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A parametrization algorithm to compute lower dimensional elliptic tori in Hamiltonian systems

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Abstract
We present an algorithm for the construction of lower dimensional elliptic tori in parametric Hamiltonian systems by means of the parametrization method with the tangent and normal frequencies being prescribed. This requires that the Hamiltonian system has as many parameters as the dimension of the normal dynamics, and the algorithm must adjust these parameters. We illustrate the methodology with an implementation of the algorithm computing 2–dimensional elliptic tori in a system of 4 coupled anharmonic oscillators (4 degrees of freedom).

Keywords: lower dimensional invariant tori, KAM theory, parametrization method.

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1. Introduction

Lower dimensional elliptic tori are invariant manifolds under the action of a Hamiltonian on which the internal motion conjugates to rigid quasi-periodic dynamics with a number of frequencies less than the number of degrees of freedom, and the complementary linear dynamics is linearly conjugated by harmonic oscillators. The second order normal form around an elliptic torus is

$$N = e + \sum_{i=1}^{d} \omega_i y_i + \frac{1}{2} \sum_{j=1}^{n-d} \beta_j \left(u_j^2 + v_j^2\right),$$

being e a constant, d the dimension of torus, $\omega = (\omega_1, \ldots, \omega_d)$ its internal frequency vector, and $\beta = (\beta_1, \ldots, \beta_{n-d})$ the normal frequency vector.

The study of the persistence of lower dimensional tori in nearly integrable Hamiltonian systems is one of the natural extensions of KAM theory ([1, 33, 40]). One of the first studies was by Melnikov [39]; the first proofs were given by Moser [40], Eliasson [15] and Bourgain [4]; other proofs, using a quadratic perturbative method, were given by Pöschel, see [41] and [42].

In this work we present an algorithm to compute parameterizations of lower dimensional elliptic tori. The main motivation of presenting this approach is given by the necessity to improve applicability. Indeed, in original versions of KAM theory, results are more of pure mathematical than physical interest: the existence of tori is subjugated to the extreme smallness of perturbing parameters (see [31, 35] where there is a good account of how small the distance to integrability is in classical examples like the 3 body problem). However, numerical explorations and semi-analytical procedures highlight how these invariant objects persist for values of the parameter way larger than the ones analytically prescribed (see [31]). In other words, there is a huge gap between the tori we expected to exist and the tori for which we could actually prove the existence using purely analytical proofs. In the case of full dimensional KAM tori, this gap has been dramatically reduced by applying either the techniques based on the computation of the Kolmogorov normal form (see [2, 24] for a quadratic and linear convergence with respect to the small parameter) or on the parametrization method (firstly introduced in [13, 14], see [26] for a review). The main advantage of these methods is that they provide algorithms which can be implemented and, therefore, used to produce good quality approximations of the invariant solutions. The *explicit* iteration of such theoretical algorithms was crucial for the success in applications. There have been extensions of these techniques to the case of lower dimensional tori ([7, 16, 21, 25, 30, 32, 34, 38]). In particular, in [38], the authors develop a KAM theorem for obtaining elliptic lower dimensional tori in a similar setting as ours. However, there are some differences, apart from the fact that their method has not been implemented: our geometrical setting is a little bit more general but, more importantly, we differ from their framework by fixing the normal frequencies and moving parameters, while they do not fix these frequencies but move them during the Newton scheme. Our algorithm could be easily adapted to their setting (see e.g. [28] for the numerical continuation of elliptic tori in quasi-periodic systems and some analysis of bifurcations at resonances) but we want to avoid resonances (to keep the elliptic character of the invariant tori, and fixing the normal frequencies and adjusting the parameters). Moreover, we have an eye in developing computer-assisted proofs of the existence of elliptic invariant tori with specific normal frequencies; in the future, we plan to obtain Lagrangian tori with the internal and normal frequencies of the seminal elliptic tori found here, and at this point, parameters also play an important role (see more comments in the conclusive section).

One of the advantages of the parametrization method is that it manipulates expressions with a number of variables equal to the dimension of the tori, while the normal form method requires as many variables as the dimension of the phase space. While on one hand this is not a problem from an analytical point of view, it can be a computational obstacle when considering applications with a moderate number of degrees of freedom (see, e.g. [9], where the algorithm described was basically unusable when increasing the number of nodes to 16 in the FPU model, in view of the huge memory required to represent functions, even for 3-dimensional elliptic tori). Another advantage of the parametrization method is that it can be quite easily converted to a computer-assisted proof, once that an analytical proof of the convergence of the algorithm to the solution is provided. This has already been done for what concerns KAM Lagrangian tori in [19] and [20], therefore the extension to the lower dimensional case appears quite natural. Normal form approaches too can be complemented with computer-assisted proofs, as it has already been done in the case of Lagrangian tori in many cases (the first scheme was described in [11]; in [37] the scheme was adapted to a different algorithm and applied to a planetary model; also, it is available a public source code in [36]). However, the great number of variables involved in the computation of a torus, and the need to introduce a scheme of estimates, make the adaptation less straightforward. Moreover, a computer-assisted proof in this setting is not yet available in the case of lower dimensional tori. Other remarkable results in the context of CAP and KAM are, for example, in the context of spin–orbit problem [5, 6] or the restricted 3 body problem [12, 23].

In this paper the frequency vector that describes the transverse dynamics is fixed. This is particularly important when the final goal is to produce a formal proof. Indeed, in order to perform a step of the algorithm, we require a set of non-resonant frequencies. In particular, the standard non-resonance conditions in this problem are the Diophantine and the first and second Melnikov condition. From a semi-analytical point of view, that is when we compute an approximated invariant solution on a computer, it is not a problem to have different frequencies at each step, because we deal with finite Fourier expansions. Therefore, we only have to check that we are not encountering small order resonances for the frequencies that we encounter. However, for the proof of the convergence we need Diophantine and Melnikov condition. Without fixed frequencies, we would have different non-resonance conditions at each step. In such a case, the convergence of the procedure can be proved only in a measure sense (see, e.g. [7, 25, 41], and [3], where the size of the resonant region is measured to be $\mathcal{O}(\varepsilon^{b_1})$ with $b_1 < 1/2$). To fix the frequencies, we consider an initial family of Hamiltonians depending on a number of parameters equal to the dimension of the transverse dynamics. Without these free parameters, the frequencies would be slightly corrected at each step (or, in other words, at each step we would solve a different Equation). One must point out, however, that the algorithm described below could be easily modified to, instead of moving free parameters, change the frequencies at each step (see for example the projection and reducibility methods in [27]).

This article is organised as follows: in section 2 we introduce the notation and set the equations that we are going to consider; Section 3 is devoted to the description of the generic step of the algorithm; section 4 contains an application of the presented procedure to a dynamical system made of four anharmonic oscillators coupled by springs. This system is a toy model that serves the purpose of testing the algorithm developed in previous sections. Finally, in section 5 we report the conclusions and some comments about future extensions and applications. In the appendix we collect some technical propositions.

2. Setting and invariance equations

2.1. General notation

We denote by $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ the n–dimensional torus. The spaces of $n_1 \times n_2$ matrices are denoted by $\mathbb{R}^{n_1\times n_2}$. In particular we use the notation I_n and O_n to denote the $n\times n$ identity matrix and the zero matrix, respectively.

Given $v \in \mathbb{R}^n$, $\mathrm{diag}(v)$ denotes the $n\times n$ matrix with its diagonal the ordered elements of v. Also, given an $n\times n$ matrix A, we denote by $\mathrm{diag}(A)$ the matrix that contains only the diagonal elements of A, while all the other elements are put to zero; $\mathbf{diag}(A)$ is a vector with components the diagonal of A. Given a $2n\times 2n$ matrix $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}$ (B_{ij} are $n\times n$ matrices), we define $\mathcal{P}(B) = \begin{pmatrix} \mathrm{diag}(B_{11}) & \mathrm{diag}(B_{12}) \\ \mathrm{diag}(B_{21}) & \mathrm{diag}(B_{22})\end{pmatrix}$.

Given an analytic function $f\colon \mathbb{T}^d \to \mathbb{C}$, we write its Fourier expansion as

$$f(\theta) = \sum_{k\in\mathbb{Z}^d} \hat{f}_k \mathrm{e}^{\mathrm{i}k\cdot\theta}, \quad \hat{f}_k = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(\theta)\,\mathrm{e}^{-\mathrm{i}k\cdot\theta} d\theta$$

and use the notation $\langle f\rangle := \hat{f}_0$ for the average over θ.

Finally, we will assume that functions live in the (scaled) Banach space of analytic functions and denote the norm by $\|\cdot\|$. Moreover, given functions $f, g_1, \dots, g_n$, we write that f is $\mathcal{O}(g_1,\dots,g_n)$ meaning that there exists a constant C such that $\|f\| \leqslant C(\|g_1\| + \cdots + \|g_n\|)$.

2.1.1. Symplectic setting. In this paper the phase space is an open set $U \subset \mathbb{R}^{2n}$ with coordinates $z = (z_1,\dots,z_{2n})$. It is endowed with a non-degenerate exact symplectic form $\boldsymbol{\omega} = d\boldsymbol{\alpha}$ for a certain 1-form $\boldsymbol{\alpha}$. We assume U is also endowed with a Riemannian metric $\mathbf{g}$ and an anti-involutive linear isomorphism $\mathbf{J} : \mathrm{T}U \to \mathrm{T}U$, i.e. $\mathbf{J}^2 = -I$, such that $\forall\, z \in U$, $\forall u, v \in \mathrm{T}_z U$, $\boldsymbol{\omega}_z(\mathbf{J}_z u, v) = \mathbf{g}_z(u, v)$. It is said that $(\boldsymbol{\omega}, \mathbf{g}, \mathbf{J})$ is a compatible triple and that $\mathbf{J}$ endows U with an almost-complex structure. Hence, the anti-involution preserves both 2-forms $\boldsymbol{\omega}$ and $\mathbf{g}$.

We represent the Riemannian metric $\mathbf{g}$, the almost-complex structure $\mathbf{J}$, the 1-form $\boldsymbol{\alpha}$ and the 2-form $\boldsymbol{\omega}$ as the matrix maps $G : U \to \mathbb{R}^{2n\times 2n}$, $J : U \to \mathbb{R}^{2n\times 2n}$, $a : U \to \mathbb{R}^n$ and $\Omega : U \to \mathbb{R}^{2n\times 2n}$, given by $a(z) = (a_1(z),\dots,a_{2n}(z))^\top$ and $\Omega(z) = \mathrm{D}a(z)^\top - \mathrm{D}a(z)$ satisfying $\det\Omega(z) \neq 0$. Since $\boldsymbol{\omega}$ is closed, it follows that for any triplet (r,s,t)

$$\frac{\partial \Omega_{r,s}}{\partial z_t} + \frac{\partial \Omega_{s,t}}{\partial z_r} + \frac{\partial \Omega_{t,r}}{\partial z_s} = 0\,.$$

Moreover, $\Omega^\top = -\Omega$ and it is pointwise invertible. The metric condition of $\mathbf{g}$ reads as $G^\top = G$ and it is positive definite. Furthermore, $\Omega J = -G$, $J^\top \Omega J = \Omega$, $J^\top G J = G$.

