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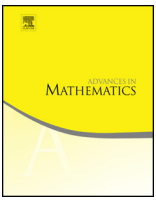

Some homological properties of category $\mathcal{O}$, VII

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We describe Calabi-Yau objects in the regular block of the (parabolic) BGG category $\mathcal{O}$ associated to a semi-simple finite dimensional complex Lie algebra. Each such object comes with a natural transformation from the Serre functor to a shifted identity whose evaluation at that object is an isomorphism.

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1. Introduction

1.1. Motivation

This paper is motivated by the ideas and the results of the two very interesting preprints [15,16] which outline the theory of categorical diagonalization and study it

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in the context of diagrammatic Soergel bimodules. The ideas of these preprints have shown to be very useful for the study of Soergel bimodules and their applications to low dimensional topology, see [21,20,17,24,40,25] and references therein. The present paper is an attempt to understand these ideas and results in the context of the BGG category $\mathcal{O}$, cf. [5,26].

Category $\mathcal{O}$ is a certain category of modules associated to a fixed triangular decomposition of a semi-simple finite dimensional complex Lie algebra. The principal block $\mathcal{O}_0$ of $\mathcal{O}$ is equivalent to the category of finite dimensional modules over a certain finite dimensional, associative, Koszul algebra A. This block is equipped with an action of the finitary bicategory $\mathscr{P}$ of projective functors. There is a combinatorial model for the action of $\mathscr{P}$ on $\mathcal{O}_0$, due to Soergel, which uses the coinvariant algebra $\mathtt{C}$ of the Weyl group of the underlying Lie algebra. This model represents the projective functors acting on $\mathcal{O}_0$ via the so-called *Soergel bimodules* (over $\mathtt{C}$). Grothendieck decategorification of the graded version of this model gives the right regular representation of the associated Hecke algebra, where the Kazhdan-Lusztig basis of the latter appears naturally as the image of the indecomposable Soergel bimodules.

Lifting the action of $\mathscr{P}$ on $\mathcal{O}_0$ to the level of the derived category (or, equivalently, to the level of the homotopy category of complexes), gives, in particular, an action of Rouquier's 2-braid group on $\mathcal{D}^b(\mathcal{O}_0)$ by the so-called *shuffling functors*. The canonical generator of the center of the braid group corresponds to the square of the longest element of the Weyl group and is sometimes referred to as the *full twist*, see [16]. At the level of the category $\mathcal{D}^b(\mathcal{O}_0)$, this full twist is a Serre functor, denoted $\mathbb{S}$, as shown in [54]. Understanding the homological properties of $\mathbb{S}$ is important for the study of the category $\mathcal{O}_0$.

The paper [16] discusses, in the setup of Soergel bimodules over the polynomial algebra, and especially in type A, the so-called *categorical diagonalization* of the full twist. This is a special kind of structure given by certain natural transformations from this full twist to the identity, shifted both in the direction of homological position and in the direction of Koszul grading.

The notion of a Serre functor $\mathbb{S}$ on a $\Bbbk$-linear category $\mathcal{C}$ was defined in [7]. The principal defining property of $\mathbb{S}$ are isomorphisms $\mathcal{C}(X,\mathbb{S}(Y)) \cong \mathcal{C}(Y,X)^*$, natural in both, X and Y, for all $X,Y \in \mathcal{C}$. In [10], an object $Y \in \mathcal{C}$ is called a *Calabi-Yau object of dimension* i, where $i \in \mathbb{Z}$, provided that $\mathcal{C}(X,Y[i]) \cong \mathcal{C}(Y,X)^*$, or, equivalently, $\mathbb{S}Y \cong Y[i]$. Understanding Calabi-Yau object provides important information about the structure and behavior of the Serre functor.

Combining the ideas of [15,16] with those of [10], given a Calabi-Yau object Y of dimension i, it seems reasonable to ask whether there is a natural transformation from $\mathbb{S}$ to the i-th shift of the identity in the homological position whose evaluation at Y is an isomorphism. That is exactly the main question we address in the present paper in the setup of the category $\mathcal{O}_0$.

1.2. Results

To be able to formulate the main results of this paper, we need to introduce some notation. Let W be the Weyl group of our Lie algebra. Then the isomorphism classes of simple objects in $\mathcal{O}_0$ are indexed by the elements of W. For $w \in W$, we denote by L_w the simple object corresponding to w. Here L_e is the trivial module. Also, the indecomposable projective endofunctors of $\mathcal{O}_0$ are indexed by the elements of W and denoted θ_w. The Kazhdan-Lusztig combinatorics splits W into left, right and two-sided Kazhdan-Lusztig cells. Each left and each right KL-cell contains a unique distinguished involution called a *Duflo element*. We denote by $\mathbf{D}$ the set of all Duflo elements in W. If $x, y \in W$ belong to the same right cell, we write $x \sim_R y$. We denote by $\mathbf{a}$ Lusztig's $\mathbf{a}$-function on W. As usual, we denote by $[i]$ the shift in homological position and by $\langle i \rangle$ the shift in the Koszul grading.

We can now state our first main result, Theorem 1 in the text, as follows:

Theorem A. *For $d \in \mathbf{D}$ and $w \in W$ such that $w \sim_R d$, the object $\theta_w L_d$ is a Calabi-Yau object in $\mathcal{D}^b(\mathcal{O}_0)$ of dimension $2\mathbf{a}(w_0 d)$. More precisely, in the graded picture, we have*

$$\mathbb{S}(\theta_w L_d) \cong \theta_w L_d \langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle [2\mathbf{a}(w_0 d)].$$

Our second main result, which is Theorem 16 in the text, asserts the following:

Theorem B. *For any $d \in \mathbf{D}$, there exists a natural transformation*

$$\alpha_d : \mathbb{S} \to \mathrm{Id}\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle [2\mathbf{a}(w_0 d)]$$

such that, for any $w \sim_R d$, the evaluation of α_d at $\theta_w L_d$ is an isomorphism.

We note that the objects of the form $\theta_w L_d$ appeared previously in [56] in the context of categorification of Wedderburn's basis in type A. They also appear as projective-injective objects for certain subquotients of $\mathcal{O}_0$, see [55]. The latter was probably the main motivation to investigate these objects as potential Calabi-Yau object. Projective-injective objects (with isomorphic top and socle) are obvious Calabi-Yau objects of dimension 0 in the very general case. Projective-injective objects for subquotient categories are, of course, not Calabi-Yau objects in general. However, the category $\mathcal{O}_0$ and, especially, the action of projective functors on this category makes the situation "nice" enough for the above results to be possible.

We propose the following conjecture:

Conjecture C. *For a fixed $d \in \mathbf{D}$, any Calabi-Yau object in $\mathcal{D}^b(\mathcal{O}_0)$ of dimension $2\mathbf{a}(w_0 d)$ admits a filtration with subquotients of the form $\theta_w L_{d'}$, where $d' \in \mathbf{D}$ is such that $\mathbf{a}(d) = \mathbf{a}(d')$ and $w \in W$ such that $w \sim_R d'$.*

1.3. Methods

Proofs in the paper involve a wide variety of various methods and techniques, most of which were developed only very recently. The general background of all arguments is, as usual,

- the Kazhdan-Lusztig combinatorics from [30] and its connection to $\mathcal{O}_0$ as established in [1,8,18];
- Soergel's combinatorial description of both $\mathcal{O}_0$ and $\mathscr{P}$ from [64,65];
- Koszul duality from [2].

Further, we extensively use the ideas and results from several of the previous papers in the "Some homological properties of category $\mathcal{O}$" series, namely, from [44,46,13,33,34]. This includes:

- explicit information on the homological invariants (for instance, on projective dimension) of the structural objects in $\mathcal{O}_0$;
- explicit information on the structure of resolutions of structural modules in $\mathcal{O}_0$;
- Auslander regularity of $\mathcal{O}_0$;
- latest advances in our understanding of Kostant's problem for simple highest weight modules.

The present paper would hardly be possible without the insight and the intuition about some very tiny special features of various complexes of modules in the category $\mathcal{O}_0$ which was developed during the work on all previous papers in this series.

Finally, we crucially employ the methods and techniques from 2-representation theory which allow us to study the category $\mathcal{O}_0$ from the point of view of the bicategory $\mathscr{P}$ acting on $\mathcal{O}_0$. Here, in particular, we heavily rely on

- properties of categories with full projective functors as developed in [31];
- abstract theory of birepresentations of finitary bicategories as developed in [48,39] and its concrete realization via the action of projective functors on category $\mathcal{O}$ as studied in [40];
- 2-categorical approach to Kostant's problem developed in [34].

1.4. Structure of the paper

The paper is organized as follows: In Section 2 we collected all necessary preliminaries on category $\mathcal{O}$ and various technical tools that are used in the paper. Section 3 is devoted to the first main result, Theorem 1, which describes a family of Calabi-Yau objects in the principal block of $\mathcal{O}$. We start by proving the result is some special cases. This eventually directs us towards the idea how to prove Theorem 1 in the general case. Section 4

is dedicated to the second main result, Theorem 16, which describes certain natural transformations from the Serre functor to the (shifted) identity, whose evaluations at the corresponding Calabi-Yau objects are isomorphisms. Section 5 contains some bonus homological properties of category $\mathcal{O}$ which are inspired by the proofs of Theorem 1 and Theorem 16. Section 6 provides generalizations of the two main results to the case of the parabolic category $\mathcal{O}$. Finally, in Section 7 we collected a number of explicit examples, both low rank and general. These examples were quite helpful to the author from the point of view of explaining, better understanding and checking the main results of the paper.

We are very far away from solving all interesting problems in the area. A number of guesses, expectations and conjectures are formulated throughout the paper.

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2. Preliminaries on category $\mathcal{O}$

2.1. Conventions

In this paper we work over the field $\mathbb{C}$ of complex numbers. In particular, all algebras are assumed to be $\mathbb{C}$-algebras and all categories and functors are assumed to be $\mathbb{C}$-linear. We denote by $*$ the usual contravariant $\mathbb{C}$-duality $\mathrm{Hom}_{\mathbb{C}}(_, \mathbb{C})$.

For a Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ its universal enveloping algebra. We have the canonical isomorphism between the category $\mathfrak{a}$-Mod of all $\mathfrak{a}$-modules and the category $U(\mathfrak{a})$-Mod of all $U(\mathfrak{a})$-modules. It restricts to an isomorphism between the category $\mathfrak{a}$-mod of all finitely generated $\mathfrak{a}$-modules and the category $U(\mathfrak{a})$-mod of all finitely generated $U(\mathfrak{a})$-modules. We denote by $Z(\mathfrak{a})$ the center of $U(\mathfrak{a})$.

It what follows, *graded* means $\mathbb{Z}$-graded.

2.2. Definition and the principal block

Let $\mathfrak{g}$ be a semi-simple, finite dimensional, complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a Cartan subalgebra. Associated to this datum, we have the corresponding BGG category $\mathcal{O}$ defined as the full subcategory of $U(\mathfrak{g})$-mod, consisting of all objects, the action of $\mathfrak{h}$ on which is diagonalizable and the action of $U(\mathfrak{n}_+)$ on which is locally finite, see [5,26].

For $\lambda \in \mathfrak{h}^*$, we denote by $\Delta(\lambda)$ the corresponding Verma module with highest weight λ and by $L(\lambda)$ the simple top of $\Delta(\lambda)$, see [14, Chapter 7].

Let $\mathbf{R}$ denote the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Our fixed triangular decomposition of $\mathfrak{g}$ induces a decomposition of $\mathbf{R}$ into a disjoint union of the set $\mathbf{R}_+$ of all positive roots

and the set $\mathbf{R}_-$ of all negative roots. Let $\boldsymbol{\pi}$ be the corresponding basis of $\mathbf{R}$. Denote by ρ the half of the sum of all positive roots.

We also have the Weyl group W of the root system $\mathbf{R}$ which acts on $\mathfrak{h}^*$ via the defining action, written $(w, \lambda) \mapsto w(\lambda)$, for $w \in W$ and $\lambda \in \mathfrak{h}^*$. We also have the dot-action of W on $\mathfrak{h}^*$ given by $w \cdot \lambda := w(\lambda + \rho) - \rho$.

Category $\mathcal{O}$ admits a decomposition

$$\mathcal{O} \cong \bigoplus_{\chi: Z(\mathfrak{g}) \to \mathbb{C}} \mathcal{O}_\chi$$

with respect to the action of the center $Z(\mathfrak{g})$ defined in the terms of a central character χ, where $\mathcal{O}_\chi$ consist of all modules $M \in \mathcal{O}$ such that, for every $m \in M$ and $z \in Z(\mathfrak{g})$, we have $(z - \chi(z))^k m = 0$, for $k \gg 0$.

Let χ_0 denote the central character of the trivial $\mathfrak{g}$-module $L(0)$. In this paper we will mostly study the principal block $\mathcal{O}_0 := \mathcal{O}_{\chi_0}$ of $\mathcal{O}$. This block coincides with the Serre subcategory of $\mathcal{O}$ generated by $L_w := L(w \cdot 0)$, where $w \in W$. The category $\mathcal{O}_0$ is equivalent to the category A-mod of finite dimensional modules over some basic, finite dimensional, associative algebra A of finite global dimension.

For $w \in W$, we set $\Delta_w := \Delta(w \cdot 0)$ and also denote by P_w and I_w the indecomposable projective cover and injective envelope of L_w, respectively.

We denote by $\star$ the usual simple preserving duality on $\mathcal{O}$ and set $\nabla_w := \Delta_w^\star$, for $w \in W$. The latter modules are usually called the *dual Verma modules*.

We have the full subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ of $\mathcal{O}_0$ consisting of all objects that admit a filtration with Verma or dual Verma subquotients, respectively. For $w \in W$, we denote by T_w the corresponding tilting module, that is the unique, up to isomorphism, indecomposable object in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ for which there is an embedding $\Delta_w \subset T_w$ such that the corresponding cokernel is in $\mathcal{F}(\Delta)$, see [11,60]. We have $T_w^\star \cong T_w$.

For $M \in \mathcal{O}_0$, we denote by

- $\mathcal{P}_\bullet(M)$ a minimal projective resolution of M;
- $\mathcal{I}_\bullet(M)$ a minimal injective coresolution of M;
- $\mathcal{T}_\bullet(M)$ a minimal complex of tilting modules that represents M in $\mathcal{D}^b(\mathcal{O}_0)$.

2.3. Grading and combinatorics

The algebra A is Koszul, see [64,2], in particular, it has the corresponding positive Koszul grading $A \cong \bigoplus_{i \geq 0} A_0$ with semi-simple A_0. The corresponding category A-fgmod of finite dimensional graded A-modules and homogeneous module homomorphisms of degree zero is called the *graded lift of* $\mathcal{O}_0$ and denoted ${}^{\mathbb{Z}}\mathcal{O}_0$. We have the obvious canonical functor Forget from ${}^{\mathbb{Z}}\mathcal{O}_0$ to A-mod $\cong \mathcal{O}_0$ which forgets the grading. For $n \in \mathbb{Z}$, we denote by $\langle n \rangle$ the usual degree shift endofunctor of ${}^{\mathbb{Z}}\mathcal{O}_0$ which maps elements of degree m to elements of degree $m - n$, for $m \in \mathbb{Z}$. As usual, we use the lowercase notation hom and

ext for homogeneous homomorphisms and extensions of degree zero as compared to the ungraded notation Hom and Ext.

All structural objects in $\mathcal{O}_0$ admit graded lifts, in the sense that they belong to the image of Forget. Moreover, for indecomposable structural objects, the corresponding graded lift (i.e. the preimage under Forget) is unique up to isomorphism and graded shift, see [68]. We fix the following graded lifts of the structural modules which we will denote in the same way as ungraded modules, for $w \in W$:

- L_w is concentrated in degree zero;
- P_w and Δ_w have tops concentrated in degree zero;
- I_w and ∇_w have socles concentrated in degree zero;
- T_w has the unique L_w-subquotient in degree zero.

The duality $\star$ also admits a graded lift, denoted by the same symbol. It satisfies $\star \circ \langle n \rangle = \langle -n \rangle \circ \star$, for $n \in \mathbb{Z}$.

Let ℓ be the length function on W and w_0 the longest element of W. Let $\mathbf{H}$ be the Hecke algebra of W. It is an algebra over $\mathbb{Z}[v, v^{-1}]$. It has the *standard basis* $\{H_w : w \in W\}$ and the *Kazhdan-Lusztig (KL) basis* $\{\underline{H}_w : w \in W\}$, see [30]. We use the normalization of [67]. The entries of the transformation matrix between the two bases are called *KL-polynomials* and denoted $p_{x,y}$, where $x, y \in W$. We have $\underline{H}_y = \sum_x p_{x,y} H_x$.

The Grothendieck group of ${}^{\mathbb{Z}}\mathcal{O}_0$ is isomorphic to $\mathbf{H}$ by sending $[\Delta_w]$ to H_w, for $w \in W$, and letting v act as $\langle -1 \rangle$. By the Kazhdan-Lusztig conjecture, proved in [8,1], see also [18], this isomorphism maps $[P_w]$ to $\underline{H}_w$, for $w \in W$.

As usual, we have the theory of left, right and two-sided KL-orders and cells. For $x, y \in W$, we have $x \leq_L y$ provided that there is $w \in W$ such that $\underline{H}_y$ appears with a non-zero coefficient in the decomposition of $\underline{H}_w \underline{H}_x$ as a linear combination of elements in the KL-basis. Then $\leq_L$ is a partial pre-order, called the *left KL-pre-order* and the equivalence classes with respect to it are called the *left KL-cells*. We write $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$. The *right* and the *two-sided* pre-orders $\leq_R$ and $\leq_J$ and the corresponding cells $\sim_R$ and $\sim_J$ are defined similarly using multiplication on the right or from both sides, respectively. We denote by $\sim_H$ the equivalence relation given by the intersection of $\sim_R$ and $\sim_L$.

Each left (and each right) KL-cell contains a unique element called *Duflo element* or *Duflo involution* (this element is an involution). We denote by $\mathbf{D}$ the set of all Duflo involutions. In type A, all involutions are Duflo involutions. We refer to [48] for details.

We also have *Lusztig's* $\mathbf{a}$*-function* $\mathbf{a} : W \to \mathbb{Z}_{\geq 0}$ introduced in [38]. For $w \in W$, the value $\mathbf{a}(w)$ is defined as the maximum v-degree, taken over all $x, y \in W$, with which $\underline{H}_w$ appears in the decomposition of $\underline{H}_x \underline{H}_y$ into a linear combination of the elements of the KL basis. Note that, to attain this maximal degree requires both $x \sim_J w$ and $y \sim_J w$. The function $\mathbf{a}$ is constant on the two-sided KL-cells of W, moreover, for $x \in W$, we have $\mathbf{a}(x) = \ell(x)$ provided that x is the longest element of some parabolic subgroup of

W. These two properties identify **a** uniquely in type A. In all types, the function **a** is strictly monotone with respect to all KL orders.

2.4. Functors

For each $w \in W$, we denote by θ_w the indecomposable projective endofunctor of $\mathcal{O}_0$ such that $\theta_w \Delta_e \cong P_w$, see [3]. Each θ_w is exact, biadjoint to $\theta_{w^{-1}}$ and admits a graded lift which we will denote by the same symbol. We denote by $\mathscr{P}$ the bicategory of projective endofunctors of $\mathcal{O}_0$. It is finitary in the sense of [48,39]. We also denote by ${}^{\mathbb{Z}}\mathscr{P}$ the bicategory of projective endofunctors of ${}^{\mathbb{Z}}\mathcal{O}_0$. It is locally finitary in the sense of [41,42]. The split Grothendieck group of ${}^{\mathbb{Z}}\mathscr{P}$ is isomorphic to $\mathbf{H}$ by sending $[\theta_w]$ to $\underline{H}_w$ and with the same convention for the action of v as above. The defining action of ${}^{\mathbb{Z}}\mathscr{P}$ on ${}^{\mathbb{Z}}\mathcal{O}_0$ decategorifies, by taking the corresponding Grothendieck groups, to a right regular $\mathbf{H}$-module.

For each simple reflection $s \in W$, we have the corresponding shuffling functor C_s defined in [9] as the cokernel of the adjunction morphism $\theta_e \to \theta_s$ and the coshuffling functor K_s defined in [9] as the kernel of the adjunction morphism $\theta_s \to \theta_e$. The functor C_s is left adjoint to K_s, in particular, the functor C_s is right exact while K_s is left exact. Furthermore, the left derived functor $\mathcal{L}C_s$ is an auto-equivalence of the bounded derived category $\mathcal{D}^b(\mathcal{O}_0)$ whose inverse is the right derived functor $\mathcal{R}K_s$. The functor $\mathcal{L}C_s$ admits a realization via tensoring with the complex $0 \to \theta_e \to \theta_s \to 0$ of functors, in which the non-zero morphism is the adjunction morphism, followed by taking the total complex. The complexes of the form $0 \to \theta_e \to \theta_s \to 0$ are known as *Rickard or Rouquier complexes*, see [59,62]. Similarly one can describe the coshuffling functors. Further, the functors $\mathcal{L}C_s$, where s runs through the set S of all simple reflections in W, satisfy the same braid relations as the corresponding simple reflections of W, see [9,52]. In particular, given a reduced decomposition $w = s_1 s_2 \dots s_k$ of some $w \in W$, we can consider the corresponding shuffling functor $C_w := C_{s_k} \circ \dots \circ C_{s_2} \circ C_{s_1}$ (here, reversing the order reflects the fact that the action of projective functors on $\mathcal{O}_0$ is the right action) and let $K_{w^{-1}}$ be its adjoint. We have $\mathcal{L}C_w := \mathcal{L}C_{s_k} \circ \dots \circ \mathcal{L}C_{s_2} \circ \mathcal{L}C_{s_1}$ and a similar decomposition for the corresponding coshuffling functors. As both adjunctions morphisms $\theta_e \to \theta_s$ and $\theta_s \to \theta_e$ are gradeable, both shuffling and coshuffling functors admit the obvious graded lifts.

