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RESEARCH ARTICLE

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Asymptotic estimates of large gaps between directions in certain planar quasicrystals

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Dedicated to Gustav Hammarhjelm (1992–2022).

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Abstract

For quasicrystals of cut-and-project type in $\mathbb{R}^d$, it was proved by Marklof and Strömbergsson [Int. Math. Res. Not. IMRN (2015), no. 15, 6588–6617; erratum, ibid. 2020] that the limit local statistical properties of the directions to the points in the set are described by certain $\mathrm{SL}_d(\mathbb{R})$-invariant point processes. In this paper, we make a detailed study of the tail asymptotics of the limiting gap statistics of the directions, for certain specific classes of planar quasicrystals.

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1 | INTRODUCTION

1.1 | Gaps between directions in planar point sets

Given a locally finite point set $\mathcal{P}$ in the plane $\mathbb{R}^2$, we consider its set of *directions*, that is, the set of points $\widehat{\boldsymbol{v}} := \|\boldsymbol{v}\|^{-1}\boldsymbol{v}$ on the unit circle S^1_1 as $\boldsymbol{v}$ runs through $\mathcal{P} \setminus \{\mathbf{0}\}$. For each $R > 0$, let Δ_R be the finite multi-set of directions to points in $\mathcal{P}$ within distance R from the origin, that is,

$$\Delta_R := \{\widehat{\boldsymbol{v}} \ : \ \boldsymbol{v} \in \mathcal{P},\ 0 < \|\boldsymbol{v}\| \leqslant R\}. \tag{1.1}$$

Throughout the paper, we will assume that $\mathcal{P}$ has an asymptotic density $c_{\mathcal{P}} > 0$, meaning that for any bounded set $\mathcal{D} \subset \mathbb{R}^2$ with boundary of measure zero,

$$\lim_{R\to\infty} \frac{\#(\mathcal{P} \cap R\mathcal{D})}{R^2} = c_{\mathcal{P}}\,\mathrm{Area}(\mathcal{D}). \tag{1.2}$$

It then follows that the multi-set Δ_R has cardinality $\sim c_{\mathcal{P}}\pi R^2$ as $R \to \infty$, and furthermore that this multi-set becomes asymptotically equidistributed along the unit circle, in the sense that for any arc $I \subset \mathrm{S}^1_1$,

$$\lim_{R\to\infty} \frac{\#(\Delta_R \cap I)}{\#\Delta_R} = \frac{|I|}{2\pi},$$

where $|I|$ denotes the length of I (in particular $|\,\mathrm{S}^1_1\,| = 2\pi$).

In this situation, it is natural to consider finer questions about the local statistics of the points in Δ_R. A particular statistics which has been much studied is the distribution of normalized gaps between the points in Δ_R, as $R \to \infty$. For example, when $\mathcal{P}$ is the set of primitive lattice points in $\mathbb{Z}^2$, the limiting distribution of normalized gaps was explicitly determined by Boca, Cobeli and Zaharescu [9]. More generally, if $\mathcal{P}$ is an arbitrary lattice (possibly translated) in $\mathbb{R}^d$, for any $d \geqslant 2$, the limiting distribution of general local statistics of directions was proved to exist by Marklof and Strömbergsson [26]; see also [14] and [23] regarding convergence of related moments. In particular it was noted in [26] that the explicit limiting distribution computed in [9] remains valid when replacing $\mathbb{Z}^2$ by an arbitrary lattice in $\mathbb{R}^2$; furthermore, when $\mathcal{P}$ is an arbitrary 'irrational' translate of a lattice in $\mathbb{R}^2$, the limiting distribution of gaps between directions coincides with the gap distribution for the fractional parts of $\sqrt{n}$ calculated by Elkies and McMullen [15]; see [26, section 1.2] and [25].

Athreya and Chaika [2, Proposition 3.10] have proved that the limiting distribution of gaps between directions exists in the case when $\mathcal{P}$ is the set of holonomy vectors of either the saddle connections or periodic cylinders on a generic translation surface; see also [3, 8, 24, 41, 46] and [36] for later studies with more precise results in specific cases. Also in hyperbolic n-space, for any point set which is the orbit of a lattice within the group of orientation preserving isometries, the analogous limiting gap distribution and also more general local statistics of directions has been proved to exist by Marklof and Vinogradov [32]; for related work, see [10, 22, 39].

This paper concerns the case of $\mathcal{P}$ being a regular cut-and-project set (also referred to as a Euclidean model set). For this case, the limit distribution of normalized gaps between directions was proved to exist, and given a complicated but explicit description, by Marklof and Strömbergsson in [29]; see Theorem 1 below. See also Rühr, Smilansky and Weiss [40] for closely related

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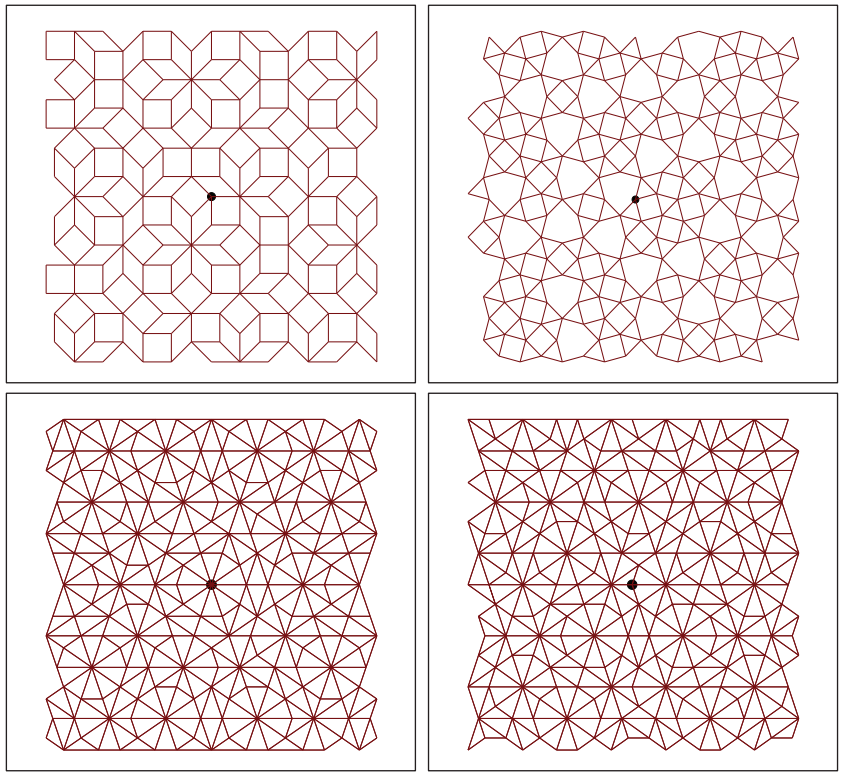

FIGURE 1 Patches of an Ammann–Beenker tiling with vertex set $\mathcal{P}_{\mathrm{AB},(0.8,0.1)}$ (top left panel), a Gähler's shield tiling with vertex set $\mathcal{P}_{\mathrm{Gh},(1.7,0.6)}$ (top right panel), Tübingen triangle tilings with vertex sets $\mathcal{P}_{\mathrm{TT},(0.001,0)}$ (bottom left panel) and $\mathcal{P}_{\mathrm{TT},(1.3,0.4)}$ (bottom right panel). The notation is explained in Section 1.5. In each patch, the origin is marked by a small black disc.

investigations, and El-Baz [13] for an extension to so-called adelic model sets. The results of [29] answered some questions which had been raised by Baake, Götze, Huck and Jakobi in [4], where a numerical investigation was carried out of the normalized gap distribution between directions for several vertex sets coming from aperiodic tilings, some of these being of cut-and-project type and others not. Building on the results of [29], Hammarhjelm [20] explicitly determined the limit of the *minimal* normalized gap between the directions to the points in $\mathcal{P}$, for $\mathcal{P}$ belonging to either of two families of planar quasicrystals, including both the Ammann–Beenker point set and the vertex sets of some rhombic Penrose tilings. Also in [20], the asymptotic density of visible points was explicitly determined for several families of planar quasicrystals, including the two families just mentioned.

The main purpose of this paper is to continue the explicit study begun in [20] of the limit distribution of normalized gaps between directions in the case of $\mathcal{P}$ belonging to certain families of planar cut-and-project sets, which include some well-known classes of quasicrystals; see Figure 1. Our focus will be on asymptotics for *large* gaps between directions. This work grew out of the initial study carried out by Hammarhjelm [19].

We remark that the methods developed in this paper can be expected to be useful also for questions related to the *Lorentz gas* on a cut-and-project scatterer configuration — namely, for the task of obtaining asymptotic estimates for the transition probabilities in the transport (Markov)

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process which arises when considering such a Lorentz gas in the limit of low scatterer density [28, 30]. For the case of a lattice scatterer configuration, such asymptotic estimates were obtained in [27], and found an important application in [31].

1.2 | The limit distribution of gaps for cut-and-project sets

To recall the precise definition of cut-and-project sets considered in [29], let $d, m \geqslant 1$, set $n = d + m$ and let $\mathcal{L}$ be a lattice of full rank in $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$. We refer to $\mathbb{R}^d$ and $\mathbb{R}^m$ as the *physical space* and *internal space*, respectively. In this paper, the dimension of the physical space will always be $d = 2$. We write π and π_{int} for the orthogonal projection of $\mathbb{R}^n$ onto the first d coordinates and last m coordinates. Let $\mathcal{A}$ be the closure of $\pi_{\mathrm{int}}(\mathcal{L})$; this is a closed abelian subgroup of $\mathbb{R}^m$. Let $\mathcal{A}^\circ$ be the identity component of $\mathcal{A}$; this is a linear subspace of $\mathbb{R}^m$, and set $m' = \dim \mathcal{A} = \dim \mathcal{A}^\circ$. Let $\mathcal{W}$, the *window*, be a bounded subset of $\mathcal{A}$ with nonempty interior (with respect to the topology of $\mathcal{A}$). We will always assume that $\mathcal{W}$ and $\mathcal{L}$ are such that

$$\forall\, \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{L} : \quad \text{if } \pi_{\mathrm{int}}(\boldsymbol{y}_1) \in \mathcal{W} \text{ and } \pi_{\mathrm{int}}(\boldsymbol{y}_2) \in \mathcal{W} \text{ and } \pi(\boldsymbol{y}_1) = \pi(\boldsymbol{y}_2), \text{ then } \boldsymbol{y}_1 = \boldsymbol{y}_2. \tag{1.3}$$

We now define the *cut-and-project set* associated to $\mathcal{W}$ and $\mathcal{L}$ to be

$$\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}) = \left\{ \pi(\boldsymbol{y}) \ : \ \boldsymbol{y} \in \mathcal{L},\ \pi_{\mathrm{int}}(\boldsymbol{y}) \in \mathcal{W} \right\} \subset \mathbb{R}^d. \tag{1.4}$$

We will always assume that $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is *regular*, meaning that $\partial\mathcal{W}$ has measure zero with respect to the Haar measure of $\mathcal{A}$. Under these assumptions, it is known that $\mathcal{P}$ has an asymptotic density, that is, (1.2) holds, with $c_{\mathcal{P}}$ being given by a simple explicit expression in terms of $\mathcal{L}$ and $\mathcal{W}$ [28, Proposition 3.2]; see also (1.11) below.

Let us view the unit circle S^1_1 as the set of $z \in \mathbb{C}$ with $|z| = 1$, and for any $z \in \mathrm{S}^1_1$, let us call the number $\frac{1}{2\pi}\arg(z)$ the *normalized angle* of z; this gives an identification between S^1_1 and $\mathbb{R}/\mathbb{Z}$. Let us order the normalized angles of the points in Δ_R in an increasing list as

$$-\tfrac{1}{2} < \xi_{R,1} \leqslant \xi_{R,2} \leqslant \cdots \leqslant \xi_{R,N(R)} \leqslant \tfrac{1}{2}, \tag{1.5}$$

where $N(R) = \#\Delta_R$. Also set $\xi_{R,0} = \xi_{R,N(R)} - 1$.

The following result was proved in [29] (see also Remark 1.7 below).

Theorem 1 [29, Corollary 3]. *Let $\mathcal{P}$ be a regular cut-and-project set in $\mathbb{R}^2$. Then there exists a decreasing function $F : \mathbb{R}_{\geqslant 0} \to [0, 1]$, which is continuous on $\mathbb{R}_{>0}$, such that for every $s \geqslant 0$,*

$$\lim_{R\to\infty} \frac{\#\{1 \leqslant j \leqslant N(R) \ : \ N(R)(\xi_{R,j} - \xi_{R,j-1}) \geqslant s\}}{N(R)} = F(s). \tag{1.6}$$

Furthermore, this limit relation (1.6) remains true, with the function F unchanged, if $\mathcal{P}$ is replaced by $\mathcal{P}\,T = \{\boldsymbol{v}T \ : \ \boldsymbol{v} \in \mathcal{P}\}$ for any fixed $T \in \mathrm{GL}_2(\mathbb{R})$.

We recall in Section 3.3 below some of the key steps in the proof of the above result; a crucial ingredient is Ratner's classification [37] of measures invariant under unipotent flows in

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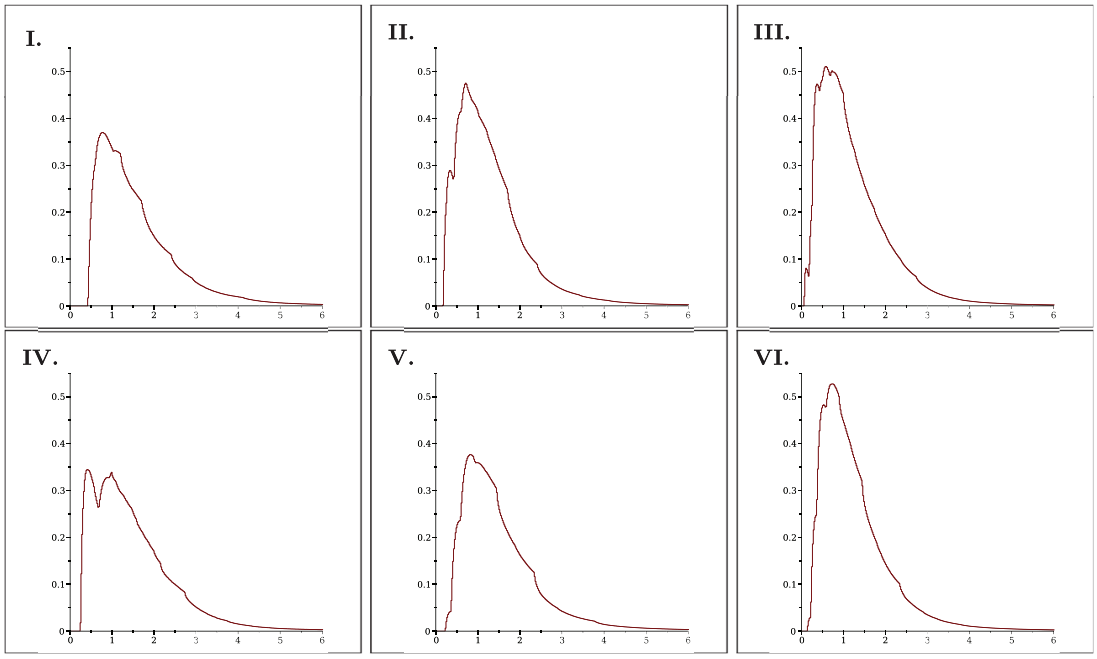

FIGURE 2 Experimental graphs of the limiting density of normalized gaps, $-F'(s)$, for the vertex sets of Ammann–Beenker tilings $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ with (I) $\boldsymbol{w} = \mathbf{0}$ and (II) $\boldsymbol{w} = (0.9, 0.3)$; Gähler's shield tilings $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ with (III) $\boldsymbol{w} = (1.7, 0.6)$ and (IV) $\boldsymbol{w} = (0.1, 0)$; Tübingen triangle tilings $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$ with (V) $\boldsymbol{w} = (0.5, 0.4)$ and (VI) $\boldsymbol{w} = (1.3, 0.4)$ (notation as in Section 1.5). These graphs were obtained from numerical computations described in Section 5.4.

homogeneous dynamics. The limit distribution function $F(s)$ in Theorem 1 is explicitly given by a formula involving the Haar measure of a certain family of subsets of a homogeneous space. For the particular planar cut-and-project sets which we consider in this paper, this homogeneous space turns out to be of the form $\Gamma\backslash H$ where Γ is a Hilbert modular subgroup of $H = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

Figure 2 shows conjectural graphs of the function $-F'(s)$, that is, the limiting density of normalized gaps, for some examples of vertex sets of Ammann–Beenker, Gähler's shield and Tübingen triangle tilings.

Remark 1.7. Theorem 1 is a special case of [29, Corollary 3], since in [29, Corollary 3] we allow $\mathcal{L}$ in $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ to be a *translate* of a lattice in $\mathbb{R}^n$, whereas in this paper we always require $\mathcal{L}$ to be a genuine lattice, that is, $\mathbf{0} \in \mathcal{L}$. In fact, in Theorem 2 below, we will also assume $\mathbf{0} \in \mathcal{W}$, and so $\mathbf{0} \in \mathcal{P}$.

We also point out that the proof of [29, Corollary 3] immediately extends (by utilizing the freedom of choice of the measure 'λ' in [29, Theorem 2]) to show that the limiting gap distribution for $\mathcal{P}$ remains the same if we restrict attention to the directions lying in any fixed subinterval of S^1_1. That is, the following more general version of (1.6) holds: For any fixed $-\frac{1}{2} \leqslant \alpha_1 < \alpha_2 \leqslant \frac{1}{2}$,

$$\lim_{R\to\infty} \frac{\#\{1 \leqslant j \leqslant N(R) \ : \ \xi_{R,j} \in (\alpha_1, \alpha_2],\ N(R)(\xi_{R,j} - \xi_{R,j-1}) \geqslant s\}}{(\alpha_2 - \alpha_1) \cdot N(R)} = F(s). \tag{1.8}$$

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The last statement of Theorem 1, regarding the invariance of $F(s)$ when replacing $\mathcal{P}$ by $\mathcal{P}T$, is easily derived from (1.8) by a limit argument letting $\alpha_2 - \alpha_1 \to 0$. (In the special cases when T is simply a scaling by a constant or a rotation or a reflection, the statement even follows directly from (1.6).)

Remark 1.9. Our notation differs from the notation used in [4, 29] and [20]: In this paper, we do not remove the points in $\mathcal{P}$ which are 'invisible' from the origin; thus we allow Δ_R to be a multi-set, and we allow equalities in the list in (1.5). Also, our function $F(s)$ equals the function in the right-hand side of [29, (1.15)], and is thus in general not the same as '$F(s)$' in [29, Corollary 3]. Clearly by (1.6), our function $F(s)$ satisfies $F(0) = 1$. In general F has a jump discontinuity at 0; it follows from [29, Corollary 3] that $\lim_{s\to 0+} F(s) = \kappa_{\mathcal{P}}$ where $\kappa_{\mathcal{P}}$ is the relative density of visible points in $\mathcal{P}$, a quantity which was defined and proved to exist in [29]. Note that it is immediate to translate between the two versions of '$F(s)$', except that it requires knowledge of the constant $\kappa_{\mathcal{P}}$, which is non-trivial to compute. See [20] for explicit formulas for $\kappa_{\mathcal{P}}$ in special cases. See also the discussion at the end of Section 5.4 below regarding the effect of the different normalizations when comparing the graph in the top left panel of Figure 2 with the graphs in [4, fig. 9] and [20, fig. 2].

1.3 | Main result tail asymptotics

Our main goal in this paper is to describe the tail asymptotics of the distribution function $F(s)$ in Theorem 1, for a particular class of cut-and-project sets in $\mathbb{R}^2$, which includes several classical examples. We expect that the methods which we develop can be extended to more general cases as well (in particular see Remark 1.33 below). We now give the description of the class which we will consider. Let K be a real quadratic field, let $\mathcal{O}_K$ be its ring of integers, let σ be the unique non-trivial automorphism of K and let $\lambda \in \mathcal{O}_K^\times$ be the fundamental unit (thus $\lambda > 1$ and $\mathcal{O}_K^\times = \{\pm\lambda^n \ : \ n \in \mathbb{Z}\}$). Let $\mathcal{L}_K$ be the Minkowski embedding of $\mathcal{O}_K^2$ in $\mathbb{R}^4$, namely,

$$\mathcal{L}_K := \left\{(\alpha, \beta, \sigma(\alpha), \sigma(\beta)) \in \mathbb{R}^4 \ : \ (\alpha, \beta) \in \mathcal{O}_K^2\right\}. \tag{1.10}$$

We set $d = m = 2$, that is, we view $\mathbb{R}^4$ as the product of a 2-dimensional physical space and a 2-dimensional internal space, and π and π_{int} denote the orthogonal projection of $\mathbb{R}^4$ onto the first 2 coordinates and the last 2 coordinates, respectively. Note that in this case $\pi_{\text{int}}(\mathcal{L}_K)$ is dense in $\mathbb{R}^2$, that is, we have $\mathcal{A} = \mathbb{R}^2$, and so we take the window $\mathcal{W}$ to be a bounded subset of $\mathbb{R}^2$ with non-empty interior. (In this case, the restriction of π to $\mathcal{L}_K$ is injective, and so the property (1.3) is automatically fulfilled.) We will always assume $\mathcal{P} := \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$ is regular, that is, that $\partial\mathcal{W}$ has Lebesgue measure zero.

For this class of cut-and-project sets, the formula for the asymptotic density of $\mathcal{P}$, [28, (1.7)], becomes (see also Section 2 below):

$$c_{\mathcal{P}} = \Delta_K^{-1} \operatorname{Area}(\mathcal{W}), \tag{1.11}$$

where Δ_K is the discriminant of K.

As we prove in Remark 3.31 below, the limiting gap distribution function is unchanged if $\mathcal{W}$ is modified by any measure zero set. In particular, without loss of generality we may assume $\mathcal{W}$

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is open (indeed, otherwise replace $\mathcal{W}$ by its interior). Finally, in this paper, we will make the key assumption that $\mathbf{0} \in \mathcal{W}$.

In our approach, we actually work with the integral of the distribution function F, that is, with

$$G(s) := \int_s^\infty F(t)\,\mathrm{d}t. \tag{1.12}$$

As we will note below (see Lemma 3.2), using the fact that F is decreasing, an easy interpolation argument allows us to deduce an asymptotic formula for F once we know an asymptotic formula for G.

Our main result is the following:

Theorem 2. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$ where K is a real quadratic field and $\mathcal{W}$ is a bounded open subset of $\mathbb{R}^2$ such that $\mathbf{0} \in \mathcal{W}$ and $\partial\mathcal{W}$ has Lebesgue measure zero. Let $F(s)$ be the associated limiting gap distribution function as in Theorem 1, and let $G(s) = \int_s^\infty F(t)\,\mathrm{d}t$. Then there exists a positive constant $a_{\mathcal{P}}$ such that*

$$G(s) \sim a_{\mathcal{P}} s^{-1} \quad \text{and} \quad F(s) \sim a_{\mathcal{P}} s^{-2} \qquad \text{as } s \to \infty. \tag{1.13}$$

If we further assume $\mathcal{W}$ to be convex, then we have

$$G(s) = a_{\mathcal{P}} s^{-1} + O\big(s^{-2}\log s\big) \quad \text{and} \quad F(s) = a_{\mathcal{P}} s^{-2} + O\Big(s^{-\frac{5}{2}}\sqrt{\log s}\Big) \qquad \text{as } s \to \infty. \tag{1.14}$$

Remark 1.15. As mentioned above, in order to prove Theorem 2, it suffices to prove the two asymptotic estimates for $G(s)$ in (1.13) and (1.14). Both these estimates will in fact follow from a more precise, general asymptotic formula for $G(s)$ with an explicit error term; see (3.7) below. We also mention that our analysis leads to an explicit formula for the leading coefficient $a_{\mathcal{P}}$; see (3.8).

1.4 | Formula for $a_{\mathcal{P}}$

As we will now describe, after imposing one more assumption on the window set, we can further evaluate the aforementioned formula for $a_{\mathcal{P}}$. To state our result, we need to introduce some more notation. For $\theta \in \mathbb{R}$, we write $\mathrm{k}_\theta := \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Then for any subset $\mathcal{W} \subset \mathbb{R}^2$, we let $\ell_{\mathcal{W}}(\theta)$ be the projection of $\mathcal{W}\mathrm{k}_{-\theta}$ on the x-axis. On top of the assumptions in Theorem 2, we will now assume that for each θ, $\ell_{\mathcal{W}}(\theta)$ is an *interval*. Note that this assumption is always satisfied if $\mathcal{W}$ is *connected*, but it also holds for many non-connected window sets $\mathcal{W}$.

Note that since $\mathcal{W}$ is open and $\mathbf{0} \in \mathcal{W}$, $\ell_{\mathcal{W}}(\theta)$ is an open subset of $\mathbb{R}$ containing 0. Hence if $\ell_{\mathcal{W}}(\theta)$ is assumed to be an interval, then we can parameterize it by two positive numbers $r(\theta), \nu(\theta)$ via

$$\ell_{\mathcal{W}}(\theta) = r(\theta)(-\nu(\theta), 1). \tag{1.16}$$

Theorem 3. *Retain the notation and assumptions in Theorem 2 and further assume that K is of class number one, and that $\ell_{\mathcal{W}}(\theta)$ is an interval for each θ. Then there exist a finite partition $\mathbb{R}_{>0} = \bigsqcup_{j=1}^{l} S_j$ of $\mathbb{R}_{>0}$ into intervals and non-negative constants A_j, B_j $(1 \leqslant j \leqslant l)$ depending only on K,*

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such that

$$a_{\mathcal{P}} = \frac{\mathrm{Area}(\mathcal{W})}{4\Delta_K^2 \zeta_K(2)} \sum_{j=1}^{l} \int_{\tilde{S}_j} r(\theta)^{-2} \big(A_j + B_j \nu(\theta)^{-2}\big)\, \mathrm{d}\theta, \tag{1.17}$$

where ζ_K *is the Dedekind zeta function attached to* K, $\tilde{S}_j := \{\theta \in [0, 2\pi) \ : \ \nu(\theta) \in S_j\}$, *and* $r(\theta)$ *and* $\nu(\theta)$ *are defined by (1.16).*

Remark 1.18. We stress that the intervals S_j are allowed to be open, closed or half-open, and may be degenerate, that is, of the form $[a, a] = \{a\}$ for some $a > 0$.

Remark 1.19. Our analysis applies for any real quadratic field K and the class number one assumption in Theorem 3 is only for simplicity of presentation; see Theorem 5 in Section 4 for the most general version. We also mention that the partition $\mathbb{R}_{>0} = \bigsqcup_{j=1}^{l} S_j$ and the constants A_j, B_j in the above theorem are all computable and we will illustrate it in the next section for three well-known classes of quasicrystals, namely, the vertex sets of the Ammann–Beenker, Gähler's shield and Tübingen triangle tilings.

Remark 1.20. Among the sets $\tilde{S}_j$, there is one that is always non-empty, namely, the unique $\tilde{S}_i$ such that $1 \in S_i$. In certain cases, this $\tilde{S}_i$ equals the whole interval $[0, 2\pi)$ while all other $\tilde{S}_j$s are empty. In this case, the formula for $a_{\mathcal{P}}$ can be simplified into

$$a_{\mathcal{P}} = C_K \,\mathrm{Area}(\mathcal{W}) \,\mathrm{Area}(\mathcal{W}^*),$$

where C_K is some positive constant depending only on K, and $\mathcal{W}^*$ is the *polar set* of $\mathcal{W}$, that is,

$$\mathcal{W}^* := \{\boldsymbol{z} \in \mathbb{R}^2 \ : \ \boldsymbol{z} \cdot \boldsymbol{w} \leqslant 1 \ \ \forall\, \boldsymbol{w} \in \mathcal{W}\}. \tag{1.21}$$

Here $\cdot$ is the standard scalar product in $\mathbb{R}^2$. See Section 4.1 for more details. If $\mathcal{W}$ is also convex and centrally symmetric, the product $\mathrm{Area}(\mathcal{W})\,\mathrm{Area}(\mathcal{W}^*)$ appearing in the above formula is known as the *Mahler volume* of $\mathcal{W}$.

Remark 1.22. For any $\mathcal{W}$ as in Theorem 3, replacing $\mathcal{W}$ by its *convex hull* does not affect the intervals $\ell_{\mathcal{W}}(\theta)$, hence does not affect the functions $r(\theta)$ and $\nu(\theta)$. Therefore, in the explicit formula (1.17), only the factor $\mathrm{Area}(\mathcal{W})$ is affected when replacing $\mathcal{W}$ by its convex hull. In this connection, it should also be noted that the polar set of the convex hull of $\mathcal{W}$ equals the polar set of $\mathcal{W}$.

1.5 | Examples

We next illustrate how our results apply in three cases of well-known planar quasicrystals. The three propositions below are all proved in Section 4.4. More details, and comparison with numerics, are provided in Section 5.

