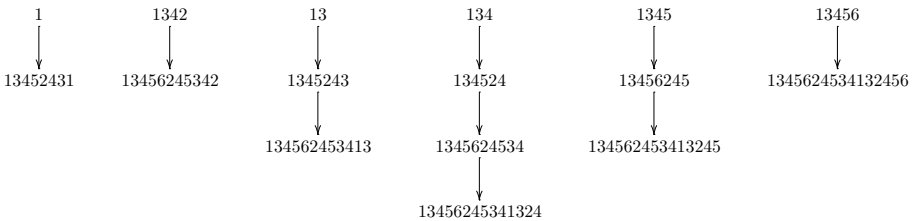


Figure 6. The posets ${}_1\mathbf{J}_j$, for $j = 1, 2, \dots, 6$, in type E_6 .

In Figure 7 and Figure 8, we present all the remaining $i\mathbf{J}\mathbf{I}_j$, up to symmetries, with the non socle-killing arrows dashed. Moreover, to the right of each element $x \in i\mathbf{J}\mathbf{I}_j$ in the diagrams, we assign a symbol of the form $cL\langle -d \rangle$, which means the following:

- the maximal degree component of Δ_e/Δ_x is isomorphic to $L_{w_{ij}}\langle -d \rangle$ (calculated by computer),
- $[\Delta_e : L_{w_{ij}}\langle -d \rangle] = c$ (see Table 1).

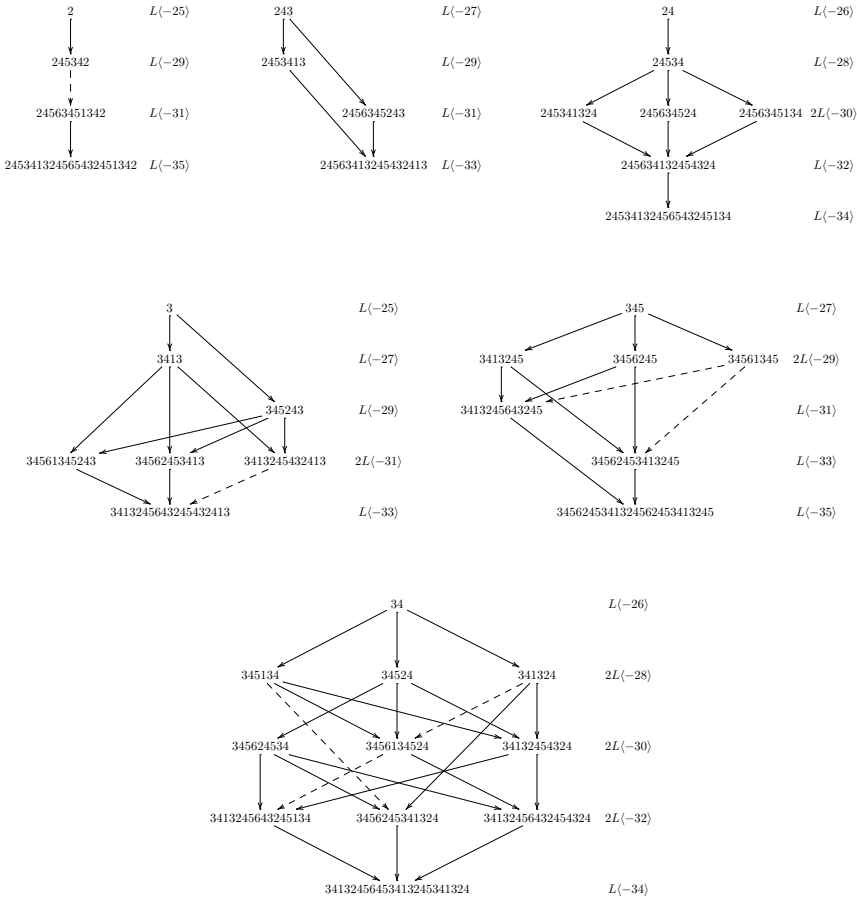


Figure 7. The posets $2\mathbf{J}\mathbf{I}_2$, $2\mathbf{J}\mathbf{I}_3$ and $2\mathbf{J}\mathbf{I}_4$ in the first row, $3\mathbf{J}\mathbf{I}_3$ and $3\mathbf{J}\mathbf{I}_5$ in the second row, and $3\mathbf{J}\mathbf{I}_4$ in the third row, in type E_6 .

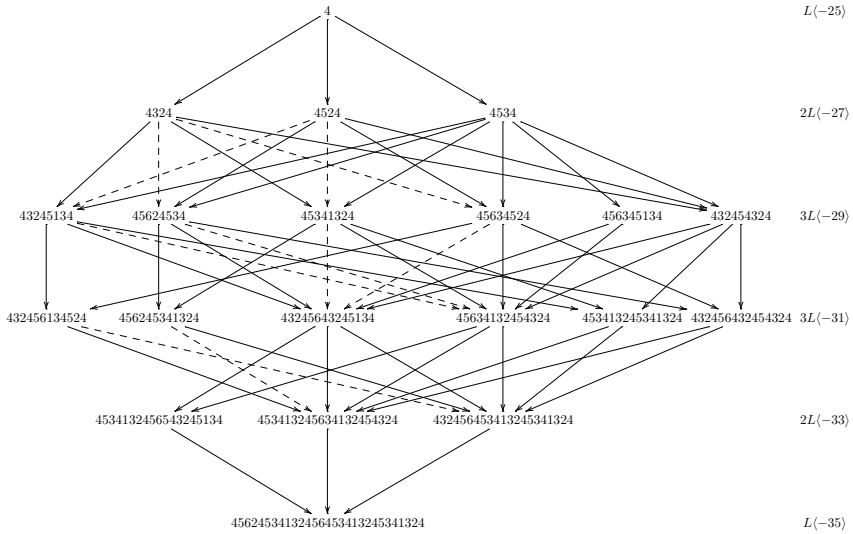


Figure 8. The poset $4\mathbf{J}_4$ in type E_6 .

PROPOSITION 5.41: For each $x \in \mathbf{J} \setminus \{24563451342\}$, the socle of Δ_e/Δ_x is simple, and hence equal to the maximal degree component of Δ_e/Δ_x . For $x = 24563451342$, we have

$$(65) \quad \text{soc } \Delta_e/\Delta_x \cong L_{w_{22}}\langle -31 \rangle \quad \text{or} \quad \text{soc } \Delta_e/\Delta_x \cong L_{w_{22}}\langle -31 \rangle \oplus L_{w_{22}}\langle -29 \rangle.$$

Proof. Comparing the diagrams in Figure 6 with the corresponding Kazhdan-Lusztig polynomials in Table 1, one immediately gets the socles of Δ_e/Δ_x , for all x in Figure 6.

