Received: 25 May 2024 | Revised: 5 July 2024 | Accepted: 14 July 2024

IET Quantum Communication

DOI: 10.1049/qtc2.12107

The Institution of Engineering and Technology WILEY

ORIGINAL RESEARCH

Majorisation-minimisation algorithm for optimal state discrimination in quantum communications

Neel Kanth Kundu[1,2,3] | **Prabhu Babu**[1] | **Petre Stoica**[4]

[1]Centre for Applied Research in Electronics (CARE), Indian Institute of Technology Delhi, New Delhi, India

[2]Bharti School of Telecommunication Technology and Management, Indian Institute of Technology Delhi, New Delhi, India

[3]Department of Electrical and Electronic Engineering, University of Melbourne, Melbourne, Victoria, Australia

[4]Division of Systems and Control, Department of Information Technology, Uppsala University, Uppsala, Sweden

Correspondence

Neel Kanth Kundu, CARE, Indian Institute of Technology Delhi, Room 208, Block-III, Hauz Khas, New Delhi 110016, India.
Email: neelkanth@iitd.ac.in

Funding information

Department of Science and Technology, Ministry of Science and Technology, India, Grant/Award Number: IFA22-ENG 34; Indian Institute of Technology Delhi, Grant/Award Number: New Faculty Seed Grant; Vetenskapsrådet, Grant/Award Numbers: 2017-04610, 2016-06079, 2021-05022

Abstract

Designing optimal measurement operators for quantum state discrimination (QSD) is an important problem in quantum communications and cryptography applications. Prior works have demonstrated that optimal quantum measurement operators can be obtained by solving a convex semidefinite program (SDP). However, solving the SDP can represent a high computational burden for many real-time quantum communication systems. To address this issue, a majorisation-minimisation (MM)-based algorithm, called Quantum Majorisation-Minimisation (QMM) is proposed for solving the QSD problem. In QMM, the authors reparametrise the original objective, then tightly upper-bound it at any given iterate, and obtain the next iterate as a closed-form solution to the upper-bound minimisation problem. Our numerical simulations demonstrate that the proposed QMM algorithm significantly outperforms the state-of-the-art SDP algorithm in terms of speed, while maintaining comparable performance for solving QSD problems in quantum communication applications.

KEYWORDS

computational complexity, Hilbert spaces, matrix algebra, quantum communication, quantum information, quantum optics, telecommunication channels

1 | INTRODUCTION AND MOTIVATION

Quantum communications and computing are emerging disruptive technologies that are poised to shape the evolution of next-generation information technologies [1]. Harnessing the quantum mechanical properties for computation and communication could bring significant improvements in the security, data rates, and reliability of future information processing technologies. Optimal quantum state discrimination (QSD) is an important problem primarily encountered in quantum communication system applications, which also has applications in quantum sensing and cryptography [2–8]. The security of quantum key distribution (QKD) is based on the QSD problem for non-orthogonal quantum states. The eavesdropper may employ QSD to attack the QKD systems and steal some information of the shared key between the legitimate parties [9, 10]. Apart from the eavesdropper, the legitimate party in a QKD system may also utilise optimum state-discrimination receivers to improve the secret key rate of QKD systems [8]. Therefore, QSD plays an important role also in the security of QKD systems [2].

A schematic of the QSD application in quantum communications is shown in Figure 1. In the quantum communications problem, the transmitter encodes the classical information into a quantum state represented as a pure state. The prepared quantum state is then transmitted to the receiver through a noisy quantum-mechanical channel, which outputs a

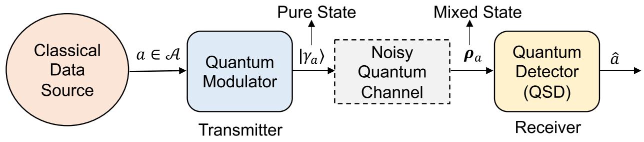

FIGURE 1 Schematic of QSD application in quantum communications (the symbols used in the Figure are explained in Section 2). QSD, quantum state discrimination.

mixed quantum state [11, 12]. At the receiver, a quantum detection problem needs to be solved in order to detect the transmitted information. This is accomplished by subjecting the received quantum states at the output of the quantum channel to a quantum measurement operation [13, 14]. In general, the quantum states at the output of the quantum channel are not orthogonal, which means that the states cannot be distinguished or discriminated with certainty. In such a case, the receiver designs a quantum measurement scheme that minimises the probability of discrimination error.

The optimal measurement operator for QSD is given by a generalised quantum measurement, also known as the positive operator valued measure (POVM) [15]. Prior works on QSD have derived the necessary and sufficient conditions that the optimal measurement operators should satisfy [16, 17]. Although the optimal measurement set is completely characterised by these conditions, analytical (or closed-form) expressions for the optimal measurement operators are not available in general QSD applications [13].

Prior works have demonstrated that the optimal POVM for QSD with minimum probability of error can be obtained by numerically solving a convex semidefinite program (SDP) [18, 19]. However, the computational complexity of SDP can be prohibitive for large problem dimensions, which hinders its applicability in real-time quantum communication applications. Therefore, designing efficient algorithms for QSD with the same performance as that of SDP is an important open research problem in the quantum communications literature. We address this important quantum communications problem in this work. We propose a Majorisation-Minimisation (MM) algorithm, which is guaranteed to converge to a minimiser of the QSD objective and which is also computationally much faster than solving the SDP. The numerical simulation results demonstrate that the performance of the proposed Quantum Majorisation-Minimisation (QMM) algorithm matches the probability of error achieved by solving the SDP, while being more than two orders of magnitude faster than the state-of-the-art SDP solver.