Finally, for each $w \in W$ we have the corresponding twisting functor $\top_w$ and its right adjoint $E_{w^{-1}}$, the Enright completion functor, see [22,32,29]. The functor $\top_w$ is right exact while E_w is left exact. The left derived functor $\mathcal{L}\top_w$ is an auto-equivalence of $\mathcal{D}^b(\mathcal{O}_0)$ whose inverse is the right derived functor $\mathcal{R}E_w$, see [22]. Further, the functors $\mathcal{L}\top_s$, where $s \in S$, satisfy the same braid relations as the corresponding simple reflections of W, see [32]. Both $\top_w$ and E_w are gradeable, see the appendix of [50]. Last, but not least, is that both twisting and Enright completion functors functorially commute with projective functors, see [22]. Consequently, both twisting and Enright completion functors commute with shuffling and coshuffling functors.

2.5. Soergel's combinatorial description

Let $\mathtt{C}$ denote the coinvariant algebra of W. As was shown in [64], the commutative algebra $\mathtt{C}$ is isomorphic to $\mathrm{End}_{\mathfrak{g}}(P_{w_0})$, moreover, the so-called *Soergel's combinatorial functor*

$$\mathbb{V} := \mathrm{Hom}_{\mathfrak{g}}(P_{w_0}, _) : \mathcal{O}_0 \to \mathtt{C}\text{-mod}$$

is full and faithful on projective objects. Consequently, the algebra A is isomorphic to the (opposite of the) endomorphism algebra of the $\mathtt{C}$-module given by the direct sum of all $\mathbb{V}(P_w)$, where $w \in W$.

For a simple reflection s, let $\mathtt{C}^s$ be the algebra of all s-invariants in $\mathtt{C}$. We can now define the corresponding $\mathtt{C}$-$\mathtt{C}$-bimodule $\mathtt{B}_s := \mathtt{C} \otimes_{\mathtt{C}^s} \mathtt{C}$. Given a reduced expression $w = s_1 s_2 \dots s_m$ of some $e \neq w \in W$, consider the $\mathtt{C}$-$\mathtt{C}$-bimodule $\mathtt{B}_w$ defined as

$$\mathtt{B}_w := \mathtt{B}_{s_1} \otimes_{\mathtt{C}} \mathtt{B}_{s_2} \otimes_{\mathtt{C}} \cdots \otimes_{\mathtt{C}} \mathtt{B}_{s_m}.$$

Set $\mathtt{B}_e := \mathtt{C}$. The additive closure $\mathscr{B}$ inside $\mathtt{C}$-mod-$\mathtt{C}$ of all $\mathtt{B}_w$, where $w \in W$, is a monoidal subcategory of $\mathtt{C}$-mod-$\mathtt{C}$. Moreover, the category $\mathscr{B}$ is monoidally equivalent to $\mathscr{P}$, see [65]. The bimodule $\mathtt{B}_e$ is, clearly, indecomposable. Each $\mathtt{B}_w$ has a unique indecomposable direct summand, denoted $\underline{\mathtt{B}}_w$, which is not isomorphic to any $\underline{\mathtt{B}}_x$, where $x < w$ with respect to the Bruhat order on W.

This allows us to construct the $\mathtt{C}$-modules $\mathbb{V}(P_w)$ inductively. To start with, the $\mathtt{C}$-module $\mathbb{V}(P_e)$ is simple. Furthermore, for $w \in W$, the $\mathtt{C}$-module $\mathbb{V}(P_w)$ is isomorphic to $\underline{\mathtt{B}}_w \otimes_{\mathtt{C}} \mathbb{V}(P_e)$.

The algebra $\mathtt{C}$ is graded, with generators in degree 2, and all the story above admits a natural graded lift.

The bimodules in the additive closure of $\underline{\mathtt{B}}_w$ are usually called *Soergel bimodules* over the coinvariant algebra. A similar definition works over the polynomial algebra, that is the symmetric algebra of $\mathfrak{h}$, and provides a combinatorial description for projective functors on the so-called *thick category* $\mathcal{O}$, where the usual category $\mathcal{O}$ is a subcategory, see [65]. Projective functors on the thick category $\mathcal{O}$ act on the usual category $\mathcal{O}$ by restriction, which defines a monoidal functor from the monoidal category of Soergel bimodules over the polynomial algebra to the monoidal category of Soergel bimodules over the coinvariant algebra. This monoidal functor maps the full twist of the former category to the full twist of the latter.

2.6. Harish-Chandra bimodules

Yet another way to look at the category $\mathscr{P}$ is through the prism of Harish Chandra bimodules. Consider the category $\mathscr{H}$ of Harish Chandra $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules, that is finitely generated $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules, the adjoint action of $\mathfrak{g}$ on which is locally finite

and has finite multiplicities for all simple subquotients. A typical example of such a bimodule would be the quotient of $U(\mathfrak{g})$ by the two-sided ideal generated be some ideal in $Z(\mathfrak{g})$ of finite codimension.

Let $\mathbf{m}$ be the maximal ideal of $Z(\mathfrak{g})$ given by the kernel of χ_0. Consider the full subcategory ${}^{\infty}_{0}\mathscr{H}^{1}_{0}$ of $\mathscr{H}$ consisting of all object on which the right action of $\mathbf{m}$ is zero while the left action of $\mathbf{m}$ is locally nilpotent. Then ${}^{\infty}_{0}\mathscr{H}^{1}_{0}$ is equivalent to $\mathcal{O}_0$ by [3].

Let $\mathbf{n}$ be the ideal of $Z(\mathfrak{g})$ given by the kernel of the projection of $Z(\mathfrak{g})$ onto the endomorphism algebra of P_{w_0}, see [64]. In particular, we have $Z(\mathfrak{g})/\mathbf{n} \cong \mathtt{C}$. Note that the latter algebra is isomorphic to the center of $\mathcal{O}_0$ and hence also of the algebra A. In particular, $\mathbf{n}$ has finite codimension, in fact, it contains $\mathbf{m}^{\ell(w_0)}$. Also, the identity functor on $\mathcal{O}_0$ can be realized as tensoring with the Harish Chandra bimodule $U(\mathfrak{g})/U(\mathfrak{g})\mathbf{n}$. Consider the full subcategory ${}^{\infty}_{0}\mathscr{H}^{\mathtt{C}}_{0}$ of $\mathscr{H}$ consisting of all object on which the right action of $\mathbf{n}$ is zero while the left action of $\mathbf{m}$ is locally nilpotent. Then ${}^{\infty}_{0}\mathscr{H}^{\mathtt{C}}_{0}$ is a monoidal category (in which the monoidal structure is given by tensoring over $U(\mathfrak{g})$) and it is monoidally equivalent to $\mathscr{P}$.

2.7. Serre functor

Given a $\mathbb{C}$-linear additive category $\mathcal{C}$ with finite dimensional morphism spaces, a *right Serre functor* on $\mathcal{C}$ is an endofunctor $\mathbb{S}$ of $\mathscr{C}$, for which there are isomorphisms $\mathcal{C}(i, \mathbb{S}(j)) \cong \mathcal{C}(j, i)^*$, for all $i, j \in \mathbb{C}$, natural in both i and j. If exists, a right Serre functor is unique, up to isomorphism, and it commutes, up to isomorphism, with any auto-equivalence of $\mathcal{C}$. A right Serre functor which is itself an equivalence is called a *Serre functor*. We refer to [7] for details.

If B is a finite dimensional associative algebra of finite global dimension, then the corresponding bounded derived category $\mathcal{D}^b(B)$ of B-mod has a Serre functor given by the left derived of the Nakayama endofunctor $B^* \otimes_B -$ of B-mod, see [23]. In particular, in the setup of this paper, both $\mathcal{D}^b(A)$ and $\mathcal{D}^b(\mathcal{O}_0)$ have a Serre functor.

In [54] it was shown that the Serre functor $\mathbb{S}$ on $\mathcal{D}^b(\mathcal{O}_0)$ can be alternatively described both as $(\mathcal{L}\top_{w_0})^2$ and as $(\mathcal{L}C_{w_0})^2$.

2.8. Auslander regularity

Recall that a finite dimensional associative algebra B of finite global dimension is called *Auslander regular* provided that the regular B-module ${}_BB$ admits a minimal injective coresolution

$$0 \to {}_BB \to Q_0 \to Q_1 \to \cdots \to Q_m \to 0$$

such that the projective dimension of each component Q_i of this resolution is at most i, where $i = 0, 1, 2, \dots, m$. In [33] it was shown that the algebra A is Auslander regular.

More detailed information about the injective dimension of the indecomposable projective modules in $\mathcal{O}_0$ was obtained in [44,46]. For $w \in W$, the injective dimension of P_w equals $2\mathbf{a}(w_0 w)$.

As explained in [57, Subsection 4.3], the notion of Auslander regularity is closely connected to certain homological properties of the Serre functor. Indeed, the fact that some I_x appears as a summand of some Q_i in the above resolution is equivalent to $\mathrm{Ext}^i(L_x, B) \neq 0$. Applying the Serre functor $\mathbb{S}$, we get $\mathrm{Ext}^i(\mathbb{S}L_x, B^*) \neq 0$ which, in turn, is equivalent to the evaluation of $\mathcal{L}_i\mathbb{S}$ at L_x being non-zero. Therefore, Auslander regularity of B can be reformulated via a certain vanishing property for the cohomology of the Serre functor on simple objects.

2.9. Koszul and Ringel self-dualities

The category $\mathcal{O}_0$ is Ringel self-dual, in particular, the functor $\top_{w_0}$ that maps P_w to $T_{w_0 w}$, for $w \in W$, induces an equivalence between the categories of projective and tilting objects in $\mathcal{O}_0$. We refer to [66] for details.

The category $\mathcal{O}_0$ is also Koszul self-dual, see [64,2]. This means the following, see [45]: Denote by $\mathcal{LP}(\mathcal{O}_0)$ the category of linear complexes of projective objects in $\mathcal{O}_0$, that is, those complexes $\mathcal{X}_\bullet$ of projective objects, for which each $\mathcal{X}_i$ is generated in degree $-i$, for $i \in \mathbb{Z}$. Then this category is equivalent to the category of graded A-modules. This equivalence maps $\mathcal{P}_\bullet(L_w)$ to $I_{w^{-1}w_0}$, for $w \in W$.

Similarly, denote by $\mathcal{LI}(\mathcal{O}_0)$ the category of linear complexes of injective objects in $\mathcal{O}_0$, that is, those complexes $\mathcal{Y}_\bullet$ of injective objects, for which the socle of each $\mathcal{Y}_i$ is concentrated in degree $-i$, for $i \in \mathbb{Z}$. Then this category is equivalent to the category of graded A-modules. This equivalence maps $\mathcal{I}_\bullet(L_w)$ to $P_{w^{-1}w_0}$, for $w \in W$.

The Koszul self-duality of $\mathcal{O}_0$ swaps the derived twisting and the derived shuffling functors, see [51, Subsection 6.5].

Combining the Ringel and Koszul self-dualities, we get the following: Denote by $\mathcal{LT}(\mathcal{O}_0)$ the category of linear complexes of tilting objects in $\mathcal{O}_0$, that is, those complexes $\mathcal{Z}_\bullet$ of tilting objects, for which the middle of each $\mathcal{Z}_i$ is concentrated in degree $-i$, for $i \in \mathbb{Z}$. Then this category is equivalent to the category of graded A-modules. This equivalence maps $\mathcal{T}_\bullet(L_w)$ to $T_{w_0 w^{-1} w_0}$, for $w \in W$.

3. Calabi-Yau objects in the principal block

3.1. Calabi-Yau objects in triangulated categories

Let $\mathcal{C}$ be a triangulated category with a shift functor denoted by [1] and a Serre functor $\mathbb{S}$. Following [10], an object $M \in \mathcal{C}$ is called a *Calabi-Yau object of dimension* i, where $i \in \mathbb{Z}$, provided that $\mathbb{S}M \cong M[i]$.

For example, if B is a finite dimensional associative algebra of finite global dimension and P is a projective-injective B-module whose top is isomorphic to the socle, then P

is a Calabi-Yau object of dimension 0 in $\mathcal{D}^b(B)$. This is because $\mathbb{S}P$ is isomorphic, by the definition of the Nakayama functor, to the injective envelope of the top of P. In the setup of the category $\mathcal{O}_0$, it follows that P_{w_0} is a Calabi-Yau object of dimension 0 in $\mathcal{D}^b(\mathcal{O}_0)$.

Another fairly straightforward example for the category $\mathcal{O}_0$ is that L_e is a Calabi-Yau object of dimension $2\ell(w_0)$. Indeed, the Serre functor $\mathbb{S}$ is the composition of $2\ell(w_0)$ functors of the form $\mathcal{L}\top_s$, where s is a simple reflection. And it is well-known that $\mathcal{L}\top_s L_e \cong L_e\langle -1\rangle[1]$, see [22].

3.2. Calabi-Yau objects in $\mathcal{D}^b(\mathcal{O}_0)$

Our main result in this section is the following theorem.

Theorem 1. *For $d \in \mathbf{D}$ and $w \in W$ such that $w \sim_R d$, the object $\theta_w L_d$ is a Calabi-Yau object in $\mathcal{D}^b(\mathcal{O}_0)$ of dimension $2\mathbf{a}(w_0 d)$.*

More precisely, in the graded picture, we have

$$\mathbb{S}(\theta_w L_d) \cong \theta_w L_d\langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)]. \tag{1}$$

Since Soergel bimodules over the polynomial algebra act on the usual category $\mathcal{O}$ via restriction, the second claim of Theorem 1 remains true if one replaces $\mathbb{S}$ with the full twist for Soergel bimodules over the polynomial algebra.

The remainder of this section is devoted to the proof of this theorem, first in some special cases, and then in general. We will also derive some homological consequences from this result at the end of the section.

Note that, in type A, the modules $\{\theta_w L_d \,:\, d \in \mathbf{D}, w \in W, w \sim_R d\}$ described in Theorem 1 categorify Wedderburn's basis of the complex group algebra of the symmetric group, see [56, Lemma 11].

3.3. Parabolic subcategories

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$. Associated to $\mathfrak{p}$, we have the corresponding parabolic category $\mathcal{O}^{\mathfrak{p}}$ defined as the full subcategory of $\mathcal{O}$ consisting of all objects, the action of $U(\mathfrak{p})$ on which is locally finite, see [61]. The category $\mathcal{O}^{\mathfrak{p}}$ is a Serre subcategory of $\mathcal{O}$ and it inherits from $\mathcal{O}$ a block decomposition. We denote by $A^{\mathfrak{p}}$ the quotient of A corresponding to $\mathcal{O}_0^{\mathfrak{p}}$. Then $A^{\mathfrak{p}}$ is a graded quotient of A and, moreover, it turns out to be Koszul, see [2]. In particular, we have the graded version ${}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}}$ of $\mathcal{O}_0^{\mathfrak{p}}$ defined as $A^{\mathfrak{p}}$-fgmod.

Let $W^{\mathfrak{p}}$ be the parabolic subgroup of W corresponding to $\mathfrak{p}$, that is, the subgroup generated by all simple reflections for which the corresponding negative root subspace belongs to $\mathfrak{p}$. Consider the set $(W^{\mathfrak{p}} \setminus W)_{\mathrm{short}}$ of all shortest coset representatives in $W^{\mathfrak{p}} \setminus W$. Then the block $\mathcal{O}_0^{\mathfrak{p}}$ is the Serre subcategory of $\mathcal{O}_0$ generated by all simples L_w, where $w \in (W^{\mathfrak{p}} \setminus W)_{\mathrm{short}}$.

The exact embedding $\iota_{\mathfrak{p}} : \mathcal{O}_0^{\mathfrak{p}} \subset \mathcal{O}_0$ has both left and right adjoints. The left adjoint $Z^{\mathfrak{p}} : \mathcal{O} \to \mathcal{O}_0^{\mathfrak{p}}$, also called *Zuckerman functor*, see [53], takes a module from $\mathcal{O}$ to its maximal quotient that belongs to $\mathcal{O}_0^{\mathfrak{p}}$. The right adjoint $Z_{\mathfrak{p}} : \mathcal{O} \to \mathcal{O}_0^{\mathfrak{p}}$, also called *dual Zuckerman functor*, takes a module from $\mathcal{O}$ to its maximal submodule that belongs to $\mathcal{O}_0^{\mathfrak{p}}$. Note that the duality $\star$ restricts to $\mathcal{O}_0^{\mathfrak{p}}$ and, moreover, we have $Z^{\mathfrak{p}} \cong \star \circ Z_{\mathfrak{p}} \circ \star$. The action of $\mathscr{P}$ restricts to $\mathcal{O}_0^{\mathfrak{p}}$, in particular, both $Z^{\mathfrak{p}}$ and $Z_{\mathfrak{p}}$ functorially commute with projective functors.

For $w \in (W^{\mathfrak{p}} \setminus W)_{\mathrm{short}}$, we have the corresponding

- *parabolic Verma module* $\Delta_w^{\mathfrak{p}} := Z^{\mathfrak{p}}(\Delta_w)$;
- indecomposable projective cover $P_w^{\mathfrak{p}} := Z^{\mathfrak{p}}(P_w)$ of L_w in $\mathcal{O}_0^{\mathfrak{p}}$;
- indecomposable injective envelope $I_w^{\mathfrak{p}} := Z_{\mathfrak{p}}(I_w)$ of L_w in $\mathcal{O}_0^{\mathfrak{p}}$;
- *parabolic dual Verma module* $\nabla_w^{\mathfrak{p}} := (\Delta_w^{\mathfrak{p}})^\star = Z_{\mathfrak{p}}(\nabla_w)$.

We will use the usual conventions for the graded version of these modules.

Let $w_0^{\mathfrak{p}}$ denote the longest element in $W^{\mathfrak{p}}$. Consider the element $w_0^{\mathfrak{p}} w_0$ and let $\mathcal{R}^{\mathfrak{p}}$ be the KL-right cell of this element. Let $d^{\mathfrak{p}}$ be the Duflo element in $\mathcal{R}^{\mathfrak{p}}$. The module $\Delta_e^{\mathfrak{p}}$ has simple socle which is isomorphic to $L_{d^{\mathfrak{p}}}$ (in the graded version, to $L_{d^{\mathfrak{p}}}\langle -\mathbf{a}(d^{\mathfrak{p}})\rangle$). Moreover, all other composition subquotients of $\Delta_e^{\mathfrak{p}}$ are of the form L_x, where $x <_J d^{\mathfrak{p}}$ and hence $\theta_w L_x = 0$, for every $w \in \mathcal{R}^{\mathfrak{p}}$, see [48, Lemma 12]. For $w \in \mathcal{R}^{\mathfrak{p}}$, the projective module $P_w^{\mathfrak{p}} \cong \theta_w \Delta_e^{\mathfrak{p}} \cong \theta_w L_{d^{\mathfrak{p}}}$ is isomorphic to $I_w^{\mathfrak{p}}$ and any indecomposable projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$ is of such form, see [27].

3.4. Special cases of Theorem 1 related to parabolic subcategories

Proposition 2. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Then, for any $w \in \mathcal{R}^{\mathfrak{p}}$, the module $\theta_w L_{d^{\mathfrak{p}}}$ is a Calabi-Yau object in $\mathcal{D}^b(\mathcal{O}_0)$ of dimension $2\mathbf{a}(w_0 d^{\mathfrak{p}})$.*

Proof. The proof is based on an extension of the computation in the proof of [54, Proposition 4.4]. The parabolic Verma module $\Delta_e^{\mathfrak{p}}$ has a resolution $\mathcal{X}_\bullet$ by Verma modules which is obtained by parabolically inducing the BGG resolution, see [4], of the trivial module over the semi-simple subalgebra of the Levi factor of $\mathfrak{p}$. In more detail, for $i \in \mathbb{Z}$, we have

$$\mathcal{X}_i \cong \begin{cases} \bigoplus_{w \in W^{\mathfrak{p}},\, \ell(w)=|i|} \Delta_w, & \text{if } \ 0 \leq -i \leq \ell(w_0^{\mathfrak{p}}); \\ 0, & \text{else.} \end{cases}$$

Applying $\mathcal{L}\top_{w_0}$, maps each Δ_w appearing in this resolution to $\nabla_{w_0 w}$ which results in a coresolution of $L_{w_0 w_0^{\mathfrak{p}}}[\ell(w_0^{\mathfrak{p}})] \cong \nabla_{w_0 w_0^{\mathfrak{p}}}^{w_0(\mathfrak{p})}[\ell(w_0^{\mathfrak{p}})]$ by dual Verma modules. Here the fact that we hit the correct coresolution follows, for example, from the uniqueness of the latter, see, for example, [49, Theorem 33]. Note that here we change the parabolic subalgebra $\mathfrak{p}$ to another parabolic subalgebra which we denote by $w_0(\mathfrak{p})$. The simple reflections defining

$w_0(\mathfrak{p})$ are exactly the w_0-conjugates of the simple reflections that define $\mathfrak{p}$. In particular, we have

$$w_0 w_0^{\mathfrak{p}} = (w_0 w_0^{\mathfrak{p}} w_0) w_0 = w_0^{w_0(\mathfrak{p})} w_0.$$

Note that here $w_0^{w_0(\mathfrak{p})}$ is the longest element of the parabolic subgroup $W^{w_0(\mathfrak{p})}$ of W. For example, in type A, the action of w_0 corresponds to the unique non-trivial automorphism of the Dynkin diagram of the root system.

