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Estimating transmission noise on networks from stationary local order

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Abstract – We study networks of nodes characterised by binary traits that change both endogenously and through nearest-neighbour interaction. Our analytical results show that those traits can be ranked according to the noisiness of their transmission using only measures of order in the stationary state. Crucially, this ranking is independent of network topology. As an example, we explain why, in line with a long-standing hypothesis, the relative stability of the structural traits of languages can be estimated from their geospatial distribution. We conjecture that similar inferences may be possible in a more general class of Markovian systems. Consequently, in many empirical domains where longitudinal information is not easily available the propensities of traits to change could be estimated from spatial data alone.

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Introduction. – Many complex systems can be represented as networks of nodes, each characterised by a set of traits which can change endogenously and through interaction. Examples are abundant in evolutionary biology, gene regulatory systems, and the dynamics of languages [1–4]. The spreading of opinions or diseases are further instances of noisy transmission processes on networks [5,6].

We may naturally wish to estimate the relative propensities of different traits to change. The most direct way is to resort to longitudinal data: in population genetics, for example, cladistics and genetic sequencing have enabled the reconstruction of phylogenetic trees reaching to the beginnings of evolutionary time [7]. More indirect approaches are, however, possible [8]. Mutation rates can also be estimated using mechanistic models together with summary statistics observed in natural populations [7,9–11]. In fact, these indirect methods offer the only avenue in fields where access to longitudinal information is restricted: the historical evolution of languages, for instance, can be inferred only to a very shallow temporal depth.

Against this background, Greenberg surmised fifty years ago that the geospatial distribution of the structural traits of languages carries useful information about those traits' temporal dynamics [12]. Motivated by a recent implementation of this intuition [13], we suggest that Greenberg's hypothesis applies to Markovian processes more widely. One can conceive of many such processes as involving the interplay of transmission noise and a tendency to order. We propose that the relative strength of these two, the noise-to-order ratio, can be estimated from a small set of measurable quantities in the stationary state. We call this the *transmission-noise conjecture*.

In this letter we show that the conjecture does indeed hold in networked systems with binary traits that change both spontaneously within nodes and through nearest-

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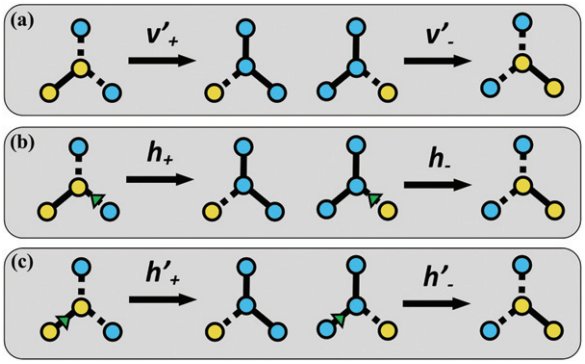

Fig. 1: Illustration of the model dynamics. We show small networks of four nodes which possess (blue) or lack (yellow) a specific trait. Links that connect nodes in the same/different state are solid/dashed. The focal node is the node being considered for update. Panel (a): node changes state endogenously. Panel (b): node correctly copies a neighbour. Panel (c): node erroneously copies a neighbour.

neighbour interaction. Moreover, detailed knowledge of the network is not needed to rank traits according to their noise-to-order ratios.

Model. – Our model describes N nodes i of an undirected network [14,15]. At time t, each node is in one of two states, indicated by variables $s_i(t) = \pm 1$. In the context of horizontal gene transfer, the binary state could stand for the presence or absence of a mutation in an organism. In a model of opinion dynamics the two states could represent two distinct views on a particular question.

The system evolves through several processes. 1) The state of a node can change endogenously from s to $-s$ with rate v'_s, as shown in fig. 1(a). This reflects transmission errors in time, for example from parent to offspring. 2) A node can change state by faithfully copying the state of a neighbour. The rate coefficients for these events are $h_\pm$ (fig. 1(b)). 3) Finally, a node can change state due to transmission error in an unfaithful copying event (fig. 1(c)); this occurs with rate coefficients $h'_\pm$. A full definition of the dynamics can be found in the Supplemental Material `Supplementarymaterial.pdf` (SM)[1]. The additional references only used in the SM are [16–40].