The main contributions of this paper can be summarised as follows:

1) We propose an MM iterative algorithm (named QMM) for finding the optimal quantum measurement operators for the QSD problem in quantum communication applications.
2) We discuss the computational complexity, convergence properties and initialisation of the proposed QMM algorithm.
3) Through numerical simulations, we compare the performance and average computation time when using QMM and SDP to solve the QSD problem in quantum communications with pure and mixed states.

In Section 2, we state the QSD problem, and in Section 3 we present the proposed algorithm and discuss its computational complexity and convergence property. In Section 4, we present the numerical simulation results. Finally, Section 5 concludes the paper.

Notations: The conjugate transpose and trace of a matrix $\boldsymbol{A}$ are denoted $\boldsymbol{A}^H$ and $\mathrm{Tr}(\boldsymbol{A})$, respectively. A positive semidefinite (PSD) matrix $\boldsymbol{A}$ is denoted as $\boldsymbol{A} \succeq 0$ and $\mathrm{Re}(z)$ denotes the real part of z. We use Dirac's bra-ket notation of quantum mechanics for pure states, where a column vector in a Hilbert space $\mathcal{H}$ is denoted by 'ket' $|x\rangle$, its conjugate transpose vector is denoted by 'bra' $\langle x|$, and their outer product is denoted $\langle x|\langle x|$. General mixed quantum states are denoted by density matrices $\boldsymbol{\rho}$.

2 | QSD PROBLEM STATEMENT

We consider the QSD problem for a quantum communications application, see Figure 1. In a quantum communication system, the classical message $a \in \mathcal{A}$ (where $\mathcal{A}$ is alphabet) emitted by a message source is encoded into a quantum state by the transmitter and is then transmitted to the receiver over a noisy quantum channel. The transmitter encodes the messages using pure quantum states, which can be represented as $|\gamma_a\rangle$ for the classical message a. The pure quantum state emitted by the transmitter $|\gamma_a\rangle$ is transformed into a mixed quantum state $\boldsymbol{\rho}_a$ at the output of the noisy quantum channel.

In quantum information processing, a general quantum state is characterised by a unit trace, PSD density operator $\boldsymbol{\rho}$ in a complex Hilbert space. In this work, we consider a finite-dimensional complex Hilbert space $\mathcal{H}$ of dimension d. The QSD problem is characterised by m PSD density operators $\{\boldsymbol{\rho}_i, 1 \leq i \leq m\}$ s.t. $\forall\ i \in \{1, \ldots, m\}$ $\mathrm{Tr}(\boldsymbol{\rho}_i) = 1$ with prior probabilities $\{p_i > 0, 1 \leq i \leq m\}$ s.t. $\sum_{i=1}^{m} p_i = 1$. The objective in QSD is to design m PSD Hermitian $d \times d$ measurement operators $\{\boldsymbol{\Pi}_i, 1 \leq i \leq m\}$ that resolve the identity operator in $\mathcal{H}$ (i.e. $\sum_{i=1}^{m} \boldsymbol{\Pi}_i = \boldsymbol{I}$) such that the probability of correct decision is maximised. According to the Born's rule ([20], Equation (3.51)), the probability of correct detection for the quantum state $\boldsymbol{\rho}_j$ using the measurement operator $\{\boldsymbol{\Pi}_j, 1 \leq j \leq m\}$ is $\mathrm{Tr}(\boldsymbol{\rho}_j\boldsymbol{\Pi}_j)$. Then, the probability of error for QSD is given by the following:

$$1 - \sum_{i=1}^{m} p_i \mathrm{Tr}\left(\boldsymbol{\rho}_i \boldsymbol{\Pi}_i\right) \quad (1)$$

Consequently, minimising the probability of QSD error amounts to solving the following problem [18, 19]:

$$\begin{aligned}\min_{\{\boldsymbol{\Pi}_i\}} \quad & 1-\sum_{i=1}^{m} p_i \mathrm{Tr}\left(\boldsymbol{\rho}_i \boldsymbol{\Pi}_i\right) \\ \text{s.t} \quad & \boldsymbol{\Pi}_i \geq 0 \quad \forall\ 1 \leq i \leq m, \\ & \sum_{i=1}^{m} \boldsymbol{\Pi}_i = \boldsymbol{I}\end{aligned} \tag{2}$$

The problem in Equation (2) is an SDP [21] that can be solved using off-the-shelf solvers, such as SeDuMi or SDPT3 and CVX as a modelling framework [22]. However, the computational complexity of solving the SDP in Equation (2) becomes prohibitive as the number of POVM matrices and the dimension of each matrix increase. For instance, solving Equation (2) via an SDP solver for the choice of $m = 100$, $d = 10$, takes more than 10 s on a standard desktop PC, which would constitute a significant bottleneck in any practical quantum communications application.

In this paper, we reformulate the problem in Equation (2) and propose an MM-based iterative algorithm that is significantly faster than an SDP solver.

3 | PROPOSED ALGORITHM

In this section, we present the proposed algorithm QMM for solving the problem (2). Because QMM uses the principle of majorisation-minimisation in the following, we briefly discuss the core steps of the MM approach [23, 24].

