



Existence of Strong Solutions for a Perfect Elastic Beam Interacting with Navier–Stokes Equations

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Abstract: A perfectly elastic beam is situated on top of a two dimensional fluid canister. The beam is deforming in accordance to an interaction with a Navier–Stokes fluid. Hence a hyperbolic equation is coupled to the Navier–Stokes equation. The coupling is partially of geometric nature, as the geometry of the fluid domain is changing in accordance to the motion of the beam. Here the existence of a unique strong solution for large initial data and all times up to geometric degeneracy is shown. For that an a-priori estimate on the time-derivative of the coupled solution is introduced. For the Navier–Stokes part it is a critical estimate in the spirit of Ladyzhenskaya applied directly to the in-time differentiated system.

1. Introduction

When a viscous fluid is interacting with a perfectly elastic solid, this yields a dissipative system. The dissipation of the system is however reduced to the fluid, while the solid evolution alone would be hyperbolic. It results in a coupling between a dissipative and non-dissipative partial differential equation, where the coupling is inherently non-linear. To determine whether the dissipation (or the irreversibility) of a system of PDE's suffices to produce regular solutions is an important and much studied problem in continuum mechanics. Indeed, many prominent open questions connect to this issue. Accordingly, in the field of fluid–structure interactions it is essential to understand to what extent the fluid dissipation is damping the solid motion [16, 33]. In the physical scenario considered here, where the solid deformation is actually changing the Eulerian domain of definition of the fluid, the system becomes immanently non-linear. Hence, as the evolution of perfectly elastic solids is hyperbolic, the fluid dissipation becomes crucial to show any regularity beyond energy bounds.

In this paper we demonstrate that in some cases the non-linear coupling of a perfect elastic solid with a viscous fluid allows for strong solutions. Indeed, here the existence of a smooth solution for large times and large data is shown for an elastic beam interacting

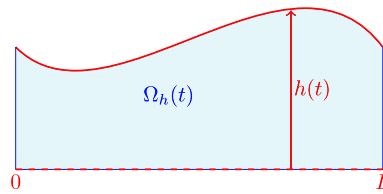


Fig. 1. 1D beam interacting with a 2D fluid

with the Navier–Stokes equation. It is probably the simplest case of such a non-linearly coupled fluid–solid interaction where the solid possesses *no dissipation*. As we will explain below, it seems that the proposed strategy is quite general and has the potential to be applied to other more complex situations.

We consider the interaction between a perfectly elastic beam and a two-dimensional fluid governed by the incompressible Navier–Stokes equation. See Fig. 1 for the typical setting and (1.1)–(1.7) for the coupled system of partial differential equations.

For this setting, we show the existence of a strong solution for arbitrary large times up to the point where the fluid domain faces a topological change and with natural assumptions on the initial data, see Theorem 1.1. In particular the present work extends the available theory on strong solutions with vertical displacement of the plate in the following ways:

- In the seminal paper [28] strong solutions for visco-elastic solids and arbitrary times are constructed with no contact in finite time. However, the regularity theory there [28, Section 4.3] is restricted to the case of viscous solids. In the present paper regularity estimates are presented that include purely elastic solids.
- The here presented results show the short and long time-existence of the missing case in [29], namely the case of an *elastic beam*. Interestingly, while the existence for short times for solids governed by the wave equation is shown there, the case of an elastic beam seems to require different methods.
- Further, it extends and improves the existence of strong solutions for short times shown in [4, 5] for the *elastic beam equation* to large time and to less regular conditions on the (large) initial data.

Further the work shows that the *global weak solution* for an elastic beam interacting with the Navier–Stokes equation that was constructed in [17] is a *strong solution* up to its first contact.

The *method proposed in this work* is independent of the previous works in this setting. We wish to mention that the central regularity estimate presented here is of *critical type* for the fluid equation, due to the scaling of the two dimensional Navier–Stokes equation. However, with regard to the elastic structure the estimates allow to speculate for further generalizations.

For an overview on previous efforts in fluid–structure interactions please see for instance [16, 33]. For the here considered coupling between a hyperbolic solid equation with a viscous fluid equation, many results showing the existence for *strong solutions* involving fixed geometries are available [2, 3, 6, 7, 35, 47]. Further results, including *variable geometries* for short times and/or small data can be found here [4, 5, 8, 10, 18, 20, 21, 29, 31, 32, 38]. Previous results for *variable geometries* on long-time and large data for hyperbolic solids are reduced to the frame-work of weak-solutions. This includes three-dimensional fluids [27, 36, 40], non-linear shells [42], tangential deformations [34],

global solutions [17] and other types of fluids [13, 14] for instance. Finally we wish to mention that there are many results when the solid is considered to be a rigid body inside the Navier–Stokes fluid, see [22–26, 30, 41, 43, 45, 46] and the references therein.

If the solid is assumed to be *viscous* more results are available. We focus here on the question of regularity. For a beam that is elastic and viscous interacting with the 2D Navier–Stokes equations, global smooth solution exist for arbitrary times [28]. Smoothness was shown recently for a viscous elastic shell interacting with Navier–Stokes equations once the solution overcomes some regularity threshold [12]. Further we mention some results on the existence of weak solutions for the full physical setting of a deforming visco-elastic solid inside a fluid with the same dimension [9].

The major technical improvement in this paper is an a-priori estimate on the spatial gradient of the time-derivative. This is achieved by analyzing the “in-time-differentiated system”. Only in this way the hyperbolic structure equation is conserved. However, differentiating the coupled system creates multiple nonlinear terms with critical order. We succeeded to estimate all of them in such a way that a Grönwall argument could be applied. It is worth noticing that criticality is already true for the 2D Navier–Stokes equations without interaction. Indeed, the existence proof of Ladyzhenskaya for a strong solution depended on a sharp critical estimate in two dimensions. This estimate known as “Ladyzhenskaya estimate” is an interpolation, which we use in this paper at many instances. What was not clear to us at the beginning of our investigations, is that an estimate on the time-derivative of the solution (combined with its energy estimate) does directly allow to show the existence of strong solutions; once the initial data is smooth enough to perform this estimate. This seems to be a different path than the one originally developed and later adapted to fluid–structure interactions [11, 28]. The approach introduced here to fluid–structure interactions with time-changing geometry using “in-time-differentiation” seems to allow for further generalizations of the solid equation. Indeed, the here presented estimates are not of critical type with regard of the solid impact on the fluid. This has a clear technical reason as it circumvents the well known fact that the second-order time-derivative of a solution is not a good test-function for hyperbolic equations.

1.1. Statement of the problem. We consider a two-dimensional canister filled with a viscous incompressible fluid and its top surface is formed of an elastic beam. As usual, we assume that the beam only deforms in the vertical direction on the surface. The fluid domain, denoted by Ω_h , is defined as

$$\Omega_h = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in (0, L), y \in (0, h(t, x)) \right\},$$

where $h(t, x)$ is the height of the fluid column. Let the fluid depth be 1 when the system is at the equilibrium state. We introduce the displacement of the beam $\eta(t, x)$, thereby the fluid column is $h(t, x) = 1 + \eta(t, x)$. The fluid is supposed to be homogeneous with density ρ_f and constant viscosity μ . The velocity and the pressure in the fluid are denoted by $u(t, x)$ and $p(t, x)$, respectively. With the above notation, the viscous incompressible Navier–Stokes equations in Ω_h , for all $t \in (0, T)$, read

$$\begin{cases} \rho_f (\partial_t u + u \cdot \nabla u) - \operatorname{div} \sigma(u, p) = 0 & \text{for all } (x, y) \in \Omega_h, \\ \operatorname{div} u = 0 & \text{for all } (x, y) \in \Omega_h. \end{cases} \tag{1.1}$$

The Cauchy stress tensor $\sigma(u, p)$ is defined by

$$\sigma(u, p) = 2\mu D(u) - p\mathbb{I}_{2 \times 2},$$

where $\mathbb{I}_{2 \times 2}$ is identity matrix of order 2 and $D(u)$ is the deformation tensor:

$$D(u) = \frac{1}{2} (\nabla u + (\nabla u)^\top).$$

Let the density of the beam be ρ_s (constant), the linearly elastic beam equation is

$$\rho_s \partial_t^2 h - \beta \partial_x^2 h + \alpha \partial_x^4 h = \phi(u, p, h) \quad \text{for all } (t, x) \in (0, T) \times (0, L), \tag{1.2}$$

where β and α are positive coefficients describing the properties of the beam. For more information on the derivation of this equation see [19, 40]. The kinematic condition means that the velocity of the fluid on the surface is consistent with the motion of the beam, i.e.

$$u(t, x, h(t, x)) = \partial_t h \mathbf{e}_2 \quad \text{for all } (t, x) \in (0, T) \times (0, L). \tag{1.3}$$

The dynamic condition is to balance the forces on the surface:

$$\phi(u, p, h)(t, x) = -\mathbf{e}_2 \cdot \sigma(u, p)(t, x, h(t, x))(-\partial_x h \mathbf{e}_1 + \mathbf{e}_2). \tag{1.4}$$

In the above expressions, $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis in \mathbb{R}^2 . To make the estimates easier to follow, we assume that the fluid and the structure are L -periodic in x -direction, i.e. for every $(t, x, y) \in (0, T) \times \Omega_h$

$$u(t, x, y) = u(t, x + L, y), \quad h(t, x) = h(t, x + L), \quad (\partial_x h)(t, x) = (\partial_x h)(t, x + L). \tag{1.5}$$

Moreover, we consider no-slip boundary condition on the bottom of the fluid domain:

$$u(t, x, 0) = 0 \quad \text{for all } (t, x) \in (0, T) \times (0, L). \tag{1.6}$$

To close the system, we propose the initial data as follows:

$$u(0, x, y) = u_0(x, y), \quad h(0, x) = h_0(x), \quad (\partial_t h)(0, x) = h_1(x) \quad \text{for all } (x, y) \in \Omega_{h_0}. \tag{1.7}$$

Further, we assume that the initial data (u_0, h_0, h_1) satisfy the following compatibility conditions:

$$\begin{aligned} u_0(x, 0) = 0, \quad u_0(x, h_0(x)) &= h_1(x) \mathbf{e}_2 && \text{for all } x \in (0, L), \\ \operatorname{div} u_0 = 0 &&& \text{for all } (x, y) \in \Omega_{h_0}, \\ \min_{x \in (0, L)} h_0(x) > \delta > 0, \quad \int_0^L h_1(x) dx &= 0, \end{aligned} \tag{1.8}$$

which allow a strong solution in accordance to the compatibility conditions derived in Sect. 2.2 below.

1.2. Main results. The main result of the paper is an estimate that implies the existence of a strong solution for smooth but arbitrary large data up to the point of collision between the beam and the bottom of the fluid.

Theorem 1.1. *Let (u_0, h_0, h_1) satisfy (1.8) and $h_0 \in H^4(0, L)$, $h_1 \in H^2(0, L)$, $u_0 \in H^2(\Omega_{h_0})$, there exists unique strong solution to the system (1.1)–(1.7) that satisfies the following a-priori estimate:*

$$\begin{aligned} & \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \|\nabla \partial_t u\|_{L^2(0,T;L^2(\Omega_h))}^2 \\ & + \|\partial_t^2 h\|_{L^\infty(0,T;L^2(0,L))}^2 + \|\partial_t h\|_{L^\infty(0,T;H^2(0,L))}^2 \\ & + \|u\|_{L^\infty(0,T;H^2(\Omega_h))}^2 + \|u\|_{L^4(0,T;H^{\frac{5}{2}}(\Omega_h))}^2 + \|p\|_{L^\infty(0,T;H^1(\Omega_h))}^2 \\ & + \|p\|_{L^4(0,T;H^{\frac{3}{2}}(\Omega_h))}^2 + \|h\|_{L^\infty(0,T;H^4(0,L))}^2 \\ & \leq C \left(h_{\min}, \|u_0\|_{L^2(\Omega_{h_0})}, \|h_0\|_{L^2(0,L)}, \|h_1\|_{L^2(0,L)} \right) \\ & \quad \left(\|u_0\|_{H^2(\Omega_{h_0})}^2 + \|h_0\|_{H^4(0,L)}^2 + \|h_1\|_{H^2(0,L)}^2 \right), \end{aligned}$$

as long as $h(t, x) \geq h_{\min}$ for all $(t, x) \in (0, T) \times (0, L)$.