Our starting point is deliberately general. The events in fig. 1 encompass all networked models with binary states in which nodes can flip either spontaneously or through pairwise interaction with one nearest neighbour. This generalises the model in [13] as well as instances of the so-called "noisy voter model" [41–43]. The subsequent analysis assumes $h_+ + h'_- = h_- + h'_+$. This is a mild requirement, indicating that nodes do not preferentially interact with nodes in a particular state (see SM). We stress that, in contrast to most existing literature on the voter model, spontaneous flipping can occur asymmetrically in our model ($v'_+ \neq v'_-$), and that the copying of a state from a neighbour can be subject to error ($h'_+ \neq 0$, $h'_- \neq 0$).

[1]The SM contains further details of the model and the theoretical and numerical analysis.

Summary statistics for local order. – We characterise the stationary state using two summary statistics. One is the trait frequency, ρ. This is the proportion of nodes which possess that trait (*i.e.*, nodes with $s_i = +1$). The second is the proportion, σ, of links in the network that connect two nodes in opposite states. In line with voter model terminology we will call these active links [44,45].

The average stationary trait frequency is given by

$$\langle \rho \rangle_{\rm st} = \frac{h'_+ + v'_+}{h'_+ + h'_- + v'_+ + v'_-}. \tag{1}$$

We can show that this expression is valid for any undirected network (see SM).

We also define the ratio

$$\mathbb{H} \equiv \frac{\sigma}{2\rho(1-\rho)}. \tag{2}$$

If, for a given trait frequency ρ, traits were distributed randomly across the nodes, without correlations, one would have $\sigma = 2\rho(1-\rho)$, and thus $\mathbb{H} = 1$. When $\mathbb{H} < 1$ there are fewer active links than random. Thus, $\mathbb{H}$ quantifies the amount of scatter in the system relative to a random configuration with the same value of ρ. We note that $\mathbb{H}$ can take values larger than one for configurations in which the states of neighbouring spins are anti-correlated (see SM).

The typical time course from simulations of the model is illustrated in the (ρ, σ)-plane in fig. 2(a). As indicated by the wiggly lines, the system approaches a parabolic curve of constant $\mathbb{H}$ quickly, and then fluctuates in the region near the parabola. This has previously been pointed out for voter models in [42,44]. For finite systems, the point defined by the time averages $\langle \rho \rangle_{\rm st}$ and $\langle \sigma \rangle_{\rm st}$ in the stationary state may not lie on this parabola. This will be discussed further below.

Transmission noise determines local order. – Our aim is to demonstrate the validity of the transmission noise conjecture for our model.

As a first step we use the correlation function of spin states at different nodes to show that the stationary density of active interfaces $\langle \sigma \rangle_{\rm st}$ can be expressed in terms of the stationary trait frequency $\langle \rho \rangle_{\rm st}$ and the parameter combination (see SM)

$$\tau \equiv \frac{(h'_+ + h'_-) + (v'_+ + v'_-)}{\frac{1}{2}[(h_+ + h_-) - (h'_+ + h'_-)]}. \tag{3}$$

This holds for any network. Further, for a wide range of networks we have $\langle \sigma \rangle_{\rm st} = 2H(\tau)\,\langle \rho \rangle_{\rm st}\,(1 - \langle \rho \rangle_{\rm st})$, where $H(\tau)$ is dependent upon the network. Importantly, the model parameters enter only through the combination τ. In simulations we confirm that $\langle \mathbb{H} \rangle_{\rm st} \approx H(\tau)$, provided τ is above some cutoff set by the network size (see SM).