3.1 | MM for a minimisation problem

Consider the following constrained minimisation problem:

$$\min_{\mathbf{x} \in \chi} f(\mathbf{x}) \tag{3}$$

where $f(\mathbf{x})$ is the objective function, $\mathbf{x}$ denotes the optimisation variable and χ is a constraint set. The MM algorithm [23, 25] solves the problem in Equation (3) in two steps. In the first step, it constructs a surrogate function $g(\mathbf{x}\,|\,\mathbf{x}^k)$ that is a tight upperbound of $f(\mathbf{x})$ at $\mathbf{x}^k$. In the second step, the surrogate function is minimised to get the next iterate that is,

$$\mathbf{x}^{k+1} \in \underset{\mathbf{x} \in \chi}{\operatorname{argmin}}\, g\left(\mathbf{x} | \mathbf{x}^k\right) \tag{4}$$

The aforementioned two steps are repeated until the sequence iterates $\left\{f\left(\mathbf{x}^k\right)\right\}_k$ converges. The surrogate function constructed in the first step of the MM procedure $g\left(\mathbf{x}|\mathbf{x}^k\right)$ satisfies the following conditions [23]:

$$g(\mathbf{x}|\mathbf{x}^k) \geq f(\mathbf{x}),\ \forall\ \mathbf{x} \in \chi \tag{5}$$

$$g(\mathbf{x}^k|\mathbf{x}^k) = f(\mathbf{x}^k) \tag{6}$$

The sequence of iterates obtained via the MM algorithm monotonically decreases the objective, that is, [23].

$$f(\mathbf{x}^{k+1}) \leq g(\mathbf{x}^{k+1}|\mathbf{x}^k) \leq g(\mathbf{x}^k|\mathbf{x}^k) = f(\mathbf{x}^k)$$

The first inequality and the equality follow from Equations (5) and (6) and second inequality from Equation (4).

3.2 | QMM

We first reparametrise the measurement matrices as $\boldsymbol{\Pi}_i = \boldsymbol{A}_i \boldsymbol{A}_i^H$ to eliminate the PSD constraints in Equation (2). Next, we reformulate the design problem in Equation (2) as (leaving out a constant term):

$$\begin{aligned}\min_{\{\boldsymbol{A}_i\}} \quad & \left\{ f(\{\boldsymbol{A}_i\}) = -\sum_{i=1}^{m} \mathrm{Tr}\left(\boldsymbol{P}_i \boldsymbol{A}_i \boldsymbol{A}_i^H\right) \right\} \\ \text{s.t.} \quad & \sum_{i=1}^{m} \boldsymbol{A}_i \boldsymbol{A}_i^H = \boldsymbol{I}\end{aligned} \tag{7}$$

where $\boldsymbol{P}_i = p_i \boldsymbol{\rho}_i$. Let

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1^H \\ \boldsymbol{A}_2^H \\ \vdots \\ \boldsymbol{A}_m^H \end{bmatrix} \tag{8}$$

and observe that the constraint in Eqaution (7) can be equivalently expressed as $\boldsymbol{A}^H \boldsymbol{A} = \boldsymbol{I}$. Using a trace property, the objective function can be expressed as follows:

$$f(\boldsymbol{A}) = -\sum_{i=1}^{m} \mathrm{Tr}\left(\boldsymbol{A}_i^H \boldsymbol{P}_i \boldsymbol{A}_i\right) \tag{9}$$

which is a concave quadratic function of $\{\boldsymbol{A}_i\}$.

As mentioned in Section 3.1, the first step of an MM algorithm consists of devising a tight surrogate function $g(\boldsymbol{A}|\boldsymbol{A}^k)$ for $f(\boldsymbol{A})$ at a given $\boldsymbol{A} = \boldsymbol{A}^k$. Each of the terms in Equation (9), that is, $-\mathrm{Tr}\left(\boldsymbol{A}_i^H \boldsymbol{P}_i \boldsymbol{A}_i\right)$, is a concave quadratic function, thus $f(\boldsymbol{A})$ is concave quadratic function. Using first-order Taylor series expansion at $\boldsymbol{A}^k$ (tangent hyperplane passing through $\boldsymbol{A}^k$), we get a tight upper bound (see ref. [23], Equation (10)]):

$$\begin{aligned} f(\boldsymbol{A}) = -\sum_{i=1}^{m} \mathrm{Tr}\left(\boldsymbol{A}_i^H \boldsymbol{P}_i \boldsymbol{A}_i\right) \leq -2\,\mathrm{Re}\left(\sum_{i=1}^{m} \mathrm{Tr}\left(\boldsymbol{A}_i^H \boldsymbol{B}_i^k\right)\right) \\ + \sum_{i=1}^{m} \mathrm{Tr}\left(\left(\boldsymbol{A}_i^k\right)^H \boldsymbol{P}_i \left(\boldsymbol{A}_i^k\right)\right) = g\left(\boldsymbol{A}|\boldsymbol{A}^k\right)\end{aligned} \tag{10}$$

where $\boldsymbol{B}_i^k$ depends on the current iterate $\boldsymbol{A}_i^k$ via

$$\boldsymbol{B}_i^k = \boldsymbol{P}_i \boldsymbol{A}_i^k \tag{11}$$

Alternatively, the bound in (10) can be verified as follows:

