REGULARITY OF OSCILLATORY INTEGRAL OPERATORS

ANDERS ISRAELSSON, TOBIAS MATTSSON, AND WOLFGANG STAUBACH

Abstract. In this paper, we establish the global boundedness of oscillatory integral operators on Besov-Lipschitz and Triebel-Lizorkin spaces, with amplitudes in general $S^{m}_{\rho,\delta}(\mathbb{R}^n)$-classes and non-degenerate phase functions in the class ᚠ^k. Our results hold for a wide range of parameters $0 \leq \rho \leq 1$, $0 \leq \delta < 1$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $k > 0$. We also provide a sufficient condition for the boundedness of operators with amplitudes in the forbidden class $S^{m}_{1,1}(\mathbb{R}^n)$ in Triebel-Lizorkin spaces.

Contents

1. Introduction

This paper is devoted to the investigation of the global regularity of oscillatory integral operators (here referred to as OIOs) of the form

$$T^{\varphi}_{a} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\, a(x,\xi)\, \widehat{f}(\xi)\, \mathrm{d}\xi,$$

with amplitudes in the general Hörmander class $S^{m}_{\rho,\delta}(\mathbb{R}^n)$ (Definition 2.1), on Besov-Lipschitz $B^{s}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin $F^{s}_{p,q}(\mathbb{R}^n)$ spaces of order $s \in \mathbb{R}$ with $0 < p < \infty$ and $0 < q \leq \infty$, (Definitions 2.14 and 2.15). Throughout the paper, we are assuming that the phase function $\varphi(x,\xi)$ is in the class ᚠ^k for some $k > 0$ (Definition

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2.3) and is strongly non-degenerate, meaning that $|\det\left(\partial^2_{x_j\xi_k}\varphi(x,\xi)\right)|$ has a uniform lower bound (Definition 2.5). Here we note that we are not assuming any homogeneity in the ξ variable of the phase function φ, as is the case for Fourier integral operators where $\varphi(x,\xi)$ is assumed to be positively homogeneous of degree one in ξ. It turns out that one needs to put further constraints on the phase functions to deal with the issues related to the (possible) singularity at the origin, and the L^2-boundedness of the associated OIOs. However a very simple example to keep in mind, is the phase function $\varphi(x,\xi) = x\cdot\xi + |\xi|^k$ which for $0 < k < \infty$ is in ᚠ^k and is SND, to which all the results of this paper are applicable without any further restrictions.

The motivation for the study of the OIOs that are investigated in this paper comes from the theory of partial differential equations where phase functions $\varphi(x,\xi) = x\cdot\xi+\phi(\xi)$ frequently appear in the study of dispersive equations. Indeed $\phi(\xi) = |\xi|^{1/2}$ corresponds to the water-wave equation, $\phi(\xi) = |\xi|^2$ corresponds to the Schrödinger equation, while $\phi(\xi) = |\xi|^3$ and $\phi(\xi) = \xi|\xi|$ (both in dimension one) correspond to Airy and linear versions of Benjamin-Ono equations respectively.
The results that are obtained in this paper also accommodate for instance the case of variable coefficient Schrödinger equations through the earlier investigations of B. Helffer and D. Robert [14], Helffer [13] and the work of E. Cordero F. Nicola and L. Rodino [5], [6].

In [4], the first and the third author in collaboration with A. Castro and M. Yerlanov established an $L^p - L^q$, $(1 < p \leq q < \infty)$ regularity theory for OIOs in the following two cases: (1) when the amplitude is in $S^m_{1,0}(\mathbb{R}^n)$ and the phase function is in ᚠ^k with $0 < k < 1$. (2) when the amplitude is in $S^m_{0,0}(\mathbb{R}^n)$ and the phase function is in ᚠ^k with $k \geq 1$. The authors of [4] also went beyond the scope of L^p-spaces and investigated the regularity of OIOs in classical function spaces such as Besov-Lipschitz and Triebel-Lizorkin spaces.

In [22] M. Pramanik, K. Rogers and A. Seeger (see Theorem 2.23 below), proved a Calderón-Zygmund-type estimate with far-reaching applications, including the regularity of Radon transforms and Fourier integral operators. In that paper, the authors also considered local Fourier integral operators T^φ_a where $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$ is compactly supported in x and $\varphi(x,\xi)$ is non-degenerate on the support of $a(x,\xi)$. Using their Calderón-Zygmund estimate in [22], they showed that if $n \geq 2$, $2 < p < \infty$, $q > 0$, then $T^\varphi_a : F^0_{p,p}(\mathbb{R}^n) \to F^0_{p,q}(\mathbb{R}^n)$, provided that $m = -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$.

In this paper, in Theorem 7.3, we prove a variation of the aforementioned result of Pramanik-Rogers-Seeger, which is on the one hand more suitable for extensions to the case of quasi-Banach scales of Besov-Lipschitz and Triebel-Lizorkin spaces, and on the other also suitable for applications to the regularity theory of OIOs. More specifically using certain decomposition in the frequency space and rather intricate kernel estimates for oscillatory integral operators, a composition theorem for the action of parameter-dependent pseudodifferential operators on OIOs, atomic and molecular decompositions of Triebel-Lizorkin spaces in the spirit of Frazier-Jawerth [9, 10, 11], and vector-valued inequalities of the type provided in Theorem 7.3, we manage to get a significant extension of the results in [4]. These extensions are both in terms of the types of oscillatory integral operators, and also the scales of the function spaces on which the operators act. Figure 2 illustrates the extensions that are obtained here.

One of our main results is Theorem 8.1 in which we prove the following:
Let $0 \leq \rho \leq 1$ and $0 \leq \delta < 1$, $0 < \mu \leq 1$, $k > 0$, $\varkappa = \min(\rho, 1-k)$ and set

$$m(p) := -n(1-\varkappa)\Big|\frac{1}{p} - \frac{1}{2}\Big| + \min\Big(0, \frac{n}{2}(\rho - \delta)\Big).$$

Assume that $a \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$, and that $\varphi \in \digamma^k$ is SND and satisfies the L^2-condition of Definition 2.6 and the LF(μ)-condition of Definition 2.8. Let T^{φ}_a be the associated OIO.
If $s \in \mathbb{R}$, $0 < q \leq \infty$, and either one of the following cases holds

$$\begin{aligned} &(i) \quad 2 \leq p < \infty \text{ when } 0 < q \leq p, \\ &(ii) \quad \frac{n}{n+\mu} < p \leq 2 \text{ when } p \leq q \end{aligned}$$

then the OIO T^{φ}_a is bounded from $F^s_{p,q}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$. Moreover in Theorem 9.1 we also show the boundedness of T^{φ}_a on Besov-Lipschitz spaces $B^s_{p,q}(\mathbb{R}^n)$, for any $\frac{n}{n+\mu} < p < \infty$ and any $0 < q \leq \infty$. These results are currently the most general regularity results for oscillatory integral operators with amplitudes in a general Hörmander class, which include the majority of OIOs that appear in the theory of partial differential equations.

The case of operators with amplitudes in the forbidden Hörmander class $S^{-kn|1/p-1/2|}_{1,1}(\mathbb{R}^n)$ is excluded in the results above since these operators do not in general even allow L^2-boundedness. However, for $s > n\big(\frac{1}{\min\{1,p,q\}} - 1\big)$ we are able to show (Theorem 8.2) the boundedness of OIOs on $F^s_{p,q}(\mathbb{R}^n)$ under either of the cases (i) and (ii) above.

The paper is organized as follows; in Section 2 we provide the necessary preliminaries from the theory of oscillatory integral operators and the theory of function spaces. Here the reader will also find some of the basic results that are used throughout the paper. In Section 3 we prove a basic kernel estimate which lies at the ground for the establishment of Besov-Lipschitz boundedness of OIOs. This is done by utilizing a particular frequency-space decomposition adapted to the OIOs with phase functions in the class $\digamma^k$. In Section 4 we prove the basic global L^2-boundedness result for OIOs with strongly non-degenerate phase functions and amplitudes in general Hörmander classes. This is done by using a continuous version of the almost orthogonality method and Cotlar-Stein's lemma. In Section 5 we prove the $h^p \to L^p$ $(p < 1)$ and $L^\infty \to$ bmo boundedness of OIOs. In Sections 6 and 7 the Hardy space results are transferred to Triebel-Lizorkin spaces. Section 8 contains our main result concerning the boundedness of OIOs on Triebel-Lizorkin spaces. This section includes the results for both classical and forbidden amplitudes. In Section (9) we conclude the paper by proving our main results on the regularity of OIOs in Besov-Lipschitz spaces.

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2. Preliminaries

In this section, we gather the necessary background material from Fourier- and microlocal analysis, and also recall certain results in these fields that will be required for the development of the machinery of this paper.

As is common practice, we will denote positive constants in the inequalities by C, which can be determined by known parameters in a given situation but whose value is not crucial to the problem at hand. Such parameters in this paper would be, for example, m, p, s, n, and the constants connected to the seminorms of various amplitudes or phase functions. The value of C may differ from line to line, but in each instance could be estimated if necessary. We also write $a \lesssim b$ as shorthand for $a \leq Cb$ and moreover will use the notation $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

2.1. **Basic facts related to oscillatory integral operators.**

Oscillatory integral operators play a central role in a wide range of mathematical fields, including harmonic analysis, partial differential equations, and mathematical physics. In this section, we provide a brief overview of some basic facts related to oscillatory integral operators that will be used throughout the rest of our paper.

A standard reference for the theory of oscillatory integral operators of the form (4), is the classical book by E.M. Stein [23], from which almost all of the terminology that is used in this paper stem from.

We begin by recalling the definition of an oscillatory integral operator and discussing some of its basic properties.

Definition 2.1. *Let $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$. An amplitude (or symbol) $a(x,\xi)$ in the class $S^m_{\rho,\delta}(\mathbb{R}^n)$ is a function $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that verifies the estimate*

$$\left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \lesssim \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

for all multi-indices α and β and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$.

One refers to m as the order of the amplitude and ρ, δ as its type. Following [23] *and* [15]*, we will refer to the class $S^m_{\rho,\delta}(\mathbb{R}^n)$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$ as classical, to the class $S^m_{0,\delta}(\mathbb{R}^n)$ with $0 \leq \delta < 1$ as the exotic class, and to $S^m_{\rho,1}(\mathbb{R}^n)$ with $0 \leq \rho \leq 1$ as the forbidden class of amplitudes.*

Occasionally and in certain instances one does not require any regularity in the x-variable of the amplitudes, and in this connection we introduce the following class of amplitudes which was defined by C. Kenig and W. Staubach in [19].

Definition 2.2. *Let* $m \in \mathbb{R}$ *and* $\rho \in [0,1]$. *An amplitude* (*symbol*) $a(x,\xi)$ *in the class* $L^\infty S^m_\rho(\mathbb{R}^n)$ *is a function* $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *that verifies the estimate*

$$\|\partial_\xi^\alpha a(x,\xi)\|_{L^\infty_x(\mathbb{R}^n)} \lesssim \langle\xi\rangle^{m-\rho|\alpha|},$$

for all multi-indices α *and* $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. *Thus we only assume measurability in the* x*-variable.*

In our treatment of oscillatory integral operators, the phase functions take center stage, and the OIOs are classified according to their phases. In accordance with this treatment, the following class of phase function where first given in [4].

Definition 2.3. *For* $0 < k < \infty$, *we say that a real-valued phase function* $\varphi(x,\xi)$ *belongs to the class* ᚠ^k, *if* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ *and satisfies the following estimates* (*depending on the range of* k):

- *for* $k \geq 1$,
$$|\partial_\xi^\alpha(\varphi(x,\xi) - x\cdot\xi)| \leq c_\alpha |\xi|^{k-1}, \quad |\alpha| \geq 1,$$
- *for* $0 < k < 1$,
$$|\partial_\xi^\alpha \partial_x^\beta(\varphi(x,\xi) - x\cdot\xi)| \leq c_{\alpha,\beta} |\xi|^{k-|\alpha|}, \quad |\alpha+\beta| \geq 1,$$

for all $x \in \mathbb{R}^n$ *and* $|\xi| \geq 1$.

Remark 2.4. *A well known and typical example of a phase in* ᚠ^k *is the phases* $|\xi|^k + x\cdot\xi$, $k > 0$, *which are related to operators* $e^{i(\Delta)^{k/2}}$.

An important condition on the phase function in the context of global regularity of OIOs, is the so-called strong non-degeneracy condition, which is a stronger version of the non-degeneracy condition for oscillatory integral operators as stated for example in Stein's book [23] page 394, in connection to the investigation of the boundedness of Fourier integral operators.

Definition 2.5. *One says that the phase function* $\varphi(x,\xi)$ *satisfies the strong non-degeneracy condition* (*or* φ *is* SND *for short*) *if*

$$\left|\det(\partial^2_{x_j\xi_k}\varphi(x,\xi))\right| \geq \delta, \qquad \text{for some } \delta > 0 \text{ and all } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}. \tag{1}$$

In order to guarantee that the operators at hand are globally L^2-bounded, one should also put yet another condition on the phase, which we shall henceforth simply refer to as the L^2-condition. This condition is essential for the validity of our main results, since it enables us to control the L^2-behavior of the oscillatory integral operator and is thereby crucial in proving boundedness of the operators in either of the ranges $1 < p \leq 2$ or $2 \leq p < \infty$, through interpolation.

Definition 2.6. *One says that the phase function* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *satisfies the weak* L^2*-condition if*

$$|\partial_x^\alpha \partial_\xi \varphi(x,\xi)| \leq C_\alpha, \quad |\partial_x \partial_\xi^\beta \varphi(x,\xi)| \leq C_\beta \tag{2}$$

for $|\alpha| \geq 1$, $|\beta| \geq 1$, *all* $x \in \mathbb{R}^n$ *and* $|\xi| \geq 1$.
We say that $\varphi(x,\xi)$ *satisfies the strong* L^2*-condition if*

$$|\partial_x^\alpha \partial_\xi^\beta \varphi(x,\xi)| \leq C_{\alpha,\beta}, \tag{3}$$

for $|\alpha| \geq 1$, $|\beta| \geq 1$, *all* $x \in \mathbb{R}^n$ *and* $|\xi| \geq 1$.

Having the definitions of the amplitudes and the phase functions at hand, one has

Definition 2.7. *An oscillatory integral operator* (OIO) T_a^φ *with amplitude* $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ *and a real valued phase function* φ*, is defined* (*once again a-priori on* $\mathscr{S}(\mathbb{R}^n)$) *by*

$$T_a^\varphi f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\, a(x,\xi)\, \widehat{f}(\xi)\, \text{đ}\xi. \tag{4}$$

If $\varphi \in \digamma^k$ *and is* SND*, then these operators will be referred to as k-type oscillatory integral operators.*

The LF(μ)-condition which was introduced in [4], is a natural requirement which from the point of view of the applications into PDE's, will always be satisfied and would not cause any loss of generality. It ensures that the operator behaves in a predictable and well-behaved manner, which is necessary for the analysis the low frequency portions of the operators.

Definition 2.8. *Assume that* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ *is real-valued and* $0 < \mu \leq 1$*. We say that* φ *satisfies the low frequency phase condition of order* μ*,* (φ *satisfies* LF(μ)*-condition for short*)*, if one has*

$$|\partial_\xi^\alpha \partial_x^\beta (\varphi(x,\xi) - x\cdot\xi)| \leq c_\alpha |\xi|^{\mu-|\alpha|}, \tag{5}$$

for all $x \in \mathbb{R}^n$*,* $0 < |\xi| \leq 2$ *and all multi-indices* α, β*.*

Remark 2.9. *As an example, note that the phase function associated to the water-wave equation is* $x\cdot\xi + |\xi|^{1/2}$*, which satisfies the* LF(μ) *condition with* $\mu = \frac{1}{2}$*. The phase function associated to the capillary wave equation is* $x\cdot\xi + |\xi|^{3/2}$*, which is in* LF(μ) *for any* $\mu \in (0,1]$*.*

We shall also need the following lemma to estimate the phase in the proofs for Sobolev-boundedness of OIOs with forbidden symbols (whenever the composition Theorem 2.12 below is used), the proof of this lemma can be found in [4, Lemma 4.1].