Now we observe that $L_{w_0 w_0^{\mathfrak{p}}}[\ell(w_0^{\mathfrak{p}})] \cong \nabla^{w_0(\mathfrak{p})}_{w_0 w_0^{\mathfrak{p}}}[\ell(w_0^{\mathfrak{p}})] \cong \Delta^{w_0(\mathfrak{p})}_{w_0 w_0^{\mathfrak{p}}}[\ell(w_0^{\mathfrak{p}})]$ is exactly the simple parabolic Verma module in $\mathcal{O}_0^{w_0(\mathfrak{p})}$. Therefore, similarly to $\Delta^{\mathfrak{p}}_e$, the module $\Delta^{w_0(\mathfrak{p})}_{w_0 w_0^{\mathfrak{p}}}[\ell(w_0^{\mathfrak{p}})]$ has a resolution, which we denote by $\mathcal{Y}_\bullet$, by Verma modules. In more detail, for $i \in \mathbb{Z}$, we have

$$\mathcal{Y}_i \cong \begin{cases} \bigoplus_{w \in W^{w_0(\mathfrak{p})},\, \ell(w) = |i| - \ell(w_0^{\mathfrak{p}})} \Delta_{w w_0 w_0^{\mathfrak{p}}}, & \text{if } \ell(w_0^{\mathfrak{p}}) \leq -i \leq 2\ell(w_0^{\mathfrak{p}}); \\ 0, & \text{else.} \end{cases}$$

Applying $\mathcal{L}\top_{w_0}$, maps each Δ_w appearing in this resolution to $\nabla_{w_0 w}$ which results in a coresolution of $\nabla^{\mathfrak{p}}_e[2\ell(w_0^{\mathfrak{p}})]$ by dual Verma modules. Note that here we come back to the parabolic subalgebra $\mathfrak{p}$ since $w_0^2 = e$.

The above shows that $\mathbb{S}(\Delta^{\mathfrak{p}}_e) \cong \nabla^{\mathfrak{p}}_e[2\ell(w_0^{\mathfrak{p}})]$. Recall that $\mathbb{S}$ is a composition of derived twisting functors and the latter commute with projective functors. Hence, $\mathbb{S}$ commutes with projective functors. For $w \in \mathcal{R}^{\mathfrak{p}}$, we note that

$$\theta_w(\Delta^{\mathfrak{p}}_e) \cong \theta_w(\nabla^{\mathfrak{p}}_e) \cong \theta_w L_{d^{\mathfrak{p}}} \cong I^{\mathfrak{p}}_w \cong P^{\mathfrak{p}}_w.$$

Therefore, applying such θ_w to $\mathbb{S}(\Delta^{\mathfrak{p}}_e) \cong \nabla^{\mathfrak{p}}_e[2\ell(w_0^{\mathfrak{p}})]$, we obtain $\mathbb{S}(P^{\mathfrak{p}}_w) \cong P^{\mathfrak{p}}_w[2\ell(w_0^{\mathfrak{p}})]$.

Finally, note that $\ell(w_0^{\mathfrak{p}}) = \mathbf{a}(w_0^{\mathfrak{p}})$ as $w_0^{\mathfrak{p}}$ is the longest element of a parabolic subalgebra. Also, we have $\mathbf{a}(w_0^{\mathfrak{p}}) = \mathbf{a}(w_0 w_0^{\mathfrak{p}} w_0)$ as conjugation by w_0 is an automorphism of the root system. We can write $w_0 w_0^{\mathfrak{p}} w_0 = w_0(w_0^{\mathfrak{p}} w_0)$ and note that $w_0^{\mathfrak{p}} w_0 \sim_R d^{\mathfrak{p}}$ implies that $w_0(w_0^{\mathfrak{p}} w_0) \sim_R w_0 d^{\mathfrak{p}}$, see [6, Proposition 6.2.9]. Hence $\mathbf{a}(w_0^{\mathfrak{p}}) = \mathbf{a}(w_0 d^{\mathfrak{p}})$ and the claim follows. □

Proposition 3. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. In the graded picture, we have*

$$\mathbb{S}(\theta_w L_{d^{\mathfrak{p}}}) \cong \theta_w L_{d^{\mathfrak{p}}} \langle 2\big(\mathbf{a}(d^{\mathfrak{p}}) - \mathbf{a}(w_0 d^{\mathfrak{p}})\big)\rangle [2\mathbf{a}(w_0 d^{\mathfrak{p}})].$$

Proof. Recall that twisting functors are acyclic on Verma modules, see [22, Theorem 2.2]. Therefore application of derived twisting functors to complexes whose components have Verma filtration reduces to application of the ordinary twisting functors. We will use this frequently in the following proof.

In $\Delta_e^{\mathfrak{p}}$, we have the subquotient $L_{d^{\mathfrak{p}}}\langle -\mathbf{a}(d^{\mathfrak{p}})\rangle$ of interest and the top of each $\mathcal{X}_i$ lives in degree $-i$. The application of $\top_{w_0}$ to $\mathcal{X}_i$ results in a module that has its socle in degree $-i$. In particular, we have that $\top_{w_0}\mathcal{X}_{-\ell(w_0^{\mathfrak{p}})}$, which is isomorphic to $\nabla_{w_0 w_0^{\mathfrak{p}}}$ as an ungraded module, has its socle $L_{w_0 w_0^{\mathfrak{p}}}$ in degree $\ell(w_0^{\mathfrak{p}})$, that is, this subquotient is $L_{w_0 w_0^{\mathfrak{p}}}\langle -\ell(w_0^{\mathfrak{p}})\rangle$.

If we now take $\mathcal{Y}_\bullet$ as the graded Verma resolution of $L_{w_0 w_0^{\mathfrak{p}}}\langle -\ell(w_0^{\mathfrak{p}})\rangle[\ell(w_0^{\mathfrak{p}})]$, then, again, the top of each $\mathcal{Y}_i$ lives in degree $-i$. Again, the application of $\top_{w_0}$ to $\mathcal{Y}_i$ results in a module that has its socle in degree $-i$. So, now we have $\top_{w_0}\mathcal{Y}_{-2\ell(w_0^{\mathfrak{p}})}$, which is isomorphic to ∇_e as an ungraded module, has its socle L_e in degree $2\ell(w_0^{\mathfrak{p}})$, that is, comes with a $-2\ell(w_0^{\mathfrak{p}})$ shift. Taking this shift into account, the subquotient $L_{d^{\mathfrak{p}}}$ of interest now lives in degree $2\ell(w_0^{\mathfrak{p}}) - \mathbf{a}(d^{\mathfrak{p}})$, in other words, this subquotient is $L_{d^{\mathfrak{p}}}\langle \mathbf{a}(d^{\mathfrak{p}}) - 2\ell(w_0^{\mathfrak{p}})\rangle$. Adding all this up, we get the total shift of $2\big(\mathbf{a}(d^{\mathfrak{p}}) - \ell(w_0^{\mathfrak{p}})\big)$ for our $L_{d^{\mathfrak{p}}}$ subquotient.

During the proof of Proposition 2 we established that $\ell(w_0^{\mathfrak{p}}) = \mathbf{a}(w_0 d^{\mathfrak{p}})$. Hence the claim of our proposition follows. □

3.5. Serre functor on simple modules

We make the following general observation on the value of the Serre functor $\mathbb{S}$ on simple modules.

Lemma 4. *Let $x, w \in W$ and $i \in \mathbb{Z}$ be such that $\mathcal{D}^b(\mathcal{O}_0)(P_x[i], \mathbb{S}(L_w)) \neq 0$. Then $x \sim_H w$ or $x <_J w$.*

Proof. From [22, Section 6] it follows that, applying $\mathcal{L}\top_s$, where $s \in S$, to a simple module L_a, for some $a \in W$, results in a complex whose homology only contains simple subquotients of the form L_b, where $b \leq_L a$. Consequently, any finite composition of functors of the form $\mathcal{L}\top_s$, for various $s \in S$, has a similar property: applying such composition to L_a results in a complex whose homology only contains simple subquotients of the form L_b, where $b \leq_L a$. As we know that $\mathbb{S}$ is a composition of functors of the form $\mathcal{L}\top_s$, where $s \in S$, the property $x \leq_L w$ follows.

On the other hand, we also know that $\mathbb{S}$ is a composition of the functors of the form $\mathcal{L}C_s$, where $s \in S$. As, for any projective functor θ, any simple subquotient of θL_w is of the form L_y, for some $y \leq_R w$, see [48, Lemma 13], it follows that $x \leq_R w$.

Combining $x \leq_L w$ and $x \leq_R w$, we get the claim of the lemma. □

3.6. Kazhdan-Lusztig combinatorics of type A

Kazhdan-Lusztig combinatorics in type A is especially nice, see [30,19]. Let $\mathfrak{g} = \mathfrak{sl}_n$, then $W = S_n$. The Robinson-Schensted correspondence, see [63], gives rise to a bijection between the elements of S_n and pairs of standard Young tableaux of the same shape, where the shape is a partition of n:

$$\mathbf{RS} : S_n \to \coprod_{\lambda \vdash n} \mathbf{SYT}_\lambda \times \mathbf{SYT}_\lambda,$$

where $\mathbf{SYT}_\lambda$ denotes the set of all standard Young tableaux of shape λ. Let us write $\mathbf{RS}(\pi) = (\mathbf{p}_\pi, \mathbf{q}_\pi)$. Then $\pi \sim_L \sigma$ if and only if $\mathbf{q}_\pi = \mathbf{q}_\sigma$ and $\pi \sim_R \sigma$ if and only if $\mathbf{p}_\pi = \mathbf{p}_\sigma$. Consequently, $\sim_H$ is the equality relation. Furthermore, $\pi \sim_J \sigma$ if and only if the shapes of $\mathbf{p}_\pi$ and $\mathbf{p}_\sigma$ coincide. This means that the two-sided cells of S_n are in bijection with partitions of n. Additionally, the two-sided order $\leq_J$ is given by the dominance order on partitions.

As a consequence of the above description, any two-sided KL-cell in type A contains both the longest element $w_0^{\mathfrak{p}}$ of some parabolic subalgebra $\mathfrak{p}$; an element of the form $w_0 w_0^{\mathfrak{q}}$, for some parabolic subalgebra $\mathfrak{q}$; and an element of the form $w_0^{\mathfrak{r}} w_0$, for some parabolic subalgebra $\mathfrak{r}$.

Another consequence is that the Duflo elements in type A are exactly the involutions.

Let $\mathcal{R}$ be a right KL-cell and d a Duflo element in $\mathcal{R}$. Then the additive closure $\mathbf{C}_{\mathcal{R}}$ of all modules in $\mathcal{O}$ of the form $\theta_w L_d$, where $w \in \mathcal{R}$, is stable under the action of $\mathscr{P}$. It is called the *cell birepresentation* of $\mathscr{P}$ corresponding to $\mathcal{R}$, see [48] for detail. A very special feature of type A is the following: given two right KL-cells $\mathcal{R}$ and $\mathcal{R}'$ inside the same two-sided KL-cell, the corresponding cell birepresentations $\mathbf{C}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}'}$ are biequivalent, see [55,48]. The biequivalence can be constructed fairly explicitly using recursive application of the derived twisting functors $\mathcal{L}\top_s$, where $s \in S$, which functorially commute with projective functors, followed by projections onto the appropriate cell birepresentations, see [55] for details.

As a consequence of this biequivalence, we have the following in type A: given a two-sided cell $\mathcal{J}$ and two elements $x, y \in \mathcal{J}$, the module $\theta_x L_y$ is either zero or indecomposable. Moreover, in the latter case, we have $\theta_x L_y \cong \theta_w L_d$, where d is the Duflo element in the right KL-cell $\mathcal{R}$ of y and w is the unique element in the intersection of $\mathcal{R}$ with the left KL-cell of x. This gives us substantial flexibility allowing to rewrite the module $\theta_w L_d$, which appears in the formulation of Theorem 1, in a different form $\theta_x L_y$ that will make the proof in the next subsection work.

3.7. Proof of Theorem 1 in type A

In this subsection, we will prove Theorem 1 in type A.

Let d be an involution in S_n (i.e. a Duflo element in S_n). Consider its left KL-cell $\mathcal{L}$, its right KL-cell $\mathcal{R}$ and its two-sided KL-cell $\mathcal{J}$. We want to prove Formula (1), for any $w \in \mathcal{R}$.

There is a parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$ such that $w_0^{\mathfrak{p}} w_0$ belongs to $\mathcal{J}$. Let $\mathcal{R}'$ be the right KL-cell of this element $w_0^{\mathfrak{p}} w_0$ and let d' be the Duflo element in $\mathcal{R}'$. From Proposition 3 we know that Formula (1) is true for $d = d'$ and for any $w \in \mathcal{R}'$.

As mentioned in the previous paragraph, the cell birepresentations $\mathbf{C}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}'}$ are biequivalent. Let $\Phi : \mathbf{C}_{\mathcal{R}} \to \mathbf{C}_{\mathcal{R}'}$ be such a biequivalence and Φ^{-1} be its inverse. Then both Φ and Φ^{-1} functorially commute with all projective functors.

Now recall that our Serre functor $\mathbb{S}$ is a composition of the derived shuffling functors $\mathcal{L}C_s$, where $s \in S$. The functor $\mathcal{L}C_s$ is represented by the complex $0 \to \theta_e \to \theta_s \to 0$, where the map $\theta_e \to \theta_s$ is the adjunction morphism. In other words, this is a complex consisting of projective functors and morphisms between them. Consequently, both Φ and Φ^{-1} commute with the functor given by this complex. This means that both Φ and Φ^{-1} commute with $\mathbb{S}$. Here we note that Φ and Φ^{-1} are only equivalences between certain subquotients of $\mathcal{O}_0$, but not of the whole $\mathcal{O}_0$, so we cannot directly use that Serre functors commute with any auto-equivalence of $\mathcal{O}_0$.

Since $\Phi(\theta_w L_d) \cong \theta_x L_{d'}$, for some $x \in \mathcal{R}'$ and Φ^{-1} commutes with $\mathbb{S}$, using Proposition 3 we have:

$$\begin{aligned}
\mathbb{S}(\theta_w L_d) &\cong \mathbb{S}(\Phi^{-1}(\theta_x L_{d'})) \\
&\cong \Phi^{-1}(\mathbb{S}(\theta_x L_{d'})) \\
&\cong \Phi^{-1}(\theta_x L_{d'}\langle 2\big(\mathbf{a}(d') - \mathbf{a}(w_0 d')\big)\rangle[2\mathbf{a}(w_0 d')]) \\
&\cong \theta_w L_d\langle 2\big(\mathbf{a}(d') - \mathbf{a}(w_0 d')\big)\rangle[2\mathbf{a}(w_0 d')].
\end{aligned}$$

Note that $\mathbf{a}(d) = \mathbf{a}(d')$ and $\mathbf{a}(w_0 d) = \mathbf{a}(w_0 d')$ since $d \sim_J d'$. This completes the proof of Theorem 1 in type A.

3.8. Subcategories $\mathcal{O}^{\hat{\mathcal{R}}}$

For a right KL-cell $\mathcal{R}$ in W, denote by $\hat{\mathcal{R}}$ the set of all elements $w \in W$ such that $w \leq_R \mathcal{R}$. Denote by $\mathcal{O}^{\hat{\mathcal{R}}}$ the Serre subcategory of $\mathcal{O}_0$ generated by all simples L_w, where $w \in \hat{\mathcal{R}}$. The category $\mathcal{O}^{\hat{\mathcal{R}}}$ appears in [53,46]. This category is stable under the action of $\mathscr{P}$. The inclusion $\mathcal{O}^{\hat{\mathcal{R}}} \subset \mathcal{O}_0$ is exact and hence has two adjoints: the left adjoint $Z^{\hat{\mathcal{R}}}$ given by taking the maximal quotient in $\mathcal{O}^{\hat{\mathcal{R}}}$ and the right adjoint $Z_{\hat{\mathcal{R}}}$ given by taking the maximal submodule in $\mathcal{O}^{\hat{\mathcal{R}}}$.

As the natural inclusion $\mathcal{O}^{\hat{\mathcal{R}}} \subset \mathcal{O}_0$ is a morphism of birepresentations of $\mathscr{P}$, both $Z^{\hat{\mathcal{R}}}$ and $Z_{\hat{\mathcal{R}}}$ are also morphisms of birepresentations of $\mathscr{P}$, that is, they both functorially commute with projective functors.

3.9. Further special cases of Theorem 1

The proof in Subsection 3.7 can be (substantially) refined to establish the following.

Proposition 5. *Assume that $\mathfrak{g}$ is any semi-simple finite dimensional complex Lie algebra and $d \in \mathbf{D}$ is such that there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with the property $d \sim_J w_0^{\mathfrak{p}} w_0$. Then, for any $w \in W$ such that $w \sim_R d$, the object $\theta_w L_d$ is a Calabi-Yau object in $\mathcal{D}^b(\mathcal{O}_0)$ of dimension $2\mathbf{a}(w_0 d)$. Moreover, in the graded picture, we have*

$$\mathbb{S}(\theta_w L_d) \cong \theta_w L_d\langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)].$$

3.10. Proof of Proposition 5

Let $\mathcal{R}$ be the right KL-cell of d and $\mathcal{J}$ the two-sided KL-cell of d. Let d' be a Duflo element such that $d \sim_J d'$ and $d' \sim_R w_0^{\mathfrak{p}} w_0$, as assumed in the formulation. Let $\mathcal{R}'$ be the right KL-cell of d'. Then, from Proposition 3 we know that Formula (1) is true for $d = d'$ and for any $w \in \mathcal{R}'$. Assume now that $d \neq d'$.

Since we are outside type A, the birepresentations $\mathbf{C}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}'}$ are not biequivalent, in general. However, we still do have homomorphisms between these birepresentations, that is functors between the underlying additive categories that commute with the action of projective functors. The problem is that these homomorphisms are no longer equivalences, for example, they might send indecomposable objects to decomposable objects etc.

Note that both birepresentations are *transitive* in the sense that, for any non-zero object X of this birepresentation, the additive closure of $\mathscr{P}X$ coincides with the whole birepresentation. Consequently, any non-zero homomorphism from such a birepresentation must be non-zero on any non-zero object. Also, the additive closure of the image of any non-zero homomorphism to a transitive birepresentation coincides with the whole birepresentation.

Lemma 6. *In the situation described above, there exist non-zero homomorphisms* Φ *of birepresentations of* $\mathscr{P}$ *from* $\mathbf{C}_{\mathcal{R}'}$ *and* $\mathbf{C}_{\mathcal{R}}$.

Proof. Such homomorphism can be constructed using twisting functors similar to the corresponding equivalences in type A that are constructed in the proof of [54, Theorem 5.4(ii)]. Let us now describe the details of how this works.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two different right KL-cells inside the same two-sided KL-cell $\mathcal{J}$. For a simple reflection s, consider the corresponding functor $\top_s$. This functor commutes with the action of $\mathscr{P}$ and hence the composition $Z_{\hat{\mathcal{R}}_2} \circ \top_s$ commutes with the action of $\mathscr{P}$ as well.

Now, choose s such that $sw < w$, for $w \in \mathcal{R}_1$ (we note that all elements in $\mathcal{R}_1$ have the same left descent). Then, for $w \in \mathcal{R}_1$, the module $\top_s L_w$ has top L_w and a semi-simple radical described in [22, Sections 6 and 7]. As $\mathcal{R}_1 \neq \mathcal{R}_2$, we have $L_w \notin \mathcal{O}^{\hat{\mathcal{R}}_2}$ since any two right KL-cells inside the same two-sided KL-cell are not comparable with respect to the right KL-order. This implies that $Z_{\hat{\mathcal{R}}_2} \circ \top_s(L_w) = Z_{\hat{\mathcal{R}}_2}(\mathrm{Rad}(\top_s(L_w)))$.