For $2\mathbf{J}_2$, we have the same argument for all elements except for $x = 24563451342$ which is hit by a non-socle-killing relation from 245342. We thus have the two possibilities as stated.

In $2\mathbf{J}_3$, the only possibility for a non-simple socle is

$$\Delta_e/\Delta_{2456345243} = L_{w_{23}}\langle -31 \rangle \oplus L_{w_{23}}\langle -29 \rangle.$$

But in such a case, Lemma 5.4 would imply that there is an arrow (dashed) between the two middle elements in $2\mathbf{J}_3$, a contradiction.

The claim for ${}_2\mathbf{J}_4$ follows from the Kazhdan–Lusztig polynomials. Note that, in this case, the three non-comparable elements in the same degree have isomorphic socles, and that any two of the three socles generate the full isotypic component $2L_{w_{24}}\langle -30 \rangle$ in Δ_e .

In ${}_3\mathbf{J}_3$, the element 345243 has a simple socle again because of Lemma 5.4. The biggest element there has a simple socle because it is hit by two socle-killing arrows, and the socles corresponding to the sources of the two arrows generate the full isotypic component $2L_{w_{33}}\langle -31 \rangle$ in Δ_e . Similar arguments apply to ${}_3\mathbf{J}_5$.

Let us consider now ${}_3\mathbf{J}_4$. For $x = 345624534$, the quotient Δ_e/Δ_x contains the composition factor $L_{w_{34}}\langle -28 \rangle$ with multiplicity one, as one can check by a Kazhdan–Lusztig computation. This composition factor must be the one coming from the socle of $x' = 34524$. Since the arrow $x' \rightarrow x$ is socle-killing, this composition factor is not in the socle of Δ_e/Δ_x , and therefore Δ_e/Δ_x has a simple socle. An analogous argument applies to 3456245341324.

Finally, consider now ${}_4\mathbf{J}_4$. One can check that the socles are simple for all the elements in degrees 25, 27, and 29 by the same arguments as above. In degree 31, the first two elements from the left, as well as the first two elements from the right contain $L_{w_{44}}\langle -29 \rangle$ with multiplicity 2, and therefore it is enough to observe that each is hit by at least two solid arrows.

Consider now $x = 43245643245134$ in degree 31. There are four socle-killing relations $x' < x$ for x' in degree 29 and it is enough to show that the sum of Δ_e/Δ_{x_i} , where we name these four elements x_1, x_2, x_3, x_4 from the left, is the isotypic component $3L\langle -29 \rangle$. Suppose not. Then the sum is a subquotient

$$X \cong 2L\langle -29 \rangle$$

in $3L\langle -29 \rangle$. However, computation shows that the joins $y = x_1 \vee x_2$ and $z = x_1 \vee x_4$ exist and that $y \neq z$. By Proposition 4.19, the socles of Δ_e/Δ_y and Δ_e/Δ_z are contained in X . At the same time, by $x_1 < y$ and $x_1 < z$, these socles strictly contain $\text{soc } \Delta_e/\Delta_{x_1}$. It follows that $\text{soc } \Delta_e/\Delta_y = X = \text{soc } \Delta_e/\Delta_z$ which contradicts $y \neq z$.

Analogous arguments cover the rest of the elements in ${}_4\mathbf{J}_4$. ■

Example 5.42: Let $b \in W$ be the join of the six elements in Figure 8 whose socle is contained in $3L\langle -29 \rangle$ (computations confirm that this join exists). Computation also shows that $\mathbf{JM}(b)$ consists exactly of these six elements, while we also have $b = x \vee y$ where $x, y \in \mathbf{JM}(b)$ are the two length 9 elements among

these six. Since

$$\text{soc } \Delta_e / \Delta_z \cong L\langle -29 \rangle,$$

for each $z \in \mathbf{JM}(b)$, by Proposition 5.41, we have

$$[\Delta_x\langle -\ell(x) \rangle \cap \Delta_y\langle -\ell(y) \rangle : L\langle -29 \rangle] = 1,$$

while a Kazhdan–Lusztig computation shows

$$[\Delta_b\langle -\ell(b) \rangle : L\langle -29 \rangle] = 0.$$

It follows that

$$\Delta_x\langle -\ell(x) \rangle \cap \Delta_y\langle -\ell(y) \rangle \neq \Delta_b\langle -\ell(b) \rangle,$$

which provides an example where the answer to Question 4.9 is negative.

5.6. TYPE E . Let (W, S) be of type E . The polynomials $p_{e,w}$, for $w \in \mathcal{J}$, in types E_7 and E_8 are listed in Appendix (see Subsection 5.5 for E_6).

Computer computations confirm the following fact:

PROPOSITION 5.43: *For any $i, j \in S$, there exists a chain of length $p_{e,w_{ij}}(1)$ in ${}_i\mathbf{BG}_j$.*

As a consequence of Proposition 5.43, we have:

COROLLARY 5.44: *Let $w \in W$. The socle of Δ_e / Δ_w is contained in the sum of $\text{soc } \Delta_e / \Delta_x$ taken over $x \in \mathbf{JM}(w)$.*

Proof. Proposition 5.43 and Proposition 4.19 proves the claim. ■

CONJECTURE 5.45: *For each $x \in \mathbf{JI}$, the socle of Δ_e / Δ_x is simple.*

Remark 5.46: Conjecture 5.45 is verified in a majority of cases in types E_7 and E_8 by the bounds from Subsection 5.1 and various computer computations. Completely proving Conjecture 5.45, however, seems to require new ideas, as already seen in (65).