Lemma 2.10. *Assume* $a(x,\xi)$ *is an amplitude, and* $\varphi(x,\xi)$ *is a* SND *phase function satisfying*

$$|\partial_\xi \partial_x^\beta \varphi(x,\xi)| \leq c_\beta, \quad |\beta| \geq 1 \text{ and } |\xi| \geq 1. \tag{6}$$

Then for all $|\beta| \geq 1$*, the following estimates*

$$|\nabla_x \varphi(x,\xi)| \sim |\xi| \tag{7}$$

$$|\partial_x^\beta \varphi(x,\xi)| \lesssim \langle\xi\rangle \tag{8}$$

hold true for the phase function φ*, on the support of* $a(x,\xi)$*, provided that the* ξ*-support of* $a(x,\xi)$ *lies outside the ball* $B(0,R)$ *for some large enough* $R \gg 1$ *and* $\partial_x^\beta \varphi(x,\xi) \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$*, for* $|\beta| \geq 1$*.*

From this lemma it readily follows that

Corollary 2.11. *The phase functions in* $\digamma^k$ *that also satisfy the* L^2*-condition* (3)*, all verify the estimates* (7) *and* (8) *of* Lemma 2.10*.*

We also recall a composition result proved in [15]. This will be essential in the proof of the boundedness of OIOs with forbidden amplitudes on Sobolev spaces as well as in the proof of the Triebel-Lizorkin boundedness of oscillatory integral operators and in other situations.

Theorem 2.12 ([15])**.** *Let* $m, s \in \mathbb{R}$, $\rho \in [0,1]$, $\delta \in [0,1)$. *Suppose that* $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $b(x,\xi) \in S^s_{1,0}(\mathbb{R}^n)$ *and* $\varphi(x,\xi)$ *is a phase function that is smooth on* $\operatorname{supp} a$ *and verifies the conditions*

$$|\xi| \lesssim |\nabla_x \varphi(x,\xi)| \lesssim |\xi|, \qquad |\partial^\alpha_\xi \partial^\beta_x \varphi(x,\xi)| \lesssim \langle \xi \rangle^{1-|\alpha|} \tag{9}$$

for all $(x,\xi) \in \operatorname{supp} a$ *and for all* $|\alpha| \geq 0$, *and all* $|\beta| \geq 1$. *For* $0 < t \leq 1$ *consider the parameter-dependent pseudodifferential operator*

$$b(x,tD)f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, b(x,t\xi)\, \widehat{f}(\xi)\, đ\xi,$$

and the oscillatory integral operator

$$T^\varphi_a f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\, a(x,\xi)\, \widehat{f}(\xi)\, đ\xi.$$

Let σ_t *be the amplitude of the composition operator* $T^\varphi_{\sigma_t} := b(x,tD)T^\varphi_a$ *given by*

$$\sigma_t(x,\xi) := \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(y,\xi)\, b(x,t\eta)\, e^{i(x-y)\cdot\eta + i\varphi(y,\xi) - i\varphi(x,\xi)}\, đ\eta\, \mathrm{d}y.$$

Then for any $M \geq 1$ *and all* $0 < \varepsilon < 1 - \max(\delta, 1/2)$, *one can write* σ_t *as*

$$\sigma_t(x,\xi) = b(x, t\nabla_x\varphi(x,\xi))\, a(x,\xi) + \sum_{0<|\alpha|<M} \frac{t^{\varepsilon|\alpha|}}{\alpha!}\sigma_\alpha(t,x,\xi) + t^{\varepsilon M} r(t,x,\xi), \tag{10}$$

where, for all multi-indices β, γ *one has*

$$|\partial^\beta_\xi \partial^\gamma_x \sigma_\alpha(t,x,\xi)| \lesssim t^{\min(s,0)} \langle\xi\rangle^{s+m-(1-\max(\delta,1/2)-\varepsilon)|\alpha| - \rho|\beta| + \delta|\gamma|},$$
$$|\partial^\beta_\xi \partial^\gamma_x r(t,x,\xi)| \lesssim t^{\min(s,0)} \langle\xi\rangle^{s+m-(1-\max(\delta,1/2)-\varepsilon)M - \rho|\beta| + \delta|\gamma|}.$$

Proof. For a proof of this result see [15]. □

2.2. **Some facts from the theory of classical function spaces.**

Classical function spaces, such as Triebel-Lizorkin spaces and Besov-Lipschitz spaces, are central to the study of partial differential equations and harmonic analysis. In this section, we review some basic facts and results from the theory of classical function spaces that will be used throughout the paper.

We begin by recalling the definitions of classical Littlewood-Paley operators, Triebel-Lizorkin spaces, and Besov-Lipschitz spaces and discussing some of their basic properties. Thereafter, we present some important lemmas and theorems that provide useful estimates and bounds for functions in these spaces.

A standard reference in the theory of function spaces is H. Triebel's treatise [24], to which we refer the reader for further information.

Definition 2.13. *Let* $\psi_0 \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ *be equal to* 1 *on* $B(0,1)$ *and have its support in* $B(0,2)$. *Then let*

$$\psi_j(\xi) := \psi_0\left(2^{-j}\xi\right) - \psi_0\left(2^{-(j-1)}\xi\right),$$

where $j \geq 1$ *is an integer and* $\psi(\xi) := \psi_1(\xi)$. *Then* $\psi_j(\xi) = \psi\left(2^{-(j-1)}\xi\right)$ *and one has the following Littlewood-Paley partition of unity*

$$\sum_{j=0}^{\infty} \psi_j(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

It is sometimes also useful to define a sequence of smooth and compactly supported functions Ψ_j *with* $\Psi_j = 1$ *on the support of* ψ_j *and* $\Psi_j = 0$ *outside a slightly larger compact set. One could for instance set*

$$\Psi_j := \psi_{j+1} + \psi_j + \psi_{j-1},$$

with $\psi_{-1} := \psi_0$.

In what follows we define the *Littlewood-Paley operators* by

$$\psi_j(D)\, f(x) = \int_{\mathbb{R}^n} \psi_j(\xi)\, \widehat{f}(\xi)\, e^{ix\cdot\xi}\, \text{đ}\xi,$$

where đξ denotes the normalised Lebesgue measure $\mathrm{d}\xi/(2\pi)^n$ and

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, f(x)\, \mathrm{d}x,$$

is the Fourier transform of f.

Define $\|\cdot\|_{L^p(\ell^q)}$ and $\|\cdot\|_{\ell^q(L^p)}$ to be the quasi-norms

$$\|(f_k)\|_{L^p(\ell^q)} := \Big\| \Big(\sum_k |f_k(\cdot)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)},$$

$$\|(f_k)\|_{\ell^q(L^p)} := \Big(\sum_k \|f_k\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q}.$$

Using the Littlewood-Paley decomposition of Definition 2.13 the *Triebel-Lizorkin space* $F^s_{p,q}(\mathbb{R}^n)$ and *Besov-Lipschitz space* $B^s_{p,q}(\mathbb{R}^n)$ are defined as follows, see e.g. [24].

Definition 2.14. *Let* $s \in \mathbb{R}$ *and* $0 < p < \infty$, $0 < q \leq \infty$. *The Triebel-Lizorkin space is defined by*

$$F^s_{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) \,:\, \|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \|(2^{js}\psi_j(D)f)\|_{L^p(\ell^q)} < \infty \Big\},$$

where $\mathscr{S}'(\mathbb{R}^n)$ *denotes the space of tempered distributions.*

Definition 2.15. *Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$. *The Besov-Lipschitz spaces are defined by*

$$B^s_{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) \,:\, \|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|(2^{js}\psi_j(D)f)\|_{\ell^q(L^p)} < \infty \Big\}.$$

Remark 2.16. *Different choices of the basis* $\{\psi_j\}_{j=0}^{\infty}$ *give equivalent norms of* $F^s_{p,q}(\mathbb{R}^n)$ *in* Definition 2.14, *see e.g.* [24]. *We will use either* $\{\psi_j\}_{j=0}^{\infty}$ *or* $\{\Psi_j\}_{j=0}^{\infty}$ *to define the norm of* $F^s_{p,q}(\mathbb{R}^n)$.

We note that for $p = q = \infty$ and $0 < s \leq 1$ we obtain the familiar Lipschitz space $\Lambda^s(\mathbb{R}^n)$, i.e. $B^s_{\infty,\infty}(\mathbb{R}^n) = \Lambda^s(\mathbb{R}^n)$. For $-\infty < s < \infty$ and $1 \leq p < \infty$, $F^s_{p,2}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$ (various L^p-based Sobolev and Sobolev-Slobodeckij spaces) and for $0 < p < \infty$, $F^0_{p,2}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$ (the local Hardy spaces). Moreover the dual space of $F^0_{1,2}(\mathbb{R}^n)$ is bmo (the local version of BMO).

The following estimate will be useful in the $L^1(\mathbb{R}^n)$ to $bmo(\mathbb{R}^n)$ boundedness of Oscillatory integral operators. Let $X^s_{p,q}(\mathbb{R}^n)$ be either $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$. Then for all $-\infty < s < \infty$ and $0 < p, q \leq \infty$ one has

$$\|fg\|_{X^s_{p,q}(\mathbb{R}^n)} \lesssim \Big(\sum_{|\alpha|\leq M} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)|\Big) \|g\|_{X^s_{p,q}(\mathbb{R}^n)}. \tag{11}$$

Two other useful facts which will be useful to us is that for $-\infty < s < \infty$ and $0 < p \leq \infty$ one has

$$B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n), \tag{12}$$

and

$$F^{s+\varepsilon}_{p,q_0}(\mathbb{R}^n) \hookrightarrow F^s_{p,q_1}(\mathbb{R}^n) \quad \text{and} \quad B^{s+\varepsilon}_{p,q_0}(\mathbb{R}^n) \hookrightarrow B^s_{p,q_1}(\mathbb{R}^n) \tag{13}$$

for $-\infty < s < \infty$, $0 < p < \infty$, $0 < q_0, q_1 \leq \infty$ and all $\varepsilon > 0$.

Furthermore, for $s' \in \mathbb{R}$, the operator $(1-\Delta)^{s'/2}$ maps $F^s_{p,q}(\mathbb{R}^n)$ isomorphically into $F^{s-s'}_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ isomorphically into $B^{s-s'}_{p,q}(\mathbb{R}^n)$, see [24, p. 58].

In connection to estimates for linear operators in Triebel-Lizorkin spaces, one often encounters the well-known Hardy-Littlewood's maximal function

$$\mathcal{M}f(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\,\mathrm{d}y,$$

where the supremum is taken over all balls B containing x. For $0 < p < \infty$, one also defines

$$(\mathcal{M}_p f(x) := \left(\mathcal{M}\left(|f|^p\right)\right)^{1/p}.$$

We now present the following abstract lemma which we will make use of in relation to the boundedness of OIOs with amplitudes in the class $S^m_{1,1}(\mathbb{R}^n)$.

Lemma 2.17. *Let $X^s_{p,q}$ be either $F^s_{p,q}(\mathbb{R}^n)$ or $B^s_{p,q}(\mathbb{R}^n)$, and let $f \in X^s_{p,q}$ for $0 < p, q < \infty$ and $s > n(\frac{1}{\min\{p,q,1\}} - 1)$, and assume that $(u_k)_{k\geq 1} \subset X^s_{p,q}$ is an arbitrary sequence such that*

$$\left\|\{2^{ks}u_k\}_{k=0}^\infty\right\|_{L^p(l^q)} \lesssim \|f\|_{X^s_{p,q}}. \tag{14}$$

Moreover let $\{h_{k,l}(x)\}_{k,l\geq 0}$ be a sequence such that the spectrum of $h_{k,l}(x)$ (i.e. the support of its Fourier transform) is in $B(0, 2^{k+l+2})$ and that there is some $N > s$ such that $h_{k,l}$ satisfies the pointwise estimate,

$$|h_{k,l}(x)| \lesssim 2^{-Nl}\,\mathcal{M}_r u_k(x), \quad \text{for some } r \in (0, \min\{p,q\}). \tag{15}$$

Assume that $g_l = \sum_{k\geq 0} h_{k,l}$, then

$$\Big\|\sum_{l\geq 0} g_l\Big\|_{X^s_{p,q}} \lesssim \|f\|_{X^s_{p,q}}. \tag{16}$$

Proof. For a proof see [15], Lemma 2.8 in that paper. □

Another important and useful fact about Besov-Lipschitz and Triebel-Lizorkin spaces is the following:

Theorem 2.18. *Let $\eta : \mathbb{R}^n \to \mathbb{R}^n$ with $\eta(x) = (\eta_1(x), \dots, \eta_n(x))$ be a diffeomorphism such that $\Big|\det D\eta(x)\Big| \geq c > 0$, $\forall x \in \mathbb{R}^n$ ($D\eta$ denotes the Jacobian matrix of η), and $\|\partial^\alpha \eta_j(x)\|_{L^\infty(\mathbb{R}^n)} \lesssim 1$ for all $j \in \{1, \dots, n\}$ and $|\alpha| \geq 1$. Then for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$ one has*

$$\|f \circ \eta\|_{F^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

The same invariance estimate is also true for Besov-Lipschitz spaces $B^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$.

For a proof see J. Johnsen, S. Munch Hansen and W. Sickel [18, Corollary 25], and H. Triebel [25, Theorem 4.3.2].

The rest of this section is dedicated to setting up some definitions which will be used in relation to transference to Triebel-Lizorkin spaces (section 6) and $h^p \to L^p$ estimates of oscillatory integral operators (section 5).

For Definition 2.19 and Definition 2.20, let $[x]$ denote the integer part of x.

Definition 2.19. *For a closed cube Q, define*

- (*i*) c_Q *as the centre of* Q,
- (*ii*) l_Q *as the side length of* Q,
- (*iii*) $k_Q := [1 - \log_2(l_Q)]$,
- (*iv*) χ_Q *as the characteristic function of* Q, *i.e.* $\chi_Q(x) := \begin{cases} 1, & x \in Q \\ 0, & x \notin Q \end{cases}$.
- (*v*) cQ *as the cube with centre* c_Q *and length* $l_{cQ} = c\, l_Q$, *where* $c \in \mathbb{R}_{>0}$.

Observe that k_Q is the unique integer such that

$$2^{-k_Q} < l_Q \leq 2^{-(k_Q - 1)}.$$

Definition 2.20. *A function $\mathfrak{a}$ is called a h^p-atom if there exists a cube Q such that the following three conditions are satisfied:*

- (*i*) $\operatorname{supp} \mathfrak{a} \subset Q$,
- (*ii*) $\sup_{x \in Q} |\mathfrak{a}(x)| \leq 2^{k_Q n/p}$,
- (*iii*) *If* $k_Q \geq 1$, $\mathfrak{M}_{\mathfrak{a}} = \Big[n\Big(\frac{1}{p} - 1\Big)\Big]$, *then* $\int_{\mathbb{R}^n} x^\alpha \mathfrak{a}(x)\, \mathrm{d}x = 0$, *for* $|\alpha| \leq \mathfrak{M}_{\mathfrak{a}}$. *No further condition is assumed if* $k_Q \leq 0$.

It is well known (see [24]) that a distribution $f \in h^p(\mathbb{R}^n)$ has an atomic decomposition

$$f = \sum_{j=0}^{\infty} \lambda_j \mathfrak{a}_j, \tag{17}$$

where the λ_j are constants such that

$$\inf_{\{\lambda_j\}} \sum_{j=0}^{\infty} |\lambda_j|^p \sim \|f\|^p_{h^p(\mathbb{R}^n)} = \|f\|^p_{F^0_{p,2}(\mathbb{R}^n)}$$

and the $\mathfrak{a}_j$ are h^p-atoms.

In the analysis of the boundedness of OIOs on classical function spaces, one typically decomposes the operator into a low- and a high-frequency part, where the low frequency part corresponds to an amplitude that is smooth and compactly supported in the ξ variable. The following theorem, which was proven in [4], addresses the boundedness of the low-frequency portion of the OIOs. The main boundedness result for the low frequency part of OIOs.

Theorem 2.21 ([4])**.** *For $\rho, \delta \in [0,1]$ let $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ be an amplitude which is compactly supported in ξ. Suppose also that $0 < q_1, q_2 \leq \infty$, and $s_1, s_2 \in \mathbb{R}$. If $\varphi(x,\xi)$ verifies the* LF(μ)*-condition, then for $\frac{n}{n+\mu} < p \leq \infty$, T^φ_a maps $F^{s_1}_{p,q_1}(\mathbb{R}^n)$ continuously to $F^{s_2}_{p,q_2}(\mathbb{R}^n)$. Moreover, all the Triebel-Lizorkin estimates above may be replaced by the corresponding Besov-Lipschitz estimates.*

Proof. See [4, Lemma 6.3]. □

Here we recall a theorem that allows one to lift $h^p \to L^p$ boundedness to Besov-Lipschitz boundedness. This result can be applied to a wide class of oscillatory integral operators. Observe that this lift is only valid for the classical amplitudes when $\delta < 1$.