The theorem above shows in particular that for any $h_0 > \delta$ satisfying the regularity assumptions above, there exists a minimal time interval $[0, T_0]$ for which a strong solution is guaranteed; this is demonstrated in Remark 2.3 below. The estimate *conserves* all quantities appearing on the left-hand-side and *improves* the regularity of the fluid due to its viscous term. Hence, the estimate holds up to the point of a collision between the beam and the bottom of the fluid canister of dimension 2. We follow here the convention that a strong solution means that all quantities in the above PDEs are valid almost everywhere; actually all quantities are at least in $L^\infty(L^2)$.

We wish to point out that the theory in [28] implies that *viscous beams* can not touch the bottom of the fluid-domain. This is not known for the hyperbolic problem and does not follow from the smoothness shown in this paper directly. It therefore remains a deep and difficult question, whether contact is possible or not, when a hyperbolic beam is considered.

1.3. Organization of the paper. The paper is organized as follows. In Sect. 2 we introduce some notation and function spaces and preliminary analysis used later in the work. This is followed by the main part of the paper, i.e. Sect. 3, which is the formal a-priori estimate using the time-derivative of the coupled system transferred to a fixed geometry. This rather involving estimate follows a technical Sect. 4, where a strong solution *satisfying the formal a-priori estimate* is constructed using a Galerkin approximation for which the formal a-priori estimate is valid rigorously. Due to the geometrically coupled setting this construction has to be done with particular care, which itself could well be of independent interest.

2. Preparation and Preliminaries

2.1. Notation. We introduce some notation in this part which will be used throughout this paper. For two non-negative quantities f and g , we write $f \lesssim g$ if there is a $c > 0$ such that $f \leq cg$. If necessary, we specify particular dependencies.

We consider function spaces that are periodic in the first spacial variable x . For an open set $\mathcal{O} \subset \mathbb{R}^2$ we denote by $L^p(\mathcal{O})$ and $W^{k,p}(\mathcal{O})$ for $p \in [1, \infty]$ and $k \in \mathbb{N}$, the usual Lebesgue and Sobolev spaces over \mathcal{O} . For $p \in [1, \infty)$, the fractional Sobolev space (Sobolev-Slobodeckij space) with differentiability $s > 0$ with $s \notin \mathbb{N}$ will be denoted by $W^{s,p}(\mathcal{O})$. As usual, we use the notation $H^s(\mathcal{O}) := W^{s,2}(\mathcal{O})$ for the case $p = 2$. The function spaces of continuous or α -Hölder-continuous functions, $\alpha \in (0, 1)$, are denoted by $C(\overline{\mathcal{O}})$ or $C^{0,\alpha}(\overline{\mathcal{O}})$ respectively, where $\overline{\mathcal{O}}$ is the closure of \mathcal{O} . Similarly, we write $C^1(\overline{\mathcal{O}})$ and $C^{1,\alpha}(\overline{\mathcal{O}})$.

We denote $\mathbf{y} = (x, y)$ for the space variable in the time changing domain Ω_h and $\mathbf{z} = (x, z)$ for the space variable in $\Omega_1 = [0, L] \times [0, 1]$, respectively. For a Banach space X , we use the shorthand $L_t^p X$ for $L^p(I; X)$ for $s \in \mathbb{R}$. For instance, we write $L_t^p(W^{1,p})$ for $L^p(I; W^{1,p}(\mathcal{O}))$. Similarly, $W_t^{s,p}(X)$ stands for $W^{s,p}(I; X)$. We will use the shorthand notations L_y^p (or L_z^p) and $W_y^{s,p}$ (or $W_z^{s,p}$) in the case of 2-dimensional domains (typically spaces defined over $\Omega_h \subset \mathbb{R}^2$ or $\Omega_1 \subset \mathbb{R}^2$). Finally, for vector-valued function f , we use f^1 and f^2 represent its first and second component, respectively. Hence we denote $f = [f^1 \ f^2]^\top$.

2.2. Compatibility conditions. Note that by using the divergence-free condition of the fluid-velocity and the boundary conditions (1.3) and (1.6), we derive that

$$\begin{aligned} \partial_t h(t, x) &= \partial_y \int_0^{h(t,x)} u^2(t, x, y) dy = - \int_0^{h(t,x)} \partial_x u^1(t, x, y) dy \\ &= -\partial_x \left(\int_0^{h(t,x)} u^1(t, x, y) dy \right) + \underbrace{\partial_x h(t, x) u^1(t, x, h(t, x))}_{=0}. \end{aligned}$$

According to the x -periodic setting (1.5), we thereby obtain that

$$\int_0^L \partial_t h(t, x) dx = 0 \quad \text{for all } t \in (0, T).$$

This together with the beam equation (1.2) further implies that

$$\int_0^L \phi(u, p, h)(t, x) dx = 0 \quad \text{for all } t \in (0, T). \tag{2.1}$$

2.3. Change of variables. We notice that, via a change of variables, the fluid-beam system can be rewritten into a fixed spatial domain. For every function f , we set

$$\hat{f}(t, x, z) = f(t, x, h(t, x)z). \tag{2.2}$$

The system (1.1)–(1.7) can be transformed in the domain Ω_1 :

$$\Omega_1 = \left\{ (x, z) \in \mathbb{R}^2 \mid x \in (0, L), z \in (0, 1) \right\}.$$

Using the relation (2.2) and the equations (1.1)–(1.2), it is not difficult to derive that the new system for (\hat{u}, \hat{p}, h) defined in Ω_1 satisfies

$$\rho_f h \partial_t \hat{u} + \rho_f (\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \hat{u} - \mu \operatorname{div} \left((A_h \nabla) \hat{u} \right) + (B_h \nabla) \hat{p} = 0, \tag{2.3a}$$

$$\operatorname{div} (B_h^\top \hat{u}) = 0, \quad (2.3b)$$

$$\rho_s \partial_t^2 h - \beta \partial_x^2 h + \alpha \partial_x^4 h = \hat{\phi}(\hat{u}, \hat{p}, h)(t, x), \quad (2.3c)$$

with the boundary conditions

$$\hat{u}(t, x, 1) = \partial_t h \mathbf{e}_2 \quad \text{for all } (t, x) \in (0, T) \times (0, L), \quad (2.4)$$

and the periodic conditions, for every $(t, x, z) \in (0, T) \times \Omega_1$,

$$\hat{u}(t, x, z) = \hat{u}(t, x + L, z), \quad h(t, x) = h(t, x + L). \quad (2.5)$$

To derive the structure equation, by using the condition $u^1(t, x, h(t, x)) = 0$ and $\operatorname{div} u = 0$, we note that

$$(\nabla u)^\top(t, x, h(t, x))(-\partial_x h \mathbf{e}_1 + \mathbf{e}_2) \cdot \mathbf{e}_2 = 0.$$

This implies that the force term $\phi(u, p, h)$ can be simplified as

$$\phi(u, p, h)(t, x) = p(t, x, h(t, x)) - \mu \mathbf{e}_2 \cdot \nabla u(t, x, h(t, x))(-\partial_x h \mathbf{e}_1 + \mathbf{e}_2).$$

Hence, after change of variables the force term $\hat{\phi}(\hat{u}, \hat{p}, h)$ in (2.3c), for every $(t, x) \in (0, T) \times (0, L)$, reads

$$\hat{\phi}(\hat{u}, \hat{p}, h)(t, x) = -\mathbf{e}_2 \cdot (\mu(A_h \nabla) \hat{u} - \hat{p} B_h)(t, x, 1) \mathbf{e}_2.$$

The corresponding initial data are thereby given by (\hat{u}_0, h_0, h_1) where $\hat{u}_0(x, z) = u_0(x, h_0 z)$. The matrices A_h, B_h and χ_h in (2.3) are as follows:

$$A_h = \begin{bmatrix} h & -z \partial_x h \\ -z \partial_x h \frac{1}{h} + \frac{(z \partial_x h)^2}{h} \end{bmatrix}, \quad B_h = \begin{bmatrix} h & -z \partial_x h \\ 0 & 1 \end{bmatrix}, \quad \chi_h = [x \ hz]^\top. \quad (2.6)$$

For the derivation of the system (2.3)–(2.5), please see for instance [29, Section 2] and [37, Section 4] for more details.

For the above change of variables, we have the following regularity relation between the old (without hat) and new functions (with hat).

Proposition 2.1. *Assume that $\min_{(t,x) \in (0,T) \times (0,L)} h(t, x) > h_{\min} > 0$. Then we have:*

- the mapping $f \mapsto \hat{f}$ introduced in (2.2) is a linear homeomorphism from $L^2((0, T) \times \Omega_h)$ onto $L^2(0, T; L^2(\Omega_1))$;
- For every $f \in W^{1,\infty}([0, T]; H^1(\Omega_h)) \cap L^\infty((0, T); H^2(\Omega_h))$, the following inequalities hold:

$$\begin{aligned} \|\nabla \hat{f}\|_{L^2(\Omega_1)} &\leq C(\|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty}) \|\nabla f\|_{L^2(\Omega_h)}, \\ \|\partial_t \hat{f}\|_{L^2(\Omega_1)} &\leq C(\|h^{-1}\|_{L_x^\infty}) \|\partial_t f\|_{L^2(\Omega_h)} + C(\|h^{-1}\|_{L_x^\infty}, \|\partial_t h\|_{L_x^\infty}) \|\nabla f\|_{L^2(\Omega_h)}, \\ \|\nabla^2 \hat{f}\|_{L^2(\Omega_1)} &\leq C(\|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty}, \|\partial_x^2 h\|_{L_x^2}) \|\nabla^2 f\|_{L^2(\Omega_h)} \\ \|\nabla \partial_t \hat{f}\|_{L^2(\Omega_1)} &\leq C(\|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty}) \|\nabla \partial_t f\|_{L^2(\Omega_h)} \\ &\quad + C(\|\partial_t h\|_{L_x^\infty}, \|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty}) \|\nabla^2 f\|_{L^2(\Omega_h)} \\ &\quad + C(\|\partial_t \partial_x h\|_{L_x^\infty}, \|\partial_t h\|_{L_x^\infty}, \|h^{-1}\|_{L_x^\infty}) \|\nabla f\|_{L^2(\Omega_h)}. \end{aligned}$$

Conversely,

$$\begin{aligned} \|\nabla f\|_{L^2(\Omega_h)} &\leq C(\|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty})\|\nabla \hat{f}\|_{L^2(\Omega_1)}, \\ \|\partial_t f\|_{L^2(\Omega_h)} &\leq C(\|h\|_{L_x^\infty})\|\partial_t \hat{f}\|_{L^2(\Omega_1)} + C(\|h^{-1}\|_{L_x^\infty}, \|\partial_t h\|_{L_x^\infty})\|\nabla \hat{f}\|_{L^2(\Omega_1)}, \\ \|\nabla^2 f\|_{L^2(\Omega_h)} &\leq C(\|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty}, \|\partial_x^2 h\|_{L_x^2})\|\nabla^2 \hat{f}\|_{L^2(\Omega_1)}, \\ \|\nabla \partial_t f\|_{L^2(\Omega_h)} &\leq C(\|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty})\|\nabla \partial_t \hat{f}\|_{L^2(\Omega_1)} \\ &\quad + C(\|\partial_t h\|_{L_x^\infty}, \|h\|_{W_x^{1,\infty}}, \|h^{-1}\|_{L_x^\infty})\|\nabla^2 \hat{f}\|_{L^2(\Omega_1)} \\ &\quad + C(\|\partial_t \partial_x h\|_{L_x^\infty}, \|\partial_t h\|_{L_x^\infty}, \|h^{-1}\|_{L_x^\infty})\|\nabla \hat{f}\|_{L^2(\Omega_1)}. \end{aligned}$$