$$\sum_{i=1}^{m} \left(\boldsymbol{A}_i - \boldsymbol{A}_i^k\right)^H \boldsymbol{P}_i \left(\boldsymbol{A}_i - \boldsymbol{A}_i^k\right) \geq 0 \tag{12}$$

since each of the matrices in the summation is positive semidefinite. Taking the trace operation and expanding, we obtain

$$\mathrm{Tr}\left[\sum_{i=1}^{m} \boldsymbol{A}_i^H \boldsymbol{P}_i \boldsymbol{A}_i - \boldsymbol{A}_i^H \boldsymbol{P}_i \boldsymbol{A}_i^k - \left(\boldsymbol{A}_i^k\right)^H \boldsymbol{P}_i \boldsymbol{A}_i + \left(\boldsymbol{A}_i^k\right)^H \boldsymbol{P}_i \left(\boldsymbol{A}_i^k\right) \geq 0\right. \tag{13}$$

Using Equation (11) and after some rearrangement of the terms, we obtain the inequality:

$$-\sum_{i=1}^{m} \mathrm{Tr}\left(\boldsymbol{A}_i^H \boldsymbol{P}_i \boldsymbol{A}_i\right) \leq -2\,\mathrm{Re}\left(\sum_{i=1}^{m} \mathrm{Tr}\left(\boldsymbol{A}_i^H \boldsymbol{B}_i^k\right)\right) + \sum_{i=1}^{m} \mathrm{Tr}\left(\left(\boldsymbol{A}_i^k\right)^H \boldsymbol{P}_i \left(\boldsymbol{A}_i^k\right)\right) \tag{14}$$

It follows from Equation (10) that the surrogate minimisation problem is defined as follows:

$$\begin{aligned} \min_{\boldsymbol{A}} \quad & -2\,\mathrm{Re}\left(\mathrm{Tr}\left(\boldsymbol{A}^H \boldsymbol{B}^k\right)\right) \\ \text{s.t} \quad & \boldsymbol{A}^H \boldsymbol{A} = \boldsymbol{I} \end{aligned} \tag{15}$$

where $\boldsymbol{B}^k = \left[\left(\boldsymbol{B}_1^k\right), \left(\boldsymbol{B}_2^k\right), \ldots, \left(\boldsymbol{B}_m^k\right)\right]^H$. The optimisation problem in Equation (15) can be reformulated as follows:

$$\begin{aligned} \min_{\boldsymbol{A}} \quad & \|\boldsymbol{B}^k - \boldsymbol{A}\|_F^2 \\ \text{s.t.} \quad & \boldsymbol{A}^H \boldsymbol{A} = \boldsymbol{I} \end{aligned} \tag{16}$$

Let $\boldsymbol{B}^k = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H$ be the singular value decomposition of $\boldsymbol{B}^k$ (where $\boldsymbol{U}$ is a semi-unitary matrix of size $md \times d$, $\boldsymbol{\Sigma}$ is a diagonal matrix of size $d \times d$ and $\boldsymbol{V}$ is a unitary matrix of size $d \times d$), then the next MM iterate can be obtained as the solution of the orthogonal Procrustes problem in Equation (16) [26]:

$$\boldsymbol{A}^{k+1} = \boldsymbol{U}\boldsymbol{V}^H \tag{17}$$

The pseudocode of the QMM algorithm for solving the QSD problem is presented in Algorithm 1.

Algorithm 1 QMM.

Input: $\boldsymbol{\rho}_i,\ p_i\ \forall\, i \in \{1, \ldots, m\}$
Output: $\boldsymbol{\Pi}_i\ \forall\, i \in \{1, \ldots, m\}$
1 Set $k = 0$
2 Initialize $\boldsymbol{A}^0$ s.t. $(\boldsymbol{A}^0)^H \boldsymbol{A}^0 = \boldsymbol{I}$
3 **repeat**
4 $\quad \boldsymbol{B}_i^k = \boldsymbol{P}_i \boldsymbol{A}_i^k, \quad i = 1, \ldots, m$
5 $\quad \boldsymbol{B}^k = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H$
6 $\quad \boldsymbol{A}^{k+1} = \boldsymbol{U}\boldsymbol{V}^H$
7 $\quad$ Set $k = k + 1$
8 **until** *stopping criteria*
9 $\boldsymbol{\Pi}_i = \boldsymbol{A}_i \boldsymbol{A}_i^H$

3.3 | Computational complexity, convergence, initialisation and stopping criterion

We first discuss the computational complexity of QMM. Each iteration of QMM requires computing m matrix multiplications $\boldsymbol{P}_i \boldsymbol{A}_i^k$ which has a computational complexity of $\mathcal{O}(md^3)$ flops, the SVD of $\boldsymbol{B}^k$ with complexity $\mathcal{O}(md^3)$ flops and the matrix multiplication $\boldsymbol{U}\boldsymbol{V}^H$ with complexity $\mathcal{O}(md^3)$ flops. Therefore, the computational complexity of each iteration of QMM is on the order of $\mathcal{O}(md^3)$. On the other hand, the computational complexity of the SeDuMi SDP solver is $\mathcal{O}(m^2 d^9)$ [27]. Therefore, the computational complexity of the proposed QMM algorithm is several orders of magnitude lower than that of the SeDuMi solver, a fact that will be illustrated via several numerical experiments in Section 4. Regarding the convergence of the proposed algorithm, as QMM is based on the MM principle, the objective in Equation (7) (and hence (2)) decreases monotonically with the iteration; moreover, the objective is bounded below, which guarantees that the QMM will converge in terms of the objective [23].