Theorem 2.22 ([16])**.** *Let $0 < p, q < \infty$, $m \leq 0$, and assume that $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ is such it is supported in $\mathbb{R}^n \times \mathbb{R}^n \setminus B(0,1)$. Suppose φ is a phase function that verifies the conditions* (9) *of* Theorem 2.12*. If T^φ_a is $h^p \to L^p$-bounded, then T^φ_a is bounded from $B^s_{p,q} \to B^s_{p,q}$ for all $s \in \mathbb{R}$.*

Proof. For a proof see [16]. □

The following theorem due to M. Pramanik, K. Rogers and A. Seeger could be found in [22, Theorem 2.1].

Theorem 2.23 ([22])**.** *Let $1 < q < p < \infty$, and $0 < b < n$. Assume that the sequence of operators S_j satisfy*

$$\sup_{j>0} 2^{jb/p} \|S_j\|_{L^p \to L^p} \leq A_0, \tag{18}$$

$$\sup_{j>0} 2^{jb/q} \|S_j\|_{L^q \to L^q} \leq B_0. \tag{19}$$

Furthermore, assume that for each cube Q there is a measurable set $\mathcal{E}_Q$ and a constant $\Gamma \geq 1$ such that

$$|\mathcal{E}_Q| \leq \Gamma \max\{l_Q^{n-b}, l_Q^n\}, \tag{20}$$

where l_Q is the side length of Q as in Definition 2.19*, and assume further that for every $j \in \mathbb{N}$ and every cube Q with $2^j l_Q \geq 1$, one has*

$$\sup_{x \in Q} \int_{\mathbb{R}^n \setminus \mathcal{E}_Q} |K_j(x,y)| \,\mathrm{d}y \leq B_1 \max\{2^{-j\varepsilon} l_Q^{-\varepsilon}, 2^{-j\varepsilon}\}, \tag{21}$$

for some $\varepsilon > 0$.
Then if $0 < r \leq p$ and

$$B_2 := B_0^{q/p} \left(A_0 \Gamma^{1/p} + B_1\right)^{1-q/p} \tag{22}$$

one has

$$\Big\| \Big(\sum_{j=0}^\infty 2^{jbr/p} |\Psi_j(D) S_j f_j|^r \Big)^{1/r} \Big\|_{L^p(\mathbb{R}^n)} \lesssim A_0 \left[\log\left(3 + \frac{B_2}{A_0}\right) \right]^{1/r - 1/p} \Big(\sum_{j=0}^\infty \|f_j\|^p_{L^p(\mathbb{R}^n)} \Big)^{1/p}.$$

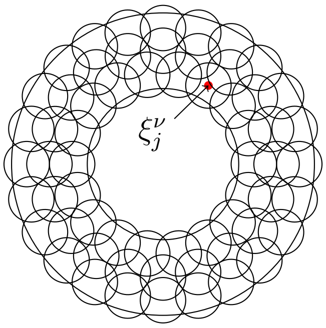

FIGURE 1. Covering of $\operatorname{supp}\psi_j \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ with balls of radii $2^{j(1-k)}$ and centres ξ_j^ν.

where $\Psi \in \mathscr{S}(\mathbb{R}^n)$*,* $\Psi_j(D) := \Psi(2^{-j}D)$ *and* f_j *is a sequence of functions.*

3. BASIC KERNEL ESTIMATES FOR OSCILLATORY INTEGRAL OPERATORS

In order to show the Besov-Lipschitz boundedness of OIOs related to forbidden amplitudes we prove some preliminary estimates on the kernel of the operators. What follows is a decomposition which, in contrast to the second angular-radii frequency decomposition of C. Fefferman (see [8]), decomposes the annuli of the standard Littlewood-Paley decomposition into balls. Furthermore, this decomposition generalizes the decomposition in [4], which used balls of constant radii to decompose the case corresponding to the OIOs of Schrödinger type phase functions in ᚠ^2, to the more general case of balls of varying radii adapted to all OIOs with phases in the class ᚠ^k.

Definition 3.1. *Let* $k > 0$*. We make the following decomposition of the integral kernel*

$$K(x,y) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\varphi(x,\xi)-iy\cdot\xi}\, \text{đ}\xi.$$

Take a standard Littlewood-Paley partition of unity $\sum_{j=0}^\infty \psi_j(\xi) = 1$ *with* $\operatorname{supp}\psi_0 \subset B(0,2)$*, and* $\operatorname{supp}\psi_j \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ *for* $j \geq 1$ *as in* Definition 2.13*. Then for every* $j \geq 0$ *we cover* $\operatorname{supp}\psi_j$ *with open balls* C_j^ν *with radii* $2^{j(1-k)}$ *and centres* ξ_j^ν*, where* ν *runs from* 1 *to* $\mathscr{N}_j := O(2^{njk})$*. See* Figure 1 *for an illustration. Observe that* $|C_j^\nu| \lesssim 2^{nj(1-k)}$ *uniformly in* ν*. Now take* $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$*, with* $0 \leq u \leq 1$ *and supported in* $B(0,2)$ *with* $u = 1$ *on* $\overline{B(0,1)}$*.*

Next set

$$\chi_j^\nu(\xi) := \frac{u(2^{-(1-k)j}(\xi - \xi_j^\nu))}{\sum_{\kappa=1}^{\mathscr{N}_j} u(2^{-(1-k)j}(\xi - \xi_j^\kappa))},$$

and note that

$$\sum_{j=0}^\infty \sum_{\nu=1}^{\mathscr{N}_j} \chi_j^\nu(\xi)\, \psi_j(\xi) = 1.$$

Now we define the second frequency localized pieces of the kernel above as

$$K_j^\nu(x,y) := \int_{\mathbb{R}^n} \psi_j(\xi)\, \chi_j^\nu(\xi)\, e^{i\varphi(x,\xi)-iy\cdot\xi}\, a(x,\xi)\, \text{đ}\xi.$$

Lemma 3.2. *Let $0 \leq \rho \leq 1$, $0 \leq \delta < 1$, $n \geq 1$, and $0 < k < \infty$. Assume that $\varphi \in \text{ᚠ}^k$ is* SND, *satisfies the L^2-condition* (3)*. Then if $a(x,\xi) \in L^\infty S^m_\rho(\mathbb{R}^n)$ (see* Definition 2.2*), and $K^\nu_j(x,y)$ is given as in* Definition 3.1*, then setting*

$$w(k,\rho) = \begin{cases} \min\{\rho, 1-k\} & 0 < k < 1, \\ k-1 & k \geq 1, \end{cases}$$

one has that

$$|\partial^\alpha_y K^\nu_j(x,y)| \lesssim \frac{2^{j(m+|\alpha|)} 2^{jn(1-k)}}{\left\langle 2^{jw(k,\rho)}(\nabla_\xi \varphi(x,\xi^\nu_j) - y)\right\rangle^M}, \tag{23}$$

for all multi-indices α, all $j \geq 0$ and $M \geq 0$.

Proof. Observe that we have

$$|\partial^\gamma_\xi \chi^\nu_j(\xi)| \lesssim 2^{j(k-1)|\gamma|}$$

Therefore, for any multi-index α and any $j \geq 0$ we have

$$\begin{aligned} \partial^\alpha_y K^\nu_j(x,y) &= \int_{\mathbb{R}^n} \psi_j(\xi)\, \chi^\nu_j(\xi)\, \partial^\alpha_y e^{i(\varphi(x,\xi) - y\cdot\xi)}\, a(x,\xi)\, \text{đ}\xi \\ &= \int_{\mathbb{R}^n} e^{i(\varphi(x,\xi)-y\cdot\xi)}\, \sigma^{\alpha,\nu}_j(x,\xi)\, \text{đ}\xi, \end{aligned}$$

where

$$\sigma^{\alpha,\nu}_j(x,\xi) := \psi_j(\xi)\, \chi^\nu_j(\xi)\, (-i\xi)^\alpha\, a(x,\xi).$$

Using the assumption that $a(x,\xi) \in L^\infty S^m_\rho(\mathbb{R}^n)$, we deduce that for any multi-index γ, any $j \geq 0$ and any ν one has

$$|\partial^\gamma_\xi \sigma^{\alpha,\nu}_j(x,\xi)| \lesssim 2^{j(m+|\alpha| - \min\{\rho, 1-k\}|\gamma|)}, \tag{24}$$

If we now set $\vartheta^\nu_j(x,\xi) := \varphi(x,\xi) - \xi \cdot \nabla_\xi \varphi(x,\xi^\nu_j)$, then we can write

$$\partial^\alpha_y K^\nu_j(x,y) = \int_{\mathbb{R}^n} e^{i(\nabla_\xi\varphi(x,\xi^\nu_j)-y)\cdot\xi}\, e^{i\vartheta^\nu_j(x,\xi)}\, \sigma^{\alpha,\nu}_j(x,\xi)\, \text{đ}\xi.$$

Now we estimate the derivatives of ϑ in ξ on the support of $\sigma^{\alpha,\nu}_j(x,\xi)$. To this end, the mean-value theorem and Definition 2.3 yields for $k \geq 1$

$$\begin{aligned} |\partial_{\xi_l} \vartheta^\nu_j(x,\xi)| &= |\partial_{\xi_l}\varphi(x,\xi) - \partial_{\xi_l}\varphi(x,\xi^\nu_j)| = \left|(\xi - \xi^\nu_j)\cdot \int_0^1 (\nabla_\xi \partial_{\xi_l}\varphi)(x, t\xi + (1-t)\xi^\nu_j)\,\mathrm{d}t\right| \\ &\lesssim 2^{j(1-k)} \sup_{t\in[0,1]} |t\xi + (1-t)\xi^\nu_j|^{k-1} \\ &\lesssim 2^{j(1-k)} 2^{j(k-1)} = 1 \end{aligned}$$

and

$$|\partial^\alpha_\xi \vartheta^\nu_j(x,\xi)| = |\partial^\alpha_\xi \varphi(x,\xi)| \lesssim 2^{j|\alpha|\max\{(k-1),0\}}, \qquad \text{for all } |\alpha| \geq 2.$$

Therefore by Faa di Bruno's formula we obtain that

$$|\partial^\gamma_\xi e^{i\vartheta^\nu_j(\xi,x)}| \lesssim \sum_{\gamma_1+\dots+\gamma_r=\gamma} |\partial^{\gamma_1}_\xi \vartheta^\nu_j| \dots |\partial^{\gamma_r}_\xi \vartheta^\nu_j| \lesssim 2^{j\max\{(k-1),0\}|\gamma|}. \tag{25}$$

Observe the simple estimate

$$|\partial^\alpha_y K^\nu_j(x,y)| \lesssim 2^{j(m+|\alpha|)} 2^{jn(1-k)} \tag{26}$$

together with integration by parts yield

$$|\partial_y^\alpha K_j^\nu(x,y)| \lesssim \frac{2^{j(m+|\alpha|+w(k,\rho)M)}2^{jn(1-k)}}{|\nabla_\xi\varphi(x,\xi_j^\nu)-y|^M}. \tag{27}$$

where $w(k,\rho)=\max\{-\min\{\rho,1-k\},\max\{(k-1),0\}\}$. These equalities, the observation that $\operatorname{supp}\sigma_j^{\alpha,\nu}\subset C_j^\nu$, with $|C_j^\nu|=O(2^{jn(1-k)})$ uniformly in ν and j, the estimates for the derivatives of ϑ, and (24) yield

$$|\partial_y^\alpha K_j^\nu(x,y)| \lesssim \frac{2^{j(m+|\alpha|)}2^{jn(1-k)}}{\langle 2^{jw(k,\rho)}(\nabla_\xi\varphi(x,\xi_j^\nu)-y)\rangle^M},$$

for all multi-indices α, all $j\geq 0$ and $M\geq 0$. ☐

4. L^2-BOUNDEDNESS OF OSCILLATORY INTEGRAL OPERATORS

The traditional way to prove end-point L^2-boundedness results for oscillatory integral operators is to utilize the almost orthogonality principle, as proved in the continuous version of the Cotlar-Stein lemma due to A. Calderón and R. Vaillancourt [2] given below.

Lemma 4.1. *Let $\mathscr{H}$ be a Hilbert space, and $A(\xi)$ a family of bounded linear endomorphisms of $\mathscr{H}$ depending on $\xi\in\mathbb{R}^n$. Assume the following three conditions hold:*

- (*i*) *the operator norm of $A(\xi)$ is less than a number C independent of ξ.*
- (*ii*) *for every $u\in\mathscr{H}$ the function $\xi\mapsto A(\xi)u$ from $\mathbb{R}^n\mapsto\mathscr{H}$ is continuous for the norm topology of $\mathscr{H}$.*
- (*iii*) *for all ξ_1 and ξ_2 in $\mathbb{R}^n$*

$$\|A^*(\xi_1)A(\xi_2)\|\leq h(\xi_1,\xi_2)^2,\ \ \textit{and}\ \ \|A(\xi_1)A^*(\xi_2)\|\leq h(\xi_1,\xi_2)^2, \tag{28}$$

with $h(\xi_1,\xi_2)\geq 0$ is the kernel of a bounded linear operator on L^2 with norm K.

Then for every $E\subset\mathbb{R}^n$, with $|E|<\infty$, the operator $A_E=\int_E A(\xi)\,\mathrm{d}\xi$ defined by $\langle A_E u,v\rangle_{\mathscr{H}}=\int_E\langle A(\xi)u,v\rangle_{\mathscr{H}}\,\mathrm{d}\xi$, is a bounded linear operator on $\mathscr{H}$ with norm less than or equal to K.

Another useful fact that will aid us in the estimate of the oscillatory integrals is the following lemma whose proof could be found in [7].

Lemma 4.2. *Let $s(x)$ and $F(x)$ be real-valued smooth functions in $\mathbb{R}^n$, and*

$$Lu(x):=D^{-2}(1-is(x)\langle\nabla_x F,\nabla_x\rangle)u(x), \tag{29}$$

with $D:=(1+s(x)|\nabla_x F|^2)^{1/2}$. Then

- (*i*) $L(e^{iF(x)})=e^{iF(x)}$
- (*ii*) *if ${}^tL$ denotes the formal transpose of L, then for any positive integer N, $({}^tL)^N u(x)$ is a finite linear combination of terms of the form*

$$CD^{-k}\Big\{\prod_{\mu=1}^{p}\partial_x^{\alpha_\mu}s(x)\Big\}\Big\{\prod_{\nu=1}^{q}\partial_x^{\beta_\nu}F(x)\Big\}\partial_x^\gamma u(x), \tag{30}$$

with

$$2N \leq k \leq 4N;\ k-2N \leq p \leq k-N;\ |\alpha_\mu| \geq 0;\ \sum_{\mu=1}^{p} |\alpha_\mu| \leq N \tag{31}$$

$$k-2N \leq q \leq k-N;\ |\beta_\nu| \geq 1;\ \sum_{\mu=1}^{q} |\beta_\nu| \leq q+N;\ |\gamma| \leq N.$$

Theorem 4.3. *If* $m = \min(0, \frac{n}{2}(\rho-\delta))$, $0 \leq \rho \leq 1$, $0 \leq \delta < 1$, $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ *and* $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *is* SND *for all* $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ *and satisfies the weak* L^2*-condition* (2) *on the support of* $a(x,\xi)$*, then the operator*

$$T^\varphi_a f(x) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\varphi(x,\xi)} \hat{f}(\xi)\, đ\xi$$

is bounded on $L^2(\mathbb{R}^n)$.

Proof. Let $\chi(x,\xi) \in \mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\chi(x,\xi) = 1$ for $|x|^2 + |\xi|^2 \leq 1$ and define $a_\varepsilon(x,\xi) := \chi(\varepsilon x, \varepsilon\xi)\, a(x,\xi)$. Since a_ε converges to a in $S^m_{\rho,\delta}(\mathbb{R}^n)$ so that for any $f \in \mathscr{S}(\mathbb{R}^n)$, $T^\varphi_{a_\varepsilon} f$ converges, as $\varepsilon \to 0$, to $T^\varphi_a f$ in $\mathscr{S}(\mathbb{R}^n)$. Since the seminorms of a_ε are bounded by a constant (depending on χ) times the seminorms of a, we can therefore assume from now on that $a(x,\xi) \in \mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{R}^n)$. Later on, of course, the estimates that we obtain won't depend on the support of a.