Proof. The proof is standard and it follows directly from the calculations by using the change of variables and the chain rule for taking derivatives. We just present one proof for $\|\nabla^2 \hat{f}\|_{L^2(\Omega_1)}$ here. Based on (2.2), we see that

$$\begin{aligned} \partial_x^2 \hat{f} &= \partial_x^2 f + 2\partial_x \partial_y f \partial_x h z + \partial_y^2 f (\partial_x h z)^2 + \partial_y f \partial_x^2 h z, \\ \partial_x \partial_z \hat{f} &= \partial_x \partial_y f h + \partial_y f \partial_x h + \partial_y^2 f h \partial_x h z, \\ \partial_z^2 \hat{f} &= \partial_y^2 f h^2. \end{aligned}$$

We derive from the above expressions that

$$\begin{aligned} \|\nabla^2 \hat{f}\|_{L^2(\Omega_1)}^2 &= \int_0^L \int_0^h \left(|\partial_x^2 \hat{f}|^2 + 2|\partial_x \partial_y \hat{f}|^2 + |\partial_y^2 \hat{f}|^2 \right) \frac{1}{h} dy dx \\ &\leq C(\|h^{-1}\|_{L_x^\infty}, \|h\|_{W_x^{1,\infty}})\|\nabla^2 f\|_{L^2(\Omega_h)}^2 + \|h^{-1}\|_{L_x^\infty} \int_0^L |\partial_x^2 h|^2 \|\partial_y f\|_{L_y^2}^2 dx \\ &\leq C(\|h^{-1}\|_{L_x^\infty}, \|h\|_{W_x^{1,\infty}})\|\nabla^2 f\|_{L^2(\Omega_h)}^2 + \|h^{-1}\|_{L_x^\infty} \|\partial_x^2 h\|_{L_x^2}^2 \|\partial_y f\|_{L_x^\infty L_y^2}^2 \\ &\leq C(\|h^{-1}\|_{L_x^\infty}, \|h\|_{W_x^{1,\infty}}, \|\partial_x^2 h\|_{L_x^2})\|\nabla^2 f\|_{L^2(\Omega_h)}^2. \end{aligned}$$

It is worth noting that we keep the norm $\|\partial_x^2 h\|_{L_x^2}$ since h is only in $H^2(0, L)$ with respect to the space variable. Similarly, we obtain the remaining estimates. \square

Remark 2.2. Let $h \in L^\infty(0, T; H^2(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L))$, according to Proposition 2.1, we obtain that

$$\|\hat{f}\|_{L^\infty(0,T;L^2(\Omega_1))} \sim \|f\|_{L^\infty(0,T;L^2(\Omega_h))},$$

and

$$\|\hat{f}\|_{L^2(0,T;H^1(\Omega_1))} \sim \|f\|_{L^2(0,T;H^1(\Omega_h))},$$

which means that each one can be controlled by the other one multiplied by a positive constant. Moreover, Proposition 2.1 will be used later with the higher regularity for h .

2.4. Energy estimate. We introduce here the standard energy estimate for the system (1.1)–(1.7). For that the total mechanical energy of the fluid-beam system (1.1)–(1.7) is denoted by E_{tot} , as follows:

$$E_{\text{tot}}(t) = \frac{1}{2} \int_0^L \left(\rho_s |\partial_t h(t, x)|^2 + \beta |\partial_x h(t, x)|^2 + \alpha |\partial_x^2 h(t, x)|^2 \right) dx + \frac{1}{2} \int_{\Omega_h} \rho_f |u(t, x, y)|^2 dy.$$

For strong solutions the energy inequality is known to satisfy the following identity:

$$E_{\text{tot}}(t) + \int_0^t \int_{\Omega_h} |\nabla u|^2 dy dt = E_{\text{tot}}(0) \quad \text{for all } t \in (0, T). \tag{2.7}$$

It can be derived by multiplying the beam equation with $\partial_t h$ and the fluid equation with u . Recalling Remark 2.2, the energy estimate (2.7) implies that

$$\begin{aligned} & \|\partial_t h\|_{L^\infty(0,T);L^2(0,L)}^2 + \|\partial_x h\|_{L^\infty(0,T);L^2(0,L)}^2 + \|\partial_x^2 h\|_{L^\infty(0,T);L^2(0,L)}^2 \\ & + \|\hat{u}\|_{L^\infty(0,T);L^2(\Omega_1)}^2 + \|\nabla \hat{u}\|_{L^2(0,T);L^2(\Omega_1)}^2 \\ & \leq \|h_1\|_{L^2(0,L)}^2 + \|h_0\|_{H^2(0,L)}^2 + \|\hat{u}_0\|_{L^2(\Omega_1)}^2 := C_0, \end{aligned} \tag{2.8}$$

under the assumption $h(t, x) > h_{\min}$ for every $(t, x) \in (0, T) \times (0, L)$.

2.5. Hölder continuity in time-space. We will also use the following interpolation to gain continuity in space and in time. Let $|x - y| < \frac{r}{2}$ and $0 < t_2 - t_1 \leq r^{\frac{3}{2}}$, then

$$\begin{aligned} |h(t_1, x) - h(t_2, y)| & \leq \left| h(t_1, x) - \frac{1}{r} \int_x^{x+r} h(t_1, s) ds \right| \\ & + \left| \frac{1}{r} \int_x^{x+r} h(t_1, s) ds - \frac{1}{r} \int_x^{x+r} h(t_2, s) ds \right| \\ & + \left| h(t_2, y) - \frac{1}{r} \int_x^{x+r} h(t_2, s) ds \right| \\ & \leq 2\sqrt{C_0}r + \frac{1}{r} \int_x^{x+r} |h(t_1, s) - h(t_2, s)| ds \\ & \leq 2\sqrt{C_0}r + \frac{1}{r} \int_x^{x+r} \int_{t_1}^{t_2} |\partial_\tau h(\tau, s)| d\tau ds \\ & \leq 2\sqrt{C_0}r + \left(\frac{|t_2 - t_1|}{r} \right)^{\frac{1}{2}} \left(\int_x^{x+r} \int_{t_1}^{t_2} |\partial_\tau h(\tau, s)|^2 d\tau ds \right)^{\frac{1}{2}} \\ & \leq \sqrt{C_0} \left(2r + \frac{|t_2 - t_1|}{\sqrt{r}} \right) \leq 3\sqrt{C_0}r. \end{aligned} \tag{2.9}$$

This implies that we have the continuous embedding:

$$L^\infty(0, T; H^2(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L)) \hookrightarrow C^{0,\alpha}([0, T]; C^1[0, L]),$$

for $0 < \alpha \leq \frac{2}{3}$. The estimate also implies that necessarily for some time no contact between the beam and the bottom can occur.

Remark 2.3 (Minimal interval of existence). The energy estimate implies the uniform bounds of

$$\|\partial_t h\|_{L^\infty(0,T;L^2(0,L))}^2 + \|h\|_{L^\infty(0,T;H^2(0,L))}^2 \leq C_0,$$

just in dependence of the initial data. This allows the following interpolation estimate. Assume that $h_0(x) \geq \delta$ for all $x \in (0, L)$, then for $r > 0$ to be chosen later, using Höder’s inequality we find that

$$\begin{aligned} h(t, x) &\geq h(t, x) - \frac{1}{r} \int_x^{x+r} h(t, s) \, ds + \frac{1}{r} \int_x^{x+r} h(t, s) \, ds - \frac{1}{r} \int_x^{x+r} h_0(s) \, ds + \delta \\ &\geq \frac{1}{r} \int_x^{x+r} \int_s^x \partial_y h(t, y) \, dy \, ds + \frac{1}{r} \int_x^{x+r} \int_0^t \partial_\tau h(\tau, s) \, d\tau \, ds + \delta \\ &\geq \delta - \sqrt{C_0} r - \left(\frac{t}{r}\right)^{\frac{1}{2}} \left(\int_x^{x+r} \int_0^t |\partial_\tau h(\tau, s)|^2 \, d\tau \, ds\right)^{\frac{1}{2}} \\ &\geq \delta - \sqrt{C_0} \left(r + \frac{t}{\sqrt{r}}\right) \geq \delta - \sqrt{C_0} t^{\frac{2}{3}}, \end{aligned}$$

for $r = t^{\frac{2}{3}}$. Hence as long as $T < (\delta/\sqrt{C_0})^{\frac{3}{2}}$ no contact is possible, thereby there exists $h_{\min}, h_{\max} > 0$ such that $h \in [h_{\min}, h_{\max}]$ for every $(t, x) \in (0, T) \times (0, L)$.

3. Formal Time-Derivative Estimate

In this section, we formally present the time-derivative estimate, which will be rigorously verified in Sect. 4 by the Galerkin approximation. The proof corresponds exactly to the proof of Theorem 4.4 in the discrete level. After passing to the limit, this eventually holds in the continuous level.

In this central part of the paper we shall obtain the estimate for $\partial_t \hat{u}$ based on the system (2.3)–(2.5). The estimate we aim for is

$$\begin{aligned} &\rho_f \|\partial_t \hat{u}\|_{L^\infty(0,T;L^2(\Omega_1))}^2 + \mu \|\nabla \partial_t \hat{u}\|_{L^2(0,T;L^2(\Omega_1))}^2 + \rho_s \|\partial_t^2 h\|_{L^\infty(0,T;L^2(0,L))}^2 \\ &\quad + \beta \|\partial_t \partial_x h\|_{L^\infty(0,T;L^2(0,L))}^2 + \alpha \|\partial_t \partial_x^2 h\|_{L^\infty(0,T;L^2(0,L))}^2 \\ &\leq C(h_{\min}, C_0) \left(\|(\partial_t \hat{u})(0)\|_{L^2(\Omega_1)}^2 + \|(\partial_t^2 h)(0)\|_{L^2(0,L)}^2 + \|h_0\|_{H^2(0,L)}^2 \right). \end{aligned} \tag{3.1}$$

Rigorously it follows from Theorem 4.4, where the estimate is performed on the Galerkin level.

3.1. Excluding the pressure. Note that $\partial_t \hat{u}$ is not a suitable test function since it does not satisfy the modified divergence free condition (2.3b). We overcome this by using $\partial_t \hat{u} + G$ with appropriate corrector G which is constructed in an explicit way in the following. Note that this strategy differs from the strategy in [28, Section 4.3], where the modified material derivative was introduced.

Based on the modified divergence free condition (2.3b), we have

$$0 = \operatorname{div} (\partial_t (B_h^\top \hat{u})) = \operatorname{div} (\partial_t B_h^\top \hat{u}) + \operatorname{div} (B_h^\top \partial_t \hat{u}),$$

which gives that $-\operatorname{div}(B_h^\top \partial_t \hat{u}) = \operatorname{div}(\partial_t B_h^\top \hat{u})$. Recalling the matrix B_h defined in (2.6), we compute that

$$\partial_t B_h^\top \hat{u} = \begin{bmatrix} \partial_t h & 0 \\ -\partial_t \partial_x h z & 0 \end{bmatrix} \begin{bmatrix} \hat{u}^1 \\ \hat{u}^2 \end{bmatrix} = \begin{bmatrix} \partial_t h \hat{u}^1 \\ -\partial_t \partial_x h z \hat{u}^1 \end{bmatrix}.$$

Let $G = B_h^{-\top} \partial_t B_h^\top \hat{u}$, then we get

$$G = \begin{bmatrix} \frac{1}{h} & 0 \\ \frac{\partial_x h z}{h} & 1 \end{bmatrix} \begin{bmatrix} \partial_t h \hat{u}^1 \\ -\partial_t \partial_x h z \hat{u}^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{h} \partial_t h \hat{u}^1 \\ \frac{z}{h} \partial_t h \partial_x h \hat{u}^1 - \partial_t \partial_x h z \hat{u}^1 \end{bmatrix}. \quad (3.2)$$

With G defined in (3.2), we have

$$\operatorname{div}(B_h^\top (\partial_t \hat{u} + G)) = \operatorname{div}(B_h^\top \partial_t \hat{u}) + \operatorname{div}(B_h^\top G) = 0,$$

which implies that $\partial_t \hat{u} + G$ is a suitable test function for the fluid equation (2.3a).