Regarding initialisation, we initialise the proposed algorithm via a random choice of $\{\boldsymbol{A}_i\}$, which satisfies the constraint in Equation (7). To do so, first, we generate a random matrix (with the same dimension as $\boldsymbol{A}$) whose entries are independent and identically distributed random variables with a standard normal distribution $\mathcal{N}(0,1)$. Then, the matrix $\boldsymbol{A}^0$ in line 2 of Algorithm 1 is built from the left singular vectors of the random matrix. Finally, the stopping criterion is: $\frac{|f(\boldsymbol{A}^{k+1}) - f(\boldsymbol{A}^k)|}{|f(\boldsymbol{A}^k)|} \leq 10^{-3}$.

4 | NUMERICAL RESULTS

The numerical simulations are carried out using MATLAB on a desktop computer with Intel i7 processor @ 2.5 GHz and 32 GB RAM. We first consider a QSD problem in quantum communications consisting of pure states with $m = 3$ and $d = 2$, similar to ref. [28]. The quantum states consists of rank-one density matrices $\boldsymbol{\rho}_i = |\psi_i\rangle\langle\psi_i|$, $i \in \{1, 2, 3\}$ where

$$|\psi_1\rangle = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \ |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix}, \ |\psi_3\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{18}$$

with prior probabilities $p_1 = p_2 = p$ and $p_3 = 1 - 2p$ where $0 < p < 1/2$. In this example, we will use $\theta = \pi/16$ and $p = 0.3$. Figure 2 shows the objective in Equation (2) minimised by the QMM algorithm as a function of the iteration number for 10 different random initialisations (solid lines). For comparison, the optimum value of the objective in Equation (2) obtained from the SeDuMi SDP solver is also shown (dashed line). It can be observed that QMM quickly converges to the optimum SDP solution from all initialisation points. Furthermore, it was observed that the SeDuMi solver and QMM return the same $\{\mathbf{\Pi}_i\}_{i=1}^{3}$ for all the 10 different random initialisation points up to the numerical precision of MATLAB. Further, the average runtime and mean squared error (MSE) of the optimum objective value of QMM with respect to SDP is shown in Table 1. It can be observed that the run time of QMM is two orders of magnitude lower than that of the SeDuMi SDP solver while achieving an MSE of the order 10^{-6}. The runtime of QMM is significantly lower than that of the SeDuMi SDP solver since the theoretical computational complexity for each iteration of QMM is $\mathcal{O}(md^3)$ while the computational complexity of the SeDuMi SDP solver is $\mathcal{O}(m^2d^9)$ as discussed in Section 3.3.

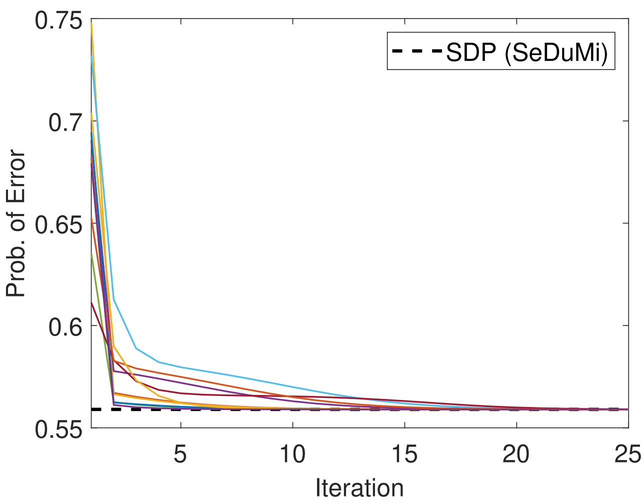

FIGURE 2 The objective function minimised by the proposed algorithm as a function of iteration for 10 different initialisation points. The results are for the QSD problem with the pure quantum states in Equation (18). For comparison, the optimum objective value obtained from the solution of the SDP using the SeDuMi solver is also shown. QSD, quantum state discrimination.

TABLE 1 Performance comparison of QMM and SDP for QSD with pure states.

Algorithm	Average run time (s)	MSE
QMM	0.0021	4.79×10^{-6}
SDP (SeDuMi)	0.36	-

Next, we consider a QSD problem with mixed quantum states, which is usually encountered in a practical quantum communication system. We consider the quantum m-PSK modulation scheme, for which the entries of the k−th density matrix are given by (for $1 \le i \le j$) [13]

$$\rho_{i,j}(\alpha_k) = (1-v)v^j \sqrt{\frac{i!}{j!}} \left(\frac{\alpha_k^*}{\mathcal{N}}\right)^{j-i} \times e^{-(1-v)|\alpha_k|^2} L_i^{j-i}\left(-\frac{|\alpha_k|^2}{\mathcal{N}(\mathcal{N}+1)}\right) \tag{19}$$