If Conjecture 5.45 is true, all $\text{soc } \Delta_e / \Delta_z$, for $z \in \mathbf{JI}$, are determined by Kazhdan–Lusztig computations and follow a pattern similar to the one described in Subsection 5.5 for type E_6 . For example, consider the case of a ‘large’ ${}_i\mathbf{JI}_j$, for example, ${}_4\mathbf{JI}_4$ in type E_8 . Recall our notation (c_r) for the coefficients of KL polynomials, see (16), (e.g., we have $(1, 2, 3, 4, 5, 6, 6, 6, 6, 6, 5, 4, 3, 2, 1)$

in type E_8 at $4\mathbf{J}_4$). Then, for each degree r , the isotypic component $c_r L\langle d_r \rangle$ contains

$$\frac{(c_r + 1)c_r}{2}$$

distinct simple subquotients which are the socles of $\frac{(c_r+1)c_r}{2}$ elements in ${}_i\mathbf{J}_j$. It would be nice to explain why the number $\frac{(c+1)c}{2}$ appears.

Remark 5.47: It is interesting to point out that, in the simply laced types, the maximal number appearing as a coefficient of the KL polynomial between e and a penultimate element coincides with the maximal coefficient of a root. For non-simply laced types this fails.

5.7. TYPE F_4 . Assume that (W, S) is of type F_4 . We denote the simple reflections by



We let

$${}^i\mathcal{H}^j = \{u_{ij}, w_{ij}\},$$

where either $u_{ij} = w_{ij}$ or u_{ij} is strictly shorter than w_{ij} .

The Kazhdan–Lusztig polynomials $p_{e,w}$ for $w \in \mathcal{J}$ are collected in Table 2. We put $p_{e,w}$ in the position (i, j) of the table, where i is the left, and j the right ascent for w . The computation was performed in SageMath v.9.0.

Table 2. Kazhdan–Lusztig polynomials $p_{e,w}$, for $w \in \mathcal{J}$, in F_4 .

	1	2	3	4
1	$v^{19} + v^{13}$ $v^{23} + v^{17}$	$v^{20} + v^{18} + v^{14}$ $v^{22} + v^{18} + v^{16}$	$v^{21} + v^{19} + v^{17} + v^{15}$	$v^{20} + v^{16}$
2	$v^{20} + v^{18} + v^{14}$ $v^{22} + v^{18} + v^{16}$	$v^{21} + 2v^{19} + v^{17} + v^{15} + v^{13}$ $v^{23} + v^{21} + v^{19} + 2v^{17} + v^{15}$	$v^{22} + 2v^{20} + 2v^{18} + 2v^{16} + v^{14}$	$v^{21} + v^{19} + v^{17} + v^{15}$
3	$v^{21} + v^{19} + v^{17} + v^{15}$	$v^{22} + 2v^{20} + 2v^{18} + 2v^{16} + v^{14}$	$v^{21} + 2v^{19} + v^{17} + v^{15} + v^{13}$ $v^{23} + v^{21} + v^{19} + 2v^{17} + v^{15}$	$v^{20} + v^{18} + v^{14}$ $v^{22} + v^{18} + v^{16}$
4	$v^{20} + v^{16}$	$v^{21} + v^{19} + v^{17} + v^{15}$	$v^{20} + v^{18} + v^{14}$ $v^{22} + v^{18} + v^{16}$	$v^{19} + v^{13}$ $v^{23} + v^{17}$

Figure 9 lists all $i\mathbf{J}\mathbf{I}_j$, up to symmetries, where the dashed arrows denote non-sockle killing relations.

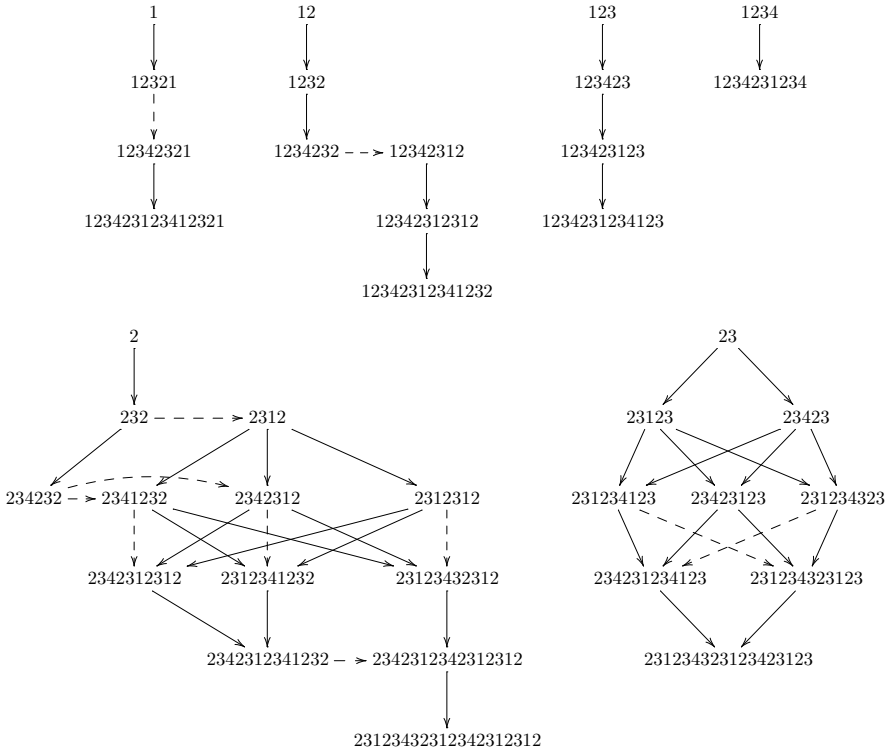


Figure 9. The Bruhat graph of $i\mathbf{J}\mathbf{I}_j$ in type F_4 , for $(i, j) = (1, 1), (1, 2), (1, 3), (1, 4), (2, 2)$ and $(2, 3)$, with the non-sockle-killing arrows dashed.

A Kazhdan–Lusztig computation determines the maximal degree components of Δ_e/Δ_x . Using this information, we calculate the socles of Δ_e/Δ_x .