Assume that L_x is a summand of $\mathrm{Rad}(\top_s(L_w))$. If $x \notin \mathcal{J}$, then $\theta_y L_x = 0$, for any $y \in \mathcal{J}$. This implies that, for $y \in \mathcal{J}$, the module $Z_{\hat{\mathcal{R}}_2} \circ \top_s(\theta_y L_w)$ is a direct sum of modules of the form $\theta_y L_x$, where $x \in \mathcal{R}_2$ is such that L_x appears in $\mathrm{Rad}(\top_s(L_w))$. In other words, the functor $Z_{\hat{\mathcal{R}}_2} \circ \top_s$ maps objects of $\mathbf{C}_{\mathcal{R}_1}$ to objects of $\mathbf{C}_{\mathcal{R}_2}$.

It remains to recall that the left KL order is generated by the relation connecting w to all x such that L_x appears in the radical of $\top_s L_w$, for some $s \in S$, cf. [30] and [22, Sections 7]. Therefore, starting from $\mathcal{R}_1$ we can find some s and some $\mathcal{R}_2 \neq \mathcal{R}_1$ such that the above gives a non-zero functor from $\mathbf{C}_{\mathcal{R}_1}$ to $\mathbf{C}_{\mathcal{R}_2}$. Then, repeating this finitely many

times, we eventually construct a non-zero functor from any $\mathbf{C}_{\mathcal{R}_1}$ to any other $\mathbf{C}_{\mathcal{R}_2}$, for $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\mathcal{J}$. □

Let $\Phi : \mathbf{C}_{\mathcal{R}'} \to \mathbf{C}_{\mathcal{R}}$ be a functor given by Lemma 4. If we apply it to the direct sum X of all $\theta_u L_{d'}$, where $u \in \mathcal{R}'$, we will get a direct sum of all $\theta_w L_d$, where $w \in \mathcal{R}$, each appearing with a non-zero multiplicity. Applying Φ (which commutes with $\mathbb{S}$) to

$$\mathbb{S}(X) \cong X\langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)],$$

and using the additivity of $\mathbb{S}$, we obtain that $\mathbb{S}(\theta_w L_d)$ is a direct sum the objects of the form

$$\theta_v L_d\langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)],$$

where $v \in \mathcal{R}$, with some multiplicities.

To complete the proof, we only need to argue that all of these have multiplicity 0 apart from $v = w$, which has multiplicity 1. Recall that $\mathbb{S}$ is an auto-equivalence. This implies that $\mathbb{S}(\theta_w L_d)$ is indecomposable, since $\theta_w L_d$ is indecomposable. In other words,

$$\mathbb{S}(\theta_w L_d) \cong \theta_v L_d\langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)],$$

for some $v \in \mathcal{R}$, and it remains to show that $v = w$.

Next we observe that the Serre functor $\mathbb{S}$ defines the identity map at the level of the ungraded Grothendieck group of $\mathcal{D}^b(\mathcal{O}_0)$. This is because $\mathbb{S}$ maps each P_w to I_w and these two modules have the same character as they are connected by the simple preserving duality $\star$. Moreover, the representatives of the modules P_w, for $w \in W$, form a basis of the ungraded Grothendieck group as $\mathcal{O}_0$ has finite global dimension.

Therefore the representatives of the objects $\mathbb{S}(\theta_w L_d)$ and $\theta_w L_d$ in the ungraded Grothendieck group of $\mathcal{D}^b(\mathcal{O}_0)$ must coincide. Next we note that $\theta_d L_d$ is the only one of the modules $\theta_w L_d$, where $w \in \mathcal{R}$, which has L_e as a simple subquotient. Consequently, $\mathbb{S}(\theta_d L_d)$ is isomorphic to $\theta_d L_d\langle 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)]$.

Lemma 7. *Assume that we are under the same assumptions as Proposition 5, that is, there exists a parabolic subalgebra $\mathfrak{p}$ such that $d \sim_J w_0^{\mathfrak{p}} w_0$. Then, for $w \in W$ such that $w \sim_H d$ and $i, j \in \mathbb{Z}$, we have*

$$\mathcal{D}^b(\mathcal{O}_0)\big(P_w\langle i\rangle[j], \mathbb{S}(L_d)\big) = \begin{cases} \mathbb{C}, & w = d, i = 2\big(\mathbf{a}(d) - \mathbf{a}(w_0 d)\big), j = 2\mathbf{a}(w_0 d); \\ 0, & \textit{else.} \end{cases}$$

Proof. Assume that some $L_w\langle i\rangle[j]$ appears as a subquotient of a homology of $\mathbb{S}(L_d)$. By Lemma 4, we either have $w \sim_H d$ or $w <_J d$. In the case $w <_J d$, we have that $\theta_d L_w = 0$. In the case $w \sim_H d$, we have $\theta_d L_w \cong \theta_w L_d$, combining [40, Proposition 4.15] with [40, Formulae (4.11) and (4.12)]. Now the claim of the lemma follows from the fact

that $\mathbb{S}(\theta_d L_d)$ is isomorphic to $\theta_d L_d\langle 2\big(\mathbf{a}(d)-\mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)]$ which we established above. □

Lemma 7 says that, modulo simple modules indexed by cells that are strictly smaller than $\mathcal{J}$ with respect to the two-sided KL-order, the only simple constituent of the homology of the complex $\mathbb{S}(L_d)$ is $L_d\langle 2\big(\mathbf{a}(d)-\mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)]$. Now we can complete the proof of Theorem 5 as, for $w\in\mathcal{R}$, we have $\theta_w L_x=0$, for all $x\leq_J w$.

Indeed, for $w\in\mathcal{R}$, applying θ_w to $\mathbb{S}(L_d)$, we obtain that $\mathbb{S}(\theta_w L_d)\cong\theta_w\mathbb{S}(L_d)$ is isomorphic to $\theta_w L_d\langle 2\big(\mathbf{a}(d)-\mathbf{a}(w_0 d)\big)\rangle[2\mathbf{a}(w_0 d)]$. This proves Proposition 5.

3.11. Proof of Theorem 1 in the general case

Our proof of Proposition 5 leads to a strategy to prove Theorem 1 in the general case. We start by the following generalization of Lemma 7.

Lemma 8. *Under the assumptions of Theorem 1, the claim of Lemma 7 is true for any $d\in\mathbf{D}$. Namely, for any $d\in\mathbf{D}$ and $w\in W$ such that $w\sim_H d$ and any $i,j\in\mathbb{Z}$, we have*

$$\mathcal{D}^b(\mathcal{O}_0)\big(P_w\langle i\rangle[j],\mathbb{S}(L_d)\big)=\begin{cases}\mathbb{C}, & w=d, i=2\big(\mathbf{a}(d)-\mathbf{a}(w_0 d)\big), j=2\mathbf{a}(w_0 d);\\ 0, & \textit{else.}\end{cases}$$

Proof. We start by noticing that

$$\mathcal{D}^b(\mathcal{O}_0)\big(P_w\langle i\rangle[j],\mathbb{S}(L_d)\big)\cong\mathcal{D}^b(\mathcal{O}_0)\big(\mathbb{S}(L_d),I_w\langle i\rangle[j]\big).$$

Recall that $\mathbb{S}$ is an equivalence and let $\mathbb{S}^{-1}$ be the inverse equivalence. Then we have

$$\mathcal{D}^b(\mathcal{O}_0)\big(\mathbb{S}(L_d),I_w\langle i\rangle[j]\big)\cong\mathcal{D}^b(\mathcal{O}_0)\big(L_d,\mathbb{S}^{-1}(I_w\langle i\rangle[j])\big).$$

Also, recall that $\mathbb{S}$, being the derived of the Nakayama functor, maps P_w to I_w. Hence

$$\mathcal{D}^b(\mathcal{O}_0)\big(\mathbb{S}(L_d),I_w\langle i\rangle[j]\big)\cong\mathcal{D}^b(\mathcal{O}_0)\big(L_d\langle -i\rangle[-j],P_w\big).$$

Now we recall from [46] that the injective dimension of P_w equals $2\mathbf{a}(w_0 w)$. At the same time, A is Auslander regular by [33]. Therefore, assuming

$$\mathcal{D}^b(\mathcal{O}_0)\big(L_d\langle -i\rangle[-j],P_w\big)\neq 0 \tag{2}$$

implies $j\geq 2\mathbf{a}(w_0 w)$ since d and w belong to the same two-sided cell and hence the projective dimension of I_d equals $2\mathbf{a}(w_0 w)$ as well. Consequently, Formula (2) implies $j=2\mathbf{a}(w_0 w)$.

Next we look at the complex $\theta_d\mathbb{S}(L_d)\cong\mathbb{S}(\theta_d L_d)$. The application of θ_d will kill all simple constituents of the homology of $\mathbb{S}(L_d)$ which come from the cell that are two-sided smaller than the two-sided cell $\mathcal{J}$ containing d (note that $\mathcal{J}$ contains w as well).

Therefore the previous paragraph implies that all homology of $\theta_d\mathbb{S}(L_d)$ is concentrated at the same homological position, namely at the position $-2\mathbf{a}(w_0w)$. In other words, $\theta_d\mathbb{S}(L_d)$ is just a module shifted by $2\mathbf{a}(w_0w)$ in the homological position.

As $\theta_d L_d$ is indecomposable and $\mathbb{S}$ is an equivalence, it follows that the homology of $\theta_d\mathbb{S}(L_d)$ must be indecomposable. At the level of the ungraded Grothendieck group, the application of $\mathbb{S}$ gives the identity map (since $w_0^2 = e$ in W). Therefore the ungraded homology of $\theta_d\mathbb{S}(L_d)$ at position $-2\mathbf{a}(w_0w)$ is isomorphic to $\theta_d L_d$. As the latter is not isomorphic to any $\theta_d L_u$, where $u \sim_H d$ and $d \neq u$ (for example, since $[\theta_d L_d : L_e] \neq 0$ while $[\theta_d L_u : L_e] = 0$), it follows that Formula (2) implies $w = d$.

It remains to show that $i = 2\big(\mathbf{a}(d) - \mathbf{a}(w_0d)\big)$. For this, we recall from [46, Theorem 16] that the collection $\{\theta_x L_y : x, y \in W\}$ of modules is Koszul-Ringel self-dual and this Koszul-Ringel self-duality swaps $\theta_x L_y$ with $\theta_{y^{-1}w_0} L_{w_0x^{-1}}$. This Koszul-Ringel self-duality, denoted K, provides two different realizations of ${}^{\mathbb{Z}}\mathcal{O}_0$ inside $\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)$ and $\mathbb{S}$ is a Serre functor on the latter category. The functor K is a self-equivalence of $\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)$ and hence commutes with $\mathbb{S}$. Moreover, the functor K commutes with the shifts in the following way: $\mathrm{K}\langle a\rangle[b] = \langle -a\rangle[a+b]\mathrm{K}$, see [51, Proposition 20]. For $w \in W$ and d_w the Duflo element in the right KL-cell of w, denote by $i_w \in \mathbb{Z}$ the shift such that

$$\mathbb{S}(\theta_w L_{d_w}) \cong \theta_w L_{d_w}\langle i_w\rangle[2\mathbf{a}(w_0d_w)].$$

Applying K, we get

$$\mathbb{S}(\theta_{d_ww_0} L_{w_0w^{-1}}) \cong \theta_{d_ww_0} L_{w_0w^{-1}}\langle -i_w\rangle[i_w + 2\mathbf{a}(w_0d_w)].$$

As we already know from the above that the homological shift $i_w + 2\mathbf{a}(w_0d_w)$ on the right hand side of the latter isomorphism should be equal to $2\mathbf{a}(w_0(w_0w^{-1})) = 2\mathbf{a}(w)$, we deduce that $i_w = 2\mathbf{a}(w) - 2\mathbf{a}(w_0d_w)$. As we always have $\mathbf{a}(w) = \mathbf{a}(d_w)$, the claim that $i = 2\big(\mathbf{a}(d) - \mathbf{a}(w_0d)\big)$ follows. This completes the proof of our lemma. □

To prove Theorem 1, we apply θ_w to $\mathbb{S}(L_d)$. Lemma 8 tells us that $\mathbb{S}(L_d)$ has a unique simple subquotient L_d in homology in the correct homological position with the correct grading shift, while all other simple subquotients of the homology of $\mathbb{S}(L_d)$ are killed by θ_w. Also, $\mathbb{S}$ functorially commutes with θ_w and hence $\theta_w\mathbb{S}(L_d) \cong \mathbb{S}(\theta_w L_d)$. Therefore the claim of Theorem 1 follows.

3.12. Homological consequences of Theorem 1

The following corollary is inspired by [33, Theorem A].

Corollary 9. *Let $x, y \in W$ and $i \in \mathbb{Z}$. Then we have:*

(a) $\mathrm{Ext}^i(I_x, L_y) = 0$ *unless* $i \leq 2\mathbf{a}(w_0x)$ *and* $y \geq_L x$.
(b) $\mathrm{Ext}^i(I_x, L_y) = 0$ *if* $i < 2\mathbf{a}(w_0x)$ *and* $y \sim_L x$.

(c) $\mathrm{Ext}^{2\mathbf{a}(w_0x)}(I_x, L_y) = 0$ *if* $y \sim_L x$ *and* $y \neq x$.
(d) $\mathrm{Ext}^{2\mathbf{a}(w_0x)}(I_x, L_x) = \mathbb{C}$, *more precisely, in the graded picture, we have*

$$\mathrm{ext}^{2\mathbf{a}(w_0x)}(I_x, L_x\langle 2(\mathbf{a}(x) - \mathbf{a}(w_0x))\rangle) = \mathbb{C}.$$

(e) $\mathrm{Ext}^{2\mathbf{a}(w_0x)}(I_x, L_y) = 0$ *provided that* $y >_L x$.

Proof. In Claim (a), the restriction $i \leq 2\mathbf{a}(w_0x)$ follows from [44, Theorem 16] and [46, Theorem 20]. To prove the second restriction, let $\mathcal{X}_\bullet = \mathcal{P}_\bullet(I_e)$. Then $\theta_x I_e \cong I_x$ and hence $\theta_x\mathcal{X}_\bullet$ is a projective resolution (not necessarily minimal) of I_x. For any $w \in W$, all summands of $\theta_x I_w$ are of the form I_z, where $z \geq_L x$. This means that only such summands appear in $\theta_x\mathcal{X}_\bullet$ and implies the second restriction in Claim (a).

Claim (b) follows from [33, Theorem A].

To prove the rest, for $w \in W$, consider the Duflo element d in the left KL-cell of w and the Duflo element d' in the right KL-cell of w. Then $\theta_d L_w \cong \theta_w L_{d'}$ by [40, Proposition 4.15].

Consider the complex $\mathbb{S}(L_w)$. Let L_z, for $z \in W$, be a simple constituent of some homology of $\mathbb{S}(L_w)$. Then either $z \sim_H w$ or $z <_J w$ by Lemma 4. In the case $z <_J w$, we have $\theta_d L_z = 0$ by [48, Lemma 12]. If $z \sim_H w$, then, by the previous paragraph, $\theta_d L_z \cong \theta_z L_{d'}$.

Therefore, from Theorem 1 it follows that, among all homologies of $\mathbb{S}(L_w)$, only the homology at the homological position $-2\mathbf{a}(w_0w)$ contains a simple constituent L_z such that $z \sim_H w$, moreover, such z is unique and, in fact, for this unique z we have $z = w$.

Since this constituent can be detected by taking homomorphism from $\mathbb{S}(L_w)$ to the complex $I_w[2\mathbf{a}(w_0w)]$, applying the inverse of $\mathbb{S}$ to this hom-space, we get the dual of Claim (c) and the dual of the ungraded formula in Claim (d). The graded formula in Claim (d) follows from Lemma 7.

Finally, to prove Claim (e), let us assume $\mathrm{Ext}^{2\mathbf{a}(w_0x)}(I_x, L_y) \neq 0$ for some $y >_L x$. Since the projective dimension of I_x equals $2\mathbf{a}(w_0x)$, this non-zero extension induces a non-zero extension $\mathrm{Ext}^{2\mathbf{a}(w_0x)}(I_x, P_y) \neq 0$. But this, in turn, implies that the injective dimension $2\mathbf{a}(w_0y)$ of P_y is at least $2\mathbf{a}(w_0x)$. This, however, contradicts the strict monotonicity of the $\mathbf{a}$ function as we assumed $y >_L x$. □

3.13. Degrees

Denote by $\mathbf{S}$ the set of all integers of the form $2\mathbf{a}(w_0d)$, where $d \in \mathbf{D}$. For $i \in \mathbf{S}$, consider the following subcategories of $\mathcal{O}_0$:

- the full subcategory $\mathbf{CY}_i$ of $\mathcal{O}_0$ consisting of all objects $M \in \mathcal{O}_0$ such that $\mathbb{S}(M) \cong M[i]$;
- the full subcategory $\mathbf{PCY}_i$ of $\mathcal{O}_0$ consisting of all objects $M \in \mathcal{O}_0$, for which there exists an object $M' \in \mathcal{O}_0$ such that $\mathbb{S}(M) \cong M'[i]$;

- the full subcategory $\mathbf{F}_i$ of $\mathcal{O}_0$ consisting of all objects $M \in \mathcal{O}_0$ which admit a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = M \tag{3}$$

such that each subquotient M_j/M_{j-1} is isomorphic to $\theta_w L_d$, for some $d \in \mathbf{D}$ such that $\mathbf{a}(w_0 d) = i$ and some $w \in W$ such that $w \sim_R d$ (both w and d might depend on j).

The notation **PCY** abbreviates *provisionally Calabi-Yau*. We record the following observations.

Proposition 10. *For $i \in \mathbf{S}$, we have:*

(a) *both $\mathbf{CY}_i \subset \mathbf{PCY}_i$ and $\mathbf{F}_i \subset \mathbf{PCY}_i$;*
(b) *both $\mathbf{CY}_i$, $\mathbf{PCY}_i$ and $\mathbf{F}_i$ are closed with respect to finite direct sums;*
(c) *both $\mathbf{CY}_i$, $\mathbf{PCY}_i$ and $\mathbf{F}_i$ are closed with respect to the action of $\mathscr{P}$;*
(d) *$\mathbf{PCY}_i$ is closed with respect to extensions, kernels of epimorphisms and cokernels of monomorphisms;*
(e) *both $\mathbf{PCY}_i$ and $\mathbf{F}_i$ are exact categories in which the exact structure is defined by the usual short exact sequences.*

Proof. To start with, $\mathbf{CY}_i \subset \mathbf{PCY}_i$ follows from the definitions combined with the fact that $\theta_w L_d \in \mathbf{CY}_i$, for $d \in \mathbf{D}$ such that $\mathbf{a}(w_0 d) = i$ and $w \in W$ such that $w \sim_R d$, given by Theorem 1. Also, Claim (b) follows from the definitions and the additivity of the Serre functor $\mathbb{S}$.

Assume $M, N \in \mathbf{PCY}_i$ and

$$0 \to M \to X \to N \to 0 \tag{4}$$

be a short exact sequence. This sequence corresponds to a distinguished triangle $N[-1] \to M \to X \to N$ in $\mathcal{D}^b(\mathcal{O}_0)$. As $\mathbb{S}$ is triangulated, we have a distinguished triangle

$$\mathbb{S}(N)[-1] \to \mathbb{S}(M) \to \mathbb{S}(X) \to \mathbb{S}(N). \tag{5}$$

We have $\mathbb{S}(M) \cong M'[i]$ and $\mathbb{S}(N) \cong N'[i]$, for some $M', N' \in \mathcal{O}_0$, due to our assumption that $M, N \in \mathbf{PCY}_i$. The distinguished triangle given by (5) implies existence of some $X' \in \mathcal{O}_0$ such that we both have $\mathbb{S}(X) \cong X'[i]$ and also an exact sequence

$$0 \to M' \to X' \to N' \to 0. \tag{6}$$

Consequently, $X \in \mathbf{PCY}_i$. This proves that $\mathbf{PCY}_i$ is extension closed. In particular, it follows that $\mathbf{F}_i$, being the extension closure of the modules of the form $\theta_w L_d \in \mathbf{CY}_i$, for

$d \in \mathbf{D}$ such that $\mathbf{a}(w_0 d) = i$ and $w \in W$ such that $w \sim_R d$, by definition, is a subcategory of $\mathbf{PCY}_i$. This proves Claim (a).

The above proves the first assertion of Claim (d). The two other assertions are proved by the same argument.

Next, Claim (e) for $\mathbf{F}_i$ follows from the definition of $\mathbf{F}_i$, while Claim (e) for $\mathbf{PCY}_i$ follows from Claim (d).

Finally, let us prove Claim (c). If $\mathbb{S}(M) \cong M'[i]$ and θ is a projective functor, then $\theta\mathbb{S}(M) \cong \mathbb{S}(\theta M) \cong \theta M'[i]$ as θ commutes with $\mathbb{S}$. This proves Claim (c) for both $\mathbf{PCY}_i$ and $\mathbf{CY}_i$. To prove Claim (c) for $\mathbf{F}_i$, we just need to show that $\theta\theta_w L_d \in \mathbf{F}_i$, for any $d \in \mathbf{D}$ such that $\mathbf{a}(w_0 d) = i$ and $w \in W$ such that $w \sim_R d$. For this, we note that $\theta\theta_w$ is a direct sum of functors of the form $\theta_{w'}$, where $w' \in W$ such that $w' \sim_R d$, and projective functors from strictly greater two-sided KL-cells. The latter projective functors annihilate L_d, see [48, Lemma 12]. This completes the proof. □

Now we can describe all possible degrees for Calabi-Yau objects in $\mathcal{O}_0$.