where $v = \mathcal{N}/(1+\mathcal{N})$, $\mathcal{N}$ denotes the average number of thermal photons, $|\alpha_k|^2$ denotes the average number of signal photons, and $L_i^{j-i}(x)$ are the generalised Laguerre polynomials. The entries for $i > j$ are obtained using the Hermitian property $\rho_{j,i}(\alpha_k) = \rho_{i,j}^*(\alpha_k)$. For m-PSK modulation, the parameter α_k satisfies $\alpha_k = \alpha_0 W_m^{k-1}$, $k = 1, \ldots, m$ where $W_m = e^{i2\pi/m}$, and α_0 is assumed to be a positive real number without loss of generality [13]. In general, the density matrices have infinite dimensions. However, in practice, the dimension of the density matrix is truncated to a finite value in order to apply the SDP or the QMM algorithms. We consider a 4-PSK modulation ($m = 4$) with prior probabilities $p_1 = 0.2$, $p_2 = 0.2$, $p_3 = 0.1$, $p_4 = 0.5$, $\alpha_0 = 1$, $\mathcal{N} = 2$, and with density matrices of truncated dimension $d = 10$ [13]. Figure 3 shows the objective in Equation (2) minimised by the QMM algorithm as a function of iteration for 10 different random initialisation points. As in the previous example, the QMM algorithm quickly converges to the optimal SDP solution from all initialisation points. As mentioned earlier, it was observed that the optimum $\{\mathbf{\Pi}_i\}_{i=1}^{4}$

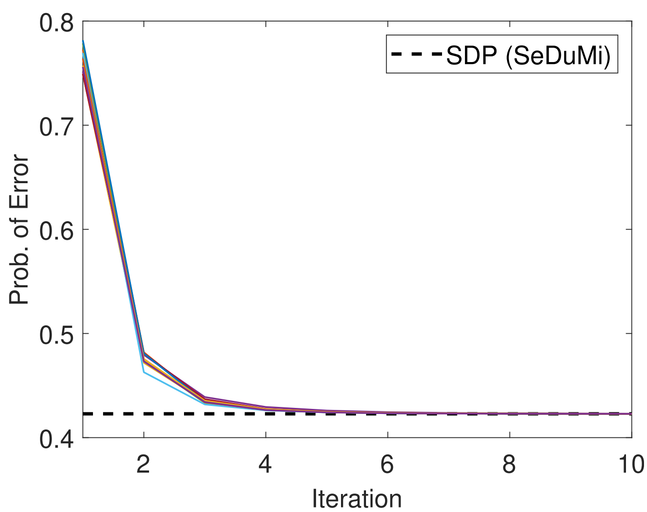

FIGURE 3 The objective function minimised by the proposed algorithm as a function of iteration for 10 different initialisations. The results are for the QSD problem with mixed quantum states corresponding to a 4-PSK modulation scheme. For comparison, the optimum objective value obtained from the solution of the SDP is also shown. QSD, quantum state discrimination.

obtained from the SeDuMi solver and QMM are the same for all the 10 different random initialisation points up to the numerical precision of MATLAB. Furthermore, the average runtime and MSE of the optimum objective value of QMM with respect to SDP is shown in Table 2. As mentioned earlier, it can be observed that the run time of QMM is two orders of magnitude lower than that of the SeDuMi SDP solver while achieving an MSE of the order 10^{-7}.

TABLE 2 Performance comparison of QMM and SDP for QSD with mixed states in 4 − PSK.

Algorithm	Average run time (s)	MSE
QMM	0.0026	5.06×10^{-7}
SDP (SeDuMi)	0.2133	-

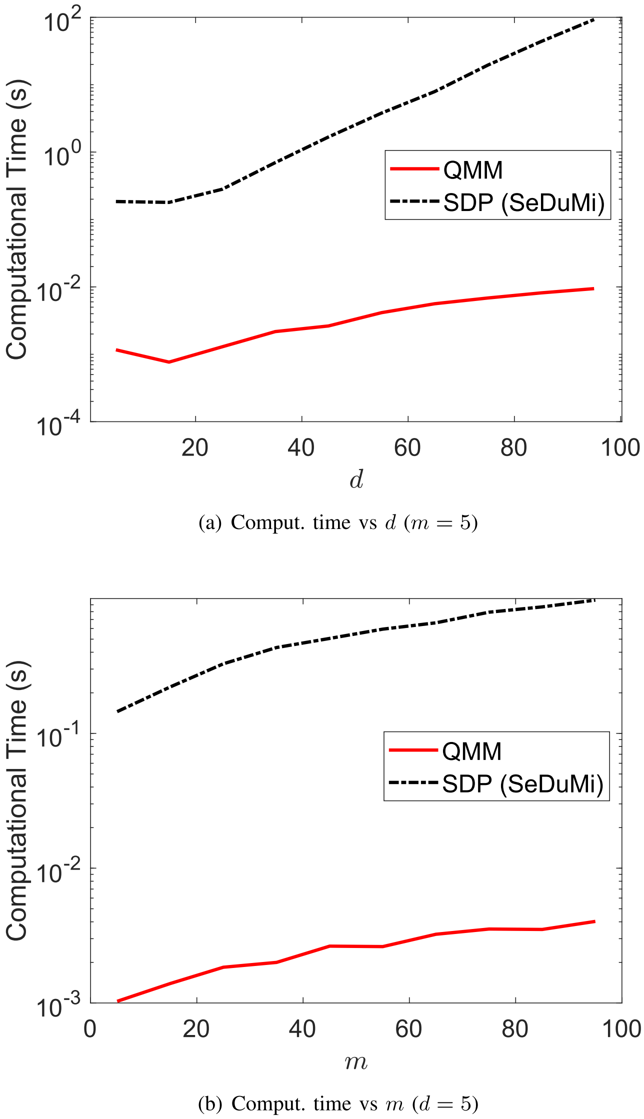

FIGURE 4 The plots compare the average computation time of the proposed QMM algorithm with that of the SeDuMi SDP solver for varying (a) Hilbert space dimension d and (b) number of states m. QMM, quantum majorisation-minimisation.