Lemma 3.1. *Assume that $\min_{(t,x) \in (0,T) \times (0,L)} h(t,x) > h_{\min} > 0$. For G given in (3.2), there exists φ determined by \hat{u} and G such that $\varphi(x, 1) = 0$ and*

$$\operatorname{div}(\partial_t B_h^\top (\partial_t \hat{u} + G)) = \operatorname{div}(B_h^\top \varphi),$$

where φ is

$$\varphi = \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{h} \partial_t h \partial_t \hat{u}^1 + \frac{1}{h^2} |\partial_t h|^2 \hat{u}^1 \\ \frac{z}{h} \partial_t h \partial_x h \partial_t \hat{u}^1 + \frac{z}{h^2} |\partial_t h|^2 \partial_x h \hat{u}^1 - z \partial_t \partial_x h \partial_t \hat{u}^1 - \frac{z}{h} \partial_t h \partial_t \partial_x h \hat{u}^1 \end{bmatrix}. \quad (3.3)$$

Proof. The above φ can be obtained directly by setting $\varphi = B_h^{-\top} \partial_t B_h^\top (\partial_t \hat{u} + G)$ with G from (3.2). \square

In the following we collect terms of an a-priori estimate that naturally relates to the time-derivative system. For that we collect various signed terms for the left hand side and error terms we put on the right hand side. The main effort will be to estimate the error terms. We start by taking the derivative of (2.3a) with respect to the time variable and multiplying the resulting equation by $\partial_t \hat{u} + G$. We then pick $t \in (0, T)$ and integrate over $(0, t) \times \Omega_1$,

$$\begin{aligned} \int_0^t \int_{\Omega_1} \left\{ \rho_f \partial_t h \partial_t \hat{u} + \rho_f h \partial_t^2 \hat{u} + \rho_f \partial_t ((\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \hat{u}) - \mu \operatorname{div}((\partial_t A_h \nabla) \hat{u}) \right. \\ \left. - \mu \operatorname{div}((A_h \nabla) \partial_t \hat{u}) + \partial_t B_h \nabla \hat{p} + B_h \nabla \partial_t \hat{p} \right\} (\partial_t \hat{u} + G) \, dz dt = 0. \end{aligned} \quad (3.4)$$

Note that we have

$$\begin{aligned} \int_0^t \int_{\Omega_1} \rho_f \partial_t h \partial_t \hat{u} (\partial_t \hat{u} + G) \, dz dt &= \underbrace{\int_0^t \int_{\Omega_1} \rho_f \partial_t h |\partial_t \hat{u}|^2 \, dz dt}_{:= 2R_1} \\ &+ \underbrace{\int_0^t \int_{\Omega_1} \rho_f \partial_t h \partial_t \hat{u} \cdot G \, dz dt}_{:= R_2}, \end{aligned}$$

and

$$\begin{aligned} \int_0^t \int_{\Omega_1} \rho_f h \partial_t^2 \hat{u} (\partial_t \hat{u} + G) \, dz dt &= \underbrace{\int_0^t \int_{\Omega_1} \frac{\rho_f}{2} \frac{d}{dt} (|\partial_t \hat{u}|^2 h) \, dz dt}_{:=L_1} \\ &+ \underbrace{\int_0^t \int_{\Omega_1} \left(\rho_f h \partial_t^2 \hat{u} \cdot G - \frac{\rho_f}{2} \partial_t h |\partial_t \hat{u}|^2 \right) \, dz dt}_{:=R_3 - R_1}. \end{aligned}$$

For the terms on A_h , we derive by Gauss theorem using the boundary conditions of \hat{u} and G ; in particular the fact that $(\partial_t \hat{u} + G)(t, x, 1) = \partial_t^2 h e_2$ (as $G(t, x, 1) = 0$ since it only depends on \hat{u}^1 , see (3.2)). We find that

$$\begin{aligned} & - \mu \int_0^t \int_{\Omega_1} \operatorname{div} ((\partial_t A_h \nabla) \hat{u}) \cdot (\partial_t \hat{u} + G) \, dz dt \\ &= - \mu \underbrace{\int_0^t \int_0^L ((\partial_t A_h \nabla) \hat{u})(t, x, 1) \partial_t^2 h e_2 \cdot e_2 \, dx dt}_{:=(*)} \\ &+ \underbrace{\mu \int_0^t \int_{\Omega_1} ((\partial_t A_h \nabla) \hat{u}) : (\nabla \partial_t \hat{u} + \nabla G) \, dz dt}_{:=R_4}, \end{aligned}$$

and

$$\begin{aligned} & - \mu \int_0^t \int_{\Omega_1} \operatorname{div} ((A_h \nabla) \partial_t \hat{u}) \cdot (\partial_t \hat{u} + G) \, dz dt \\ &= - \mu \underbrace{\int_0^t \int_0^L ((A_h \nabla) \partial_t \hat{u})(t, x, 1) \partial_t^2 h e_2 \cdot e_2 \, dx dt}_{:=(**)} \\ &+ \underbrace{\mu \int_0^t \int_{\Omega_1} ((A_h \nabla) \partial_t \hat{u}) : \nabla \partial_t \hat{u} \, dz dt}_{:=L_2} + \underbrace{\mu \int_0^t \int_{\Omega_1} ((A_h \nabla) \partial_t \hat{u}) : \nabla G \, dz dt}_{:=R_5}. \end{aligned}$$

For the two pressure terms in (3.4), using integration by parts with respect to the spatial variable, we derive from $\operatorname{div} (B_h^T (\partial_t \hat{u} + G)) = 0$ and $\hat{u}(t, x, 1) = \partial_t h e_2$ that

$$\int_0^t \int_{\Omega_1} B_h \nabla \partial_t \hat{p} \cdot (\partial_t \hat{u} + G) \, dz dt = \underbrace{\int_0^t \int_0^L ((\partial_t \hat{p} B_h)(t, x, 1) \partial_t^2 h e_2) \cdot e_2 \, dx dt}_{:=(***)}.$$

Moreover, integration by parts implies (using the structure of G)

$$\begin{aligned} & \int_0^t \int_{\Omega_1} \partial_t B_h \nabla \hat{p} \cdot (\partial_t \hat{u} + G) \, dz dt \\ &= \underbrace{\int_0^t \int_0^L \left((\hat{p} \partial_t B_h)(t, x, 1) \partial_t^2 h e_2 \right) \cdot e_2 \, dx dt}_{=0} - \int_0^t \int_{\Omega_1} \hat{p} \operatorname{div} (\partial_t B_h^\top (\partial_t \hat{u} + G)) \, dz dt. \end{aligned} \quad (3.5)$$

Note that $\operatorname{div} (\partial_t B_h^\top (\partial_t \hat{u} + G)) \neq 0$, we thus need to estimate the second term on the right hand side of (3.5). Multiplying the momentum equation of the fluid by φ introduced in (3.3), we find using integration by parts and Lemma 3.1, that

$$\begin{aligned} \int_0^t \int_{\Omega_1} \hat{p} \operatorname{div} (\partial_t B_h^\top (\partial_t \hat{u} + G)) \, dz dt &= \underbrace{\int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot \varphi \, dz dt}_{:=R_6} \\ &+ \underbrace{\int_0^t \int_{\Omega_1} \rho_f (\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \hat{u} \cdot \varphi \, dz dt}_{:=R_7} \\ &+ \underbrace{\int_0^t \int_{\Omega_1} \mu (A_h \nabla) \hat{u} : \nabla \varphi \, dz dt}_{:=R_8}. \end{aligned}$$

We combine the above by taking the time-derivative of the beam equation (2.3c), which gives that for every $(t, x) \in (0, T) \times (0, L)$,

$$\begin{aligned} & \rho_s \partial_t^3 h - \beta \partial_t \partial_x^2 h + \alpha \partial_t \partial_x^4 h \\ &= -e_2 \cdot \left\{ \mu (\partial_t A_h \nabla) \hat{u} + \mu (A_h \nabla) \partial_t \hat{u} - \partial_t \hat{p} B_h - \hat{p} \partial_t B_h \right\} (t, x, 1) e_2. \end{aligned}$$

Taking the inner product of the above equation with $\partial_t^2 h$, we have (using that $e_2 \cdot \partial_t B_h e_2 = 0$)

$$\begin{aligned} & \int_0^t \int_0^L \left\{ \rho_s \partial_t^3 h - \beta \partial_t \partial_x^2 h + \alpha \partial_t \partial_x^4 h + e_2 \cdot (\mu (\partial_t A_h \nabla) \hat{u} \right. \\ & \quad \left. + \mu (A_h \nabla) \partial_t \hat{u} - \partial_t \hat{p} B_h) (t, x, 1) e_2 \right\} \partial_t^2 h \, dx dt = 0. \end{aligned} \quad (3.6)$$

By taking intergration by parts and using the boundary conditions, we obtain that

$$\begin{aligned} & \int_0^t \int_0^L \rho_s \partial_t^3 h \partial_t^2 h \, dx dt = \frac{\rho_s}{2} \int_0^t \int_0^L \frac{d}{dt} |\partial_t^2 h|^2 \, dx dt, \\ & - \int_0^t \int_0^L \beta \partial_t \partial_x^2 h \partial_t^2 h \, dx dt = \int_0^t \int_0^L \beta \partial_t \partial_x h \partial_t^2 \partial_x h \, dx dt \\ & = \frac{\beta}{2} \int_0^t \int_0^L \frac{d}{dt} |\partial_t \partial_x h|^2 \, dx dt, \end{aligned}$$

$$\begin{aligned} \int_0^t \int_0^L \alpha \partial_t \partial_x^4 h \partial_t^2 h \, dx dt &= \alpha \int_0^t \int_0^L \partial_t \partial_x^2 h \partial_t^2 \partial_x^2 h \, dx dt \\ &= \frac{\alpha}{2} \int_0^t \int_0^L \frac{d}{dt} \left| \partial_t \partial_x^2 h \right|^2 \, dx dt. \end{aligned}$$

We add the equations (3.4) and (3.6) together. Note that this implies that the terms (*), (**) and (***) cancel with the beam equation. Hence we find, for every $t \in (0, T)$, that

$$\begin{aligned} &\frac{\rho_s}{2} \int_0^L \left| (\partial_t^2 h)(t) \right|^2 \, dx + \frac{\beta}{2} \int_0^L \left| (\partial_t \partial_x h)(t) \right|^2 \, dx + \frac{\alpha}{2} \int_0^L \left| (\partial_t \partial_x^2 h)(t) \right|^2 \, dx \\ &\quad + \underbrace{\frac{\rho_f}{2} \int_{\Omega_1} h(t) |\partial_t \hat{u}(t)|^2 \, dz}_{=L_1} + \underbrace{\mu \int_0^t \int_{\Omega_1} (A_h \nabla) \partial_t \hat{u} : \nabla \partial_t \hat{u} \, dz dt}_{=L_2} \\ &= - \underbrace{\int_0^t \int_{\Omega_1} \frac{\rho_f}{2} \partial_t h |\partial_t \hat{u}|^2 \, dz dt}_{=R_1} - \underbrace{\int_0^t \int_{\Omega_1} \rho_f \partial_t h \partial_t \hat{u} \cdot G \, dz dt}_{=R_2} - \underbrace{\int_0^t \int_{\Omega_1} \rho_f h \partial_t^2 \hat{u} \cdot G \, dz dt}_{=R_3} \\ &\quad - \underbrace{\mu \int_0^t \int_{\Omega_1} ((\partial_t A_h \nabla) \hat{u}) : (\nabla \partial_t \hat{u} + \nabla G) \, dz dt}_{=R_4} - \underbrace{\mu \int_0^t \int_{\Omega_1} (A_h \nabla) \partial_t \hat{u} : \nabla G \, dz dt}_{=R_5} \\ &\quad + \underbrace{\int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot \varphi \, dz dt}_{=R_6} + \underbrace{\int_0^t \int_{\Omega_1} \rho_f (\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \hat{u} \cdot \varphi \, dz dt}_{=R_7} \\ &\quad + \underbrace{\mu \int_0^t \int_{\Omega_1} (A_h \nabla) \hat{u} : \nabla \varphi \, dz dt}_{=R_8} \\ &\quad - \underbrace{\int_0^t \int_{\Omega_1} \partial_t (\rho_f (\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \hat{u}) \cdot (\partial_t \hat{u} + G) \, dz dt}_{:=R_9} \\ &\quad + \frac{\rho_s}{2} \int_{\Omega_1} h_0 \left| (\partial_t \hat{u})(0) \right|^2 \, dz + \frac{\rho_s}{2} \int_0^L \left| (\partial_t^2 h)(0) \right|^2 \, dx \\ &\quad + \frac{\beta}{2} \int_0^L \left| \partial_x h_1 \right|^2 \, dx + \frac{\alpha}{2} \int_0^L \left| \partial_x^2 h_1 \right|^2 \, dx, \end{aligned} \tag{3.7}$$

where φ and G have been introduced in (3.3) and (3.2), respectively. We note that R_9 is a term that is left from (3.4). Under the no-contact assumption $\min_{(t,x) \in (0,T) \times (0,L)} h(t, x) > h_{\min}$, together with the energy estimate (2.8), we notice that there exists $c_1, c_2 > 0$ such that

$$c_1 \mathbb{1}_{2 \times 2} \leq A_h(t, x) \leq c_2 \mathbb{1}_{2 \times 2} \quad \text{for all } (t, x) \in (0, T) \times \Omega_1.$$

Hence, if the left hand side of (3.7) is bounded it implies the same bounds for the left hand side of (3.1). Moreover, together with the continuous embedding $H_t^1 \hookrightarrow L_t^\infty$ we can additionally put $\|\nabla \hat{u}\|_{L^\infty(0,T;L^2(\Omega_1))}^2$ on the left side of (3.7).