Corollary 11. *Let $i \in \mathbb{Z}$ and $0 \neq M \in \mathcal{O}_0$ be such that $\mathbb{S}(M) \cong M[i]$. Then there exists $d \in \mathbf{D}$ such that $i = 2\mathbf{a}(w_0 d)$.*

Proof. Let $x \in W$ be such that L_x is a subquotient of M and, for any $y \in W$ such that L_y is a subquotient of M, we have $y \not\geq_J x$. Let d be the Duflo element in the left KL-cell of x. Then, for any $y \in W$ such that L_y is a subquotient of M, we either have $\theta_d L_y = 0$ or $y \sim_L x$ by [48, Lemma 12].

For $y \sim_L x$, let d' be the Duflo element in the right KL-cell of y. Then $\theta_d L_y$ belongs to the additive closure of the modules of the form $\theta_w L_{d'}$, for $w \sim_R d'$, see [40, Proposition 4.15]. This implies that, for $i = 2\mathbf{a}(w_0 d')$, we have $\theta_d M \in \mathbf{F}_i$. Now the claim of our corollary follows from Proposition 10. □

Corollary 12. *Let $d \in \mathbf{D}$ and $0 \neq M \in \mathcal{O}_0$ be such that $\mathbb{S}(M) \cong M[2\mathbf{a}(w_0 d)]$. Then, if M is graded, we have $\mathbb{S}(M) \cong M\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$.*

Proof. This is proved similarly to the proof of Corollary 11. □

Remark 13. Some further information about the categories $\mathbf{F}_i$, $\mathbf{CY}_i$ and $\mathbf{PCY}_i$ will be obtained latter in Subsection 4.6.

4. Homomorphisms from $\mathbb{S}$ to shifts of the identity

4.1. Motivation from categorical diagonalization

One of the main ideas of categorical diagonalization described in the papers [15,16] is that isomorphisms similar to $\mathbb{S}(M) \cong M\langle 2(\mathbf{a}(x) - \mathbf{a}(w_0 x))\rangle[2\mathbf{a}(w_0 d)]$ should not only exist as abstract isomorphisms, but they should come as evaluations at M of natural

transformations from $\mathbb{S}$ to $\mathrm{Id}\langle 2(\mathbf{a}(x)-\mathbf{a}(w_0x))\rangle[2\mathbf{a}(w_0d)]$. Here the collection $\{\mathrm{Id}\langle t\rangle[s] : s,t\in\mathbb{Z}\}$ is referred to as a *choice of scalars* for the setup in question.

In this section we will take a closer look at the possibility of such phenomena in the context of category $\mathcal{O}$.

4.2. Two extreme cases

Proposition 14. *There is a unique, up to scalar, non-zero natural transformation* $\alpha_{w_0}:\mathbb{S}\to\mathrm{Id}\langle 2(\mathbf{a}(w_0))\rangle$. *Moreover, the evaluation of* α_{w_0} *at* $P_{w_0}\cong\theta_{w_0}L_{w_0}$ *is an isomorphism.*

Proof. We will use that $\mathbb{S}=(\mathcal{L}\top_{w_0})^2$, see Subsection 2.7. For a simple reflection s, consider $\mathcal{L}\top_s$. By definition, the 0-th homology of this functor is $\top_s$. By [32, Theorem 4], see also [53, Proposition 2.3], there is a natural transformation $\top_s\to\mathrm{Id}\langle 1\rangle$ whose evaluation at P_{w_0} is an isomorphism. Composing these natural transformations along some reduced expression of w_0, we get α_{w_0}.

Uniqueness of α_{w_0}, up to scalar, follows from [53, Theorem 6]. □

Proposition 15. *There exists a unique, up to scalar, non-zero natural transformation* $\alpha_e:\mathbb{S}\to\mathrm{Id}\langle -2(\mathbf{a}(w_0))\rangle[2(\mathbf{a}(w_0)]$. *Moreover, the evaluation of* α_e *at* $L_e\cong\theta_eL_e$ *is an isomorphism.*

Proof. We will use that $\mathbb{S}=(\mathcal{L}\top_{w_0})^2$, see Subsection 2.7. For a simple reflection s, let $\mathfrak{p}=\mathfrak{p}_s$ be the (minimal) parabolic subcategory of $\mathfrak{g}$ corresponding to s. Consider the functor $\mathcal{L}\top_s$.

By [22, Theorem 2.2], only the first two homologies of $\mathcal{L}\top_s$ are non-zero. By [53, Theorem 2], the 1-st homology of $\mathcal{L}\top_s$ is isomorphic to the dual Zuckerman functor $Z_{\mathfrak{p}}$. The embedding $\iota_{\mathfrak{p}}$ of $\mathcal{O}_0^{\mathfrak{p}}$ into $\mathcal{O}_0$ gives rise to a natural transformation from $\iota_{\mathfrak{p}}Z_{\mathfrak{p}}$ to $\mathrm{Id}\langle -1\rangle$ whose evaluation at L_e is an isomorphism. Since $\mathcal{L}\top_s$ has no higher homologies, this induces a natural transformation β_s from $\mathcal{L}\top_s$ to $\mathrm{Id}\langle -1\rangle[1]$. Composing these β_s along some reduced expression of w_0, we get α_e.

By adjunction, in the category of functors, we have $\mathrm{Hom}(\iota_{\mathfrak{p}}Z_{\mathfrak{p}},\mathrm{Id})\cong\mathrm{Hom}(Z_{\mathfrak{p}},Z_{\mathfrak{p}})$. The algebra $\mathrm{Hom}(Z_{\mathfrak{p}},Z_{\mathfrak{p}})$ is isomorphic to the endomorphism algebra of the identity endofunctor of $\mathcal{O}_0^{\mathfrak{p}}$, that is, to the center of $A^{\mathfrak{p}}$, by [53, Theorem 6]. The algebra $A^{\mathfrak{p}}$ is positively graded and hence its center is positively graded as well. This center is local as $\mathcal{O}_0^{\mathfrak{p}}$ is connected. This means that the homogeneous part of degree zero in this center is one-dimensional (as $\mathbb{C}$ is algebraically closed). This implies uniqueness of α_e, up to scalar. □

4.3. General case

Theorem 16. *For any* $d\in\mathbf{D}$ *and* $w\in W$ *such that* $w\sim_R d$, *there exists a natural transformation*

$$\alpha_d : \mathbb{S} \to \mathrm{Id}\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$$

such that the evaluation of α_d at $\theta_w L_d$ is an isomorphism.

We expect α_d to be a scalar multiple of $\alpha_{d'}$ if (and, probably, only if) $d \sim_J d'$.

Remark 17. The function $\mathbf{a}$ is constant on two-sided cells and strictly monotone with respect to the two-sided order. However, the two-sided order is not linear in general. It is easy to find examples of W and $x, y \in W$ such that $\mathbf{a}(x) = \mathbf{a}(y)$ but $\mathbf{a}(w_0 x) \neq \mathbf{a}(w_0 y)$. For instance, we can take $W = S_7$ with x in the two-sided cell corresponding to the partition $(5, 1, 1)$ and y in the two-sided cell corresponding to the partition $(4, 3)$. Then $\mathbf{a}(x) = \mathbf{a}(y) = 3$ while $\mathbf{a}(w_0 x) = 10 \neq 9 = \mathbf{a}(w_0 y)$. Thus we might have homomorphisms from $\mathbb{S}$ to Id corresponding to different two-sided cells which live in the same homological position but in different degrees. More details on this example can be found in Subsection 7.3.

I do not know how to extend Theorem 16 to Soergel bimodules over the polynomial algebra.

4.4. Preparation

To prove Proposition 16, we will need some preparation. Essentially, we need to go over the proof of [33, Theorem 3] with a fine-tooth comb.

For any $w \in W$, denote by $\widehat{w}$ the Duflo element in the left KL-cell of w.

Lemma 18. *For any $x, w \in W$, $d \in \mathbf{D}$, $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}$, we have:*

(*i*) $\mathrm{Ext}^k(T_w, L_x) \neq 0$ *implies* $k \leq \mathbf{a}(w)$.
(*ii*) $\mathrm{ext}^k(T_w, L_x\langle i\rangle) \neq 0$ *implies* $-\mathbf{a}(x) - k \leq i \leq \mathbf{a}(x) - k$.
(*iii*) $\mathrm{Ext}^{\mathbf{a}(w)}(T_w, L_x) \neq 0$ *implies* $x \sim_L w_0 w$.
(*iv*) $\mathrm{Ext}^k(T_w, L_x) \neq 0$ *and* $k < \mathbf{a}(w)$ *implies* $\mathbf{a}(x) > \mathbf{a}(w_0 w)$.
(*v*) $\mathrm{ext}^{\mathbf{a}(d)}(T_d, L_{\widehat{w_0 d}}\langle \mathbf{a}(w_0 d) - \mathbf{a}(d)\rangle) \cong \mathbb{C}$.
(*vi*) *If $x \sim_L w_0 d$, the inequality $\mathrm{ext}^{\mathbf{a}(d)}(T_d, L_x\langle i\rangle) \neq 0$ implies both $i = \mathbf{a}(w_0 d) - \mathbf{a}(d)$ and $x = \widehat{w_0 d}$.*

Proof. Claim (i) follows from [46, Theorem 17]. As $T_w = \theta_{w_0 w} T_{w_0}$, Claim (iii) follows from [34, Theorem A].

To prove Claim (iv), consider $\mathcal{P}_\bullet(T_{w_0})$ and note that $T_{w_0} = L_{w_0}$. Therefore the complex $\mathcal{P}_\bullet(T_{w_0}) \in \mathcal{LP}(\mathcal{O}_0)$ is sent to ∇_e via the Koszul duality. By the KL-combinatorics, if some P_y appears as a summand of $\mathcal{P}_{-k}(T_{w_0})$, for $k < \mathbf{a}(w)$, then $\mathbf{a}(w_0 y) < \mathbf{a}(w)$, that is $\mathbf{a}(y) > \mathbf{a}(w_0 w)$. Applying $\theta_{w_0 w}$ to $\mathcal{P}_\bullet(T_{w_0})$, we get some, not necessarily minimal, projective resolution of T_w, as projective functors are exact and send projective objects to

projective objects. Now Claim (ii) follows from the definition of the $\mathbf{a}$-function, see Subsection 2.3. Further, any summand P_x of $\mathcal{P}_{-k}(T_w)$ is a summand of $\theta_{w_0w}\mathcal{P}_{-k}(T_{w_0})$, that is of some $\theta_{w_0w}P_y$, for y as above. Therefore we have $\mathbf{a}(x) \geq \mathbf{a}(y)$, implying Claim (iv).

As $T_d \cong \theta_{w_0d}T_{w_0}$, by adjunction, we have

$$\mathrm{ext}^{\mathbf{a}(d)}(T_d, L_{\widehat{w_0d}}\langle i\rangle) \cong \mathrm{ext}^{\mathbf{a}(d)}(T_{w_0}, \theta_{dw_0}L_{\widehat{w_0d}}\langle i\rangle).$$

Since all Duflo elements are involutions, the element $\widehat{w_0d}$, being the Duflo element in the left KL-cell of w_0d, is also the Duflo element in the right KL-cell of $dw_0 = (w_0d)^{-1}$. The module $\theta_{dw_0}L_{\widehat{w_0d}}$ is Koszul-Ringel dual to the module $\theta_{\widehat{w_0dw_0}}L_d$ by [46, Theorem 16]. As $\widehat{w_0d} \sim_R dw_0$, we have $\widehat{w_0d}w_0 \sim_R d$ by [6, Page 179]. It follows that $\theta_{\widehat{w_0dw_0}}L_d$ has simple top $L_{\widehat{w_0dw_0}}$. The latter corresponds to $\theta_{w_0\widehat{w_0dw_0}}L_{w_0} \cong T_{\widehat{w_0dw_0}}$ under the Koszul-Ringel duality. This means that the $\mathbf{a}(d)$-position of the complex $\mathcal{T}_\bullet(\theta_{dw_0}L_{\widehat{w_0d}}) \in \mathcal{LT}(\mathcal{O}_0)$ is isomorphic to $T_{\widehat{w_0dw_0}}\langle\mathbf{a}(d)\rangle$.

The module $T_{\widehat{w_0dw_0}}$ is the image of $P_{w_0\widehat{w_0dw_0}}$ under $\top_{w_0}$, moreover, any costandard filtration of $T_{\widehat{w_0dw_0}}$ is the image of a standard filtration of $P_{w_0\widehat{w_0dw_0}}$ under $\top_{w_0}$. Homomorphisms from T_{w_0} to $T_{\widehat{w_0dw_0}}$ correspond to costandard subquotients of the form ∇_{w_0} of $T_{\widehat{w_0dw_0}}$, which, in turn, correspond to standard subquotients of the form Δ_e in $P_{w_0\widehat{w_0dw_0}}$.

By the BGG reciprocity, the standard subquotients of the form Δ_e in $P_{w_0\widehat{w_0dw_0}}$ correspond to simple subquotients of the form $L_{w_0\widehat{w_0dw_0}}$ in Δ_e. Since conjugation by w_0 corresponds to an automorphism of the Dynkin diagram, $w_0\widehat{w_0dw_0} \in \mathbf{D}$. Clearly, $\mathbf{a}(w_0\widehat{w_0dw_0}) = \mathbf{a}(dw_0)$. From the definition of the $\mathbf{a}$-function, the simple subquotients $L_{w_0\widehat{w_0dw_0}}$ in Δ_e appear in degrees between $\mathbf{a}(dw_0)$ and $\ell(\widehat{w_0d})$, with $\mathbf{a}(dw_0) \leq \ell(\widehat{w_0d})$.

To establish Claim (v), it remains to compare with Claim (ii). The shift by $\mathbf{a}(dw_0)$ is the minimal possible by the previous paragraph, but is the maximal one allowed by Claim (ii). Hence Claim (v) follows from $[\Delta_e : L_{w_0\widehat{w_0dw_0}}\langle -\mathbf{a}(dw_0)\rangle] = 1$ and the next paragraph, which, together with Claim (iii), implies that $\mathrm{Ext}^{\mathbf{a}(d)}(T_d, L_{\widehat{w_0d}}) \neq 0$.

The same arguments applied to some $x \sim_L w_0d$, different from $\widehat{w_0d}$, will give strictly larger shifts as $[\Delta_e : L_x\langle -i\rangle] \neq 0$ implies $i > \mathbf{a}(dw_0)$ and this is not allowed by Claim (ii). This proves Claim (vi) and completes the proof of the lemma. □

Lemma 19. *For any $w \in W$, $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}$, we have:*

(i) $\mathrm{Ext}^k(\nabla_e, L_w) \neq 0$ *implies* $k \geq 2\mathbf{a}(w_0w)$.
(ii) $\mathrm{ext}^k(\nabla_e, L_w\langle i\rangle) \neq 0$ *implies* $-k \leq i \leq \mathbf{a}(x) - k$.
(iii) $\mathrm{ext}^{2\mathbf{a}(w_0w)}(\nabla_e, L_w\langle i\rangle) \neq 0$ *implies* $i = \mathbf{a}(w) - 2\mathbf{a}(w_0w)$.
(iv) $\mathrm{ext}^{2\mathbf{a}(w_0w)}(\nabla_e, L_w\langle \mathbf{a}(w) - 2\mathbf{a}(w_0w)\rangle) \neq 0$ *implies* $w \in \mathbf{D}$.
(v) $\mathrm{ext}^{2\mathbf{a}(w_0w)}(\nabla_e, L_w\langle \mathbf{a}(w) - 2\mathbf{a}(w_0w)\rangle) \cong \mathbb{C}$, *if* $w \in \mathbf{D}$.

Proof. Claim (i) follows from [33, Theorem 3] and [46, Theorem 20].

To prove the rest, consider $\mathcal{T}_\bullet(\nabla_e) \in \mathcal{LT}(\mathcal{O}_0)$. Under the Koszul-Ringel self-duality, this corresponds to ∇_e. From the Kazhdan-Lusztig combinatorics, if some $T_x\langle -k\rangle$ is a summand of some $\mathcal{T}_{-k}(\nabla_e)$, then $\mathbf{a}(x) \leq k \leq \ell(x)$. Moreover, if $k = \mathbf{a}(x)$, then $x \in \mathbf{D}$. To construct a (not necessarily minimal) projective resolution of ∇_e, we need to glue minimal projective resolutions of all such $T_x\langle -k\rangle$. In particular, the right inequality in Claim (ii) follows directly from Lemma 18(ii). The left inequality in Claim (ii) follows from the Koszulity of $\mathcal{O}_0$. Indeed, due to this Koszulity, extensions between simple modules of $\mathcal{O}_0$ live on the main diagonal (i.e. the diagonal defined by the "homological position equals the graded shift" condition). Since the module ∇_e lives in non-positive degrees, it follows that all extensions from it to simple modules live in non-negative shifts of the main diagonal.

A minimal projective resolution $\mathcal{P}_\bullet(T_x)$ of T_x has length $\mathbf{a}(x)$. From Lemma 18(iv), we see that, if some T_y appears as a summand of some $\mathcal{P}_m(T_x)$, for $m > -\mathbf{a}(x)$, then $\mathbf{a}(y) > \mathbf{a}(w_0 x)$. At the same time, if $x \in \mathbf{D}$ and T_y is a summand of $\mathcal{P}_{-\mathbf{a}(x)}(T_x)$, then $y = \widehat{w_0 x}$ by Lemma 18(vi). Now, Claims (iii), (iv) and (v) follow from Lemma 18(vi) and Lemma 18(v). □

4.5. Proof of Theorem 16

For $d \in \mathbf{D}$, consider the minimal projective resolution $\mathcal{P}_\bullet(I_d)$ of I_d. We know that $\mathcal{P}_i(I_d) = 0$, for $i < -2\mathbf{a}(w_0 d)$, by [46, Theorem 20]; and also that $\mathcal{P}_{-2\mathbf{a}(w_0 d)}(I_d) \cong P_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle$, by Corollary 9. We also know that any P_x appearing as a summand of $\mathcal{P}_i(I_d)$, for $i > -2\mathbf{a}(w_0 d)$, satisfies $x >_L d$, by Corollary 9.

The natural projection $P_d \twoheadrightarrow L_d$ gives rise to a non-zero homomorphism from $\mathcal{P}_\bullet(I_d)$ to $L_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$ in the derived category. In fact, we have

$$\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)\big(\mathcal{P}_\bullet(I_d), L_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]\big) \cong \mathbb{C}.$$

As $\mathcal{P}_\bullet(I_d) \cong \theta_d \mathcal{P}_\bullet(I_e)$ and θ_d is self-adjoint, by adjunction we have

$$\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)\big(\mathcal{P}_\bullet(I_e), \theta_d L_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]\big) \cong \mathbb{C}. \tag{7}$$

Now, recall that, up to graded shift, all simple subquotients of $\theta_d L_d$ are of the form L_x, where $x \leq_R d$, see [48, Lemma 13]. At the same time, from Lemma 19 it follows that the only summand of $\mathcal{P}_{-2\mathbf{a}(w_0 d)}(I_e)$ that is an indecomposable projective cover of a simple module of the latter form is the summand $P_d\langle \mathbf{a}(d) - 2\mathbf{a}(w_0 d)\rangle$. This means that the map which realizes a non-zero element in the space (7) comes from a projection π of $P_d\langle \mathbf{a}(d) - 2\mathbf{a}(w_0 d)\rangle$ onto $L_d\langle \mathbf{a}(d) - 2\mathbf{a}(w_0 d)\rangle$ which is the socle of $\theta_d L_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$, see [46, Corollary 3].

Since $d \in \mathbf{D}$, there is a non-zero morphism φ from $\Delta_e\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$ to $\theta_d L_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$ see [48, Proposition 17], and so the image of π belongs to the image of φ.

From the definition of the $\mathbf{a}$-function, we also have a unique, up to scalar, non-zero homomorphism $\psi : P_d \twoheadrightarrow \Delta_e\langle \mathbf{a}(d)\rangle$. This homomorphism has the property that any simple L_w appearing in the cokernel is killed by θ_d and hence satisfies $d \not\leq_L w$, see [48, Proposition 17] and [48, Lemma 12]. In particular, there are no non-zero homomorphism from $\mathcal{P}_{1-2\mathbf{a}(w_0 d)}(I_d)$ to the cokernel of ψ. This implies that

$$\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)\big(\mathcal{P}_\bullet(I_d), \Delta_e\langle 3\mathbf{a}(d) - 2\mathbf{a}(w_0 d)\rangle[2\mathbf{a}(w_0 d)]\big) \cong \mathbb{C}.$$

As $\mathcal{P}_\bullet(I_d) \cong \theta_d \mathcal{P}_\bullet(I_e)$, $\theta_d P_e \cong P_d$ and θ_d is self-adjoint, by adjunction we have

$$\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)\big(\mathcal{P}_\bullet(I_e), P_d\langle 3\mathbf{a}(d) - 2\mathbf{a}(w_0 d)\rangle[2\mathbf{a}(w_0 d)]\big) \cong \mathbb{C}. \tag{8}$$

The module $\theta_d L_d\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle$ is a quotient of $P_d\langle 3\mathbf{a}(d) - 2\mathbf{a}(w_0 d)\rangle$, in fact, this quotient map is the image, under θ_d, of the projection from the image $\mathrm{Im}(\psi)$ of ψ onto the simple top of $\mathrm{Im}(\psi)$, as the cokernel of ψ is killed by the exact functor θ_d.