Next, we compare the average computation time (in seconds) of the QMM algorithm with that of the SDP for varying problem dimensions. We consider a simulation setting where the density matrices and the prior probabilities are randomly generated for varying m and d. Figure 4a plots the average computation time as a function of d (for $m = 5$), and Figure 4b plots the average computation time as a function of m (for $d = 5$). It can be observed that the average computation time of the proposed QMM algorithm is more than two orders of magnitude smaller than that of the SDP. The MSE of the QMM objective with respect to the optimum SDP objective as a function of d, m for the two examples considered in Figure 4 is shown in Figure 5. It can be observed that the MSE of the QMM objective and the optimum SDP objective is of the order 10^{-6} for each (m, d) test case in Figure 4. Although not shown, the difference between the QMM and SDP optimal solution of the measurement matrices is of the order of numerical precision of MATLAB.

5 | CONCLUSION

In this paper, we have proposed a computationally efficient algorithm (called QMM) for solving the QSD problem, which has applications in quantum communications. The proposed algorithm is based on the technique of majorisation-minimisation that monotonically decreases the design objective. In order to derive the QMM algorithm, we reparametrised the original objective, tightly upper-bounded it at any given iterate, and then obtained the next iterate as a closed-form solution to the upper-bound minimisation problem. Our numerical simulations demonstrate that the proposed QMM algorithm significantly outperforms the state-of-the-art SDP

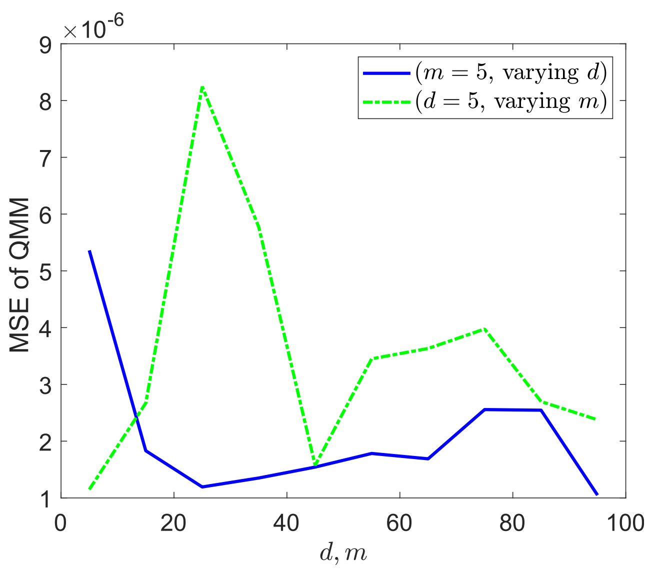

FIGURE 5 The plots show the MSE of the QMM objective with respect to the optimum SDP objective as a function of d, m for the two examples considered in Figure 4. MSE, mean squared error; QMM, quantum majorisation-minimisation.

algorithm in terms of speed, while maintaining comparable performance for solving QSD problems in quantum communication applications.

AUTHOR CONTRIBUTIONS

Neel Kanth Kundu: Conceptualization; formal analysis; funding acquisition; investigation; methodology; project administration; software; validation; visualization; writing – original draft; writing – review & editing. **Prabhu Babu**: Conceptualization; formal analysis; investigation; methodology; project administration; software; supervision; validation; writing – original draft; writing – review & editing. **Petre Stoica**: Conceptualization; formal analysis; funding acquisition; investigation; methodology; project administration; supervision; writing – original draft; writing – review & editing.

ACKNOWLEDGEMENTS

The work of Neel Kanth Kundu was supported in part by the INSPIRE Faculty Fellowship awarded by the Department of Science and Technology, Government of India (Reg. No.: IFA22-ENG 344), and the New Faculty Seed Grant (NFSG) from the Indian Institute of Technology Delhi. Petre Stoica's work was supported in part by the Swedish Research Council (VR grants 2017-04610, 2016–06079, and 2021–05022).

CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Neel Kanth Kundu https://orcid.org/0000-0002-6439-4024