Remark 2.1. The standard 1-form $\boldsymbol{\alpha}_z$ is $\sum_{i=1}^n z_{n+i}\mathrm{d}z_i$ the 2-form $\boldsymbol{\omega}_z$ is $\sum_{i=1}^n \mathrm{d}z_{n+i}\wedge \mathrm{d}z_i$; so $a(z) = (z_{n+1},\dots,z_{2n},0,\dots,0)$ and $\Omega(z) = \Omega_n := \begin{pmatrix} O_n & -I_n \\ I_n & O_n\end{pmatrix}$.

Given a function $h : U \to \mathbb{R}$ the corresponding Hamiltonian vector field $X_h : U \to \mathbb{R}^{2n}$ satisfies $i_{X_h}\boldsymbol{\omega} = -\mathrm{d}h$ ($\boldsymbol{\omega}(\cdot, X_h) = \mathrm{d}h$). In coordinates, it satisfies

$$X_h(z) = (\Omega(z))^{-1}(\mathrm{D}_z h(z))^\top\,.$$

2.2. Invariance equations for lower dimensional invariant elliptic tori

Consider a family of Hamiltonians $h_\lambda : U \subset \mathbb{R}^{2n} \to \mathbb{R}$, depending on a parameter $\lambda \in \mathbb{R}^{n-d}$. We look for a lower dimensional elliptic torus of dimension $d < n$ whose tangent and normal frequency vectors are $\omega \in \mathbb{R}^d$ and $\beta \in \mathbb{R}^{n-d}$, respectively. Both ω and β are fixed and $\beta_i \neq \beta_j \neq 0$ for $i \neq j$.

A parametrization of an invariant torus $K : \mathbb{T}^d \to \mathbb{R}^{2n}$ and of its normal bundle $W : \mathbb{T}^d \to \mathbb{R}^{2n\times 2(n-d)}$ must satisfy the following equations:

$$\begin{cases} \mathcal{L}_\omega K + X_h \circ (K;\lambda) = 0 \\ \mathcal{L}_\omega W + \mathrm{D}X_h \circ (K;\lambda)\, W - W\Gamma_{0,\beta} = 0 \end{cases}, \tag{1}$$

where $\mathcal{L}_\omega$ is the *Lie derivative* operator in the direction of ω, that given a function $u : \mathbb{R}^d \to \mathbb{R}$ applies as $\mathcal{L}_\omega u = -\mathrm{D}u \cdot \omega$, while $\Gamma_{0,\beta}$ is the following $2(n-d)$-dimensional matrix

$$\Gamma_{0,\beta} = \begin{pmatrix} O_{n-d} & -\mathrm{diag}\,(\beta) \\ \mathrm{diag}\,(\beta) & O_{n-d} \end{pmatrix}.$$

The first equation in (1) is the standard *invariance equation* and it tells us that the torus is invariant for the Hamiltonian flow associated to h and the motion on it is the rigid rotation with frequency vector ω. Indeed, for every $\theta_0 \in \mathbb{T}^d$, $\phi(t) = K(\omega t + \theta_0)$ satisfies $X_h(K(\omega t + \theta_0)) = X_h(\phi(t)) = \dot{\phi}(t) = \mathrm{D}K(\omega t + \theta_0)\,\omega = -\mathcal{L}_\omega K_{|\omega t + \theta_0}$.

The second equation comes from the study of the *linear dynamics* normal to the torus, that we impose to be conjugated to a constant linear vector field with matrix $\Gamma_{0,\beta}$. Again, if we consider $\phi(t) = K(\theta + \omega t)$ with $\theta \in \mathbb{T}^d$ and we take $z = \phi + v$ with v small, then

$$\dot{z} = \dot{\phi} + \dot{v} = X_h(\phi + v) = X_h(\phi) + \mathrm{D}X_h(\phi)\, v + h.o.t.,$$

where $h.o.t.$ denotes *high order terms*. Since $\dot{\phi} = X_h(\phi)$ because of the invariance, at first order v satisfies the following equation:

$$\dot{v} = \mathrm{D}X_h(\phi)\, v\,.$$

The particular equation for W comes from the reducibility property that we require to solve the equation. The shape of the equation is explained in section 2.2 of [38] and will be more clear in the decription of the algorithm.

In order to design our algorithm to solve the invariance equation (1), we introduce a dummy parameter $\alpha \in \mathbb{R}^{n-d}$ and denote

$$\Gamma_{\alpha,\beta} = \begin{pmatrix} \mathrm{diag}\,(\alpha) & -\mathrm{diag}\,(\beta) \\ \mathrm{diag}\,(\beta) & \mathrm{diag}\,(\alpha) \end{pmatrix}.$$

Then, we define a Newton-like method to construct the exact solution, starting from the approximate solution (K, W, λ, α) satisfying

$$\begin{cases} \mathcal{L}_\omega K + X_h \circ (K;\lambda) = E_K \\ \mathcal{L}_\omega W + \mathrm{D}X_h \circ (K;\lambda)\, W - W\Gamma_{\alpha,\beta} = E_W \end{cases}, \tag{2}$$

where the invariance errors E_K and E_W are 'small'. The role of the parameter λ is to introduce shear in the normal dynamics (non-vanishing normal torsion), making it possible the existence of tori with the normal dynamics's frequency β. The role of α is to facilitate the step by making it possible to cancel some averages (more details are deferred to Subsection 3.2.1). An important issue is that symplectic properties will force *a posteriori* that in fact the extra-parameter α in $\Gamma_{\alpha,\beta}$ must be zero if both invariance errors equal to zero, see proposition 2.3.

For the success of the procedure (and, eventually, in order to produce a proof of convergence), some hypotheses are required. First, we need to assume some non-degeneracy condition. One condition is usually requested in KAM theorems and relies on the fact that we are fixing the frequency vector ω. This condition in the parametrization method is equivalent to the invertibility of a torsion matrix, defined later. The second condition is about how to choose the parameter λ. In order to do that, we must assume that the family of Hamiltonians is non-degenerate in some sense, to be detailed during the presentation of the algorithm.

The standard non-resonance hypotheses on the frequency vectors ω and β that we require are the following.

Definition 2.2. We say that $(\omega,\beta)\in\mathbb{R}^d\times\mathbb{R}^{n-d}$ satisfies Diophantine-Melnikov conditions of type (γ,τ), with $\gamma>0$, $\tau>d-1$, if for all $k\in\mathbb{Z}^d\setminus\{0\}$, $i,j=1,\dots,n-d$ with $i\neq j$, $|\beta_i|\neq|\beta_j|\neq 0$, and:

0) $|k\cdot\omega|\geqslant\frac{\gamma}{|k|_1^\tau}$ (Diophantine condition);
1) $|k\cdot\omega-\beta_i|\geqslant\frac{\gamma}{|k|_1^\tau}$ (first Melnikov condition);
2) $|k\cdot\omega-\beta_i\pm\beta_j|\geqslant\frac{\gamma}{|k|_1^\tau}$ (second Melnikov condition).

Let us remind that these conditions are quite generic, in the sense that the set of non-resonant frequencies has positive Lebesgue measure.

2.3. Reducibility

Given the matrix maps $V_1:\mathbb{T}^d\to\mathbb{R}^{2n\times m_1}, V_2:\mathbb{T}^d\to\mathbb{R}^{2n\times m_2}$ we define the maps $\Omega_{V_1V_2},G_{V_1V_2}:\mathbb{T}^d\to\mathbb{R}^{m_1\times m_2}$ as

$$\Omega_{V_1V_2}=V_1^\top\Omega\circ KV_2$$

and

$$G_{V_1V_2}=V_1^\top G\circ KV_2.$$

We define the map $P:\mathbb{T}^d\to\mathbb{R}^{2n\times 2n}$ as the three juxtaposed matrix maps as follows

$$P(\theta)=\left(L(\theta)\ N(\theta)\ W(\theta)\right), \tag{3}$$

where $L,\ N:\mathbb{T}^d\to\mathbb{R}^{2n\times d}$ are defined as

$$L=\mathrm{D}K, \tag{4}$$

$$N=LA+(J\circ K)LB+WC, \tag{5}$$

with $A,B:\mathbb{T}^d\to\mathbb{R}^{d\times d}$ and $C:\mathbb{T}^d\to\mathbb{R}^{2(n-d)\times d}$ are defined as

$$B=G_{LL}^{-1},\quad C=\Omega_{WW}^{-1}G_{WL}B,\quad A=\frac{1}{2}C^\top\,\Omega_{WW}C. \tag{6}$$

Finally, define the operator

$$\mathcal{X}_V=\mathcal{L}_\omega V+\mathrm{D}_zX_h\circ(K;\lambda)\,V$$

with $V:\mathbb{T}^d\to\mathbb{R}^{2n\times m}$.

In the following, we discuss properties of P that are a consequence of the invariance of K and W.

Proposition 2.3. *Under the hypothesis that both K and W are invariant (i.e. the invariance errors E_K and E_W appearing in (2) are zero) then*

1. $\alpha = 0$;
2. *P is symplectic, i.e.*
$$P^\top \Omega \circ K P = \Omega_{PP} = \begin{pmatrix} \Omega_d & O_{2d\times 2(n-d)} \\ O_{2(n-d)\times 2d} & \Omega_{WW} \end{pmatrix}, \tag{7}$$
is invertible and
$$P^{-1} = P_{inv} := \begin{pmatrix} N^\top \Omega \circ K \\ -L^\top \Omega \circ K \\ \Omega_{WW}^{-1} W^\top \Omega \circ K \end{pmatrix};$$
3. *the linearized dynamics is reducible, that is*
$$P^{-1} \mathcal{X}_P = \Lambda = \begin{pmatrix} O_d & T & O_{d\times 2(n-d)} \\ O_d & O_d & O_{d\times 2(n-d)} \\ O_{2(n-d)\times d} & O_{2(n-d)\times d} & \Gamma_{0,\beta} \end{pmatrix}, \tag{8}$$
where T is a matrix called torsion matrix and it is defined as
$$T := N^\top \Omega \circ K \mathcal{X}_N = N^\top \Omega \circ K \left(\mathcal{L}_\omega N + D_z X_h \circ (K; \lambda) N \right). \tag{9}$$

The proof of this proposition is a particular case of the more general ones appearing in the appendix, where both errors there are considered to be small but possibly non-zero.

Notice that the symplecticity of P provides a simple way to compute its inverse. Moreover, when the torus is *almost* invariant (E_K, E_W are small) then: α is small, P is *almost* symplectic and its inverse is approximately given by (2), and the linear dynamics are approximately conjugated to (8), all of these up to $\mathcal{O}(E_K, E_W)$. From now on, $P_{inv} := P^\top \Omega \circ K$ denotes the approximation of the inverse of P given in Proposition 2.3. Notice that P_{inv} differs from P^{-1} in the case of K or W not being invariant. See lemma A.4 and Corollary A.5 in the appendix for more details.