The module P_d has a Verma filtration. By the BGG-reciprocity and the definition of $\mathbf{a}$, the module $\Delta_e\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle$ appears exactly once as a subquotient of this filtration. Using the usual long exact sequence arguments, our observation in the previous paragraph and the fact that we already established that the image of π belongs to the image of φ, we obtain that any non-zero element in (8) induces a non-zero element η in

$$\mathcal{D}^b({}^{\mathbb{Z}}\mathcal{O}_0)\big(\mathcal{P}_\bullet(I_e), \Delta_e\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]\big). \tag{9}$$

In particular, it follows that

$$\mathrm{ext}^{2\mathbf{a}(w_0 d)}(\nabla_e, \Delta_e\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle) \neq 0.$$

Now let us recall that $\mathbb{S}$, being a composition of derived twisting functors, commutes with the action of the bicategory of projective functors. The identity functor obviously commutes with the action of the bicategory of projective functors. When evaluated at the dominant object Δ_e, the functor $\mathbb{S}$ outputs ∇_e, while the identity outputs Δ_e. By [31, Theorem 1], this determines out functors uniquely, up to isomorphism. Furthermore, by [31, Theorem 2], our η induces a non-zero natural transformation from $\mathbb{S}$ to $\mathrm{Id}\langle 2(\mathbf{a}(d) - \mathbf{a}(w_0 d))\rangle[2\mathbf{a}(w_0 d)]$. This is our α_d.

It remains to show that α_d evaluates at $\theta_w L_d$ to a non-zero map. Indeed, $\theta_w L_d$ is indecomposable as a quotient of (shifted) P_w, moreover, the latter also gives that the endomorphism algebra of $\theta_w L_d$ is a quotient of the endomorphism algebra of P_w and thus is positively graded. Consequently, any non-zero endomorphism of $\theta_w L_d$ of degree zero is an automorphism.

By our above construction of α_d, we see that it is defined in terms of the evaluation of the canonical map $\theta_d \to \theta_e\langle \mathbf{a}(d)\rangle$ as constructed in [48, Proposition 17] at $\theta_w L_d$. This canonical map is non-zero when evaluated at L_d by construction. The fact that it is non-zero when evaluated on $\theta_w L_d$ follows from the adjunction axioms as $\theta_w L_d \neq 0$.

This completes the proof of Theorem 16.

4.6. Consequences

Using Theorem 16, we can now strengthen the assertion of Proposition 10:

Corollary 20. *For $i \in \mathbf{S}$, we have $\mathbf{F}_i \subset \mathbf{CY}_i$.*

We do expect that $\mathbf{F}_i = \mathbf{CY}_i$, for all $i \in \mathbf{S}$, however, at the moment we can only prove this in the two extreme cases, see Subsection 5.5.

To prove Corollary 20, we would need the following lemma. Denote by α some linear combination of all α_d, where $d \in \mathbf{D}$ is such that $2\mathbf{a}(w_0 d) = i$.

Lemma 21. *We can choose α such that, for any $d \in \mathbf{D}$ with $2\mathbf{a}(w_0 d) = i$ and for any $w \in W$ with $w \sim_R d$, the evaluation of α at $\theta_w L_d$ is an isomorphism.*

Proof. Recall that $\theta_w L_d$ is an indecomposable module whose endomorphism algebra is positively graded. In particular, any homogeneous endomorphism of $\theta_w L_d$ of degree zero is a scalar multiple of the identity. For $d' \in \mathbf{D}$ such that $2\mathbf{a}(w_0 d') = i$, consider the corresponding $\alpha_{d'}$. Then we have the corresponding scalar $\lambda_{w,d,d'}$ with which the evaluation of $\alpha_{d'}$ acts on $\theta_w L_d$.

Write $\alpha = \sum_{d'} c_{d'} \alpha_{d'}$. We know from Theorem 16 that $\lambda_{w,d,d} \neq 0$. Therefore the set of all those coefficient vectors $(c_{d'})$ for which the evaluation of α at $\theta_w L_d$ is zero (i.e. $\sum_{d'} c_{d'} \lambda_{w,d,d'} = 0$) is a proper subspace of the space of all coefficient vectors. Note that the set of all coefficient vectors is just a non-zero finite dimensional complex vector space. Recall that a non-zero finite dimensional complex vector space is not the union of a finite set of its proper subspaces (e.g. because any such union has measure zero). Therefore there is a vector of coefficients for which the corresponding α, when evaluated at any $\theta_w L_d$, is non-zero and hence an isomorphism, as asserted. □

Proof of Corollary 20. For $M \in \mathbf{F}_i$, it is enough to show that the evaluation of the natural transformation α given by Lemma 21 at M is an isomorphism. We proceed by induction on the length k of the filtration (3). If $k = 1$, then $M \in \mathbf{CY}_i$ by Theorem 1.

Assume now that we have a short exact sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} \theta_w L_d \longrightarrow 0 \ ,$$

for some $d \in \mathbf{D}$ and $w \in W$ such that $w \sim_R d$, with $N \in \mathbf{F}_i$. By induction, we may assume that the evaluation of α at N is an isomorphism. This short exact sequence corresponds to a distinguished triangle

$$N \xrightarrow{f} M \xrightarrow{g} \theta_w L_d \xrightarrow{h} N[1]$$

in the derived category. Since α is a natural transformation, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{S}N & \xrightarrow{\mathbb{S}f} & \mathbb{S}M & \xrightarrow{\mathbb{S}g} & \mathbb{S}\theta_w L_d & \xrightarrow{\mathbb{S}h} & \mathbb{S}N[1] \\
\downarrow{\scriptstyle \alpha_N} & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_{\theta_w L_d}} & & \downarrow{\scriptstyle \alpha_N[1]} \\
N & \xrightarrow{f} & M & \xrightarrow{g} & \theta_w L_d & \xrightarrow{h} & N[1]
\end{array}$$

In this diagram, α_N, and hence also $\alpha_N[1]$, are isomorphisms by the inductive assumption and $\alpha_{\theta_w L_d}$ is an isomorphism by the basis of the induction. Now the fact that α_M is an isomorphism follows from the Five Lemma, completing the proof. □

5. Various additional results

5.1. Some extensions in the singular case

For appropriate singular blocks, our results on extensions from the dominant dual Verma module to the dominant Verma module can be made more precise as follows:

Lemma 22. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Let λ be a dominant integral weight whose dot-stabilizer is $W^{\mathfrak{p}}$. Then*

$$\operatorname{ext}^{2\mathbf{a}(w_0 w_0^{\mathfrak{p}})}\big(\nabla(\lambda), \Delta(\lambda)\langle i\rangle\big) \cong \begin{cases} \mathbb{C}, & i = 2(\mathbf{a}(w_0^{\mathfrak{p}}) - \mathbf{a}(w_0 w_0^{\mathfrak{p}})); \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let θ^{on} be the projective functor of translation onto the $w_0^{\mathfrak{p}}$-wall from $\mathcal{O}_0$ to $\mathcal{O}_\lambda$. Let θ^{out} be the projective functor of translation out of the $w_0^{\mathfrak{p}}$-wall from $\mathcal{O}_\lambda$ to $\mathcal{O}_0$. These two functors are biadjoint, the composition $\theta^{\mathrm{out}}\theta^{\mathrm{on}}$ is isomorphic to $\theta_{w_0^{\mathfrak{p}}}$, and the composition $\theta^{\mathrm{on}}\theta^{\mathrm{out}}$ is isomorphic to a direct sum of $|W^{\mathfrak{p}}|$ copies of the identity functor on $\mathcal{O}_\lambda$, see [12, Proposition 4.1]. The more detailed graded decomposition is:

$$\theta^{\mathrm{on}}\theta^{\mathrm{out}} \cong \bigoplus_{i\geq 0} \mathrm{Id}_{\mathcal{O}_\lambda}\langle 2i - \ell(w_0^{\mathfrak{p}})\rangle^{\oplus|\{w\in W^{\mathfrak{p}} : \ell(w)=i\}|}.$$

We have $\theta^{\mathrm{on}}\nabla_e \cong \nabla(\lambda)\langle \ell(w_0^{\mathfrak{p}})\rangle$, then $\theta^{\mathrm{out}}\nabla(\lambda)\langle \ell(w_0^{\mathfrak{p}})\rangle = I_{w_0^{\mathfrak{p}}}$ and, finally,

$$\theta^{\mathrm{on}} I_{w_0^{\mathfrak{p}}} \cong \bigoplus_{i\geq 0} \nabla(\lambda)\langle 2i\rangle^{\oplus|\{w\in W^{\mathfrak{p}} : \ell(w)=i\}|}.$$

As projective functors are exact and map projectives to projectives, it follows that the projective dimensions of $\nabla(\lambda)$ and $I_{w_0^{\mathfrak{p}}}$ coincide. The latter module has projective dimension $2\mathbf{a}(w_0 w_0^{\mathfrak{p}})$ by [46, Theorem 20]. Moreover, Corollary 9 describes explicitly the last non-zero term of a minimal projective resolution of $I_{w_0^{\mathfrak{p}}}$. Namely, this term is $P_{w_0^{\mathfrak{p}}}\langle 2(\mathbf{a}(w_0^{\mathfrak{p}}) - \mathbf{a}(w_0 w_0^{\mathfrak{p}}))\rangle$.

Note that we dually have $\theta^{\mathrm{on}}\Delta_e \cong \Delta(\lambda)\langle -\ell(w_0^{\mathfrak{p}})\rangle$, then $\theta^{\mathrm{out}}\Delta(\lambda)\langle -\ell(w_0^{\mathfrak{p}})\rangle = P_{w_0^{\mathfrak{p}}}$ and, finally,

$$\theta^{\mathrm{on}} P_{w_0^{\mathfrak{p}}} \cong \bigoplus_{i\geq 0} \Delta(\lambda)\langle -2i\rangle^{\oplus |\{w\in W^{\mathfrak{p}}\,:\,\ell(w)=i\}|}.$$

By matching the summands, it follows that the last non-zero term of a minimal projective resolution of $\nabla(\lambda)$ is $\Delta(\lambda)\langle 2(\mathbf{a}(w_0^{\mathfrak{p}}) - \mathbf{a}(w_0 w_0^{\mathfrak{p}}))\rangle$. Since all Verma modules have trivial endomorphism algebras, the claim of the lemma follows. □

5.2. Self-extensions of the simple $\mathtt{C}$*-modules*

Consider the coinvariant algebra $\mathtt{C}$ with its natural positive grading in which the generators have degree 1. Note that this differs from the grading associated to the interpretation of $\mathtt{C}$ as $\mathrm{End}(P_{w_0})$ as in the latter the generators have degree 2. So, we need to be careful with rescaling when interpreting the content of this subsection in the setup of category $\mathcal{O}$.

Let $\mathbb{C}$ be the unique simple $\mathtt{C}$-module which we consider as the graded module concentrated in degree 0. Recall, see [70, Subsection 1.1], the description of $\mathrm{Ext}^*_{\mathtt{C}}(\mathbb{C},\mathbb{C})$.

Here is the list of the facts which we will need:

- as an algebra, $\mathrm{Ext}^*_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ is generated by $\mathrm{Ext}^1_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ and $\mathrm{Ext}^2_{\mathtt{C}}(\mathbb{C},\mathbb{C})$;
- as a vector space, $\mathrm{Ext}^*_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ is isomorphic to the tensor product of the exterior algebra $\bigwedge \mathrm{Ext}^1_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ and the symmetric algebra of a certain subspace V of $\mathrm{Ext}^2_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ of dimension $\dim(\mathfrak{h})$;
- $\mathrm{Ext}^1_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ is a homogeneous space of dimension $\dim(\mathfrak{h})$ and is concentrated in degree 1;
- V is the direct sum of homogeneous subspaces whose degrees are the degrees of the algebraically independent generators of the algebra of W-invariant polynomials (note that these degrees are connected, using the ± 1 shift, to the exponents of the root system, see [37]), moreover, those subspaces have dimension 1, if the root system is irreducible.

This information allows us to efficiently compute $\mathrm{Ext}^*_{\mathtt{C}}(\mathbb{C},\mathbb{C})$, especially, for small values of $*$. For example, in the case of a root system of type A_n, the degrees in question are $2,3,\dots,n+1$. Consequently, in this case, we have

$$\dim \mathrm{ext}^2_{\mathtt{C}}(\mathbb{C},\mathbb{C}\langle -i\rangle) = \begin{cases} \binom{n}{2}+1, & i=2;\\ 1, & i=3,4,\dots,n+1;\\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

Here we see the contribution $\binom{n}{2}$ from $\bigwedge \mathrm{Ext}^1_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ and all the 1's, including that in degree 2, add up to the contribution of V.

5.3. Some stability phenomena

Relevance of $\mathrm{Ext}^*_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ for the problems considered in the previous section is explained by the following proposition:

Proposition 23. *For all $k,m\in\mathbb{Z}_{\geq 0}$ with $m\leq k$, if $k=0$, and $m\leq k+1$, if $k>0$, and all $i\in\mathbb{Z}$, we have:*

$$\mathrm{ext}^m_{\mathbb{Z}_{\mathcal{O}_0}}(\mathbb{S}^k\Delta_e,\Delta_e\langle i\rangle)\cong\begin{cases}\mathrm{ext}^m_{\mathtt{C}}(\mathbb{C},\mathbb{C}\langle\frac{i}{2}-k\ell(w_0)\rangle), & i\equiv 0 \mod 2;\\ 0, & i\equiv 1 \mod 2.\end{cases}\tag{11}$$

Proof. For $k=0$, we have $m=0$ and

$$\mathrm{ext}^0_{\mathbb{Z}_{\mathcal{O}_0}}(\Delta_e,\Delta_e\langle i\rangle)\cong\mathrm{ext}^0_{\mathtt{C}}(\mathbb{C},\mathbb{C}\langle i\rangle)\cong\begin{cases}\mathbb{C}, & i=0;\\ 0, & i\neq 0;\end{cases}$$

is, obviously, true.

Consider the linear tilting resolution $\mathcal{T}_\bullet(\nabla_e)$ of ∇_e. To form a projective resolution of ∇_e, we need to glue projective resolutions of each tilting summand T_w appearing in $\mathcal{T}_\bullet(\nabla_e)$. Let $\mathcal{X}^1_\bullet$ be the resulting complex. Note that $\mathcal{P}_\bullet(\nabla_e)$ is a summand of $\mathcal{X}^1_\bullet$ and can be obtained from the latter by removing subcomplexes homotopic to zero.

Now, for $k>1$, define $\mathcal{X}^k_\bullet$ recursively from $\mathcal{X}^{k-1}_\bullet$ as follows: apply $\mathbb{S}$ to $\mathcal{X}^{k-1}_\bullet$ to get a complex of injective modules and then glue projective resolutions of all individual indecomposable summands of that complex. Again, $\mathcal{P}_\bullet(\mathbb{S}^k\Delta_e)$ is a summand of $\mathcal{X}^k_\bullet$ and can be obtained from the latter by removing subcomplexes homotopic to zero.

Note that each tilting and each injective module has a projective cover by a sum of copies of P_{w_0}. Also note that the latter module is unchanged under $\mathbb{S}$ (up to isomorphism and apart from the shift in grading by $2\ell(w_0)$). Consequently, all $\mathcal{X}^k_m$, for $m\leq k$, are sums of copies of P_{w_0}. And all these components and the differentials between them are unchanged, up to isomorphism and shift of grading, when m is fixed and k increases.

Now we note that

$$\mathrm{ext}^m_{\mathbb{Z}_{\mathcal{O}_0}}(\mathbb{S}^k\Delta_e,\Delta_e\langle i\rangle)=\mathrm{hom}(\top_{w_0}\mathcal{X}^k_{-m},\top_{w_0}\Delta_e\langle i\rangle)/N_m,\tag{12}$$

where N_m is the subspace generated by all maps that factor through $\top_{w_0}\mathcal{X}^k_{1-m}$. Also note that $\top_{w_0}\Delta_e=L_{w_0}$. For a fixed $m\leq k$, the previous paragraph implies that the right hand side of (12) does not depend on k. Since $\mathbb{V}$ is exact and sends Δ_e to the simple $\mathtt{C}$-module $\mathbb{C}$ and P_{w_0} to the regular $\mathtt{C}$-module $\mathtt{C}$, we obtain that the right hand side of (12) is computed by the corresponding right hand side of (11).

It remains to consider the case $m=k+1$, for $k>0$, in which we need a bit more care. The component $\mathcal{X}^k_{-k-1}$ might have, up to shift of grading, indecomposable projective summands of the form P_w, where $w\neq w_0$. Applying $\top_{w_0}$ maps such P_w to T_{w_0w} and the

latter has a projective cover whose all summands are of the form P_{w_0}, up to shift. This cover also covers $I_w \cong \mathbb{S}P_w$. This implies that the above arguments extend to the case $m = k+1$, but this is the absolute limit of these arguments. The claim follows. □

5.4. Homological position −2 in type A

Consider the type A_n, for $n \geq 3$. In this case we have the following:

- the minimum partition with respect to the two-sided order is (n), the **a**-value of the corresponding two-sided KL-cell is 0;
- after removing (n), the remaining minimum partition with respect to the two-sided order is $(n-1,1)$, the **a**-value of the corresponding two-sided KL-cell is 1;
- the maximum partition with respect to the two-sided order is (1^n), the **a**-value of the corresponding two-sided KL-cell is $\binom{n+1}{2}$;
- after removing (1^n), the remaining maximum partition with respect to the two-sided order is $(2,1^{n-2})$, the **a**-value of the corresponding two-sided KL-cell is $\binom{n}{2}$.

Note that, in types A_1 and A_2 some of the partitions listed above coincide.

Recall the shifts $\langle 2\big(\mathbf{a}(d^{\mathfrak{p}}) - \mathbf{a}(w_0 d^{\mathfrak{p}})\big)\rangle[2\mathbf{a}(w_0 d^{\mathfrak{p}})]$ in (1). If our Duflo element $d^{\mathfrak{p}}$ equals w_0 (i.e. belongs to the two-sided cell corresponding to (1^n)), this formula outputs $\langle n(n+1)\rangle[0]$. If our Duflo element $d^{\mathfrak{p}}$ belongs to the two-sided cell corresponding to $(2,1^{n-2})$, this formula outputs $\langle n(n-1)-2\rangle[2]$.

Now let us look at the degree shift between $n(n+1)$ and $n(n-1)-2$. The difference is $2(n+1)$. At the same time, Formula (10) says that the maximal expected shift between a non-trivial $\mathrm{ext}^0_{\mathsf{C}}(\mathbb{C}, \mathbb{C}\langle -i\rangle)$ and a non-trivial $\mathrm{ext}^2_{\mathsf{C}}(\mathbb{C}, \mathbb{C}\langle -j\rangle)$ is $n+1$, which becomes $2(n+1)$ in the setup of category $\mathcal{O}$. Therefore the extension given by Theorem 16 corresponds exactly to this extreme shift.

5.5. Classification of Calabi-Yau objects in the two extreme cases

Proposition 24. *We have* $\mathbf{F}_0 = \mathbf{CY}_0 = \mathrm{add}(P_{w_0})$.

Proof. Note that $\mathbf{F}_0$ consists of those modules in $\mathcal{O}_0$ which have a filtration whose subquotients are isomorphic to the projective-injective module $P_{w_0} = \theta_{w_0} L_{w_0}$. This means that $\mathbf{F}_0 = \mathrm{add}(P_{w_0})$, that is the category of all projective-injective objects in $\mathcal{O}_0$. So, due to Corollary 20, we just need to prove $\mathbf{CY}_0 \subset \mathbf{F}_0$.

Let $M \in \mathcal{O}_0$ be a non-zero object such that $\mathbb{S}M \cong M$. As ∇_e is a quotient of P_{w_0}, the 0-th component of $\mathcal{P}_\bullet(\nabla_e)$ is P_{w_0}, which implies that M is quotients of $\theta_{w_0} M$. This means that each simple constituent of the top of M is isomorphic to L_{w_0}. In particular, for any simple reflection s, the evaluation of the natural transformation $\top_s \to \theta_e$ at M is surjective. Hence, by our construction of α_{w_0} in Proposition 14, the evaluation of α_{w_0} at M is surjective, and hence an isomorphism as $\mathbb{S}M \cong M$.

Let P be a minimal projective cover of M. Then each summand of P is isomorphic to P_{w_0}. In particular, $P \in \mathbf{F}_0 \subset \mathbf{CY}_0$. Consider a short exact sequence

$$0 \to N \to P \to M \to 0.$$

Here N is isomorphic, in the derived category and up to shift, to the cone of the projection $P \to M$. We know that both P and M are in $\mathbf{CY}_0$ and that the evaluation of α_{w_0} at both these modules is an isomorphism. Now the same argument as in the proof of Corollary 20 shows that the evaluation of α_{w_0} at N is an isomorphism. In particular, $N \in \mathbf{CY}_0$.