We first consider the *Ammann–Beenker tiling*, which was discovered by Robert Ammann in the 70s and first described in [18] and [1]. Specifically, we consider the 'A5 set, variant (b)', in the notation of [1]; see Figure 1 (top left panel) above for a small patch of this tiling. It is well-known

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that the set of vertices of an arbitrary Ammann–Beenker tiling can be generated using the cut-and-project construction. Specifically, let $K = \mathbb{Q}(\sqrt{2})$, and let $\mathcal{W}_{\mathrm{AB}} \subset \mathbb{R}^2$ be the open regular octagon centered at the origin of edge length 1, oriented so that four of the edges are perpendicular to a coordinate axis. For each $\boldsymbol{w} \in \mathbb{R}^2$ we set

$$\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{AB})} := (\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w})g_1 \subset \mathbb{R}^2 \qquad \text{and} \qquad \mathcal{P}_{\mathrm{AB},\boldsymbol{w}} := \mathcal{P}(\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{AB})}, \mathcal{L}_K)g_2, \tag{1.23}$$

where $g_1 := \left(\begin{smallmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{smallmatrix}\right)$ and $g_2 := \left(\begin{smallmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{smallmatrix}\right)$. Then for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{AB}}$ with the property that $\pi_{\mathrm{int}}(\mathcal{L}_K) \cap \partial\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{AB})} = \emptyset$, the cut-and-project set $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ is the vertex set of an Ammann–Beenker tiling with one vertex at $\mathbf{0}$, and conversely, the vertex set of *any* Ammann–Beenker tiling having a vertex at $\mathbf{0}$ is (up to scaling and rotation) either equal to such a point set $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$, or can be obtained as a limit of such point sets $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ in an appropriate topology. (See [5, Chapter 7.3] and Section 5.1 below.)

Note that because of (1.23) and the last statement in Theorem 1, the formula (1.14) in Theorem 2 applies to the cut-and-project set $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{AB}}$. Here the constant $a_{\mathcal{P}}$ is given by a quite simple formula, also valid for much more general window sets:

Proposition 1.1. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$ with $K = \mathbb{Q}(\sqrt{2})$ and with $\mathcal{W}$ as in Theorem 3, and let $\ell_{\mathcal{W}}(\theta)$ be parameterized as in (1.16). Then*

$$\begin{aligned} a_{\mathcal{P}} = \frac{3\sqrt{2}}{16\pi^4}\operatorname{Area}(\mathcal{W})\Bigg(&(7+8\sqrt{2})\int_{\tilde{S}_1} r(\theta)^{-2}\,\mathrm{d}\theta + (4+6\sqrt{2})\int_{\tilde{S}_2} r(\theta)^{-2}\,\mathrm{d}\theta \\ &+ (2+2\sqrt{2})\int_{\tilde{S}_3} r(\theta)^{-2}\,\mathrm{d}\theta + \int_{\tilde{S}_4} r(\theta)^{-2}\,\mathrm{d}\theta\Bigg), \end{aligned} \tag{1.24}$$

where $\tilde{S}_j := \{\theta \in [0,2\pi) \ : \ \nu(\theta) \in S_j\}$, with

$$S_1 = \big(0, \sqrt{2}-1\big], \quad S_2 = \big(\sqrt{2}-1, \tfrac{1}{2}\sqrt{2}\big), \quad S_3 = \big[\tfrac{1}{2}\sqrt{2}, \sqrt{2}\big] \quad \textit{and} \quad S_4 = \big(\sqrt{2}, 1+\sqrt{2}\big).$$

When $\tilde{S}_3 = [0,2\pi)$, the above formula simplifies to

$$a_{\mathcal{P}} = \frac{3\sqrt{2}(1+\sqrt{2})}{4\pi^4}\operatorname{Area}(\mathcal{W})\operatorname{Area}(\mathcal{W}^*). \tag{1.25}$$

Note that Proposition 1.1 applies when $\mathcal{W} = \mathcal{W}_{\boldsymbol{w}}^{(\mathrm{AB})}$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{AB}}$, and the formula (1.25) holds whenever $\boldsymbol{w}$ lies sufficiently near $\mathbf{0}$. In particular, (1.25) implies

$$a_{\mathcal{P}_{\mathrm{AB},\mathbf{0}}} = 24/\pi^4 = 0.24638\ldots. \tag{1.26}$$

In Section 5.4, we present a comparison, for a few examples of points $\boldsymbol{w} \in \mathcal{W}_{\mathrm{AB}}$, between the exact values of $a_{\mathcal{P}}$ computed using Proposition 1.1, and numerically computed approximate values of $s^2F(s)$ for large s.

Next we consider *Gähler's shield tiling*, which was discovered in [17]; these tilings are built up of a certain triple of tiles (a triangle, a square and a hexagon called a 'shield'), equipped with

local matching rules. The vertex set of a Gähler's shield tiling can be obtained using the cut-and-project construction in the following way: Let $K = \mathbb{Q}(\sqrt{3})$, and let $\mathcal{W}_{\mathrm{Gh}} \subset \mathbb{R}^2$ be the open regular dodecagon centered at the origin of edge length 1, so that four of the edges are perpendicular to a coordinate axis. For each $\boldsymbol{w} \in \mathbb{R}^2$, we set

$$\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{Gh})} := (\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w})g_1 \subset \mathbb{R}^2 \qquad \text{and} \qquad \mathcal{P}_{\mathrm{Gh},\boldsymbol{w}} := \mathcal{P}(\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{Gh})}, \mathcal{L}_K)g_2, \tag{1.27}$$

where $g_1 := \left(\begin{smallmatrix} 1 & 0 \\ \sqrt{3} & 2 \end{smallmatrix}\right)$ and $g_2 := \left(\begin{smallmatrix} 1 & 0 \\ \sqrt{3}/2 & 1/2 \end{smallmatrix}\right)$. Then for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{Gh}}$ with the property that $\pi_{\mathrm{int}}(\mathcal{L}_K) \cap \partial\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{Gh})} = \emptyset$, the cut-and-project set $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ is the vertex set of a Gähler's shield tiling with one vertex at $\mathbf{0}$. (See [5, Chapter 7.3] and Section 5.2 below.)

Again, (1.14) in Theorem 2 applies to $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{Gh}}$, and regarding $a_{\mathcal{P}}$ we have the following:

Proposition 1.2. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$ with $K = \mathbb{Q}(\sqrt{3})$ and with $\mathcal{W}$ as in Theorem 3, and let $\ell_{\mathcal{W}}(\theta)$ be parameterized as in (1.16). Then*

$$\begin{aligned} a_{\mathcal{P}} = \frac{\sqrt{3}}{16\pi^4}\,\mathrm{Area}(\mathcal{W})\Bigg(& (17+16\sqrt{3})\int_{\tilde{S}_1} r(\theta)^{-2}\,\mathrm{d}\theta + (9+12\sqrt{3})\int_{\tilde{S}_2} r(\theta)^{-2}\,\mathrm{d}\theta \\ & + (6+6\sqrt{3})\int_{\tilde{S}_3} r(\theta)^{-2}\,\mathrm{d}\theta + (3+4\sqrt{3})\int_{\tilde{S}_4} r(\theta)^{-2}\,\mathrm{d}\theta + (3+2\sqrt{3})\int_{\tilde{S}_5} r(\theta)^{-2}\,\mathrm{d}\theta \\ & + 2\int_{\tilde{S}_6} r(\theta)^{-2}\,\mathrm{d}\theta\Bigg), \end{aligned} \tag{1.28}$$

where $\tilde{S}_j := \{\theta \in [0, 2\pi) \ : \ \nu(\theta) \in S_j\}$, with $S_1 = \left(0, \frac{1}{1+\sqrt{3}}\right]$, $S_2 = \left(\frac{1}{1+\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $S_3 = \left[\frac{1}{\sqrt{3}}, \sqrt{3}-1\right]$, $S_4 = \left(\sqrt{3}-1, \frac{1}{\sqrt{3}-1}\right)$, $S_5 = \left[\frac{1}{\sqrt{3}-1}, \sqrt{3}\right]$ and $S_6 = \left(\sqrt{3}, 1+\sqrt{3}\right)$. When $\tilde{S}_4 = [0, 2\pi)$, the above formula simplifies to

$$a_{\mathcal{P}} = \frac{12+3\sqrt{3}}{8\pi^4}\,\mathrm{Area}(\mathcal{W})\,\mathrm{Area}(\mathcal{W}^*). \tag{1.29}$$

Note that Proposition 1.2 applies when $\mathcal{W} = \mathcal{W}_{\boldsymbol{w}}^{(\mathrm{Gh})}$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{Gh}}$, and the formula (1.29) holds whenever $\boldsymbol{w}$ lies sufficiently near $\mathbf{0}$. Again see Section 5.4 for numerical computations related to the values of $a_{\mathcal{P}}$ for $\mathcal{P} = \mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$.

Finally, we consider the *Tübingen triangle tiling*, which was discovered and studied in [6]; these tilings are built up of a certain pair of isosceles triangles. The set of vertices of a Tübingen triangle tiling can be obtained using the cut-and-project construction in the following way. Let $K = \mathbb{Q}(\sqrt{5})$ and $\tau = \frac{1}{2}(1+\sqrt{5})$, and let $\mathcal{W}_{\mathrm{TT}} \subset \mathbb{R}^2$ be the open regular decagon centered at the origin of edge length $\sqrt{(2+\tau)/5}$, oriented so that two of the edges are perpendicular to the first coordinate axis. For each $\boldsymbol{w} \in \mathbb{R}^2$ we set

$$\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{TT})} := (\mathcal{W}_{\mathrm{TT}} + \boldsymbol{w})g_1 \subset \mathbb{R}^2 \qquad \text{and} \qquad \mathcal{P}_{\mathrm{TT},\boldsymbol{w}} := \mathcal{P}(\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{TT})}, \mathcal{L}_K)g_2, \tag{1.30}$$

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where $g_1 := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(2+\tau)/5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau & 2 \end{pmatrix}$ and $g_2 := \begin{pmatrix} 1 & 0 \\ (\tau-1)/2 & \sqrt{2+\tau}/2 \end{pmatrix}$. Then for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{TT}}$ with the property that $\frac{1}{5}\pi_{\mathrm{int}}(\mathcal{L}_K) \cap \partial \mathcal{W}_{\boldsymbol{w}}^{(\mathrm{TT})} = \emptyset$, the cut-and-project set $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$ is the vertex set of a Tübingen triangle tiling with one vertex at $\mathbf{0}$. (See [5, Chapter 7.3], [6, Section 4] and Section 5.3 below.)

Again, (1.14) in Theorem 2 applies to $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{TT}}$, and regarding $a_{\mathcal{P}}$ we have the following:

Proposition 1.3. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$ with $K = \mathbb{Q}(\sqrt{5})$ and with $\mathcal{W}$ as in Theorem 3, and let $\ell_{\mathcal{W}}(\theta)$ be parameterized as in (1.16). Then*

$$a_{\mathcal{P}} = \frac{3\sqrt{5}}{16\pi^4} \operatorname{Area}(\mathcal{W}) \left((5+3\sqrt{5}) \int_{\tilde{S}_1} r(\theta)^{-2}\, \mathrm{d}\theta + (1+\sqrt{5}) \int_{\tilde{S}_2} r(\theta)^{-2}\, \mathrm{d}\theta \right), \tag{1.31}$$

where $\tilde{S}_j := \{\theta \in [0, 2\pi) \ : \ \nu(\theta) \in S_j\}$, with $S_1 = (0, \frac{\sqrt{5}-1}{2}]$ and $S_2 = (\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2})$. When $\tilde{S}_2 = [0, 2\pi)$, the above formula simplifies to

$$a_{\mathcal{P}} = \frac{3(5+\sqrt{5})}{8\pi^4} \operatorname{Area}(\mathcal{W}) \operatorname{Area}(\mathcal{W}^*). \tag{1.32}$$

Proposition 1.3 applies when $\mathcal{W} = \mathcal{W}_{\boldsymbol{w}}^{(\mathrm{TT})}$ for any $\boldsymbol{w} \in \mathcal{W}_{\mathrm{TT}}$, and the formula (1.32) holds whenever $\boldsymbol{w}$ lies sufficiently near $\mathbf{0}$. Again see Section 5.4 for related numerical computations.

Remark 1.33. As a concluding remark of the introduction, we mention that our main result, Theorem 2, does not apply to the vertex sets of the classical rhombic Penrose tilings, as these can only be realized as *unions* of (four) translates of the cut-and-project type sets considered in Theorem 2; see [28, section 2.5]. Nevertheless, in preliminary work, using similar methods as in this paper we have proved analogous tail asymptotic formulas for the limiting gap distribution function for point sets in this generality, thus in particular covering the rhombic Penrose tilings.

2 | PRELIMINARIES

2.1 | Real quadratic fields

In this section, we give a brief review on backgrounds on real quadratic fields. Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with $d \in \mathbb{N}$ square-free. Let $\mathcal{O}_K = \mathbb{Z}[\tau] \subset K$ be its *ring of integers*, where

$$\tau = \begin{cases} \frac{1+\sqrt{d}}{2} & d \equiv 1 \pmod 4, \\ \sqrt{d} & d \equiv 2, 3 \pmod 4. \end{cases}$$

Let J_K be the (abelian) group of *fractional ideals* of $\mathcal{O}_K$, and let P_K be the subgroup of *principal ideals*. We let $C_K = J_K/P_K$ be the *ideal class group* of K, and we call elements in C_K *ideal classes* of K. For any non-zero $a_1, \dots, a_m \in K$, we denote by $\langle a_1, \dots, a_m \rangle$ the fractional ideal generated by $a_1, \dots, a_m$.

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Let σ be the unique non-trivial automorphism of K and consider the embedding

$$\iota : K \to \mathbb{R}^2, \qquad k \mapsto (k, \sigma(k)). \tag{2.1}$$

We note that $\iota(I)$ is a lattice in $\mathbb{R}^2$ for any $I \in J_K$. In particular, $\iota(\mathcal{O}_K)$ is the lattice generated by (1,1) and $(\tau, \sigma(\tau))$, which is of covolume $\Delta_K^{1/2}$, where

$$\Delta_K := \begin{cases} d & d \equiv 1 \pmod 4, \\ 4d & d \equiv 2, 3 \pmod 4, \end{cases} \tag{2.2}$$

is the *discriminant* of K. For any fractional ideal $I \in J_K$, its *absolute norm* is defined by

$$\mathrm{Nr}(I) := \frac{\mathrm{covol}(\iota(I))}{\mathrm{covol}(\iota(\mathcal{O}_K))} = \Delta_K^{-1/2} \mathrm{covol}(\iota(I)). \tag{2.3}$$

Note that Nr is multiplicative in J_K, that is, $\mathrm{Nr}(I_1 I_2) = \mathrm{Nr}(I_1)\mathrm{Nr}(I_2)$ for any $I_1, I_2 \in J_K$. Moreover, for any $\alpha \in K \setminus \{0\}$, we have $\mathrm{Nr}(\alpha I) = |\mathrm{N}(\alpha)|\mathrm{Nr}(I)$, where $\mathrm{N} : K \to \mathbb{R}$ is the standard norm on K given by $\mathrm{N}(\alpha) := \alpha\sigma(\alpha)$.

2.2 | Hilbert modular group

Let $\mathbb{H}$ be the upper half plane. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via the Möbius transformation: $g \cdot z = \frac{az+b}{cz+d}$ for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$. Let $\mathrm{H} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ (which we view as a subgroup of $\mathrm{SL}_4(\mathbb{R})$ via the block diagonal embedding) and let $\mathbb{H}^2$ be the product of two upper half planes. The group H naturally acts on $\mathbb{H}^2$ via

$$(h_1, h_2)(z_1, z_2) := (h_1 \cdot z_2, h_2 \cdot z_2), \qquad \forall\, h_1, h_2 \in \mathrm{SL}_2(\mathbb{R}),\ z_1, z_2 \in \mathbb{H}.$$

Denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ which can be identified with the boundary of $\mathbb{H}$. The action of H on $\mathbb{H}^2$ naturally extends to $\overline{\mathbb{R}}^2$ via the same formula. We also denote by $\overline{K} := K \cup \{\infty\}$. It embeds into $\overline{\mathbb{R}}^2$ via the natural extension of the embedding $\iota : K \to \mathbb{R}^2$ (by sending $\infty \in \overline{K}$ to $(\infty, \infty) \in \overline{\mathbb{R}}^2$). With slight abuse of notation, we also denote this embedding from $\overline{K}$ to $\overline{\mathbb{R}}^2$ by ι. We will *also* write ι for the group homomorphism $\mathrm{SL}_2(K) \to \mathrm{H}$ given by

$$\iota\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{smallmatrix}\right)\right). \tag{2.4}$$

The *Hilbert modular group* for the field K is given by

$$\mathrm{SL}_2(\mathcal{O}_K) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(K) \ :\ a, b, c, d \in \mathcal{O}_K \right\}.$$

We write Γ_K for the embedding of $\mathrm{SL}_2(\mathcal{O}_K)$ in H:

$$\Gamma_K := \iota\big(\mathrm{SL}_2(\mathcal{O}_K)\big).$$

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The discreteness of $\iota(\mathcal{O}_K)$ in $\mathbb{R}^2$ implies that Γ_K is a discrete subgroup of H. Indeed, it is a non-uniform lattice in H, that is, the homogeneous space $\Gamma_K\backslash \mathrm{H}$ is non-compact and has finite volume with respect to a Haar measure of H. As a subgroup of H, Γ_K naturally acts on $\overline{\mathbb{R}}^2$, and this action preserves $\iota(\overline{K})$. The orbits of $\iota(\overline{K})$ under Γ_K are called the *cusps* of Γ_K. We will often represent a cusp by an element in the corresponding Γ_K-orbit, or by an element in $\overline{K}$ (via the natural identification between $\iota(\overline{K})$ and $\overline{K}$). The following lemma, together with the obvious identification between $\iota(\overline{K})/\Gamma_K$ and $\overline{K}/\mathrm{SL}_2(\mathcal{O}_K)$, shows that the number of cusps of Γ_K equals the number of ideal classes of K.

Lemma 2.1 [16, Lemma 3.5]. *The map sending $k \in \overline{K}$ to $[\langle k, 1\rangle] \in C_K$ induces a bijection between $\overline{K}/\mathrm{SL}_2(\mathcal{O}_K)$ and C_K, where $\langle k, 1\rangle$ is the ideal generated by $k, 1$ if $k \in K$ and $\langle k, 1\rangle := \mathcal{O}_K$ if $k = \infty$. In particular,*

$$\kappa := \#C_K = \textit{number of cusps of } \Gamma_K.$$

Throughout the remainder of this paper, we fix $k_1 = \infty, k_2, \dots, k_\kappa \in \overline{K}$ to be a complete list of cusps of Γ_K. Fix $a_1 = 1, a_2, \dots, a_\kappa \in \mathcal{O}_K$ and $c_1 = 0, c_2, \dots, c_\kappa \in \mathcal{O}_K$ so that $k_i = \frac{a_i}{c_i}$ for each i; then set $I_i := \langle a_i, c_i\rangle$. Then $I_1 = \mathcal{O}_K, I_2, \dots, I_\kappa$ are integral ideals which by Lemma 2.1 form a system of representatives of the ideal classes of K. (Note that I_i depends on the choice of a_i, c_i; however the class $[I_i]$ depends only on k_i.)

For each $1 \leqslant i \leqslant \kappa$, let

$$\Gamma_i := \{\gamma \in \Gamma_K \ : \ \gamma\iota(k_i) = \iota(k_i)\} \tag{2.5}$$

be the *isotropy group* of the cusp k_i. Below we give a more precise description of these isotropy groups. First, the isotropy group of ∞ is easy to compute: For $\gamma = \iota\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right) \in \Gamma_K$, by direct computation $\gamma \cdot \iota(\infty) = \iota(\infty)$ if and only if $c = 0$, that is,

$$\Gamma_1 = \Gamma_K \cap B = \left\{\iota\left(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)\right) \ : \ a \in \mathcal{O}_K^\times,\ b \in \mathcal{O}_K\right\},$$

where $B < \mathrm{H}$ is the subgroup of upper triangular matrices in H.

To compute the other isotropy groups, we will translate the cusp k_i to ∞. For any integral ideal $I \subset \mathcal{O}_K$, let

$$\Gamma_{K,I} := \left\{\iota\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right) \ : \ a, d \in \mathcal{O}_K,\ b \in I^{-2},\ c \in I^2,\ ad - bc = 1\right\}. \tag{2.6}$$

For each $1 \leqslant i \leqslant \kappa$ since $I_i = \langle a_i, c_i\rangle$, we can find $b_i, d_i \in I_i^{-1}$ such that $a_i d_i - b_i c_i = 1$. Throughout the paper, we fix the *scaling matrix*

$$\xi_i = \iota\left(\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i\end{smallmatrix}\right)\right) \in \mathrm{H}. \tag{2.7}$$

Note that ξ_i satisfies $\xi_i\iota(\infty) = \iota(k_i)$. We then have the following description of Γ_i.

Lemma 2.2. *We have for each $1 \leqslant i \leqslant \kappa$,*

$$\Gamma_i = \xi_i\left(\Gamma_{K,I_i} \cap B\right)\xi_i^{-1} = \xi_i\left\{\iota\left(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)\right) \ : \ a \in \mathcal{O}_K^\times,\ b \in I_i^{-2}\right\}\xi_i^{-1}. \tag{2.8}$$

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Proof. By definition, $\gamma \in \Gamma_i$ if any only if $\gamma\iota(k_i) = \iota(k_i)$. Since $\xi_i\iota(\infty) = \iota(k_i)$, this is equivalent to $\xi_i^{-1}\gamma\xi_i\iota(\infty) = \iota(\infty)$. This shows

$$\xi_i^{-1}\Gamma_i\xi_i = (\xi_i^{-1}\Gamma_K\xi_i) \cap B.$$

By direction computation, we have $\xi_i^{-1}\Gamma_K\xi_i \subset \Gamma_{K,I_i}$ and $\xi_i\Gamma_{K,I_i}\xi_i^{-1} \subset \Gamma_K$, implying that $\xi_i^{-1}\Gamma_K\xi_i = \Gamma_{K,I_i}$. Hence

$$\xi_i^{-1}\Gamma_i\xi_i = \Gamma_{K,I_i} \cap B = \left\{\iota\left(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)\right) \ : \ a \in \mathcal{O}_K^\times,\ b \in I_i^{-2}\right\}.$$

We can then finish the proof by conjugating both sides of the above equation by ξ_i. □

2.3 | Coordinates and measures

Let

$$\mathcal{L}_K := \left\{(\alpha, \beta, \sigma(\alpha), \sigma(\beta)) \in \mathbb{R}^4 \ : \ (\alpha, \beta) \in \mathcal{O}_K^2\right\}$$

be the Minkowski embedding of $\mathcal{O}_K^2$ in $\mathbb{R}^4$. Let X be the space of lattices of the form $\mathcal{L}_K h$ with $h \in \mathrm{H}$. Since $\mathcal{L}_K h = \mathcal{L}_K$ if and only if $h \in \Gamma_K$, X can be parameterized by the homogeneous space $\Gamma_K\backslash\mathrm{H}$ via $\mathcal{L}_K h \in X \leftrightarrow \Gamma_K h \in \Gamma_K\backslash\mathrm{H}$. We thus identify X with $\Gamma_K\backslash\mathrm{H}$. Let μ_K be the unique invariant probability measure on $X = \Gamma_K\backslash\mathrm{H}$.

Since we will be working with this measure extensively when computing $G(s)$ later in Section 3, here we give a more explicit description of it in terms of coordinates from an Iwasawa decomposition of H. It is well known that the group $\mathrm{SL}_2(\mathbb{R})$ has an Iwasawa decomposition saying that any element $g \in \mathrm{SL}_2(\mathbb{R})$ can be written uniquely as $g = \mathrm{n}_x\mathrm{a}_y\mathrm{k}_\theta$ for some $x \in \mathbb{R}, y > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, where $\mathrm{n}_x := \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $\mathrm{a}_y := \left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)$ and $\mathrm{k}_\theta := \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. This thus induces an Iwasawa decomposition of H: Every $h \in \mathrm{H}$ can be written uniquely in the form $h = \mathrm{n}_{\mathbf{x}}\mathrm{a}_{\mathbf{y}}\mathrm{k}_{\boldsymbol{\theta}}$ with $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{y} \in (\mathbb{R}_{>0})^2$ and $\boldsymbol{\theta} \in (\mathbb{R}/2\pi\mathbb{Z})^2$, where

$$\mathrm{n}_{\mathbf{x}} := \left(\mathrm{n}_{x_1}, \mathrm{n}_{x_2}\right), \quad \mathrm{a}_{\mathbf{y}} := \left(\mathrm{a}_{y_1}, \mathrm{a}_{y_2}\right) \quad \text{and} \quad \mathrm{k}_{\boldsymbol{\theta}} := (\mathrm{k}_{\theta_1}, \mathrm{k}_{\theta_2}).$$

Under these coordinates, the Haar measure of H (up to scalars) is given by

$$\mathrm{d}\mu_{\mathrm{H}}(h) = y_1^{-3}y_2^{-3}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\boldsymbol{\theta}. \tag{2.9}$$

Note that μ_K essentially comes from a Haar measure of H. Indeed, we may identify X with a fundamental domain inside H; then μ_K is the restriction of a certain Haar measure to this fundamental domain normalized to be a probability measure. Thus μ_K is given by

$$\mathrm{d}\mu_K(h) = c_K y_1^{-3}y_2^{-3}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\boldsymbol{\theta} \tag{2.10}$$

for some normalizing factor $c_K > 0$. This normalizing factor was computed by Siegel [44, 45] and is given by the following formula (see [47, p. 59]):

$$c_K = \Delta_K^{-3/2}\zeta_K(2)^{-1}, \tag{2.11}$$

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where $\zeta_K(s) := \sum_{I\subset\mathcal{O}_K} \mathrm{Nr}(I)^{-s}$ ($\Re\mathfrak{e}(s) > 1$) is the Dedekind zeta function attached to K. Here the summation is over all the non-zero integral ideals of $\mathcal{O}_K$.

Remark 2.12. To compare our formula for $a_{\mathcal{P}}$ in (1.17) with numerical computations, we need to express $\zeta_K(2)$ in more explicit terms. Let χ_K be the Dirichlet character defined by $\chi_K(n) := \left(\frac{\Delta_K}{n}\right)$ with $\left(\frac{\cdot}{\cdot}\right)$ the Kronecker symbol. Note that χ_K is an even primitive quadratic character of modulus Δ_K; see [33, pp. 296–297]. Using the fact that for any prime number p, the ideal $(p) \subset \mathcal{O}_K$ is inert, ramified or split if and only if $\chi_K(p)$ equals $-1, 0, 1$, respectively, we have the following well-known formula for ζ_K:

$$\zeta_K(s) = \zeta(s)L(s,\chi_K),$$

where $\zeta(s)$ is the Riemann zeta function and $L(s,\chi_K)$ is the Dirichlet L-function associated to χ_K. Now by [48, Proposition 4.1 and Theorem 4.2] we have for any $n \in \mathbb{N}$,

$$L(1-n,\chi_K) = -\frac{\Delta_K^{n-1}}{n}\sum_{a=1}^{\Delta_K}\chi_K(a)B_n\Big(\frac{a}{\Delta_K}\Big),$$

where $B_n(X)$ is the nth Bernoulli polynomial given by the formula $B_n(X) = \sum_{i=0}^{n}\binom{n}{i}B_iX^{n-i}$ with B_i the ith Bernoulli number. Combined with the functional equation of $\zeta_K(s)$ (see [47, p. 59]), we get

$$\zeta_K(2) = 4\pi^4\Delta_K^{-\frac{3}{2}}\zeta_K(-1) = 4\pi^4\Delta_K^{-\frac{3}{2}}\zeta(-1)L(-1,\chi_K) = \frac{\pi^4}{6\sqrt{\Delta_K}}\sum_{a=1}^{\Delta_K}\chi_K(a)B_2\Big(\frac{a}{\Delta_K}\Big). \tag{2.13}$$

For instance, when $K = \mathbb{Q}(\sqrt{2})$ we have $\Delta_K = 8$ and $\chi_K(n) = \left(\frac{8}{n}\right)$ is the quadratic character of modulus 8 with $\chi_K(1) = \chi_K(7) = 1$ and $\chi_K(3) = \chi_K(5) = -1$. We also have $B_2(1/8) = B_2(7/8) = \frac{11}{192}$ and $B_2(3/8) = B_2(5/8) = -\frac{13}{192}$. Plugging in all these relations, we get in this case $\zeta_K(2) = \frac{\pi^4}{48\sqrt{2}}$. Similarly, one can apply (2.13) to get $\zeta_{\mathbb{Q}(\sqrt{3})}(2) = \frac{\pi^4}{36\sqrt{3}}$ and $\zeta_{\mathbb{Q}(\sqrt{5})}(2) = \frac{2\pi^4}{75\sqrt{5}}$.

2.4 | Siegel domain and cusp neighborhoods

In this section, we give a more precise description of the homogeneous space $X = \Gamma_K\backslash\mathrm{H}$ in terms of the coordinates given in the previous section.

First, note that under these coordinates and in view of Lemma 2.2, for each $1 \leqslant i \leqslant \kappa$ the set

$$\mathcal{F}_i := \left\{\xi_i \mathrm{n}_{\mathbf{x}}\mathrm{a}_{\mathbf{y}}\mathrm{k}_\theta \in \mathrm{H} \ : \ \mathbf{x}\in\mathfrak{F}_i,\ y_1/y_2 \in [1,\lambda^2),\ \theta\in[0,\pi)\times[0,2\pi)\right\} \tag{2.14}$$

is a fundamental domain for $\Gamma_i\backslash\mathrm{H}$. Here $\mathfrak{F}_i \subset \mathbb{R}^2$ is a fundamental domain for $\mathbb{R}^2/\iota(I_i^{-2})$. (Recall that $\iota(I)$ is a lattice in $\mathbb{R}^2$ for any fractional ideal I of $\mathcal{O}_K$.)