PROPOSITION 5.48: *Table 3 gives the socles of Δ_e/Δ_x . In Table 3, we denote a simple subquotient by its parameter y , instead of L_y . If a socle component is not a (graded) isotypic component in Δ_e , but a simple submodule in an isotypic component of multiplicity 2, we write y in gray.*

Table 3. The socle of Δ_e/Δ_x for x from Figure 9

$x \in \mathbf{JI}$	$\text{soc } \Delta_e/\Delta_x$	$x \in \mathbf{JI}$	$\text{soc } \Delta_e/\Delta_x$
1	$u_{11}\langle -13 \rangle$	2	$u_{22}\langle -13 \rangle$
12321	$w_{11}\langle -17 \rangle$	232	$w_{22}\langle -15 \rangle$
12342321	$u_{11}\langle -19 \rangle \oplus w_{11}\langle -17 \rangle$	2312	$(u_{22} \oplus w_{22})\langle -15 \rangle$
123423123412321	$w_{11}\langle -23 \rangle$	234232	$u_{22}\langle -17 \rangle$
12	$u_{12}\langle -14 \rangle$	2312312	$w_{22}\langle -17 \rangle$
1232	$w_{12}\langle -16 \rangle$	2342312	$(u_{22} \oplus w_{22})\langle -17 \rangle$
1234232	$u_{12}\langle -18 \rangle$	2341232	$(u_{22} \oplus w_{22})\langle -17 \rangle$
12342312	$(u_{12} \oplus w_{12})\langle -18 \rangle$	2312341232	$u_{22}\langle -19 \rangle$
12342312312	$u_{12}\langle -20 \rangle$	2342312312	$u_{22}\langle -19 \rangle$
12342312341232	$w_{12}\langle -22 \rangle$	23123432312	$(u_{22} \oplus w_{22})\langle -19 \rangle$
123	$u_{13}\langle -15 \rangle$	2342312341232	$w_{22}\langle -21 \rangle$
123423	$u_{13}\langle -17 \rangle$	2342312342312312	$(u_{22} \oplus w_{22})\langle -21 \rangle$
123423123	$u_{13}\langle -19 \rangle$	23123432312342312312	$w_{22}\langle -23 \rangle$
1234231234123	$u_{13}\langle -21 \rangle$	23	$u_{23}\langle -14 \rangle$
1234	$u_{14}\langle -16 \rangle$	23423	$u_{23}\langle -16 \rangle$
1234231234	$u_{14}\langle -20 \rangle$	23123	$u_{23}\langle -16 \rangle$
		23423123	$u_{23}\langle -18 \rangle$
		231234123	$u_{23}\langle -18 \rangle$
		231234323	$u_{23}\langle -18 \rangle$
		234231234123	$u_{23}\langle -20 \rangle$
		231234323123	$u_{23}\langle -20 \rangle$
		231234323123423123	$u_{23}\langle -22 \rangle$

Proof. Note that $\text{soc } \Delta_e/\Delta_x$ always contains the maximal degree component of Δ_e/Δ_x , but, other than that, all other summands, if any, must be of strictly higher degree. Using the diagrams in Figure 9, the Kazhdan–Lusztig computation of the maximal degree components, and the property of socle-killing relations (Lemma 2.7), we can verify Table 3 except maybe in the following three cases, each of which has two possibilities:

$$(66) \quad \text{soc } \Delta_e/\Delta_{12342321} = L_{u_{11}}\langle -19 \rangle \text{ or } L_{u_{11}}\langle -19 \rangle \oplus L_{w_{11}}\langle -17 \rangle,$$

$$(67) \quad \text{soc } \Delta_e/\Delta_{234232} = L_{u_{22}}\langle -17 \rangle \text{ or } L_{u_{22}}\langle -17 \rangle \oplus L_{u_{22}}\langle -15 \rangle,$$

$$(68) \quad \text{soc } \Delta_e/\Delta_{2342312341232} = L_{u_{22}}\langle -21 \rangle \text{ or } L_{u_{22}}\langle -21 \rangle \oplus L_{w_{22}}\langle -19 \rangle.$$

For (67) and (68), one can check by calculating the Kazhdan–Lusztig polynomials that the hypothetical summand above the maximal degree does not appear in the corresponding Δ_e/Δ_x , and therefore cannot appear in the socle.

Now we consider (66). For $x = 12342321$, we claim that $L_{w_{11}}\langle -17 \rangle$ does appear in Δ_e/Δ_x . Assume that it does not appear in the socle. Then $L_{w_{11}}\langle -17 \rangle$ must extend to $L_{u_{11}}\langle -19 \rangle$ inside Δ_e , which implies that the projective cover $P_{w_{11}}$ of $L_{w_{11}}$ must contain $L_{u_{11}}\langle -21 \rangle$ as a composition factor. However, this is not the case, as one can check by Kazhdan–Lusztig computations. The claim follows. ■

COROLLARY 5.49: *Let $w \in W$. The socle of Δ_e/Δ_w is contained in the sum of $\text{soc } \Delta_e/\Delta_x$ taken over $x \in \mathbf{JM}(w)$.*

Proof. The socles of Δ_e/Δ_z for $z \in \mathbf{JI}$ determined in Proposition 5.48 satisfy the assumption of Proposition 4.18, thus the claim follows from Proposition 4.18. ■

Example 5.50: Let

$$x = 32341232, \quad y = 234231234, \quad z = 2312312, \quad w = 23423123432.$$

We have $x, y, z \in \mathbf{JI}$ and $w \in W$ are such that

$$\mathbf{JM}(w) = \{x, y, z\}, \quad \mathbf{JM}''(w) = \{x, y\}$$

and $w = x \vee y \vee z = x \vee y$. We claim

$$(69) \quad \begin{aligned} \text{soc } \Delta_e/\Delta_w &= \text{soc } \Delta_e/\Delta_x \oplus \text{soc } \Delta_e/\Delta_y \\ &\subsetneq \text{soc } \Delta_e/\Delta_x + \text{soc } \Delta_e/\Delta_y + \text{soc } \Delta_e/\Delta_z. \end{aligned}$$

In particular, the socle-sum property does not hold for w .

To prove the claim, we need to consider the elements $x', y' \in {}_2\mathbf{JI}_2$ given by

$$x' = 2341232, \quad y' = 2342312$$

which satisfy $x' < x$ and $y' < y$, necessarily socle-killing, and $x' \vee y' > z$. The sum of socles of $\Delta_e/\Delta_{x'}$ and $\Delta_e/\Delta_{y'}$ thus contains $\text{soc } \Delta_e/\Delta_z$ while it is not contained in $\text{soc } \Delta_e/\Delta_w$. On the other hand, by Corollary 5.49, the socle of Δ_e/Δ_w is contained in $\text{soc } \Delta_e/\Delta_x \oplus \text{soc } \Delta_e/\Delta_y \oplus \text{soc } \Delta_e/\Delta_z$ (the sum is direct since $x, y, z \in \mathbf{JI}$ with distinct descents). We conclude that

$$\text{soc } \Delta_e/\Delta_w \subseteq \text{soc } \Delta_e/\Delta_x \oplus \text{soc } \Delta_e/\Delta_y,$$

but a strict inclusion would imply $w = x$ or $w = y$ which is not true. This proves Formula (69).