We can now repeat the same argument for N and continue recursively. This must, however, stop after finitely many steps since $\mathcal{O}_0$ has finite global dimension. It follows that M has a projective resolution consisting of injective modules. Therefore M itself is both projective and injective, that is belongs to $\mathbf{F}_0$. The claim follows. □

Proposition 25. *We have* $\mathbf{F}_{\mathbf{2a}(w_0)} = \mathbf{CY}_{\mathbf{2a}(w_0)} = \mathrm{add}(L_e)$.

Proof. By Corollary 20, we know that $\mathbf{F}_{\mathbf{2a}(w_0)} \subset \mathbf{CY}_{\mathbf{2a}(w_0)}$. Let $M \in \mathbf{CY}_{\mathbf{2a}(w_0)}$ and $w \in W$ be such that $w \neq e$. We claim that $[M : L_w] = 0$.

Indeed, if $[M : L_w] \neq 0$, then $\mathcal{D}^b(\mathcal{O}_0)(\mathbb{S}M, I_w) \neq 0$. Applying $\mathbb{S}^{-1}$, we get the inequality $\mathrm{Ext}^{\mathbf{2a}(w_0)}(M, P_w) \neq 0$. This, however, is not possible as the injective dimension of P_w is $\mathbf{2a}(w_0 w) < \mathbf{2a}(w_0)$ by [46, Theorem 20].

Therefore the only composition subquotients of M are L_e, in particular, $M \in \mathbf{F}_{\mathbf{2a}(w_0)}$. Also, by Weyl's theorem on complete reducibility, it follows that $M \in \mathrm{add}(L_e)$. This completes the proof. □

6. Parabolic category $\mathcal{O}$ and its generalizations

6.1. The category $\mathcal{O}_0^{\hat{\mathcal{R}}}$

Fix a right KL-cell $\mathcal{R}$ in W and denote by $\hat{\mathcal{R}}$ the ideal which $\mathcal{R}$ generates with respect to the right order, that is

$$\hat{\mathcal{R}} = \{w \in W \,:\, w \leq_R x \text{ for some } x \in \mathcal{R}\}.$$

Denote by $\mathcal{O}_0^{\hat{\mathcal{R}}}$ the Serre subcategory of $\mathcal{O}_0$ generated by all L_w, where $w \in \hat{\mathcal{R}}$. If $\mathcal{R}$ contains the element $w_0^{\mathfrak{p}} w_0$, for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$, then $\mathcal{O}_0^{\hat{\mathcal{R}}} = \mathcal{O}_0^{\mathfrak{p}}$. Hence, the categories of the form $\mathcal{O}_0^{\hat{\mathcal{R}}}$ are natural generalizations of the parabolic category $\mathcal{O}$, see [55].

In the general case, the structure of $\mathcal{O}_0^{\hat{\mathcal{R}}}$ is not as nice as that of blocks of parabolic category $\mathcal{O}$, for example, $\mathcal{O}_0^{\hat{\mathcal{R}}}$ might fail to be a highest weight category, see [56, Lemma 11].

The category $\mathcal{O}_0^{\hat{\mathcal{R}}}$ is stable with respect to the action of projective functors and inherits from $\mathcal{O}_0$ a natural $\mathbb{Z}$-grading. For $w \in \hat{\mathcal{R}}$, we denote by $P_w^{\hat{\mathcal{R}}}$ and $I_w^{\hat{\mathcal{R}}}$ the indecomposable

projective cover and injective envelope of L_w in $\mathcal{O}_0^{\hat{\mathcal{R}}}$, respectively. The module $P_w^{\hat{\mathcal{R}}}$ is injective if and only if $w \in \mathcal{R}$. In the latter case, $P_w^{\hat{\mathcal{R}}} \cong I_w^{\hat{\mathcal{R}}}$.

6.2. Kostant's problem and dominant dimension

For $w \in W$, we have two natural birepresentations of $\mathscr{P}$ associated with L_x. The first one is $\mathscr{P}/\mathrm{Ann}_{\mathscr{P}}(L_w)$ and the other one is $\mathrm{add}(\bigoplus_{x \in W} \theta_x L_w)$. There is the obvious morphism of birepresentations from the former to the latter, given by sending θ to θL_w. Recall, see [34, Corollary 7.6], that Kostant's problem, in the sense of [28], has positive solution for L_w if and only if above morphism of birepresentations is an equivalence.

For $d \in \mathbf{D}$, the Duflo element in a KL right cell $\mathcal{R}$, [35, Theorem 5] says that Kostant's problem has positive solution for L_d if and only if the cokernel of the natural embedding $P_e^{\hat{\mathcal{R}}} \hookrightarrow P_d^{\hat{\mathcal{R}}}$ embeds into a projective-injective object in $\mathcal{O}_0^{\hat{\mathcal{R}}}$. This is equivalent to the property that $\mathcal{O}_0^{\hat{\mathcal{R}}}$ has dominant dimension at least two with respect to projective-injective objects, see [36] for the details on the latter notion.

One particular case is the following: Kostant's problem has positive solution for L_d, where $d \in \mathbf{D}$ is the Duflo element in the right KL-cell $\mathcal{R}$ which contains $w_0^{\mathfrak{p}} w_0$, for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$, see [35, Corollary 18].

6.3. Applying $\mathbb{S}$ to $\mathcal{O}_0^{\hat{\mathcal{R}}}$

Proposition 26. *Let $\mathcal{R}$ be a right KL-cell and $d \in \mathbf{D}$ be the Duflo element in $\mathcal{R}$. Then, for any $-2\mathbf{a}(w_0 d) < i \leq 0$ and $M \in \mathcal{O}_0^{\hat{\mathcal{R}}}$, the i-th homology of $\mathbb{S}M$ is zero.*

Proof. Each composition subquotient of M is of the form L_x, for some $x \leq_R d$. Since $-2\mathbf{a}(w_0 d) < i$, each summand P_y of $\mathcal{P}_i(\nabla_e)$ satisfies $\mathbf{a}(y) > \mathbf{a}(d)$. Therefore $\theta_y L_x = 0$. The claim follows. □

Proposition 27. *Let $\mathcal{R}$ be a KL-right cell and $d \in \mathbf{D}$ be the Duflo element in $\mathcal{R}$. Then the condition that Kostant's problem has positive solution for L_d is equivalent to the condition that the homology of $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ at position $-2\mathbf{a}(w_0 d)$ is isomorphic to $I_e^{\hat{\mathcal{R}}}$.*

Proof. Let M denote the homology of $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ at position $-2\mathbf{a}(w_0 d)$.

The only element $w \in W$ such that P_w is a summand of $\mathcal{P}_{-2\mathbf{a}(w_0 d)}(\nabla_e)$ and $\theta_w L_d \neq 0$ is $w = d$. Therefore M is a quotient of $\theta_d P_e^{\hat{\mathcal{R}}} \cong \theta_d L_d$. At the same time, from $\theta_d M \cong \theta_d L_d$, which is true by Theorem 1, it follows that $[M : L_d] = 1$ and $[M : L_u] = 0$, for any $u \sim_H d$ different from d. Moreover, if $d' \in \mathbf{D}$ is such that $d' \sim_J d$ and $d' \neq d$, then $\theta_{d'} P_e^{\hat{\mathcal{R}}} = 0$ and thus $\theta_{d'} M = 0$. This implies that $[M : L_x] = 0$, for any $x \sim_R d$ different from d.

This means that M is a quotient of the module N defined as the quotient of $P_d^{\hat{\mathcal{R}}}$ by the trace of all $P_x^{\hat{\mathcal{R}}}$, where $x \sim_R d$, in the radical of $P_d^{\hat{\mathcal{R}}}$.

The kernel of the map $N \twoheadrightarrow M$ contains only simples of the form L_x, where $x <_R d$. However, the corresponding projectives P_x cannot be summands of $\mathcal{P}_{-2\mathbf{a}(w_0d)-1}(\nabla_e)$ as $w_0x >_R w_0d$ and therefore $-2\mathbf{a}(w_0x) < -2\mathbf{a}(w_0d) - 1$, since $\mathbf{a}(w_0x) > \mathbf{a}(w_0d)$. This means that N is isomorphic to M.

The fact that $N \cong I_e^{\hat{\mathcal{R}}}$ is equivalent to the condition that Kostant's problem has positive solution for L_d follows from [35, Theorem 5], see Subsection 6.2. The claim follows. □

Corollary 28. *Let $\mathcal{R}$ be a KL-right cell and $d \in \mathbf{D}$ be the Duflo element in $\mathcal{R}$. Then $\mathbb{S}P_e^{\hat{\mathcal{R}}}[-2\mathbf{a}(w_0d)] \cong I_e^{\hat{\mathcal{R}}}$ if and only if Kostant's problem has positive solution for L_d and the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ equals $2\mathbf{a}(w_0d)$.*

Proof. Assume that the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ equals m and let P_w be a summand of $\mathcal{P}_{-m}(P_e^{\hat{\mathcal{R}}})$. Then the identity on P_w gives rise to a non-zero extension in $\mathcal{O}$, of degree m, from $P_e^{\hat{\mathcal{R}}}$ to P_w. Applying $\mathbb{S}$, we get a non-zero homomorphism from $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ to $I_w[m]$, that is, the homology of $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ at the homological position $-m$ is non-zero.

We know that the homology of $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ at the homological position $-2\mathbf{a}(w_0d)$ is non-zero. Therefore the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ is at least $2\mathbf{a}(w_0d)$. If the inequality is strict, then $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ has a non-zero homology at some position different from $-2\mathbf{a}(w_0d)$ and therefore $\mathbb{S}P_e^{\hat{\mathcal{R}}}[-2\mathbf{a}(w_0d)] \cong I_e^{\hat{\mathcal{R}}}$ is not possible.

If the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ is exactly $2\mathbf{a}(w_0d)$, then the homology of $\mathbb{S}P_e^{\hat{\mathcal{R}}}$ is concentrated at the homological position $-2\mathbf{a}(w_0d)$. Now the claim of the corollary follows from Proposition 27. □

Corollary 29. *Let $\mathcal{R}$ be a KL-right cell and $d \in \mathbf{D}$ be the Duflo element in $\mathcal{R}$. Then $\mathbb{S}[-2\mathbf{a}(w_0d)]$ is an ungraded Serre functor on the category of perfect complexes in $\mathcal{O}_0^{\hat{\mathcal{R}}}$ if and only if the following conditions are satisfied:*

(a) *Kostant's problem has positive solution for L_d;*
(b) *the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ equals $2\mathbf{a}(w_0d)$;*
(c) *the projective dimension of $I_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}_0^{\hat{\mathcal{R}}}$ is finite.*

Proof. If $\mathbb{S}[-2\mathbf{a}(w_0d)]$ is a Serre functor on the category of perfect complexes in $\mathcal{O}_0^{\hat{\mathcal{R}}}$, then $\mathbb{S}P_e^{\hat{\mathcal{R}}}[-2\mathbf{a}(w_0d)] \cong I_e^{\hat{\mathcal{R}}}$ which implies both (a) and (b) by Corollary 28. The necessity of (c) follows from the assumption that category of perfect complexes in $\mathcal{O}_0^{\hat{\mathcal{R}}}$ has a Serre functor.

Conversely, if we assume (c), we know that the category of perfect complexes in $\mathcal{O}_0^{\hat{\mathcal{R}}}$ has a Serre functor. Additionally, assuming (a), (b), we have that $\mathbb{S}P_e^{\hat{\mathcal{R}}}[-2\mathbf{a}(w_0d)] \cong I_e^{\hat{\mathcal{R}}}$ by Corollary 28. Note that the homology functor of $\mathbb{S}$ at position $-2\mathbf{a}(w_0d)$ naturally commutes with projective functors by [31, Lemma 8]. Since this homology functor maps $P_e^{\hat{\mathcal{R}}}$ to $I_e^{\hat{\mathcal{R}}}$, it thus must be isomorphic to the Nakayama functor on $\mathcal{O}_0^{\hat{\mathcal{R}}}$.

Therefore $\mathbb{S}[-2\mathbf{a}(w_0 d)]$ is a triangulated endofunctor of the category of perfect complexes in $\mathcal{O}_0^{\hat{\mathcal{R}}}$ which is isomorphic to the Serre functor when restricted to the category of projective modules. Then the two functors are isomorphic, completing the proof. □

Remark 30. The conditions (a), (b) and (c) of Corollary 29 are satisfied provided that $\mathcal{R}$ contains the element $w_0^{\mathfrak{p}} w_0$, for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$. Indeed, in this case $\mathcal{O}_0^{\hat{\mathcal{R}}} \cong \mathcal{O}_0^{\mathfrak{p}}$ has finite global dimension, which implies Corollary 29(c). Condition Corollary 29(a) follows from [35, Corollary 18]. Finally, condition Corollary 29(b) follows from [54, Proposition 4.4] using the arguments in the proof of Corollary 29.

It is possible that the categories $\mathcal{O}_0^{\mathfrak{p}}$ are the only categories of the form $\mathcal{O}_0^{\hat{\mathcal{R}}}$ that satisfy all the conditions (a), (b) and (c) of Corollary 29. A detailed example in Subsection 7.5 points in that direction. If this turns out to be the case, then the results of this subsection are not stronger than the corresponding results of [54], however, our proof that $\mathbb{S}[-2\mathbf{a}(w_0 d)]$ is a Serre functor is different (our proof uses [31] while the proof of [54] is a computation based on the self-duality of derived Zuckerman functors).

6.4. Calabi-Yau objects for $\mathcal{O}_0^{\hat{\mathcal{R}}}$

Corollary 31. *Assume that the conditions* (a), (b) *and* (c) *of Corollary 29 are satisfied. Then we have:*

(a) $\mathbb{S}\langle 2\mathbf{a}(w_0 d)\rangle[-2\mathbf{a}(w_0 d)]$ *is a graded Serre functor for the category of perfect complexes over* $\mathcal{O}_0^{\hat{\mathcal{R}}}$.
(b) *For* $d' \in \mathbf{D}$ *and* $w \in W$ *such that* $d' \leq_R d$ *and* $w \sim_R d'$, *the object* $\theta_w L_{d'}$ *is a Calabi-Yau object in* $\mathcal{O}_0^{\hat{\mathcal{R}}}$ *of dimension* $2(\mathbf{a}(w_0 d') - \mathbf{a}(w_0 d))$.
(c) *The evaluation of* $\alpha_{d'}[-2\mathbf{a}(w_0 d)]\langle 2\mathbf{a}(w_0 d)\rangle$ *at* $\theta_w L_{d'}$ *gives rise to an isomorphism between* $\mathbb{S}\langle 2\mathbf{a}(w_0 d)\rangle[-2\mathbf{a}(w_0 d)]\theta_w L_{d'}$ *and*

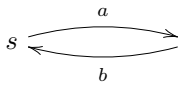
$$\theta_w L_{d'}\langle 2(\mathbf{a}(d') - \mathbf{a}(w_0 d') + \mathbf{a}(w_0 d))\rangle[2(\mathbf{a}(w_0 d') - \mathbf{a}(w_0 d))].$$

Proof. The graded shift for the Serre functor on the category of perfect complexes over $\mathcal{O}_0^{\hat{\mathcal{R}}}$ can be determined by the condition that it sends the projective object $\theta_d L_d\langle -\mathbf{a}(d)\rangle$ to the injective object $\theta_d L_d\langle \mathbf{a}(d)\rangle$. Now the shift in (a) follows from the formulae in Corollary 29 and Theorem 1. The remaining claims follow by combining Corollary 29 with Theorem 1 and Theorem 16. □

7. Examples

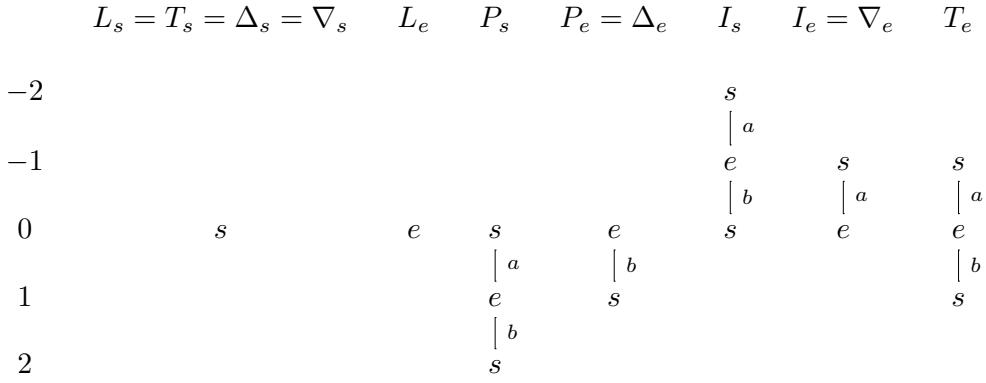
7.1. Principal block of $\mathfrak{sl}_2$

In the case $\mathfrak{g} = \mathfrak{sl}_2$, the principal block $\mathcal{O}_0$ is equivalent to the category of modules over the following quiver with relations, see [47, Theorem 5.3.1]:



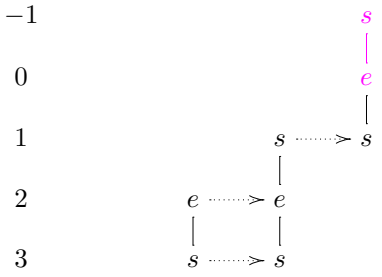
$ab = 0.$

Here are the graded diagrams of all structural modules in this case (they are well-defined as all graded composition multiplicities are either 0 or 1). For simplicity, the simple L_s is displayed as s and the simple L_e as e:



Note that, up to a shift of grading, all indecomposable objects in $\mathcal{O}_0$ appear above.

A minimal graded projective resolution of ∇_e looks as follows:



(13)

Here we directly see that

$$\mathrm{ext}^0(\nabla_e, \Delta_e\langle 2\rangle) = \mathrm{ext}^1(\nabla_e, \Delta_e) = \mathrm{ext}^2(\nabla_e, \Delta_e\langle -2\rangle) = \mathbb{C},$$

where the natural transformation given by $\mathrm{ext}^0(\nabla_e, \Delta_e\langle 2\rangle)$ is constructed in Proposition 14 while the natural transformation given by $\mathrm{ext}^2(\nabla_e, \Delta_e\langle -2\rangle)$ is constructed in Proposition 15.

We have $\mathbf{D} = W = \{e, s\}$. The object $L_e = \theta_e L_e$ is a Calabi-Yau object of dimension 2 and the object $T_e = \theta_s L_s$ is a Calabi-Yau object of dimension 0. These exhaust indecomposable Calabi-Yau objects.