Remark 3.2. Here we provide more details on the ellipticity of A_h . Recalling the definition of A_h in (2.6), we note that for $v = [v_1 \ v_2]^T \in \mathbb{R}^2$,

$$\begin{aligned} A_h v \cdot v &= h v_1^2 - 2z \partial_x h v_1 v_2 + \frac{1}{h} (1 + (z \partial_x h)^2) v_2^2 \\ &\geq h(1 - \varepsilon^2) v_1^2 + \frac{1}{h} \left(1 + (z \partial_x h)^2 (1 - \frac{1}{\varepsilon^2}) \right) v_2^2, \end{aligned}$$

where we used the inequality

$$2z \partial_x h v_1 v_2 \leq \varepsilon^2 h v_1^2 + \frac{1}{h \varepsilon^2} (z \partial_x h)^2 v_2^2.$$

From the energy estimate (2.8) we have $\|\partial_x h\|_{L^\infty((0,T) \times (0,L))}^2 \leq C_0$. Now we choose $\varepsilon > 0$ such that $(1 + C_0^{-1})^{-1/2} < \varepsilon < 1$. In this way, we find that for $c_1 = \min \left\{ h_{\min}(1 - \varepsilon^2), h_{\max}^{-1} (1 + C_0(1 - \varepsilon^{-2})) \right\}$, we have

$$A_h v \cdot v \geq c_1 |v|^2.$$

3.2. Estimating the right hand side of (3.7). In the remaining part, we focus on the estimate of the right hand side of (3.7). To present the proof clearly, we divide it into four steps below. In the following we use without further notice that $\partial_x h$, h and $\frac{1}{h}$ are uniformly bounded in space time.

Step 1. The estimate of the first three terms on the right side of (3.7). For the first term on the right side of (3.7), we have for arbitrary $\varepsilon > 0$,

$$\begin{aligned} |R_1| &\lesssim \int_0^t \int_0^1 \|\partial_t h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^4}^2 \, dz \, dt \\ &\lesssim \int_0^t \int_0^1 \|\partial_t h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^2}^{\frac{3}{2}} \|\partial_t \partial_x \hat{u}\|_{L_x^2}^{\frac{1}{2}} \, dz \, dt \\ &\lesssim \|\partial_t h\|_{L_t^\infty(L_x^2)} \left(\int_0^t \int_0^1 \left(\varepsilon \|\partial_t \partial_x \hat{u}\|_{L_x^2}^2 + C(\varepsilon) \|\partial_t \hat{u}\|_{L_x^2}^2 \right) \, dz \, dt \right) \\ &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \, dt, \end{aligned} \tag{3.8}$$

where we used the Young's inequality and the boundedness of $\partial_t h$ in $L_t^\infty(L_x^2)$ (see (2.8)), as well as the following interpolation inequality:

$$\|\partial_t \hat{u}\|_{L_x^4} \lesssim \|\partial_t \hat{u}\|_{L_x^2}^{\frac{3}{4}} \|\partial_t \partial_x \hat{u}\|_{L_x^2}^{\frac{1}{4}}.$$

For R_2 we use the formula of G in (3.2) and derive that

$$R_2 = \int_0^t \int_{\Omega_1} \left(\frac{1}{h} |\partial_t h|^2 \hat{u}^1 \partial_t \hat{u}^1 + \frac{z}{h} |\partial_t h|^2 \partial_x h \hat{u}^1 \partial_t \hat{u}^2 - \partial_t h \partial_t \partial_x h z \hat{u}^1 \partial_t \hat{u}^2 \right) \, dz \, dt. \tag{3.9}$$

Observe that we need to estimate the first and the last integrals on the right side of (3.9). Firstly,

$$\begin{aligned}
 \int_0^t \int_{\Omega_1} \frac{1}{h} |\partial_t h|^2 \hat{u}^1 \partial_t \hat{u}^1 \, dz dt &\lesssim \int_0^t \|\partial_t h\|_{L_x^\infty}^2 \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \|\partial_t h\|_{L_x^2} \|\partial_t \partial_x h\|_{L_x^2} \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \|\partial_t h\|_{L_t^\infty(L_x^2)} \|\hat{u}\|_{L_t^\infty(L_z^2)} \int_0^t \|\partial_t \partial_x h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \left(\|\partial_t \partial_x h\|_{L_x^2}^2 + \|\partial_t \hat{u}\|_{L_z^2}^2 \right) \, dt,
 \end{aligned} \tag{3.10}$$

where we used the interpolation inequality as follows:

$$\|\partial_t h\|_{L_x^\infty} \lesssim \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x h\|_{L_x^2}^{\frac{1}{2}}. \tag{3.11}$$

This 1D interpolation (3.11) we will use frequently below and please see [1, Theorem 5.9] or [15] for reference. Then we estimate the last integral in (3.9):

$$\begin{aligned}
 \int_0^t \int_{\Omega_1} \partial_t h \partial_t \partial_x h \hat{u}^1 \partial_t \hat{u}^2 \, dz dt &\leq \int_0^t \|\partial_t h\|_{L_x^\infty} \|\partial_t \partial_x h\|_{L_x^\infty} \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x h\|_{L_x^2} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}} \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \|\partial_t h\|_{L_x^2} \|\partial_t \partial_x^2 h\|_{L_x^2} \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \right) \, dt.
 \end{aligned} \tag{3.12}$$

In the above estimate, we used in particular the interpolation inequality:

$$\|\partial_t \partial_x h\|_{L_x^2} \leq C \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}}.$$

Combining with the estimate (3.10) and (3.12), we thereby obtain the estimate for (3.9) as follows:

$$|R_2| \lesssim \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \right) \, dt.$$

To estimate R_3 we first take an integration by parts with respect to the time variable, which gives that

$$\begin{aligned}
 R_3 &= \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot G \, dz dt - \int_0^t \int_{\Omega_1} \rho_f \partial_t h \partial_t \hat{u}(t) \cdot G(t) \, dz dt \\
 &\quad - \int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot \partial_t G \, dz dt - \int_{\Omega_1} \rho_f h_0(\partial_t \hat{u})(0) \cdot G(0) \, dz.
 \end{aligned} \tag{3.13}$$

Using the expression of G in (3.2), we start to estimate the right side of (3.13) by using interpolation inequalities. We first have

$$\int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot G \, d\mathbf{z} = \int_{\Omega_1} \rho_f \left(\partial_t h \hat{u}^1 \partial_t \hat{u}^1 + z \partial_t h \partial_x h \hat{u}^1 \partial_t \hat{u}^2 - z h \partial_t \partial_x h \hat{u}^1 \partial_t \hat{u}^2 \right) d\mathbf{z}. \quad (3.14)$$

and then for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \int_{\Omega_1} \rho_f \partial_t h \hat{u}^1 \partial_t \hat{u}^1 \, d\mathbf{z} &\lesssim \|\partial_t h\|_{L_x^\infty} \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \\ &\lesssim \|\partial_t h\|_{L_z^2}^{\frac{3}{4}} \|\partial_t \partial_x h\|_{L_x^2}^{\frac{1}{4}} \|\partial_t \hat{u}\|_{L_z^2} \\ &\leq \varepsilon \|\partial_t \hat{u}\|_{L_z^2}^2 + C(\varepsilon) \|\partial_t h\|_{L_x^2}^{\frac{3}{2}} \|\partial_t \partial_x h\|_{L_x^2}^{\frac{1}{2}} \\ &\leq \varepsilon \|\partial_t \hat{u}\|_{L_t^\infty(L_z^2)}^2 + \delta \|\partial_t \partial_x h\|_{L_t^\infty(L_x^2)}^2 + C(\varepsilon, \delta) \|\partial_t h\|_{L_t^\infty(L_x^2)}^2, \end{aligned} \quad (3.15)$$

where we used twice Young's inequality. The second term on the right side of (3.14) can be estimated in a similar way as (3.15). For the last term in (3.14), we also have for arbitrary $\varepsilon > 0$

$$\begin{aligned} \int_{\Omega_1} \rho_f z h \partial_t \partial_x h \hat{u}^1 \partial_t \hat{u}^2 \, d\mathbf{z} &\lesssim \|\partial_t \partial_x h\|_{L_x^\infty} \|\hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2} \\ &\lesssim \|\partial_t h\|_{L_x^2}^{\frac{1}{4}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{3}{4}} \|\partial_t \hat{u}\|_{L_z^2} \\ &\leq \varepsilon \|\partial_t \hat{u}\|_{L_z^2}^2 + C(\varepsilon) \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{3}{2}} \\ &\leq \varepsilon \|\partial_t \hat{u}\|_{L_t^\infty(L_z^2)}^2 + \delta \|\partial_t \partial_x^2 h\|_{L_t^\infty(L_x^2)}^2 + C(\varepsilon, \delta) \|\partial_t h\|_{L_t^\infty(L_x^2)}^2. \end{aligned} \quad (3.16)$$

Combining with the estimate (3.15) and (3.16), we obtain that

$$\int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot G \, d\mathbf{z} \leq \varepsilon \|\partial_t \hat{u}\|_{L_t^\infty(L_z^2)}^2 + \delta \|\partial_t \partial_x^2 h\|_{L_t^\infty(L_x^2)}^2 + C(\varepsilon, \delta) \|\partial_t h\|_{L_t^\infty(L_x^2)}^2. \quad (3.17)$$

The second term on the right of (3.13) is the same with R_2 term, which has been estimated in (3.9). Recalling the expression of G in (3.2), the third term on the right side of (3.13) is

$$\begin{aligned} &\int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot \partial_t G \, d\mathbf{z} dt \\ &= \int_0^t \int_{\Omega_1} \left\{ \rho_f h \partial_t \hat{u}^1 \left(-\frac{1}{h^2} |\partial_t h|^2 \hat{u}^1 + \frac{1}{h} \partial_t^2 h \hat{u}^1 + \frac{1}{h} \partial_t h \partial_t \hat{u}^1 \right) \right. \\ &\quad + \rho_f h \partial_t \hat{u}^2 \left(-\frac{z}{h^2} |\partial_t h|^2 \partial_x h \hat{u}^1 + \frac{z}{h} \partial_t^2 h \partial_x h \hat{u}^1 + \frac{z}{h} \partial_t h \partial_t \partial_x h \hat{u}^1 \right. \\ &\quad \left. \left. + \frac{z}{h} \partial_t h \partial_x h \partial_t \hat{u}^1 - \partial_t^2 \partial_x h z \hat{u}^1 - \partial_t \partial_x h z \partial_t \hat{u}^1 \right) \right\} d\mathbf{z} dt. \end{aligned}$$