Furthermore, we observe that since for $\delta \leq \rho$, $S^0_{\rho,\delta}(\mathbb{R}^n) \subset S^0_{\rho,\rho}(\mathbb{R}^n)$, it is enough to show the theorem for $0 \leq \rho \leq \delta < 1$ and $m = \frac{n}{2}(\rho - \delta)$. Using the unitarity of the Fourier transform in $L^2(\mathbb{R}^n)$ and a TT^* argument, it is enough to show that the operator

$$T_b f(x) = \iint b(x,y,\xi)\, e^{i\varphi(x,\xi) - i\varphi(y,\xi)}\, f(y)\, \mathrm{d}y\, đ\xi, \tag{32}$$

where $b(x,y,\xi) := a(x,\xi)\overline{a(y,\xi)}$ satisfies the estimate

$$|\partial^\alpha_\xi \partial^\beta_x \partial^\gamma_y b(x,y,\xi)| \leq C_{\alpha\beta\gamma} \langle\xi\rangle^{m_1 - \rho|\alpha| + \delta(|\beta|+|\gamma|)}, \tag{33}$$

with $m_1 = n(\rho-\delta)$ and $0 \leq \rho \leq \delta < 1$, is bounded on $L^2(\mathbb{R}^n)$. Moreover due to assumption of compact support of $a(x,\xi)$ we can also assume that $b \in \mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, under the understanding that the norm estimates that we obtain will be independent of the support of b.

Now we introduce a differential operator

$$L := D^{-2}\Big\{1 - i\langle\xi\rangle^\rho\big(\langle\nabla_\xi\varphi(x,\xi) - \nabla_\xi\varphi(y,\xi), \nabla_\xi\rangle\big)\Big\},$$

with $D = (1 + \langle\xi\rangle^\rho|\nabla_\xi\varphi(x,\xi) - \nabla_\xi\varphi(y,\xi)|^2)^{\frac{1}{2}}$. It follows from Lemma 4.2 that

$$L\big(e^{i\varphi(x,\xi)-i\varphi(y,\xi)}\big) = e^{i\varphi(x,\xi)-i\varphi(y,\xi)}$$

and that $({}^tL)^N b(x,y,\xi)$ is a finite sum of terms of the form

$$D^{-k}\Big\{\prod_{\mu=1}^{p} \partial^{\alpha_\mu}_\xi \langle\xi\rangle^\rho\Big\}\Big\{\prod_{\nu=1}^{q} \big(\partial^{\beta_\nu}_\xi \varphi(x,\xi) - \partial^{\beta_\nu}_\xi \varphi(y,\xi)\big)\Big\}\, \partial^\gamma_\xi b(x,y,\xi), \tag{34}$$

with p, α_μ, β_ν, q and γ quantified by (31). Furthermore since φ is SND, we can use Proposition 1.11 in [7] to show that

$$|\nabla_\xi\varphi(x,\xi) - \nabla_\xi\varphi(y,\xi)| \geq c_1|x-y| \tag{35}$$

$$|\nabla_z\varphi(z,\xi_1) - \nabla_z\varphi(z,\xi_2)| \geq c_2|\xi_1 - \xi_2|. \tag{36}$$

Using (35), (31), (33) and (34), we have

$$|\partial_x^\sigma ({}^tL)^N b(x,y,\xi)| \leq C\Lambda(\langle\xi\rangle^\rho (x-y))\langle\xi\rangle^{m_1+\delta|\sigma|}, \tag{37}$$

where Λ is a function with $\int_{\mathbb{R}^n} \Lambda(\xi)\, \mathrm{đ}\xi \lesssim 1$. Integration by parts using L, N times, in (32) one has

$$T_b f(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y,\xi)\, e^{i\varphi(x,\xi)-i\varphi(y,\xi)}\, f(y)\,\mathrm{d}y\,\mathrm{đ}\xi, \tag{38}$$

with $c(x,y,\xi) = ({}^tL)^N b(x,y,\xi)$ and

$$|\partial_x^\sigma c(x,y,\xi)| \leq C\Lambda(\langle\xi\rangle^\rho (x-y))\langle\xi\rangle^{m_1+\delta|\sigma|} \tag{39}$$

and the same estimate is valid for $\partial_y^\sigma c(x,y,\xi)$. From this we get the representation

$$T_b = \int_{\mathbb{R}^n} A(\xi)\,\mathrm{đ}\xi, \tag{40}$$

where

$$A(\xi)f(x) := \int_{\mathbb{R}^n} c(x,y,\xi)\, e^{i\varphi(x,\xi)-i\varphi(y,\xi)}\, f(y)\,\mathrm{d}y.$$

Noting that $A(\xi)=0$ for ξ outside some compact set, we observe that condition (i) of Lemma 4.1 follows from Young's inequality and (39) with $\sigma=0$, and condition (ii) of Lemma 4.1 follows from the assumption of the compact support of the amplitude. To verify condition (iii) we confine ourselves to the estimate of $\|A^*(\xi_1)A(\xi_2)\|$, since the one for $\|A(\xi_1)A^*(\xi_2)\|$ is similar. To this end, a calculation shows that the kernel of $A^*(\xi_1)A(\xi_2)$ is given by

$$K(x,y,\xi_1,\xi_2) := \int_{\mathbb{R}^n} \overline{c}(z,x,\xi_1)\, c(z,y,\xi_2)\, e^{i[\varphi(z,\xi_2)-\varphi(z,\xi_1)+\varphi(x,\xi_1)-\varphi(y,\xi_2)]}\,\mathrm{d}z. \tag{41}$$

The estimate (39) yields

$$|K(x,y,\xi_1,\xi_2)| \lesssim \langle\xi_1\rangle^{m_1}\,\langle\xi_2\rangle^{m_1} \int_{\mathbb{R}^n} \Lambda(\langle\xi_1\rangle^\rho(x-z))\,\Lambda(\langle\xi_2\rangle^\rho(y-z))\,\mathrm{d}z. \tag{42}$$

Therefore by choosing N large enough, Young's inequality and using the fact that $\int_{\mathbb{R}^n}\Lambda(x)\,dx \lesssim 1$ yield

$$\|A^*(\xi_1)A(\xi_2)\| \lesssim \langle\xi_1\rangle^{m_1-n\rho}\,\langle\xi_2\rangle^{m_1-n\rho}. \tag{43}$$

At this point we introduce another first order differential operator $M := G^{-2}\{1 - i(\langle\nabla_z\varphi(z,\xi_2)-\nabla_z\varphi(z,\xi_1),\nabla_z\rangle)\}$, with $G=(1+|\nabla_z\varphi(z,\xi_2)-\nabla_z\varphi(z,\xi_1)|^2)^{\frac{1}{2}}$. Using the fact that $Me^{i(\varphi(z,\xi_2)-\varphi(z,\xi_1))} = e^{i(\varphi(z,\xi_2)-\varphi(z,\xi_1))}$, integration by parts in (41) yields

$$\int_{\mathbb{R}^n} ({}^tM)^{N'}\{\overline{c}(z,x,\xi_1)\,c(z,y,\xi_2)\}\, e^{i[\varphi(z,\xi_2)-\varphi(z,\xi_1)+\varphi(x,\xi_1)-\varphi(y,\xi_2)]}\,\mathrm{d}z. \tag{44}$$

Using the second part of Lemma 4.2, we find that $({}^tM)^{N'}\{\overline{c}(z,x,\xi_1)\,c(z,y,\xi_2)\}$ is a linear combination of terms of the form

$$G^{-k}\Big\{\prod_{\nu=1}^{q}(\partial_z^{\beta_\nu}\varphi(z,\xi_2)-\partial_z^{\beta_\nu}\varphi(z,\xi_1))\Big\}\partial_z^{\gamma_1}\overline{c}(z,x,\xi_1)\,\partial_z^{\gamma_2}c(z,y,\xi_2), \tag{45}$$

where k, q, β_ν satisfy the inequalities in (31) and $|\gamma_1|+|\gamma_2|\leq N'$.

Now we observe that (36) yields that

$$G^{-k} \lesssim (1+|\xi_1-\xi_2|)^{-k}.$$

Moreover using (2) we can also deduce that

$$|\partial_z^{\beta_\nu}\varphi(z,\xi_2)-\partial_z^{\beta_\nu}\varphi(z,\xi_1)|\lesssim|\xi_1-\xi_2|\leq 1+|\xi_1-\xi_2|.$$

Moreover (31) yields that $q-k\leq -N'$ and therefore we obtain

$$G^{-k}\Big|\prod_{\nu=1}^{q}(\partial_z^{\beta_\nu}\varphi(z,\xi_2)-\partial_z^{\beta_\nu}\varphi(z,\xi_1))\Big|\lesssim(1+|\xi_1-\xi_2|)^{q-k}\lesssim(1+|\xi_1-\xi_2|)^{-N'}\lesssim|\xi_1-\xi_2|^{-N'}.$$

Thus (39) and (45), yield the following estimate for $K(x,y,\xi_1,\xi_2)$

$$\begin{aligned}|K(x,y,\xi_1,\xi_2)|&\lesssim|\xi_1-\xi_2|^{-N'}\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_1}(1+|\xi_1|+|\xi_2|)^{\delta N'}\\&\times\int_{\mathbb{R}^n}\Lambda(\langle\xi_1\rangle^{\rho}(x-z))\,\Lambda(\langle\xi_2\rangle^{\rho}(y-z))\,\mathrm{d}z.\end{aligned}\tag{46}$$

Once again, choosing N large enough, Young's inequality yields

$$\|A^*(\xi_1)A(\xi_2)\|\lesssim\langle\xi_1\rangle^{m_1-n\rho}\langle\xi_2\rangle^{m_1-n\rho}\frac{(1+|\xi_1|+|\xi_2|)^{\delta N'}}{|\xi_1-\xi_2|^{N'}}.\tag{47}$$

Using the fact that for $x>0$, $\inf(1,x)\sim(1+\frac{1}{x})^{-1}$, one optimizes the estimates (43) and (47) by

$$\begin{aligned}\|A^*(\xi_1)A(\xi_2)\|&\lesssim\langle\xi_1\rangle^{m_1-n\rho}\langle\xi_2\rangle^{m_1-n\rho}\left(1+\frac{|\xi_1-\xi_2|^{N'}}{(1+|\xi_1|+|\xi_2|)^{\delta N'}}\right)^{-1}\\&:=h^2(\xi_1,\xi_2).\end{aligned}\tag{48}$$

Therefore recalling that $m_1=n(\rho-\delta)$, in applying Lemma 4.1, we need to show that

$$K(\xi_1,\xi_2)=(1+|\xi_1|)^{\frac{-n\delta}{2}}(1+|\xi_2|)^{-\frac{n\delta}{2}}\left(1+\frac{|\xi_1-\xi_2|^{N'}}{(1+|\xi_1|+|\xi_2|)^{\delta N'}}\right)^{-\frac{1}{2}}\tag{49}$$

is the kernel of a bounded operator in L^2. At this point we use Schur's lemma, which yields the desired conclusion provided that

$$\sup_{\xi_1}\int_{\mathbb{R}^n}K(\xi_1,\xi_2)\,\mathrm{d}\xi_2,\quad\sup_{\xi_2}\int_{\mathbb{R}^n}K(\xi_1,\xi_2)\,\mathrm{d}\xi_1$$

are both finite. Due to the symmetry of the kernel, we only need to show the finiteness of one of these quantities.

To this end, we fix ξ_1 and consider the domains $\mathcal{A}=\{(\xi_1,\xi_2);\,|\xi_2|\geq 2|\xi_1|\}$, $\mathcal{B}=\{(\xi_1,\xi_2);\,\frac{|\xi_1|}{2}\leq|\xi_2|\leq 2|\xi_1|\}$, and $\mathcal{C}=\{(\xi_1,\xi_2);\,|\xi_2|\leq\frac{|\xi_1|}{2}\}$. Now we observe that on the set $\mathcal{A}$, $K(\xi_1,\xi_2)$ is dominated by

$$(1+|\xi_1|)^{-\frac{n\delta}{2}}(1+|\xi_2|)^{-\frac{n\delta}{2}+\frac{N'}{2}(\delta-1)},$$

on $\mathcal{B}$, $K(\xi_1,\xi_2)$ is dominated by

$$(1+|\xi_1|)^{-n\delta}\left(1+\frac{|\xi_1-\xi_2|^{N'}}{(1+|\xi_1|)^{\delta N'}}\right)^{-\frac{1}{2}},$$

and on $\mathcal{C}$, $K(\xi_1,\xi_2)$ is dominated by

$$(1+|\xi_2|)^{-\frac{n\delta}{2}}(1+|\xi_1|)^{-\frac{n\delta}{2}+\frac{N'}{2}(\delta-1)}.$$

Therefore, if $I_\Omega:=\int_\Omega K(\xi_1,\xi_2)\,\text{đ}\xi_2$, then choosing $\frac{N'}{2}(\delta-1)<-n$, which is only possible if $\delta<1$, we have that $I_{\mathcal{A}}<\infty$ uniformly in ξ_1. Also,

$$I_{\mathcal{C}}\leq(1+|\xi_1|)^{n-\frac{n\delta}{2}+\frac{N'}{2}(\delta-1)}\leq C,\tag{50}$$

which is again possible by the fact that $\delta < 1$ and a suitable choice of N'. In $I_{\mathcal{B}}$ let us make a change of variables to set $\xi_2 - \xi_1 = (1 + |\xi_1|)^{\delta}\eta$, then

$$I_{\mathcal{B}} \leq \int_{\mathbb{R}^n} (1 + |\eta|^{N'})^{-\frac{1}{2}} \,\text{đ}\eta < \infty, \tag{51}$$

by taking N' large enough. These estimates yield the desired result and the proof of their theorem is therefore complete. □

5. A $h^p \to L^p$ ESTIMATE FOR OSCILLATORY INTEGRAL OPERATORS

In this section, we shall deal with the boundedness of OIOs on local Hardy spaces h^p for $0 < p < \infty$ and also the boundedness from L^{∞} to bmo. Apart from the distinction caused by the types of the amplitudes (i.e. ρ and δ), the values of k (related to the phase functions) also play a decisive role in the analysis of the regularity properties of OIOs. When $0 < k < 1$, it turns out that the type of the amplitude can be incorporated in the analysis in such a way that the critical order of decay of the amplitude can be improved compared to the case of $k \geq 1$ where the order of the amplitude is

$$-kn\Big|\frac{1}{p} - \frac{1}{2}\Big| + \min\Big\{0, \frac{n}{2}(\rho - \delta)\Big\}. \tag{52}$$

To this end, let

$$\varkappa = \min(\rho, 1 - k) \tag{53}$$

and set

$$m(p) := m_{\varkappa}(p) + \zeta. \tag{54}$$

where

$$m_{\varkappa}(p) := -n(1 - \varkappa)\Big|\frac{1}{p} - \frac{1}{2}\Big|$$

and

$$\zeta := \min\Big\{0, \frac{n}{2}(\rho - \delta)\Big\}.$$

Observe that for $k \geq 1$ one recovers (52) from (54), since in that case, $\varkappa = 1 - k$.

Lemma 5.1. *Let $0 \leq \rho \leq 1$ and $0 \leq \delta < 1$. Assume that $\varphi \in \digamma^k$ for $0 < k < \infty$ is* SND *and satisfies the L^2-condition* (3)*. Then if $a(x, \xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ for some $m \in \mathbb{R}^n$, let*

$$K_j(x, y) = \int_{\mathbb{R}^n} e^{i(\varphi(x,\xi) - y\cdot\xi)}\, \sigma_j(x, \xi) \,\text{đ}\xi.$$

where $\sigma_j(x, \xi) = \psi_j(\xi)\, a(x, \xi)$. For $M \geq 0$, $y \in \mathbb{R}^n$ and $j \geq 1$ we have

$$\|(1 + 2^{j\varkappa}|x - y|)^M \partial_y^{\beta} K_j(x, y)\|_{L^2_x(\mathbb{R}^n)} \lesssim 2^{j(\tilde{m} + n/2 + |\beta|)}, \tag{55}$$

where $\tilde{m} := m - \zeta$. Moreover, under the extra assumption of $\varphi(x, 0) = 0$ then estimate (55) *is also valid when $K_j(x, y)$ is replaced by the kernel of the adjoint $K_j^*(x, y)$.*

Proof. Observe that for any multi-index β and any $j \geq 0$ we have

$$\begin{aligned}\partial_y^\beta K_j(x,y) &= \int_{\mathbb{R}^n} \psi_j(\xi)\, \partial_y^\beta e^{i(\varphi(x,\xi)-y\cdot\xi)}\, a(x,\xi)\, \text{đ}\xi \\ &= \int_{\mathbb{R}^n} e^{i(\varphi(x,\xi)-y\cdot\xi)}\, \sigma_j^\beta(x,\xi)\, \text{đ}\xi,\end{aligned}$$

where

$$\sigma_j^\beta(x,\xi) := \sigma_j(x,\xi)\,(-i\xi)^\beta.$$

Therefore, since $|\partial_\xi^\alpha \partial_x^\gamma \sigma_j^\beta(x,\xi)| \lesssim 2^{j(m+|\beta|-\rho|\alpha|+\delta|\gamma|)}$ we can reduce ourselves to the case when $\beta = 0$.