Remark 2.4. By a suitably scaling W one can make the antisymmetric matrix Ω_{WW} that appears in equation (7) equal to $\Omega_{2(n-d)}$.

Remark 2.5. The numerical computation of the term $\mathcal{L}_\omega N$ that appears in the definition of the torsion T in formula (9) can be rather demanding. This definition can be replaced by another one that does not involve the Lie derivative, as it has been done in [17].

3. Newton step

Given (approximated solutions) (K, W, λ, α) satisfying equation (2) with $E_K : \mathbb{T}^d \to \mathbb{R}^{2n}$ and $E_W : \mathbb{T}^d \to \mathbb{R}^{2n\times 2(n-d)}$ small, we want to obtain better approximations $(K_{\text{new}}, W_{\text{new}}, \lambda_{\text{new}}, \alpha_{\text{new}})$. For doing so, we design one step of the Newton method to look for corrections $\Delta K : \mathbb{T}^d \to \mathbb{R}^{2n}$, $\Delta W : \mathbb{T}^d \to \mathbb{R}^{2n\times 2(n-d)}$, $\Delta\lambda, \Delta\alpha \in \mathbb{R}^{n-d}$ such that

$$K_{\text{new}} := K + \Delta K,$$
$$W_{\text{new}} := W + \Delta W,$$

$$\lambda_{\text{new}} := \lambda + \Delta\lambda,$$
$$\alpha_{\text{new}} := \alpha + \Delta\alpha$$

satisfy equation (2) with errors of size $\mathcal{O}(E_K, E_W)^2$.
Some important facts are:

(i) the dummy parameter α will be of order of the errors;
(ii) the corrections of the new solutions $(\Delta K, \Delta W, \Delta\lambda, \Delta\alpha)$ will be of the order of the errors;
(iii) the invariance equation for K depends only on (K, λ);
(iv) under Diophantine-Melnikov conditions on (ω, β), given any fixed λ the invariance equation for K can always be solved;
(v) under Diophantine-Melnikov conditions on (ω, β) the linear dynamics equation for W admits a solution for fixed values of the vectors λ and α.

Remark 3.1. All these previous statements are without proofs but they will appear in a forthcoming paper [8]. Here our focus is on the algorithm.

One starts with a set of equations for all the unknowns and the first goal is to adjust the parameters $\Delta\lambda$ and $\Delta\alpha$. Once these parameters have been carefully chosen (see the detailed discussion in section 3.2.1), we get with a (solvable) set of cohomological equations of the type:

$$\begin{cases} \mathcal{L}_\omega u^L + Tu^N & = v^L \\ \mathcal{L}_\omega u^N & = v^N \\ \mathcal{L}_\omega u^W + \Gamma_{\alpha,\beta} u^W & = v^W \\ & \\ \mathcal{L}_\omega U^L + TU^N - U^L\Gamma_{\alpha,\beta} & = V^L \\ \mathcal{L}_\omega U^N - U^N\Gamma_{\alpha,\beta} & = V^N \\ \mathcal{L}_\omega U^W + \Gamma_{\alpha,\beta} U^W - U^W\Gamma_{\alpha,\beta} & = V^W \end{cases} \tag{10}$$

where T is defined in (9), v, V are given (there $\Delta\lambda$ and $\Delta\alpha$ have been absorbed) and u, U are the unknowns. $u, v : \mathbb{T}^d \to \mathbb{R}^{2n}$ and $(\cdot^L, \cdot^N, \cdot^W)$ denote their components that are d, d and $2(n-d)$ dimensional; while $U, V : \mathbb{T}^d \to \mathbb{R}^{2n\times 2(n-d)}$ decompose in $(\cdot^L, \cdot^N, \cdot^W)$ with d, d and $2(n-d)$ rows, respectively. The first two equations are the same that appear in standard KAM statements for Lagrangian tori (see for example [17]) and they require a Diophantine frequency vector and the invertibility of the d-dimensional square matrix $\langle T\rangle$ (also known as non-degeneracy or twist conditions). The third, fourth and fifth equations in equation (10), where also the matrix $\Gamma_{\alpha,\beta}$ appears, require first Melnikov condition on the frequency vectors (ω, β) in order to be solved. Finally, the last cohomological equation is the one that requires the second Melnikov non-resonance condition.

Remark 3.2. In appendix we prove that the parameter α is of the size of the invariance errors and, so, all $\Gamma_{\alpha,\beta}$ in equation (10) can be substituted by $\Gamma_{0,\beta}$. From now on, we do so.

In the following subsections, we describe in detail how we encounter these Equations and which are the known terms v, V appearing there.

3.1. Correction of the approximate solution of the invariance equation for K

We look for $(K_{\text{new}}, \lambda_{\text{new}})$ that satisfies the equation

$$\mathcal{L}_\omega (K+\Delta K) + X_h \circ (K+\Delta K; \lambda+\Delta\lambda) = 0.$$

By Taylor expanding and discarding supralinear terms we get the following linear equation

$$\mathcal{L}_\omega \Delta K + \mathrm{D}_z X_h \circ (K;\lambda)\,\Delta K + \mathrm{D}_\lambda X_h \circ (K;\lambda)\,\Delta\lambda = -E_K. \tag{11}$$

In order to solve equation (11) more easily we use the almost symplectic change of coordinates P defined in (3) to write the unknown ΔK as

$$\Delta K(\theta) = P(\theta)\,\xi_K(\theta), \tag{12}$$

with $\xi_K : \mathbb{T}^d \to \mathbb{R}^{2n}$. By substituting (12) in equation (11) we get

$$\mathcal{L}_\omega P \xi_K + P \mathcal{L}_\omega \xi_K + \mathrm{D}_z X_h \circ (K;\lambda)\, P\xi_K + \mathrm{D}_\lambda X_h \circ (K;\lambda)\,\Delta\lambda = -E_K,$$

that can be rearranged as

$$\begin{aligned}
&\mathcal{L}_\omega \xi_K + P^{-1}\left(\mathcal{L}_\omega P + \mathrm{D}_z X_h \circ (K;\lambda)\,P\right)\xi_K + P^{-1}\mathrm{D}_\lambda X_h \circ (K;\lambda)\,\Delta\lambda + \mathcal{O}\left(E_K, E_W\right)^2 \\
&\quad = \mathcal{L}_\omega \xi_K + \Lambda \xi_K + P_{inv} \mathrm{D}_\lambda X_h \circ (K;\lambda)\,\Delta\lambda + \mathcal{O}\left(E_K, E_W\right)^2 = -P_{inv} E_K,
\end{aligned} \tag{13}$$

where Λ is the matrix defined in equation (8).

Remark 3.3. The $\mathcal{O}(E_K, E_W)^2$ term in equation (13) comes from both the errors in the reducibility and the invertibility of P times the norm of ξ_K, that is $\mathcal{O}(E_K, E_W)$.

Discarding all supralinear terms and using equation (A6) the equation *reduces* to

$$\mathcal{L}_\omega \xi_K + \Lambda \xi_K + b\cdot \Delta\lambda = \eta_K, \tag{14}$$

where

$$\begin{aligned}
\eta_K(\theta) &= -P_{inv}(\theta)\,E_K(\theta) \in \mathbb{R}^{2n}, \\
b(\theta) &= P_{inv}(\theta)\,\mathrm{D}_\lambda X_h \circ (K(\theta);\lambda) \in \mathbb{R}^{2n\times(n-d)}.
\end{aligned}$$

3.1.1. Resolution of the invariance equation for K. We split the solution ξ_K of equation (14) in two components:

$$\xi_K(\theta) = \xi_{\eta_K}(\theta) - \xi_b(\theta)\,\Delta\lambda, \tag{15}$$

where ξ_{η_K} and ξ_b satisfy $\mathcal{L}_\omega \xi_{\eta_K} + \Lambda \xi_{\eta_K} = \eta_K$ and $\mathcal{L}_\omega \xi_b + \Lambda \xi_b = b$. Moreover, by expressing the vectors ξ_{η_K}, ξ_b in components $(\cdot^L, \cdot^N, \cdot^W)$ and in view of the special shape of the matrix Λ defined in equation (8), we obtain that both triples satisfy the equations

$$\begin{cases}
\mathcal{L}_\omega \xi^L + T\xi^N & = \eta_K^L,\ b^L \\
\mathcal{L}_\omega \xi^N & = \eta_K^N,\ b^N \\
\mathcal{L}_\omega \xi^W + \Gamma_{0,\beta}\xi^W & = \eta_K^W,\ b^W
\end{cases}. \tag{16}$$

The first couple of cohomological equations are the same as in the Lagrangian case (see [17]). In particular, the equations with the $\cdot^N$ component can be solved provided that η_K^N and b^N have zero average. This is the case because:

- By definition we have $\eta_K^N = L^\top \Omega \circ K E_K$. Therefore,

$$\begin{aligned}\eta_K^N &= L^\top \Omega\circ K\left(\mathcal{L}_\omega K + X_h\circ(K;\lambda)\right)\\ &= L^\top \Omega\circ K\left(-\mathrm{D}K\cdot\omega + (\Omega\circ K)^{-1}\left(\mathrm{D}_z h\circ(K;\lambda)\right)^\top\right)\\ &= -\Omega_{LL}\cdot\omega + \mathrm{D}_\theta\left(h\circ(K;\lambda)\right)^\top.\end{aligned}$$

  Finally, Ω_{LL} has zero average since the 2-form is exact, see e.g. the proof in Lemma 4.3 of [29]), while the second term has zero average since it is a derivative of periodic functions.
- Similarly for b^N, by using its definition and expressing it as a derivative of periodic functions:

$$\begin{aligned}b^N &= -L^\top \Omega\circ K\,\mathrm{D}_\lambda X_h\circ(K;\lambda)\\ &= -\mathrm{D}K^\top \Omega\circ K \mathrm{D}_\lambda\left((\Omega\circ K)^{-1}\left(\mathrm{D}_z h\circ(K;\lambda)\right)^\top\right)\\ &= -\mathrm{D}K^\top \mathrm{D}_\lambda\left(\mathrm{D}_z h\circ(K;\lambda)\right)^\top = -\mathrm{D}_\lambda\left(\mathrm{D}_\theta\left(h\circ(K;\lambda)\right)\right)^\top.\end{aligned}$$

We start by solving the $\cdot^N$ equations in (16) (up to the average). Since it is of the form $\mathcal{L}_\omega u = v$ its solutions are of the form[4]

$$u(\theta) = \langle u\rangle + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{\mathbf{i}\,\hat{v}_k}{k\cdot\omega}\mathrm{e}^{\mathbf{i}k\cdot\theta}.$$

Then, solving the $\cdot^L$ equations is the same once the average of the $\cdot^N$ terms is fixed. This is done by making the cohomological equations solvable, that is, with zero average. This is accomplished by selecting

$$\begin{aligned}\langle\xi_{\eta_K}^N\rangle &= \langle T\rangle^{-1}\left(\langle\eta_K^L - T\xi^N\rangle\right)\\ \langle\xi_b^N\rangle &= \langle T\rangle^{-1}\left(\langle b^L - T\xi^N\rangle\right)\end{aligned}.$$

Here, we use the non-degeneracy of T (the invertibility of its average).