5.8. TYPE G . Assume that (W, S) is of type G_2 . We denote the simple reflections by



The KL polynomials are trivial (see Table 4) and the submodule structure of the dominant Verma module is well-known. For $x \neq e, w_0$, we have

$$\text{soc } \Delta_e / \Delta_x = L_{\sigma(x)} \langle -\ell(x) \rangle$$

where σ is the diagram automorphism swapping the two elements in S , and the socle-sum property and Conjecture 4.1 hold, for all $w \in W$.

Table 4. Kazhdan–Lusztig polynomials $p_{e,w}$ for $w \in \mathcal{J}$ in G_2

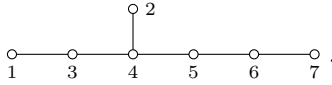
	1	2
1	v v^3 v^5	v^2 v^4
2	v^2 v^4	v v^3 v^5

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Appendix A. Kazhdan–Lusztig polynomials for E_7 and E_8

In the tables below we list the Kazhdan–Lusztig polynomials $p_{e,w}$, for $w \in \mathcal{J}$, for the exceptional types E_7 and E_8 . We put $p_{e,w}$ in the position (i, j) of the table, where i is the left, and j the right ascent for w . The computations for E_7 were performed in SageMath v.9.0. For E_8 , some additional tricks were necessary.

A.1. TYPE E_7 . We denote the simple reflections by



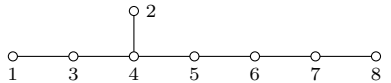
See Table 5 for the results.

Table 5. Kazhdan–Lusztig polynomials $p_{e,w}$ for $w \in \mathcal{J}$ in E_7

	1	2	3	4
1	$v^{62} + v^{56} + v^{52} + v^{46}$	$v^{59} + v^{55} + v^{53} + v^{49}$	$v^{61} + v^{57} + v^{55} + v^{53} + v^{51} + v^{47}$	$v^{60} + v^{58} + v^{56} + 2v^{54} + v^{52} + v^{50} + v^{48}$
2	$v^{59} + v^{55} + v^{53} + v^{49}$	$v^{62} + v^{58} + v^{56} + v^{54} + v^{52} + v^{50} + v^{46}$	$v^{60} + v^{58} + v^{56} + 2v^{54} + v^{52} + v^{50} + v^{48}$	$v^{61} + v^{59} + 2v^{57} + 2v^{55} + 2v^{53} + 2v^{51} + v^{49} + v^{47}$
3	$v^{61} + v^{57} + v^{55} + v^{53} + v^{51} + v^{47}$	$v^{60} + v^{58} + v^{56} + 2v^{54} + v^{52} + v^{50} + v^{48}$	$v^{62} + v^{60} + v^{58} + 2v^{56} + 2v^{54} + 2v^{52} + v^{50} + v^{48} + v^{46}$	$v^{61} + 2v^{59} + 2v^{57} + 3v^{55} + 3v^{53} + 2v^{51} + 2v^{49} + v^{47}$
4	$v^{60} + v^{58} + v^{56} + 2v^{54} + v^{52} + v^{50} + v^{48}$	$v^{61} + v^{59} + 2v^{57} + 2v^{55} + 2v^{53} + 2v^{51} + v^{49} + v^{47}$	$v^{61} + 2v^{59} + 2v^{57} + 3v^{55} + 3v^{53} + 2v^{51} + 2v^{49} + v^{47}$	$v^{62} + 2v^{60} + 3v^{58} + 4v^{56} + 4v^{54} + 4v^{52} + 3v^{50} + 2v^{48} + v^{46}$
5	$v^{59} + v^{57} + v^{55} + v^{53} + v^{51} + v^{49}$	$v^{60} + v^{58} + 2v^{56} + v^{54} + 2v^{52} + v^{50} + v^{48}$	$v^{60} + 2v^{58} + 2v^{56} + 2v^{54} + 2v^{52} + 2v^{50} + v^{48}$	$v^{61} + 2v^{59} + 3v^{57} + 3v^{55} + 3v^{53} + 3v^{51} + 2v^{49} + v^{47}$
6	$v^{58} + v^{56} + v^{52} + v^{50}$	$v^{59} + v^{57} + v^{55} + v^{53} + v^{51} + v^{49}$	$v^{59} + 2v^{57} + v^{55} + v^{53} + 2v^{51} + v^{49}$	$v^{60} + 2v^{58} + 2v^{56} + 2v^{54} + 2v^{52} + 2v^{50} + v^{48}$
7	$v^{57} + v^{51}$	$v^{58} + v^{54} + v^{50}$	$v^{58} + v^{56} + v^{52} + v^{50}$	$v^{59} + v^{57} + v^{55} + v^{53} + v^{51} + v^{49}$

	5	6	7
1	$v^{59} + v^{57} + v^{55} + v^{53} + v^{51} + v^{49}$	$v^{58} + v^{56} + v^{52} + v^{50}$	$v^{57} + v^{51}$
2	$v^{60} + v^{58} + 2v^{56} + v^{54} + 2v^{52} + v^{50} + v^{48}$	$v^{59} + v^{57} + v^{55} + v^{53} + v^{51} + v^{49}$	$v^{58} + v^{54} + v^{50}$
3	$v^{60} + 2v^{58} + 2v^{56} + 2v^{54} + 2v^{52} + 2v^{50} + v^{48}$	$v^{59} + 2v^{57} + v^{55} + v^{53} + 2v^{51} + v^{49}$	$v^{58} + v^{56} + v^{52} + v^{50}$
4	$v^{61} + 2v^{59} + 3v^{57} + 3v^{55} + 3v^{53} + 3v^{51} + 2v^{49} + v^{47}$	$v^{60} + 2v^{58} + 2v^{56} + 2v^{54} + 2v^{52} + 2v^{50} + v^{48}$	$v^{59} + v^{57} + v^{55} + v^{53} + v^{51} + v^{49}$
5	$v^{62} + v^{60} + 2v^{58} + 2v^{56} + 3v^{54} + 2v^{52} + 2v^{50} + v^{48} + v^{46}$	$v^{61} + v^{59} + v^{57} + 2v^{55} + 2v^{53} + v^{51} + v^{49} + v^{47}$	$v^{60} + v^{56} + v^{54} + v^{52} + v^{48}$
6	$v^{61} + v^{59} + v^{57} + 2v^{55} + 2v^{53} + v^{51} + v^{49} + v^{47}$	$v^{62} + v^{60} + v^{56} + 2v^{54} + v^{52} + v^{48} + v^{46}$	$v^{61} + v^{55} + v^{53} + v^{47}$
7	$v^{60} + v^{56} + v^{54} + v^{52} + v^{48}$	$v^{61} + v^{55} + v^{53} + v^{47}$	$v^{62} + v^{54} + v^{46}$