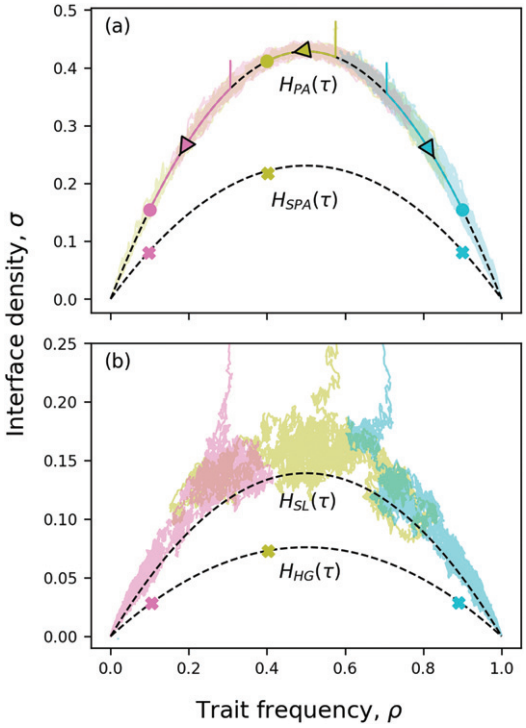

Fig. 2: Model dynamics in the plane spanned by trait frequency ρ and density of active interfaces σ. (a) is a Barabási-Albert network ($N = 2500$), (b) is the periodic square lattice ($N = 40000$). The wiggly lines show realisations from simulations, each realisation is shown in a different colour. Each wiggly line is for a different set of model parameters, but with a common value of $\tau = 1 \times 10^{-3}$ in (a), and $\tau = 1 \times 10^{-4}$ in (b). The solid lines in (a) show the trajectories within the pair approximation, derived in the limit $N \to \infty$ (eqs. (S25) and (S42) in the SM). The fixed points of these equations are shown by filled circles. The crosses in each panel indicate the points $(\langle\rho\rangle_{\rm st}, \langle\sigma\rangle_{\rm st})$ for the different parameter sets. Dashed lines are parabolas of constant $\mathbb{H}$, obtained from different analytical results, as indicated.

The numerator in eq. (3) describes the average rate of events in the system starting from a fully ordered state ($\rho = 0$ or $\rho = 1$). The only state changes that are possible in such a configuration are those due to transmission noise. Therefore, the numerator can be interpreted as a strength of transmission noise. The denominator in the expression for τ is the rate with which correlations build up between neighbouring spins, starting from a random initial condition with $\rho = 1/2$ (see SM). Thus, we can think of τ as a noise-to-order ratio, and hence a measure of instability[2].

This interpretation, together with the observation that local order is set by τ (*i.e.*, H is a function of τ), confirms the transmission noise conjecture in our model.

Inference of τ from observed data: analytical results. – We now turn to the question of inferring the noise-to-order ratio τ. This is possible if $\langle\mathbb{H}\rangle_{\rm st}$ can be estimated from observations, and, if the functional form of $H(\tau)$ can be calculated either exactly or as an approximation. We now discuss how to do the latter, and then turn to an empirical example.

The pair approximation (PA) is a standard tool for the analysis of interacting dynamics on infinite uncorrelated networks [44,46]. Further, the so-called stochastic pair approximation (SPA) captures leading-order corrections in the network size [43]. For our model we find (see SM)

$$H_{\rm PA}(\tau) = \frac{\mu - 2 - \mu\tau + \sqrt{(\tau+1)^2\mu^2 - 4(\mu-1)}}{2(\mu-1)}, \quad (4)$$

and

$$H_{\rm SPA}(\tau) = H_{\rm PA}(\tau)\left[1 - \frac{\mu_2}{N\mu^2}\frac{\tau + H_{\rm PA}(\tau)}{\tau}\right] + \mathcal{O}(N^{-2}), \quad (5)$$

where μ is the mean degree of the network and μ_2 is the second moment of the degree distribution.

In addition to these approximations, we can calculate $H(\tau)$ exactly for networks which have the following properties for all fixed integers ℓ: The number of distinct walks of length ℓ starting at any node is the same. Additionally, the fraction $\Omega^{(\ell)}$ of those walks ending at the starting point is also the same for all nodes. For such "homogeneous" networks (HG) we show that (see SM)

$$H_{\rm HG}(\tau) = \frac{1+\tau}{\sum_{\ell=0}^{\infty}\Omega^{(\ell)}/(1+\tau)^{\ell}} - \tau. \quad (6)$$

Simulations show that this expression works well when the degree distribution is sufficiently tight, even if the network is not strictly homogeneous (see SM).

The coefficients $\Omega^{(\ell)}$ can either be calculated in closed form (see the SM for examples), or be obtained by direct enumeration of walks.

Interpretation and test in simulations. – Figure 2(a) illustrates the relation between the different approximations for Barabási-Albert networks [47]. We show simulations for different parameter combinations, but keeping τ in eq. (3) fixed. The trajectories in the (ρ, σ)-space obtained from the PA for infinite systems converge to fixed points on the parabola $\sigma = 2H_{\rm PA}(\tau)\rho(1-\rho)$. Trajectories from simulations of finite systems fluctuate about these fixed points and remain near the parabola. The time averages in the stationary state, $\langle\rho\rangle_{\rm st}$ and $\langle\sigma\rangle_{\rm st}$, are indicated by crosses in fig. 2(a), and lie below the parabola set by $H_{\rm PA}(\tau)$. This deviation is a consequence of finite-size fluctuations. As seen in the figure, the SPA captures this effect to good accuracy.