Remark 3.4. We notice that fixing the average of ξ^L is unimportant (plays no role except a shift on the parameterization). Hence, we fix it to be equal to zero.

The $\cdot^W$ equations are of the form

$$\mathcal{L}_\omega u + \Gamma_{0,\beta} u = v.$$

By denoting $u = (u^1,\dots,u^{2(n-d)})$ this equation in terms of its Fourier coefficients is expressed as follows

$$\begin{pmatrix}-\mathbf{i}k\cdot\omega & -\beta_j\\ \beta_j & -\mathbf{i}k\cdot\omega\end{pmatrix}\begin{pmatrix}\hat{u}_k^j\\ \hat{u}_k^{j+n-d}\end{pmatrix} = \begin{pmatrix}\hat{v}_k^j\\ \hat{v}_k^{j+n-d}\end{pmatrix}\quad\text{with}\quad j=1,\dots,n-d. \tag{17}$$

Notice that solutions exist as long as there are no resonances of the form $k\cdot\omega \neq \pm\beta_j$. Again, by imposing further the first Melnikov condition (see definition 2.2), we obtain analytic solutions.

Finally, according to equation (15), the new parametrization is

$$K_{\text{new}} = K + \Delta K = K + P\left(\xi_{\eta_K} - \xi_b\Delta\lambda\right). \tag{18}$$

[4] By explicitly writing the Lie derivative $\mathcal{L}_\omega u$ and the Fourier expansions of u and v, the equation reads as $-\partial_\theta(\sum_k \hat{u}_k \mathrm{e}^{\mathbf{i}k\cdot\theta})\cdot\omega = \sum_{k\neq 0}\hat{v}_k\mathrm{e}^{\mathbf{i}k\cdot\theta}$ and one can solve it term by term. Clearly the term $\langle u\rangle$ is free.

The term $\Delta\lambda$ will be determined in the next step, when we solve the invariance equation for W.

3.2. Correction of the approximate solution for the invariance equation for W

As before, we put in $(K_{\rm new}, W_{\rm new}, \lambda_{\rm new}, \alpha_{\rm new})$ in the invariance equation for W in (2) obtaining

$$\mathcal{L}_\omega (W + \Delta W) + \mathrm{D}_z X_h \circ (K + \Delta K; \lambda + \Delta\lambda)(W + \Delta W) - (W + \Delta W)\Gamma_{\alpha+\Delta\alpha,\beta} = 0.$$

By Taylor expanding and discarding supralinear terms (recall that also $\Gamma_{\alpha,\beta}$ is then replaced by $\Gamma_{0,\beta}$) we get

$$\begin{aligned}\mathcal{L}_\omega \Delta W + \mathrm{D}_z X_h \circ (K;\lambda)\,\Delta W + \mathrm{D}_{\lambda z} X_h \circ (K;\lambda)\,[\Delta\lambda, W] \\ + \mathrm{D}_{zz} X_h \circ (K;\lambda)\,[\Delta K, W] - \Delta W \Gamma_{0,\beta} - W\Gamma_{\Delta\alpha,0} = -E_W \end{aligned} \tag{19}$$

where

$$\mathrm{D}_{\lambda z} X_h \circ (K;\lambda)\,[\cdot, W] : \Delta\lambda \to \Delta\lambda^\top \mathrm{D}_{\lambda z} X_h \circ (K;\lambda)\,W,$$

being $\mathrm{D}_{\lambda z} X_h \circ (K;\lambda)$ a collection of $n-d$ squared matrices of dimension $2n$ with[5]

$$(\mathrm{D}_{\lambda_k z} X_h \circ (K;\lambda))_{i,j} = \partial_{\lambda_k}\partial_{z_i}(X_h \circ (K;\lambda))_j, \ \forall\, i,j = 1,\ldots,2n,\ k = 1,\ldots,n-d,$$

and

$$\mathrm{D}_{zz} X_h \circ (K;\lambda)\,[\cdot, W] : \Delta K \to \Delta K^\top \mathrm{D}_{zz} X_h \circ (K;\lambda)\,W$$

being $\mathrm{D}_{zz} X_h \circ (K;\lambda)$ a collection of $2n$ squared matrices of dimension $2n$ with

$$(\mathrm{D}_{z_k z} X_h \circ (K;\lambda))_{i,j} = \partial_{z_k}\partial_{z_i}(X_h \circ (K;\lambda))_j \ \ \forall\, i,j,k = 1,\ldots,2n.$$

By applying the almost symplectic change of coordinates P,

$$\Delta W(\theta) = P(\theta)\,\xi_W(\theta)$$

with $\xi_W : \mathbb{T}^d \to \mathbb{R}^{2n\times 2(n-d)}$, and removing supralinear terms in (19) we obtain

$$\mathcal{L}_\omega \xi_W + \Lambda \xi_W - \xi_W \Gamma_{0,\beta} - P^{-1} W \Gamma_{\Delta\alpha,0} + \mathcal{B}(\Delta\lambda) + \mathcal{C}(\Delta K) = \eta_W, \tag{20}$$

where $\eta_W, \mathcal{B}(\Delta\lambda), \mathcal{C}(\Delta K) : \mathbb{T}^d \to \mathbb{R}^{2n\times 2(n-d)}$ are defined as

$$\begin{aligned} \eta_W(\theta) &= -P_{inv}(\theta)\,E_W(\theta), \\ \mathcal{B}(\Delta\lambda)(\theta) &= P_{inv}(\theta)\,\mathrm{D}_{\lambda z} X_h(K(\theta);\lambda)\,[\Delta\lambda, W(\theta)], \\ \mathcal{C}(\Delta K)(\theta) &= P_{inv}(\theta)\,\mathrm{D}_{zz} X_h(K(\theta);\lambda)\,[\Delta K(\theta), W(\theta)]. \end{aligned}$$

Since $\Delta K = P\xi_{\eta_K} - P\xi_b \Delta\lambda$ with the unknown $\Delta\lambda$, we split the contribution of $\mathcal{C}(\Delta K)$ as follows:

1. the known terms, i.e. $\mathcal{C}(P\xi_{\eta_K})$, are moved to the r.h.s. of equation (20), thus replacing η_W by

$$\hat{\eta}_W = \eta_W - \mathcal{C}(P\xi_{\eta_K});$$

2. the terms depending on $\Delta\lambda$, i.e. $-\mathcal{C}(P\xi_b\Delta\lambda)$, are collected in

$$\hat{\mathcal{B}}(\Delta\lambda) = \mathcal{B}(\Delta\lambda) - \mathcal{C}(P\xi_b\Delta\lambda).$$

[5] $(X_h)_j$ denotes the jth component of the vector field.

Therefore, we obtain

$$\mathcal{L}_\omega \xi_W + \Lambda \xi_W - \xi_W \Gamma_{0,\beta} - P^{-1} W \Gamma_{\Delta\alpha,0} + \hat{\mathcal{B}}(\Delta\lambda) = \hat{\eta}_W,$$

which splits into the set of equations

$$\begin{cases} \mathcal{L}_\omega \xi_W^L + T\xi_W^N - \xi_W^L \Gamma_{0,\beta} & = \hat{\eta}_W^L - \hat{\mathcal{B}}^L(\Delta\lambda) \\ \mathcal{L}_\omega \xi_W^N - \xi_W^N \Gamma_{0,\beta} & = \hat{\eta}_W^N - \hat{\mathcal{B}}^N(\Delta\lambda) \\ \mathcal{L}_\omega \xi_W^W + \Gamma_{0,\beta}\xi_W^W - \xi_W^W \Gamma_{0,\beta} & = \hat{\eta}_W^W - \hat{\mathcal{B}}^W(\Delta\lambda) + \Gamma_{\Delta\alpha,0} \end{cases}. \tag{21}$$

We start by solving the third equation in (21), that is of the form

$$\left(\mathcal{L}^2_{\omega,\Gamma_{0,\beta}}\right)(u) = \mathcal{L}_\omega u + \Gamma_{0,\beta} u - u\Gamma_{0,\beta} = v, \tag{22}$$

with $u : \mathbb{T}^d \to \mathbb{R}^{2(n-d)\times 2(n-d)}$. In terms of Fourier coefficient and components ($\hat{u}_k^{i,j}$ denotes the kth Fourier coefficient of the element i,j of u) this equation is equivalent to

$$\begin{pmatrix} -\mathbf{i}k\cdot\omega & -\beta_j & -\beta_i & 0 \\ \beta_j & -\mathbf{i}k\cdot\omega & 0 & -\beta_i \\ \beta_i & 0 & -\mathbf{i}k\cdot\omega & -\beta_j \\ 0 & \beta_i & \beta_j & -\mathbf{i}k\cdot\omega \end{pmatrix} \cdot \begin{pmatrix} \hat{u}_k^{i,j} \\ \hat{u}_k^{i,j+n-d} \\ \hat{u}_k^{i+n-d,j} \\ \hat{u}_k^{i+n-d,j+n-d} \end{pmatrix} = \begin{pmatrix} \hat{v}_k^{i,j} \\ \hat{v}_k^{i,j+n-d} \\ \hat{v}_k^{i+n-d,j} \\ \hat{v}_k^{i+n-d,j+n-d} \end{pmatrix}, \tag{23}$$

with $i,j = 1,\dots,n-d$. When $i \neq j$, and if $\beta_i \neq \pm\beta_j$, the equation admits a solution provided that the determinant of the matrix is not zero, i.e. assuming again a non-resonance condition $k\cdot\omega \neq \pm(\beta_i \pm \beta_j)$, in this case the second Melnikov. Therefore, we have to discuss the existence of a solution to the third equation in (21) only in the case $i = j$ and $k = 0$. This means that we have to consider the averaged terms in the diagonal of the four $n-d$ dimensional blocks. In matrix form, we have the following equation

$$\begin{pmatrix} 0 & -\beta_i & -\beta_i & 0 \\ \beta_i & 0 & 0 & -\beta_i \\ \beta_i & 0 & 0 & -\beta_i \\ 0 & \beta_i & \beta_i & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{u}_0^{i,i} \\ \hat{u}_0^{i,i+n-d} \\ \hat{u}_0^{i+n-d,i} \\ \hat{u}_0^{i+n-d,i+n-d} \end{pmatrix} = \begin{pmatrix} \hat{v}_0^{i,i} \\ \hat{v}_0^{i,i+n-d} \\ \hat{v}_0^{i+n-d,i} \\ \hat{v}_0^{i+n-d,i+n-d} \end{pmatrix}, \tag{24}$$

that has a solution if and only if $\hat{v}_0^{i,i} = -\hat{v}_0^{i+n-d,i+n-d}$ and $\hat{v}_0^{i,i+n-d} = \hat{v}_0^{i+n-d,i}$, i.e. $\langle v\rangle$ should be, in its diagonal and antidiagonal terms, of the form

$$\begin{pmatrix} \hat{v}_0^{i,i} & \hat{v}_0^{i,i+n-d} \\ \hat{v}_0^{i,i+n-d} & -\hat{v}_0^{i,i} \end{pmatrix}. \tag{25}$$

In our equation $\hat{v}_0 = \langle \hat{\eta}_W^W - \hat{\mathcal{B}}^W(\Delta\lambda) + \Gamma_{\Delta\alpha,0}\rangle$. In the next subsection, we discuss how to choose $\Delta\lambda$ and $\Delta\alpha$ such that $\hat{\eta}_W^W - \hat{\mathcal{B}}^W(\Delta\lambda) + \Gamma_{\Delta\alpha,0} \in \Im(\mathcal{L}^2_{\omega,\Gamma_{0,\beta}})$, that is, how to make the equation solvable. Clearly, in view of the degenerate nature of the matrix, the solution is not unique. Indeed, every matrix of type

$$\begin{pmatrix} \hat{u}_0^{i,i} & \hat{u}_0^{i,i+n-d} \\ -\hat{u}_0^{i,i+n-d} & \hat{u}_0^{i,i} \end{pmatrix}$$

has zero image and therefore the solution will depend on 2 parameters ($\hat{u}_0^{i,i}$ and $\hat{u}_0^{i,i+n-d}$) for each $i = 1,\dots,n-d$.