Now, using (24) and (25) and letting Ψ_j be a Littlewood-Paley function that is supported on a larger annulus, in the sense of Definition 2.13, we have

$$\begin{aligned}(x-y)^\alpha K_j(x,y) &= \int_{\mathbb{R}^n} (-i)^{|\alpha|}\partial_\xi^\alpha\, e^{i(x-y)\cdot\xi}\, \sigma_j(x,\xi)\, e^{i(\varphi(x,\xi)-x\cdot\xi)}\, \text{đ}\xi \\ &= \int_{\mathbb{R}^n} i^{|\alpha|}\partial_\xi^\alpha \Big[\, e^{i(\varphi(x,\xi)-x\cdot\xi)}\, \sigma_j(x,\xi)\Big]\, e^{i(x-y)\cdot\xi}\, \Psi_j(\xi)\, \text{đ}\xi \\ &= \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} \int_{\mathbb{R}^n} \partial_\xi^{\alpha_1}\sigma_j(x,\xi)\, \partial_\xi^{\alpha_2} e^{i(\varphi(x,\xi)-x\cdot\xi)}\; e^{i(x-y)\cdot\xi}\Psi_j(\xi)\, \text{đ}\xi \\ &= \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \lambda_1+\dots+\lambda_r=\alpha_2}} C_{\alpha_1,\alpha_2,\lambda_1,\dots,\lambda_r} 2^{j(\tilde{m}-\varkappa|\alpha|)} \int_{\mathbb{R}^n} b_j^{\alpha_1,\alpha_2,\lambda_1,\dots\lambda_r}(x,\xi)\, e^{i\varphi(x,\xi)}\, e^{-iy\cdot\xi}\, \Psi_j(\xi)\, \text{đ}\xi \\ &=: \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \lambda_1+\dots+\lambda_r=\alpha_2}} C_{\alpha_1,\alpha_2,\lambda_1,\dots,\lambda_r} 2^{j(\tilde{m}-\varkappa|\alpha|)} S_j^{\alpha_1,\alpha_2,\lambda_1,\dots\lambda_r}\big(\tau_{-y}\widehat{\Psi}_j\big)(x),\end{aligned}$$

where τ_{-y} is a translation by $-y$, $|\lambda_j| \geq 1$ and

$$\begin{aligned}b_j^{\alpha_1,\alpha_2,\lambda_1,\dots,\lambda_r}(x,\xi) &:= 2^{-j(\tilde{m}-\varkappa|\alpha|)} \\ &\qquad \times \partial_\xi^{\alpha_1}\sigma_j(x,\xi)\, \partial_\xi^{\lambda_1}(\varphi(x,\xi)-x\cdot\xi)\dots\partial_\xi^{\lambda_r}(\varphi(x,\xi)-x\cdot\xi),\end{aligned}$$

Now we claim that $b_j^{\alpha_1,\alpha_2,\lambda_1,\dots,\lambda_r} \in S^\zeta_{0,\delta}(\mathbb{R}^n)$, uniformly in j. Indeed, since $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ and $\varphi \in \digamma^k$, we can write

$$\begin{aligned}|b_j^{\alpha_1,\alpha_2,\alpha_3,\lambda_1,\dots,\lambda_r}(x,\xi)| &\lesssim 2^{-j(\tilde{m}-\varkappa|\alpha|)}\, 2^{j\tilde{m}+j\zeta-j\rho|\alpha_1|}\, 2^{j(k-1)|\alpha_2|} \\ &\lesssim 2^{j\zeta}2^{j\varkappa|\alpha|}\, 2^{-j\rho|\alpha_1|}\, 2^{j(k-1)|\alpha_2|} \\ &\lesssim 2^{j\zeta}.\end{aligned}$$

Moreover, observe also that by the $\digamma^k$-condition for $k<1$,

$$|\partial_x^{\beta_j}\partial_\xi^{\lambda_{j-1}+\gamma_j}(\varphi(x,\xi)-x\cdot\xi)| \lesssim 2^{j(k-|\lambda_{j-1}+\gamma_j|)} \text{ when } |\beta_j| \geq 0$$

and for $k \geq 1$ the L^2-condition (3) yields that

$$|\partial_x^{\beta_j}\partial_\xi^{\lambda_{j-1}+\gamma_j}(\varphi(x,\xi)-x\cdot\xi)| \lesssim 1 \text{ when } |\beta_j| \geq 1.$$

Thus, using these estimates, we have for all $k>0$ and all $1 \leq j \leq r+1$ that

$$|\partial_x^{\beta_j}\partial_\xi^{\lambda_{j-1}+\gamma_j}(\varphi(x,\xi)-x\cdot\xi)| \lesssim 2^{j(k-1)|\lambda_{j-1}+\gamma_j|}, \text{ when } |\beta_j| \geq 1 \tag{56}$$

Now using (56) we can also check that, for any multi-indices γ and β,

$$
\begin{aligned}
(57)\quad & |\partial_\xi^\gamma \partial_x^\beta b_j^{\alpha_1,\alpha_2,\lambda_1,\dots,\lambda_r}(x,\xi)| \\
&\lesssim \sum_{\substack{\gamma_1+\dots+\gamma_{r+1}=\gamma\\ \beta_1+\dots+\beta_{r+1}=\beta}} 2^{-j(\tilde m-\varkappa|\alpha|)} |\partial_x^{\beta_1}\partial_\xi^{\alpha_1+\gamma_1}\sigma_j(x,\xi)|\, |\partial_x^{\beta_2}\partial_\xi^{\lambda_1+\gamma_2}(\varphi(x,\xi)-x\cdot\xi)| \dots \\
&\qquad \times |\partial_x^{\beta_{r+1}}\partial_\xi^{\lambda_r+\gamma_{r+1}}(\varphi(x,\xi)-x\cdot\xi)| \\
&\lesssim \sum_{\substack{\gamma_1+\dots+\gamma_{r+1}=\gamma\\ \beta_1+\dots+\beta_{r+1}=\beta}} 2^{-j(\tilde m-\varkappa|\alpha|)} 2^{j(\tilde m+\zeta+\delta|\beta_1|-\rho|\alpha_1+\gamma_1|)} 2^{j(k-1)|\alpha_2|} \\
&\lesssim \sum_{\substack{\gamma_1+\dots+\gamma_{r+1}=\gamma\\ \beta_1+\dots+\beta_{r+1}=\beta}} 2^{-j(\tilde m-\varkappa|\alpha|)} 2^{j(\tilde m+\zeta+\delta|\beta_1|)} 2^{-j\varkappa|\alpha|} \\
&\lesssim 2^{j(\zeta+\delta|\beta|)},
\end{aligned}
$$

hence $b_j^{\alpha_1,\alpha_2,\lambda_1,\dots,\lambda_r} \in S^{\zeta}_{0,\delta}(\mathbb{R}^n)$.

Therefore, Theorem 4.3 yields that

$$
\begin{aligned}
&\|(x-y)^\alpha K_j(x,y)\|_{L^2_x(\mathbb{R}^n)} \\
&\lesssim \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \lambda_1+\dots+\lambda_r=\alpha_2}} 2^{j(\tilde m-\varkappa|\alpha|)} \big\| S_j^{\alpha_1,\alpha_2,\lambda_1,\dots\lambda_r}\big(\tau_y \widehat{\Psi}_j\big)\big\|_{L^2(\mathbb{R}^n)} \\
&\lesssim 2^{j(-|\alpha|\varkappa+\tilde m)} \big\|\widehat{\Psi}_j\big\|_{L^2(\mathbb{R}^n)} \\
&\lesssim 2^{j(-|\alpha|\varkappa+\tilde m+n/2)}.
\end{aligned}
$$

From this and the discussion at the beginning of the proof, one can deduce that

$$
(58)\qquad \|(x-y)^\alpha \partial_y^\beta K_j(x,y)\|_{L^2_x(\mathbb{R}^n)} \lesssim 2^{j(-|\alpha|\varkappa+|\beta|+\tilde m+n/2)}.
$$

Thus by summing over all $|\alpha| \leq M$ for any integer M one obtains

$$
\|(1+2^{j\varkappa}|x-y|)^M \partial_y^\beta K_j(x,y)\|_{L^2_x(\mathbb{R}^n)} \lesssim 2^{j(\tilde m+n/2+|\beta|)}.
$$

For the kernel of the adjoint, we have that

$$
K_j^*(x,y) = \int_{\mathbb{R}^n} e^{-i(\varphi(y,\xi)-x\cdot\xi)}\, \overline{\sigma_j(y,\xi)}\, \text{đ}\xi,
$$

therefore for any multi-index α we have

$$
\begin{aligned}
(x-y)^\alpha \partial_y^\beta K_j^*(x,y) &= (x-y)^\alpha \int_{\mathbb{R}^n} \partial_y^\beta\Big(e^{-i(\varphi(y,\xi)-x\cdot\xi)}\, \overline{\sigma_j(y,\xi)}\Big)\, \text{đ}\xi \\
&= (x-y)^\alpha \int_{\mathbb{R}^n} e^{-i(\varphi(y,\xi)-x\cdot\xi)}\, \sigma_j^{\beta,*}(y,\xi)\, \text{đ}\xi \\
&= \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, i^{|\alpha|}\partial_\xi^\alpha\Big[e^{-i(\varphi(y,\xi)-y\cdot\xi)}\, \sigma_j^{\beta,*}(y,\xi)\Big]\, \text{đ}\xi,
\end{aligned}
$$

where

$$
\sigma_j^{\beta,*}(y,\xi) := \sum_{\substack{\beta_1+\beta_2=\beta\\ \lambda_1+\dots+\lambda_r=\beta_2}} C_{\beta_1,\beta_2,\lambda_1,\dots\lambda_r}\, \partial_y^{\beta_1}\overline{\sigma_j(y,\xi)}\, \partial_y^{\lambda_1}\varphi(y,\xi)\cdots\partial_y^{\lambda_r}\varphi(y,\xi),
$$

and $|\lambda_j| \geq 1$. Now, for $|\lambda_j + \beta| \geq 2$,

$$|\partial_y^{\lambda_j}\partial_\xi^\beta \varphi(y,\xi)| \lesssim 1,$$

and using that $\varphi(y,0) = 0$, the mean-value theorem and L^2-condition (3), we obtain

$$|\nabla_y \varphi(y,\xi)| \lesssim |\xi|.$$

From these estimates, we deduce that for any multi-index γ one has $|\partial_y^\alpha \partial_\xi^\gamma \sigma_j^{\beta,*}(y,\xi)| \lesssim 2^{j(m+|\beta|-\rho|\gamma|+\delta|\alpha|))}$. Therefore, following the same line of reasoning as in the case of $K_j(x,y)$, we obtain the estimate given in (55) for $K_j^*(x,y)$. ☐

Now we are ready to show the main result of this section.

Theorem 5.2. *Let $n \geq 1$, $0 < k < \infty$, and $0 < p < \infty$. Assume that $\varphi \in \text{ᚠ}^k$ is* SND, *satisfies the* LF(μ)*-condition for some $0 < \mu \leq 1$, and the L^2-condition* (3)*. Assume also that $a(x,\xi) \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$, for $0 \leq \rho \leq 1$ and $0 \leq \delta < 1$. Then the* OIO *T_a^φ is bounded from*

(i) $h^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, and

(ii) $L^\infty(\mathbb{R}^n) \to \mathrm{bmo}(\mathbb{R}^n)$ provided that one also has $|\nabla \varphi_x(x,0)| \in L^\infty(\mathbb{R}^n)$.

Proof. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be supported in $\{\xi : |\xi| \lesssim 1\}$, and write

$$\begin{aligned} T_a^\varphi f(x) &:= \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)\widehat{f}(\xi)(1-\chi(\xi))\,\text{đ}\xi + \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)\widehat{f}(\xi)\chi(\xi)\,\text{đ}\xi \\ &= T_{\text{high}}f(x) + T_{\text{low}}f(x). \end{aligned}$$

The boundedness of T_{low} follows from the low frequency result Theorem 2.21. Thus for the remainder of the proof we only consider the high frequency portion of the operator.

Let T_j^ν be the operator associated to the kernel in Definition 3.1, so that

$$T_{\text{high}} = \sum_{j=1}^{\infty}\sum_{\nu=1}^{\mathscr{N}_j} T_j^\nu.$$

Case when $\boldsymbol{0 < p < 1}$:

First we consider the case when $0 < p < 1$. Let $\mathfrak{a}$ be a p-atom supported in a cube Q with side length l_Q and let $2Q$ be the cube with the same center and twice the side length. Since $0 < p < 1$ we have

$$\|T_{\text{high}}\mathfrak{a}\|^p_{L^p(\mathbb{R}^n)} \lesssim \|T_{\text{high}}\mathfrak{a}\|^p_{L^p(2Q)} + \sum_{j=1}^{\infty} \|T_j\mathfrak{a}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \tag{59}$$

Observe that by Hölder's inequality and the L^2-boundedness we have,

$$\begin{aligned} \|T_{\text{high}}\mathfrak{a}\|^p_{L^p(2Q)} &\lesssim \|T_{\text{high}}\mathfrak{a}\|^2_{L^2(2Q)}\|1\|^p_{L^{2p/(2-p)}(2Q)} \lesssim \|\mathfrak{a}\|^2_{L^2(2Q)} l_Q^{n(2-p)/2p} \\ &\lesssim l_Q^{-n(2-p)/2p} l_Q^{n(2-p)/2p} \lesssim 1 \end{aligned} \tag{60}$$

By Lemma 5.1 we have for $0 < k < 1$ that

$$\|(1+2^{j\varkappa}|x-y|)^M K_j(x,y)\|_{L^2_x(\mathbb{R}^n)} \lesssim 2^{j(m_\varkappa(p)+n/2)},$$

Observe that for $t \in [0,1]$ and $x \in \mathbb{R}^n \setminus 2Q$, one has

$$|x-\bar{y}| \lesssim |x-y+t(y-\bar{y})| \tag{61}$$

Now, setting

$$g(x) = \langle 2^{j\varkappa}|x-\bar{y}|\rangle^{-M}$$

Observe that we have for $\tau \geq 1$,

$$\|g\|_{L^\tau(\mathbb{R}^n)} \lesssim 2^{-nj\varkappa/\tau}. \tag{62}$$

By Hölder's inequality and Lemma 5.1 and (62) we have for $l_Q > 1$ and $\tau \geq 1$ that

$$\begin{aligned} \|T_j\mathfrak{a}\|_{L^p(\mathbb{R}^n\setminus 2Q)} &\lesssim \Big\|\frac{1}{g(x)}\int_Q K_j(x,y)\mathfrak{a}(y)\,\mathrm{d}y\Big\|_{L^2(\mathbb{R}^n)}\|g\|_{L^\tau(\mathbb{R}^n)} \\ &\lesssim \int_Q \Big\|\frac{1}{g(x)}K_j(x,y)\Big\|_{L^2(\mathbb{R}^n)}|\mathfrak{a}(y)|\,\mathrm{d}y\,\|g\|_{L^\tau(\mathbb{R}^n)} \\ &\lesssim 2^{-nj\varkappa/\tau}\int_Q \Big\|\frac{1}{g(x)}K_j(x,y)\Big\|_{L^2(\mathbb{R}^n)}|\mathfrak{a}(y)|\,\mathrm{d}y \\ &\lesssim l_Q^{n-n/p}2^{j(m_\varkappa(p)+n/2-n\varkappa(\frac{1}{p}-\frac{1}{2}))} \lesssim l_Q^{n-n/p}2^{j(n-n/p)} \end{aligned} \tag{63}$$

where $\frac{1}{\tau}=\frac{1}{p}-\frac{1}{2}$.