Now for each $1 \leqslant i \leqslant \kappa$ and any $t > 0$, define

$$\mathcal{F}_i(t) := \left\{\xi_i \mathrm{n}_{\mathbf{x}}\mathrm{a}_{\mathbf{y}}\mathrm{k}_\theta \in \mathcal{F}_i \ : \ y_1y_2 \geq t\right\}. \tag{2.15}$$

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It follows from Shimizu's lemma [43, Lemma 5] that there exists some constant $t_1 > 0$ depending only on K such that

$$\forall \gamma \in \Gamma_K, 1 \leqslant i, j \leqslant \kappa \ : \ \gamma \mathcal{F}_i(t_1) \cap \mathcal{F}_j(t_1) \neq \emptyset \quad \Leftrightarrow \quad i = j \text{ and } \gamma = \mathrm{id}. \tag{2.16}$$

Moreover, by the reduction theory of Borel and Harish-Chandra [11], we have a Siegel fundamental domain of the form

$$\mathcal{F}_{\Gamma_K} := \mathfrak{C} \bigcup \left(\bigsqcup_{i=1}^{\kappa} \mathcal{F}_i(t_1) \right). \tag{2.17}$$

Here $\mathfrak{C} \subset \mathrm{H}$ is compact and the natural map from $\mathcal{F}_{\Gamma_K}$ to $\Gamma_K \backslash \mathrm{H}$ is surjective and finite-to-one. Note, on the other hand, it follows from (2.16) that the projection from $\bigsqcup_{i=1}^{\kappa} \mathcal{F}_i(t_1)$ to $\Gamma_K \backslash \mathrm{H}$ is injective.

2.5 | Geometry of lattices avoiding large balls

In this section, we give necessary conditions for lattices $\mathcal{L}_K h \in X$ avoiding large balls. These lattices will be the main objects we deal with when computing $G(s)$. We have the following description of these lattices.

Proposition 2.3. *There exists some $R_0 > 0$ depending only on K such that for any $R > R_0$ and for any lattice $\mathcal{L}_K h \in X$ avoiding a ball of radius R, we have $\mathcal{L}_K h = \mathcal{L}_K \xi_i \mathrm{n}_{\mathrm{x}} \mathrm{a}_{\mathrm{y}} \mathrm{k}_\theta \in \Gamma_K \mathcal{F}_i(t_1)$ for some $1 \leqslant i \leqslant \kappa$. Moreover, we have $y_1 \asymp y_2 \gg R$ with the bounding constants depending only on K.*

Remark 2.18. Regarding the notation '$\asymp$', '$\gg$' and '$\ll$': For two positive quantities A and B, we write $A \ll B$ or $B \gg A$ to mean that there is a constant $C > 0$ such that $A \leqslant CB$, and we will write $A \asymp B$ for $A \ll B \ll A$. We will sometimes use subscripts to indicate the dependence of the bounding constant C on parameters.

Proof. Let $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (\tau, 0, \sigma(\tau), 0)$, $\mathbf{v}_3 = (0, 1, 0, 1)$ and $\mathbf{v}_4 = (0, \tau, 0, \sigma(\tau))$. Note that $\{\mathbf{v}_j \ : \ j = 1, 2, 3, 4\}$ is a basis for $\mathcal{L}_K$. Let $R_0 > 0$ be sufficiently large such that

$$2\|\mathbf{v}_j h\| \leqslant R_0, \qquad \forall\, 1 \leqslant j \leqslant 4,\ h \in \mathfrak{C}. \tag{2.19}$$

We note that R_0 depends only on the compact set $\mathfrak{C}$ (hence only depends on K). Now take $R > R_0$ and suppose $x \in X$ avoids a ball B_R of radius R. To prove the first claim, in view of the description of the Siegel fundamental domain $\mathcal{F}_{\Gamma_K}$ in (2.17), we want to show there does not exist any $h \in \mathfrak{C}$ such that $x = \mathcal{L}_K h$. We prove by contradiction. Suppose there exists such $h \in \mathfrak{C}$. Then $\{\mathbf{v}_j h \ : \ 1 \leqslant j \leqslant 4\}$ is a basis for $x = \mathcal{L}_K h$. Let $\mathbf{x} \in \mathbb{R}^4$ be the center of B_R and write $\mathbf{x} = \sum_{j=1}^4 a_j \mathbf{v}_j h$ ($a_j \in \mathbb{R}$) as an $\mathbb{R}$-linear combination of these basis vectors. For each $1 \leqslant j \leqslant 4$, take $n_j \in \mathbb{Z}$ the closest integer to a_j so that we have $|n_j - a_j| \leqslant \frac{1}{2}$. Consider the lattice vector $\mathbf{x}' := \sum_{j=1}^4 n_j \mathbf{v}_j h \in \mathcal{L}_K h$. Since $\mathcal{L}_K h \cap B_R = \emptyset$, we have

$$R \leqslant \|\mathbf{x}' - \mathbf{x}\| \leqslant \textstyle\sum_{j=1}^4 |n_j - a_j| \|\mathbf{v}_j h\| \leqslant \frac{1}{2} \sum_{i=j}^4 \|\mathbf{v}_j h\|,$$

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implying that there exists $1 \leqslant j \leqslant 4$ such that $\|\mathbf{v}_j h\| \geq \frac{R}{2} > \frac{R_0}{2}$, contradicting (2.19). This proves the first claim, that is, $\mathcal{L}_K h \in \Gamma_K \mathcal{F}_i(t_1)$ for some $1 \leqslant i \leqslant \kappa$. For the second claim, without loss of generality we may assume $h = \xi_i \mathrm{n}_{\mathbf{x}} \mathrm{a}_{\mathbf{y}} \mathrm{k}_\theta \in \mathcal{F}_i(t_1)$. To show $y_1 \asymp y_2 \asymp R$, first note that $y_1 \asymp_\lambda y_2$ and $y_1 y_2 \geqslant t_1 \gg 1$. Thus,

$$y_1 \asymp y_2 \asymp \max\{y_1, y_1^{-1}, y_2, y_2^{-1}\} = \|\mathrm{a}_{\mathbf{y}}\|_{\mathrm{op}}.$$

Here $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm (with respect to the Euclidean norm). On the other hand, for any $1 \leqslant j \leqslant 4$,

$$\|\mathbf{v}_j h\| \leqslant \|\mathbf{v}_j\| \|\xi_i\|_{\mathrm{op}} \|\mathrm{n}_{\mathbf{x}}\|_{\mathrm{op}} \|\mathrm{a}_{\mathbf{y}}\|_{\mathrm{op}} \ll_{\mathfrak{F}_i, \xi_i} \|\mathrm{a}_{\mathbf{y}}\|_{\mathrm{op}} \asymp y_1 \asymp y_2.$$

This, together with the bound $\sum_{j=1}^4 \|\mathbf{v}_j h\| \geq 2R$ implies $y_1 \asymp y_2 \gg R$ as claimed. □

For later reference, note that for each $1 \leqslant i \leqslant \kappa$,

$$\mathcal{L}_K = \left\{(\alpha, \beta, \sigma(\alpha), \sigma(\beta))\xi_i^{-1} \ : \ \alpha \in I_i,\ \beta \in I_i^{-1}\right\} \tag{2.20}$$

with $I_i = \langle a_i, c_i \rangle$ as before. Let $c_i = (-c_i, a_i) \in \mathcal{O}_K^2$. We denote by

$$\Pi_i := \mathbb{R}\iota(c_i) \oplus \mathbb{R}\iota(\tau c_i) = (\mathbb{R}(0,1,0,1) \oplus \mathbb{R}(0,\tau,0,\sigma(\tau)))\,\xi_i^{-1} = (\{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R})\xi_i^{-1}, \tag{2.21}$$

and

$$\mathcal{L}_i := \Pi_i \cap \mathcal{L}_K = ((\{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}) \cap \mathcal{L}_K \xi_i)\xi_i^{-1} = \left\{(0, \beta, 0, \sigma(\beta))\xi_i^{-1} \ : \ \beta \in I_i^{-1}\right\}. \tag{2.22}$$

By Proposition 2.3 if $x \in X$ avoids some ball of radius $R > R_0$, then $x = \mathcal{L}_K h$ with $h = \xi_i \mathrm{n}_{\mathbf{x}} \mathrm{a}_{\mathbf{y}} \mathrm{k}_\theta \in \mathcal{F}_i(t_1)$ for some $1 \leqslant i \leqslant \kappa$ and $y_1, y_2 > 0$ with $y_1 \asymp y_2 \gg R$. Then the two vectors

$$\iota(c_i)h = (0, y_1^{-1}, 0, y_2^{-1})\mathrm{k}_\theta \quad \text{and} \quad \iota(\tau c_i)h = (0, \tau y_1^{-1}, 0, \sigma(\tau) y_2^{-1})\mathrm{k}_\theta$$

are of length $\ll R^{-1}$. Note that $\iota(c_i)h$ and $\iota(\tau c_i)h$ are linearly independent and $\mathcal{L}_i h$ is the primitive rank two sublattice of x containing them. Thus, $\mathcal{L}_i h$ lies $O(R^{-1})$-densely in the plane $\Pi_i h$. For this reason, we call $\Pi_i h$ the *filled plane* of $x = \mathcal{L}_K h$.

3 | ASYMPTOTICS OF $G(s)$

3.1 | Main result

The main goal of this Section 3 is to prove Theorem 2, which gives the tail asymptotics of both the limiting gap distribution function $F(s)$ and its integral $G(s)$, for the particular class of cut-and-project sets which we consider. In fact we will prove Theorem 4 below, which will be shown (in Section 3.2) to imply Theorem 2. This Theorem 4 gives an explicit bound on the error term in the asymptotics for $G(s)$, and also an explicit formula for the leading coefficient $a_{\mathcal{P}}$. As we mentioned in the introduction, the asymptotics for $F(s)$ will be derived as a consequence of those for $G(s)$; see Lemma 3.2 below.

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In order to state Theorem 4, we introduce some further notation. For any $\theta \in \mathbb{R}$, we write $\mathrm{k}_\theta := \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Then for any $\theta, x \in \mathbb{R}$ and any subset $\mathcal{W} \subset \mathbb{R}^2$, we set

$$R_{\mathcal{W}}(\theta, x) := \{y \in \mathbb{R} \ : \ (x, y)\mathrm{k}_\theta \in \mathcal{W}\}. \tag{3.1}$$

It follows that $\ell_{\mathcal{W}}(\theta)$, which we defined in Section 1.4, can be expressed as

$$\ell_{\mathcal{W}}(\theta) = \big\{x \in \mathbb{R} \ : \ R_{\mathcal{W}}(\theta, x) \neq \emptyset\big\}. \tag{3.2}$$

Note also that $\ell_{\mathcal{W}}(\theta)$ satisfies the relation $\ell_{\mathcal{W}}(\theta) = -\ell_{\mathcal{W}}(\pi + \theta)$ for any $\theta \in \mathbb{R}$. For any integral ideal $I \subset \mathcal{O}_K$, any bounded Lebesgue measurable set $J \subset \mathbb{R}$ with non-empty interior, and any $y > 0$, we set

$$\alpha_{I,J}(y) := \min\big\{\alpha \in I \cap \mathbb{R}_{>0} \ : \ y\,\sigma(\alpha) \in J\big\}. \tag{3.3}$$

For later reference, we record here a useful identity for the function $\alpha_{I,J}$ which can be checked directly from its definition: For any $\beta \in I^{-1}$ with $\sigma(\beta) > 0$ and for any $y > 0$,

$$\alpha_{\beta I,J}(y) = |\beta|\alpha_{I,\mathrm{sgn}(\beta)J}(\sigma(\beta)y). \tag{3.4}$$

Let us write m for the Lebesgue measure on $\mathbb{R}$. For any Lebesgue measurable set $A \subset \mathbb{R}$ and $a > 0$, we define $\mathfrak{f}(A, a)$ to be the infimum of $a^{-1}m(J \setminus A)$ when J ranges over all intervals of length a:

$$\mathfrak{f}(A, a) := a^{-1} \inf\big\{m\big((x, x + a) \setminus A\big) \ : \ x \in \mathbb{R}\big\}. \tag{3.5}$$

Note that $0 \leqslant \mathfrak{f}(A, a) \leqslant 1$ always, and $\mathfrak{f}(A, a) = 0$ whenever A contains some interval of length a. Let us also define

$$\widetilde{\mathfrak{f}}(A, a) := \begin{cases} \mathfrak{f}(A, a) & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases} \tag{3.6}$$

Recall from below Lemma 2.1 that we have made a fixed choice of ideals $I_1, \dots, I_\kappa$ of $\mathcal{O}_K$, and that these form a system of representatives of the ideal classes of K.

Theorem 4. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$ where K is a real quadratic field and $\mathcal{W}$ is a bounded open subset of $\mathbb{R}^2$ such that $\mathbf{0} \in \mathcal{W}$ and $\partial\mathcal{W}$ has Lebesgue measure zero. Let $F(s)$ be the associated limiting gap distribution function as in Theorem 1, and let $G(s) = \int_s^\infty F(t)\,\mathrm{d}t$. Then there exists a positive constant $C = C_{\mathcal{P}}$ such that for all sufficiently large s,*

$$0 \leqslant G(s) - \frac{a_{\mathcal{P}}}{s} \ll s^{-2} \int_0^{Cs} \int_0^{2\pi} \int_{\mathbb{R}} \widetilde{\mathfrak{f}}\Big(R_{\mathcal{W}}(\theta, y), u^{-1}\Big)\,\mathrm{d}y\,\mathrm{d}\theta\,\mathrm{d}u, \tag{3.7}$$

where

$$a_{\mathcal{P}} = \frac{\mathrm{Area}(\mathcal{W})}{4\Delta_K^2 \zeta_K(2)} \sum_{i=1}^{\kappa} \mathrm{Nr}(I_i)^{-2} \int_0^{2\pi} \int_1^{\lambda^2} \alpha_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2\,\frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta. \tag{3.8}$$

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We here make a couple of remarks regarding Theorem 4; for two more remarks, see the end of Section 3.2.

Remark 3.9. In the formula (3.8), it should be noted that for any integral ideal I of $\mathcal{O}_K$, we have

$$\int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3} = \int_a^{a\lambda^2} \alpha_{I,\ell_{\mathcal{W}}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3}, \qquad \forall\, a>0,\ \theta\in\mathbb{R}, \tag{3.10}$$

and furthermore, the product

$$\mathrm{Nr}(I)^{-2} \int_0^{2\pi}\int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta \tag{3.11}$$

depends only on the ideal class $[I]$ of I. To prove (3.10), it suffices to note that the differential form $\alpha_{I,\ell_{\mathcal{W}}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3}$ is invariant under the map sending y to $\lambda^2 y$, a fact which follows by applying (3.4) for the special case when $\beta=\lambda^{-2}$. Next, to prove the invariance of (3.11), assume that I' is another integral ideal belonging to the same ideal class as I. Then $I'=aI$ for some non-zero $a\in I^{-1}$, which we may take to satisfy $\sigma(a)>0$ (otherwise replace a by $-a$). Then the absolute norm $\mathrm{Nr}(I_i)^{-2}$ scales by $\mathrm{N}(a)^{-2}$ when replacing I by $I'=aI$. On the other hand, using (3.4) and the fact that $\ell_{\mathcal{W}}(\theta+\pi)=-\ell_{\mathcal{W}}(\theta)$ for any $\theta\in\mathbb{R}$, we have

$$\alpha_{I',\ell_{\mathcal{W}}(\theta)}(y) = \begin{cases} a\alpha_{I,\ell_{\mathcal{W}}(\theta)}(\sigma(a)y) & \text{if } a>0,\\ |a|\alpha_{I,\ell_{\mathcal{W}}(\theta+\pi)}(\sigma(a)y) & \text{if } a<0,\end{cases}$$

from which it follows (via the obvious substitution and then using (3.10) with $\sigma(a)$ in place of a) that the double integral in (3.11) scales by a factor $\mathrm{N}(a)^2$ when replacing I by $I'=aI$. Hence the product in (3.11) is invariant as claimed.

Remark 3.12. The right-hand side of (3.8) is invariant under replacing $\mathcal{W}$ by $\mathcal{W}g$, for any $g\in \mathrm{GL}_2(\mathbb{R})$.

Proof. We will prove the claim by showing that for any integral ideal I of $\mathcal{O}_K$, and any $g\in\mathrm{GL}_2(\mathbb{R})$,

$$\mathrm{Area}(\mathcal{W}g)\int_0^{2\pi}\int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}g}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta = \mathrm{Area}(\mathcal{W})\int_0^{2\pi}\int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta. \tag{3.13}$$

Write $\boldsymbol{e}_1=(1,0)$, $\boldsymbol{e}_2=(0,1)$. For fixed $g\in\mathrm{GL}_2(\mathbb{R})$, define the smooth maps $\omega:\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{R}/2\pi\mathbb{Z}$ and $\rho:\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{R}_{>0}$ through

$$\boldsymbol{e}_2 \mathrm{k}_\theta g^{-1} = \rho(\theta)\boldsymbol{e}_2 \mathrm{k}_{\omega(\theta)}. \tag{3.14}$$

Note that ω is a diffeomorphism of $\mathbb{R}/2\pi\mathbb{Z}$ onto itself. Differentiating both sides of (3.14) with respect to θ, and using the fact that $\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{k}_\theta = \mathrm{k}_{\theta+\frac{\pi}{2}}$, we have

$$\boldsymbol{e}_1 \mathrm{k}_\theta g^{-1} = \big(f(\theta),\rho'(\theta)\big)\mathrm{k}_{\omega(\theta)}, \qquad \text{where } f(\theta):=\rho(\theta)\omega'(\theta). \tag{3.15}$$

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It also follows from (3.15) and (3.14) that

$$f(\theta) = \boldsymbol{e}_1 \mathrm{k}_{\omega(\theta)} \cdot \boldsymbol{e}_1 \mathrm{k}_\theta g^{-1} = \boldsymbol{e}_2 \mathrm{k}_{\omega(\theta)} \mathrm{k}_{\frac{\pi}{2}} \cdot \boldsymbol{e}_1 \mathrm{k}_\theta g^{-1} = \rho(\theta)^{-1} \boldsymbol{e}_2 \mathrm{k}_\theta g^{-1} \mathrm{k}_{\frac{\pi}{2}} \cdot \boldsymbol{e}_1 \mathrm{k}_\theta g^{-1}$$
$$= \rho(\theta)^{-1} \det(\mathrm{k}_\theta g^{-1}) = \rho(\theta)^{-1} \det(g)^{-1}, \quad (3.16)$$

where in the fourth equality we used the fact that $\boldsymbol{w}\mathrm{k}_{\frac{\pi}{2}} \cdot \boldsymbol{v} = v_1 w_2 - v_2 w_1$ for any $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ in $\mathbb{R}^2$. Note that (3.15) and (3.16) imply that if $\det(g) > 0$, then $f(\theta) > 0$ and $\omega'(\theta) > 0$ for all θ, while if $\det(g) < 0$, then $f(\theta) < 0$ and $\omega'(\theta) < 0$ for all θ.

Using (3.14) and (3.15), we have $(x, y)\mathrm{k}_\theta g^{-1} = \big(xf(\theta), x\rho'(\theta) + y\rho(\theta)\big)\mathrm{k}_{\omega(\theta)}$ for any $(x, y) \in \mathbb{R}^2$, and hence via (3.1) and (3.2), one gets $\ell_{\mathcal{W}g}(\theta) = f(\theta)^{-1}\ell_{\mathcal{W}}(\omega(\theta))$. We also have $\omega(\theta + \pi) = \omega(\theta) + \pi$, $f(\theta + \pi) = f(\theta)$ and $\ell_{\mathcal{W}}(\omega(\theta) + \pi) = -\ell_{\mathcal{W}}(\omega(\theta))$, and hence:

$$\ell_{\mathcal{W}g}(\theta) = |f(\tilde{\theta})|^{-1}\ell_{\mathcal{W}}(\omega(\tilde{\theta})), \qquad \text{where } \tilde{\theta} := \theta \text{ if } \det(g) > 0, \tilde{\theta} := \theta + \pi \text{ if } \det(g) < 0.$$

Using also the simple relation $\alpha_{I,aJ}(y) = \alpha_{I,J}(a^{-1}y)$ $(\forall\, a > 0)$, it follows that

$$\mathrm{Area}(\mathcal{W}g) \int_0^{2\pi} \int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}g}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta = \mathrm{Area}(\mathcal{W}g) \int_{\mathbb{R}/2\pi\mathbb{Z}} \int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}}(\omega(\tilde{\theta}))}\big(|f(\tilde{\theta})|y\big)^2 \frac{\mathrm{d}y}{y^3}\,\mathrm{d}\tilde{\theta}$$
$$= |\det g| \,\mathrm{Area}(\mathcal{W}) \int_{\mathbb{R}/2\pi\mathbb{Z}} f(\tilde{\theta})^2 \int_1^{\lambda^2} \alpha_{I,\ell_{\mathcal{W}}(\omega(\tilde{\theta}))}(u)^2 \frac{\mathrm{d}u}{u^3}\,\mathrm{d}\tilde{\theta}, \quad (3.17)$$

where we substituted $u = |f(\tilde{\theta})|y$ and then used (3.10). The two formulas for $f(\theta)$ in (3.15) and (3.16) imply $f(\tilde{\theta})^2 = \det(g)^{-1}\omega'(\tilde{\theta})$. Using this in the last expression in (3.17), and then taking ω as a new variable of integration, we obtain the identity (3.13).

Let us also note that for any $g \in \mathrm{SL}_2(\mathbb{R})$, the invariance statement in Remark 3.12 may alternatively be deduced as a consequence of (3.7) in Theorem 4 and Remark 3.37 below. Once the invariance is known for all $g \in \mathrm{SL}_2(\mathbb{R})$, in order to extend it to all $g \in \mathrm{GL}_2(\mathbb{R})$ it suffices to verify that it also holds for $g = \mathrm{diag}(1, -1)$ and $g = \mathrm{diag}(a, a)$ $(\forall\, a > 0)$; for these cases, the above proof applies in a significantly simplified form. □

3.2 | Proof of Theorem 2 using Theorem 4

In this section, we prove Theorem 2 assuming Theorem 4. We will first prove the two tail asymptotics of $G(s)$ in (1.13) and (1.14); then, by a standard general argument using the convexity of $G(s)$, we will deduce the tail asymptotics of $F(s)$.

We start with (1.13). We need to show that for any $\mathcal{W}$ as in Theorem 2, the error bound in (3.7) is $o(s^{-1})$ as $s \to \infty$. Take $R > 0$ so that $\mathcal{W} \subset \mathcal{B}_R^2$, where $\mathcal{B}_R^2 \subset \mathbb{R}^2$ is the open disc with center $\mathbf{0}$ and radius R. Then for every $y \in \mathbb{R}$ with $|y| \geqslant R$, and for every θ, we have $R_{\mathcal{W}}(\theta, y) = \emptyset$ and hence $\widetilde{\mathrm{f}}\big(R_{\mathcal{W}}(\theta, y), u^{-1}\big) = 0$ for all $u > 0$. It follows that the integral over $\mathbb{R}$ in (3.7) can be replaced by $\int_{-R}^{R}$. Hence, if we take $C = C_{\mathcal{P}}$ as in Theorem 4, and set

$$F_{\theta,y}(s) := s^{-1} \int_0^{Cs} \widetilde{\mathrm{f}}\big(R_{\mathcal{W}}(\theta, y), u^{-1}\big)\,\mathrm{d}u,$$

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then our task is to prove that $\int_0^{2\pi}\int_{-R}^{R} F_{\theta,y}(s)\,\mathrm{d}y\,\mathrm{d}\theta \to 0$ as $s\to\infty$. Using $0\leqslant \widetilde{\mathfrak{f}}\big(R_{\mathcal{W}}(\theta,y),u^{-1}\big)\leqslant 1$, we have $0\leqslant F_{\theta,y}(s)\leqslant C$ for all θ, y and s; hence by the Lebesgue bounded convergence theorem, it suffices to prove that $F_{\theta,y}(s)\to 0$ as $s\to\infty$ for any fixed θ, y. But if $R_{\mathcal{W}}(\theta,y)=\emptyset$, then $F_{\theta,y}(s)=0$ for all s, by the definition (3.6), while if $R_{\mathcal{W}}(\theta,y)\neq\emptyset$, then since $\mathcal{W}$ is open, $R_{\mathcal{W}}(\theta,y)$ contains a non-empty interval, and so $\widetilde{\mathfrak{f}}\big(R_{\mathcal{W}}(\theta,y),u^{-1}\big)=0$ for all sufficiently large u; therefore $F_{\theta,y}(s)\to 0$ as $s\to\infty$. This concludes the proof that the error bound in (3.7) is $o(s^{-1})$ as $s\to\infty$, that is, we have proved that $G(s)\sim a_{\mathcal{P}}s^{-1}$, as stated in (1.13).

For the proof of (1.14), we need the following simple lemma regarding the function $\widetilde{\mathfrak{f}}(A,a)$.

Lemma 3.1. *Let $C>0$. Then for any interval A of length $b>0$ and for any $s>0$,*

$$\int_0^{Cs}\widetilde{\mathfrak{f}}(A,u^{-1})\,\mathrm{d}u \asymp_C \min(s,b^{-1}).$$

Proof. Note that $\mathfrak{f}(A,a)=0$ for $a\in(0,b]$ and $\mathfrak{f}(A,a)=1-(b/a)$ for $a\in[b,\infty)$. Also $\widetilde{\mathfrak{f}}(A,a)=\mathfrak{f}(A,a)$. Hence for $s\leqslant (Cb)^{-1}$:

$$\int_0^{Cs}\widetilde{\mathfrak{f}}(A,u^{-1})\,\mathrm{d}u=\int_0^{Cs}\big(1-bu\big)\,\mathrm{d}u\asymp Cs\asymp_C s,$$

while for $s\geqslant (Cb)^{-1}$:

$$\int_0^{Cs}\widetilde{\mathfrak{f}}(A,u^{-1})\,\mathrm{d}u=\int_0^{b^{-1}}\big(1-bu\big)\,\mathrm{d}u+0\asymp b^{-1}.$$

The desired estimate then follows from these two estimates noting that $\min(s,b^{-1})\asymp_C s$ for $s\leqslant (Cb)^{-1}$ and $\min(s,b^{-1})\asymp_C b^{-1}$ for $s\geqslant (Cb)^{-1}$. □

Now let us further assume that $\mathcal{W}$ is convex. Then for all θ and y, $R_{\mathcal{W}}(\theta,y)$ is either empty or an interval. Hence, using Lemma 3.1 and the fact that $m(R_{\mathcal{W}}(\theta+\pi,y))=m(R_{\mathcal{W}}(\theta,-y))$, it follows that the error bound in (3.7) is

$$\asymp s^{-2}\int_0^{2\pi}\int_0^{\infty} I\big(R_{\mathcal{W}}(\theta,y)\neq\emptyset\big)\cdot\min\Big(s,m(R_{\mathcal{W}}(\theta,y))^{-1}\Big)\,\mathrm{d}y\,\mathrm{d}\theta. \tag{3.18}$$

Choose $0<r_0\leqslant R_0$ such that $\mathcal{B}^2_{r_0}\subset\mathcal{W}\subset\mathcal{B}^2_{R_0}$. Let us temporarily fix $\theta\in[0,2\pi]$, and set $y_0:=\sup\{y>0\ :\ R_{\mathcal{W}}(\theta,y)\neq\emptyset\}$; then the function $f(y):=m(R_{\mathcal{W}}(\theta,y))$ vanishes for $y>y_0$, but is positive and concave for $y\in[0,y_0)$. Hence for our fixed θ, the inner integral in (3.18) equals $\int_0^{y_0}\min\big(s,f(y)^{-1}\big)\,\mathrm{d}y$. We also have $r_0\leqslant y_0\leqslant R_0$ and $f(0)\geqslant r_0$, and so, by the concavity, $f(y)\geqslant (r_0/R_0)\cdot(y_0-y)$ for all $y\in[0,y_0]$. Hence

$$\int_0^{y_0}\min\big(s,f(y)^{-1}\big)\,\mathrm{d}y\leqslant\int_0^{R_0}\min\Big(s,\frac{R_0}{r_0\,t}\Big)\,\mathrm{d}t,$$

and by a simple computation, the last integral is seen to equal $\frac{R_0}{r_0}\big(\log(r_0s)+1\big)$ for all $s\geqslant 1/r_0$. Since this holds for every $\theta\in[0,2\pi]$, we conclude that the expression in (3.18) is $\ll s^{-2}\log s$ for all large s, that is, we have proved the asymptotics for $G(s)$ stated in (1.14).

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The next lemma shows how to use the convexity of $G(s)$ to deduce the tail asymptotics of $F(s) = -G'(s)$ from that of $G(s)$.

Lemma 3.2. *Let $G : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a differentiable function with $F(s) := -G'(s)$ continuous and decreasing. Suppose further that there exist constants $a > 0$ and $C_1, C_2 > 1$ and a function $E : (C_1, \infty) \to \mathbb{R}_{>0}$, such that*

$$\left|G(s) - as^{-1}\right| \leqslant E(s) \leqslant \frac{1}{s}, \qquad \forall\, s > C_1, \tag{3.19}$$

and

$$C_2^{-1} \leqslant \frac{E(s)}{E(s')} \leqslant C_2 \qquad \text{whenever } C_1 < s \leqslant s' \leqslant 2s. \tag{3.20}$$

Then we have

$$F(s) = as^{-2} + O(s^{-\frac{3}{2}} E(s)^{\frac{1}{2}}) \quad \text{as } s \to \infty, \tag{3.21}$$

where the implied constant depends only on a and C_2.