A.2. TYPE E_8 . We did not manage to get the polynomials directly from the computer, due to the complexity of the Weyl group, and the length of the elements from \mathcal{J} , so we use a different approach here. Denote the simple reflections by



The element w_0 is central, and we have $\ell(w_0) = 120$. According to [GP, Table C.6], we have $\mathbf{a}(\mathcal{J}) = 91$. Since \mathcal{J} is strongly regular, i.e., all \mathbb{H} -cells inside \mathcal{J} are singletons, we denote the unique element ${}^i\mathcal{H}^j$ as w_{ij} . These elements can be given explicitly as $w_{ij} = s_{ij} \cdot w_0$, where $s_{ij} \in W$ denotes the product $i \cdots j$ of simple reflections along the unique shortest path starting in i and ending in j in the Dynkin diagram. The Bruhat graph of \mathcal{J} is given on Figure 10. The rows are right cells, and the columns are left cells. We will simplify the notation and use $p_{ij} := p_{e, w_{ij}}$ for $i, j \in S$.

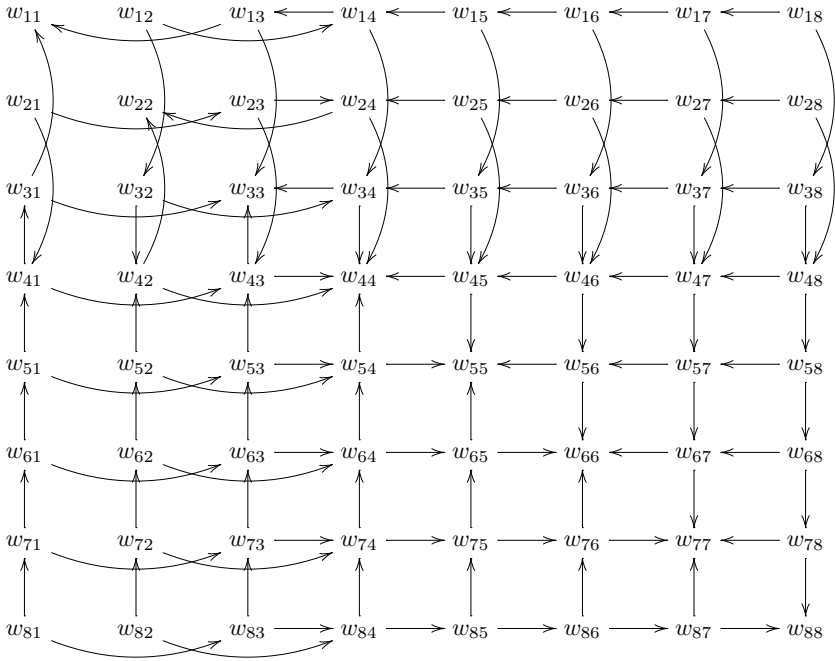


Figure 10. Bruhat graph of the penultimate two-sided cell in type E_8 .

PROPOSITION A.1: *The polynomials p_{ij} are given in Table 6.*

Proof. By applying Lemma 5.2 to non-diagonal elements of \mathcal{J} , one can see that all p_{ij} 's can be reconstructed from p_{18} (similarly to the proofs from Subsection 5.4.1), see Figure 10. In fact, it is convenient to use the computer to calculate each p_{ij} from p_{18} . So it is enough to prove that $p_{18} = v^{113} + v^{107} + v^{103} + v^{97}$.

Table 6. Kazhdan–Lusztig polynomials $p_{e,w}$ for $w \in \mathcal{J}$ in E_8 (continued on next page)