7.2. Principal block of $\mathfrak{sl}_3$

Consider the principal block $\mathcal{O}_0$ for the Lie algebra $\mathfrak{sl}_3$. In this case $W = S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$. The minimal graded tilting resolution of ∇_e looks as follows:

$$0 \to T_{w_0}\langle -3\rangle \to T_{st}\langle -2\rangle \oplus T_{ts}\langle -2\rangle \to T_s\langle -1\rangle \oplus T_t\langle -1\rangle \to T_e \to \nabla_e \to 0.$$

The minimal projective resolutions of the indecomposable tilting modules look as follows:

$$\begin{gathered}
0 \to P_{w_0}\langle 3\rangle \to T_e \to 0;\\
0 \to P_t \to P_{w_0}\langle 2\rangle \to T_s \to 0;\\
0 \to P_s \to P_{w_0}\langle 2\rangle \to T_t \to 0;\\
0 \to P_{ts} \to P_{w_0}\langle 1\rangle \to T_{st} \to 0;\\
0 \to P_{st} \to P_{w_0}\langle 1\rangle \to T_{ts} \to 0;\\
0 \to P_e\langle -3\rangle \to P_s\langle -2\rangle \oplus P_t\langle -2\rangle \to P_{st}\langle -1\rangle \oplus P_{ts}\langle -1\rangle \to P_{w_0} \to T_{w_0} \to 0.
\end{gathered}$$

Here T_{w_0} is simple so the last linear complex is just computed using the Koszul duality. All other are obtained from the last one applying indecomposable projective functors (and removing summands that are homotopic to zero). This implies that ∇_e has a projective resolution of the form:

$$\begin{aligned}
0 \to P_e\langle -6\rangle \to P_s\langle -5\rangle \oplus P_t\langle -5\rangle \to\\
\to P_{st}\langle -4\rangle \oplus P_{ts}\langle -4\rangle \to P_{w_0}\langle -3\rangle \oplus P_{ts}\langle -2\rangle \oplus P_{st}\langle -2\rangle\\
\to P_{w_0}\langle -1\rangle \oplus P_{w_0}\langle -1\rangle \oplus P_t\langle -1\rangle \oplus P_s\langle -1\rangle \to\\
\to P_{w_0}\langle 1\rangle \oplus P_{w_0}\langle 1\rangle \to P_{w_0}\langle 3\rangle \to \nabla_e \to 0.
\end{aligned}$$

As there are no potential cancellations, this resolution is minimal, that is, coincides with $\mathcal{P}_\bullet(\nabla_e)$. The Duflo elements are $\mathbf{D} = \{e, s, t, w_0\}$ and the corresponding $\mathbf{a}$-values are:

d	e	s	t	w_0
$\mathbf{a}(d)$	0	1	1	3

The summands of the $\mathcal{P}_\bullet(\nabla_e)$ which are relevant for Theorem 16 are:

- $P_e\langle -6\rangle[6]$ which corresponds to a non-zero element in $\mathrm{ext}^6(\nabla_e, \Delta_e\langle -6\rangle)$;
- $P_s\langle -1\rangle[2]$ which corresponds to a non-zero element in $\mathrm{ext}^2(\nabla_e, \Delta_e)$;
- $P_t\langle -1\rangle[2]$ which corresponds to a non-zero element in $\mathrm{ext}^2(\nabla_e, \Delta_e)$;
- $P_{w_0}\langle 3\rangle$ which corresponds to a non-zero element in $\mathrm{ext}^0(\nabla_e, \Delta_e\langle 6\rangle)$

Note that $\mathrm{ext}^2(\nabla_e, \Delta_e)$ has dimension 1 in this case, due to Proposition 23 (in other words, the two non-zero elements of this space corresponding to $P_s\langle -1\rangle[2]$ and $P_t\langle -1\rangle[2]$ are linearly dependent). To see how this works, we need to apply $\top_{w_0}$ to $\mathcal{P}_\bullet(\nabla_e)$ and then construct a projective resolution of the outcome. In homological positions -3, -2, -1 and 0, we get:

$$P_{w_0} \oplus P_{w_0} \oplus P_{w_0} \oplus P_{st}\langle -1\rangle \oplus P_{ts}\langle -1\rangle \\ \to P_{w_0}\langle 2\rangle \oplus P_{w_0}\langle 2\rangle \oplus P_{w_0} \oplus P_{w_0} \to P_{w_0}\langle 4\rangle \oplus P_{w_0}\langle 4\rangle \to P_{w_0}\langle 6\rangle \to 0$$

And here we see some potential for cancellation of summands P_{w_0} in homological positions -3 and -2. By Proposition 23, one copy of P_{w_0} must survive in homological position -2, so exactly one such summand should be removed in each of these two positions to obtain a minimal projective resolution. The remaining part

$$\to P_{w_0}\langle 2\rangle \oplus P_{w_0}\langle 2\rangle \oplus P_{w_0} \to P_{w_0}\langle 4\rangle \oplus P_{w_0}\langle 4\rangle \to P_{w_0}\langle 6\rangle \to 0$$

mimics the beginning

$$\mathtt{C}\langle -2\rangle \oplus \mathtt{C}\langle -2\rangle \oplus \mathtt{C}\langle -3\rangle \to \mathtt{C}\langle -1\rangle \oplus \mathtt{C}\langle -1\rangle \to \mathtt{C} \to \mathbb{C} \to 0$$

of a minimal projective resolution of the simple $\mathtt{C}$-module $\mathbb{C}$ (note the overall grading shift and the fact that the generators of $\mathtt{C}$ have degree 2 when related to $\mathcal{O}_0$ via $\mathrm{End}(P_{w_0})$).

7.3. Shifts in type A_6

In type A_6, we have $W = S_7$. Two-sided cells of S_7 are in bijection with partitions of 7 and have the following values of the **a**-function:

λ	(7)	$(6,1)$	$(5,2)$	$(5,1^2)$	$(4,3)$	$(4,2,1)$	$(4,1^3)$	$(3^2,1)$	$(3,2^2)$
$\mathbf{a}(\lambda)$	0	1	2	3	3	4	6	5	6

λ	$(3,2,1^2)$	$(3,1^4)$	$(2^3,1)$	$(2^2,1^3)$	$(2,1^5)$	(1^7)
$\mathbf{a}(\lambda)$	7	10	9	11	15	21

Fig. 1 represents graded shifts appearing in Theorem 1 plotted in the coordinate plane. As usual, the horizontal axis depicts homological position with positive shifts going to the left while the vertical axis depicts the grading with positive shifts going up. Each magenta dot represents a two-sided cell and the partition corresponding to that cell is written next to that dot. The position of the dot in the coordinate plane represents the shifts. One can note that there are two homological positions with two dots, representing two pairs of two-sided cells with the same **a**-value. The area between the two dashed lines is the area of potential tops of P_{w_0} summands as predicted by $\mathrm{Ext}^*_{\mathtt{C}}(\mathbb{C},\mathbb{C})$ via Proposition 23. Note that the magenta dot for $(2,1^5)$ is on the lower dashed line.

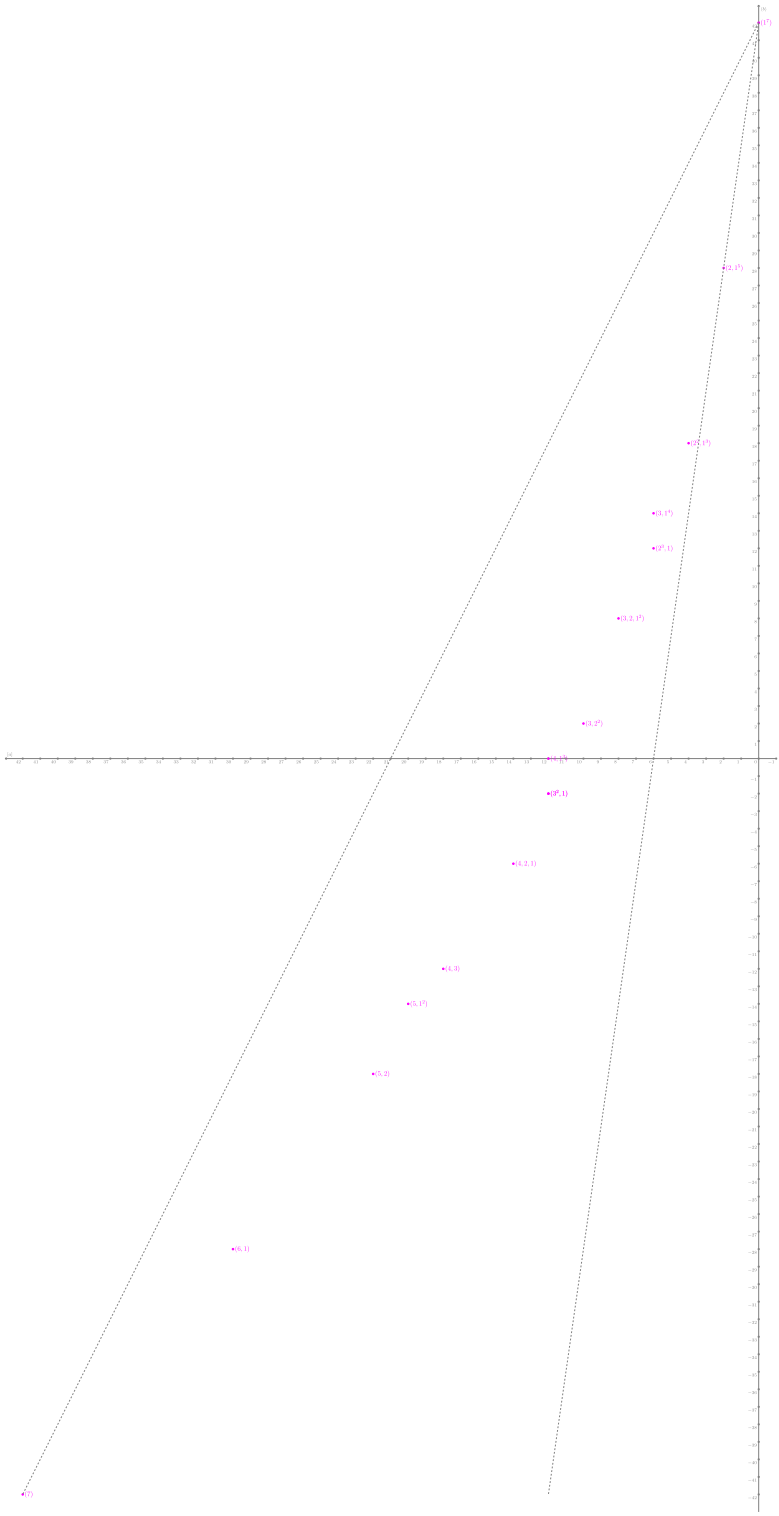


Fig. 1. Shifts for S_7. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

7.4. Parabolic category of type A_{n-1}/A_{n-2}

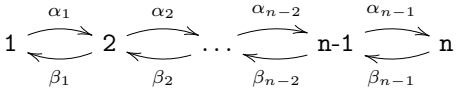
Consider $\mathfrak{g} = \mathfrak{sl}_n$, for $n \geq 2$, and a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$, the semi-simple part of the Levi quotient of which is isomorphic to $\mathfrak{sl}_{n-1}$, embedded into the upper left corner. Then $W \cong S_n$, the Weyl group of type A_{n-1}, with Dynkin diagram

$$s_1 \text{ —— } s_2 \text{ —— } s_3 \text{ —— } \dots \text{ —— } s_{n-1} \ ,$$

and the group $W^{\mathfrak{p}}$ is the parabolic subgroup of W generated by the simple reflections $s_1, s_2, \dots, s_{n-2}$. We have $W^{\mathfrak{p}} \cong S_{n-1}$, the Weyl group of type A_{n-2}, and

$$(W^{\mathfrak{p}} \setminus W)_{\text{short}} = \{e, s_{n-1}, s_{n-1}s_{n-2}, \dots, s_{n-1}s_{n-2}\dots s_2 s_1\}.$$

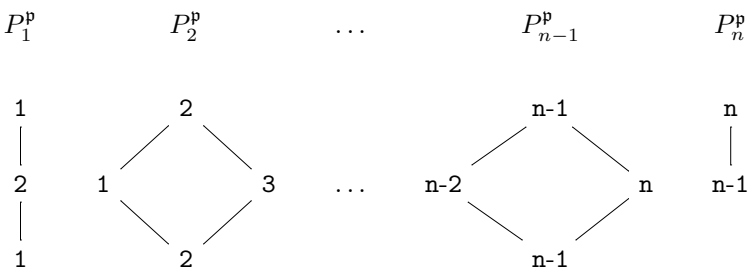
Set $\mathtt{1} := s_{n-1}s_{n-2}\dots s_2 s_1$, $\mathtt{2} := s_{n-1}s_{n-2}\dots s_2$, and so on up to $\mathtt{n} := e$. Then the category $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to the category of modules over the following quiver:



with the following relations, for $i = 1, 2, \dots, n-2$:

$$\alpha_{i+1}\alpha_i = 0, \quad \beta_i\beta_{i+1} = 0, \quad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1}, \quad \alpha_{n-1}\beta_{n-1} = 0.$$

For further properties of this algebra, see [58]. Here are the graded diagrams of projective objects in $\mathcal{O}_0^{\mathfrak{p}}$:



In particular, the projectives $P_1^{\mathfrak{p}}, P_2^{\mathfrak{p}}, \dots, P_{n-1}^{\mathfrak{p}}$ are also injective. As all these projective-injective modules have isomorphic top and socle, it follows that they all are Calabi-Yau objects of dimension 0 for $\mathcal{O}^{\mathfrak{p}}$.

The set $(W^{\mathfrak{p}} \setminus W)_{\text{short}}$ splits into two KL-right cells: $\mathcal{R}_0 := \{e\}$ and $\mathcal{R}_1$, containing all the remaining elements. The only element e of $\mathcal{R}_0$ is a Duflo element. The Duflo element in $\mathcal{R}_1$ is s_{n-1}. We have $P_{n-1}^{\mathfrak{p}} = \theta_{s_{n-1}} L_{s_{n-1}}$, $P_{n-2}^{\mathfrak{p}} = \theta_{s_{n-2}}\theta_{s_{n-1}} L_{s_{n-1}}$, and so on. Note that $\theta_{s_{n-2}}\theta_{s_{n-1}} = \theta_{s_{n-1}s_{n-2}}$, $\theta_{s_{n-3}}\theta_{s_{n-2}}\theta_{s_{n-1}} = \theta_{s_{n-1}s_{n-2}s_{n-3}}$ and so on. We have $\mathbf{a}(e) = 0$, $\mathbf{a}(s_{n-1}) = 1$, $\mathbf{a}(w_0) = \frac{n(n-1)}{2}$ and $\mathbf{a}(w_0 s_{n-1}) = \frac{(n-1)(n-2)}{2}$.

The injective module $I_n^{\mathfrak{p}} \cong (P_n^{\mathfrak{p}})^\star$ has the following projective resolution:

$$0 \to P_n^{\mathfrak{p}}\langle -2-2n\rangle \to P_{n-1}^{\mathfrak{p}}\langle -1-2n\rangle \cdots \to P_2^{\mathfrak{p}}\langle -4-n\rangle \to P_1^{\mathfrak{p}}\langle -3-n\rangle \to$$
$$\to P_1^{\mathfrak{p}}\langle -1-n\rangle \to \cdots \to P_{n-2}^{\mathfrak{p}}\langle 0\rangle \to P_{n-1}^{\mathfrak{p}}\langle 1\rangle \to I_n^{\mathfrak{p}} \to 0.$$

Note that the only non-linear step in this resolution is the step between the positions $P_1^{\mathfrak{p}}\langle -3-n\rangle$ and $P_1^{\mathfrak{p}}\langle -1-n\rangle$, where the difference in degree shifts is 2 instead of 1 in all other steps.

The unique, up to scalar, non-zero map $P_{n-1}^{\mathfrak{p}}\langle 1\rangle \to P_n^{\mathfrak{p}}\langle 2\rangle$ gives rise to a natural transformation from the Serre functor to the identity (shifted by $2 = 2\mathbf{a}(s_{n-1})$) and the evaluation of this natural transformation at all projective-injective objects (i.e. Calabi-Yau objects of dimension 0) is an isomorphism.

The remaining Calabi-Yau object is $L_n = \theta_e L_e$. The unique, up to scalar, non-zero map $P_n^{\mathfrak{p}}\langle -2-2n\rangle \to P_n^{\mathfrak{p}}\langle -2-2n\rangle$ (both at the homological position $2-2n$) gives rise to a natural transformation from the Serre functor to the appropriately shifted identity. The evaluation of this natural transformation at L_n is an isomorphism. Note that

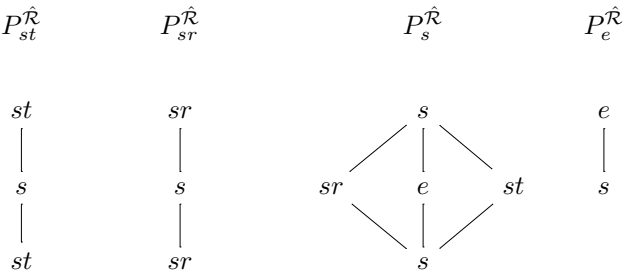
$$2(n-1) = 2\left(\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2}\right) = 2(\mathbf{a}(w_0) - \mathbf{a}(w_0 s_{n-1})).$$

7.5. A non-parabolic example of $\mathcal{O}_0^{\hat{\mathcal{R}}}$ in type A_3

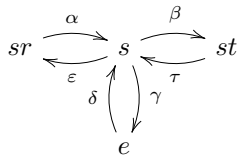
Let $\mathfrak{g} = \mathfrak{sl}_4$. Then $W = S_4$, the Weyl group of type A_3 with Dynkin diagram

$$r \text{ —— } s \text{ —— } t \ .$$

Let $\mathcal{R}$ be the KL-right cell in W containing the simple reflection s. Then $\mathcal{R} = \{s, sr, st\}$ and $\hat{\mathcal{R}}$ contains one additional element, namely e. Consider $\mathcal{O}_0^{\hat{\mathcal{R}}}$. The projective objects in this category have the following graded diagrams:



Consequently, the category $\mathcal{O}_0^{\hat{\mathcal{R}}}$ is equivalent to the category of modules over the following quiver:



with the following relations:

$$\beta\alpha = \gamma\alpha = \varepsilon\tau = \gamma\tau = \beta\delta = \varepsilon\delta = \gamma\delta = 0, \quad \alpha\varepsilon = \tau\beta = \delta\gamma.$$

We see that $P_s^{\hat{\mathcal{R}}}$, $P_{st}^{\hat{\mathcal{R}}}$ and $P_{sr}^{\hat{\mathcal{R}}}$ are injective. The injective $I_e^{\hat{\mathcal{R}}} = (P_e^{\hat{\mathcal{R}}})^\star$ has the following projective resolution:

$$0 \to P_e^{\hat{\mathcal{R}}}\langle -2\rangle \to P_s^{\hat{\mathcal{R}}}\langle -1\rangle \to P_{sr}^{\hat{\mathcal{R}}} \oplus P_{st}^{\hat{\mathcal{R}}} \to P_s^{\hat{\mathcal{R}}}\langle 1\rangle \to I_e^{\hat{\mathcal{R}}} \to 0.$$

In particular, the projective dimension of $I_e^{\hat{\mathcal{R}}}$ is finite. That is, the condition (c) of Corollary 29 is satisfied.

Kostant's problem has positive solution for L_s by [43, Theorem 1]. That is, the condition (a) of Corollary 29 is satisfied. Unfortunately, the condition (b) of Corollary 29 is **not** satisfied in this example. Let us see why.

Just as in the previous example, we have $\mathbf{a}(e) = 0$, $\mathbf{a}(s) = 1$, $\mathbf{a}(w_0) = 6$, $\mathbf{a}(w_0 s) = 3$. So, the condition (b) of Corollary 29 requires that the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ equals 6. We claim that this is not the case and, in fact, the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ equals 10.

To see this, consider the short exact sequence

$$0 \to L_s\langle 1\rangle \to P_e^{\hat{\mathcal{R}}} \to L_e \to 0. \tag{14}$$

The module L_e has a minimal projective resolution in $\mathcal{O}$ of length $12 = 2\mathbf{a}(w_0)$, see [44, Proposition 6]. Under the Koszul duality, this resolution is mapped to the indecomposable projective-injective object I_{w_0}. The latter has a (dual) Verma flag in which each Verma module appears with multiplicity 1.

The module L_s has a minimal projective resolution of length 11, see [44, Proposition 6]. Under the Koszul duality, this resolution is mapped to the indecomposable injective object $I_{w_0 s}$. Note that $[\Delta_e : L_{w_0 s}] = 2$, see [69, Page 344]. Therefore, by the BGG reciprocity, $I_{w_0 s}$ has a dual Verma flag in which ∇_e, ∇_r, ∇_t and ∇_{rt} appear twice and all other ∇_w, where $w \leq w_0 s$ with respect to the Bruhat order, appear once.

We can use now the sequence in (14) to construct a projective resolution of $P_e^{\hat{\mathcal{R}}}$: there is a unique, up to scalar, non-zero homomorphism φ from I_{w_0} to $I_{w_0 s}\langle 1\rangle$, which corresponds, via Koszul duality, to a non-zero homomorphism ψ from L_e to $L_s\langle -1\rangle[1]$. The cone of the latter is isomorphic to $P_e^{\hat{\mathcal{R}}}$.

As the functor $\mathbb{V}$ is full and faithful on injective modules, let us look at the images of I_{w_0} and $I_{w_0 s}\langle 1\rangle$ under this functor. The graded dimension vectors of the corresponding images are:

$$v = (1, 0, 3, 0, 5, 0, 6, 0, 5, 0, 3, 0, 1) \quad \text{and} \quad w = (0, 0, 1, 0, 4, 0, 7, 0, 7, 0, 4, 0, 1).$$

Here the leftmost degree is 0. The vector v just records the number of elements in S_4 of a fixed length: S_4 has 1 element of length 0, then 3 elements of length 1 and so on. The vector w is the sum

$$w = (0, 0, 1, 0, 4, 0, 6, 0, 5, 0, 3, 0, 1) + (0, 0, 0, 0, 0, 0, 1, 0, 2, 0, 1, 0, 0),$$

where the first summand describes the contribution, similar to v, of the Bruhat interval $[e, w_0 s]$ while the second summand describes the contribution of the additional four dual Verma modules ∇_e, ∇_r, ∇_t and ∇_{rt}.

Note the violet entries 3 and 4 in degree 10. As $4 > 3$, the cokernel of φ will be non-zero in that degree. Via Koszul duality, this means that the minimal projective resolution of $P_e^{\hat{\mathcal{R}}}$ will have a non-zero entry at homological position -10. This implies that the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ is at least 10.

As the map φ is bijective in both degrees 11 and 12, it follows that the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ is exactly 10.

Remark 32. If one instead takes as $\mathcal{R}$ the right cell of r, then this cell contains $\{r, sr, tsr\}$. Similarly to the above, we can compute the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ in this case. The difference will be that $[\Delta_e : L_{w_0 r}] = 1$ and, consequently, the new vector w will be $(0, 0, 1, 0, 3, 0, 5, 0, 5, 0, 3, 0, 1)$, where we only have singleton contributions by each element in $[e, w_0 r]$. Therefore the difference between v and w is $(1, 0, 2, 0, 2, 0, 1, 0, 0, 0, 0, 0, 0)$. Here the maximal non-zero degree is 6. This implies that the projective dimension of $P_e^{\hat{\mathcal{R}}}$ in $\mathcal{O}$ in this case equals 6, as required by the condition (b) of Corollary 29.

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