Proof. Since by assumption $F(s)$ is continuous and decreasing, we have for all $0 < s_1 < s_2$, $G(s_1) - G(s_2) = \int_{s_1}^{s_2} F(s)\,\mathrm{d}s \leqslant (s_2 - s_1)F(s_1)$. Hence, assuming $C_1 < s_1 < s_2 \leqslant 2s_1$, writing $h := s_2 - s_1$, and using (3.19) and (3.20), we have

$$F(s_1) - \frac{a(s_1^{-1} - s_2^{-1})}{h} \geqslant -\frac{E(s_1) + E(s_2)}{h} \geqslant -\frac{C_2 + 1}{h} E(s_1).$$

From this, using also $s_1^{-1} - s_2^{-1} = h/(s_1 s_2)$, we get

$$F(s_1) \geqslant \frac{a}{s_1 s_2} - \frac{C_2 + 1}{h} E(s_1) \geqslant \frac{a}{s_1^2} - \frac{ah}{s_1^3} - \frac{C_2 + 1}{h} E(s_1).$$

Here we optimize by choosing $h = \sqrt{E(s_1)s_1^3}$; note that this choice of h is admissible, that is, yields $s_2 \leqslant 2s_1$, because of our assumption $E(s) \leqslant s^{-1}$. The conclusion is that $F(s_1) - as_1^{-2} \geqslant -(a + C_2 + 1)s_1^{-\frac{3}{2}} E(s_1)^{\frac{1}{2}}$ holds for any $s_1 > C_1$.

Similarly, using $G(s_1) - G(s_2) = \int_{s_1}^{s_2} F(s)\,\mathrm{d}s \geqslant (s_2 - s_1)F(s_2)$, we have for all $C_1 < s_1 < s_2 \leqslant 2s_1$:

$$F(s_2) \leqslant \frac{a}{s_1 s_2} + \frac{E(s_1) + E(s_2)}{h} \leqslant \frac{a}{s_2^2} + \frac{2ah}{s_2^3} + \frac{C_2 + 1}{h} E(s_2).$$

Here choose $h = \sqrt{4^{-1}E(s_2)s_2^3}$; then $h \leqslant \sqrt{4^{-1}s_2^2} = 2^{-1}s_2$, so that our choice is admissible for any $s_2 > 2C_1$. For these s_2, we obtain $F(s_2) - as_2^{-2} \leqslant (a + 2C_2 + 2)s_2^{-\frac{3}{2}} E(s_2)^{\frac{1}{2}}$. Combining this with the bound proved above, we have proved (3.21) (with an implied constant $C = a + 2C_2 + 2$). □

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Let us now use Lemma 3.2 to conclude the proof of (1.13). Recall that in the situation of Theorem 2, $F(s)$ is continuous and decreasing for $s > 0$ (see Theorem 1), and $G(s) = \int_s^\infty F(t)\,\mathrm{d}t$. Furthermore, we have proved the first half of (1.13), that is, that $G(s) \sim a_{\mathcal{P}} s^{-1}$ as $s \to \infty$. Hence, for any given constant $0 < c < 1$, Lemma 3.2 applies with $E(s) = cs^{-1}$, $C_2 = 2$ and an appropriate constant C_1 (depending on c), to yield $F(s) = a_{\mathcal{P}} s^{-2} + O(c^{1/2} s^{-2})$ as $s \to \infty$. Here the implied constant is independent of c, and by taking c arbitrarily small, we conclude that $F(s) = a_{\mathcal{P}} s^{-2} + o(s^{-2})$. This completes the proof of (1.13).

Similarly, the last part of (1.14) follows from the first part of (1.14), by Lemma 3.2 applied with appropriate constants C_1, C_2 and $E(s) = Cs^{-2}\log s$ with an appropriate $C > 0$. This concludes the proof of Theorem 2.

Remark 3.22. Let us note that there exist quite 'nice' (but non-convex) windows $\mathcal{W}$ for which the relative error bound in (3.7) in Theorem 4 tends to zero more slowly than any prescribed rate, as $s \to \infty$. One way in which this can happen is if $\mathcal{W}$ has a *cusp* of an appropriate asymptotic shape.

To give a concrete statement, let us define

$$H_{\mathcal{W}}(s) := \frac{1}{s}\int_0^{C_{\mathcal{P}} s} \int_{\mathbb{R}/2\pi\mathbb{Z}} \int_{\mathbb{R}} \widetilde{\mathfrak{f}}\big(R_{\mathcal{W}}(\theta, y), u^{-1}\big)\,\mathrm{d}y\,\mathrm{d}\theta\,\mathrm{d}u. \tag{3.23}$$

Then (3.7) says that $G(s) = a_{\mathcal{P}} s^{-1}\big(1 + O(H_{\mathcal{W}}(s))\big)$ as $s \to \infty$, namely, $H_{\mathcal{W}}(s)$ is the relative error bound in this asymptotics, and our proof of the relation $G(s) \sim a_{\mathcal{P}} s^{-1}$ in (1.13) shows $H_{\mathcal{W}}(s) \to 0$ as $s \to \infty$. Now let $H_0 : [1, \infty) \to (0, \infty)$ be an arbitrary decreasing function with $\lim_{s\to\infty} H_0(s) = 0$. Choose a sequence $1 > t_2 > t_3 > \cdots$ satisfying both $\lim_{n\to\infty} t_{n+1}/H_0(n) = \infty$ and $\lim_{n\to\infty} t_n = 0$,[†] and then let $f : [0,1] \to [0,1]$ be a continuous, increasing function satisfying $f(0) = 0$, $f(1) = 1$, and $f(t_n) = n^{-1}$ for all $n \geqslant 2$. (Note that we may even take $f(s)$ to be smooth for $0 < s \leqslant 1$.) Set

$$\mathcal{W} := \mathcal{W}_0 + \boldsymbol{w} \quad \text{with} \quad \mathcal{W}_0 := \{(w_1, w_2) \in \mathbb{R}^2 \ : \ 0 < w_1 < 1,\ 0 < w_2 < f(w_1)\},$$

and with $\boldsymbol{w}$ being a fixed vector in $\mathbb{R}^2$ chosen so that $\mathbf{0} \in \mathcal{W} = \mathcal{W}^\circ$. Finally let $\mathcal{P}$ be the cut-and-project set $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}_K)$, for any fixed real quadratic field K. We claim that in this case, $\lim_{s\to\infty} H_{\mathcal{W}}(s)/H_0(s) = \infty$, namely, $H_{\mathcal{W}}(s)$ tends to zero more slowly than the given function $H_0(s)$!

To prove the claim, we first note that for every $-\frac{\pi}{4} < \theta < 0$, writing $w_{\theta,1}$ for the first coordinate of $\boldsymbol{w}\mathrm{k}_{-\theta}$, one easily verifies that for every $0 < y < 1/2$, $R_{\mathcal{W}}(\theta, y + w_{\theta,1})$ is a non-empty open interval of length $\leqslant (\cos\theta)^{-1} f(y/\cos\theta) \leqslant \sqrt{2} f(y/\cos\theta)$. Hence by Lemma 3.1, for each such θ,

$$\int_{\mathbb{R}}\int_0^{C_{\mathcal{P}} s} \widetilde{\mathfrak{f}}(R_{\mathcal{W}}(\theta, y), u^{-1})\,\mathrm{d}u\,\mathrm{d}y \gg_{\mathcal{P}} \int_0^{1/2} \min(s, f(y/\cos\theta)^{-1})\,\mathrm{d}y \gg \int_0^{1/2} \min(s, f(y)^{-1})\,\mathrm{d}y.$$

This implies

$$H_{\mathcal{W}}(s) \gg_{\mathcal{P}} \frac{1}{s}\int_0^{1/2} \min(s, f(y)^{-1})\,\mathrm{d}y = \int_0^{1/2} \min\Big(1, \frac{1}{sf(y)}\Big)\,\mathrm{d}y =: H_1(s). \tag{3.24}$$

[†] For example, one may choose $t_{n+1} := 3^{-1}(1 + n^{-1})\sqrt{H_0(n)/H_0(1)}$.

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Clearly $H_1(s)$ is a decreasing function of $s > 0$. Also, for each $n \geqslant 2$, we have $f(y) \leqslant f(t_n) = n^{-1}$ for all $y \in [0, t_n]$, and so $H_1(n) \geqslant \int_0^{t_n} 1\,\mathrm{d}y = t_n$. Hence by our choice of $\{t_n\}$, we have $\lim_{n\to\infty} H_1(n+1)/H_0(n) = \infty$. Note also that for every $n \geqslant 1$ and every $s \in [n, n+1]$, we have $H_1(s)/H_0(s) \geqslant H_1(n+1)/H_0(n)$; hence $\lim_{s\to\infty} H_1(s)/H_0(s) = \infty$. In view of (3.24), this implies $\lim_{s\to\infty} H_{\mathcal{W}}(s)/H_0(s) = \infty$.

As an interesting open problem, we mention that it seems plausible to us that, by using windows $\mathcal{W}$ as above (perhaps with some further restrictions), also the *actual* relative difference, $G(s)s - a_{\mathcal{P}}$, can be proved to tend to zero more slowly than any prescribed rate.

Remark 3.25. On the other hand, there exist $\mathcal{W}$ with *fractal* boundary for which (3.7) in Theorem 4 implies as strong an error bound as for convex window sets, that is, $G(s) = a_{\mathcal{P}} s^{-1} + O\big(s^{-2}\log s\big)$ as in (1.14) in Theorem 2. For example, this holds if $\partial\mathcal{W}$ is the standard Koch snowflake; and it appears to also hold for the window sets appearing in, for example, [12, 34] and for several of the specific examples in [35]. Indeed, the proof of Lemma 3.1 shows that if the set $A \subset \mathbb{R}$ *contains* an interval of length $b > 0$, then $\int_0^{Cs} \widetilde{\mathfrak{f}}(A, u^{-1})\,\mathrm{d}u \ll_C \min(s, b^{-1})$ for all $s > 0$. Hence by the argument leading to (3.18), the error bound in (3.7) is

$$\ll s^{-2} \int_0^{2\pi} \int_0^{\infty} I\big(m(R_{\mathcal{W}}(\theta, y)) > 0\big) \cdot \min\Big(s, j(R_{\mathcal{W}}(\theta, y))^{-1}\Big)\,\mathrm{d}y\,\mathrm{d}\theta,$$

where for $A \subset \mathbb{R}$ we write $j(A)$ for the supremum of the lengths of all intervals contained in A. Now temporarily fix $\theta \in [0, 2\pi]$ and write $y_0 := \sup\{y > 0 \ : \ R_{\mathcal{W}}(\theta, y) \neq \emptyset\}$. One verifies that, if $\mathcal{W}$ is an open set (with $\mathbf{0} \in \mathcal{W}$) such that $\partial\mathcal{W}$ is the standard Koch snowflake, then there exists a constant $c > 0$ independent of θ such that $j(R_{\mathcal{W}}(\theta, y)) > c \cdot (y_0 - y)$ for all $y \in [0, y_0]$. This leads to $G(s) = a_{\mathcal{P}} s^{-1} + O\big(s^{-2}\log s\big)$, as claimed.

3.3 | A preliminary integral formula for $G(s)$

We will now recall from [29] some of the key steps in the proof of Theorem 1, and in particular an expression for the function $G(s) := \int_s^{\infty} F(t)\,\mathrm{d}t$ as the Haar measure of a certain set in a homogeneous space.

Let $\mathcal{P}$ be as in Theorem 1; thus $\mathcal{P}$ is given by (1.4) where $d = 2$ and where $\mathcal{L}$ is a lattice in $\mathbb{R}^n$ with $n = 2 + m$. Set $\mathrm{G} = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and choose $g \in \mathrm{G}$ and $\delta > 0$ so that $\mathcal{L} = \delta^{1/n}\mathbb{Z}^n g$. Let φ_g be the embedding of $\mathrm{SL}_2(\mathbb{R})$ in G given by

$$\varphi_g(A) = g \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} g^{-1}.$$

It follows from Ratner's work [37, 38] that there exists a unique closed connected subgroup H_g of G such that $\Gamma \cap \mathrm{H}_g$ is a lattice in H_g, $\varphi_g(\mathrm{SL}_2(\mathbb{R})) \subset \mathrm{H}_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}_2(\mathbb{R}))$ in $\Gamma\backslash\mathrm{G}$ equals $\Gamma\backslash\Gamma\mathrm{H}_g$. Let μ_g be the Haar measure on H_g normalized so that $\mu_g(\Gamma\backslash\Gamma\mathrm{H}_g) = 1$.

Recall that for any $R > 0$, we have defined Δ_R in (1.1), and we have ordered the normalized angles of the points in Δ_R as

$$-\tfrac{1}{2} < \xi_{R,1} \leqslant \xi_{R,2} \leqslant \cdots \leqslant \xi_{R,N(R)} \leqslant \tfrac{1}{2},$$

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where $N(R) = \#\Delta_R$; also $\xi_{R,0} := \xi_{R,N(R)} - 1$. The proof of Theorem 1 in [29] starts from the simple geometric observation that the events $\xi_{R,j} - \xi_{R,j-1} \geqslant s/N(R)$ can be detected by rotating the following open sector:

$$S_R(s) := \left\{ (r\cos\theta, r\sin\theta) \,:\, 0 < r < R,\ |\theta| < \frac{\pi s}{N(R)} \right\} \qquad (s > 0).$$

Indeed, we have

$$\sum_{j=1}^{N(R)} \left(\xi_{j,R} - \xi_{j-1,R} - \frac{s}{N(R)} \right)^+ = \frac{1}{2\pi} \int_0^{2\pi} I(\mathcal{P} \cap S_R(s)\mathrm{k}_\theta = \emptyset)\, \mathrm{d}\theta, \tag{3.26}$$

where $x^+ := \max(0, x)$. Noticing here that $\mathcal{P} \cap S_R(s)\mathrm{k}_\theta = \emptyset$ holds if and only if $\mathcal{P}\mathrm{k}_{-\theta}\mathrm{a}_{1/R} \cap S_R(s)\mathrm{a}_{1/R} = \emptyset$, and then using (1.4), it follows that the last expression is

$$= \frac{1}{2\pi} \int_0^{2\pi} I\Big(\delta^{1/n}\mathbb{Z}^n \varphi_g(\mathrm{k}_{-\theta}\mathrm{a}_{1/R})g \cap (S_R(s)\mathrm{a}_{1/R} \times \mathcal{W}) = \emptyset \Big)\, \mathrm{d}\theta. \tag{3.27}$$

Recalling that $N(R) \sim c_{\mathcal{P}}\pi R^2$ as $R \to \infty$ with $c_{\mathcal{P}}$ the asymptotic density of $\mathcal{P}$, one verifies immediately that as $R \to \infty$, the rescaled sector $S_R(s)\mathrm{a}_{1/R}$ approaches the open triangle $\mathfrak{C}(s)$ in $\mathbb{R}^2$ with vertices at (0,0) and $(1, \pm s/c_{\mathcal{P}})$;

$$\mathfrak{C}(s) := \left\{ (x_1, x_2) \,:\, 0 < x_1 < 1,\ |x_2| < \frac{s}{c_{\mathcal{P}}} x_1 \right\},$$

in the sense that the area of the symmetric difference of $S_R(s)\mathrm{a}_{1/R}$ and $\mathfrak{C}(s)$ tends to zero. Also as $R \to \infty$, the expanding SO(2)-orbit appearing in (3.27) equidistributes in the homogeneous space $\Gamma\backslash\Gamma\mathrm{H}_g$ [28, Theorem 4.1]; the proof of this fact makes crucial use of Shah [42, Theorem 1.4] and ultimately hinges on Ratner's classification of measures invariant under unipotent flows [37]. As is shown in [29], these facts imply that the expression in (3.27) tends to

$$\begin{aligned} G(s) &:= \mu_g\big(\{ \Gamma h \in \Gamma\backslash\Gamma\mathrm{H}_g \,:\, \delta^{1/n}\mathbb{Z}^n hg \cap (\mathfrak{C}(s) \times \mathcal{W}) = \emptyset \} \big) \\ &= \mu_g\big(\{ \Gamma h \in \Gamma\backslash\Gamma\mathrm{H}_g \,:\, \mathcal{P}(\mathcal{W}, \delta^{1/n}\mathbb{Z}^n hg) \cap \mathfrak{C}(s) = \emptyset \} \big) \end{aligned} \tag{3.28}$$

as $R \to \infty$. It is verified in [29, section 11] that this fact in turn leads to the conclusions of Theorem 1, with the limit distribution $F(s)$ satisfying

$$F(s) = -G'(s) \qquad (s > 0). \tag{3.29}$$

(In particular the proof in [29, section 11] shows that $G(s)$ is C^1 on $\mathbb{R}_{>0}$, and so F is continuous on $\mathbb{R}_{>0}$ as stated in Theorem 1.)

Remark 3.30. The relation (3.29) corresponds to [29, (3.6) and (11.1)]; however recall from Remark 1.9 that our function $F(s)$ equals the function in the right-hand side of [29, (1.15)]. This is the reason why we have $\mathfrak{C}(s)$ in (3.28), and not $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}s)$ as in [29, (3.6)].

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Remark 3.31. It follows from the formula (3.28) that $G(s)$ (and hence also $F(s)$) remains unchanged if $\mathcal{W}$ is modified by a null set (with respect to the Haar measure $\mu_{\mathcal{A}}$ of $\mathcal{A} = \overline{\pi_{\mathrm{int}}(\mathcal{L})} \subset \mathbb{R}^m$). Indeed, let $\mathcal{W}_1 \subset \mathcal{A}$ be any bounded subset of $\mathcal{A}$ with boundary of measure zero, satisfying $\mu_{\mathcal{A}}(\mathcal{W} \triangle \mathcal{W}_1) = 0$, and let $G_1(s)$ be given by (3.28) but with $\mathcal{W}_1$ in place of $\mathcal{W}$. Let $s > 0$ be given, and let $f_s : \mathbb{R}^d \times \mathcal{A} \to \{0,1\}$ be the indicator function of the set $\delta^{-\frac{1}{n}}\big(\mathfrak{C}(s) \times (\mathcal{W} \triangle \mathcal{W}_1)\big)$. Then

$$
\begin{aligned}
&|G_1(s) - G(s)| \\
&\quad \leqslant \mu_g\big(\{\Gamma h \ : \ \text{exactly one of the sets } \mathcal{P}(\mathcal{W}, \delta^{1/n}\mathbb{Z}^n hg) \text{ and } \mathcal{P}(\mathcal{W}_1, \delta^{1/n}\mathbb{Z}^n hg) \text{ intersects } \mathfrak{C}(s)\}\big) \\
&\quad \leqslant \int_{\Gamma\backslash\Gamma \mathrm{H}_g} \sum_{\boldsymbol{m} \in \mathbb{Z}^n hg} f_s(\boldsymbol{m})\, \mathrm{d}\mu_g(h) = 0,
\end{aligned} \tag{3.32}
$$

where the final equality holds by the Siegel formula [28, Theorem 5.1],[†] since $\mu_{\mathcal{A}}(\mathcal{W} \triangle \mathcal{W}_1) = 0$.

Because of the H_g-invariance of μ_g, the formula (3.28) remains valid if in the right-hand side we replace '$\delta^{1/n}\mathbb{Z}^n hg$' by '$\delta^{1/n}\mathbb{Z}^n h\varphi_g(A)g$', for any given $A \in \mathrm{SL}_2(\mathbb{R})$; and applying this with $A = \mathrm{diag}[(s/c_{\mathcal{P}})^{-1/2}, (s/c_{\mathcal{P}})^{1/2}]$, we obtain:

$$
G(s) = \mu_g\big(\{\Gamma h \in \Gamma\backslash\Gamma \mathrm{H}_g \ : \ \mathcal{P}(\mathcal{W}, \delta^{1/n}\mathbb{Z}^n hg) \cap T(s) = \emptyset\}\big), \tag{3.33}
$$

where $T(s) \subset \mathbb{R}^2$ is the open triangle with vertices at (0,0) and $(s/c_{\mathcal{P}})^{1/2}(1, \pm 1)$.

Next we specialize to the setting of Theorem 4. Thus we take $m = 2$ and let $\mathcal{L}_K$ be as in (1.10); this means that $n = 4$, $\mathrm{G} = \mathrm{SL}_4(\mathbb{R})$ and $\Gamma = \mathrm{SL}_4(\mathbb{Z})$, and that $g \in \mathrm{G}$ and $\delta > 0$ are such that $\mathcal{L}_K = \delta^{1/4}\mathbb{Z}^4 g$. Then by [28, section 2.2.1] we have $\mathrm{H}_g = g\mathrm{H}g^{-1}$, where H is as in Section 2.2. Furthermore,

$$
\Gamma_K = g^{-1}\Gamma g \cap \mathrm{H}. \tag{3.34}
$$

Indeed, we have $\Gamma_K \subset \mathrm{H}$, and every $\gamma \in \Gamma_K$ satisfies $\mathcal{L}_K\gamma = \mathcal{L}_K$, hence $\mathbb{Z}^4 g\gamma = \mathbb{Z}^4 g$ and so $\gamma \in g^{-1}\Gamma g$. Conversely, assume $\gamma \in g^{-1}\Gamma g \cap \mathrm{H}$, and write $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right)$. Then $\gamma \in g^{-1}\Gamma g$ implies $\mathbb{Z}^4 g\gamma g^{-1} = \mathbb{Z}^4$, and so $\mathcal{L}_K\gamma = \mathcal{L}_K$. It follows in particular that $(1,0,1,0)\gamma$ and $(0,1,0,1)\gamma$ lie in $\mathcal{L}_K$, and this in turn implies $a, b, c, d \in \mathcal{O}_K$ and $a' = \sigma(a)$, $b' = \sigma(b)$, $c' = \sigma(c)$, $d' = \sigma(d)$. Hence $\gamma \in \Gamma_K$, and (3.34) is proved.

It follows that the map $\Gamma_K h \mapsto \Gamma g h g^{-1}$ ($h \in \mathrm{H}$) is a diffeomorphism from $\Gamma\backslash\Gamma \mathrm{H}_g$ onto $\Gamma_K\backslash\mathrm{H}$, carrying μ_g to μ_K. Hence (3.33) can be rewritten as

$$
G(s) = \mu_K\big(\{\Gamma_K h \in \Gamma_K\backslash\mathrm{H} \ : \ \mathcal{P}(\mathcal{W}, \mathcal{L}_K h) \cap T(s) = \emptyset\}\big). \tag{3.35}
$$

Here let us also note that $\mathcal{P}(\mathcal{W}, \mathcal{L}_K h) \cap T(s) = \emptyset$ is equivalent with $\mathcal{L}_K h \cap \mathcal{T}'(s) = \emptyset$, where

$$
\mathcal{T}'(s) := T(s) \times \mathcal{W} \subset \mathbb{R}^4.
$$

[†] Note the correction of this formula given in the erratum to [28]; note also that we may restrict the summation over $\boldsymbol{m}$ in (3.32) by requiring $\pi(\boldsymbol{m}) \neq \mathbf{0}$, since $\mathbf{0} \notin \mathfrak{C}(s)$.

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Hence (3.35) can be rewritten as follows:

$$G(s) := \int_X I\big(\mathcal{L}_K h \cap \mathcal{T}'(s) = \emptyset\big)\, \mathrm{d}\mu_K(h). \tag{3.36}$$

This formula will be the starting point of our proof of Theorem 4. Since we will be only concerned about the asymptotics of $G(s)$ for large s, in the remainder of this section, we will always assume $s > 1$.

Remark 3.37. Using (3.36) and the fact that $(I_2, g) \in \mathrm{H}$ for any $g \in \mathrm{SL}_2(\mathbb{R})$, it is easy to verify that $G(s)$ is unchanged if $\mathcal{W}$ is replaced by $\mathcal{W}g$ for any $g \in \mathrm{SL}_2(\mathbb{R})$.

3.4 | Separating the main term and error term

The remainder of Section 3 is devoted to proving Theorem 4. We will start by applying results from Section 2.5 to further analyze the condition $\mathcal{L}_K h \cap \mathcal{T}'(s) = \emptyset$ in (3.36). For this, we first renormalize the sets $\mathcal{T}'(s)$ to produce sets containing large balls (for large s).

Lemma 3.3. *For $s > 1$ let $r \in \mathbb{N}$ be such that $\lambda^{2r} \leqslant s^{1/4} < \lambda^{2r+2}$, and let*

$$\mathcal{T}(s) := (\lambda^{-2r} T(s)) \times (\lambda^{2r} \mathcal{W}).$$

Then for any $h \in \mathrm{H}$, $\mathcal{L}_K h \cap \mathcal{T}'(s) = \emptyset$ holds if and only if $\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset$.

Proof. Let $d_{2r} = \mathrm{diag}(\lambda^{-2r}, \lambda^{-2r}, \lambda^{2r}, \lambda^{2r})$. The lemma then follows immediately by noting that $\mathcal{T}'(s) d_{2r} = \mathcal{T}(s)$ and $\mathcal{L}_K h d_{2r} = \mathcal{L}_K d_{2r} h = \mathcal{L}_K h$. □

Since $\mathcal{T}(s)$ contains a ball of radius $\gg s^{1/4}$, Proposition 2.3 gives that for s sufficiently large, the condition $\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset$ forces $\Gamma_K h \in \Gamma_K \mathcal{F}_i(t_1)$ for some $1 \leqslant i \leqslant \kappa$, that is, the point $\Gamma_K h$ in X belongs to the ith cuspidal neighborhood. Using also the fact that these cuspidal neighborhoods are pairwise disjoint, by (2.16), it follows that for s sufficiently large we have

$$G(s) = \sum_{i=1}^{\kappa} G_i(s),$$

where for each $1 \leqslant i \leqslant \kappa$,

$$G_i(s) := \int_X I\big(\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset \text{ and } \mathcal{L}_K h \in \Gamma_K \mathcal{F}_i(t_1)\big)\, \mathrm{d}\mu_K(h). \tag{3.38}$$

For the remainder of this section, we fix an index $i \in \{1, \dots, \kappa\}$, and seek an asymptotic formula for the function $G_i(s)$ as $s \to \infty$.

First we note that since $\mathcal{F}_i(t_1)$ projects injectively into X (by (2.16) applied with $j = i$), we may express $G_i(s)$ as an integral over $\mathcal{F}_i(t_1)$. In view of (2.10), (2.14) and (2.15), we get:

$$G_i(s) = c_K \int_Z \int_{Y_{t_1}} \int_{\mathfrak{F}_i} I\left(\mathcal{L}_K \xi_i \mathrm{n}_{\mathbf{x}} \mathrm{a}_{\mathbf{y}} \mathrm{k}_\theta \cap \mathcal{T}(s) = \emptyset\right) y_1^{-3} y_2^{-3}\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\theta, \tag{3.39}$$

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where $c_K = \Delta_K^{-3/2}\zeta_K(2)^{-1}$ as in (2.11),

$$Z := (0,\pi)\times(0,2\pi),$$

and

$$Y_{t_1} := \left\{(y_1,y_2)\in(\mathbb{R}_{>0})^2 \ : \ 1\leqslant y_1/y_2<\lambda^2,\ y_1y_2>t_1\right\}. \tag{3.40}$$

Now for s sufficiently large, let $h=\xi_i\mathrm{n}_{\mathbf{x}}\mathrm{a}_{\mathbf{y}}\mathrm{k}_\theta\in\mathcal{F}_i(t_1)$ be such that $\mathcal{L}_Kh\cap\mathcal{T}(s)=\emptyset$. Then by Proposition 2.3 we have $y_1,y_2\gg s^{1/4}$. As discussed in Section 2.5, since $\mathcal{L}_Kh$ avoids a large ball and $h\in\mathcal{F}_i(t_1)$, it contains a rank two sublattice $\mathcal{L}_ih$ (cf. (2.22)) which sits densely in the corresponding filled plane Π_ih (cf. (2.21)). Let us write

$$\tilde{\mathcal{L}}_i := \bigcup_{\mathbf{v}\in\mathcal{L}_K}(\Pi_i+\mathbf{v}) = \{(\alpha,t_1,\sigma(\alpha),t_2)\ :\ \alpha\in I_i,\ t_1,t_2\in\mathbb{R}\}\,\xi_i^{-1}. \tag{3.41}$$

Then $\mathcal{L}_Kh\subset\widetilde{\mathcal{L}}_ih$, and $\widetilde{\mathcal{L}}_ih$ is the union of all $\mathcal{L}_Kh$-translates of the filled plane Π_ih. Here $I_i=\langle a_i,c_i\rangle$ is as before and the second equality in (3.41) follows from (2.20) and (2.21). Our strategy of computing $G_i(s)$ is to replace the condition $\mathcal{L}_Kh\cap\mathcal{T}(s)=\emptyset$ by the slightly stronger (since $\widetilde{\mathcal{L}}_ih$ is very close to $\mathcal{L}_Kh$ when s is sufficiently large) and more manageable condition $\widetilde{\mathcal{L}}_ih\cap\mathcal{T}(s)=\emptyset$. Thus we define, for $s>1$:

$$G_{M,i}(s) := c_K\int_Z\int_{Y_{t_1}}\int_{\mathfrak{F}_i} I\left(\tilde{\mathcal{L}}_ih\cap\mathcal{T}(s)=\emptyset\right)y_1^{-3}y_2^{-3}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\theta \tag{3.42}$$

and

$$G_{E,i}(s) := c_K\int_Z\int_{Y_{t_1}}\int_{\mathfrak{F}_i} I\left(\mathcal{L}_Kh\cap\mathcal{T}(s)=\emptyset,\ \tilde{\mathcal{L}}_ih\cap\mathcal{T}(s)\neq\emptyset\right)y_1^{-3}y_2^{-3}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\theta, \tag{3.43}$$

where in both integrals we use the notation $h=\xi_i\mathrm{n}_{\mathbf{x}}\mathrm{a}_{\mathbf{y}}\mathrm{k}_\theta$. It is clear from the definitions that $G_i(s)=G_{M,i}(s)+G_{E,i}(s)$.