A similar effect is found for square lattices (fig. 2(b)). Trajectories for finite systems fluctuate near the parabola $\sigma = 2H_{\rm SL}(\tau)\rho(1-\rho)$ (originally derived in [13], where an analytical formula for $H_{\rm SL}(\tau)$ was given). The time average $(\langle\rho\rangle_{\rm st}, \langle\sigma\rangle_{\rm st})$, however, is not on this parabola, but is captured by eq. (6).

[2]The quantity τ can take negative values. This describes systems with anti-correlations between neighbouring spins. For details see sect. S8 B in the SM.

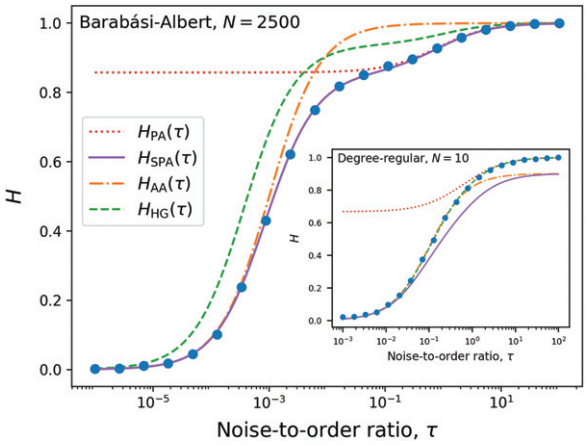

Fig. 3: Order parameter $H(\tau)$ within the different theories discussed in the text. Main axis: Barabási-Albert network ($N = 2500$, mean degree $\mu \approx 8$). Insert: degree-regular network ($N = 10$, $\mu = 4$). Lines show the analytical results as indicated. Blue markers are from simulations, averaged over 100–10000 independent samples in the steady state. Model parameters for a given choice of τ can be randomly generated via the algorithm described in the SM. $H_{AA}(\tau)$ is the "annealed approximation" (see SM).

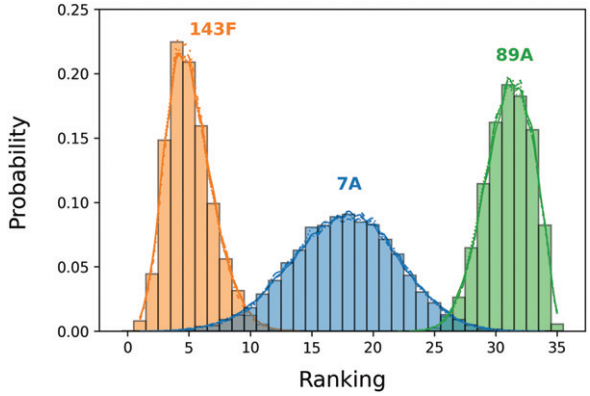

Fig. 4: Rankograms for three linguistic traits 143F, 7A, and 89A from [48], each obtained from 10000 bootstrap samples. For each trait we show rankograms obtained from $\mathbb{H}$ directly, and from τ obtained using the functions $H_{AA}(\tau), H_{SL}(\tau), H_{HG}(\tau)$ and $H_{SPA}(\tau)$. These are virtually indistinguishable from each other on the network. For clarity we show Gaussian kernel density estimates, for H_{HG} we show the full histogram. Further details are given in the SM along with rankograms for the remaining traits.

We test the different approximations further in fig. 3, focusing on Barabási-Albert networks of size $N = 2500$, and a small random degree-regular network ($N = 10$). The PA fails for $\tau \to 0$, because complete ordering can only occur on finite networks. We find $\lim_{\tau\to 0} H_{PA} = \frac{\mu-2}{\mu-1}$, describing long-lived partially ordered states of the voter model [44]. For the Barabási-Albert network the assumption of homogeneity is not valid and eq. (6) is inaccurate. The SPA, on the contrary, describes simulations well in the main panel, but becomes inaccurate for large τ on small networks (inset) due to finite-size effects beyond leading order. For random degree-regular networks (inset) the HG result in eq. (6), on the other hand, produces satisfactory results for all τ. We have deliberately chosen a small network to demonstrate that the HG approach does not require the network to be large.