1	2	3	4
1	$v^{116} + v^{112} + v^{110} + v^{108} + v^{106} + v^{104} + v^{102} + v^{100} + v^{98} + v^{94}$	$v^{118} + v^{114} + v^{112} + v^{110} + 2v^{108} + v^{106} + v^{104} + 2v^{102} + v^{100} + v^{98} + v^{96} + v^{92}$	$v^{117} + v^{115} + v^{113} + 2v^{111} + 2v^{109} + 2v^{107} + 2v^{105} + 2v^{103} + 2v^{101} + 2v^{99} + v^{97} + v^{95} + v^{93}$
2	$v^{119} + v^{112} + v^{110} + v^{108} + v^{106} + v^{104} + v^{102} + v^{100} + v^{98} + v^{94}$	$v^{119} + v^{115} + v^{113} + v^{111} + 2v^{109} + v^{107} + 2v^{105} + v^{103} + 2v^{101} + v^{99} + v^{97} + v^{95} + v^{91}$	$v^{118} + v^{116} + 2v^{114} + 2v^{112} + 3v^{110} + 3v^{108} + 3v^{106} + 4v^{104} + 4v^{102} + 4v^{100} + 3v^{98} + 3v^{96} + 2v^{94} + v^{92}$
3	$v^{118} + v^{114} + v^{112} + v^{110} + 2v^{108} + v^{106} + v^{104} + 2v^{102} + v^{100} + v^{98} + v^{96} + v^{92}$	$v^{119} + v^{117} + v^{115} + 2v^{113} + 2v^{111} + 3v^{109} + 3v^{107} + 2v^{105} + 3v^{103} + 3v^{101} + 2v^{99} + v^{97} + v^{95} + v^{93}$	$v^{118} + 2v^{116} + 2v^{114} + 3v^{112} + 4v^{110} + 4v^{108} + 4v^{106} + 4v^{104} + 4v^{102} + 4v^{100} + 3v^{98} + 2v^{96} + 2v^{94} + v^{92}$
4	$v^{117} + v^{115} + v^{113} + 2v^{111} + 2v^{109} + 2v^{107} + 2v^{105} + 2v^{103} + 2v^{101} + 2v^{99} + v^{97} + v^{95} + v^{93}$	$v^{118} + 2v^{116} + 2v^{114} + 3v^{112} + 4v^{110} + 4v^{108} + 4v^{106} + 4v^{104} + 4v^{102} + 4v^{100} + 3v^{98} + 2v^{96} + 2v^{94} + v^{92}$	$v^{119} + 2v^{117} + 3v^{115} + 4v^{113} + 5v^{111} + 6v^{109} + 6v^{107} + 6v^{105} + 6v^{103} + 6v^{101} + 5v^{99} + 4v^{97} + 3v^{95} + 2v^{93} + v^{91}$
5	$v^{116} + v^{114} + v^{112} + 2v^{110} + v^{108} + 2v^{106} + 2v^{104} + v^{102} + 2v^{100} + v^{98} + v^{96} + v^{94}$	$v^{117} + v^{115} + 2v^{113} + 2v^{111} + 2v^{109} + 3v^{107} + 2v^{105} + 3v^{103} + 2v^{101} + 2v^{99} + 2v^{97} + v^{95} + v^{93}$	$v^{118} + 2v^{116} + 3v^{114} + 4v^{112} + 4v^{110} + 5v^{108} + 5v^{106} + 5v^{104} + 5v^{102} + 4v^{100} + 4v^{98} + 3v^{96} + 2v^{94} + v^{92}$
6	$v^{115} + v^{113} + v^{111} + v^{109} + v^{107} + 2v^{105} + v^{103} + v^{101} + v^{99} + v^{97} + v^{95}$	$v^{116} + 2v^{114} + 2v^{112} + v^{110} + 2v^{108} + 2v^{106} + 2v^{104} + 2v^{102} + v^{100} + 2v^{98} + v^{96} + v^{94}$	$v^{117} + 2v^{115} + 3v^{113} + 3v^{111} + 3v^{109} + 3v^{107} + 4v^{105} + 3v^{103} + 3v^{101} + 3v^{99} + 2v^{97} + v^{95} + v^{93}$
7	$v^{114} + v^{112} + v^{108} + v^{106} + v^{104} + v^{102} + v^{98} + v^{96}$	$v^{115} + 2v^{113} + v^{111} + v^{109} + v^{107} + 2v^{105} + v^{103} + v^{101} + v^{99} + v^{97} + v^{95}$	$v^{116} + 2v^{114} + 2v^{112} + 2v^{110} + 2v^{108} + 2v^{106} + 3v^{104} + 2v^{102} + 2v^{100} + 2v^{98} + 2v^{96} + v^{94}$
8	$v^{113} + v^{107} + v^{103} + v^{97}$	$v^{114} + v^{112} + v^{108} + v^{106} + v^{104} + v^{102} + v^{98} + v^{96}$	$v^{115} + v^{113} + v^{111} + v^{109} + v^{107} + 2v^{105} + v^{103} + v^{101} + v^{99} + v^{97} + v^{95}$

Table 6 (continued). Kazhdan–Lusztig polynomials $p_{e,w}$ for $w \in \mathcal{J}$ in E_8 .

	5	6	7	8
1	$v^{116} + v^{114} + v^{112} + 2v^{110} + v^{108} + 2v^{106} + 2v^{104} + v^{102} + 2v^{100} + v^{98} + v^{96} + v^{94}$	$v^{115} + v^{113} + v^{111} + v^{109} + v^{107} + 2v^{105} + v^{103} + v^{101} + v^{99} + v^{97} + v^{95}$	$v^{114} + v^{112} + v^{108} + v^{106} + v^{104} + v^{102} + v^{98} + v^{96}$	$v^{113} + v^{107} + v^{103} + v^{97}$
2	$v^{117} + v^{115} + 2v^{113} + 2v^{111} + 2v^{109} + 3v^{107} + 2v^{105} + 3v^{103} + 2v^{101} + 2v^{99} + 2v^{97} + v^{95} + v^{93}$	$v^{116} + v^{114} + 2v^{112} + v^{110} + 2v^{108} + 2v^{106} + 2v^{104} + 2v^{102} + v^{100} + 2v^{98} + v^{96} + v^{94}$	$v^{115} + v^{113} + v^{111} + v^{109} + v^{107} + 2v^{105} + v^{103} + v^{101} + v^{99} + 2v^{97} + v^{95}$	$v^{114} + v^{110} + v^{106} + v^{104} + v^{100} + v^{96}$
3	$v^{117} + 2v^{115} + 2v^{113} + 3v^{111} + 3v^{109} + 3v^{107} + 4v^{105} + 3v^{103} + 3v^{101} + 3v^{99} + 2v^{97} + 2v^{95} + v^{93}$	$v^{116} + 2v^{114} + 2v^{112} + 2v^{110} + 2v^{108} + 3v^{106} + 3v^{104} + 2v^{102} + 2v^{100} + 2v^{98} + 2v^{96} + v^{94}$	$v^{115} + 2v^{113} + v^{111} + v^{109} + 2v^{107} + 2v^{105} + 2v^{103} + v^{101} + v^{99} + 2v^{97} + v^{95}$	$v^{114} + v^{112} + v^{108} + v^{106} + v^{104} + v^{102} + v^{98} + v^{96}$
4	$v^{118} + 2v^{116} + 3v^{114} + 4v^{112} + 4v^{110} + 5v^{108} + 5v^{106} + 5v^{104} + 5v^{102} + 4v^{100} + 4v^{98} + 3v^{96} + 2v^{94} + v^{92}$	$v^{117} + 2v^{115} + 3v^{113} + 3v^{111} + 3v^{109} + 4v^{107} + 4v^{105} + 4v^{103} + 3v^{101} + 3v^{99} + 3v^{97} + 2v^{95} + v^{93}$	$v^{116} + 2v^{114} + 2v^{112} + 2v^{110} + 2v^{108} + 3v^{106} + 3v^{104} + 2v^{102} + 2v^{100} + 2v^{98} + 2v^{96} + v^{94}$	$v^{115} + v^{113} + v^{111} + v^{109} + v^{107} + 2v^{105} + v^{103} + v^{101} + v^{99} + v^{97} + v^{95}$
5	$v^{119} + v^{117} + 2v^{115} + 3v^{113} + 3v^{111} + 4v^{109} + 4v^{107} + 4v^{105} + 4v^{103} + 4v^{101} + 3v^{99} + 3v^{97} + 2v^{95} + v^{93} + v^{91}$	$v^{118} + v^{116} + 2v^{114} + 2v^{112} + 3v^{110} + 3v^{108} + 3v^{106} + 3v^{104} + 3v^{102} + 3v^{100} + 2v^{98} + 2v^{96} + v^{94} + v^{92}$	$v^{117} + v^{115} + v^{113} + 2v^{111} + 2v^{109} + 2v^{107} + 2v^{105} + 2v^{103} + 2v^{101} + 2v^{99} + v^{97} + v^{95} + v^{93}$	$v^{116} + v^{112} + v^{110} + v^{108} + v^{106} + v^{104} + v^{102} + v^{100} + v^{98} + v^{94}$
6	$v^{118} + v^{116} + 2v^{114} + 2v^{112} + 3v^{110} + 3v^{108} + 3v^{106} + 3v^{104} + 3v^{102} + 3v^{100} + 2v^{98} + 2v^{96} + v^{94} + v^{92}$	$v^{119} + v^{117} + v^{115} + v^{113} + 2v^{111} + 3v^{109} + 2v^{107} + 2v^{105} + 2v^{103} + 3v^{101} + 2v^{99} + v^{97} + v^{95} + v^{93} + v^{91}$	$v^{118} + v^{116} + v^{112} + 2v^{110} + 2v^{108} + v^{106} + v^{104} + 2v^{102} + 2v^{100} + v^{98} + v^{94} + v^{92}$	$v^{117} + v^{111} + v^{109} + v^{107} + v^{103} + v^{99} + v^{93}$
7	$v^{117} + v^{115} + v^{113} + 2v^{111} + 2v^{109} + 2v^{107} + 2v^{105} + 2v^{103} + 2v^{101} + 2v^{99} + v^{97} + v^{95} + v^{93}$	$v^{118} + v^{116} + v^{112} + 2v^{110} + 2v^{108} + v^{106} + v^{104} + 2v^{102} + 2v^{100} + v^{98} + v^{94} + v^{92}$	$v^{119} + v^{117} + v^{111} + 2v^{109} + v^{107} + v^{103} + 2v^{101} + v^{99} + v^{93} + v^{91}$	$v^{118} + v^{110} + v^{108} + v^{102} + v^{100} + v^{92}$
8	$v^{116} + v^{112} + v^{110} + v^{108} + v^{106} + v^{104} + v^{102} + v^{100} + v^{98} + v^{94}$	$v^{117} + v^{111} + v^{109} + v^{107} + v^{103} + v^{101} + v^{99} + v^{93}$	$v^{118} + v^{110} + v^{108} + v^{102} + v^{100} + v^{92}$	$v^{119} + v^{109} + v^{101} + v^{91}$