We start with a few auxiliary lemmas. For $\theta\in\mathbb{R}$, let $\ell(\theta)$ be the projection of $T(1)\mathrm{k}_{-\theta}$ on the x-axis. Recall also that $\ell_{\mathcal{W}}(\theta)$ is the projection of $\mathcal{W}\mathrm{k}_{-\theta}$ on the x-axis.

Lemma 3.4. *Fix $s>1$ and let $r\in\mathbb{N}$ be such that $\lambda^{2r}\leqslant s^{1/4}<\lambda^{2r+2}$ as before. For $h=\xi_i\mathrm{n}_{\mathbf{x}}\mathrm{a}_{\mathbf{y}}\mathrm{k}_\theta\in\mathcal{F}_i(t_1)$, the condition $\widetilde{\mathcal{L}}_ih\cap\mathcal{T}(s)=\emptyset$ holds if and only if*

$$\iota(I_i)\cap(y_1^{-1}s^{1/2}\lambda^{-2r}\ell(\theta_1))\times(y_2^{-1}\lambda^{2r}\ell_{\mathcal{W}}(\theta_2))=\emptyset, \tag{3.44}$$

where $\iota(I_i)\subset\mathbb{R}^2$ is the Minkowski embedding of I_i as before.

Proof. Note that

$$\tilde{\mathcal{L}}_ih=\{(\alpha y_1,t_1,\sigma(\alpha)y_2,t_2)\mathrm{k}_\theta\ :\ \alpha\in I_i,\ t_1,t_2\in\mathbb{R}\},$$

and recall that $\mathcal{T}(s)=(\lambda^{-2r}T(s))\times(\lambda^{2r}\mathcal{W})$. Hence $\widetilde{\mathcal{L}}_ih\cap\mathcal{T}(s)=\emptyset$ holds if and only if

$$\forall\,\alpha\in I_i,\ \forall\,(t_1,t_2)\in\mathbb{R}^2:\quad(\alpha y_1,t_1,\sigma(\alpha)y_2,t_2)\notin s^{1/2}\lambda^{-2r}T(1)\mathrm{k}_{-\theta_1}\times\lambda^{2r}\mathcal{W}\mathrm{k}_{-\theta_2},$$

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which is easily seen to be equivalent to (3.44). □

Remark 3.45. Similarly, using $\mathcal{L}_K = \{(\alpha, \beta, \sigma(\alpha), \sigma(\beta))\xi_i^{-1} \ : \ \alpha \in I_i, \ \beta \in I_i^{-1}\}$ (see (2.20)), one verifies that $\mathcal{L}_K h \cap \mathcal{T}(s) \neq \emptyset$ holds if and only if there exist $\alpha \in I_i$ and $\beta \in I_i^{-1}$ such that

$$(y_1\alpha, y_1^{-1}(\alpha x_1 + \beta), y_2\sigma(\alpha), y_2^{-1}(\sigma(\alpha)x_2 + \sigma(\beta))) \in (s^{1/2}\lambda^{-2r}T(1)\mathrm{k}_{-\theta_1}) \times (\lambda^{2r}\mathcal{W}\mathrm{k}_{-\theta_2}). \quad (3.46)$$

As a consequence of Lemma 3.4, we have a simple necessary condition for $\widetilde{\mathcal{L}}_i h \cap \mathcal{T}(s) = \emptyset$.

Lemma 3.5. *Keep the assumptions as in Lemma 3.4. If* $(\theta_1, \theta_2) \in (\frac{\pi}{4}, \frac{3\pi}{4}) \times [0, 2\pi)$*, then* $\widetilde{\mathcal{L}}_i h \cap \mathcal{T}(s) \neq \emptyset$.

Proof. Recall that we are assuming $\mathbf{0} \in \mathcal{W}^\circ = \mathcal{W}$, that is, that $\mathcal{W}$ contains a small disc centered at the origin; therefore $0 \in \ell_{\mathcal{W}}(\theta_2)$ for every θ_2. Now if $\theta_1 \in (\frac{\pi}{4}, \frac{3\pi}{4})$, then $0 \in \ell(\theta_1)$, and hence $\mathbf{0} \in \iota(I_i) \cap (y_1^{-1}s^{1/2}\lambda^{-2r}\ell(\theta_1)) \times (y_2^{-1}\lambda^{2r}\ell_{\mathcal{W}}(\theta_2))$, and so $\widetilde{\mathcal{L}}_i h \cap \mathcal{T}(s) \neq \emptyset$ by Lemma 3.4. □

Because of Lemma 3.5, we will often be able to reduce our discussion to the case $\theta_1 \in J$, where

$$J := (0, \pi) \setminus \left[\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right] = \left(0, \tfrac{\pi}{4}\right) \cup \left(\tfrac{3\pi}{4}, \pi\right).$$

In other words, we will be able to reduce the domain for $\boldsymbol{\theta}$ to be

$$Z_J := J \times (0, 2\pi) \subset Z.$$

3.5 | Computing the main term

The goal of this section is to obtain simpler formulas for the function $G_{M,i}(s)$ for large s. We first record the following useful and simple sufficient condition for a rectangle in the plane to intersect a lattice generated by an ideal of $\mathcal{O}_K$.

Lemma 3.6. *For any fractional ideal* I *of* $\mathcal{O}_K$*, there exists a constant* $C = C(I) > 0$ *such that for any two intervals* $R_1, R_2 \subset \mathbb{R}$ *with* $|R_1| \cdot |R_2| \geqslant C$*, we have* $\iota(I) \cap (R_1 \times R_2) \neq \emptyset$*. Here* $|R_i|$ *denotes the length of the interval* R_i*.*

Proof. This is a direct consequence of the fact that $\iota(I)$ is invariant under $\mathrm{diag}(u, \sigma(u))$ for any $u \in \mathcal{O}_K^\times$, so that we can renormalize the rectangle $R_1 \times R_2$ to be such that $|R_1|/|R_2| \in [1, \lambda)$. □

We next give a first explicit formula for $G_{M,i}(s)$. Recall that $\ell(\theta)$ denotes the projection of $T(1)\mathrm{k}_{-\theta}$ on the x-axis.

Proposition 3.7. *For all sufficiently large* $s > 1$*, we have*

$$G_{M,i}(s) = \tfrac{c_K \mathrm{Area}(\mathfrak{F}_i)}{2s} \int_{Z_J} \int_Y I\left(\iota(I_i) \cap (y_1^{-1}\ell(\theta_1) \times y_2^{-1}\ell_{\mathcal{W}}(\theta_2)) = \emptyset\right) y_1^{-3}y_2^{-3} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\boldsymbol{\theta}, \quad (3.47)$$

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where $Y := \{(y_1, y_2) \in (\mathbb{R}_{>0})^2 \ : \ y_2 \in [1, \lambda^2)\}$.

Proof. Using (3.42), Lemma 3.4 and Lemma 3.5, and the fact that $\iota(I_i)$ is invariant under the action of $\mathrm{diag}(\lambda^{-2r}, \lambda^{2r})$, we have

$$G_{M,i}(s) := c_K Area(\mathfrak{F}_i) \int_{Z_J} \int_{Y_{t_1}} I\left(\iota(I_i) \cap (y_1^{-1} s^{1/2} \ell(\theta_1)) \times (y_2^{-1} \ell_{\mathcal{W}}(\theta_2)) = \emptyset\right) \, y_1^{-3} y_2^{-3} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\boldsymbol{\theta}. \tag{3.48}$$

Because of $\mathbf{0} \in \mathcal{W}^\circ = \mathcal{W}$, there is an $r_{\mathcal{W}} > 0$ such that $(-r_{\mathcal{W}}, r_{\mathcal{W}}) \subset \ell_{\mathcal{W}}(\theta_2)$ for all θ_2. Furthermore, $\ell(\theta_1)$ is an open interval of length $|\ell(\theta_1)| \asymp 1$ for all θ_1. Hence by Lemma 3.6, there is a constant $C > 0$ such that $\iota(I_i) \cap (y_1^{-1} s^{1/2} \ell(\theta_1)) \times (y_2^{-1} \ell_{\mathcal{W}}(\theta_2)) \neq \emptyset$ holds for all $\boldsymbol{\theta} \in Z_J$ and all $s, y_1, y_2 \in \mathbb{R}_{>0}$ subject to $y_1^{-1} s^{1/2} \cdot y_2^{-1} > C$. It follows that if s is sufficiently large, then the formula in (3.48) remains valid if the domain of integration Y_{t_1} (defined in (3.40)) is replaced by the larger set

$$Y' := \{(y_1, y_2) \in (\mathbb{R}_{>0})^2 \ : \ y_1/y_2 \in [1, \lambda^2)\},$$

that is, we have (after also changing the order of integration)

$$G_{M,i}(s) := c_K \, \mathrm{Area}(\mathfrak{F}_i) \int_{Y'} \int_{Z_J} I\Big(\iota(I_i) \cap (y_1^{-1} s^{1/2} \ell(\theta_1)) \times (y_2^{-1} \ell_{\mathcal{W}}(\theta_2)) = \emptyset\Big) \, \mathrm{d}\boldsymbol{\theta} \, \frac{\mathrm{d}y_1}{y_1^3} \, \frac{\mathrm{d}y_2}{y_2^3}. \tag{3.49}$$

Going through the same arguments but replacing the condition $y_1/y_2 \in [1, \lambda^2)$ by $y_1/y_2 \in [\lambda^2, \lambda^4)$ in the definition of $\mathcal{F}_i$ in (2.14) (this amounts to choosing a different fundamental domain for $\Gamma_i \backslash \mathrm{H}$), one verifies that (3.49) remains valid for all sufficiently large $s > 1$ with Y' replaced by the set

$$\{(y_1, y_2) \in (\mathbb{R}_{>0})^2 \ : \ y_1/y_2 \in [\lambda^2, \lambda^4)\}.$$

Equivalently, we have for $s > 1$ sufficiently large

$$G_{M,i}(s) = \frac{1}{2} c_K \, \mathrm{Area}(\mathfrak{F}_i) \int_{Y''} \int_{Z_J} I\Big(\iota(I_i) \cap (y_1^{-1} s^{1/2} \ell(\theta_1)) \times (y_2^{-1} \ell_{\mathcal{W}}(\theta_2)) = \emptyset\Big) \, \mathrm{d}\boldsymbol{\theta} \, \frac{\mathrm{d}y_1}{y_1^3} \, \frac{\mathrm{d}y_2}{y_2^3}, \tag{3.50}$$

where

$$Y'' := \{(y_1, y_2) \in (\mathbb{R}_{>0})^2 \ : \ y_1/y_2 \in [1, \lambda^4)\}.$$

Next, using the fact that the lattice $\iota(I_i)$ is invariant under the action of $S = \mathrm{diag}(\lambda^2, \lambda^{-2})$, it follows that the integrand in (3.50) is invariant under $\boldsymbol{y} \mapsto \boldsymbol{y}S$, for each fixed $\boldsymbol{\theta}$. Also the measure $y_1^{-3} y_2^{-3} \, \mathrm{d}\boldsymbol{y}$ is invariant under this map and both Y'' and Y are fundamental domains for $(\mathbb{R}_{>0})^2 / \langle \boldsymbol{y} \mapsto \boldsymbol{y}S \rangle$. Hence the formula (3.50) remains valid if the domain of integration Y'' is replaced by Y, giving us (3.47). □

Next we further simplify the above empty intersection condition to get an even more explicit formula in terms of the function $\alpha_{I,J}$ defined in (3.3).

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Proposition 3.8. *For all sufficiently large $s > 1$, we have*

$$G_{M,i}(s) = \frac{c_K \operatorname{Area}(\mathfrak{F}_i)\, c_{\mathcal{P}}}{4s} \int_0^{2\pi} \int_1^{\lambda^2} \alpha_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2 \, \frac{\mathrm{d}y}{y^3} \, \mathrm{d}\theta. \tag{3.51}$$

In particular, we have for all sufficiently large $s > 1$,

$$\sum_{i=1}^{\kappa} G_{M,i}(s) = \frac{a_{\mathcal{P}}}{s}, \tag{3.52}$$

where $a_{\mathcal{P}}$ is as in (3.8).

Proof. The formula in Proposition 3.7 can be expressed as

$$G_{M,i}(s) = \frac{c_K \operatorname{Area}(\mathfrak{F}_i)}{2s} \int_0^{2\pi} \int_1^{\lambda^2} \mathcal{I}(y_2, \theta_2) \, \frac{\mathrm{d}y_2}{y_2^3} \, \mathrm{d}\theta_2, \tag{3.53}$$

where

$$\mathcal{I}(y_2, \theta_2) := \int_J \int_0^{\infty} I\big(\iota(I_i) \cap (y_1^{-1}\ell(\theta_1) \times y_2^{-1}\ell_{\mathcal{W}}(\theta_2)) = \emptyset\big) \, \frac{\mathrm{d}y_1}{y_1^3} \, \mathrm{d}\theta_1.$$

Here recall that $\ell(\theta_1)$ equals the projection of $T(1)\mathrm{k}_{-\theta_1}$ on the x-axis, which we can compute to be:

$$\ell(\theta_1) = \begin{cases} \big(0, c_{\mathcal{P}}^{-1/2}(\cos\theta_1 + \sin\theta_1)\big) & \text{if } 0 \leqslant \theta_1 \leqslant \pi/4, \\ \big(c_{\mathcal{P}}^{-1/2}(\cos\theta_1 - \sin\theta_1), 0\big) & \text{if } 3\pi/4 \leqslant \theta_1 \leqslant \pi. \end{cases} \tag{3.54}$$

Recall the definition of $\alpha_{I,J}(y)$ in (3.3). We also give a companion definition here. For any integral ideal $I \subset \mathcal{O}_K$ and any bounded Lebesgue measurable set $J \subset \mathbb{R}$ with non-empty interior, we set for any $y > 0$,

$$\tilde{\alpha}_{I,J}(y) := \max\big\{\alpha \in I \cap \mathbb{R}_{<0} \ : \ y\,\sigma(\alpha) \in J\big\}. \tag{3.55}$$

With these definitions, the formula for $\mathcal{I}(y_2, \theta_2)$ can be rewritten as

$$\begin{aligned} &\int_0^{\pi/4} \int_0^{\infty} I\Big(\alpha_{I_i,\ell_{\mathcal{W}}(\theta_2)}(y_2) \notin y_1^{-1}\ell(\theta_1)\Big) \, \frac{\mathrm{d}y_1}{y_1^3} \, \mathrm{d}\theta_1 \\ &+ \int_{3\pi/4}^{\pi} \int_0^{\infty} I\Big(\tilde{\alpha}_{I_i,\ell_{\mathcal{W}}(\theta_2)}(y_2) \notin y_1^{-1}\ell(\theta_1)\Big) \, \frac{\mathrm{d}y_1}{y_1^3} \, \mathrm{d}\theta_1 \\ &= \frac{1}{2} c_{\mathcal{P}} \alpha_{I_i,\ell_{\mathcal{W}}(\theta_2)}(y_2)^2 \int_0^{\pi/4} \frac{\mathrm{d}\theta_1}{(\cos\theta_1 + \sin\theta_1)^2} + \frac{1}{2} c_{\mathcal{P}} \tilde{\alpha}_{I_i,\ell_{\mathcal{W}}(\theta_2)}(y_2)^2 \int_{3\pi/4}^{\pi} \frac{\mathrm{d}\theta_1}{(\cos\theta_1 - \sin\theta_1)^2} \\ &= \frac{c_{\mathcal{P}}}{4} \big(\alpha_{I_i,\ell_{\mathcal{W}}(\theta_2)}(y_2)^2 + \tilde{\alpha}_{I_i,\ell_{\mathcal{W}}(\theta_2)}(y_2)^2\big), \end{aligned}$$

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implying that

$$\int_0^{2\pi}\int_1^{\lambda^2} \mathcal{I}(y_2,\theta_2)\,\frac{\mathrm{d}y_2}{y_2^3}\,\mathrm{d}\theta_2 = \frac{c_{\mathcal{P}}}{4}\int_0^{2\pi}\int_1^{\lambda^2}\left(\alpha_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2+\tilde{\alpha}_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2\right)\frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta.$$

Now note that $\ell_{\mathcal{W}}(\theta+\pi)=-\ell_{\mathcal{W}}(\theta)$ from which we can deduce that $\alpha_{I_i,\ell_{\mathcal{W}}(\theta+\pi)}(y)=-\tilde{\alpha}_{I_i,\ell_{\mathcal{W}}(\theta)}(y)$. Hence

$$\begin{aligned}\int_0^{2\pi}\int_1^{\lambda^2}\tilde{\alpha}_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2\,\frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta &= \int_0^{2\pi}\int_1^{\lambda^2}\alpha_{I_i,\ell_{\mathcal{W}}(\theta+\pi)}(y)^2\,\frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_1^{\lambda^2}\alpha_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2\,\frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta,\end{aligned}$$

implying that

$$\int_0^{2\pi}\int_1^{\lambda^2} \mathcal{I}(y_2,\theta_2)\,\frac{\mathrm{d}y_2}{y_2^3}\,\mathrm{d}\theta_2 = \frac{c_{\mathcal{P}}}{2}\int_0^{2\pi}\int_1^{\lambda^2}\alpha_{I_i,\ell_{\mathcal{W}}(\theta)}(y)^2\,\frac{\mathrm{d}y}{y^3}\,\mathrm{d}\theta.$$

Plugging this back to (3.53), we get the desired formula for $G_{M,i}(s)$, (3.51).

The formula (3.52) follows directly from (3.51) by using

$$\mathrm{Area}(\mathfrak{F}_i)=\mathrm{covol}(\iota(I_i^{-2}))=\mathrm{Nr}(I_i)^{-2}\Delta_K^{1/2}$$

(which holds by (2.3)) and since $\mathfrak{F}_i$ is a fundamental domain for $\mathbb{R}^2/\iota(I_i^{-2})$, and the formulas for c_K and $c_{\mathcal{P}}$, respectively; see (2.11) and (1.11). □

3.6 | Estimating the error term

Recall that $G(s)=\sum_{i=1}^{\kappa}G_i(s)=\sum_{i=1}^{\kappa}\big(G_{M,i}(s)+G_{E,i}(s)\big)$ for all sufficiently large s, and by definition we have $G_{E,i}(s)\geqslant 0$; see (3.43). Recalling also Proposition 3.8, it follows that, in order to complete the proof of Theorem 4, it only remains to prove that $G_{E,i}(s)$ is majorized by the bound in (3.7) for all sufficiently large s.

We first prove two auxiliary lemmas.

Lemma 3.9. *For any fractional ideal I of $\mathcal{O}_K$, there exists a constant $C=C(I)>0$ such that for any two intervals $R_1,R_2\subset\mathbb{R}$ with $|R_1|\cdot|R_2|\geqslant C$, and for any Lebesgue measurable subset $R_2'\subset\mathbb{R}$, we have*

$$\mathrm{Area}(X_I\setminus\pi(R_1\times R_2'))\leqslant|R_1|\cdot m(R_2\setminus R_2'),$$

where $X_I:=\mathbb{R}^2/\iota(I)$, and $\pi:\mathbb{R}^2\to X_I$ is the quotient map.

(Recall that m denotes Lebesgue measure on $\mathbb{R}$; also for an interval R, we write $|R|=m(R)$ for its length.)

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Proof. Let us fix a fundamental parallelogram $\mathfrak{F}$ for the lattice $\iota(I)$ in $\mathbb{R}^2$, and then fix intervals $J_1, J_2 \subset \mathbb{R}$ such that $\mathfrak{F} \subset J_1 \times J_2$. We claim that the statement of the lemma holds with $C := \lambda |J_1||J_2|$. Indeed, assume that R_1, R_2 are intervals with $|R_1| \cdot |R_2| \geqslant \lambda |J_1||J_2|$. Then there exists some $m \in \mathbb{Z}$ such that $\lambda^m |R_1| \geqslant |J_1|$ and $\lambda^{-m}|R_2| \geqslant |J_2|$, namely, $|\sigma(\lambda^m)R_2| \geqslant |J_2|$. This means that a translate of $(R_1 \times R_2)\,\mathrm{diag}(\lambda, \sigma(\lambda))^m$ contains F; and hence $R_1 \times R_2$ contains a translate $\mathfrak{F}'$ of $\mathfrak{F}\,\mathrm{diag}(\lambda, \sigma(\lambda))^{-m}$. But using the fact that $\iota(I)$ is invariant under $\mathrm{diag}(\lambda, \sigma(\lambda))$, it follows that $\mathfrak{F}'$ is again a fundamental parallelogram for the lattice $\iota(I)$ in $\mathbb{R}^2$. Hence

$$\mathrm{Area}(X_I \setminus \pi(R_1 \times R_2')) = \mathrm{Area}(\mathfrak{F}' \setminus \pi^{-1}(\pi(R_1 \times R_2'))).$$

Using $\mathfrak{F}' \subset R_1 \times R_2$ and $\pi^{-1}(\pi(R_1 \times R_2')) \supset R_1 \times R_2'$, the above is

$$\leqslant \mathrm{Area}((R_1 \times R_2) \setminus (R_1 \times R_2')) = \mathrm{Area}(R_1 \times (R_2 \setminus R_2')) = |R_1| \cdot m(R_2 \setminus R_2'). \qquad \square$$

Lemma 3.10. *For any fixed set $A \subset \mathbb{R}$, we have*

$$\forall\, 0 < a_1 \leqslant a_2 : \qquad \mathfrak{f}(A, a_1) \leqslant 2\mathfrak{f}(A, a_2). \tag{3.56}$$

Proof. Let $0 < a_1 \leqslant a_2$ be given, and let J be an arbitrary interval of length a_2. Take $k \in \mathbb{Z}_{>0}$ so that $ka_1 \leqslant a_2 < (k+1)a_1$; then there exist k pairwise disjoint (open) intervals $J_1, \dots, J_k \subset J$ which all have length a_1. Now $m(J \setminus A) \geqslant \sum_{\ell=1}^{k} m(J_\ell \setminus A) \geqslant ka_1 \mathfrak{f}(A, a_1)$, and since this holds for all intervals J of length a_2, it follows that $\mathfrak{f}(A, a_2) \geqslant a_2^{-1} \cdot ka_1 \mathfrak{f}(A, a_1) \geqslant \frac{k}{k+1}\mathfrak{f}(A, a_1) \geqslant \frac{1}{2}\mathfrak{f}(A, a_1)$. $\square$

We now turn to bounding the error term $G_{E,i}(s)$, which we defined in (3.43), for large s.

First of all, note that if $\theta_1 \in (\frac{\pi}{4}, \frac{3\pi}{4})$ then $T(1)\mathrm{k}_{-\theta_1}$ contains the line segment between (0,0) and $(0, c_{\mathcal{P}}^{-1/2})$. Recall also that since $\mathbf{0} \in \mathcal{W}^\circ = \mathcal{W}$, there is an $r_{\mathcal{W}} > 0$ such that $\mathcal{B}^2_{r_{\mathcal{W}}} \subset \mathcal{W}$. It follows that if $\theta_1 \in (\frac{\pi}{4}, \frac{3\pi}{4})$, then the condition (3.46) is satisfied with $\alpha = 0$ whenever $(\beta, \sigma(\beta))$ belongs to the rectangle $\big(0, y_1 s^{1/2}\lambda^{-2r} c_{\mathcal{P}}^{-1/2}\big) \times \big(-y_2\lambda^{2r} r_{\mathcal{W}}, y_2\lambda^{2r} r_{\mathcal{W}}\big)$. This rectangle has area $\gg s^{1/2}$, since $\boldsymbol{y} \in Y_{t_1}$. Hence by Lemma 3.6, for s sufficiently large, there always exists some $\beta \in I_i^{-1}$ such that $(\beta, \sigma(\beta))$ belongs to the rectangle, and so, by Remark 3.45, $\mathcal{L}_K h \cap \mathcal{T}(s) \neq \emptyset$. This proves that the contribution from all $\theta_1 \in (\frac{\pi}{4}, \frac{3\pi}{4})$ to the integral in (3.43) is zero. Thus, for s sufficiently large, we may replace the range of integration for $\boldsymbol{\theta}$ in (3.43) by Z_J.

Next, as in the discussion in Section 3.4 (near (3.40)), because of the condition $\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset$ in the integrand, any $\boldsymbol{y} \in Y_{t_1}$ which contribute to the integral in (3.43) must satisfy $y_1, y_2 \gg s^{1/4}$. Furthermore, by Lemma 3.4, given any $\langle \boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{x}\rangle$ which makes the integrand in (3.43) non-zero, there exists some $\alpha \in I_i$ such that $(\alpha, \sigma(\alpha))$ belongs to the set

$$\mathfrak{A}_{\boldsymbol{\theta},\boldsymbol{y}}(s) := (y_1^{-1} s^{1/2}\lambda^{-2r}\ell(\theta_1)) \times (y_2^{-1}\lambda^{2r}\ell_{\mathcal{W}}(\theta_2)). \tag{3.57}$$

Since we have restricted to $\theta_1 \in J$, we have $0 \notin \ell(\theta_1)$, and hence $\alpha \neq 0$. Note that $\ell(\theta_1)$ and $\ell_{\mathcal{W}}(\theta_2)$ are contained in bounded intervals independent of θ_1, θ_2; hence for $y_1, y_2 \gg s^{1/4}$, the set $\mathfrak{A}_{\boldsymbol{\theta},\boldsymbol{y}}(s)$ is contained in a fixed bounded region $B \subset \mathbb{R}^2$. Set

$$\Lambda := \{\alpha \in I_i \setminus \{0\} \ : \ (\alpha, \sigma(\alpha)) \in B\}.$$

This is a finite subset of $I_i \setminus \{0\}$, and our discussion shows that for any $\langle \boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{x} \rangle \in Z_J \times Y_{t_1} \times \mathfrak{F}_i$ which makes the integrand in (3.43) non-zero, there must exist some $\alpha \in \Lambda$ satisfying $(\alpha, \sigma(\alpha)) \in \mathfrak{A}_{\boldsymbol{\theta},\boldsymbol{y}}(s)$, or equivalently, $\iota(\Lambda) \cap \mathfrak{A}_{\boldsymbol{\theta},\boldsymbol{y}}(s) \neq \emptyset$. Using now the fact that both the minima of $|\alpha|$ and $|\sigma(\alpha)|$ for $\alpha \in \Lambda$ are bounded away from zero, and also the fact that both $|\ell(\theta_1)|$ and $|\ell_{\mathcal{W}}(\theta_2)|$ are bounded away from zero independently of θ_1, θ_2, it follows that we must have $y_1, y_2 \ll s^{1/4}$. We have thus proved that there exist some constants $c_2 > c_1 > 0$ (which depend on K and $\mathcal{W}$, but not on s), such that for all sufficiently large s, every $\langle \boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{x} \rangle \in Z_J \times Y_{t_1} \times \mathfrak{F}_i$ which makes the integrand in (3.43) non-zero, must satisfy $c_1 s^{1/4} \leqslant y_1, y_2 \leqslant c_2 s^{1/4}$.