Now, if $l_Q < 1$, Taylor expansion of K in the y-variable around $\bar{y}$, using the moment conditions of $\mathfrak{a}$ yields for all $t \in [0,1]$ and $N := [n(1/p-1)]$ yield that

$$\|T_j\mathfrak{a}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim \sum_{|\alpha|=N+1}\int_{\mathbb{R}^n\setminus 2Q}\Big(\int_Q |\partial_y^\alpha K_j(x,\bar{y}+t(y-\bar{y}))||y-\bar{y}|^{N+1}|\mathfrak{a}(y)|\,\mathrm{d}y\Big)^p\,\mathrm{d}x.$$

Moreover, by using that $|y-\bar{y}|^{N+1} \lesssim r^{N+1}$ with (61) and Lemma 5.1 we obtain the following estimate by a similar calculation as in (63),

$$\|T_j\mathfrak{a}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim 2^{jm_\varkappa(p)p+jnp/2-jnp\varkappa(1/p-1/2)}2^{jp(N+1)}l_Q^{np-n+p(N+1)}. \tag{64}$$

Now, since $l_Q < 1$, take the unique integer $k_Q \in \mathbb{Z}_{\geq 1}$ such that $2^{-k_Q} < l_Q \leq 2^{-(k_Q-1)}$. Then using (63) and (64) we have

$$\begin{aligned} &\sum_{j=1}^{\infty}\|T_j\mathfrak{a}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \\ &\quad\lesssim \sum_{j=k_Q}^{\infty} 2^{jp(m_\varkappa(p)+n/2+n\varkappa(\frac{1}{p}-\frac{1}{2}))}l_Q^{np-n} \\ &\qquad + \sum_{j=1}^{k_Q-1} 2^{jm_\varkappa(p)p+np/2+jnp\varkappa(1/p-1/2)}2^{jp(N+1)}l_Q^{np-n+p(N+1)} \\ &\quad\lesssim \sum_{j=k_Q}^{\infty} 2^{j(np-n)}l_Q^{np-n} + \sum_{j=1}^{k_Q-1} 2^{j(np-n+p(N+1))}l_Q^{np-n+p(N+1)} \\ &\quad\lesssim 2^{k_Q(np-n)}l_Q^{np-n} + 2^{k_Q(np-n+p(N+1))}l_Q^{np-n+p(N+1)} \\ &\quad\lesssim l_Q^{-(np-n)}l_Q^{np-n} + l_Q^{-(np-n+p(N+1))}l_Q^{np-n+p(N+1)} \\ &\quad\sim 1. \end{aligned}$$

Now, if $l_Q \geq 1$ we do the same calculation as above but with $k_Q = 0$, and do not consider the case $j < k_Q$. Thus we conclude that

$$\|T_{\mathrm{high}}\mathfrak{a}\|^p_{L^p(\mathbb{R}^n)} \lesssim \|T_{\mathrm{high}}\mathfrak{a}\|^p_{L^p(2Q)} + \sum_{j=1}^{\infty} \|T_j\mathfrak{a}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim 1. \tag{65}$$

Interpolation and arguments for the adjoint operator:

Now, interpolating the result above with the L^2-boundedness of Theorem 4.3 for $0 \leq \rho \leq 1$ and $0 \leq \delta < 1$ using Riesz-Thorin's interpolation theorem, one obtains the $h^p - L^p$–boundedness of T^φ_a (with the decay $m(p)$), in the range $0 < p \leq 2$.

Next we observe that one may write the phase as $\varphi(x,\xi) = \varphi(x,\xi) - \varphi(x,0) + \varphi(x,0) =: \psi(x,\xi) + \varphi(x,0)$. This would certainly yield that

$$\|T^\varphi_a f\|_{L^p(\mathbb{R}^n)} = \|T^\psi_a f\|_{L^p(\mathbb{R}^n)},$$

for $2 \leq p < \infty$. Moreover observe that $\psi(x,0) = 0$, and therefore we can without loss of generality assume that $\varphi(x,0) = 0$ in T^φ_a. Now in order to prove the $h^p - L^p$–boundedness of T^φ_a for $2 \leq p < \infty$, using duality and interpolation, it is enough to show that the adjoint operator $(T^\varphi_a)^*$ is bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $0 < p \leq 2$. However, this can be shown by the same argument as in the proof of the $h^p - L^p$–boundedness of (T^φ_a), replacing the L^2-inequality for the kernel with the corresponding inequality for the kernel of the adjoint (given in Lemma 5.1).

Now for the boundedness of T^φ_a from $L^\infty(\mathbb{R}^n)$ to $\mathrm{bmo}(\mathbb{R}^n)$ one can write $T^\varphi_a = e^{i\varphi(x,0)}T^\psi_a$ with $\psi(x,0) = 0$. Then given the assumption that $\varphi \in \mathfrak{F}^k$, that φ satisfies the L^2-condition of Definition 2.6, and the extra assumption $|\nabla_x\varphi(x,0)| \in L^\infty(\mathbb{R}^n)$ on the phase function, one can use (11) to reduce matters to the boundedness of T^ψ_a. But the boundedness of T^ψ_a from $L^\infty(\mathbb{R}^n)$ to $\mathrm{bmo}(\mathbb{R}^n)$ is a consequence of the boundedness of $(T^\psi_a)^*$ from $h^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ which was achieved above, and therefore the proof of the theorem is concluded.

□

6. TRANSFERENCE TO TRIEBEL-LIZORKIN SPACES FOR $0 < p < 1$ AND $q \geq p$

This section is devoted to one-half of the process of going from $h^p \to L^p$ to Triebel-Lizorkin boundedness. In particular, we state and prove the global boundedness of oscillatory integral operators with classical and exotic amplitudes on Triebel-Lizorkin spaces $F^s_{p,q}$ for $0 < p < 2$ and $q \geq p$. To this end, we define a molecular representation of the Triebel-Lizorkin spaces. Similar to the atomic representation of the Hardy spaces, this can be used to prove boundedness results for OIOs.

Before we show the main results of this section, we recall a number of useful lemmas from [15], and since the proofs are very similar or exactly the same as those in [15], we leave them out. In particular, this goes for Lemma 6.3 all the way through Lemma 6.7.

Definition 6.1 (Notation). *Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbb{R}^n$ and define the following sets:*

(*i*) $\mathcal{D}_j := \{Q \in \mathcal{D} : k_Q = j\}$
(*ii*) $\mathcal{D}_+ := \{Q \in \mathcal{D} : k_Q \geq 0\}$

(iii) $\mathcal{D}_j(Q) := \{\tilde{Q} \in \mathcal{D}_j : \tilde{Q} \subseteq Q\}$.

Observe that for $j < k_Q$, $D_j(Q) = \emptyset$.

We start with defining a space of sequences that is easier to handle than the Triebel-Lizorkin spaces themselves.

Definition 6.2. *For a sequence of complex numbers* $b = \{b_Q\}_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}}$ *we define*

$$g^{s,q}(b)(x) := \Big(\sum_{\tilde{Q}\in\mathcal{D}_+} \big(2^{k_{\tilde{Q}}(s+n/2)}|b_{\tilde{Q}}|\chi_{\tilde{Q}}(x)\big)^q\Big)^{1/q}.$$

We say that $b \in f^s_{p,q}$ *is*

$$\|b\|_{f^s_{p,q}} := \|g^{s,q}(b)\|_{L^p(\mathbb{R}^n)} < \infty.$$

In what follows, take $\check{\Psi}^{\tilde{Q}}(x) := 2^{-nk_{\tilde{Q}}/2}\check{\Psi}_{k_{\tilde{Q}}}(x - c_{\tilde{Q}})$, where $\Psi_{k_{\tilde{Q}}}$ as given in Definition 2.13 is a Littlewood-Paley cut-off. The following Lemma is a corollary of [9, Theorem II B].

Lemma 6.3. *Suppose* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. *For any sequence* $b = \{b_{\tilde{Q}}\}_{\tilde{Q}\in\mathcal{D}}$ *of complex numbers satisfying* $\|b\|_{f^s_{p,q}} < \infty$, *one has*

$$f(x) := \sum_{\tilde{Q}\in\mathcal{D}_+} b_{\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x)$$

belongs to $F^s_{p,q}(\mathbb{R}^n)$ *and*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \lesssim \|b\|_{f^s_{p,q}}.$$

Proof. This follows by combining [9, Theorem II A i] and [12, Theorem 1]. It is worth noting that this result can also be applied to inhomogeneous Triebel-Lizorkin spaces, as discussed in Chapter 12 of [10]. □

We now discuss the converse of Lemma 6.3, namely when a Triebel-Lizorkin function can be expressed in terms of molecules and so-called "∞-atoms", here denoted $b_{\iota,\tilde{Q}}$.

Lemma 6.4 ([15])**.** *Suppose* $0 < p \leq 1$, $p \leq q \leq \infty$. *Every* $f \in F^0_{p,q}(\mathbb{R}^n)$ *has an atomic decomposition*

$$f(x) = \sum_{\iota=1}^{\infty} \lambda_\iota \sum_{\tilde{Q}\in\mathcal{D}_+} b_{\iota,\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x), \qquad b_{\iota,\tilde{Q}} \in f^0_{\infty,q},$$

where $\{b_{\iota,\tilde{Q}}\}_{\tilde{Q}\in\mathcal{D}_+} =: b_\iota$ *satisfies*

$$\|b_\iota\|_{f^0_{\infty,q}} \leq 2^{k_{\tilde{Q}}n/p}.$$

Moreover

$$\|f\|_{F^0_{p,q}(\mathbb{R}^n)} \sim \inf_{\{\lambda_\iota\}} \Big(\sum_{\iota=1}^{\infty} |\lambda_\iota|^p\Big)^{1/p}$$

Next, we define the analog of the Hardy space atoms, which will be used to prove the boundedness results of our OIO's.

Lemma 6.5 ([15])**.** *Let* $Q \in \mathcal{D}$, $0 < p \leq 1$, $0 < q \leq \infty$, $j \geq 1$ *and* $b = \big\{b_{\tilde{Q}}\big\}_{\tilde{Q}\in\mathcal{D}_+} \in f^0_{\infty,q}$ *with* $\|b\|_{f^0_{\infty,q}} \leq 2^{k_Q n/p}$. *Define*

$$R_{Q,j}(x) := \sum_{\tilde{Q}\in\mathcal{D}_j(Q)} b_{\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x). \tag{66}$$

Then $R_{Q,j}(x) = 0$ *for* $j < k_{\tilde{Q}}$. *Moreover, for* $0 < p < \infty$,

$$\|R_{Q,j}\|_{L^2(\mathbb{R}^n)} \lesssim 2^{k_Q n(1/p-1/2)}.$$

Now we start with the lifting results for OIO's to Triebel-Lizorkin spaces. In order to do that, we need to introduce a partition of unity to $R_{Q,j}$. We estimate the pieces separately. The first estimate is given in Lemma 6.6.

Lemma 6.6 ([15])**.** *Suppose that* $0 < p < 1$ *and* $q \geq p$. *Let* $a(x,\xi) \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$ *be supported outside a neighborhood of the origin* (*in the* ξ *variable*) *and assume that* $\varphi \in \mathfrak{F}^k$ *with* $k > 0$ *is* SND, *satisfies the* LF(μ)*-condition for some* $0 < \mu \leq 1$, *and satisfies the* L^2*-condition* (3). *Then*

$$\|T^\varphi_a\psi_j(D)\chi_{\mathbb{R}^n\setminus 2\sqrt{n}Q}R_{Q,j}\|_{L^p(\mathbb{R}^n)} \lesssim 2^{n(k_Q-j)}. \tag{67}$$

for $j \geq k_Q$.

Lemma 6.7 ([15])**.** *Let* $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ *and that* T^φ_a *is an oscillatory integral operator that is bounded from* $F^s_{p,p}(\mathbb{R}^n)$ *to* $F^s_{p,p}(\mathbb{R}^n)$. *Assume also that the phase function* φ *satisfies the conditions of* Theorem 2.12. *Then* T^φ_σ *is bounded from* $F^s_{p,q}(\mathbb{R}^n)$ *to* $F^s_{p,q}(\mathbb{R}^n)$ *for* $\sigma(x,\xi) \in S^{m-\varepsilon}_{\rho,\delta}(\mathbb{R}^n)$ *where* $\varepsilon > 0$ *is arbitrary.*

Proof. This lemma is proven in a similar manner as [15, Lemma 5.8], with some minor modifications. One replaces the $h^p \to L^p$ boundedness result in that paper (Proposition 5.1) with Theorem 5.2 in this paper, apart from this abstract modification, the proof remains the same and therefore one only substitutes the hypothesis on T^φ_a in [15, Lemma 5.8], by the hypothesis necessary for the $h^p \to L^p$ boundedness of T^φ_a in Theorem 5.2. ☐

Now we are prepared to show the main lifting results of this section.

Lemma 6.8. *Let* T^φ_a *be an* OIO *with an amplitude* $a \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$ *for* $\rho \in [0,1]$, $\delta \in [0,1)$. *Assume* $\varphi \in \mathfrak{F}^k$ *with* $k > 0$ *is* SND *and satisfies the* L^2*-condition* (3), *and set* $T_j := T^\varphi_a\psi_j(D)$, *where* T^φ_a *is the* OIO *associated to* a *and* φ, *and* $\psi_j(D)$ *is a Littlewood-Paley operator as in* Definition 2.13. *Furthermore, suppose that* f *is supported in a cube* Q *with* $\|f\|_{L^1(\mathbb{R}^n)} \lesssim 2^{k_Q n(1/p-1)}$. *Then the following statements hold true*

(*i*) *If* $0 < p < 1$, *then*

$$\sum_{j=\max\{k_Q+1,1\}}^{\infty} \|T_j f\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim 1. \tag{68}$$

(*ii*) *If* $0 < p \leq 1$ *and* f *is an* h^p*-atom, see* Definition 2.20, *then*

$$\sum_{j=1}^{\max\{k_Q,0\}} \|T_j f\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim 1. \tag{69}$$

Moreover, the same estimates hold true for the adjoint operator $(T_j)^*$.

Proof. (i) Using (63) we have

$$\begin{aligned}\sum_{j=\max\{k_Q+1,1\}}^{\infty} \|T_j f\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} &\lesssim \sum_{j=\max\{k_Q+1,1\}}^{\infty} 2^{j(np-n)} l_Q^{np-n}\\ &\lesssim 2^{k_Q(np-n)} l_Q^{np-n}\\ &\lesssim l_Q^{-(np-n)} l_Q^{np-n}\\ &\sim 1.\end{aligned}$$

(ii) Observe that for $l_Q > 1$, $k_Q \leq 0$ and in this case the statement is trivially true. Hence we assume that $l_Q \leq 1$.
Now, using (64) we have

$$\begin{aligned}&\sum_{j=1}^{\max\{k_Q,0\}} \|T_j f\|^p_{L^p(\mathbb{R}^n\setminus 2Q)}\\ &\quad\lesssim \sum_{j=1}^{\max\{k_Q,0\}} 2^{jm_\varkappa(p)p+np/2+jnp\varkappa(1/p-1/2)} 2^{jp(N+1)} l_Q^{np-n+p(N+1)}\\ &\quad\lesssim \sum_{j=1}^{\max\{k_Q,0\}} 2^{j(np-n+p(N+1))} l_Q^{np-n+p(N+1)}\\ &\quad\lesssim 2^{k_Q(np-n+p(N+1))} l_Q^{np-n+p(N+1)}\\ &\quad\lesssim l_Q^{-(np-n+p(N+1))} l_Q^{np-n+p(N+1)}\\ &\quad\sim 1.\end{aligned}$$

We observe that the same estimates are also valid for the adjoint T_j^*, because (63) and (64) are valid for the adjoint. $\square$

Proposition 6.9. *Suppose that* $0 < p < 1$, $q \geq p$ *and* $k > 0$. *Let* $a(x,\xi) \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$ *be supported outside a neighborhood of the origin and assume that* $\varphi \in \digamma^k$ *is* SND, *and satisfies the* L^2*-condition* (3). *Then the* OIO T_a^φ *is bounded from* $F^s_{p,q}(\mathbb{R}^n)$ *to* $F^s_{p,q}(\mathbb{R}^n)$.

Proof. Observe that it is enough to show the result for $s = 0$ and by the inclusion $F^0_{p,p} \hookrightarrow F^0_{p,q}$ it is enough to show that T_a^φ is bounded from $F^0_{p,q}(\mathbb{R}^n)$ to $F^0_{p,p}(\mathbb{R}^n)$.