Once we solve for $\Delta\lambda$ and $\Delta\alpha$ we can also solve the first and second equations.

3.2.1. Choice of $\Delta\lambda$ *and* $\Delta\alpha$. The values of $\Delta\lambda$ and $\Delta\alpha$ are determined while solving the third equation in formula (21). As mentioned before, we must concentrate on the $i=j$ and $k=0$ terms of the family of linear systems in (23), i.e. equation (24). In particular, this is equivalent to study the equation

$$\mathcal{P}\left(\Gamma_{0,\beta}\langle\xi_W^W\rangle - \langle\xi_W^W\rangle\Gamma_{0,\beta} + \langle\hat{\mathcal{B}}^W\rangle(\Delta\lambda) - \Gamma_{\Delta\alpha,0}\right) = \mathcal{P}\left(\langle\hat{\eta}_W^W\rangle\right) \tag{26}$$

where $\mathcal{P}$ is defined in section 2.1. Notice that

$$\mathcal{P}\left(\Gamma_{0,\beta}\langle\xi_W^W\rangle - \langle\xi_W^W\rangle\Gamma_{0,\beta}\right) = \begin{pmatrix}\operatorname{diag}(s) & \operatorname{diag}(t)\\ \operatorname{diag}(t) & -\operatorname{diag}(s)\end{pmatrix} \tag{27}$$

being

$$\operatorname{diag}(s) = -\operatorname{diag}(\beta)\operatorname{diag}\left(\langle\xi_W^W\rangle_{21}\right) - \operatorname{diag}\left(\langle\xi_W^W\rangle_{12}\right)\operatorname{diag}(\beta) \tag{28}$$

and

$$\operatorname{diag}(t) = -\operatorname{diag}(\beta)\operatorname{diag}\left(\langle\xi_W^W\rangle_{22}\right) + \operatorname{diag}\left(\langle\xi_W^W\rangle_{11}\right)\operatorname{diag}(\beta), \tag{29}$$

where

$$\langle\xi_W^W\rangle = \begin{pmatrix}\langle\xi_W^W\rangle_{11} & \langle\xi_W^W\rangle_{12}\\ \langle\xi_W^W\rangle_{21} & \langle\xi_W^W\rangle_{22}\end{pmatrix}.$$

Equation (26) rewrites as the system of equations for the unknowns $(s,t,\Delta\alpha,\Delta\lambda)$:

$$\begin{pmatrix} I_{n-d} & O_{n-d} & -I_{n-d} & \langle\hat{\mathcal{B}}^W\rangle_{11}\\ O_{n-d} & I_{n-d} & O_{n-d} & \langle\hat{\mathcal{B}}^W\rangle_{12}\\ O_{n-d} & I_{n-d} & O_{n-d} & \langle\hat{\mathcal{B}}^W\rangle_{21}\\ -I_{n-d} & O_{n-d} & -I_{n-d} & \langle\hat{\mathcal{B}}^W\rangle_{22}\end{pmatrix}\begin{pmatrix}s\\ t\\ \Delta\alpha\\ \Delta\lambda\end{pmatrix} = \begin{pmatrix}\mathbf{diag}\left(\langle\hat{\eta}_W^W\rangle_{11}\right)\\ \mathbf{diag}\left(\langle\hat{\eta}_W^W\rangle_{12}\right)\\ \mathbf{diag}\left(\langle\hat{\eta}_W^W\rangle_{21}\right)\\ \mathbf{diag}\left(\langle\hat{\eta}_W^W\rangle_{22}\right)\end{pmatrix}.$$

Notice that it admits a solution under the *trasversality condition:*

$$\langle\hat{\mathcal{B}}^W\rangle_{12} - \langle\hat{\mathcal{B}}^W\rangle_{21}$$

is invertible. Looking at the definition of the matrix $\hat{\mathcal{B}}$, this is a condition on the family of Hamiltonians. Under this assumption its solution is

$$\begin{aligned}
s &= \frac{\mathbf{diag}\left(\langle\hat{\eta}^W_{W11}\rangle - \langle\hat{\eta}^W_{W22}\rangle\right)}{2} - \frac{\left(\langle\hat{\mathcal{B}}^W_{11}\rangle - \langle\hat{\mathcal{B}}^W_{22}\rangle\right)}{2}\Delta\lambda,\\
t &= \frac{\mathbf{diag}\left(\langle\hat{\eta}^W_{W12}\rangle + \langle\hat{\eta}^W_{W21}\rangle\right)}{2} - \frac{\left(\langle\hat{\mathcal{B}}^W_{12}\rangle + \langle\hat{\mathcal{B}}^W_{21}\rangle\right)}{2}\Delta\lambda,\\
\Delta\alpha &= -\frac{\mathbf{diag}\left(\langle\hat{\eta}^W_{W11}\rangle + \langle\hat{\eta}^W_{W22}\rangle\right)}{2} + \frac{\left(\langle\hat{\mathcal{B}}^W_{11}\rangle + \langle\hat{\mathcal{B}}^W_{22}\rangle\right)}{2}\Delta\lambda,\\
\Delta\lambda &= \left(\langle\hat{\mathcal{B}}^W_{12}\rangle - \langle\hat{\mathcal{B}}^W_{21}\rangle\right)^{-1}\mathbf{diag}\left(\langle\hat{\eta}^W_{W12}\rangle - \langle\hat{\eta}^W_{W21}\rangle\right).
\end{aligned}$$

Furthermore, we emphasize again that once we have a solution $(s,t,\Delta\alpha,\Delta\lambda)$ there is a $2(n-d)$-dimensional family of solutions for $\langle\xi_W^W\rangle$ that satisfies (27): given a solution $\begin{pmatrix}\langle\xi_W^W\rangle_{11} & \langle\xi_W^W\rangle_{12}\\ \langle\xi_W^W\rangle_{21} & \langle\xi_W^W\rangle_{22}\end{pmatrix}$ of the equations (28) and (29), we obtain another solution by adding $\begin{pmatrix}\operatorname{diag}(v_1) & \operatorname{diag}(v_2)\\ -\operatorname{diag}(v_2) & \operatorname{diag}(v_1)\end{pmatrix}$, with $v_1, v_2 \in \mathbb{R}^{n-d}$. The existence of these extra parameters v_1, v_2

reflects the fact that the operator defined in (22) has a kernel and the solution for W is symmetric for rotations. We, then, fix these two vectors to be equal to zero.

Remark 3.5. By adding the extra condition of fixing that the diagonal and antidiagonal terms of Ω_{WW} are equal to the ones in Ω_{n-d}, we could fix the vector v_1.

Remark 3.6. The dummy correction $\Delta\alpha$ has the role to control the diagonal terms of the r.h.s. of the third equation in equation (21) and make them in the form of the image (25). However, if the term $\langle \hat{\eta}_W^W - \hat{\mathcal{B}}^W(\Delta\lambda)\rangle$ is antisymmetric (e.g. when the symplectic matrix Ω is constant) the $\Delta\alpha$ correction is not needed.

The $\Delta\lambda$ correction acts on the antidiagonal terms. Without considering this extra-correction, the linear transverse motion would still be elliptic (for a positive measure set of λs) but with frequency vector slightly different from β.

3.2.2. Resolution of the first and second equation. The first and second equations in (21) are very similar to each other and are of the form

$$\mathcal{L}_\omega u - u\Gamma_{0,\beta} = v.$$

In Fourier components it becomes

$$\begin{pmatrix} -\mathbf{i}k\cdot\omega & -\beta_j \\ \beta_j & -\mathbf{i}k\cdot\omega \end{pmatrix} \begin{pmatrix} \hat{u}_k^{i,j} \\ \hat{u}_k^{i,j+n-d} \end{pmatrix} = \begin{pmatrix} \hat{v}_k^{i,j} \\ \hat{v}_k^{i,j+n-d} \end{pmatrix},$$

with $i = 1,\dots,d$ and $j = 1,\dots,n-d$. Notice it is the same as equation (17), so solvable provided first Melnikov condition is satisfied.

With this final resolution, we have determined all the corrections needed. We can use $\Delta\lambda$ to completely determine K_{new} as in (18) and $W_{\text{new}} = W + P\xi_W$.

4. Application: coupled anharmonic oscillators

We tested the algorithm described in the previous section for computing lower dimensional tori in a toy system of 4 degrees of freedom. Its Hamiltonian is

$$h(y,x) = \sum_{j=1}^{4}\left(\frac{y_j^2}{2\ell_j^2} - \ell_j\cos x_j\right) + \frac{k_1}{2}(\ell_2x_2 - \ell_1x_1)^2 + \varepsilon\sum_{j=1}^{2}\frac{k_{j+1}}{2}(\ell_{j+2}x_{j+2} - \ell_{j+1}x_{j+1})^2, \quad (30)$$

where the x_j for $j = 1,\dots,4$ represent the displacement of the jth oscillator and the y_j are the conjugated momenta. The parameters ℓ_j represents the strengths of the oscillators, while the k_j modulate the quadratic coupling potential that allows us to *connect* the oscillator jth to the $j+1$th one. When $\varepsilon = 0$ the last two oscillators are disconnected, i.e. the system consists of two uncoupled oscillators plus two oscillators coupled together with a quadratic potential. In this case we choose to continue (for different values of ε) a torus K ($d = 2$) that is a solution of the model made of the coupled oscillators and the normal dynamics W as the harmonic oscillators resulting in linearizing the uncoupled oscillators around $(x_j, y_j) = 0$ with frequencies $\beta_j = \frac{1}{\sqrt{\ell_{j+d}}}, j = 1,2$.

In the Hamiltonian (30) the dependence on a parameter λ is not yet explicit. Our choice is the following: we take the variable ℓ for the last two oscillators, that is $\ell_{j+2} = 1/(\beta_j + \lambda_j)^2$ for $j = 1,2$. In particular, the parameter vector λ at $\varepsilon = 0$ is initially set to 0.