In view of the above discussion, we have for s sufficiently large:

$$\begin{aligned} G_{E,i}(s) &\asymp s^{-3/2} \int_{Z_J} \int_{c_1 s^{1/4}}^{c_2 s^{1/4}} \int_{c_1 s^{1/4}}^{c_2 s^{1/4}} I\left(\iota(\Lambda) \cap \mathfrak{A}_{\boldsymbol{\theta},\mathbf{y}}(s) \neq \emptyset\right) \int_{\mathfrak{F}_i} I\left(\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset\right) \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\boldsymbol{\theta} \\ &\asymp s^{-3/2} \sum_{\alpha \in \Lambda} \int_{Z_J} \int_{c_1 s^{1/4}}^{c_2 s^{1/4}} \int_{c_1 s^{1/4}}^{c_2 s^{1/4}} I\left((\alpha, \sigma(\alpha)) \in \mathfrak{A}_{\boldsymbol{\theta},\mathbf{y}}(s)\right) \int_{\mathfrak{F}_i} I\left(\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset\right) \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\boldsymbol{\theta}. \end{aligned} \tag{3.58}$$

Next, the condition $\mathcal{L}_K h \cap \mathcal{T}(s) = \emptyset$ in (3.58) implies by Remark 3.45 that (3.46) fails for all $(\alpha, \beta) \in I_i \times I_i^{-1}$. This is equivalent to saying that for every $\alpha \in I_i$ we have $\iota(I_i^{-1}) \cap \mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\boldsymbol{x},\alpha}(s) = \emptyset$, where

$$\mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\boldsymbol{x},\alpha}(s) := (y_1 \mathfrak{F}_1 - \alpha x_1) \times (y_2 \mathfrak{F}_2 - \sigma(\alpha) x_2) \tag{3.59}$$

with $\mathfrak{F}_1 = \mathfrak{F}_1(s, y_1, \theta_1, \alpha)$ and $\mathfrak{F}_2 = \mathfrak{F}_2(s, y_2, \theta_2, \alpha)$ defined by

$$\begin{aligned} \mathfrak{F}_1 &:= \left\{ t \in \mathbb{R} \ : \ (y_1 \alpha, t) \in s^{1/2} \lambda^{-2r} T(1) \mathrm{k}_{-\theta_1} \right\} \quad \text{and} \\ \mathfrak{F}_2 &:= \left\{ t \in \mathbb{R} \ : \ (y_2 \sigma(\alpha), t) \in \lambda^{2r} \mathcal{W} \mathrm{k}_{-\theta_2} \right\}. \end{aligned} \tag{3.60}$$

Hence from (3.58) we get, using also the fact that if $\iota(I_i^{-1}) \cap \mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\boldsymbol{x},\alpha}(s) = \emptyset$ holds for every $\alpha \in I_i$, then in particular it holds for the $\alpha \in \Lambda$ which is our summation variable:

$$\begin{aligned} G_{E,i}(s) \ll s^{-3/2} \sum_{\alpha \in \Lambda} \int_{Z_J} \int_{c_1 s^{1/4}}^{c_2 s^{1/4}} \int_{c_1 s^{1/4}}^{c_2 s^{1/4}} I\left((\alpha, \sigma(\alpha)) \in \mathfrak{A}_{\boldsymbol{\theta},\mathbf{y}}(s)\right) \\ \times \int_{\mathfrak{F}_i} I\left(\iota(I_i^{-1}) \cap \mathfrak{B}_{\boldsymbol{\theta},\mathbf{y},\mathbf{x},\alpha}(s) = \emptyset\right) \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\boldsymbol{\theta}. \end{aligned} \tag{3.61}$$

Note here that $\mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\boldsymbol{x},\alpha}(s) = \mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\mathbf{0},\alpha}(s) - (\alpha x_1, \sigma(\alpha) x_2)$, and recall that $\mathfrak{F}_i$ is a fundamental domain for $\mathbb{R}^2/\iota(I_i^{-2})$. Note also that the map $\boldsymbol{x} \mapsto \boldsymbol{v} := (\alpha x_1, \sigma(\alpha) x_2)$ induces a diffeomorphism from $\mathbb{R}^2/\iota(I_i^{-2})$ onto $\mathbb{R}^2/\iota(\alpha I_i^{-2})$, carrying $\mathrm{d}\boldsymbol{x}$ to $|\mathrm{N}(\alpha)|^{-1} \mathrm{d}\boldsymbol{v}$. Hence the integral over $\boldsymbol{x}$ in (3.61) equals

$$\frac{1}{|\mathrm{N}(\alpha)|} \int_{\mathbb{R}^2/\iota(\alpha I_i^{-2})} I\Big(\iota(I_i^{-1}) \cap (\mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\mathbf{0},\alpha}(s) - \boldsymbol{v}) = \emptyset\Big) \mathrm{d}\boldsymbol{v}. \tag{3.62}$$

Let π be the quotient map from $\mathbb{R}^2$ onto the torus $X := \mathbb{R}^2/\iota(I_i^{-1})$. Note that the condition $\iota(I_i^{-1}) \cap (\mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\mathbf{0},\alpha}(s) - \boldsymbol{v}) = \emptyset$ is equivalent with $\pi(\boldsymbol{v}) \notin \pi(\mathfrak{B}_{\boldsymbol{\theta},\boldsymbol{y},\mathbf{0},\alpha}(s))$. Also, since $\alpha \in \Lambda \subset I_i$, $\iota(\alpha I_i^{-2})$ is a sublattice of $\iota(I_i^{-1})$; hence π induces to a covering map from $\mathbb{R}^2/\iota(\alpha I_i^{-2})$ onto X, which preserves

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Lebesgue measure. Using (2.3), it follows that this covering map has degree $|\mathrm{N}(\alpha)|/\mathrm{Nr}(I_i)$. Hence the integral in (3.62) equals

$$\mathrm{Nr}(I_i)^{-1}\,\mathrm{Area}(X \setminus \pi(\mathfrak{B}_{\theta,\boldsymbol{y},\mathbf{0},\alpha}(s))).$$

But we have $\mathfrak{B}_{\theta,\boldsymbol{y},\mathbf{0},\alpha}(s) = y_1\mathfrak{J}_1 \times y_2\mathfrak{J}_2$, where $\mathfrak{J}_1$ is an open interval which is non-empty if and only if $\alpha \in y_1^{-1}s^{1/2}\lambda^{-2r}\ell(\theta_1)$. This last condition is guaranteed to hold because of the condition $(\alpha,\sigma(\alpha)) \in \mathfrak{A}_{\theta,\boldsymbol{y}}(s)$ appearing in (3.61). It now follows from Lemma 3.9 that there is a constant $C = C(I_i^{-1})$ such that, for any interval R_2 of length $|R_2| = C/|y_1\mathfrak{J}_1|$, the integral in (3.62) is bounded from above by $\mathrm{Nr}(I_i)^{-1}|y_1\mathfrak{J}_1| \cdot m\big(R_2 \setminus y_2\mathfrak{J}_2\big)$. Furthermore, comparing (3.60) and (3.1), we see that $\mathfrak{J}_2 = \lambda^{2r}R_{\mathcal{W}}(\theta_2, \lambda^{-2r}\sigma(\alpha)y_2)$. Hence, writing $R_2 = y_2\lambda^{2r}\cdot(x, x+a)$ with $x \in \mathbb{R}$ and $a = C/\big(\lambda^{2r}y_1y_2|\mathfrak{J}_1|\big)$, it follows that the integral in (3.62) is bounded from above by

$$\mathrm{Nr}(I_i)^{-1}\lambda^{2r}y_1y_2|\mathfrak{J}_1| \cdot m\big((x, x+a) \setminus R_{\mathcal{W}}(\theta_2, \lambda^{-2r}\sigma(\alpha)y_2)\big).$$

This is true for every $x \in \mathbb{R}$. Hence, using the notation introduced in (3.5), we conclude that the integral in (3.62) is bounded from above by

$$\mathrm{Nr}(I_i)^{-1}C \cdot \mathfrak{f}\left(R_{\mathcal{W}}(\theta_2, \lambda^{-2r}\sigma(\alpha)y_2), \frac{C}{\lambda^{2r}y_1y_2|\mathfrak{J}_1|}\right).$$

Note also that $(\alpha,\sigma(\alpha)) \in \mathfrak{A}_{\theta,\boldsymbol{y}}(s)$ holds if and only if both $\alpha \in y_1^{-1}s^{1/2}\lambda^{-2r}\ell(\theta_1)$ and $\sigma(\alpha) \in y_2^{-1}\lambda^{2r}\ell_{\mathcal{W}}(\theta_2)$, and we recall that the first of these conditions is equivalent with $\mathfrak{J}_1 \neq \emptyset$, while, by (3.2), the second condition is equivalent with $R_{\mathcal{W}}(\theta_2, \lambda^{-2r}y_2\sigma(\alpha)) \neq \emptyset$. Using these facts together with our bound on the integral in (3.62), we obtain:

$$\begin{aligned} G_{E,i}(s) \ll s^{-3/2} \sum_{\alpha\in\Lambda} \int_{Z_J} \int_{c_1s^{1/4}}^{c_2s^{1/4}} I(\mathfrak{J}_1 \neq \emptyset) \int_{c_1s^{1/4}}^{c_2s^{1/4}} I\left(R_{\mathcal{W}}(\theta_2, \lambda^{-2r}y_2\sigma(\alpha)) \neq \emptyset\right) \\ \times \mathfrak{f}\left(R_{\mathcal{W}}(\theta_2, \lambda^{-2r}\sigma(\alpha)y_2), \frac{C}{\lambda^{2r}y_1y_2|\mathfrak{J}_1|}\right) \mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}\boldsymbol{\theta}. \end{aligned} \tag{3.63}$$

Here it should be recalled that $\mathfrak{J}_1 = \mathfrak{J}_1(s, y_1, \theta_1, \alpha)$; see (3.60). Now note that for any y_1, y_2 appearing in the above integral, we have $\lambda^{2r}y_1y_2 \geqslant c_1^2\lambda^{-2}s^{3/4}$; hence if we set $C' := C\lambda^2c_1^{-2}$, then $\frac{C}{\lambda^{2r}y_1y_2|\mathfrak{J}_1|} \leqslant \frac{C'}{s^{3/4}|\mathfrak{J}_1|}$, and so, using Lemma 3.10 and then substituting $y_2 = \lambda^{2r}\sigma(\alpha)^{-1}y$, we obtain (since Λ is a fixed finite subset of $I_i \setminus \{0\}$):

$$G_{E,i}(s) \ll s^{-5/4} \textstyle\sum_{\alpha\in\Lambda} \int_{Z_J} \int_{c_1s^{1/4}}^{c_2s^{1/4}} I(\mathfrak{J}_1 \neq \emptyset) \int_{\mathbb{R}} \tilde{\mathfrak{f}}\left(R_{\mathcal{W}}(\theta_2, y), \frac{C'}{s^{3/4}|\mathfrak{J}_1|}\right) \mathrm{d}y\,\mathrm{d}y_1\,\mathrm{d}\boldsymbol{\theta}, \tag{3.64}$$

where we use the notation $\tilde{\mathfrak{f}}$ defined in (3.5).

Now we have

$$\mathfrak{J}_1 \neq \emptyset \quad \Leftrightarrow \quad \begin{cases} 0 < y_1\alpha < s^{\frac{1}{2}}\lambda^{-2r}c_p^{-\frac{1}{2}}(\cos\theta_1 + \sin\theta_1) & \text{if } \theta_1 \in (0, \frac{\pi}{4}), \\ s^{\frac{1}{2}}\lambda^{-2r}c_p^{-\frac{1}{2}}(\cos\theta_1 - \sin\theta_1) < y_1\alpha < 0 & \text{if } \theta_1 \in (\frac{3\pi}{4}, \pi), \end{cases}$$

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and it follows from the definition in (3.60) that for any $\alpha \in \Lambda$, $\theta_1 \in J$ and $y_1 \geqslant c_1 s^{1/4}$ for which $\mathfrak{F}_1 \neq \emptyset$ holds, we have $|\mathfrak{F}_1| \geqslant C'' u_1$ with $u_1 := s^{\frac{1}{2}} \lambda^{-2r} c_{\mathcal{P}}^{-\frac{1}{2}} (|\cos\theta_1| + \sin\theta_1) - y_1 |\alpha| \in \mathbb{R}_{>0}$, where $C'' > 0$ is a certain absolute constant. Furthermore, $u_1 < c_3 s^{1/4}$ with $c_3 := 2c_{\mathcal{P}}^{-\frac{1}{2}} \lambda^2$. Hence, using Lemma 3.10,

$$G_{E,i}(s) \ll s^{-5/4} \sum_{\alpha\in\Lambda} \int_{Z_J} \int_0^{c_3 s^{1/4}} \int_{\mathbb{R}} \tilde{\mathfrak{f}} \left(R_{\mathcal{W}}(\theta_2, y), \frac{C'}{C'' s^{3/4} u_1} \right) \mathrm{d}y \, \mathrm{d}u_1 \, \mathrm{d}\theta.$$

Here the integrand is independent of θ_1 and α. Hence, using the fact that Λ is a fixed finite set, and substituting $u_1 = (C'/C'') s^{-3/4} u$, we conclude:

$$G_{E,i}(s) \ll s^{-2} \int_0^{(c_3 C''/C')s} \int_0^{2\pi} \int_{\mathbb{R}} \widetilde{\mathfrak{f}}\big(R_{\mathcal{W}}(\theta_2, y), u^{-1}\big) \, \mathrm{d}y \, \mathrm{d}\theta_2 \, \mathrm{d}u. \tag{3.65}$$

Hence we have proved the error bound in (3.7) (with $C := c_3 C''/C'$). As we have already noted, because of $G(s) = \sum_{i=1}^{\kappa} G_i(s) = \sum_{i=1}^{\kappa} \big(G_{M,i}(s) + G_{E,i}(s)\big)$ and Proposition 3.8, the bound (3.65) completes the proof of Theorem 4.

4 | MORE EXPLICIT FORMULA FOR THE LEADING COEFFICIENT

The main goal of this section is to prove the following more explicit formula for the leading coefficient $a_{\mathcal{P}}$ in Theorem 2, when we further assume that $\mathcal{W}$ is such that $\ell_{\mathcal{W}}(\theta)$ is an interval for every θ. Recall here that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $\ell_{\mathcal{W}}(\theta)$ is the projection of $\mathcal{W}\mathrm{k}_{-\theta}$ on the x-axis.

Theorem 5. *Keep the notation and assumptions as in Theorem 2 and further assume that $\ell_{\mathcal{W}}(\theta)$ is an interval for each θ, parameterized as in (1.16), that is, $\ell_{\mathcal{W}}(\theta) = r(\theta)(-\nu(\theta), 1)$. Then for each I_i $(1 \leqslant i \leqslant \kappa)$, there exist a finite partition $\mathbb{R}_{>0} = \bigsqcup_{j=1}^{l_{I_i}} S_{I_i,j}$ of $\mathbb{R}_{>0}$ into intervals and non-negative constants $A_{I_i,j}, B_{I_i,j}$ $(1 \leqslant j \leqslant l_{I_i})$, such that*

$$a_{\mathcal{P}} = \frac{\mathrm{Area}(\mathcal{W})}{4\Delta_K^2 \zeta_K(2)} \sum_{i=1}^{\kappa} \mathrm{Nr}(I_i)^{-2} \sum_{j=1}^{l_{I_i}} \int_{\tilde{S}_{I_i,j}} r(\theta)^{-2} \Big(A_{I_i,j} + B_{I_i,j} \nu(\theta)^{-2} \Big) \, \mathrm{d}\theta, \tag{4.1}$$

where $\tilde{S}_{I_i,j} := \{\theta \in [0, 2\pi) \ : \ \nu(\theta) \in S_{I_i,j}\}$.

Remark 4.2. For each I_i, the partition $\mathbb{R}_{>0} = \bigsqcup_{j=1}^{l_{I_i}} S_{I_i,j}$ and the constants $A_{I_i,j}, B_{I_i,j}$ are all computable from our analysis; see Section 4.4 for three explicit examples. We stress that the intervals $S_{I_i,j}$ are allowed to be open, closed or half-open, and may be degenerate, that is, of the form $[a, a] = \{a\}$ for some $a > 0$.

4.1 | Relations to the polar set of $\mathcal{W}$

Let $\mathcal{W} \subset \mathbb{R}^2$ be as above. The integral appearing in (4.1) is closely related to the polar set $\mathcal{W}^*$ of $\mathcal{W}$, which is defined by (1.21). Note that $\mathcal{W}^*$ is closed and convex; furthermore, since $\mathcal{W}$ is bounded

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and contains $\mathbf{0}$ in its interior, $\mathcal{W}^*$ is also bounded and contains $\mathbf{0}$ in its interior. Moreover, if $\mathcal{W}$ is symmetric about the origin, then so is $\mathcal{W}^*$; and if $\mathcal{W}$ is a convex polygon, then also $\mathcal{W}^*$ is a convex polygon.

The following lemma shows that $\mathcal{W}^*$ can be parameterized in polar coordinates in terms of the function $r(\theta)$.

Lemma 4.1. *Let $\mathcal{W}$ be as in Theorem 5. Then $\mathcal{W}^*$ is parameterized in polar coordinates by*

$$\mathcal{W}^* = \left\{(t,\theta) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}/2\pi\mathbb{Z} \ : \ 0 \leqslant t \leqslant r(-\theta)^{-1}\right\}.$$

Proof. Note that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$\mathcal{W}\mathrm{k}_{-\theta} = \{(w_1\cos\theta - w_2\sin\theta, w_1\sin\theta + w_2\cos\theta) \ : \ (w_1,w_2) \in \mathcal{W}\}.$$

Thus,

$$\ell_{\mathcal{W}}(\theta) = \{w_1\cos\theta - w_2\sin\theta \ : \ (w_1,w_2) \in \mathcal{W}\}.$$

It then follows from the definition of polar set and the relation $\ell_{\mathcal{W}}(\theta) \cap \mathbb{R}_{\geqslant 0} = [0, r(\theta))$, that the set

$$\left\{t \in \mathbb{R}_{\geqslant 0} \ : \ t(\cos\theta, -\sin\theta) \in \mathcal{W}^*\right\}$$

equals the interval $[0, r(\theta)^{-1}]$. The lemma then follows by a substitution $\theta \mapsto -\theta$. □

Remark 4.3. As a direct consequence of Lemma 4.1 and the relation

$$r(\theta+\pi) = r(\theta)\nu(\theta), \quad \forall\, \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tag{4.4}$$

(which in turn follows from the symmetry $\ell_{\mathcal{W}}(\theta+\pi) = -\ell_{\mathcal{W}}(\theta)$), we have

$$2\,\mathrm{Area}(\mathcal{W}^*) = \int_0^{2\pi} r(\theta)^{-2}\,\mathrm{d}\theta = \int_0^{2\pi} \nu(\theta)^{-2} r(\theta)^{-2}\,\mathrm{d}\theta. \tag{4.5}$$

For each $1 \leqslant i \leqslant \kappa$, let $1 \leqslant j_i \leqslant l_{I_i}$ be the unique index such that $1 \in S_{I_i,j_i}$. In the special case when $\tilde{S}_{I_i,j_i} = [0, 2\pi)$ for every $1 \leqslant i \leqslant \kappa$ (this happens if $\mathcal{W}$ is symmetric with respect to the origin, or a sufficiently small translate of such a set), then by (4.5), formula (4.1) simplifies to

$$a_{\mathcal{P}} = \frac{\mathrm{Area}(\mathcal{W})\,\mathrm{Area}(\mathcal{W}^*)}{\Delta_K^2\,\zeta_K(2)} \sum_{i=1}^{\kappa} C_{I_i}\mathrm{Nr}(I_i)^{-2}, \tag{4.6}$$

with $C_{I_i} := \frac{1}{2}(A_{I_i,j_i} + B_{I_i,j_i})$. More generally, the sum of integrals appearing in (4.1) can be understood as a *weighted* area of $\mathcal{W}^*$.

The remainder of this section is devoted to proving Theorem 5.

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4.2 | Some preliminary computations

We start our proof with the formula for $a_{\mathcal{P}}$ given in Theorem 4 which states that $a_{\mathcal{P}} = \sum_{i=1}^{\kappa} a_{\mathcal{P},i}$ with

$$a_{\mathcal{P},i} = \frac{\mathrm{Area}(\mathcal{W})}{4\Delta_K^2 \zeta_K(2)} \mathrm{Nr}(I_i)^{-2} \int_0^{2\pi} \int_1^{\lambda^2} \alpha_{I_i, \ell_{\mathcal{W}}(\theta)}(y)^2 \frac{\mathrm{d}y}{y^3} \, \mathrm{d}\theta. \tag{4.7}$$

It thus suffices to compute the above double integral, which we denote by $\mathcal{I}_{\mathcal{P},i}$. It is immediate from the definition (3.3) that $\alpha_{I,aJ}(ay) = \alpha_{I,J}(y)$ for all $a, y > 0$. Thus, defining $J_\nu := (-\nu, 1)$ for $\nu > 0$ and applying (3.10), we have:

$$\mathcal{I}_{\mathcal{P},i} = \int_0^{2\pi} \int_1^{\lambda^2} \alpha_{I_i, J_{\nu(\theta)}}\left(\frac{y}{r(\theta)}\right)^2 \frac{\mathrm{d}y}{y^3} \, \mathrm{d}\theta = \int_0^{2\pi} r(\theta)^{-2} \int_1^{\lambda^2} \alpha_{I_i, J_{\nu(\theta)}}(y)^2 \frac{\mathrm{d}y}{y^3} \, \mathrm{d}\theta. \tag{4.8}$$

We wish to compute the innermost integral in the last expression.

4.3 | Extremal points

Fix an integral ideal $I \subset \mathcal{O}_K$. As mentioned above, we are interested in computing the integral

$$\int_1^{\lambda^2} \alpha_{I, J_\nu}(y)^2 \frac{\mathrm{d}y}{y^3} \tag{4.9}$$

for any $\nu > 0$. For this, we introduce a notion which captures the possible values of the function α_{I,J_ν}. For any $\nu > 0$ and $\alpha \in I \cap \mathbb{R}_{>0}$, let $b_{\alpha,\nu} := \sigma(\alpha)$ if $\sigma(\alpha) > 0$ and $b_{\alpha,\nu} := \nu^{-1}\sigma(\alpha)$ if $\sigma(\alpha) < 0$. Note that for any $y > 0$, $y\sigma(\alpha) \in J_\nu$ if and only if $y < |b_{\alpha,\nu}|^{-1}$. We say $\alpha \in I \cap \mathbb{R}_{>0}$ is *ν-extremal with respect to I* if

$$\iota(I) \cap R_{\alpha,\nu} = \emptyset,$$

where

$$R_{\alpha,\nu} := \begin{cases} (0,\alpha) \times [-\nu\sigma(\alpha), \sigma(\alpha)] & \text{if } \sigma(\alpha) > 0, \\ (0,\alpha) \times [\sigma(\alpha), \nu^{-1}|\sigma(\alpha)|] & \text{if } \sigma(\alpha) < 0. \end{cases}$$

We denote by $E_{I,\nu} \subset I \cap \mathbb{R}_{>0}$ the set of all ν-extremal points with respect to I. Below we will simply call elements in $E_{I,\nu}$ ν-extremal when there is no ambiguity. Since $\iota(I)$ is a lattice in $\mathbb{R}^2$, $E_{I,\nu}$ is always nonempty. Indeed, $E_{I,\nu}$ is always infinite, as it is invariant under a certain multiplication map as shown in the following lemma.

Lemma 4.2. *The set $E_{I,\nu}$ is invariant under the map $S : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined by $S(y) := \lambda y$ if $\sigma(\lambda) > 0$ and $S(y) := \lambda^2 y$ if $\sigma(\lambda) < 0$.*

Proof. Suppose $\alpha \in I \cap \mathbb{R}_{>0}$ is ν-extremal, that is, $\iota(I) \cap R_{\alpha,\nu} = \emptyset$ with $R_{\alpha,\nu}$ the rectangle given as above. Let $d_\lambda = \mathrm{diag}(\lambda, \sigma(\lambda))$ and note that $\iota(I)$ is invariant under the right multiplication action

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of d_λ. The lemma then follows by noting that $R_{\alpha,\nu}d_\lambda = R_{\lambda\alpha,\nu}$ if $\sigma(\lambda) > 0$ and $R_{\alpha,\nu}d_\lambda^2 = R_{\lambda^2\alpha,\nu}$ if $\sigma(\lambda) < 0$. □

Remark 4.10. The set $E_{I,1}$ is invariant under the map $y \mapsto \lambda y$ even if $\sigma(\lambda) < 0$. This is because we always have $R_{\alpha,1}d_\lambda = R_{\lambda\alpha,1}$, since $R_{\alpha,1} = (0,\alpha)\times[-|\sigma(\alpha)|, |\sigma(\alpha)|]$.

Note that $\alpha \in I \cap \mathbb{R}_{>0}$ is ν-extremal if and only if

$$|b_{\beta,\nu}| > |b_{\alpha,\nu}| \text{ for any } \beta \in I \cap \mathbb{R}_{>0} \text{ with } \beta < \alpha. \tag{4.11}$$

In particular, if $\alpha < \beta$ are two ν-extremal points, then $|b_{\alpha,\nu}| > |b_{\beta,\nu}|$. Moreover, one can check from (4.11) that α_{I,J_ν} takes values only on ν-extremal points. Using these facts, we can now prove the following lemma which enables us to compute the integral (4.9).

Lemma 4.3. *For any $\nu > 0$, let $0 < \beta < \alpha$ be two consecutive elements in $E_{I,\nu}$. Then $\alpha_{I,J_\nu}(y) = \alpha$ for any $|b_{\beta,\nu}|^{-1} \leqslant y < |b_{\alpha,\nu}|^{-1}$.*

Remark 4.12. By discreteness of $\iota(I)$, one sees that $\#(E_{I,\nu}/S) < \infty$ for any $\nu > 0$; thus we can take two consecutive elements from $E_{I,\nu}$ as in the above lemma.

Proof of Lemma 4.3. Take any $|b_{\beta,\nu}|^{-1} \leqslant y < |b_{\alpha,\nu}|^{-1}$ and let $\beta' = \alpha_{I,J_\nu}(y)$. In particular, β' is ν-extremal and $y\sigma(\beta') \in J_\nu$, and the last condition is equivalent to $y < |b_{\beta',\nu}|^{-1}$. Moreover, $y < |b_{\alpha,\nu}|^{-1}$ implies $y\sigma(\alpha) \in J_\nu$. Thus $\beta' \leqslant \alpha$. If $\beta' < \alpha$, then since $\beta < \alpha$ are two consecutive ν-extremal points, we must have $\beta' \leqslant \beta$ and therefore $|b_{\beta',\nu}| \geqslant |b_{\beta,\nu}|$, which contradicts the fact that $|b_{\beta,\nu}|^{-1} \leqslant y < |b_{\beta',\nu}|^{-1}$. Hence $\beta' = \alpha$ as desired. □

We can now compute the integral $\int_1^{\lambda^2} \alpha_{I,J_\nu}(y)^2 \frac{\mathrm{d}y}{y^3}$ in terms of the set $E_{I,\nu}$.

Lemma 4.4. *For any $\nu > 0$ and $\alpha_0 \in E_{I,\nu}$, if $\alpha_0 < \alpha_1 < \cdots < \alpha_l = \lambda^2\alpha_0$ is the complete list of points in $E_{I,\nu} \cap [\alpha_0, \lambda^2\alpha_0]$, then*

$$\int_1^{\lambda^2} \alpha_{I,J_\nu}(y)^2 \frac{\mathrm{d}y}{y^3} = A_{I,\nu} + B_{I,\nu}\nu^{-2},$$

where

$$A_{I,\nu} := \frac{1}{2} \sum_{\substack{j=0 \\ (\sigma(\alpha_j)>0)}}^{l-1} \sigma(\alpha_j)^2\left(\alpha_{j+1}^2 - \alpha_j^2\right) \quad \text{and} \quad B_{I,\nu} := \frac{1}{2} \sum_{\substack{j=0 \\ (\sigma(\alpha_j)<0)}}^{l-1} \sigma(\alpha_j)^2\left(\alpha_{j+1}^2 - \alpha_j^2\right).$$

Proof. We have $|b_{\alpha_0,\nu}|^{-1} < |b_{\alpha_1,\nu}|^{-1} < \cdots < |b_{\alpha_l,\nu}|^{-1} = |b_{\alpha_0,\nu}|^{-1}\lambda^2$. Hence by (3.10) and Lemma 4.3,

$$\int_1^{\lambda^2} \alpha_{I,J_\nu}(y)^2 \frac{\mathrm{d}y}{y^3} = \sum_{j=1}^{l} \int_{|b_{\alpha_{j-1},\nu}|^{-1}}^{|b_{\alpha_j,\nu}|^{-1}} \alpha_j^2 \frac{\mathrm{d}y}{y^3} = \frac{1}{2}\sum_{j=1}^{l} \alpha_j^2\left(b_{\alpha_{j-1},\nu}^2 - b_{\alpha_j,\nu}^2\right).$$

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The lemma then follows by rewriting the above as (using the fact that $\alpha_l^2 b_{\alpha_l,\nu}^2 = \alpha_0^2 b_{\alpha_0,\nu}^2$):

$$\frac{1}{2}\sum_{j=0}^{l-1} b_{\alpha_j,\nu}^2\left(\alpha_{j+1}^2-\alpha_j^2\right) = \frac{1}{2}\sum_{\substack{j=0\\(\sigma(\alpha_j)>0)}}^{l-1} \sigma(\alpha_j)^2\left(\alpha_{j+1}^2-\alpha_j^2\right) + \frac{\nu^{-2}}{2}\sum_{\substack{j=0\\(\sigma(\alpha_j)<0)}}^{l-1} \sigma(\alpha_j)^2\left(\alpha_{j+1}^2-\alpha_j^2\right).$$

□

Remark 4.13. If $\alpha < \beta$ are two consecutive ν-extremal points, then so are $\lambda^{2n}\alpha < \lambda^{2n}\beta$ for any integer n. Using the equality $\sigma(\alpha)^2(\beta^2-\alpha^2) = \sigma(\lambda^{2n}\alpha)^2\left((\lambda^{2n}\beta)^2-(\lambda^{2n}\alpha)^2\right)$, one easily sees that both $A_{I,\nu}$ and $B_{I,\nu}$ are independent of the choice of α_0.

Next, we study how $E_{I,\nu}$ varies in ν. Our goal is to show there are only finitely many possibilities for $E_{I,\nu}$ (and hence also for the coefficients $A_{I,\nu}$ and $B_{I,\nu}$). For this we introduce another related notion. We say $\alpha \in I \cap \mathbb{R}_{>0}$ is *positively* (resp. *negatively*) *extremal* with respect to I if $\sigma(\alpha) > 0$ (resp. $\sigma(\alpha) < 0$) and $\iota(I) \cap (0,\alpha) \times [0,\sigma(\alpha)] = \emptyset$ (resp. $\iota(I) \cap (0,\alpha) \times [\sigma(\alpha),0] = \emptyset$). We denote by E_I^+ and E_I^- the set of positively and negatively extremal points in I. Recall that S is the map $y \mapsto \lambda y$ if $\sigma(\lambda) > 0$ and the map $y \mapsto \lambda^2 y$ if $\sigma(\lambda) < 0$. It is clear that $E_I^\pm$ are invariant under S, and from the discreteness of $\iota(I)$ we see that $\#(E_I^\pm/S) < \infty$. Note that a necessary condition for $\alpha \in I \cap \mathbb{R}_{>0}$ to be ν-extremal for some $\nu > 0$ is that $\alpha \in E_I^+ \sqcup E_I^-$. Indeed this is also a sufficient condition in the sense that for any $\alpha \in E_I^+ \sqcup E_I^-$ there exists some $\nu > 0$ such that $\alpha \in E_{I,\nu}$; see Lemma 4.6 below.