Stability of language traits. – The World Atlas of Language Structures (WALS) [48] reports geographical coordinates and information on language traits for 2662 of the approximately 7000 existing human languages. We extract 35 binary (or binarisable) traits, such as the presence or absence of a definiteness marker (*e.g.*, the English definite article *the*).

For each trait, we measure the fraction of languages possessing the trait, ρ. We use geographic distance between languages to construct a putative interaction network. Each node is a language and the processes operated on the node represent a coarse-grained view of the dynamics of speaker behaviour (*e.g.*, acquisition of the language by children, or the use of traits of a neighbouring language by bilingual speakers). Further details can be found in the SM. Using this network we calculate the interface density σ. To assess the uncertainty due to the incompleteness of WALS we use bootstrap sampling (*i.e.*, we sample subsets of languages from WALS). This allows us to obtain a distribution for $\mathbb{H} = \sigma/[2\rho(1-\rho)]$ for each trait. Further details of the procedure can be found in the SM.

In ref. [13] the values for $\mathbb{H}$ were converted into τ by inverting the formula $H_{SL}(\tau)$ for the infinite square lattice. This was done without proper justification, as no other analytical solution for $H(\tau)$ was available. Surprisingly, the estimates of τ constructed in this manner correlate well with other measures of the (in)stability of language traits [13,49]. This agreement, while welcome, was somewhat puzzling.

The results of the present work explain these observations. We can use the different functional forms $H_X(\tau)$ to convert the distributions of $\mathbb{H}$ into distributions of τ_X. The subscript X denotes the different theories SPA, AA, SL or HG. For a given X, we can use the distributions of τ_X to construct rankograms for the language traits. The rankogram of each trait describes the probability that the trait is ranked r-th. We illustrate this in fig. 4. The width of a trait's rankogram reflects the confidence we can have in its relative instability based on the data. In [13] the ranking was based on median values of τ only.

Strikingly, the rankogram for any given trait is the same irrespective of the choice X made to infer τ. That is to say, no matter which of the analytical forms $H_X(\tau)$ we use to obtain τ from $\mathbb{H}$, the ranking of traits will remain the same. Noting that $H_{PA}(\tau)$ and $H_{SPA}(\tau)$ only use the first two moments of the empirical degree distribution and that the different approximations $H_X(\tau)$ assume different network types, this ultimately means that the detailed

topology of the interaction network is not important. This robustness explains the surprising success of the stability estimates in [13] obtained using the analytical solution of the model on a square lattice. Figure 4 also shows that ranking on $\mathbb{H}$ itself (*i.e.*, based on the amount of spatial scatter) gives similar results, thus confirming Greenberg's hypothesis that low scattering indicates relative stability, whereas highly scattered traits would be comparatively unstable.

Discussion. – This letter has demonstrated the transmission-noise conjecture for two-state Markovian dynamics with nearest-neighbour interaction on networks. Traits can be ranked by stability from purely spatial information; neither longitudinal data nor detailed knowledge of the interaction network are required. We have used these results to put a long-standing hypothesis about the stability of linguistic traits on firm mathematical ground.

While we have explained the success of the *ad hoc* method of [13], our analysis remains subject to limitations. For example, the model in its current form does not capture temporal networks [50], dynamics with absorbing states [5,51], or cyclic behaviour and frequency-dependent selection in evolutionary games [52–56].

Despite these limitations our model has proved able to recover important facts about the temporal dynamics of language from purely spatial data. We believe that similar applications are possible in the study of opinion change, using, for example, the dataset in [57], or of a range of cases in cultural evolution for which spatial data are more readily available than longitudinal data: the evolution of ethnographic traits in anthropology [58,59], the evolution of musical styles [60–62], and further afield, bird and whale song [63–65]. In addition, further theoretical work should explore how generally it is the case that the noise-to-order ratio in the dynamics of a complex system can be inferred from configurations in the stationary state.

* * *

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Data availability statement: All data that support the findings of this study are included within the article (and any supplementary files).