To this end, we compose a Littlewood-Paley operator $\psi_j(D)$ with T_a^φ and apply Theorem 2.12. This yields for any $M \geq 1$ and $0 < \varepsilon < 1 - \max(\delta, 1/2)$ one has

$$\psi_j(D)\, T_\sigma^\varphi = \sum_{0\leq|\alpha|\leq M-1} \frac{2^{-j\varepsilon|\alpha|}}{\alpha!} T^\varphi_{\sigma_{\alpha,j}} + 2^{-j\varepsilon M} T^\varphi_{r_j} \tag{70}$$

where

$$\begin{aligned}\left|\partial_\xi^\gamma \partial_x^\beta \sigma_{\alpha,j}(x,\xi)\right| &\lesssim \langle\xi\rangle^{m(p)-(1-\max(\delta,1/2)-\varepsilon)|\alpha|-\rho|\beta|+\delta|\gamma|}\\ \left|\partial_\xi^\beta \partial_x^\gamma r_j(t,x,\xi)\right| &\lesssim \langle\xi\rangle^{m(p)-(1-\max(\delta,1/2)-\varepsilon)M-\rho|\beta|+\delta|\gamma|}.\end{aligned} \tag{71}$$

Now using Lemma 6.7 one can use the very same reasoning as in Step 2 of Proposition 5.4 in [16] to show that

$$\left\|T^\varphi_{r_j} f\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F^{-(1-\max(\delta,1/2)-\varepsilon)}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{F^0_{p,q}(\mathbb{R}^n)}.$$

Therefore from now on, we will only consider the $F^0_{p,q} \to L^p(\ell^q)$-boundedness of the first term $T_{\sigma_{0,j}} = T^\varphi_a \psi_j(\nabla_x \varphi(x,\xi))$, whose amplitude is equal to $a(x,\xi)\psi_j(\nabla_x\varphi(x,\xi))$. The reason for this is that other terms with $|\alpha| > 0$ in the sum in (70) have an amplitude with a better decay than $T_{\sigma_{0,j}}$, and so considering this case is the worst case scenario.

Let $\mathbf{t}(\xi) = \nabla_x\varphi(x,\xi)$ and $\eta : \mathbb{R}^n \to \mathbb{R}^n$ be diffeomorphisms such that $\eta(y)\cdot\mathbf{t}^{-1}(\xi) = y\cdot\xi$. Then we have

$$\begin{aligned}
&T^\varphi_a \psi_j(\nabla_x\varphi(x,D))f(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\varphi(x,\xi)-iy\cdot\xi}\psi_j(\mathbf{t}(\xi))a(x,\xi)f(y)\,\mathrm{d}y\,\text{đ}\xi\\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{|\det(\nabla t)|} e^{i\varphi(x,\mathbf{t}^{-1}(\xi))-iy\cdot t^{-1}(\xi)}\psi_j(\xi)a(x,\mathbf{t}^{-1}(\xi))f(y)\,\mathrm{d}y\,\text{đ}\xi\\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\det(\nabla\eta)|}{|\det(\nabla t)|} e^{i\varphi(x,\mathbf{t}^{-1}(\xi))-iy\cdot\xi}\psi_j(\xi)a(x,\mathbf{t}^{-1}(\xi))f(\eta(y))\,\mathrm{d}y\,\text{đ}\xi\\
&=: T_j(f\circ\eta)(x).
\end{aligned}$$

Thus in light of what we mentioned earlier, by Theorem 2.18 it is enough to analyse $T_j f$.

By Lemma 6.4, $f \in F^0_{p,q}(\mathbb{R}^n)$ has an atomic decomposition

$$f(x) = \sum_{\iota=1}^\infty \lambda_\iota \sum_{\tilde{Q}\in\mathcal{D}_+} b_{\iota,\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x), \qquad b_{\iota,\tilde{Q}} \in f^0_{p,q}\cap f^0_{\infty,q},$$

such that

$$\|f\|_{F^0_{p,q}(\mathbb{R}^n)} \sim \inf_{\{\lambda_\iota\}} \Big(\sum_{\iota=1}^\infty |\lambda_\iota|^p\Big)^{1/p}.$$

Since $\operatorname{supp}\psi_j \subset \operatorname{supp}\Psi^{\tilde{Q}}$ it is enough to consider $\tilde{Q}\in\mathcal{D}_j$ and hence

$$\begin{aligned}
\Big\|\Big(\sum_{j=1}^\infty |T_j f|^p\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^n)} &= \Big\|\Big(\sum_{j=1}^\infty\Big|\sum_{\iota=1}^\infty \lambda_\iota T_j \sum_{\tilde{Q}\in\mathcal{D}_j} b_{\iota,\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x)\Big|^p\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^n)}\\
&\lesssim \Big(\sum_{\iota=1}^\infty |\lambda_\iota|^p \int_{\mathbb{R}^n}\sum_{j=1}^\infty \Big|T_j\sum_{Q\in\mathcal{D}_j} b_{\iota,\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x)\Big|^p\,\mathrm{d}x\Big)^{1/p}\\
&\lesssim \Big(\sum_{\iota=1}^\infty |\lambda_\iota|^p\Big)^{1/p} \sup_{\iota\in\mathbb{Z}_{>0}}\Big(\sum_{j=1}^\infty \Big\|T_j\sum_{\tilde{Q}\in\mathcal{D}_j} b_{\iota,\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x)\Big\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p}
\end{aligned}$$

Therefore it is enough to show that one has an expression of the form

$$\sum_{j=1}^\infty \|T_j R_{Q,j}\|^p_{L^p(\mathbb{R}^n)} \lesssim 1 \tag{72}$$

uniformly in $Q\in\mathcal{D}$, where

$$R_{Q,j}(x) := \sum_{\tilde{Q}\in\mathcal{D}_j(Q)} b_{\tilde{Q}}\check{\Psi}^{\tilde{Q}}(x).$$

(Recall from Lemma 6.5 that $R_{Q,j}(x) = 0$ for $j < k_{\tilde{Q}}$.)

To show (72) we need to show the following three estimates for $0 < k < \infty$:

$$\sum_{j=\max\{1,k_Q\}}^{\infty} \|T_j R_{Q,j}\|^p_{L^p(2Q)} \lesssim 1, \tag{73}$$

$$\sum_{j=\max\{1,k_Q\}}^{\infty} \|T_j \chi_{2\sqrt{n}Q} R_{Q,j}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim 1, \tag{74}$$

$$\sum_{j=\max\{1,k_Q\}}^{\infty} \|T_j \chi_{\mathbb{R}^n\setminus 2\sqrt{n}Q} R_{Q,j}\|^p_{L^p(\mathbb{R}^n\setminus 2Q)} \lesssim 1, \tag{75}$$

Step 1 – Proof of (73)

To see (73) consider

$$\begin{aligned}
\|T_j R_{Q,j}\|^p_{L^p(2Q)} &\leq l_Q^{n(2-p)/2p} 2^{2jm_\varkappa(p)} \|2^{-jm_\varkappa(p)} T_j R_{Q,j}\|^2_{L^2} \\
&\lesssim l_Q^{n(2-p)/2p} 2^{2jm_\varkappa(p)} \|R_{Q,j}\|^2_{L^2} \\
&\lesssim l_Q^{n(2-p)/2p} 2^{2k_Q n(1/p-1/2)} 2^{2jm_\varkappa(p)} \\
&\lesssim 2^{-k_Q n(2-p)/2p} 2^{2k_Q n(1/p-1/2)} 2^{2jm_\varkappa(p)} \\
&= 2^{2jm_\varkappa(p)}
\end{aligned}$$

for all $0 < p < 1$, thus we obtain (73) by summing in j, since $m_\varkappa(p) < 0$.

Step 3 – Proof of (74), (75)

(74) follows directly from using Lemma 6.8 and Lemma 6.5. (75) follows immediately from Lemma 6.6. This concludes the proof. ☐

7. Transference to Triebel-Lizorkin spaces for $p > 2$

Definition 7.1. *In accordance to* Theorem 2.23, *let Q be a cube and set*

$$\mathcal{E} := 2Q.$$

Lemma 7.2. *Let $n \geq 1$, $0 < p < 1$, $0 \leq \rho \leq 1$, $0 < k < \infty$ and $0 \leq \delta < 1$. Assume that $\varphi \in \mathfrak{F}^k$ is* SND *and satisfies the L^2-condition* (3). *Let $a(x,\xi) \in S^{m_\varkappa(p)}_{\rho,\delta}(\mathbb{R}^n)$, and*

$$K_j(x,y) = \int_{\mathbb{R}^n} e^{i(\varphi(x,\xi)-y\cdot\xi)}\, \sigma_j(x,\xi)\, \text{đ}\xi,$$

be the kernel of an OIO *T_j. Then for all $\varepsilon > 0$ we have for an h^p-atom $\mathfrak{a}$ supported in Q that*

(*i*) *If $2^{-j} \leq l_Q \leq 1$, then*

$$\int_{\mathbb{R}^n} |T_j^* \Psi_j(D)\mathfrak{a}|\, \mathrm{d}x \lesssim 2^{jm_\varkappa(p)\frac{1}{\frac{2}{p}-1}} l_Q^{-\varepsilon} + l_Q^{\frac{1}{\frac{2}{p}-1}(n-n/p)} 2^{\frac{1}{\frac{2}{p}-1}j(n-n/p)}, \tag{76}$$

(*ii*) *If $l_Q \geq 1$, then*

$$\int_{\mathbb{R}^n} |T_j^* \Psi_j(D)\mathfrak{a}|\, \mathrm{d}x \lesssim \big(2^{jm_\varkappa(p)} + l_Q^{n-n/p} 2^{j(n-n/p)}\big)^{\frac{1}{\frac{2}{p}-1}}. \tag{77}$$

Proof. We begin by proving (76), to this end let w be a real number such that $1/w = 1/p - 1/2$, and split $\mathbb{R}^n = \mathcal{E} \cup (\mathbb{R}^n \setminus \mathcal{E})$. Then

$$\Big(\int_{\mathcal{E}} |T_j^* \Psi_j(D)\mathfrak{a}|^p\, \mathrm{d}x\Big)^{1/p} \lesssim \|1\|_{L^w(\mathcal{E})} \Big(\int_{\mathbb{R}^n} |T_j^* \Psi_j(D)\mathfrak{a}|^2\, \mathrm{d}x\Big)^{1/2}$$

$$\lesssim |Q|^{1/p-1/2} 2^{jm_\varkappa(p)} \Big(\int_{\mathbb{R}^n} |\mathfrak{a}(x)|^2 \,\mathrm{d}x\Big)^{1/2} \lesssim 2^{jm_\varkappa(p)}.$$

Now, recall that

$$g(x) = \big\langle 2^{j\varkappa}|x-\bar{y}|\big\rangle^{-M}$$

and that we have for $w \geq 1$,

$$\|g\|_{L^w(\mathbb{R}^n)} \lesssim 2^{-nj\varkappa/w}. \tag{78}$$

Next Hölder's and Minkowski's inequalities and Lemma 5.1 yield that

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n\setminus\mathcal{E}} |T_j^*\Psi_j(D)\mathfrak{a}|^p \,\mathrm{d}y\Big)^{1/p} \lesssim \Big\|\frac{1}{g(x)}\int_B \overline{K_j(y,x)}\Psi_j(D)\mathfrak{a}(y)\,\mathrm{d}y\Big\|_{L^2(\mathbb{R}^n)} \|g\|_{L^w(\mathbb{R}^n)}\\
&\lesssim \int_B \Big\|\frac{1}{g(x)}\overline{K_j(y,x)}\Big\|_{L^2(\mathbb{R}^n)} |\Psi_j(D)\mathfrak{a}(y)|\,\mathrm{d}y\, \|g\|_{L^w(\mathbb{R}^n)}\\
&\lesssim 2^{-nj\varkappa/w}\int_B \Big\|\frac{1}{g(x)}\overline{K_j(y,x)}\Big\|_{L^2(\mathbb{R}^n)} |\Psi_j(D)\mathfrak{a}(y)|\,\mathrm{d}y\\
&\lesssim l_Q^{n-n/p} 2^{j(m_\varkappa(p)+n/2-n\varkappa(1/p-1/2))}\\
&= l_Q^{n-n/p} 2^{j(n-n/p)}
\end{aligned}$$

where we have also used that $\mathfrak{a}$ is an h^p-atom and that $\Psi_j(D)$ is L^2-bounded uniformly in j.

Interpolation step:

Using the L^2 result (Theorem 4.1) we obtain by Riesz-Thorin interpolation and $\frac{1}{p_t} = \frac{t}{2} + \frac{1-t}{p}$ with $0 < p < 1$ and $0 < k < 1$ that

$$\begin{aligned}
\Big(\int_{\mathbb{R}^n} |T_j^*\Psi_j(D)\mathfrak{a}|^{p_t} \,\mathrm{d}x\Big)^{1/p_t} &\lesssim \big(2^{jm_\varkappa(p)} + l_Q^{n-n/p} 2^{j(n-n/p)}\big)^{(1-t)}\\
&\lesssim 2^{jm_\varkappa(p)(1-t)} + l_Q^{(1-t)(n-n/p)} 2^{(1-t)j(n-n/p)}.
\end{aligned}$$

Taking $p_t = 1$ and $t = \frac{1-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}}$ we have $1-t = \frac{1}{\frac{2}{p}-1}$ and therefore

$$\int_{\mathbb{R}^n} |T_j^*\Psi_j(D)\mathfrak{a}|\,\mathrm{d}x \lesssim \big(2^{jm_\varkappa(p)} + l_Q^{n-n/p} 2^{j(n-n/p)}\big)^{\frac{1}{\frac{2}{p}-1}}.$$

Thus for $l_Q > 1$ we have

$$\int_{\mathbb{R}^n} |T_j^*\Psi_j(D)\mathfrak{a}|\,\mathrm{d}x \lesssim 2^{jm_\varkappa(p)\frac{1}{\frac{2}{p}-1}} + 2^{\frac{1}{\frac{2}{p}-1}j(n-n/p)}$$

and for $l_Q < 1$ and $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^n} |T_j^*\Psi_j(D)\mathfrak{a}|\,\mathrm{d}x \lesssim 2^{jm_\varkappa(p)\frac{1}{\frac{2}{p}-1}} l_Q^{-\varepsilon} + l_Q^{\frac{1}{\frac{2}{p}-1}(n-n/p)} 2^{\frac{1}{\frac{2}{p}-1}j(n-n/p)}.$$

$\square$

Theorem 7.3. *Let $n \geq 1$ and $0 < k < \infty$. Assume that $\varphi \in \digamma^k$ is* SND *and satisfies the L^2-condition* (3)*. Furthermore assume that $a(x,\xi) \in S^{m_\varkappa(p)}_{\rho,\delta}(\mathbb{R}^n)$, and let T_j be* *given as in* Lemma 7.2. *If we let* $p > 2$, $0 < q \leq \infty$, *and* $b > 0$ *and if we also assume that the operators* T_j *satisfy*

$$\sup_{j>0} 2^{jb/p} \|T_j\|_{L^p \to L^p} \lesssim 1, \tag{79}$$

$$\sup_{j>0} 2^{jb/2} \|T_j\|_{L^2 \to L^2} \lesssim 1, \tag{80}$$

then, the following inequality holds:

$$\Big\| \Big(\sum_{j=0}^{\infty} 2^{jbq/p} |\Psi_j(D) T_j f_j|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\sum_{j=0}^{\infty} \|f_j\|^p_{L^p(\mathbb{R}^n)} \Big)^{1/p}.$$

where $\Psi \in \mathscr{S}(\mathbb{R}^n)$, $\Psi_j(D) := \Psi(2^{-j}D)$ *and* f_j *is a sequence of functions.*

Proof. We consider a measurable set $\mathcal{E}$ (as defined in Definition 7.1). We have the following inequality:

$$|\mathcal{E}| \lesssim l_Q^n,$$

which means that all the hypotheses of Theorem (2.23) are satisfied, with $q = 2$. Note that the condition $b < n$ in Theorem (2.23) is not necessary here, because it stems from Lemma 2.2 in [22], which we substitute with our Lemma (7.2) (and duality). We also do not need the assumption (21) about the kernel, because we show Lemma 7.2 using a different argument. Therefore, we can obtain the result by applying the same method as in [22] to the adjoint T_j^*, and then interpolating with $L^2(\mathbb{R}^n)$. □

8. TRIEBEL-LIZORKIN ESTIMATES

In this section we use the lifting results from section 6 and 7 to lift Theorem 5.2 to Triebel-Lizorkin boundedness.

Theorem 8.1. *Let* $0 \leq \rho \leq 1$, $0 \leq \delta < 1$, $k > 0$, $0 < \mu \leq 1$, *and* $0 < q \leq \infty$. *Assume furthermore that* $\varphi \in \Phi^k$ *is* SND, *and satisfies the* LF(μ) *condition* (5) *and the* L^2*-condition* (3). *For* $a \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$ *let* T_a^φ *be the associated* OIO. *If* $s \in \mathbb{R}$ *and either one of the following cases holds*

$$(i) \quad 2 \leq p < \infty \text{ when } 0 < q \leq p,$$

$$(ii) \quad \frac{n}{n+\mu} < p \leq 2 \text{ when } p \leq q,$$

then the OIO T_a^φ *is bounded from* $F^s_{p,q}(\mathbb{R}^n)$ *to* $F^s_{p,q}(\mathbb{R}^n)$

Proof. We decompose the operator into a low and a high-frequency part. The result for the low-frequency part follows from Theorem 2.21, so we only consider the high-frequency part.