For any $\nu > 0$, set $E_{I,\nu}^\pm := E_{I,\nu} \cap E_I^\pm$ so that $E_{I,\nu} = E_{I,\nu}^+ \sqcup E_{I,\nu}^-$. Note that $E_{I,\nu}^+$ (resp. $E_{I,\nu}^-$) is decreasing (resp. increasing) in ν, and there is a constant $C = C_I > 0$ such that $E_{I,\nu}^+ = \emptyset$ for all $\nu > C$ and $E_{I,\nu}^- = \emptyset$ for all $\nu < 1/C$, because of Lemma 3.6 and the fact that

$$\mathrm{Area}(R_{\alpha,\nu}) = (1+\nu^{\mathrm{sgn}(\sigma(\alpha))})|\alpha\sigma(\alpha)| > \nu^{\mathrm{sgn}(\sigma(\alpha))} \qquad (\forall\, \alpha \in I \setminus \{0\}).$$

To understand how $E_{I,\nu}$ varies in ν we study how $E_{I,\nu}^\pm$ varies in ν which is easier as it is monotone. We first study this property for a *single* element in I. The following lemma is immediate from our definitions.

Lemma 4.5. *For any $\alpha \in E_I^+$, let*

$$\nu_\alpha := \sup\{\nu > 0 \ : \ \iota(I) \cap (0,\alpha) \times [-\nu\sigma(\alpha), \sigma(\alpha)] = \emptyset\}.$$

Then α is ν-extremal if and only if $0 < \nu < \nu_\alpha$. Similarly, for any $\alpha \in E_I^-$, define

$$\nu_\alpha := \inf\{\nu > 0 \ : \ \iota(I) \cap (0,\alpha) \times [\sigma(\alpha), \nu^{-1}|\sigma(\alpha)|] = \emptyset\}.$$

Then α is ν-extremal if and only if $\nu > \nu_\alpha$.

Remark 4.14. For each $\alpha \in E_I^+ \sqcup E_I^-$, when ν attains the critical value ν_α, α *fails* to be ν_α-extremal. This fact, together with the finiteness of $(E_I^+ \sqcup E_I^-)/S$ and the fact that $E_{I,\nu}^+$ (resp. $E_{I,\nu}^-$) is decreasing (resp. increasing) in ν implies that $E_{I,\nu}^+$ (resp. E_I^-) is right (resp. left) continuous in the sense that for any $\nu > 0$, there exists some $\epsilon > 0$ such that $E_{I,\nu'}^+ = E_{I,\nu}^+$ for any $\nu' \in (\nu, \nu+\epsilon)$ (resp. $\nu' \in (\nu -$

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ϵ, ν)). Note also that

$$\nu_\alpha = \begin{cases} \inf\left\{ \dfrac{-\sigma(\beta)}{\sigma(\alpha)} \ : \ \beta \in I,\, 0 < \beta < \alpha,\, \sigma(\beta) < 0 \right\} & \text{if } \alpha \in E_I^+, \\ \sup\left\{ \dfrac{-\sigma(\alpha)}{\sigma(\beta)} \ : \ \beta \in I, 0 < \beta < \alpha,\, \sigma(\beta) > 0 \right\} & \text{if } \alpha \in E_I^-. \end{cases}$$

From this formula we see that the β attaining the above infimum (resp. supremum) is an element in E_I^- (resp. E_I^+). This gives very strict restrictions on the possible values of these ν_αs. In particular, ν_α cannot be 1 since otherwise we would have $\beta = -\alpha$, contradicting $0 < \beta < \alpha$. Moreover, the set of values $\{\nu_\alpha \ : \ \alpha \in E_I^+ \sqcup E_I^-\}$ is finite, since $\#(E_I^+ \sqcup E_I^-)/S < \infty$ and $\nu_{\lambda^2\alpha} = \nu_\alpha$ for every $\alpha \in E_I^+ \sqcup E_I^-$.

Lemma 4.6. *There exist finitely many constants* $0 < \nu_1^+ < \cdots < \nu_l^+ < \infty$ *such that* $E_I^+ \supsetneq E_{I,\nu_1^+}^+ \supsetneq E_{I,\nu_2^+}^+ \supsetneq \cdots \supsetneq E_{I,\nu_l^+}^+ = \emptyset$ *and*

$$E_{I,\nu}^+ = \begin{cases} E_I^+ & \text{if } 0 < \nu < \nu_1^+, \\ E_{I,\nu_i^+}^+ & \text{if } \nu_i^+ \leqslant \nu < \nu_{i+1}^+ \quad (1 \leqslant i < l), \\ \emptyset & \text{if } \nu \geqslant \nu_l^+. \end{cases}$$

Similarly, there exist $\infty > \nu_1^- > \cdots > \nu_{l'}^- > 0$ *such that* $E_I^- \supsetneq E_{I,\nu_1^-}^- \supsetneq E_{I,\nu_2^-}^- \supsetneq \cdots \supsetneq E_{I,\nu_{l'}^-}^- = \emptyset$ *and*

$$E_{I,\nu}^- = \begin{cases} E_I^- & \text{if } \nu > \nu_1^-, \\ E_{I,\nu_i^-}^- & \text{if } \nu_{i+1}^- < \nu \leqslant \nu_i^- \quad (1 \leqslant i < l'), \\ \emptyset & \text{if } 0 < \nu \leqslant \nu_{l'}^-. \end{cases}$$

Proof. We noted in Remark 4.14 that the set $\{\nu_\alpha \ : \ \alpha \in E_I^+\}$ is finite; let us order its elements as $0 < \nu_1^+ < \cdots < \nu_l^+$. Then if $0 < \nu < \nu_1^+$, it follows from Lemma 4.5 that every $\alpha \in E_I^+$ is ν-extremal, that is, $E_{I,\nu}^+ = E_I^+$; similarly, if $\nu \geqslant \nu_l^+$, then $E_{I,\nu}^+ = \emptyset$. Finally if $\nu_i^+ \leqslant \nu < \nu_{i+1}^+$ for some $1 \leqslant i < l$, then for every $\alpha \in E_I^+$ the condition $\nu < \nu_\alpha$ is equivalent with $\nu_\alpha \in \{\nu_{i+1}^+, \dots, \nu_l^+\}$; hence by Lemma 4.5, $E_{I,\nu}^+ = \{\alpha \in E_I^+ \ : \ \nu_\alpha \in \{\nu_{i+1}^+, \dots, \nu_l^+\}\} = E_{I,\nu_i^+}^+$. We have thus proved the part of the lemma which concerns $E_{I,\nu}^+$; the proof of the part concerning $E_{I,\nu}^-$ is entirely similar. □

Remark 4.15. From the proof, we see that the break-points in the statement of Lemma 4.6 are exactly the values ν_α with $\alpha \in E_I^+ \sqcup E_I^-$. Explicitly, $\{\nu_i^+ \ : \ 1 \leqslant i \leqslant l\} = \{\nu_\alpha \ : \ \alpha \in E_I^+\}$ and $\{\nu_i^- \ : \ 1 \leqslant i \leqslant l'\} = \{\nu_\alpha \ : \ \alpha \in E_I^-\}$.

As a corollary we have the following.

Corollary 4.7. *There exist a finite partition of* $\mathbb{R}_{>0} = \sqcup_{j=1}^{l_I} S_{I,j}$ *into intervals and* l_I *pairwise distinct subsets* $B_1 = E_I^+, B_2, \dots, B_{l_I - 1}, B_{l_I} = E_I^-$ *of* $E_I^+ \sqcup E_I^-$, *such that* $E_{I,\nu} = B_j$ *for any* $\nu \in S_{I,j}$.

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(The intervals $S_{I,j}$ are allowed to be open, closed, half-open and degenerate; see Remark 4.2.)

Proof. The existence of such a partition follows from Lemma 4.6 and the relation $E_{I,\nu} = E^+_{I,\nu} \sqcup E^-_{I,\nu}$. The pairwise disjointness of $B_1, \dots, B_{l_I} \subset E^+_I \sqcup E^-_I$ follows again from the relation $E_{I,\nu} = E^+_{I,\nu} \sqcup E^-_{I,\nu}$ and also the fact that $E^+_{I,\nu}$ and $E^-_{I,\nu}$ are decreasing and increasing in ν, respectively. □

Remark 4.16. We see from the above proof that if $\nu \notin \{\nu_\alpha \ : \ \alpha \in E^+_I \sqcup E^-_I\}$, then ν is an interior point of one of the intervals $S_{I,j}$. In particular, since $1 \notin \{\nu_\alpha \ : \ \alpha \in E^+_I \sqcup E^-_I\}$, one of the intervals $S_{I,j}$ contains 1 as an interior point.

We can now give the proof of Theorem 5.

Proof. For each I_i $(1 \leqslant i \leqslant \kappa)$, let $\mathbb{R}_{>0} = \sqcup_{j=1}^{l_{I_i}} S_{I_i,j}$ be the partition as in Corollary 4.7 applied with $I = I_i$, and define $A_{I_i,\nu} \geqslant 0$ and $B_{I_i,\nu} \geqslant 0$ for $\nu > 0$ as in Lemma 4.4. For each $1 \leqslant j \leqslant l_{I_i}$, it follows from Corollary 4.7 that both $A_{I_i,\nu}$ and $B_{I_i,\nu}$ are constant as ν varies over $S_{I_i,j}$; thus we may define $A_{I_i,j} := A_{I_i,\nu}$ and $B_{I_i,j} := B_{I_i,\nu}$ for any $\nu \in S_{I_i,j}$. Then by (4.8) and Lemma 4.4, we have

$$\mathcal{I}_{\mathcal{P},i} = \sum_{j=1}^{l_{I_i}} \int_{\tilde{S}_{I_i,j}} r(\theta)^{-2} \Big(A_{I_i,j} + B_{I_i,j} \nu(\theta)^{-2} \Big) \frac{\mathrm{d}y}{y^3} \mathrm{d}\theta,$$

where $\tilde{S}_{I_i,j} := \{\theta \in [0, 2\pi) \ : \ \nu(\theta) \in S_{I_i,j}\}$. Plugging this formula into (4.7) and applying the relation $a_{\mathcal{P}} = \sum_{i=1}^{\kappa} a_{\mathcal{P},i}$ we get the desired formula for $a_{\mathcal{P}}$. □

4.4 | Examples; proofs of Propositions 1.1–1.3

In this section, we work out the formula (4.1) explicitly in the three cases $K = \mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$, thus proving Propositions 1.1–1.3. Note that in all cases, K is of class number one; thus $\kappa = 1$ and $I_1 = \mathcal{O}_K$, and we set $I := I_1$. For details of the following computations, see also the program [21, explicit_aP.mpl].

First we treat the case when $K = \mathbb{Q}(\sqrt{2})$. Then $\lambda = 1 + \sqrt{2}$, and one verifies that

$$E^+_I = \{1, 2 + \sqrt{2}\} \cdot \lambda^{2\mathbb{Z}} \quad \text{and} \quad E^-_I = \{\sqrt{2}, 1 + \sqrt{2}\} \cdot \lambda^{2\mathbb{Z}},$$

with the associated values ν_α given by

$$\nu_1 = 1 + \sqrt{2}, \qquad \nu_{2+\sqrt{2}} = \tfrac{1}{2}\sqrt{2} \qquad \text{and} \qquad \nu_{\sqrt{2}} = \sqrt{2}, \quad \nu_{1+\sqrt{2}} = \sqrt{2} - 1.$$

Hence

$$E^+_{I,\nu} = \begin{cases} \{1, 2 + \sqrt{2}\} \cdot \lambda^{2\mathbb{Z}} & \text{if } \nu < \frac{1}{2}\sqrt{2}, \\ \lambda^{2\mathbb{Z}} & \text{if } \frac{1}{2}\sqrt{2} \leqslant \nu < 1 + \sqrt{2}, \\ \emptyset & \text{if } 1 + \sqrt{2} \leqslant \nu, \end{cases}$$

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and

$$E^-_{I,\nu} = \begin{cases} \emptyset & \text{if } \nu \leqslant \sqrt{2}-1, \\ (1+\sqrt{2})\lambda^{2\mathbb{Z}} & \text{if } \sqrt{2}-1 < \nu \leqslant \sqrt{2}, \\ \{\sqrt{2}, 1+\sqrt{2}\}\cdot\lambda^{2\mathbb{Z}} & \text{if } \sqrt{2} < \nu. \end{cases}$$

Using also the relation $E_{I,\nu} = E^+_{I,\nu} \sqcup E^-_{I,\nu}$, we get the corresponding partition $\mathbb{R}_{>0} = \sqcup_{j=1}^5 S_{I,j}$ where $S_{I,j} = S_j$ with $S_1 = (0, \sqrt{2}-1], S_2 = (\sqrt{2}-1, \frac{1}{2}\sqrt{2}), S_3 = [\frac{1}{2}\sqrt{2}, \sqrt{2}], S_4 = (\sqrt{2}, 1+\sqrt{2})$ and $S_5 = [1+\sqrt{2}, \infty)$. The corresponding constants $A_j := A_{I,j}$ and $B_j := B_{I,j}$ are

$$(A_1, B_1) = (B_5, A_5) = (\tfrac{7}{2} + 4\sqrt{2}, 0), \quad (A_2, B_2) = (B_4, A_4) = (2+3\sqrt{2}, \tfrac{1}{2}) \quad \text{and}$$
$$A_3 = B_3 = 1+\sqrt{2}.$$

Plugging these values into (4.1) and using also $\Delta_K = 8$ and $\zeta_K(2) = \frac{\pi^4}{48\sqrt{2}}$ (see Remark 2.12), we get

$$\begin{aligned} a_{\mathcal{P}} = \frac{3\sqrt{2}}{16\pi^4}\operatorname{Area}(\mathcal{W})\Bigg(& \int_{\tilde{S}_2} r(\theta)^{-2}\Big((2+3\sqrt{2}) + \tfrac{1}{2}\nu(\theta)^{-2}\Big)\,\mathrm{d}\theta \\ & + \int_{\tilde{S}_4} r(\theta)^{-2}\Big(\tfrac{1}{2} + (2+3\sqrt{2})\nu(\theta)^{-2}\Big)\,\mathrm{d}\theta \\ & + (\tfrac{7}{2}+4\sqrt{2})\Bigg(\int_{\tilde{S}_1} r(\theta)^{-2}\,\mathrm{d}\theta + \int_{\tilde{S}_5} r(\theta)^{-2}\nu(\theta)^{-2}\,\mathrm{d}\theta\Bigg) \\ & + (1+\sqrt{2})\int_{\tilde{S}_3} r(\theta)^{-2}\big(1+\nu(\theta)^{-2}\big)\,\mathrm{d}\theta\Bigg). \end{aligned} \tag{4.17}$$

Moreover, because of (4.4) and $\nu(\theta+\pi) = \nu(\theta)^{-1}$, we have the general symmetry relation that for any bounded measurable set $S \subset \mathbb{R}_{>0}$,

$$\int_{\{\theta\,:\,\nu(\theta)\in S\}} r(\theta)^{-2}\nu(\theta)^{-2}\,\mathrm{d}\theta = \int_{\{\theta\,:\,\nu(\theta)\in S^{-1}\}} r(\theta)^{-2}\,\mathrm{d}\theta. \tag{4.18}$$

Using (4.18) and the fact that $S_j^{-1} = S_{6-j}$ for $1 \leqslant j \leqslant 5$, we see that (4.17) is equivalent to the formula (1.24) in Proposition 1.1. Finally, if $\tilde{S}_3 = [0, 2\pi)$, then the sum inside the parenthesis in (1.24) equals $2(1+\sqrt{2})\int_0^{2\pi} r(\theta)^{-2}\,\mathrm{d}\theta = 4(1+\sqrt{2})\operatorname{Area}(\mathcal{W}^*)$ by (4.5), and hence (1.25) holds. This completes the proof of Proposition 1.1.

Next, we consider the case when $K = \mathbb{Q}(\sqrt{3})$. Then $\lambda = 2+\sqrt{3}$, and one verifies that

$$E^+_I = \{1, \lambda\}\cdot\lambda^{2\mathbb{Z}} = \lambda^{\mathbb{Z}} \quad \text{and} \quad E^-_I = \{\sqrt{3}, 1+\sqrt{3}, \sqrt{3}\lambda, (1+\sqrt{3})\lambda\}\cdot\lambda^{2\mathbb{Z}} = \{\sqrt{3}, 1+\sqrt{3}\}\lambda^{\mathbb{Z}}$$

with the associated values ν_α given by

$$\nu_1 = \nu_\lambda = 1+\sqrt{3}, \quad \nu_{\sqrt{3}} = \nu_{\sqrt{3}\lambda} = \sqrt{3} \quad \text{and} \quad \nu_{1+\sqrt{3}} = \nu_{(1+\sqrt{3})\lambda} = -1+\sqrt{3}.$$

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Hence

$$E^+_{I,\nu} = \begin{cases} \lambda^{\mathbb{Z}} & \text{if } \nu < 1+\sqrt{3}, \\ \emptyset & \text{if } \nu \geqslant 1+\sqrt{3}, \end{cases}$$

and

$$E^-_{I,\nu} = \begin{cases} \emptyset & \text{if } \nu \leqslant \sqrt{3}-1, \\ \{1+\sqrt{3}\}\lambda^{\mathbb{Z}} & \text{if } \sqrt{3}-1 < \nu \leqslant \sqrt{3}, \\ \{\sqrt{3}, 1+\sqrt{3}\}\lambda^{\mathbb{Z}} & \text{if } \sqrt{3} < \nu. \end{cases}$$

The corresponding partition of $\mathbb{R}_{>0}$ is $\mathbb{R}_{>0} = \sqcup_{j=1}^4 S_{I,j}$ where $S_{I,j} = S_j$ with $S_1 = (0, \sqrt{3}-1]$, $S_2 = (\sqrt{3}-1, \sqrt{3}]$, $S_3 = (\sqrt{3}, \sqrt{3}+1)$ and $S_4 = [\sqrt{3}+1, \infty)$. The corresponding constants $A_j := A_{I,j}$ and $B_j := B_{I,j}$ are

$$(A_1, B_1) = (6+4\sqrt{3}, 0), \qquad (A_2, B_2) = (3+2\sqrt{3}, 2\sqrt{3})$$

and

$$(A_3, B_3) = (2, 3+8\sqrt{3}), \qquad (A_4, B_4) = (0, 11+12\sqrt{3}).$$

Plugging these formulas into (4.1) and using also $\Delta_K = 12$ and $\zeta_K(2) = \frac{\pi^4}{36\sqrt{3}}$ (see Remark 2.12), we get

$$a_{\mathcal{P}} = \frac{\sqrt{3}}{16\pi^4}\operatorname{Area}(\mathcal{W})\Big((6+4\sqrt{3})\int_{\tilde{S}_1} r(\theta)^{-2}\,\mathrm{d}\theta + \int_{\tilde{S}_2} r(\theta)^{-2}\Big((3+2\sqrt{3}) + 2\sqrt{3}\nu(\theta)^{-2}\Big)\,\mathrm{d}\theta$$
$$+ \int_{\tilde{S}_3} r(\theta)^{-2}\Big(2 + (3+8\sqrt{3})\nu(\theta)^{-2}\Big)\,\mathrm{d}\theta + (11+12\sqrt{3})\int_{\tilde{S}_4} r(\theta)^{-2}\nu(\theta)^{-2}\,\mathrm{d}\theta\Big). \tag{4.19}$$

Again by the symmetry identity (4.18), we see that (4.19) is equivalent to the formula (1.28) in Proposition 1.2. The last statement of Proposition 1.2 is again verified using (4.5). This completes the proof of Proposition 1.2.

Finally we consider the case when $K = \mathbb{Q}(\sqrt{5})$. Then $\lambda = \frac{1+\sqrt{5}}{2}$, and one verifies that

$$E^+_I = \{1\}\cdot\lambda^{2\mathbb{Z}} = \lambda^{2\mathbb{Z}} \quad \text{and} \quad E^-_I = \{\lambda\}\cdot\lambda^{2\mathbb{Z}} = \lambda^{2\mathbb{Z}+1}$$

with the associated values ν_α given by

$$\nu_1 = \tfrac{1+\sqrt{5}}{2} \quad \text{and} \quad \nu_\lambda = \tfrac{\sqrt{5}-1}{2}.$$

Hence

$$E^+_{I,\nu} = \begin{cases} \lambda^{2\mathbb{Z}} & \text{if } \nu < \frac{1+\sqrt{5}}{2}, \\ \emptyset & \text{if } \nu \geqslant \frac{1+\sqrt{5}}{2}, \end{cases} \quad \text{and} \quad E^-_{I,\nu} = \begin{cases} \emptyset & \text{if } \nu \leqslant \frac{\sqrt{5}-1}{2}, \\ \lambda^{2\mathbb{Z}+1} & \text{if } \frac{\sqrt{5}-1}{2} < \nu. \end{cases}$$

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The corresponding partition of $\mathbb{R}_{>0}$ is $\mathbb{R}_{>0} = \sqcup_{j=1}^{3} S_{I,j}$ where $S_{I,j} = S_j$ with $S_1 = (0, \frac{\sqrt{5}-1}{2}]$, $S_2 = (\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{5}}{2})$ and $S_3 = [\frac{1+\sqrt{5}}{2}, \infty)$. The corresponding constants $A_j := A_{I,j}$ and $B_j := B_{I,j}$ are

$$(A_1, B_1) = (B_3, A_3) = (\tfrac{5+3\sqrt{5}}{4}, 0) \quad \text{and} \quad (A_2, B_2) = (\tfrac{1+\sqrt{5}}{4}, \tfrac{1+\sqrt{5}}{4}).$$

Plugging these formulas into (4.1) and using also $\Delta_K = 5$ and $\zeta_K(2) = \frac{2\pi^4}{75\sqrt{5}}$ (see Remark 2.12) and the symmetry identity (4.18), we obtain (1.31). The last statement of Proposition 1.3 is again verified using (4.5). This completes the proof of Proposition 1.3.

5 | DETAILS ON THE EXAMPLES AND NUMERICAL COMPUTATIONS

In Sections 5.1–5.3 below, we show how the formulas (1.23), (1.27) and (1.30) follow from the cut-and-project presentation of these vertex sets in [5, Chapter 7.3]. Then in Section 5.4, we discuss numerical computations of the corresponding gap distributions.

5.1 | Ammann–Beenker tilings

Recall the definition of $\mathcal{W}_{\mathrm{AB}} \subset \mathbb{R}^2$ in Section 1.5. Let $\xi_8 = e^{\pi i/4}$ (an eighth root of unity), and let $\star$ be the automorphism of the cyclotomic field $\mathbb{Q}(\xi_8)$ given by $\xi_8^\star = \xi_8^3$. From now on, we will use the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$, $z \leftrightarrow (\mathfrak{Re}(z), \mathfrak{Im}(z))$; in particular, we consider the ring $\mathbb{Z}[\xi_8]$ as a subset of $\mathbb{R}^2$. Set

$$\mathcal{T}_{\mathrm{AB}} := \{\boldsymbol{w} \in \mathcal{W}_{\mathrm{AB}} \ : \ \mathbb{Z}[\xi_8] \cap \partial(\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w}) = \emptyset\}; \tag{5.1}$$

this is the set of admissible 'generic' translates of $\mathcal{W}_{\mathrm{AB}}$. Next, for any $\boldsymbol{w} \in \mathbb{R}^2$, set

$$\mathcal{P}_{\mathrm{AB},\boldsymbol{w}} := \{z \in \mathbb{Z}[\xi_8] \ : \ z^\star \in \mathcal{W}_{\mathrm{AB}} + \boldsymbol{w}\} \qquad (\subset \mathbb{R}^2). \tag{5.2}$$

Then for every $\boldsymbol{w} \in \mathcal{T}_{\mathrm{AB}}$, $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ is the vertex set of an Ammann–Beenker tiling (see [7] and [5, Chapter 7.3; Example 7.8]); and we note that $\mathbf{0} \in \mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$. In fact *every* Ammann–Beenker tiling (appropriately scaled and rotated, and translated so that $\mathbf{0}$ is a vertex), has vertex set either equal to $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ for some $\boldsymbol{w} \in \mathcal{T}_{\mathrm{AB}}$, or equal to a *limit* of a sequence of such point sets, with respect to an appropriate metric on the family of locally finite point sets in $\mathbb{R}^2$. (In the limit case, it follows that there exists some $\boldsymbol{w} \in \overline{\mathcal{W}_{\mathrm{AB}}}$ such that the vertex set $\mathcal{P}$ agrees with $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ 'up to density zero', namely, the symmetric difference set $\mathcal{P} \triangle \mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ has asymptotic density zero in $\mathbb{R}^2$. This implies that $\mathcal{P}$ and $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ have the same limiting distribution of normalized gaps between directions, and therefore the methods of this paper apply also to these vertex sets, unless $\boldsymbol{w} \in \partial\mathcal{W}_{\mathrm{AB}}$.)

In order to rewrite (5.2) as in (1.23), let $K = \mathbb{Q}(\sqrt{2})$; then $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. Also set $g_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{smallmatrix}\right)$ and $g_2 := \left(\begin{smallmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{smallmatrix}\right)$. Using $\xi_8^2 = \sqrt{2}\xi_8 - 1$, one verifies that $\mathbb{Z}[\xi_8] = \mathcal{O}_K \oplus \mathcal{O}_K\xi_8$. Note that for any $x_1, x_2 \in \mathcal{O}_K$, we have $x_1 + x_2\xi_8 = (x_1, x_2)g_2$ under our identification of $\mathbb{C}$ with $\mathbb{R}^2$, and

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in particular, $\mathbb{Z}[\xi_8] = \mathcal{O}_K^2 g_2$ as a subset of $\mathbb{R}^2$. Furthermore, using $(\sqrt{2})^\star = (\xi_8 + \xi_8^{-1})^\star = \xi_8^3 + \xi_8^{-3} = -\sqrt{2}$, we have for any $(x_1, x_2) \in \mathcal{O}_K^2$:

$$(x_1 + x_2\xi_8)^\star = \sigma(x_1) + \sigma(x_2)\xi_8^3 = (\sigma(x_1), \sigma(x_2))g_1^{-1},$$

and this point lies in $\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w}$ if and only if $(\sigma(x_1), \sigma(x_2)) \in (\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w})g_1$. It follows that $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}} = \mathcal{P}((\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w})g_1, \mathcal{L}_K)g_2$, that is, the formula in (1.23) holds. (In the case $\boldsymbol{w} = \boldsymbol{0}$, the formula (1.23) was noted in [20, section 3.2].)

5.2 | Gähler's shield tilings

Recall the definition of $\mathcal{W}_{\mathrm{Gh}} \subset \mathbb{R}^2$ in Section 1.5. Let $\xi_{12} = e^{\pi i/6}$ (a 12th root of unity), and let $\star$ be the automorphism of the cyclotomic field $\mathbb{Q}(\xi_{12})$ given by $\xi_{12}^\star = \xi_{12}^5$. As before, we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way. For any $\boldsymbol{w} \in \mathbb{R}^2$, set

$$\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}} := \{z \in \mathbb{Z}[\xi_{12}] \ : \ z^\star \in \mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w}\} \qquad (\subset \mathbb{R}^2). \tag{5.3}$$

Then for every $\boldsymbol{w} \in \mathcal{W}_{\mathrm{Gh}}$ satisfying $\mathbb{Z}[\xi_{12}] \cap \partial(\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w}) = \emptyset$, $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ is the vertex set of a Gähler's shield tiling [5, Chapter 7.3; Example 7.12].

In order to rewrite (5.3) as in (1.27), let $K = \mathbb{Q}(\sqrt{3})$; then $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$. Also set $g_1 := \left(\begin{smallmatrix} 1 & 0 \\ \sqrt{3} & 2 \end{smallmatrix}\right)$ and $g_2 := \left(\begin{smallmatrix} 1 & 0 \\ \sqrt{3}/2 & 1/2 \end{smallmatrix}\right)$. Using $\xi_{12}^2 = \sqrt{3}\xi_{12} - 1$, one verifies that $\mathbb{Z}[\xi_{12}] = \mathcal{O}_K \oplus \mathcal{O}_K\xi_{12}$. Note that for any $x_1, x_2 \in \mathcal{O}_K$, we have $x_1 + x_2\xi_{12} = (x_1, x_2)g_2$ under our identification of $\mathbb{C}$ with $\mathbb{R}^2$, and in particular, $\mathbb{Z}[\xi_{12}] = \mathcal{O}_K^2 g_2$ as a subset of $\mathbb{R}^2$. Furthermore, using $(\sqrt{3})^\star = (\xi_{12} + \xi_{12}^{-1})^\star = \xi_{12}^5 + \xi_{12}^{-5} = -\sqrt{3}$, we have for any $(x_1, x_2) \in \mathcal{O}_K^2$:

$$(x_1 + x_2\xi_{12})^\star = \sigma(x_1) + \sigma(x_2)\xi_{12}^5,$$

and this point lies in $\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w}$ if and only if $(\sigma(x_1), \sigma(x_2)) \in (\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w})g_1$. It follows that $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}} = \mathcal{P}((\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w})g_1, \mathcal{L}_K)g_2$, that is, the formula in (1.27) holds.

5.3 | Tübingen triangle tilings

Recall the definition of $\mathcal{W}_{\mathrm{TT}} \subset \mathbb{R}^2$ in Section 1.5. Let $\xi_5 = e^{2\pi i/5}$ (a fifth root of unity), and let $\star$ be the automorphism of the cyclotomic field $\mathbb{Q}(\xi_5)$ given by $\xi_5^\star = \xi_5^2$. As in Section 5.1, we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way. For any $\boldsymbol{w} \in \mathbb{R}^2$, set

$$\mathcal{P}_{\mathrm{TT},\boldsymbol{w}} := \{z \in \mathbb{Z}[\xi_5] \ : \ z^\star \in \mathcal{W}_{\mathrm{TT}} + \boldsymbol{w}\} \qquad (\subset \mathbb{R}^2). \tag{5.4}$$

Then for every $\boldsymbol{w} \in \mathcal{W}_{\mathrm{TT}}$ satisfying $\frac{1}{5}\mathbb{Z}[\xi_5] \cap \partial(\mathcal{W}_{\mathrm{TT}} + \boldsymbol{w}) = \emptyset$, $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$ is the vertex set of a Tübingen triangle tiling [5, Chapter 7.3; Example 7.10], [6, section 4].