Using Lemma 5.2 along the vertical arrows in the last column in Figure 10, we can get

$$(70) \quad p_{78} = \frac{v^6 + v^4 - 1 + v^{-4} + v^{-6}}{v + v^{-1}} \cdot p_{18},$$

$$(71) \quad p_{88} = (v^6 - 1 + v^{-6}) \cdot p_{18}.$$

Applying again Lemma 5.2 to $w_{88} \rightarrow w_0$, we get

$$(72) \quad v \cdot p_{88} + p_{8,w_{88}} = v^{120} + p_{78}.$$

From (5) and the fact that $\Delta_8 \langle -1 \rangle \subset \Delta_e$, it follows that $p_{88} - v \cdot p_{8,w_{88}}$ has non-negative coefficients. From (70), (71) and (72) we get

$$(73) \quad p_{88} - v \cdot p_{8,w_{88}} = \frac{v^{16} + v^{14} - v^{10} - v^8 - v^6 + v^2 + 1}{v^6(v^2 + 1)} \cdot p_{18} - v^{121}.$$

Recall that w_{88} is a Duflo element, so the lowest degree monomial appearing in p_{88} is $v^{a(\mathcal{J})} = v^{91}$, and moreover, it appears with coefficient 1. Therefore, from (71) we see that we can write

$$p_{18} = v^{113} + a_1 v^{111} + a_2 v^{109} + a_3 v^{107} + a_4 v^{105} + a_5 v^{103} + a_6 v^{101} + a_7 v^{99} + v^{97}, \quad a_i \geq 0.$$

From (73) we see that p_{18} must be divisible by $v^2 + 1$, and therefore

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 = 2.$$

Write also

$$(74) \quad \frac{p_{18}}{v^6(v^2 + 1)} = v^{105} + b_1 v^{103} + b_2 v^{101} + b_3 v^{99} + b_4 v^{97} + b_5 v^{95} + b_6 v^{93} + v^{91},$$

where

$$(75) \quad \begin{aligned} a_1 &= 1 + b_1, \\ a_i &= b_{i-1} + b_i, \quad i = 2, 3, \dots, 6, \\ a_7 &= b_6 + 1. \end{aligned}$$

By multiplying (74) with the numerator in (73), and from the non-negativity of the coefficients in (73), we get the following conditions on b_i 's:

$$\begin{aligned} b_2 + b_3 - 1 &\geq 0, & b_1 - b_3 - b_4 - b_5 + 2 &\geq 0, \\ -b_1 + b_3 + b_4 - 1 &\geq 0, & b_1 + b_2 - b_4 - b_5 - b_6 &\geq 0, \\ -b_1 - b_2 + b_4 + b_5 - 1 &\geq 0, & b_2 + b_3 - b_5 - b_6 - 1 &\geq 0, \\ -b_1 - b_2 - b_3 + b_5 + b_6 &\geq 0, & b_3 + b_4 - b_6 - 1 &\geq 0, \\ -b_2 - b_3 - b_4 + b_6 + 2 &\geq 0, & b_4 + b_5 - 1 &\geq 0. \end{aligned}$$

Under (75), these conditions translate to the following conditions on a_i 's:

$$\begin{aligned} a_3 &\geq 1, & a_2 + 2 &\geq a_3 + a_5, \\ a_4 &\geq a_1, & a_2 + 1 &\geq a_5 + a_7, \\ a_5 &\geq a_2 + 1, & a_3 &\geq a_6 + 1, \\ a_6 + 1 &\geq a_1 + a_3, & a_4 &\geq a_7, \\ a_1 + a_7 &\geq a_2 + a_4, & a_5 &\geq 1. \end{aligned}$$

The fact that the latter set of conditions implies $a_1 = a_2 = a_4 = a_6 = a_7 = 0$ and $a_3 = a_5 = 1$ is left as an exercise. This finishes the proof. ■

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