Observe that when $p = q = 2$, the claim is a consequence of Theorem 4.3.
We split the proof into different ranges of p and q, the two parts of the proof correspond to the blue and the red regions in **Figure 2**, respectively.

Part 1 – Proof when $p > 2$ and $p \geq q > 0$
We use Theorem 2.12 to write

$$\psi_j(D) T_a^\varphi f(x) = T_a^\varphi \psi_j(\nabla_x \varphi(x, D)) f(x) + Rf(x),$$

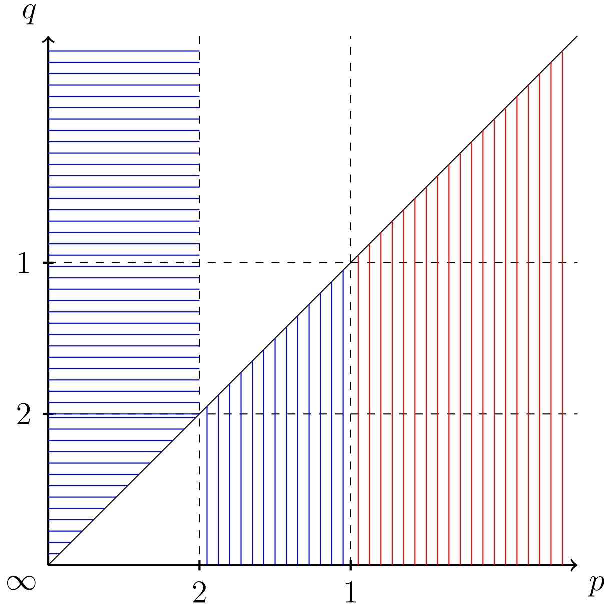

FIGURE 2. Triebel-Lizorkin boundedness for OIO:s with their respective critical decay. The blue horizontal lines illustrate Theorem 7.3. The blue vertical lines illustrate the boundedness results obtained by applying a duality argument for the case $p > 2$. The red vertical lines illustrate Proposition 6.9 together with interpolation with the vertical blue area.

The operator R is bounded by Lemma 6.7 and a reasoning similar to the Proposition (6.9). So from now on we will only consider the first term. Denote $T_j := T_a^\varphi \psi_j(\nabla_x \varphi(x,D))$.

Hence we can use Theorem 7.3 with $b := n(1-\varkappa)$ and $S_j := T_j(1-\Delta)^{-b/2p}$ to prove the desired result. Observe that the $h^p \to L^p$ boundedness (Proposition 5.2) and the L^2 boundedness (Theorem 4.3)of T_a^φ yield (79) and (80) respectively.

Theorem 7.3 immediately yields that

$$\Big\|\Big(\sum_{j=0}^{\infty} 2^{jbq/p}|\Psi_j(D)T_j f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jbq/p}|\Psi_j(D)S_j f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\lesssim \Big(\sum_{j=0}^{\infty} \|f_j\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p}.$$

Thus $S_j : B^0_{p,p} \to F^{b/p}_{p,q}$ which immediately implies that $T_j : B^{b/p}_{p,p} \to F^{b/p}_{p,q}$. Now the assertion follows from the facts that $F^0_{p,q} \hookrightarrow F^0_{p,p} = B^0_{p,p}$ and the calculus using Bessel potentials and Theorem 2.12.

Part 2 – Proof when $1 < p < 2$ and $p \leq q$
Using that the operator is self-adjoint we can also obtain Theorem (7.3) for the adjoint, then apply the arguments from part 1.

Part 3 – Proof when $0 < p < 1$ and $p \leq q$
By Proposition 6.9 one obtains the result for $0 < p < 1$ and $q \geq p$.

Part 3.1 – Proof when $0 < p \leq q < 1$
In this case using Riesz-Thorin interpolation with $F^s_{p,p} = B^s_{p,p}$, yields the result.

Part 3.2 – Proof when $0 < p < 1 \leq q \leq \infty$
Here Riesz-Thorin interpolation with **Part 2** above, yields the result.

Now notice that $(1-\Delta)^{\frac{s}{2}} T^{\varphi}_a (1-\Delta)^{-\frac{s}{2}}$ is a similar operator associated to an amplitude in $S^{m_c(p)}(\mathbb{R}^n)$ and phase φ, and hence bounded from $F^0_{p,q}(\mathbb{R}^n)$ to itself. Therefore using the fact that the operator $(1-\Delta)^{\frac{s}{2}}$ is an isomorphism from $F^s_{p,q}(\mathbb{R}^n)$ to $F^0_{p,q}(\mathbb{R}^n)$ for $0 < p \leq \infty$, we obtain the desired result. □

8.1. Triebel-Lizorkin estimates related to forbidden amplitudes.

It is well known that the oscillatory integral operators with amplitudes in $S^m_{1,1}(\mathbb{R}^n)$ fail to be L^2-bounded, and this serves as a motivation for the designation "forbidden", see e.g. Stein [23] Chapter VII, Section 1. However for pseudodifferential operators E. Stein (unpublished) and independently by Y. Meyer [20], [21] showed the Sobolev-boundedness for $H^s(\mathbb{R}^n)$ for $s > 0$. This result had a redeeming effect on the reputation of the pseudodifferential operators with forbidden symbols. But the main fact that gave the forbidden symbols their current important status is their important role in J. M. Bony's paradifferential calculus [1], which is an efficient tool in the analysis of nonlinear partial differential equations. Regarding psedodifferential operators with symbols in $S^m_{1,1}(\mathbb{R}^n)$, a refined and sophisticated analysis was made by J. Johnsen in [17], which went far beyond other known results for these operators. Indeed the boundedness of pseudodifferential operators with forbidden symbols in all classical function spaces (such as Besov-Lipschitz and Triebel-Lizorkin spaces) can be deduced from the results of Johnsen in the aforementioned paper. In the analysis that is carried out in [17], great care has been taken to make sense of operators with forbidden symbols when they act on spaces with $p = \infty$. This case is problematic because Schwartz space is no longer dense in the spaces at hand and it behoves one to give a rigorous definition for the action of the operator on a tempered distribution. This difficulty was overcome by Johnsen [17] as follows:
Let ϕ be a smooth function so that $\widehat{\phi}$ is supported in $\{\xi : 2^{-1} < |\xi| < 2\}$ and

$$\sum_{k\in\mathbb{Z}} \widehat{\phi_k}(\xi) = 1$$

for $\xi \neq 0$ where $\phi_k := 2^{kd}\phi\left(2^k\cdot\right)$. For $j,\, k \geq 0$, set $a_{j,k}(x,\xi) := \phi_j * a(\cdot,\xi)(x)\widehat{\phi_k}(\xi)$ Then the action of the pseudodifferential operator $a(x,D)$ with symbol $a(x,\xi) \in S^m_{1,1}(\mathbb{R}^n)$ is given by

$$a(x,D)f := \lim_{N\to\infty} \sum_{k=0}^{N}\sum_{j=0}^{N} a_{j,k}(x,D)f, \quad f \in \mathscr{S}',$$

whenever the limit exists in $\mathscr{S}'$.

In what follows, our main focus is on oscillatory integral operators that are outside the scope of pseudodifferential operators, and in the results that we are proving, we will be avoiding the case $p = \infty$. In this connection, the boundedness of Fourier integral operators in Besov-Lipschitz and Triebel-Lizorkin spaces with amplitudes in forbidden classes was established in [3]. In this section, we state and prove two results about the boundedness of oscillatory integral operators in Triebel-Lizorkin spaces with amplitudes in $S^m_{1,1}(\mathbb{R}^n)$.

Now we turn to the boundedness of OIOs with forbidden amplitudes in the class $S^m_{1,1}(\mathbb{R}^n)$ and ask whether they are bounded on Triebel-Lizorkin spaces.

Theorem 8.2. *Let $n \geq 1$, $k > 0$, $s > 0$, $0 < \mu \leq 1$. Assume that $a(x,\xi) \in S^{-kn|1/p-1/2|}_{1,1}(\mathbb{R}^n)$, $\varphi \in \mathfrak{F}^k$ is* SND*, satisfies the* LF(μ)*-condition and the conditions in* (9)*, and the L^2-condition* (3)*. If $s > n\big(\frac{1}{\min\{1,p,q\}} - 1\big)$, and either one of the following cases hold*

- *(i) $2 \leq p < \infty$ when $0 < q \leq p$,*
- *(ii) $\frac{n}{n+\mu} < p \leq 2$ when $p \leq q$,*

or if $s > \frac{n}{p}$ with $q = \infty$, then the OIO *T^φ_a is bounded from $F^s_{p,q}(\mathbb{R}^n) \to F^s_{p,q}(\mathbb{R}^n)$.*

Proof. The proof is similar to the proof of Theorem 5.13 in [15], and therefore we shall only highlight the differences here. The only differences appear in connection to (100). At this point in the proof one instead uses Theorem 8.1 instead of Theorem 5.15. The rest of the proof remains the same. ☐

9. Besov-Lipschitz estimates

In this section we include both the Besov-Lipschitz boundedness results of OIO's with amplitudes in $S^m_{\rho,\delta}$ for all $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$. We begin with the classical amplitudes were $\delta < 1$.

Theorem 9.1. *Let $0 \leq \rho \leq 1$ and $0 \leq \delta < 1$, $k > 0$, $0 < \mu \leq 1$, $0 < q \leq \infty$. Assume that $\varphi \in \mathfrak{F}^k$ is* SND*, satisfies the* LF(μ)*-condition and the conditions in* (9)*, and the L^2-condition* (3)*. Assume also that $a \in S^{m(p)}_{\rho,\delta}(\mathbb{R}^n)$, and let T^φ_a be the associated* OIO*. Then T^φ_a is a bounded operator from $B^s_{p,q}(\mathbb{R}^n)$ to $B^s_{p,q}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ for $\frac{n}{n+\mu} < p < \infty$.*

Proof. Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ be supported in $\{\xi : |\xi| \lesssim 1\}$, and write

$$\begin{aligned} T^\varphi_a f(x) &:= \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{f}(\xi)(1-\chi(\xi))\, đ\xi + \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{f}(\xi)\chi(\xi)\, đ\xi \\ &= T_{\mathrm{high}} f(x) + T_{\mathrm{low}} f(x). \end{aligned}$$

The boundedness of T_{low} follows from the low frequency result Theorem 2.21. The boundedness of T_{high} follows immediately by the Besov-Lipschitz lift Theorem 2.22 and Proposition 5.2. ☐

In the next theorem, we obtain an estimate for OIOs with amplitudes in $L^\infty S^m_0(\mathbb{R}^n)$ (see Definition 2.2). Observe that this class of amplitudes contains the Hörmander classes $S^m_{0,1}(\mathbb{R}^n)$ and $S^m_{1,1}(\mathbb{R}^n)$.

Theorem 9.2. *Let $n \geq 1$, $s \in \mathbb{R}$, $1 < p < \infty$, $0 < q \leq \infty$. Assume that $\varphi(x,\xi) \in \text{ᚠ}^k$ with $k > 0$ is an* SND *phase, and $a \in L^\infty S^m_\rho(\mathbb{R}^n)$ with $\rho \in [0,1]$ and*

$$m < -n\Big(1 - \Big(1 - \frac{1}{p}\Big)w(k,\rho)\Big),$$

where $w(k,\rho)$ is defined in Lemma 3.2*. Then the associated* OIO *T^φ_a is bounded from $B^s_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n)$.*

Proof. We decompose the operator into a low and a high-frequency part. The result for the low-frequency part follows from Theorem 2.21, so we only consider the high-frequency part.

Following the proof of Theorem 5.2 we obtain the kernels K^ν_j with the associated kernel estimate (23).

Let T^ν_j be the operators corresponding to the kernels K^ν_j. One observes that

$$\begin{aligned}
|T^\nu_j f_j(x)|^r &= \Big|\int_{\mathbb{R}^n} K^\nu_j(x,y) f_j(y)\,\mathrm{d}y\Big|^r \\
&= \Big|\int_{\mathbb{R}^n} K^\nu_j(x,y)\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)\frac{1}{\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)} f_j(y)\,\mathrm{d}y\Big|^r,
\end{aligned}$$

with weight functions $\sigma(y)$ which will be chosen momentarily. Therefore, Hölder's inequality with $\frac{1}{r} + \frac{1}{r'} = 1$, $r, r' > 1$ yields

$$\begin{aligned}
|T^\nu_j f_j(x)|^r \leq \Big(\int_{\mathbb{R}^n} |K^\nu_j(x,y)|^{r'} |\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)|^{r'}\,\mathrm{d}y\Big)^{\frac{r}{r'}} \\
\times \Big(\int_{\mathbb{R}^n} \frac{|f_j(y)|^r}{|\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)|^r}\,\mathrm{d}y\Big).
\end{aligned} \tag{81}$$

where σ is defined by

$$\sigma(y) = \begin{cases} 1, & |y| \leq 1; \\ |y|^\lambda, & |y| > 1, \end{cases}$$

with $\lambda > \frac{n}{r}$. Observe further that for M sufficiently large, one has

$$\begin{aligned}
&\int_{\mathbb{R}^n} |K^\nu_j(x,y)|^{r'} |\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)|^{r'}\,\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^n} \frac{2^{r'jm} 2^{r'jn(1-k)}}{\langle 2^{jw(k,\rho)}(\nabla_\xi\varphi(x,\xi^\nu_j) - y)\rangle^{Mr'}} |\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)|^{r'}\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \frac{2^{r'jm} 2^{r'jn(1-k)}}{\langle 2^{jw(k,\rho)} y\rangle^{Mr'}} |\sigma(y)|^{r'}\,\mathrm{d}y \\
&\lesssim \int_{|y|>1} \frac{2^{jr'm} 2^{-Mjr'w(k,\rho)} 2^{jnr'(1-k)}}{|y|^{(M-\lambda)r'}}\,\mathrm{d}y + \int_{|y|\leq 1} \frac{2^{jr'm} 2^{jnr'(1-k)}}{\langle 2^{jw(k,\rho)} y\rangle^{Mr}}\,\mathrm{d}y \\
&\lesssim 2^{jr'm} 2^{-Mjr'w(k,\rho)} 2^{jnr'(1-k)} + 2^{jr'm} 2^{jn(r'(1-k) - w(k,\rho))}.
\end{aligned}$$

Furthermore, using (17) in [23, p. 57], we have

$$\int_{\mathbb{R}^n} \frac{|f_j(y)|^r\,\mathrm{d}y}{|\sigma(\nabla_\xi\varphi(x,\xi^\nu_j) - y)|^r} \lesssim \big(\mathcal{M}_r f_j(\nabla_\xi\varphi(x,\xi^\nu_j))\big)^r$$

where the hidden constant in the estimate above only depends on the dimension n. Thus (81) yields

$$|T_j f_j(x)| \lesssim 2^{jm} 2^{jn(1-k)} 2^{-jnw(k,\rho)/r'} \sum_\nu \mathcal{M}_r f_j(\nabla_\xi \varphi(x, \xi_j^\nu)).$$

Now the estimate above, the SND condition, and the Hardy-Littlewood maximal theorem yield that for $1 < r < p$ we have

$$\begin{aligned} \|T_j f_j\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{jm} 2^{jn(1-k)} 2^{-jnw(k,\rho)/r'} \sum_\nu \big\|\mathcal{M}_r f_j(\nabla_\xi \varphi(\cdot, \xi_j^\nu))\big\|_{L^p(\mathbb{R}^n)} \\ &\lesssim 2^{jm} 2^{jn(1-k)} 2^{-jnw(k,\rho)/r'} 2^{njk} \|f_j\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where we have also used the number of ν's involved in the sum which is $O(2^{jnk})$. Therefore

$$\begin{aligned} \|T_a^\varphi f\|_{B^s_{p,q}} &\lesssim \Big(\sum_{j=1}^\infty 2^{jqs} \|T_j f_j\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q} \\ &\lesssim \Big(\sum_{j=1}^\infty 2^{jqs} 2^{jmq} 2^{jn(1-k)q} 2^{-jnqw(k,\rho)/r'} 2^{njqk} \|f_j\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q} \\ &\lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^n)}, \end{aligned}$$

whenever

$$m \leq -n\Big(1 - \frac{w(k,\rho)}{r'}\Big).$$

Thus since r in $(1, p)$ is arbitrary, by choosing $m < -n\Big(1 - \frac{w(k,\rho)}{p'}\Big) = -n\Big(1 - \Big(1 - \frac{1}{p}\Big) w(k,\rho)\Big)$, one obtains the desired boundedness. □

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