In order to rewrite (5.4) as in (1.30), let $K = \mathbb{Q}(\sqrt{5})$; then $\mathcal{O}_K = \mathbb{Z}[\tau]$, where we recall that $\tau = \frac{1}{2}(1 + \sqrt{5})$. Also set $g_1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{(2+\tau)/5} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ \tau & 2 \end{smallmatrix}\right)$ and $g_2 := \left(\begin{smallmatrix} 1 & 0 \\ (\tau-1)/2 & \sqrt{2+\tau}/2 \end{smallmatrix}\right)$. Using

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TABLE 1 Values of $a_{\mathcal{P}}$ and approximate values of $s^2F(s)$, for $\mathcal{P} = \mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$. The last line gives a conjectural approximate upper bound on the absolute error of each value in the corresponding column. (Thus, for example, we expect that for $\boldsymbol{w} = \mathbf{0}$ and $s = 50$, $|s^2F(s) - 0.2525| \lesssim 0.0011$.)

Experimental approximate values of $s^2F(s)$ for $\mathcal{P} = \mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$								
$\boldsymbol{w}$	$a_{\mathcal{P}}$	$s = 10$	$s = 20$	$s = 50$	$s = 100$	$s = 200$	$s = 500$	$s = 1000$
(0,0)	0.24638…	0.2810	0.2628	0.2525	0.2497	0.250	0.246	0.23
(0.8,0.1)	0.21809…	0.2460	0.2317	0.2236	0.2212	0.220	0.219	0.21
(0.9,0.3)	0.20416…	0.2286	0.2160	0.2090	0.2067	0.206	0.205	0.19
(1.2,0.5)	0.17444…	0.2211	0.2048	0.1815	0.1767	0.177	0.178	0.18
Error $\lesssim$		0.0005	0.0006	0.0011	0.0023	0.005	0.015	0.05

TABLE 2 Values of $a_{\mathcal{P}}$ and approximate values of $s^2F(s)$, for $\mathcal{P} = \mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$.

Experimental approximate values of $s^2F(s)$ for $\mathcal{P} = \mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$								
$\boldsymbol{w}$	$a_{\mathcal{P}}$	$s = 10$	$s = 20$	$s = 50$	$s = 100$	$s = 200$	$s = 500$	$s = 1000$
(1.7,0.6)	0.17790…	0.1974	0.1873	0.1817	0.180	0.180	0.176	0.19
(0.1,0)	0.21374…	0.2460	0.2289	0.2199	0.216	0.217	0.215	0.23
(1.5, −0.3)	0.19210…	0.2143	0.2027	0.1964	0.194	0.195	0.190	0.20
(0.3,1.2)	0.20870…	0.2341	0.2208	0.2136	0.211	0.211	0.205	0.23
Error $\lesssim$		0.0004	0.0007	0.001	0.004	0.006	0.019	0.05

$\xi_5^2 = (\tau - 1)\xi_5 - 1$, one verifies that $\mathbb{Z}[\xi_5] = \mathcal{O}_K \oplus \mathcal{O}_K\xi_5$. Note that for any $x_1, x_2 \in \mathcal{O}_K$, we have $x_1 + x_2\xi_5 = (x_1, x_2)g_2$ under our identification of $\mathbb{C}$ with $\mathbb{R}^2$, and in particular, $\mathbb{Z}[\xi_5] = \mathcal{O}_K^2 g_2$ as a subset of $\mathbb{R}^2$. Furthermore, using $(\sqrt{5})^\star = (1 + 2\xi_5 + 2\xi_5^4)^\star = 1 + 2\xi_5^2 + 2\xi_5^3 = -(1 + 2\xi_5 + 2\xi_5^4) = -\sqrt{5}$, we have for any $(x_1, x_2) \in \mathcal{O}_K^2$:

$$(x_1 + x_2\xi_5)^\star = \sigma(x_1) + \sigma(x_2)\xi_5^2,$$

and this point lies in $\mathcal{W}_{\mathrm{TT}} + \boldsymbol{w}$ if and only if $(\sigma(x_1), \sigma(x_2)) \in (\mathcal{W}_{\mathrm{TT}} + \boldsymbol{w})g_1$. It follows that $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}} = \mathcal{P}((\mathcal{W}_{\mathrm{TT}} + \boldsymbol{w})g_1, \mathcal{L}_K)g_2$, that is, the formula in (1.30) holds.

5.4 | Numerical computations

Next we discuss numerical computations of the gaps between directions of points in the Ammann–Beenker, Gähler's shield and Tübingen triangle tilings, $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$, $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ and $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$. In Table 1, for a few examples of points $\boldsymbol{w} \in \mathcal{W}_{\mathrm{AB}}$, we present conjectural approximate values of $s^2F(s)$, where $F(s)$ is the limiting gap distribution function associated to $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$, and, for comparison, the exact values of $a_{\mathcal{P}}$. Tables 2 and 3 present similar types of data for $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ and $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$.

The approximations of $F(s)$ in these tables were obtained by collecting all the points of $\mathcal{P}$ (where $\mathcal{P} = \mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ or $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$ or $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$) in fairly large discs $\mathcal{B}_R^2$ and evaluating the ratio in the left-hand side of Theorem 1; by repeating this for several large R-values, we also obtained conjectural error bounds; see below for a more detailed discussion. Recall that $\lim_{s\to\infty} s^2F(s) = a_{\mathcal{P}}$ by Theorem 2, in fact with a rate $O(s^{-\frac{1}{2}+\varepsilon})$. We certainly expect that the absolute difference $|s^2F(s) - a_{\mathcal{P}}|$ typically

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TABLE 3 Values of $a_{\mathcal{P}}$ and approximate values of $s^2F(s)$, for $\mathcal{P} = \mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$.

Experimental approximate values of $s^2F(s)$ for $\mathcal{P} = \mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$								
$\boldsymbol{w}$	$a_{\mathcal{P}}$	$s = 10$	$s = 20$	$s = 50$	$s = 100$	$s = 200$	$s = 500$	$s = 1000$
(0.5,0.4)	0.26135…	0.2919	0.2762	0.2673	0.2643	0.263	0.265	0.26
(0.4,0)	0.28875…	0.3283	0.3080	0.2965	0.2927	0.291	0.294	0.30
(0.15,0.3)	0.29103…	0.3329	0.3116	0.2995	0.2954	0.294	0.297	0.30
(1.3,0.4)	0.18732…	0.2195	0.1977	0.1907	0.1892	0.188	0.190	0.19
Error $\lesssim$		0.0004	0.0004	0.0007	0.0017	0.004	0.014	0.03

decreases as s runs through the values 10,20,50,100,200,500,1000, and it should be noted that the data in Tables 1–3 are consistent with this hypothesis, when considering also the conjectural error bounds displayed in the last line of the tables.

Precise numerical values of $a_{\mathcal{P}}$ such as those presented in Tables 1–3 (but also to much higher precision) are quite easy to compute using Propositions 1.1–1.3. To describe how to do this, first note that in the case $\mathcal{P} = \mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$, it follows from (1.23), the last statement in Theorem 1 and Remark 3.12, that this $a_{\mathcal{P}}$ is the same as for $\mathcal{P} = \mathcal{P}(\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w}, \mathcal{L}_K)$; similarly, for $\mathcal{P} = \mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$, $a_{\mathcal{P}}$ is the same as for $\mathcal{P} = \mathcal{P}(\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w}, \mathcal{L}_K)$ (see (1.27)), and for $\mathcal{P} = \mathcal{P}_{\mathrm{TT},\boldsymbol{w}}$, $a_{\mathcal{P}}$ is the same as for $\mathcal{P} = \mathcal{P}(\mathcal{W}_{\mathrm{TT}} + \boldsymbol{w}, \mathcal{L}_K)$ (see (1.30)). Now, somewhat more generally, assume $\mathcal{W} = \mathcal{W}_0 + \boldsymbol{w}$ where $\mathcal{W}_0$ is the open regular N-gon with vertices at the points $R_0 e^{2\pi i(\frac{1}{2N} + \frac{k}{N})}$ ($k \in \mathbb{Z}/N\mathbb{Z}$), for some integer $N \geqslant 3$ and some $R_0 > 0$, and $\boldsymbol{w} \in \mathcal{W}_0$. (We obtain $\mathcal{W}_0 = \mathcal{W}_{\mathrm{AB}}$ by taking $N = 8$ and $R_0 = \sqrt{1 + \sqrt{1/2}}$, $\mathcal{W}_0 = \mathcal{W}_{\mathrm{Gh}}$ by taking $N = 12$ and $R_0 = \sqrt{2 + \sqrt{3}}$, and $\mathcal{W}_0 = \mathcal{W}_{\mathrm{TT}}$ by taking $N = 10$ and $R_0 = \frac{\sqrt{2}}{20}(5 + \sqrt{5})^{3/2}$.) For such a window $\mathcal{W}$, the function $r(\theta)$ in (1.16) is given by

$$r(\theta) = \mathfrak{Re}\big(r_j e^{i(\theta_j + \theta)}\big) = r_j \cos(\theta_j + \theta), \qquad \forall \theta \in \big[(j-1)\tfrac{2\pi}{N}, j\tfrac{2\pi}{N}\big] + 2\pi\mathbb{Z},$$

where for each $j \in \mathbb{Z}/N\mathbb{Z}$, $r_j > 0$ and $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$ are defined by $\boldsymbol{w} + R_0 e^{2\pi i(\frac{1}{2N} - \frac{j}{N})} = r_j e^{i\theta_j}$. Using also $\nu(\theta) = r(\theta + \pi)/r(\theta)$, it is then easy to verify that all the sets $\tilde{S}_j$ in Propositions 1.1–1.3 are finite unions of intervals. For given $\mathcal{W}_0$ and $\boldsymbol{w}$, these intervals can be computed numerically, and with this, also the integrals in (1.24) and (1.28) can be computed. This is carried out in the program [21, explicit_aP.mpl].

We next describe more in detail how the experimental approximate values of $F(s)$ in Tables 1–3 were computed. (This is similar as in [4] and [20], but we have carried out more extensive computations.) With notation as in Theorem 1, set

$$F_R(s) := \frac{\#\{1 \leqslant j \leqslant N(R) \ : \ N(R)(\xi_{R,j} - \xi_{R,j-1}) \geqslant s\}}{N(R)}. \tag{5.5}$$

Then $F(s) = \lim_{R\to\infty} F_R(s)$, and so for R sufficiently large, $F_R(s)$ is a good approximation of $F(s)$. Note that computing $F_R(s)$ involves collecting all the points of $\mathcal{P}$ in the disc $\mathcal{B}_R^2$. When carrying out the computations, we noted that for fixed s, the values $F_R(s)$ typically *oscillate* as a function of R; hence if $\mathfrak{R}$ is a finite set of several sufficiently large R-values of the same order of magnitude, it seems reasonable to consider the difference

$$\Delta_{\mathfrak{R}} F(s) := \max_{R\in\mathfrak{R}} F_R(s) - \min_{R\in\mathfrak{R}} F_R(s)$$

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as an *indication* of the size of the error $|F_R(s) - F(s)|$ for any $R \in \mathfrak{R}$. To lessen the probability of getting a small difference by coincidence, we took the experimental error bound given in the last line of Tables 1–3 to be the *maximum* of $\tilde{s}^2 \Delta_{\mathfrak{R}} F(\tilde{s})$ over 100 values of $\tilde{s}$ roughly equispaced in the interval $[\frac{4}{5}s, \frac{6}{5}s]$, and over all four choices $\boldsymbol{w}$. Because of the factor $\tilde{s}^2$, it is not surprising that these experimental error bounds get worse as s increases.

In order to produce Table 1 (that is, the case of Ammann–Beenker tilings), we used

$$\mathfrak{R} = \{22000, 23000, 24000, 25000\}, \tag{5.6}$$

and for each s we chose as approximate value of $F(s)$ the mid-point $\frac{1}{2}\big(\min_{R\in\mathfrak{R}} F_R(s) + \max_{R\in\mathfrak{R}} F_R(s)\big)$. We remark that computing the functions F_R in this case involve collecting a bit over $2.3 \cdot 10^9$ points from each point set $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$. For example, for $R = 25000$ we found the number of points in $\mathcal{P}_{\mathrm{AB},\mathbf{0}} \cap \mathcal{B}_R^2 \setminus \{\mathbf{0}\}$ to be 2370148592 (this may be compared with $c_{\mathcal{P}}\pi R^2 = \frac{1+\sqrt{2}}{2}\pi R^2 = 2370148622.42\ldots$).

Similarly, for Table 2 (Gähler's shield tilings), we used

$$\mathfrak{R} = \{12000, 13000, 14000, 15000\}, \tag{5.7}$$

and computing the functions F_R involved collecting a bit over $2.6 \cdot 10^9$ points from each point set $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$. For example, for $R = 15000$ and $\boldsymbol{w} = (1.7, 0.6)$, we found the number of points in $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}} \cap \mathcal{B}_R^2 \setminus \{\mathbf{0}\}$ to be 2638031485 (which may be compared with $c_{\mathcal{P}}\pi R^2 = (2+\sqrt{3})\pi R^2 = 2638031264.97\ldots$).

Similarly, for Table 3 (Tübingen triangle tilings), we used

$$\mathfrak{R} = \{17000, 18000, 19000, 20000\}, \tag{5.8}$$

and computing the functions F_R involved collecting a bit over $2.5 \cdot 10^9$ points from each point set $\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$. For example, for $R = 20000$ and $\boldsymbol{w} = (0.4, 0)$, we found the number of points in $\mathcal{P}_{\mathrm{TT},\boldsymbol{w}} \cap \mathcal{B}_R^2 \setminus \{\mathbf{0}\}$ to be 2503118410 (which may be compared with $c_{\mathcal{P}}\pi R^2 = \frac{2}{5}(\tau+1)\sqrt{\tau+2} \cdot \pi R^2 = 2503118771.19\ldots$).

Let us next discuss the graphs in Figure 2. These were generated by distributing the normalized gaps $N(R)(\xi_{R,j} - \xi_{R,j-1})$, $j = 1, \ldots, N(R)$ — discarding the vanishing gaps — in bins of width 0.02, and then drawing the corresponding histogram, appropriately scaled. In other words, using the notation in (5.5), each panel displays the graph of the function

$$s \mapsto E_{R,h}(s) := \frac{\widetilde{F}_R(s_h) - F_R(s_h + h)}{h}, \qquad \text{with } s_h := h\lfloor s/h \rfloor \text{ and } h := 0.02, \tag{5.9}$$

where $\widetilde{F}_R(0) := F_R(0+) := \lim_{s\to 0+} F_R(s)$ while $\widetilde{F}_R(s) := F_R(s)$ for all $s > 0$.[†] We point out that the graphs of $E_{R,h}$, as they are scaled in Figure 2, look identical to the naked eye for each $R \in \mathfrak{R}$, that is, the picture looks the same if we plot all four graphs in the same coordinate system; in fact the maximum over $s \in [0, 6]$ of the difference $\max_{R\in\mathfrak{R}} E_{R,h}(s) - \min_{R\in\mathfrak{R}} E_{R,h}(s)$ was in each of the six cases found to be less than 0.0018, and the average of that same difference was less than 0.0004.

[†] The fact that we use $\widetilde{F}_R$ in place of F_R corresponds exactly to the fact that we discard those gaps $N(R)(\xi_{R,j} - \xi_{R,j-1})$ which vanish; without this modification, the graph would have a huge peak for $s \in [0, h)$; see the discussion below.

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These findings support the conjecture that, in the cases considered, the derivative $F'(s)$ exists, is continuous and is essentially correctly depicted by the graphs in Figure 2.

It is also of interest to consider the *areas under the curves* in Figure 2. Note that it follows from (5.9) that the area under the graph of $E_{R,h}(s)$ is

$$\int_0^\infty E_{R,h}(s)\,ds = F_R(0+).$$

Furthermore, as $R \to \infty$, $F_R(0+)$ tends to $\kappa_{\mathcal{P}}$, the relative density of visible points in $\mathcal{P}$, a quantity which was defined and proved to exist for general regular cut-and-project sets in [29]. (The proof of $\lim_{R\to\infty} F_R(0+) = \kappa_{\mathcal{P}}$ is immediate by comparing with the definition of $\kappa_{\mathcal{P}}$ in [29].) The values which we obtained for $F_R(0+)$ for varying R in $R \in \mathfrak{R}$ consistently agreed to within ca $3 \cdot 10^{-6}$; hence we expect that these values approximate $\kappa_{\mathcal{P}}$ to within an absolute error of similar order of magnitude. In particular for $\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$ with $\boldsymbol{w} = \mathbf{0}$ and $R = 25000$, we obtained $F_R(0+) = \frac{1368315872}{2370148592} = 0.57731227\ldots$.[†] Thus, this value is the area under the curve in panel I of Figure 2, and it may be compared with the known exact value [20, p. 10]

$$\kappa_{\mathcal{P}} = \frac{2|\sigma(\lambda)|}{\zeta_K(2)} = \frac{2(\sqrt{2}-1)}{\pi^4/(48\sqrt{2})} = 0.57731262\ldots. \tag{5.10}$$

For the other panels in Figure 2, we obtained the following values of $F_R(0+)$ (namely, area under the curve): II. 0.71023 .... III. 0.76447 .... IV. 0.62863 .... V. 0.60173 .... VI. 0.73515 ....

Finally, to avoid possible confusion, let us remind about the fact that, as explained in Remark 1.9, in this paper, we are using a different normalization of '$F(s)$' than what was used in [4, 29] and [20]. This has the effect that the graph in the top left panel of Figure 2 is quantitatively strongly different from the graphs in [4, fig. 9] and [20, fig. 2], despite the fact that each of these three graphs depict the gap distribution for directions in the point set $\mathcal{P} = \mathcal{P}_{\mathrm{AB},\mathbf{0}}$. In fact, in our notation, the plots in [4, fig. 9] and [20, fig. 2] are experimental graphs of the function $s \mapsto -\kappa_{\mathcal{P}}^{-2} F'(\kappa_{\mathcal{P}}^{-1} s)$, with $\kappa_{\mathcal{P}}$ as in (5.10). Taking this rescaling into account, the three graphs agree fairly well — one naturally expects that the more erratic small-scale behavior of the graphs in [4] and [20] is due to the fact that those were calculated using considerably smaller R-values than those used for our Figure 2.

INDEX OF NOTATION

$a_{\mathcal{P}}$	leading coefficient of the limiting gap distribution function $F(s)$ of a planar point set $\mathcal{P}$
a_y	matrix element $(\mathrm{a}_{y_1}, \mathrm{a}_{y_2})$ with $\mathrm{a}_y = \left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)$
$\mathcal{A}$	closure of $\pi_{\mathrm{int}}(\mathcal{L})$ with $\mathcal{L}$ a lattice in $\mathbb{R}^{d+m}$ and π_{int} the natural projection to the internal space
$\mathcal{A}^\circ$	identity component of $\mathcal{A}$
$\mathfrak{A}_{\theta,\boldsymbol{y}}(s)$	the set $(y_1^{-1} s^{1/2} \lambda^{-2r} \ell(\theta_1)) \times (y_2^{-1} \lambda^{2r} \ell_{\mathcal{W}}(\theta_2))$
$\alpha_{I,J}(y)$	the function $\min\{\alpha \in I \cap \mathbb{R}_{>0} \ : \ y\,\sigma(\alpha) \in J\}$ with $I \subset \mathcal{O}_K$ an integral ideal

(Continues)

[†] Similar numerics, but for considerably smaller R-values, also appears in [4, table 2] and [20, table 1].

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	and $J \subset \mathbb{R}$ bounded measurable with non-empty interior
$\tilde{\alpha}_{I,J}(y)$	the function $\max\{\alpha \in I \cap \mathbb{R}_{<0} \ : \ y\,\sigma(\alpha) \in J\}$ with I, J as above
$b_{\alpha,\nu}$	$b_{\alpha,\nu} := \sigma(\alpha)$ if $\sigma(\alpha) > 0$ and $b_{\alpha,\nu} := \nu^{-1}\sigma(\alpha)$ if $\sigma(\alpha) < 0$
$\mathcal{B}_R^2$	open disc in $\mathbb{R}^2$ with center $\mathbf{0}$ and radius R
$\mathfrak{B}_{\theta,\boldsymbol{y},\boldsymbol{x},\alpha}(s)$	the set $(y_1\mathfrak{F}_1 - \alpha x_1) \times (y_2\mathfrak{F}_2 - \sigma(\alpha)x_2)$
$\mathfrak{c}_i$	the vector $(-c_i, a_i) \in \mathcal{O}_K^2$ such that $\frac{a_i}{c_i} = k_i$ is the i-cusp of Γ_K
C_K	ideal class group of K
c_K	normalizing factor of a Haar measure of H which equals $\Delta_K^{-3/2}\zeta_K(2)^{-1}$
$c_{\mathcal{P}}$	asymptotic density of a planar point set $\mathcal{P}$
Δ_K	discriminant of K
Δ_R	directions of points of length bounded by R in a planar point set $\mathcal{P}$
$E_I^{\pm}$	set of positively or negatively extremal points with respect to an integral ideal I
$E_{I,\nu}$	set of ν-extremal points with respect to an integral ideal I
$E_{I,\nu}^{\pm}$	set of positive or negative ν-extremal points with respect to I
$F(s)$	limiting distribution function of gaps of directions of a planar cut-and-project set
$\mathfrak{f}(A,a)$	the function $a^{-1}\inf\{m((x,x+a)\setminus A) \ : \ x \in \mathbb{R}\}$ with $a > 0$ and $A \subset \mathbb{R}$ Lebesgue measurable
$\widetilde{\mathfrak{f}}(A,a)$	either 0 or $\mathfrak{f}(A,a)$ depending on whether A is empty or not
$\mathcal{F}_{\Gamma_K}$	a Siegel fundamental domain for $\Gamma_K\backslash\mathrm{H}$
$\mathcal{F}_i$	a fixed fundamental domain for $\Gamma_i\backslash H$
$\mathcal{F}_i(t)$	the set $\{\xi_i \mathrm{n_x a_y k}_\theta \in \mathcal{F}_i \ : \ y_1y_2 \geq t\}$
$\mathfrak{F}_i$	a fixed fundamental domain for $\mathbb{R}^2/\iota(I_i^{-2})$
$G(s)$	integral of the limiting gap distribution function $F(s)$
$G_i(s)$	contribution of the ith cusp of Γ_K to the function $G(s)$ for sufficiently large s
$G_{E,i}(s)$	error term of $G_i(s)$
$G_{M,i}(s)$	main term of $G_i(s)$
Γ	the group $\mathrm{SL}_4(\mathbb{Z})$
Γ_i	isotropy group of the ith cusp of Γ_K
Γ_K	the group $\iota(\mathrm{SL}_2(\mathcal{O}_K))$
H	the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
H_g	the group $g\mathrm{H}g^{-1}$ with $g \in \mathrm{SL}_4(\mathbb{R})$
I_i	ideal generated by $(a_i, c_i) \in \mathcal{O}_K^2$ with a_i, c_i such that $k_i = \frac{a_i}{c_i}$
ι	the natural embedding either from K to $\mathbb{R}^2$ or from $\mathrm{SL}_2(K)$ to H
J	the set $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$
J_ν	open interval $(-\nu, 1)$ with $\nu > 0$
$\mathfrak{F}_1$	the set $\{t \in \mathbb{R} \ : \ (y_1\alpha, t) \in s^{1/2}\lambda^{-2r}T(1)\mathrm{k}_{-\theta_1}\}$
$\mathfrak{F}_2$	the set $\{t \in \mathbb{R} \ : \ (y_2\sigma(\alpha), t) \in \lambda^{2r}\mathcal{W}\mathrm{k}_{-\theta_2}\}$
K	real quadratic field $\mathbb{Q}(\sqrt{d})$ with $d \geqslant 2$ square-free
$\overline{K}$	the set $K \cup \{\infty\}$
k_i	ith cusp of Γ_K
k_θ	matrix element $(\mathrm{k}_{\theta_1}, \mathrm{k}_{\theta_2})$ with $\mathrm{k}_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$

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κ	number of cusps of Γ_K
$\kappa_{\mathcal{P}}$	relative density of visible points in a planar point set $\mathcal{P}$
μ_K	the probability H-invariant measure on $\Gamma_K\backslash\mathrm{H}$
$\mathcal{L}_i$	the set $\Pi_i \cap \mathcal{L}_K$
$\widetilde{\mathcal{L}}_i$	the set $\{(\alpha, t_1, \sigma(\alpha), t_2) \ : \ \alpha \in I_i,\ t_1, t_2 \in \mathbb{R}\}\,\xi_i^{-1}$
$\mathcal{L}_K$	Minkowski embedding of $\mathcal{O}_K^2$ in $\mathbb{R}^4$
$\ell(\theta)$	projection of $T(1)\mathrm{k}_{-\theta}$ on the x-axis with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$
$\ell_{\mathcal{W}}(\theta)$	projection of $\mathcal{W}\mathrm{k}_{-\theta}$ on the x-axis
λ	fundamental unit of K
m	Lebesgue measure on $\mathbb{R}$
N	standard norm on K defined by $\mathrm{N}(\alpha) = \alpha\sigma(\alpha)$
Nr	absolute norm of ideals
$\mathrm{n}_{\mathbf{x}}$	matrix element $(\mathrm{n}_{x_1}, \mathrm{n}_{x_2})$ with $\mathrm{n}_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$
$\mathcal{O}_K$	ring of integers of K
$\mathcal{O}_K^\times$	unit group of $\mathcal{O}_K$
ν_α	critical value where the ν-extremality of $\alpha \in E_I^+ \sqcup E_I^-$ changes
$\nu(\theta)$	parameter in the expression $\ell_{\mathcal{W}}(\theta) = r(\theta)(-\nu(\theta), 1)$
$\mathcal{P}_{\mathrm{AB},\boldsymbol{w}}$	the Ammann–Beenker tiling corresponding to the translate $\boldsymbol{w}$
$\mathcal{P}_{\mathrm{Gh},\boldsymbol{w}}$	the Gähler's shield tiling corresponding to the translate $\boldsymbol{w}$
$\mathcal{P}(\mathcal{W}, \mathcal{L})$	cut-and-project set associated to a window set $\mathcal{W}$ and a lattice $\mathcal{L}$
π	natural projection from $\mathbb{R}^{d+m}$ to the physical space $\mathbb{R}^d$
π_{int}	natural projection from $\mathbb{R}^{d+m}$ to the internal space $\mathbb{R}^m$
Π_i	the plane $(\{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R})\xi_i^{-1}$
$R_{\alpha,\nu}$	a rectangle depending on $\alpha \in I \cap \mathbb{R}_{>0}$ and $\nu > 0$ with I an integral ideal
$R_{\mathcal{W}}(\theta, x)$	the set $\{y \in \mathbb{R} \ : \ (x, y)\mathrm{k}_\theta \in \mathcal{W}\}$
$r(\theta)$	parameter in the expression $\ell_{\mathcal{W}}(\theta) = r(\theta)(-\nu(\theta), 1)$
S_1^1	the unit circle
$S_{I_i,j}$	the j-th interval in the partition of $\mathbb{R}_{>0}$ corresponding to the i-th cusp of Γ_K
$\tilde{S}_{I_i,j}$	the set $\left\{\theta \in [0, 2\pi) \ : \ \nu(\theta) \in S_{I_i,j}\right\}$
$\mathrm{SL}_2(\mathcal{O}_K)$	Hilbert modular group
σ	the unique nontrivial automorphism of the real quadratic field K
t_1	an absolute positive number depending only on K
τ	a generator of $\mathcal{O}_K$ which is $\frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \pmod 4$ and $\sqrt{d}$ if $d \equiv 2, 3 \pmod 4$
$T(s)$	open triangle with vertices at (0,0) and $(s/c_{\mathcal{P}})^{1/2}(1, \pm 1)$
$\mathcal{T}(s)$	the set $\lambda^{-2r}T(s) \times \lambda^{2r}\mathcal{W}$ with $r \in \mathbb{N}$ such that $\lambda^{2r} \leqslant s^{1/4} < \lambda^{2(r+1)}$
$\mathcal{T}'(s)$	the set $T(s) \times \mathcal{W}$
$\mathcal{W}$	window set of the cut-and-project set $\mathcal{P}(\mathcal{W}, \mathcal{L})$
$\mathcal{W}_{\mathrm{AB}}$	an open regular octagon with particular size and position
$\mathcal{W}_{\mathrm{Gh}}$	an open regular dodecagon with particular size and position
$\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{AB})}$	the set $(\mathcal{W}_{\mathrm{AB}} + \boldsymbol{w})\left(\begin{smallmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{smallmatrix}\right)$
$\mathcal{W}_{\boldsymbol{w}}^{(\mathrm{Gh})}$	the set $(\mathcal{W}_{\mathrm{Gh}} + \boldsymbol{w})\left(\begin{smallmatrix} 1 & 0 \\ \sqrt{3} & 2 \end{smallmatrix}\right)$
$\mathcal{W}^*$	polar set of $\mathcal{W}$

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ξ_i	scaling matrix with respect to the ith cusp of Γ_K
Y	the set $\{(y_1, y_2) \in (\mathbb{R}_{>0})^2 \ : \ y_2 \in [1, \lambda^2)\}$
Y_{t_1}	the set $\{(y_1, y_2) \in (\mathbb{R}_{>0})^2 \ : \ 1 \leqslant y_1/y_2 < \lambda^2, \ y_1 y_2 > t_1\}$
Z	the set $(0, \pi) \times (0, 2\pi)$
Z_J	the set $J \times (0, 2\pi)$
ζ_K	Dedekind zeta function corresponding to K

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