4.1. Initialization

We are looking for 2–dimensional tori whose transverse elliptic dynamics is given by small oscillations around the equilibrium positions of the other two oscillators. Let us define the set of coordinates: we divide in different blocks the variables associated to the first 2 oscillators (y_1, y_2, x_1, x_2) and the coordinates for the last 2, (y_3, y_4, x_3, x_4). Therefore, the symplectic matrix Ω is constant and it takes the form $\Omega = \begin{pmatrix} \Omega_2 & 0 \\ 0 & \Omega_2 \end{pmatrix}$.

An initial parametrization of the torus K with fixed frequency ω, denoted with K^0, is given by $(K_1^0,\ K_2^0,\ K_3^0,\ K_4^0,\ 0,\ 0,\ 0,\ 0)$, where $(K_1^0,\ K_2^0,\ K_3^0,\ K_4^0)$ is the parametrization of the torus coming from the two coupled oscillators. This torus is obtained as follows: First we set $k_1 = 0$ and get the torus from the uncoupled system. Then, we perform a continuation with respect to k_1 by applying the techniques in [29].

In our computations, we use Fourier expansions to represent our functions. We consider a sample of points on a regular grid of size $N_F = (N_{F_1}, N_{F_2}) \in \mathbb{N}^2$, $\theta = (\theta_1, \theta_2) = (\frac{j_1}{N_{F_1}}, \frac{j_2}{N_{F_2}})$ and $j_1 = 0, \ldots, N_{F_1}$ and $j_2 = 0, \ldots, N_{F_2}$. In particular, (by trial and error) we set $N_{Fj} = 1024$.

As initial approximation for W^0, we first set

$$\tilde{W}^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{\ell_3^3} & 0 \\ 0 & 1 & 0 & -\sqrt{\ell_4^3} \\ 1 & 0 & \frac{1}{\sqrt{\ell_3^3}} & 0 \\ 0 & 1 & 0 & \frac{1}{\sqrt{\ell_4^3}} \end{pmatrix},$$

so as to satisfy the equation $\mathrm{D}X_h(K^0; \lambda^0)\tilde{W}^0 - \tilde{W}^0\Gamma_{0,\beta} = 0$. However, in the procedure we are also interested in keeping fixed the symplectic form, that is we want to fix $W^\top \Omega \circ K W = \Omega_{WW} = \Omega_2$. This condition is not satisfied by $\tilde{W}^0$ since

$$\tilde{W}^{0T} \Omega \circ K \tilde{W}^0 = \begin{pmatrix} O_{n-d} & -\mathrm{diag}\,(a) \\ \mathrm{diag}\,(a) & O_{n-d} \end{pmatrix},$$

where $a_j = \sqrt{\ell_{j+d}^3} + \dfrac{1}{\sqrt{\ell_{j+d}^3}}$. To fix the symplectic form Ω_{WW} we normalize by

$$W^0 = \tilde{W}^0 \begin{pmatrix} \mathrm{diag}\,(b) & O_{n-d} \\ O_{n-d} & \mathrm{diag}\,(b) \end{pmatrix}$$

where $b_j = 1/\sqrt{a_j}$ for $j = 1, 2$.

4.2. Results

Here we report some numerics done in multiprecision using the library **mpfr**, see [22], with 60 digits of precision. We fix the following parameters:

$$k_1 = 10^{-2},\ k_2 = k_3 = 1,\ \omega_1 = \sqrt{2},\ \omega_2 = \sqrt{3},\ \beta_1 = \sqrt{2.5},\ \beta_2 = \sqrt{2.8}.$$

Table 1. Norms of the errors E_K and E_W and of the strengths ℓ_3 and ℓ_4 for different values of the parameter ε and after 8 iterations of the algorithm and $N_{F_1} = N_{F_2} = 1024$. The starting values of the strengths are $\ell_3 = 0.4$ and $\ell_4 = 0.357\,142\,857$.

ε	ℓ_3	ℓ_4	$\lvert E_K \rvert$	$\lvert E_W \rvert$
10^{-6}	0.399 999 368	0.357 142 558	$1.205\,318\,45 \cdot 10^{-41}$	$1.010\,278\,95 \cdot 10^{-43}$
10^{-5}	0.399 993 676	0.357 139 869	$1.090\,240\,78 \cdot 10^{-41}$	$1.009\,110\,89 \cdot 10^{-42}$
10^{-4}	0.399 936 772	0.357 112 966	$1.015\,878\,89 \cdot 10^{-41}$	$9.937\,628\,43 \cdot 10^{-42}$
10^{-3}	0.399 369 337	0.356 843 027	$8.686\,127\,23 \cdot 10^{-40}$	$1.166\,835\,24 \cdot 10^{-36}$

Moreover, we fix ℓ_1 and ℓ_2 as

$$\ell_1 = 0.45678, \ \ell_2 = 0.325,$$

while we define the last two as $\ell_3 = 1/\beta_1^2, \ell_4 = 1/\beta_2^2$. The frequencies (and the ℓs) have been chosen so as to be non-resonant and so as to correspond to librations around the equilibrium positions. Moreover, the ℓ of the last two oscillators is such that the frequencies of the initial small oscillations are exactly β_1 and β_2.

The efficiency of the procedure is tested by looking at the norms of the invariance errors E_K and E_W, that are defined as follows:

$$|E_K| = \max_{1 \leqslant i \leqslant 2n} |(E_K)_i|$$

and

$$|E_W| = \max_{1 \leqslant i \leqslant 2n} \sum_{1 \leqslant j \leqslant 2(n-d)} |(E_W)_{i,j}|.$$

In table 1 we observe the positive correlation between the error of the invariance equations and the magnitude of ε: as it is expected, the larger ε is the worse are the errors. In particular, we experienced that the accuracy the torus could not be further improved without increasing the number of Fourier coefficients N_F.

In figure 1, we report in semi-logscale the rate of convergence of the errors $|E_W|$, $|E_K|$ and of the corrections $|\Delta\lambda|$ with respect to the number of steps. Also in this picture, one can observe how the rate of convergence becomes slower once we approach the numerical limit. Indeed, while in the first 5 steps we can observe a quadratic decay, in the last ones the convergence rate appears as linear.

A complete analysis of the breakdown threshold of the existence of these tori is beyond the scope of this article, however we emphasize how the value $\varepsilon = 10^{-3}$ is only 10 times smaller than the constant k_1, that is a limit value for considering this system a perturbed one and the chosen initial torus K^0 a good initial approximation.

In figure 2 we show 2D projections on the phase space of the obtained torus (we recall that the phase space is 8-dimensional and the constructed torus is 2-dimensional and parametrized by (θ_1, θ_2)) after 8 iterations of the algorithm with parameters $k_1 = 10^{-2}$ and $\varepsilon = 10^{-3}$, where θ is sampled in a grid of $N_{F_1} \times N_{F_2}$ points. Notice that the projections on the pairs (x_1, y_1) and (x_2, y_2) have order of magnitude one while the projections on the pairs (x_3, y_3) and (x_4, y_4)

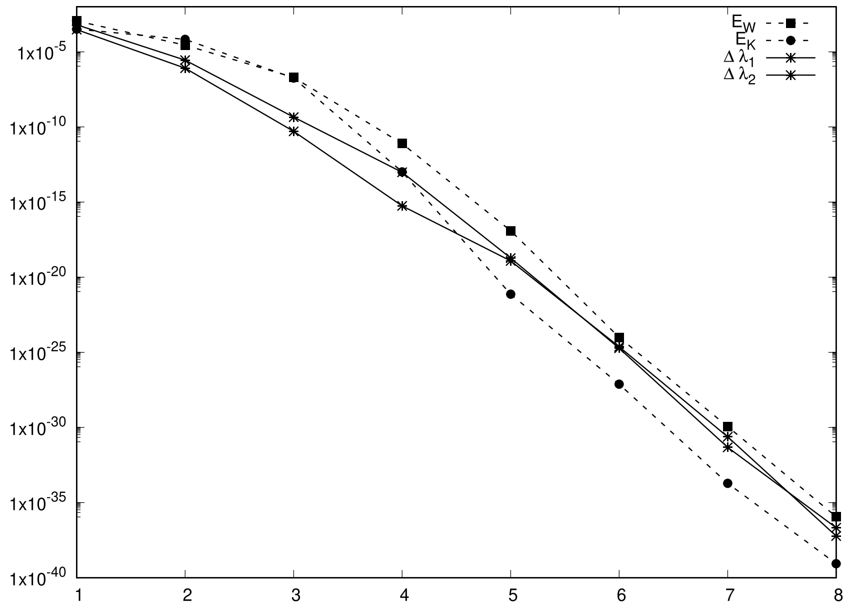

Figure 1. Decrease, in semi-logscale, of the norms of the errors E_K and E_W and $|\Delta\lambda|$ when $\varepsilon = 10^{-3}$, r denotes the number of steps.

have order ε. This reflects the fact that for the unperturbed system the torus lies in the first two pairs times the origin on the second ones.

The computational time to compute 8 steps for $N_{F_1} = N_{F_2} = 1024$ on a computer equipped with a processor `Intel(R) Core(TM) i7-8550U CPU @ 1.80 GHz` and 16 *GB* of RAM is around 10 h. The amount of time is strongly correlated to the high number of Fourier components needed to accurately represent the torus. Indeed, one deals with several computations that involve grids of 1024×1024 points. A refined implementation could improve the computational time and cost. As commented at the end of section 2, using an alternative formula to compute the torsion matrix T and avoiding to store the matrix N could help to partially reduce the memory and time needed. Also, some computations could be parallelized.