In view of (3.8) and (3.10) the terms left to be estimated are:

$$\begin{aligned} & \int_0^t \int_{\Omega_1} \partial_t^2 h \hat{u}^1 \partial_t \hat{u}^1 \, dz dt, \quad \int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u}^2 \partial_t^2 \partial_x h \hat{u}^1 \, dz dt, \\ & \int_0^t \int_{\Omega_1} \rho_f z h \partial_t \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 \, dz dt. \end{aligned} \tag{3.18}$$

Using interpolation inequality and Young’s inequality, we obtain that

$$\begin{aligned} \int_0^t \int_{\Omega_1} \partial_t^2 h \hat{u}^1 \partial_t \hat{u}^1 \, dz dt & \lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2} \int_0^1 \|\hat{u}\|_{L_x^2}^{\frac{1}{2}} \|\partial_x \hat{u}\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \hat{u}\|_{L_x^2} \, dz dt \\ & \lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2} \|\hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_x \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_t \hat{u}\|_{L_z^2} \, dt \\ & \lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2}^2 \, dt + \int_0^t \|\nabla \hat{u}\|_{L_z^2}^2 \|\partial_t \hat{u}\|_{L_z^2}^2 \, dt. \end{aligned} \tag{3.19}$$

For the other two in (3.18), we need to take the integration by parts with respect to x , which gives that

$$\begin{aligned} & \int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u}^2 \partial_t^2 \partial_x h \hat{u}^1 \, dz dt \\ & = - \int_0^t \int_{\Omega_1} \rho_f \partial_t^2 h \left(\partial_x h \partial_t \hat{u}^2 \hat{u}^1 + h \partial_t \partial_x \hat{u}^2 \hat{u}^1 + h \partial_t \hat{u}^2 \partial_x \hat{u}^1 \right) \, dz dt. \end{aligned} \tag{3.20}$$

The first term on the right of (3.20) can be estimated as (3.19). The second term on the right of (3.20) can be estimate with arbitrary $\varepsilon > 0$ as:

$$\begin{aligned} \int_0^t \int_{\Omega_1} \rho_f h \partial_t^2 h \partial_t \partial_x \hat{u}^2 \hat{u}^1 \, dz dt & \lesssim \int_0^t \int_0^1 \|\partial_t^2 h\|_{L_x^2} \|\partial_t \partial_x \hat{u}\|_{L_x^2} \|\hat{u}\|_{L_x^\infty} \, dz dt \\ & \lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2} \|\partial_t \partial_x \hat{u}\|_{L_z^2} \|\hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_x \hat{u}\|_{L_z^2}^{\frac{1}{2}} \, dt \\ & \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\nabla \hat{u}\|_{L_z^2}^2 \|\partial_t^2 h\|_{L_x^2}^2 \, dt. \end{aligned}$$

Similarly, we estimate the last term on the right side of (3.20):

$$\begin{aligned} \int_0^t \int_{\Omega_1} \rho_f h \partial_t^2 h \partial_t \hat{u}^2 \partial_x \hat{u}^1 \, dz dt & \lesssim \int_0^t \int_0^1 \|\partial_t^2 h\|_{L_x^2} \|\partial_x \hat{u}\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^\infty} \, dz dt \\ & \lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2} \|\partial_x \hat{u}\|_{L_z^2} \|\partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_t \partial_x \hat{u}\|_{L_z^2}^{\frac{1}{2}} \, dt \\ & \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_x \hat{u}\|_{L_z^2}^2 \|\partial_t^2 h\|_{L_x^2}^2 \, dt. \end{aligned}$$

We also use integration by parts for the last integral in (3.18):

$$\begin{aligned} & \int_0^t \int_{\Omega_1} \rho_f z h \partial_t \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 \, d\mathbf{z} dt \\ &= - \int_0^t \int_{\Omega_1} \rho_f z \partial_t h \left(h \partial_t \partial_x \hat{u}^1 \partial_t \hat{u}^2 + h \partial_t \hat{u}^1 \partial_t \partial_x \hat{u}^2 + \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 \right) \, d\mathbf{z} dt. \end{aligned} \quad (3.21)$$

The critical term in (3.21) is estimated for every $\varepsilon > 0$ as follows:

$$\begin{aligned} \int_0^t \int_{\Omega_1} \partial_t h \partial_t \hat{u} \partial_t \partial_x \hat{u} \, d\mathbf{z} dt &\leq \int_0^t \int_0^1 \|\partial_t h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^\infty} \|\partial_t \partial_x \hat{u}\|_{L_x^2} \, d\mathbf{z} dt \\ &\lesssim \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_t \partial_x \hat{u}\|_{L_z^2}^{\frac{3}{2}} \, dt \\ &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \, dt. \end{aligned} \quad (3.22)$$

Combining with the estimate (3.19)–(3.21), we have

$$\begin{aligned} \int_0^t \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot \partial_t G \, d\mathbf{z} dt &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t^2 h\|_{L_x^2}^2 \right) \, dt \\ &\quad + C(\varepsilon) \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t^2 h\|_{L_x^2}^2 \right) \|\nabla \hat{u}\|_{L_z^2}^2 \, dt. \end{aligned} \quad (3.23)$$

Putting together the estimate (3.17), (3.9) and (3.23), we derive that

$$\begin{aligned} |R_3| &\leq \varepsilon \|\partial_t \hat{u}\|_{L_t^\infty(L_z^2)} + \delta \|\partial_t \partial_x^2 h\|_{L_t^\infty(L_x^2)} + \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt \\ &\quad + C(\varepsilon, \delta) \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t^2 h\|_{L_x^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \right) \, dt \\ &\quad + C(\varepsilon) \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t^2 h\|_{L_x^2}^2 \right) \|\nabla \hat{u}\|_{L_z^2}^2 \, dt \\ &\quad - \int_{\Omega_1} \rho_f h_0(\partial_t \hat{u})(0) \cdot G(0) \, d\mathbf{z}. \end{aligned}$$

The last integral with the initial data above can be handled in the following way:

$$\begin{aligned} & \int_{\Omega_1} \rho_f h_0(\partial_t \hat{u})(0) \cdot G(0) \, d\mathbf{z} \\ &\lesssim \int_{\Omega_1} \left(|h_1| |\hat{u}_0| |(\partial_t \hat{u})(0)| + |h_1| |\partial_x h_0| |\hat{u}_0| |(\partial_t \hat{u})(0)| + |h_0| |\partial_x h_1| |(\partial_t \hat{u})(0)| \right) \, d\mathbf{z} \\ &\leq C(C_0, \|h_1\|_{H^2(0,L)}) (C_0 + \|(\partial_t \hat{u})(0)\|_{L_z^2}^2), \end{aligned}$$

where we used the expression (3.14) and C_0 has been introduced in (2.8).

Step 2. The estimate of the two terms R_4 and R_5 that are related to ∇G in (3.7).

Recalling the matrix A_h and G introduced in (2.6) and (3.2), we compute $\partial_t A_h \nabla \hat{u} : (\nabla \partial_t \hat{u} + \nabla G)$. With that we can list the terms appearing in R_4 below:

$$\begin{aligned}
 a_1 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x \hat{u}^1| |\partial_t \partial_x \hat{u}^1| \, dz dt, & a_2 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\hat{u}^1| |\partial_x \hat{u}^1| \, dz dt, \\
 a_3 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t \partial_x h| |\hat{u}^1| |\partial_x \hat{u}^1| \, dz dt, & a_4 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\partial_x \hat{u}^1|^2 \, dz dt, \\
 a_5 &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\partial_z \hat{u}^2| |\partial_t \partial_z \hat{u}^1| \, dz dt, & a_6 &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h|^2 |\hat{u}^1| |\partial_x \hat{u}^1| \, dz dt, \\
 a_7 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t \partial_x h| |\partial_z \hat{u}^2| |\partial_z \hat{u}^1| \, dz dt, & a_8 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\partial_x^2 h| |\hat{u}^1| |\partial_x \hat{u}^2| \, dz dt, \\
 a_9 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t \partial_x^2 h| |\hat{u}^1| |\partial_x \hat{u}^2| \, dz dt, & a_{10} &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x^2 h| |\partial_t \partial_x h| |\hat{u}^1| |\partial_x \hat{u}^1| \, dz dt, \\
 a_{11} &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\partial_t \partial_x^2 h| |\hat{u}^1| |\partial_x \hat{u}^1| \, dz dt, & a_{12} &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h|^2 |\partial_x \hat{u}^1|^2 \, dz dt
 \end{aligned}$$

We do select the most critical terms of above. First a_1 is subcritical to a_5 . Next we find that a_2 and a_8 are subcritical to a_{10} . Further a_3, a_4, a_6 and a_7 are subcritical to a_{12} . Finally a_9 is subcritical to a_{11} . Hence we need to estimate

$$\begin{aligned}
 A_5 &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\nabla \hat{u}| |\partial_t \nabla \hat{u}| \, dz dt, & A_{10} &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x^2 h| |\partial_t \partial_x h| |\hat{u}| |\nabla \hat{u}| \, dz dt, \\
 A_{11} &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\partial_t \partial_x^2 h| |\hat{u}| |\nabla \hat{u}| \, dz dt, & A_{12} &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h|^2 |\nabla \hat{u}|^2 \, dz dt.
 \end{aligned}$$

We begin with A_5 where we find that for every $\varepsilon > 0$

$$\begin{aligned}
 |A_5| &\leq \int_0^t \|\partial_t \partial_x h\|_{L_x^\infty} \|\nabla \hat{u}\|_{L_z^2} \|\partial_t \nabla \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2} \|\nabla \hat{u}\|_{L_z^2} \|\partial_t \nabla \hat{u}\|_{L_z^2} \, dt \\
 &\leq \varepsilon \int_0^t \|\partial_t \nabla \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 \, dt.
 \end{aligned}$$

Next we estimate using interpolation in x and the energy bounds we find

$$\begin{aligned}
 |A_{10}| &\leq \int_0^t \int_{\Omega_1} \|\partial_t h\|_{L_x^\infty} \|\partial_t \partial_x h\|_{L_x^\infty} \|\hat{u}\|_{L_x^\infty} \|\nabla \hat{u}\|_{L_x^2} \|\partial_x^2 h\|_{L_x^2} \, dz dt \\
 &\lesssim \int_0^t \int_{\Omega_1} \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x h\|_{L_x^2} \|\hat{u}\|_{L_x^2} \|\nabla \hat{u}\|_{L_x^2}^{\frac{3}{2}} \, dz dt \\
 &\lesssim \int_0^t \int_{\Omega_1} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{3}{2}} \|\nabla \hat{u}\|_{L_x^2}^{\frac{3}{2}} \, dz dt \\
 &\lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 \, dt + C,
 \end{aligned}$$

which is suitable for a Grönwall estimate. Further we estimate similarly

$$\begin{aligned} |A_{11}| &\leq \int_0^t \int_{\Omega_1} \|\partial_t \partial_x h\|_{L_x^\infty} \|\partial_t \partial_x^2 h\|_{L_x^2} \|\hat{u}\|_{L_x^\infty} \|\nabla \hat{u}\|_{L_x^2} \, dz \, dt \\ &\lesssim \int_0^t \int_{\Omega_1} \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{3}{2}} \|\hat{u}\|_{L_x^2}^{\frac{1}{2}} \|\nabla \hat{u}\|_{L_x^2}^{\frac{3}{2}} \, dz \, dt \\ &\lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_x^2}^2 \, dt + C. \end{aligned}$$

Finally, we estimate

$$\begin{aligned} |A_{12}| &\leq \int_0^t \|\partial_t \partial_x h\|_{L_x^\infty}^2 \|\nabla \hat{u}\|_{L_x^2}^2 \, dt \lesssim \int_0^t \|\partial_t \partial_x h\|_{L_x^2} \|\partial_t \partial_x^2 h\|_{L_x^2} \|\nabla \hat{u}\|_{L_x^2}^2 \, dt \\ &\lesssim \int_0^t (\|\partial_t \partial_x^2 h\|_{L_x^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2) \|\partial_x \hat{u}\|_{L_x^2}^2 \, dt. \end{aligned}$$

Hence we conclude (applying Poincaré's inequality) that

$$|R_4| = \int_0^t \int_{\Omega_2} \partial_t A_h \nabla \hat{u} : (\nabla \partial_t \hat{u} + \nabla G) \, dz \, dt \lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_x^2}^2 \, dt + C,$$

is suitable for a Grönwall estimate.