REFERENCES

[1] Huson D. H. and Bryant D., *Mol. Biol. Evolution*, **23** (2005) 254.
[2] Alon U., *Nat. Rev. Genet.*, **8** (2007) 450.
[3] Beckner C., Blythe R., Bybee J., Christiansen M. H., Croft W., Ellis N. C., Holland J., Ke J., Larsen-Freeman D. and Schoenemann T., *Language Learning*, **59** (2009) 1.
[4] Baxter G. J., Blythe R. A. and McKane A. J., *Phys. Rev. Lett.*, **101** (2008) 258701.
[5] Castellano C., Fortunato S. and Loreto V., *Rev. Mod. Phys.*, **81** (2009) 591.
[6] Kiss I., Miller J. C. and Simon P. L., *Mathematics of Epidemics on Networks From Exact to Approximate Models* (Springer International Publishing, New York, NY) 2017.
[7] Hey J. and Machado C. A., *Nat. Rev. Genet.*, **4** (2003) 535.
[8] Dettmer S. L. and Berg J., *J. Stat. Mech.: Theory Exp.*, **2018** (2018) 023403.
[9] Nachman M. W. and Crowell S. L., *Genetics*, **156** (2000) 297.
[10] Foster P. L., *Methods for determining spontaneous mutation rates* in *DNA Repair, Part B*, *Methods in Enzymology*, Vol. **409** (Academic Press) 2006, pp. 195–213.
[11] Rosche W. A. and Foster P. L., *Methods*, **20** (2000) 4.
[12] Greenberg J. H., *Language Typology: A Historical and Analytic Overview* (De Gruyter Mouton, Berlin, Germany) 1974.
[13] Kauhanen H., Gopal D., Galla T. and Bermúdez-Otero R., *Sci. Adv.*, **7** (2021) eabe6540.
[14] Newman M. E. J., *Networks* (Oxford University Press, Oxford, UK) 2018.
[15] Barabási A.-L., *Network Science* (Cambridge University Press, Cambridge, UK) 2016.
[16] Bermúdez-Otero R., *Glossa: J. Gen. Linguist.*, **5** (2020) 1.
[17] Hickey R., *The Handbook of Language Contact* (Blackwell Publishing Ltd, Hoboken, NJ, USA) 2010.
[18] Comrie B., Dryer M. S., Gil D. and Haspelmath M., *Locations of languages (v2020.4)*, in *The World Atlas of Language Structures Online* (Zenodo) 2013, `https://wals.info/chapter/s1#3.4._Locations_of_languages`.
[19] Johnson R. W., *Teaching Stat.*, **23** (2001) 49.
[20] Boslaugh S., *Statistics in a Nutshell: A Desktop Quick Reference* (O'Reilly Media, Inc., Sebastopol, Cal., USA) 2012.
[21] Dediu D. and Cysouw M., *PLoS ONE*, **8** (2013) e55009.
[22] Commander C. W., *Maximum cut problem, max-cut*, in *Encyclopedia of Optimization*, edited by Floudas C. A. and Pardalos P. M. (Springer US, Boston, Mass.) 2009, pp. 1991–1999.
[23] Salanti G., Ades A. E. and Ioannidis J. P. A., *J. Clin. Epidemiol.*, **64** (2011) 163.
[24] Oldham K., Myland J. and Spanier J., *An Atlas of Functions* (Springer, New York, NY) 2008.
[25] Sato K. and Iwasa Y., *Pair approximations for lattice-based ecological models*, in *The Geometry of Ecological Interactions: Simplifying Spatial Complexity*, edited by Dieckmann U., Law R. and Metz J. A. J., *Cam-*