5. Conclusion and perspectives

In this paper we have presented a method to efficiently compute elliptic lower dimensional invariant tori and their attached normal bundles; we tested it in a simple model, with the smallest (nontrivial) dimensions of the frequency vectors, that is 2. The next natural step is applying the methodology to challenging and important problems, such as the planetary three-body problem. Indeed, despite a numerical evidence of the existence of invariant tori, the existence of Lagrangian tori or lower dimensional tori is still an open problem, at least for real values of the masses (in the Solar System, the natural perturbing parameter is the ratio between the mass of the Sun and the one of Jupiter, i.e. $\varepsilon \simeq 10^{-3}$). The latest results with real observed data are numerical, see [18], but supported by the KAM theorem in [17] and for small values of the masses, eccentricities and mutual inclinations are in [3]. Such a problem presents several extra

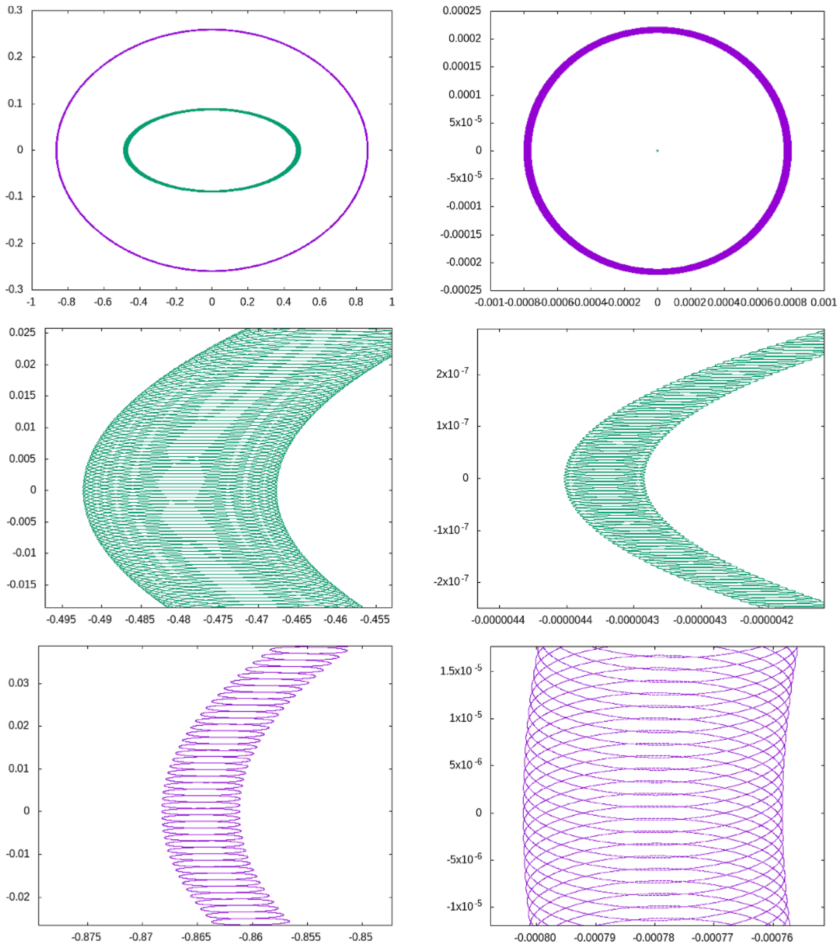

Figure 2. Approximated invariant torus when $k_1 = 10^{-2}$ and $\varepsilon = 10^{-3}$, after 8 iteration of the algorithm. Left column shows (x_1, y_1, x_2, y_2)-projection of the invariant torus. The second and third picture from top to bottom are magnifications of the first. Right column shows (x_3, y_3, x_4, y_4)-projection of the invariant torus. The second and third picture from top to bottom are magnifications of the first.

difficulties that indeed we did not encounter in the presented model. For example, the parameters involved in the problem are not free: indeed not only ε is given, but also the frequency vectors (semi-major axes). Furthermore, it is well known that the integrable system (the Kepler problem) is degenerate, making standard continuation techniques not possible. However, we believe that an adaptation of the technique of translated tori employed in [18] might be useful to overcome these difficulties. For what concerns the real values of the parameters, at the present moment there are computer-assisted proofs for the secular model only, that is a model with reduced degrees of freedom, where the fast motion, i.e. the motion of revolution of the planets, is averaged. These results are based on normal forms (see [37] for a model of the Solar System, [10] for a model of an exoplanetary system with significant eccentricities and mutual inclination).

As an intermediate step before a computer-assisted proof, the performances of the algorithm can be tested also in comparison with purely numerical procedures. In fact, a fast decrease of the error in the implemented algorithm is a necessary requirement for a computer-assisted proof of the existence of invariant tori. In this context, the conversion of the semi-analytical algorithm to a computer-assisted proof is more straightforward in the parametrization method than in the normal form technique. In the latter, the algorithm and the formal proof of the convergence have to be extended with a scheme of estimates, that have to be iterated in queue of the algorithm. This is due also to the fact that the number of explicit steps which can be performed (of the order of 10^2) is usually not yet enough to decrease the perturbation to the extreme low level of formal theorems. Even if in principle the same technique used for the KAM theory could be extended to the case of lower dimensional tori, the translation would not be straightforward. The fast convergence rate of the parametrization method, combined with the simplest interplay between the estimates needed to produce a formal proof and the ones needed for a computer-assisted proof, make this method more promising. However, for the mentioned reasons, the main step towards the direction of a computer-assisted proof is to produce a formal proof of the convergence of the procedure, that we will present in a forthcoming paper [8].

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix. Auxiliary results

As it is customary with the resolution of cohomological equations, we have the following lemma. We omit its proof since it is standard.

Lemma A.1. *Let v be $\mathcal{O}(E)$. Then both solutions to the equations $\mathcal{L}_\omega u_1 = v$ and $\mathcal{L}_\omega u_2 \pm \Gamma_{0,\beta} u_2 = v$ are $\mathcal{O}(E)$.*

Now we prove a series of lemmas that help us proving proposition 2.3.

Lemma A.2. *Under the hypothesis that both K and W are almost invariant we have that Ω_{LL}, Ω_{LW} are $\mathcal{O}(E_K, E_W)$.*

Proof. As discussed in Lemma 4.3 in [29], Ω_{LL} is $\mathcal{O}(E_K)$ in view of the almost invariance of K. For what concerns the other term, let us preliminarily compute the following quantity:

$$\mathcal{L}_\omega \Omega_{LW} - \Omega_{LW}\Gamma_{\alpha,\beta} = \mathcal{L}_\omega L^\top \Omega\circ K W + L^\top \mathcal{L}_\omega \Omega\circ K W + L^\top \Omega\circ K \mathcal{L}_\omega W - \Omega_{LW}\Gamma_{\alpha,\beta}.$$

Now using the properties (which can be obtained from the invariance equation)

$$\mathcal{L}_\omega \mathrm{D}K = \mathrm{D}E_K - \mathrm{D}X_h\circ(K;\lambda)\,\mathrm{D}K$$

and

$$\mathcal{L}_\omega \Omega\circ K = \mathrm{D}\Omega\circ K\left[E_K - X_h\circ(K;\lambda)\right]\;, \tag{31}$$

and in view of the equality (31), one gets

$$\begin{aligned}
\mathcal{L}_\omega \Omega_{LW} - \Omega_{LW}\Gamma_{\alpha,\beta} =\;& (\mathrm{D}E_K)^\top\,\Omega\circ K W - L^\top\left(\mathrm{D}X_h\circ(K;\lambda)\right)^\top \Omega\circ K W\\
&+ L^\top \mathrm{D}\Omega\circ(K;\lambda)\left[E_K\right]W - L^\top \mathrm{D}\Omega\circ(K)\left[X_h\circ(K;\lambda)\right]W\\
&- L^\top \Omega\circ K \mathrm{D}X_h\circ(K;\lambda)\,W + L^\top \Omega\circ K W\Gamma_{\alpha,\beta}\\
&+ L^\top \Omega\circ K E_W - \Omega_{LW}\Gamma_{\alpha,\beta}\\
=\;& (\mathrm{D}E_K)^\top\,\Omega\circ K W + L^\top \mathrm{D}\Omega\circ K\left[E_K\right]W + L^\top \Omega\circ K E_W\,.
\end{aligned}$$

Therefore, $\mathcal{L}_\omega \Omega_{LW} - \Omega_{LW}\Gamma_{\alpha,\beta}$ is $\mathcal{O}(E_K, E_W)$ and the result for Ω_{LW} follows by Lemma A.1. □

Lemma A.3. *The dummy parameter α is $\mathcal{O}(E_K, E_W)$. In particular, it is equal to zero if the torus K and the normal bundles W are invariant.*

Proof. One may argue that at each step we are solving an equation

$$\mathcal{L}_\omega W + \mathrm{D}X_h\circ(K;\lambda)\,W - W\Gamma_{\alpha,\beta} = E_W,$$

for a given α, while the goal is to solve the equation for $\alpha = 0$ and thus guarantee ellipticity. Here we underline that at each step α is $\mathcal{O}(E_K, E_W)$ and at the end of the procedure this parameter must be in fact zero.

Indeed, if W, K and λ satisfy

$$\mathcal{L}_\omega W + \mathrm{D}X_h\circ(K;\lambda)\,W - W\Gamma_{\alpha,\beta} = 0,$$

then, using the equality

$$\left(\mathrm{D}X_h\circ(K;\lambda)\right)^\top \Omega\circ K + \mathrm{D}\Omega\circ K\left[X_h\circ(K;\lambda)\right] + \Omega\circ K \mathrm{D}X_h\circ(K;\lambda) = O_{2n} \tag{32}$$

(see Lemma 4.3 in [29]) we obtain that Ω_{WW} satisfies the following equation

$$\begin{aligned}
\mathcal{L}_\omega \Omega_{WW} &= \mathcal{L}_\omega W^\top \Omega\circ K W + W^\top \mathcal{L}_\omega \Omega\circ K W + W^\top \Omega\circ K \mathcal{L}_\omega W\\
&= \Gamma_{\alpha,\beta}^\top \Omega_{WW} + \Omega_{WW}\Gamma_{\alpha,\beta} + E_W^\top \Omega\circ K W + W^\top \mathrm{D}\Omega\circ K\left[E_W\right]W + W^\top \Omega\circ K E_W
\end{aligned}$$

This can be rewritten as

$$\mathcal{L}_\omega \Omega_{WW} - \Gamma_{\alpha,\beta}^\top \Omega_{WW} - \Omega_{WW}\Gamma_{\alpha,\beta} = \hat{E}, \tag{33}$$

where $\hat{E}$ is $\mathcal{O}(E_K, E_W)$. Similarly as in solving equation (22) we obtain that $\Omega_{WW} - \langle \Omega_{WW}\rangle = \mathcal{O}(\hat{E})$. Moreover, since Ω_{WW} is antisymmetric it implies

$$\langle \Omega_{WW}\rangle = \begin{pmatrix} 0 & -\mathrm{diag}(b)\\ \mathrm{diag}(b) & 0\end{pmatrix} + \mathcal{O}\left(\hat{E}\right). \tag{34}$$

with $b \in \mathbb{R}^{n-d}$. Finally, we obtain that the averages of both sides in equation (33) are equal. Using (34) and writing the averages componentwise we obtain $2\alpha_i b_i = \mathcal{O}(\hat{E})$. But, since $b_i \neq 0$ (Ω_{WW} is non-degenerate), we get that $\alpha_i = \mathcal{O}(\hat{E})$. $\square$

Lemma A.4. *Under the hypothesis that both K and W are almost invariant we have that the matrix $P(\theta)$ defined as in (3)–(6) is almost symplectic, i.e.*

$$P^\top \Omega\circ K P = \begin{pmatrix} \Omega_d & O_{2d\times 2(n-d)} \\ O_{2(n-d)\times 2d} & \Omega_{WW} \end{pmatrix} + E_{sym}$$

being $E_{sym} = \mathcal{O}(E_K, E_W)$ and, in particular, with the following expression

$$E_{sym} = \begin{pmatrix} \Omega_{LL} & \Omega_{LL}A + \Omega_{LW}C & \Omega_{LW} \\ -A^\top\Omega_{LL} - C^\top\Omega_{LW}^\top & A^\top\Omega_{LL}A + B^\top\Omega_{LL}B + A^\top\Omega_{LW}C + C^\top\Omega_{LW}A & A^\top\Omega_{LW} \\ -\Omega_{LW}^\top & -\Omega_{LW}^\top A & 0 \end{pmatrix}.$$