Now we continue estimating R_5 in (3.7). It can be easily checked that the terms appearing that have not been treated before are

$$\begin{aligned} b_1 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x h| |\hat{u}| |\partial_t \partial_x \hat{u}| \, dz \, dt, & b_2 &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\hat{u}| |\partial_t \partial_x \hat{u}| \, dz \, dt, \\ b_3 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x^2 h| |\hat{u}| |\partial_t \partial_x \hat{u}| \, dz \, dt, & b_4 &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x^2 h| |\hat{u}| |\partial_t \partial_x \hat{u}| \, dz \, dt, \end{aligned}$$

where critical are only b_3 and b_4 . For every $\varepsilon > 0$ we find

$$\begin{aligned} |b_3| &\leq \int_0^t \int_0^1 \|\partial_t h\|_{L_x^\infty} \|\partial_x^2 h\|_{L_x^2} \|\hat{u}\|_{L_x^\infty} \|\partial_t \partial_x \hat{u}\|_{L_x^2} \, dz \, dt \\ &\lesssim \int_0^t \|\partial_t \partial_x h\|_{L_x^2}^{\frac{1}{2}} \|\partial_x \hat{u}\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x \hat{u}\|_{L_x^2} \, dt \\ &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_x^2}^2 \, dt + C(\varepsilon) \int_0^t \left(\|\nabla \hat{u}\|_{L_x^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2 \right) \, dt, \end{aligned}$$

and

$$|b_4| \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_x^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 (1 + \|\nabla \hat{u}\|_{L_x^2}^2) \, dt.$$

Therefore, we have the estimate:

$$\begin{aligned} \int_0^t \int_{\Omega_1} (A_h \nabla) \partial_t \hat{u} : \nabla G \, dz \, dt &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_x^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_x^2}^2 \, dt \\ &\quad + C(\varepsilon) \int_0^t \left(\|\partial_t \partial_x h\|_{L_x^2}^2 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 + \|\nabla \hat{u}\|_{L_x^2}^2 \right) \, dt. \end{aligned}$$

Step 3. The estimate of the terms R_6, R_7 and R_8 depending on φ in (3.7). Recall that φ has been introduced in (3.3). We first notice that

$$\begin{aligned} \rho_f h \partial_t \hat{u} \cdot \varphi &= \rho_f \partial_t h |\partial_t \hat{u}^1|^2 + \frac{\rho_f}{h} |\partial_t h|^2 \partial_t \hat{u}^1 \hat{u}^1 + \rho_f z \partial_t h \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 \\ &\quad + \frac{\rho_f z}{h} |\partial_t h|^2 \partial_x h \hat{u}^1 \partial_t \hat{u}^2 - \rho_f h z \partial_t \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 - \rho_f z \partial_t h \partial_t \partial_x h \partial_t \hat{u}^2 \hat{u}^1. \end{aligned}$$

Compared with the terms we have estimated before, here we only need to consider the following integral

$$\begin{aligned} \int_0^t \int_{\Omega_1} \rho_f h z \partial_t \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 \, dz dt &= - \int_0^t \int_{\Omega_1} \rho_f z \partial_t h \partial_x h \partial_t \hat{u}^1 \partial_t \hat{u}^2 \, dz dt \\ &\quad - \int_0^t \int_{\Omega_1} \rho_f h z \partial_t h \partial_t \partial_x \hat{u}^1 \partial_t \hat{u}^2 \, dz dt \quad (3.24) \\ &\quad - \int_0^t \int_{\Omega_1} \rho_f h z \partial_t h \partial_t \hat{u}^1 \partial_t \partial_x \hat{u}^2 \, dz dt, \end{aligned}$$

where we used the integration by parts with respect to the space variable x . The first term on the right side of (3.24) has been estimated in (3.8). Note that the other two terms on the right of (3.24) have the same structure and actually also have been considered in (3.22). Thus, for every $\varepsilon > 0$ we have

$$|R_6| \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L^2_z}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L^2_z}^2 \, dt.$$

For R_7 , using the structure of the matrix χ_h and B_h introduced in (2.6) and φ in (3.3), respectively, we observe that the terms appearing are:

$$\begin{aligned} c_1 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\hat{u}| |\partial_t \hat{u}| |\partial_x \hat{u}| \, dz dt, & c_2 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\hat{u}|^2 |\partial_x \hat{u}| \, dz dt, \\ c_3 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\partial_t \hat{u}| |\partial_x \hat{u}| \, dz dt, & c_4 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^3 |\hat{u}| |\partial_x \hat{u}| \, dz dt, \\ c_5 &:= \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\hat{u}| |\partial_t \hat{u}| |\partial_z \hat{u}| \, dz dt, & c_6 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t \partial_x h| |\hat{u}|^2 |\partial_z \hat{u}| \, dz dt, \\ c_7 &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t \partial_x h| |\partial_t \hat{u}| |\partial_z \hat{u}| \, dz dt, & c_8 &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\partial_t \partial_x h| |\hat{u}| |\partial_z \hat{u}| \, dz dt. \end{aligned} \tag{3.25}$$

Now, c_1 is subcritical to c_5 , c_2 to c_6 , c_3 to c_7 and c_4 to c_8 . We estimate using again the interpolation inequality in 1D for L^∞ as in (3.11) but also, the interpolation inequality in 2D for L^4 , also known as Ladyzhenskaya's estimate in Ω_1 :

$$\|\hat{u}\|_{L^4_z} \lesssim \|\hat{u}\|_{L^2_z}^{\frac{1}{2}} \|\nabla \hat{u}\|_{L^2_z}^{\frac{1}{2}}.$$

With that we find for every $\varepsilon > 0$

$$\begin{aligned} c_5 &\leq \int_0^t \|\partial_t \partial_x h\|_{L_x^\infty} \|\partial_z \hat{u}\|_{L_z^2} \|\hat{u}\|_{L_z^4} \|\partial_t \hat{u}\|_{L_z^4} dt \\ &\lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2} \|\nabla \hat{u}\|_{L_z^2}^{\frac{3}{2}} \|\partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\nabla \partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} dt \\ &\leq \varepsilon \int_0^t \|\nabla \partial_t \hat{u}\|_{L_z^2}^2 dt + \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 dt, \end{aligned}$$

where in the end we used Young's inequality (with three terms). For c_6 in (3.25), we use the same interpolations

$$c_6 \leq \int_0^t \|\partial_t h\|_{L_x^\infty} \|\partial_t \partial_x h\|_{L_x^\infty} \|\partial_z \hat{u}\|_{L_z^2} \|\hat{u}\|_{L_z^4}^2 dt \lesssim \int_0^t (1 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2) \|\nabla \hat{u}\|_{L_z^2}^2 dt.$$

Further

$$\begin{aligned} c_7 &\leq \int_0^t \int_0^1 \|\partial_t \partial_x h\|_{L_x^\infty} \|\partial_t h\|_{L_x^4} \|\partial_t \hat{u}\|_{L_x^4} \|\partial_z \hat{u}\|_{L_x^2} dz dt \\ &\lesssim \int_0^t \int_0^1 \|\partial_t \partial_x h\|_{L_x^2}^{\frac{3}{4}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t h\|_{L_x^2}^{\frac{3}{4}} \|\partial_t \hat{u}\|_{L_x^2}^{\frac{3}{4}} \|\partial_t \partial_x \hat{u}\|_{L_x^2}^{\frac{1}{4}} \|\partial_z \hat{u}\|_{L_x^2} dz dt \\ &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x h\|_{L_x^2}^{\frac{6}{7}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{4}{7}} \|\partial_t \hat{u}\|_{L_z^2}^{\frac{6}{7}} \|\partial_z \hat{u}\|_{L_z^2}^{\frac{8}{7}} dt \\ &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 dt + \delta \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 dt + C(\varepsilon, \delta) \int_0^t (1 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2) \|\partial_z \hat{u}\|_{L_z^2}^2 dt, \end{aligned}$$

where we used the uniform boundedness of $\|\partial_t h\|_{L_x^2}$, the following interpolation inequality:

$$\|\partial_t \partial_x h\|_{L_x^2} \lesssim \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}}.$$

and Young's inequality. Finally, the last term is estimated by

$$\begin{aligned} c_8 &\leq \int_0^t \|\partial_t h\|_{L_x^\infty}^2 \|\partial_t \partial_x h\|_{L_x^\infty} \|\hat{u}\|_{L_z^2} \|\partial_z \hat{u}\|_{L_z^2} dt \\ &\lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 dt + \int_0^t (1 + \|\nabla \hat{u}\|_{L_z^2}^2) \|\partial_t \partial_x h\|_{L_x^2}^2 dt. \end{aligned}$$

All together we have the estimate

$$\begin{aligned} |R_7| &\leq \varepsilon \int_0^t \|\nabla \partial_t \hat{u}\|_{L_z^2}^2 dt + C(\varepsilon) \int_0^t \left(\|\partial_t \hat{u}\|_{L_z^2}^2 dt + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \right) dt \\ &\quad + C(\varepsilon, \delta) \int_0^t \left(1 + \|\partial_t \hat{u}\|_{L_z^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 + \|\partial_t \hat{u}\|_{L_z^2}^2 \right) \|\nabla \hat{u}\|_{L_z^2}^2 dt. \end{aligned}$$

Now we continue the estimate for the last integral including φ in (3.7), namely R_8 . Evaluating this term one realizes that the terms not considered before are

$$\int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x \hat{u}| |\partial_t \hat{u}| \, d\mathbf{z} dt, \quad \int_0^t \int_{\Omega_1} |\partial_t \partial_x h| |\partial_t \hat{u}| |\partial_z \hat{u}| \, d\mathbf{z} dt,$$

$$\int_0^t \int_{\Omega_1} |\partial_t \partial_x^2 h| |\partial_t \hat{u}| |\partial_x \hat{u}| \, d\mathbf{z} dt, \quad \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x^2 h| |\partial_t \hat{u}| |\partial_x \hat{u}| \, d\mathbf{z} dt.$$

The first two terms above are subcritical to c_5 . The third and the last terms is estimated for every $\varepsilon > 0$ as follows

$$\begin{aligned} & \int_0^t \int_{\Omega_1} |\partial_t \partial_x^2 h| |\partial_t \hat{u}| |\partial_x \hat{u}| \, d\mathbf{z} dt \\ & \leq \int_0^t \int_0^1 \|\partial_t \partial_x^2 h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^\infty} \|\partial_x \hat{u}\|_{L_x^2} \, d\mathbf{z} dt \\ & \leq \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_t \partial_x \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\nabla \hat{u}\|_{L_z^2} \, dt \\ & \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \, dt. \end{aligned}$$

and

$$\begin{aligned} \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_x^2 h| |\partial_t \hat{u}| |\partial_x \hat{u}| \, d\mathbf{z} dt & \leq \int_0^t \int_0^1 \|\partial_t h\|_{L_x^\infty} \|\partial_x^2 h\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^\infty} \|\partial_x \hat{u}\|_{L_x^2} \, d\mathbf{z} dt \\ & \leq \int_0^t \|\partial_t \partial_x h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_t \partial_x \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\partial_x \hat{u}\|_{L_z^2} \, dt \\ & \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \, dt \\ & \quad + \int_0^t \|\partial_t \partial_x h\|_{L_x^2}^2 \|\partial_x \hat{u}\|_{L_z^2}^2 \, dt. \end{aligned}$$

Therefore, we have for arbitrary $\varepsilon > 0$.

$$\begin{aligned} |R_8| & \leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \, dt \\ & \quad + C(\varepsilon) \int_0^t \left(\|\partial_t \partial_x^2 h\|_{L_x^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2 \right) \|\nabla \hat{u}\|_{L_z^2}^2 \, dt. \end{aligned}$$