REFERENCES

1. Rozenman, G.G., et al.: The quantum internet: a synergy of quantum information technologies and 6G networks. IET Quan. Commun. 4(4), 147–166 (2023). https://doi.org/10.1049/qtc2.12069
2. Bae, J., Kwek, L.-C.: Quantum state discrimination and its applications. J. Phys. Math. Theor. 48(8), 083001 (2015). https://doi.org/10.1088/1751-8113/48/8/083001
3. Bergou, J.A.: Quantum state discrimination and selected applications. J. Phys. Conf. 84(1), 012001 (2007). IOP Publishing. https://doi.org/10.1088/1742-6596/84/1/012001
4. Kundu, N.K., et al.: MIMO terahertz quantum key distribution. IEEE Commun. Lett. 25(10), 3345–3349 (2021). https://doi.org/10.1109/lcomm.2021.3102703
5. Kundu, N.K., McKay, M.R., Mallik, R.K.: Wireless quantum key distribution at terahertz frequencies: opportunities and challenges. IET Quan. Commun. (2024). https://doi.org/10.1049/qtc2.12085
6. Liu, R., et al.: Towards the industrialisation of quantum key distribution in communication networks: a short survey. IET Quan. Commun. 3(3), 151–163 (2022). https://doi.org/10.1049/qtc2.12044
7. Kundu, N.K., McKay, M.R., Balaji, B.: Quantum enhanced sensing using Gaussian quantum states. In: 2023 IEEE Sensors Applications Symposium (SAS), pp. 1–6. IEEE (2023)
8. Notarnicola, M.N., et al.: Optimizing state-discrimination receivers for continuous-variable quantum key distribution over a wiretap channel. New J. Phys. 25(10), 103014 (2023). https://doi.org/10.1088/1367-2630/acfd50
9. Zhang, C.-M., et al.: Discrete-phase-randomized twin-field quantum key distribution with advantage distillation. Phys. Rev. 109(5), 052432 (2024). https://doi.org/10.1103/physreva.109.052432
10. Jiang, C., et al.: Sending-or-not-sending twin-field quantum key distribution with discrete-phase-randomized weak coherent states. Phys. Rev. Res. 2(4), 043304 (2020). https://doi.org/10.1103/physrevresearch.2.043304
11. Waghmare, C., Kothari, A., Sharma, P.K.: Performance analysis of classical data transmission over a quantum channel in the presence of atmospheric turbulence. IEEE Commun. Lett. 27(8), 2127–2131 (2023). https://doi.org/10.1109/lcomm.2023.3284900
12. Yuan, R., Cheng, J.: Closed-form density matrices of free-space optical quantum communications in turbulent channels. IEEE Commun. Lett. 24(5), 1072–1076 (2020). https://doi.org/10.1109/lcomm.2020.2974735
13. Cariolaro, G., Pierobon, G.: Performance of quantum data transmission systems in the presence of thermal noise. IEEE Trans. Commun. 58(2), 623–630 (2010). https://doi.org/10.1109/tcomm.2010.02.080013
14. Vázquez-Castro, A., Samandarov, B.: Quantum advantage of binary discrete modulations for space channels. IEEE Wireless Commun. Lett. 12(5), 903–906 (2023). https://doi.org/10.1109/lwc.2023.3249282
15. Helstrom, C.W.: Quantum detection and estimation theory. J. Stat. Phys. 1(2), 231–252 (1969). https://doi.org/10.1007/bf01007479
16. Holevo, A.S.: Statistical decision theory for quantum systems. J. Multivariate Anal. 3(4), 337–394 (1973). https://doi.org/10.1016/0047-259x(73)90028-6
17. Yuen, H., Kennedy, R., Lax, M.: Optimum testing of multiple hypotheses in quantum detection theory. IEEE Trans. Inf. Theor. 21(2), 125–134 (1975). https://doi.org/10.1109/tit.1975.1055351
18. Eldar, Y.C., Megretski, A., Verghese, G.C.: Designing optimal quantum detectors via semidefinite programming. IEEE Trans. Inf. Theor. 49(4), 1007–1012 (2003). https://doi.org/10.1109/tit.2003.809510
19. Wengang, W., Guohua, D., Mingshan, L.: Minimum-error quantum state discrimination based on semidefinite programming. In: 2008 27th Chinese Control Conference, pp. 521–524. IEEE (2008)
20. Cariolaro, G.: Quantum Communications, vol. 2. Springer (2015)
21. Boyd, S.P., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)
22. Grant, M., Boyd, S.: CVX: Matlab Software for Disciplined Convex Programming, Version 2.1. http://cvxr.com/cvx Mar 2014
23. Sun, Y., Babu, P., Palomar, D.P.: Majorization-minimization algorithms in signal processing, communications, and machine learning. IEEE Trans. Signal Process. 65(3), 794–816 (2016). https://doi.org/10.1109/tsp.2016.2601299
24. Lange, K.: MM Optimization Algorithms. SIAM, Philadelphia (2016)
25. Stoica, P., Selen, Y.: Cyclic minimizers, majorization techniques, and the expectation-maximization algorithm: a refresher. IEEE Signal Process. Mag. 21(1), 112–114 (2004). https://doi.org/10.1109/msp.2004.1267055
26. Schönemann, P.H.: A generalized solution of the orthogonal Procrustes problem. Psychometrika 31(1), 1–10 (1966). [Online]. Available:. https://doi.org/10.1007/BF02289451
27. Labit, Y., Peaucelle, D., Henrion, D.: SeDuMi interface 1.02: a tool for solving LMI problems with SeDuMi. In: Proceedings. IEEE International Symposium on Computer Aided Control System Design, pp. 272–277. IEEE (2002)
28. Andersson, E., et al.: Minimum-error discrimination between three mirror-symmetric states. Phys. Rev. 65(5), 052308 (2002). https://doi.org/10.1103/physreva.65.052308

How to cite this article: Kundu, N.K., Babu, P., Stoica, P.: Majorisation-minimisation algorithm for optimal state discrimination in quantum communications. IET Quant. Comm. 5(4), 612–618 (2024). https://doi.org/10.1049/qtc2.12107