Proof. In view of the definition of P, we get the following:

$$P^\top \Omega\circ K P = \begin{pmatrix} \Omega_{LL} & \Omega_{LN} & \Omega_{LW} \\ -\Omega_{LN}^\top & \Omega_{NN} & \Omega_{NW} \\ -\Omega_{LW}^\top & -\Omega_{NW}^\top & \Omega_{WW} \end{pmatrix}.$$

Therefore, we have to check that the blocks of this matrix are of the form

$$\begin{pmatrix} O_d & -I_d & O_{d\times 2(n-d)} \\ I_d & O_d & O_{d\times 2(n-d)} \\ O_{2(n-d)\times d} & O_{2(n-d)\times d} & \Omega_{WW} \end{pmatrix} + E_{sym}.$$

We already showed in lemma A.2 that Ω_{LL} is $\mathcal{O}(E_K)$. The second term of the first row, $\Omega_{LN} = -I_d + \Omega_{LL}A + \Omega_{LW}C$, is equal to $-I_d + \mathcal{O}(E_K, E_W)$ since we showed in lemma A.2 that Ω_{LW} and Ω_{LL} are $\mathcal{O}(E_{K,W})$. The next term to be considered is

$$\begin{aligned}
\Omega_{NN} &= \left(C^\top W^\top + B^\top L^\top J^\top + A^\top L^\top\right)\Omega\circ K\left(LA + J\circ KLB + WC\right) \\
&= -C^\top\Omega_{LW}^\top A + C^\top W^\top \Omega\circ KJ\circ KLB + C^\top\Omega_{WW}C \\
&\quad + B^\top L^\top (J\circ K)^\top \Omega\circ KLA + B^\top L^\top (J\circ K)^\top \Omega\circ KJ\circ KLB \\
&\quad + B^\top L^\top (J\circ K)^\top \Omega\circ KWC + A^\top\Omega_{LL}A + A^\top L^\top\Omega\circ KJ\circ KLB + A^\top\Omega_{LW}C \\
&= \left(A^\top\Omega_{LW}C\right) + \left(A^\top\Omega_{LW}C\right)^\top + \left(B^\top L^\top (J\circ K)^\top \Omega\circ KWC\right) - \left(B^\top L^\top (J\circ K)^\top \Omega\circ KWC\right)^\top \\
&\quad + C^\top\Omega_{WW}C + \left(A^\top L^\top\Omega\circ KJ\circ KLB\right) - \left(A^\top L^\top\Omega\circ KJ\circ KLB\right)^\top + B^\top\Omega_{LL}B + A^\top\Omega_{LL}A.
\end{aligned}$$

With our choice of A, B and C, we can rewrite

$$B^\top L^\top (J\circ K)^\top \Omega\circ KWC = B^\top G_{LW}\Omega_{WW}^{-1}G_{WL}B$$

and, therefore, we can see that it is antisymmetric in view of the antisymmetry of Ω_{WW} and of the resulting property $(\Omega_{WW}^{-1})^\top = (\Omega_{WW}^T)^{-1} = -\Omega_{WW}^{-1}$. Moreover, it is equal to $-C^\top\Omega_{WW}C$. Furthermore, by observing that $A = -A^\top$, we get

$$\Omega_{NN} = \left(A^\top\Omega_{LW}C\right) + \left(A^\top\Omega_{LW}C\right)^\top - C^\top\Omega_{WW}C + 2A + B^\top\Omega_{LL}B + A^\top\Omega_{LL}A$$

and the result follows by the definition of A and the estimates in lemma A.2.

Finally, using that $G^\top = G$ and $(\Omega_{WW}^{-1})^\top = -\Omega_{WW}^{-1}$ and again lemma A.2,

$$\begin{aligned}\Omega_{NW} &= \left(C^\top W^\top + B^\top L^\top (J\circ K)^\top + A^\top L^\top\right)\Omega\circ KW \\ &= C^\top \Omega_{WW} + B^\top L^\top (J\circ K)^\top \Omega\circ KW + A^\top \Omega_{LW} \\ &= B^\top G_{LW} \left(\Omega_{WW}^{-1}\right)^\top \Omega_{WW} + B^\top G_{LW} + A^\top \Omega_{LW},\end{aligned}$$

and the result follows. □

The simplecticity of P can be used to determine the inverse of P. However, in the case of almost simplecticity, this matrix is an approximation of the inverse of P, as it is seen in the following corollary.

Corollary A.5. *Under the hypothesis that both K and W are almost invariant with both E_K and E_W being small enough then P is invertible and has expression*

$$P^{-1} = P_{inv} + E_{inv} = \begin{pmatrix} N^\top \Omega\circ K \\ -L^\top \Omega\circ K \\ \Omega_{WW}^{-1} W^\top \Omega\circ K \end{pmatrix} + E_{inv}\,, \tag{35}$$

with E_{inv} being $\mathcal{O}(E_K, E_W)$.

In the following lemma we ensure that the problem is almost reducible if both K and W are almost invariant.

Lemma A.6. *Under the hypothesis that both K and W are almost invariant with both E_K and E_W being small enough, the reducibility matrix $P(\theta)$ defined as in (3)–(6), the problem is almost reducible, that is*

$$P^{-1}\mathcal{X}_P = P^{-1}\left(\mathcal{L}_\omega P + D_z X_h \circ (K;\lambda)\, P\right) = \Lambda + \mathcal{O}\left(E_K, E_W\right). \tag{36}$$

Proof. Under the Lemma hypotheses we have that P is almost symplectic and satisfies equation (35). So, the goal is to see that

$$P^{-1}\mathcal{X}_P = \begin{pmatrix} N^\top \Omega\circ K\mathcal{X}_L & N^\top \Omega\circ K\mathcal{X}_N & N^\top \Omega\circ K\mathcal{X}_W \\ -L^\top \Omega\circ K\mathcal{X}_L & -L^\top \Omega\circ K\mathcal{X}_N & -L^\top \Omega\circ K\mathcal{X}_W \\ \Omega_{WW}^{-1} W^\top \Omega\circ K\mathcal{X}_L & \Omega_{WW}^{-1} W^\top \Omega\circ K\mathcal{X}_N & \Omega_{WW}^{-1} W^\top \Omega\circ K\mathcal{X}_W \end{pmatrix} + E_{inv}\mathcal{X}_P \tag{37}$$

is equal to

$$\begin{pmatrix} O_d & T & O_{d\times 2(n-d)} \\ O_d & O_d & O_{d\times 2(n-d)} \\ O_{2(n-d)\times d} & O_{2(n-d)\times d} & \Gamma_{0,\beta} \end{pmatrix} + \mathcal{O}\left(E_K, E_W\right).$$

First, because of corollary A.5 we have that $E_{inv}\mathcal{X}_P$ is $\mathcal{O}(E_K, E_W)$. Notice that all terms in (37) containing $\mathcal{X}_L$ are $\mathcal{O}(E_K)$ since $\mathcal{X}_L = \mathcal{L}_\omega L + \mathrm{D}X_h \circ (K;\lambda) L = \mathrm{D}E_K$ and $\mathrm{D}E_K$ is $\mathcal{O}(E_K)$ because of Cauchy inequality for analytic functions.

For the second column, the first term is equal to T by definition. The second term is

$$-L^\top \Omega\circ K\mathcal{X}_N = -L^\top \Omega\circ K\mathcal{L}_\omega N - L^\top \Omega\circ K \mathrm{D}X_h \circ (K;\lambda)\, N.$$

Using that $\mathcal{L}_\omega \Omega_{LN} = \mathcal{L}_\omega L^\top \Omega\circ KN + L^\top \mathcal{L}_\omega \Omega\circ KN + L^\top \Omega\circ K\mathcal{L}_\omega N$ and the formulæ (31) and (32), one obtains

$$
\begin{aligned}
-L^\top \Omega{\circ}K\mathcal{X}_N &= \mathcal{L}_\omega L^\top \Omega{\circ}KN + L^\top \mathcal{L}_\omega \Omega{\circ}KN - \mathcal{L}_\omega \Omega_{LN} - L^\top \Omega{\circ}K\mathrm{D}X_h{\circ}(K;\lambda)N \\
&= -L^\top \left(\mathrm{D}X_h{\circ}(K;\lambda)\right)^\top \Omega{\circ}KN + \mathrm{D}E_K^\top \Omega{\circ}KN + L^\top \mathrm{D}\Omega{\circ}K[E_K]N \\
&\quad - L^\top \mathrm{D}\Omega{\circ}KX_h{\circ}(K;\lambda)N - \mathcal{L}_\omega \Omega_{LN} - L^\top \Omega{\circ}K\mathrm{D}X_h{\circ}(K;\lambda)N \\
&= \mathrm{D}E_K^\top \Omega{\circ}KN - \mathcal{L}_\omega \Omega_{LN} + L^\top \mathrm{D}\Omega{\circ}K[E_K]N
\end{aligned}
$$

and it is, therefore, of $\mathcal{O}(E_K, E_W)$. Similarly, we can show that

$$
\begin{aligned}
\Omega_{WW}^{-1} W^\top \Omega{\circ}K\mathcal{X}_N &= \Omega_{WW}^{-1}\big(-\mathcal{L}_\omega W^\top \Omega{\circ}KN - W^\top \mathcal{L}_\omega \Omega{\circ}KN + \mathcal{L}_\omega \Omega_{WN} \\
&\quad + W^\top \Omega{\circ}K\mathrm{D}X_h{\circ}(K;\lambda)N\big) \\
&= -\,\Omega_{WW}^{-1}\left(E_W^\top \Omega{\circ}KN - \Gamma_{\alpha,\beta}^\top \Omega_{WN} - W^\top \mathrm{D}\Omega{\circ}K[E_K]N + \mathcal{L}_\omega \Omega_{WN}\right)
\end{aligned}
$$

and, therefore, it is $\mathcal{O}(E_{K,W})$.

The third column contains the following terms:

$$
\begin{aligned}
N^\top \Omega{\circ}K\mathcal{X}_W &= N^\top \Omega\left(\mathcal{L}_\omega W + \mathrm{D}_z X_h{\circ}(K;\lambda)\,W\right) = N^\top \Omega{\circ}K\left(W\Gamma_{\alpha,\beta} + E_W\right) \\
&= \Omega_{NW}\Gamma_{\alpha,\beta} + N^\top \Omega{\circ}KE_W\,, \\
-L^\top \Omega{\circ}K\mathcal{X}_W &= -\Omega_{LW}\Gamma_{\alpha,\beta} - L^\top \Omega{\circ}KE_W\,, \\
\Omega_{WW}^{-1} W^\top \Omega{\circ}K\mathcal{X}_W &= \Gamma_{\alpha,\beta} + \Omega_{WW}^{-1} W^\top \Omega{\circ}KE_W = \Gamma_{0,\beta} + \Gamma_{\alpha,0} + \Omega_{WW}^{-1} W^\top \Omega{\circ}KE_W.
\end{aligned}
$$

Therefore, using the fact that Ω_{LW}, Ω_{NW} and α are $\mathcal{O}(E_K, E_W)$ gives us the desired smallness. □

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