bridge Studies in Adaptive Dynamics (Cambridge University Press, Cambridge, UK) 2000, pp. 341–358.
[26] Stanley R. P., *Catalan Numbers* (Cambridge University Press, Cambridge, UK) 2015.
[27] Wanless I. M., *Combinator. Probab. Comput.*, **19** (2010) 463.
[28] Borwein J. M. and Borwein P. B., *Pi and the AGM: A Study in the Analytic Number Theory and Computational Complexity* (John Wiley & Sons, Inc, Hoboken, NJ, USA) 1987, https://wayback.cecm.sfu.ca/~pborwein/TEMP_PROTECTED/pi-agm.pdf.
[29] Knuth D. E., *The art of computer programming*, 3rd edition, Vol. **1**: *Fundamental Algorithms* (Addison Wesley Longman, Boston, Mass.) 1997, https://haio.ir/wp-contentuploads/2024/12/Donald-Knuth-The-Art-of-Computer-Programming-Vol.-1[haio.ir]-Fundamental-Algorithms-3rd-Edition-Addison-Wesley-Professional-1997.pdf.
[30] Abbott J., *Laplace transforms of products of modified Bessel functions* (March 31 2022), https://math.stackexchange.com/questions/4404226/laplace-transforms-of-products-of-modified-bessel-functions.
[31] Matsumoto K., *Appell and lauricella hypergeometric functions*, in *Encyclopedia of Special Functions: The Askey-Bateman Project*, edited by Koornwinder T. H. and Stokman J. V. (Cambridge University Press, Cambridge, UK) 2020, pp. 79–100.
[32] Lauricella G., *Rendiconti del Circolo Matematico di Palermo*, **7** (1893) 111.
[33] Prudnikov A. P., Brychkov Y. A. and Marichev O. I., *Integrals and Series Volume 4: Direct Laplace Transforms*, Vol. **4** (Gordon and Breach Science Publishers, Reading, UK) 1992.
[34] Schmidt M. D., *Axioms*, **8** (2019) 62.
[35] Flajolet P. and Sedgewick R., *Analytic Combinatorics* (Cambridge University Press) 2009.
[36] Graham R. L., Knuth D. E., Patashnik O. and Liu S., *Comput. Phys.*, **3** (1989) 106.
[37] Sloane N. J. A., *The on-line encyclopedia of integer sequences*, http://oeis.org.
[38] Gleeson J. P., *Phys. Rev. X* , **3** (2013) 021004.
[39] Dorogovtsev S., *Lectures on Complex Networks* (Oxford University Press, Oxford, UK) 2010.
[40] Knoblauch A., *SIAM J. Appl. Math.*, **69** (2008) 197.
[41] Granovsky B. L. and Madras N., *Stoch. Processes Appl.*, **55** (1995) 23.
[42] Carro A., Toral R. and Miguel M. S., *Sci. Rep.*, **6** (2016) 1.
[43] Peralta A. F., Carro A., Miguel M. S. and Toral R., *New J. Phys.*, **20** (2018) 103045.
[44] Vazquez F. and Eguíluz V. M., *New J. Phys.*, **10** (2008) 063011.
[45] Redner S., *C. R. Phys.*, **20** (2019) 275.
[46] Pugliese E. and Castellano C., *EPL*, **88** (2009) 58004.
[47] Barabási A.-L. and Albert R., *Science*, **286** (1999) 509.
[48] Dryer M. S. and Haspelmath M., (Editors) *WALS Online (v2020.4)* (Zenodo) 2013, https://wals.info/.
[49] Dediu D., *Proc. R. Soc. B: Bio. Sci.*, **278** (2011) 474.
[50] Holme P. and Saramäki J., *Phys. Rep.*, **519** (2012) 97.
[51] Al Hammal O., Chaté H., Dornic I. and Muñoz M. A., *Phys. Rev. Lett.*, **94** (2005) 230601.
[52] Traulsen A. and Hauert C., *Stochastic Evolutionary Game Dynamics* (John Wiley & Sons, Ltd, Hoboken, NJ, USA) 2009, Chapt. 2, pp. 25–61.
[53] Ahern C. and Clark R., *Semantics and Pragmatics*, **10** (2017) 11.
[54] Kauhanen H., *R. Soc. Open Sci.*, **7** (2020) 201682.
[55] Baumann A. and Ritt N., *Phonology*, **34** (2017) 439.
[56] Deo A., *Seman. Pragmat.*, **8** (2015) 1.
[57] Hofmann V., Schütze H. and Pierrehumbert J. B., *The Reddit Politosphere: a large-scale text and network resource of online political discourse*, in *Proceedings of the International AAAI Conference on Web and Social Media*, Vol. **16** (AAAI) 2022, pp. 1259–1267.
[58] Murdock G. P., *Atlas of World Cultures* (University of Pittsburg Press, Pittsburg, Penn., USA) 1981.
[59] Goody J., *World Ethnographic Atlas, 1967–1971* (1974), UK Data Service.
[60] Lomax A. and Berkowitz N., *Science*, **177** (1972) 228.
[61] Savage P. E., *Palgrave Commun.*, **5** (2019) 1.
[62] Verhoef T. and Ravignani A., *Front. Psychol.*, **12** (2021) 668300.
[63] Slater P. J. B., *Trends Ecol. Evol.*, **1** (1986) 94.
[64] Aplin L. M., *Animal Behaviour*, **147** (2019) 179.
[65] Garland E. C., Goldizen A. W., Rekdahl M. L., Constantine R., Garrigue C., Hauser N. D., Poole M. M., Robbins J. and Noad M. J., *Curr. Biol.*, **21** (2011) 687.