Step 4. The estimate of the time-derivative of the convective term in (3.7). This last part is the most critical estimate as it is related to the non-linearity of the Navier–Stokes equation. We split the terms appearing in R_9 in three sub-parts:

$$\begin{aligned} \partial_t (\rho_f (\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \hat{u}) \cdot (\partial_t \hat{u} + G) & = \rho_f \underbrace{(\partial_t \hat{u} - \partial_t^2 \chi_h) \cdot (B_h \nabla) \hat{u} \cdot (\partial_t \hat{u} + G)}_{:=r_1} \\ & \quad + \underbrace{(\hat{u} - \partial_t \chi_h) \cdot (\partial_t B_h \nabla) \hat{u} \cdot (\partial_t \hat{u} + G)}_{:=r_2} + \underbrace{(\hat{u} - \partial_t \chi_h) \cdot (B_h \nabla) \partial_t \hat{u} \cdot (\partial_t \hat{u} + G)}_{:=r_3}. \end{aligned}$$

We analyze the above three forms one by one. Recalling the definition of B_h and χ_h in (2.6) we first have

$$\begin{aligned} r_1 = & \left(h \partial_t \hat{u}^1 \partial_x \hat{u}^1 - \partial_x h z \partial_t \hat{u}^1 \partial_x \hat{u}^2 + \partial_t \hat{u}^2 \partial_x \hat{u}^2 - \partial_t^2 h z \partial_x \hat{u}^2 \right) \left(\partial_t \hat{u}^1 + \frac{1}{h} \partial_t h \hat{u}^1 \right) \\ & + \left(h \partial_t \hat{u}^1 \partial_z \hat{u}^1 - \partial_x h z \partial_t \hat{u}^1 \partial_z \hat{u}^2 + \partial_t \hat{u}^1 \partial_z \hat{u}^2 - \partial_t^2 h z \partial_z \hat{u}^2 \right) \\ & \left(\partial_t \hat{u}^2 + \frac{z}{h} \partial_t h \partial_x h \hat{u}^1 - \partial_t \partial_x h z \hat{u}^1 \right). \end{aligned} \quad (3.26)$$

The terms in (3.26) that appear are summarized by

$$\begin{aligned} r_{11} &:= \int_0^t \int_{\Omega_1} |\partial_t \hat{u}|^2 |\partial_x \hat{u}| \, d\mathbf{z} dt, \quad r_{12} := \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t^2 h| |\hat{u}| |\partial_z \hat{u}| \, d\mathbf{z} dt, \\ r_{13} &:= \int_0^t \int_{\Omega_1} |\partial_t^2 h| |\partial_t \partial_x h| |\hat{u}| |\partial_z \hat{u}| \, d\mathbf{z} dt, \end{aligned}$$

where r_{12} is subcritical to r_{13} . We first find by Ladyzhenskaya's interpolation estimate that for every $\varepsilon > 0$

$$r_{11} \leq \int_0^t \|\partial_x \hat{u}\|_{L_x^2} \|\partial_t \hat{u}\|_{L_x^4}^2 \, dt \leq \varepsilon \int_0^t \|\nabla \partial_t \hat{u}\|_{L_x^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_x^2}^2 \, dt,$$

and similarly to above (using 1D interpolation for L^∞) we find

$$\begin{aligned} r_{13} &\leq \int_0^t \int_{\Omega_1} \|\partial_t^2 h\|_{L_x^2} \|\partial_z \hat{u}\|_{L_x^2} \|\partial_t \partial_x h\|_{L_x^\infty} \|\hat{u}\|_{L_x^\infty} \, d\mathbf{z} dt \\ &\lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2} \|\nabla \hat{u}\|_{L_x^2}^{\frac{3}{2}} \|\partial_t \partial_x^2 h\|_{L_x^\infty} \, dt \\ &\lesssim \int_0^t \|\partial_t^2 h\|_{L_x^2}^2 (1 + \|\nabla \hat{u}\|_{L_x^2}^2) \, dt + \int_0^t \|\nabla \hat{u}\|_{L_x^2}^2 \|\partial_t \partial_x^2 h\|_{L_x^\infty}^2 \, dt. \end{aligned}$$

This allows to conclude that for every $\varepsilon > 0$

$$\begin{aligned} \int_0^t \int_{\Omega_1} |r_1| \, d\mathbf{z} dt &\leq \varepsilon \int_0^t \|\nabla \partial_t \hat{u}\|_{L_x^2}^2 \, dt + C(\varepsilon) \int_0^t \left(\|\partial_t^2 h\|_{L_x^2}^2 + \|\partial_t \partial_x h\|_{L_x^2}^2 \right) \, dt \\ &\quad + C(\varepsilon) \int_0^t \left(1 + \|\partial_t^2 h\|_{L_x^2}^2 + \|\partial_t \partial_x^2 h\|_{L_x^2}^2 + \|\partial_t \hat{u}\|_{L_x^2}^2 \right) \|\nabla \hat{u}\|_{L_x^2}^2 \, dt. \end{aligned}$$

Next we compute

$$\begin{aligned} r_2 = & \left(\hat{u}^1 \partial_t h \partial_x \hat{u}^1 - \partial_t \partial_x h z \hat{u}^1 \partial_x \hat{u}^2 \right) \left(\partial_t \hat{u}^1 + \frac{1}{h} \partial_t h \hat{u}^1 \right) \\ & + \left(\partial_t h \hat{u}^1 \partial_z \hat{u}^1 - \partial_t \partial_x h z \hat{u}^1 \partial_z \hat{u}^2 \right) \left(\partial_t \hat{u}^2 + \frac{z}{h} \partial_t h \partial_x h \hat{u}^1 - \partial_t \partial_x h z \hat{u}^1 \right), \end{aligned}$$

which can be estimated as c_5 and c_7 . Finally

$$\begin{aligned} r_3 = & \left(h \hat{u}^1 \partial_t \partial_x \hat{u}^1 - \partial_x h z \hat{u}^1 \partial_t \partial_z \hat{u}^1 + \hat{u}^2 \partial_t \partial_z \hat{u}^1 - \partial_t h z \partial_t \partial_z \hat{u}^1 \right) \left(\partial_t \hat{u}^1 + \frac{1}{h} \partial_t h \hat{u}^1 \right) \\ & + \left(h \hat{u}^1 \partial_t \partial_x \hat{u}^2 - \partial_x h z \hat{u}^1 \partial_t \partial_z \hat{u}^2 + \hat{u}^2 \partial_t \partial_z \hat{u}^2 - \partial_t h z \partial_t \partial_z \hat{u}^2 \right) \\ & \left(\partial_t \hat{u}^2 + \frac{z}{h} \partial_t h \partial_x h \hat{u}^1 - \partial_t \partial_x h z \hat{u}^1 \right) \end{aligned}$$

In this part, we need to estimate:

$$\begin{aligned}
 r_{31} &:= \int_0^t \int_{\Omega_1} |\hat{u}| |\partial_t \nabla \hat{u}| |\partial_t \hat{u}| \, d\mathbf{z} \, dt, & r_{32} &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\hat{u}|^2 |\partial_t \nabla \hat{u}| \, d\mathbf{z} \, dt, \\
 r_{33} &:= \int_0^t \int_{\Omega_1} |\hat{u}|^2 |\partial_t \partial_x h| |\partial_t \nabla \hat{u}| \, d\mathbf{z} \, dt, & r_{34} &:= \int_0^t \int_{\Omega_1} |\partial_t h| |\partial_t \partial_x h| |\hat{u}| |\partial_t \nabla \hat{u}| \, d\mathbf{z} \, dt, \\
 r_{35} &:= \int_0^t \int_{\Omega_1} |\partial_t h|^2 |\hat{u}| |\partial_t \partial_z \hat{u}| \, d\mathbf{z} \, dt.
 \end{aligned}$$

Notice that r_{32} and r_{35} is subcritical to r_{33} and r_{34} , respectively. We estimate (using Ladyzhenskaya’s interpolation estimate once more) that for every $\varepsilon > 0$

$$\begin{aligned}
 r_{31} &\leq \int_0^t \|\partial_t \nabla \hat{u}\|_{L_z^\infty} \|\partial_t \hat{u}\|_{L_z^4} \|\hat{u}\|_{L_z^4} \, dt \\
 &\lesssim \int_0^t \|\nabla \partial_t \hat{u}\|_{L_z^2}^{\frac{3}{2}} \|\partial_t \hat{u}\|_{L_z^2}^{\frac{1}{2}} \|\nabla \hat{u}\|_{L_z^2}^{\frac{1}{2}} \, dt \\
 &\leq \varepsilon \int_0^t \|\nabla \partial_t \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \hat{u}\|_{L_z^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 \, dt.
 \end{aligned}$$

Next

$$\begin{aligned}
 r_{33} &\leq \int_0^t \|\partial_t \partial_x h\|_{L_x^\infty} \|\hat{u}\|_{L_x^4}^2 \|\partial_t \nabla \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x h\|_{L_x^2}^{\frac{1}{2}} \|\hat{u}\|_{L_z^2} \|\nabla \hat{u}\|_{L_z^2} \|\partial_t \nabla \hat{u}\|_{L_z^2} \, dt \\
 &\leq \varepsilon \int_0^t \|\partial_t \partial_x \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \|\nabla \hat{u}\|_{L_z^2}^2 \, dt,
 \end{aligned}$$

where we used Young’s and Poincaré’s inequality in the last step. Finally

$$\begin{aligned}
 r_{34} &\leq \int_0^t \|\hat{u}\|_{L_z^2} \|\partial_t \nabla \hat{u}\|_{L_z^2} \|\partial_t h\|_{L_x^\infty} \|\partial_t \partial_x h\|_{L_x^\infty} \, dt \\
 &\lesssim \int_0^t \|\partial_t h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \partial_x h\|_{L_x^2} \|\partial_t \partial_x^2 h\|_{L_x^2}^{\frac{1}{2}} \|\partial_t \nabla \hat{u}\|_{L_z^2} \, dt \\
 &\lesssim \int_0^t \|\partial_t h\|_{L_x^2} \|\partial_t \partial_x^2 h\|_{L_x^2} \|\partial_t \nabla \hat{u}\|_{L_z^2} \, dt \\
 &\leq \varepsilon \int_0^t \|\partial_t \nabla \hat{u}\|_{L_z^2}^2 \, dt + C(\varepsilon) \int_0^t \|\partial_t \partial_x^2 h\|_{L_x^2}^2 \, dt.
 \end{aligned}$$

The remaining integrals can be estimated directly.

Putting all the estimates together and taking the supremum with respect to time on both sides of (3.7), we obtain from Grönwall’s lemma that for $T > 0$ and for $\varepsilon > 0$

small enough,

$$\begin{aligned} & \sup_{t \in (0, T)} \int_{\Omega_1} |\partial_t \hat{u}|^2 \, d\mathbf{z} + \int_0^T \int_{\Omega_1} |\nabla \partial_t \hat{u}|^2 \, d\mathbf{z} dt \\ & \quad + \sup_{t \in (0, T)} \int_0^L \left(\rho_s |\partial_t^2 h|^2 + |\beta \partial_t \partial_x h|^2 + \alpha |\partial_t \partial_x^2 h|^2 \right) dx \\ & \lesssim \int_{\Omega_1} |(\partial_t \hat{u})(0)|^2 \, d\mathbf{z} + \int_0^L \left(|(\partial_t^2 h)(0)|^2 + |\partial_x h_1|^2 + |\partial_x^2 h_1|^2 \right) dx + 1 \end{aligned}$$

where the bound depends on C_0 defined in (2.8) and $\min_{(t,x) \in [0, T] \times [0, L]} h(t, x) =: h_{\min}$ only.

3.3. Formally obtaining the initial values for $\partial_t \hat{u}$ and $\partial_t^2 h$. Observe that we only need to know

$$\|(\partial_t \hat{u})(0)\|_{L^2(\Omega_1)} + \|(\partial_t^2 h)(0)\|_{L_x^2}.$$

Assume for the moment that $\Omega(0) = \Omega_1$. Then (formally) multiplying the equation at zero with $((\partial_t \hat{u})(0), (\partial_t^2 h)(0))$ we find that

$$\begin{aligned} & \rho_s \|(\partial_t \hat{u})(0)\|_{L^2(\Omega_1)}^2 + \rho_f \|(\partial_t^2 h)(0)\|_{L_x^2}^2 \\ & = \int_{\Omega_1} (\mu \Delta \hat{u}_0 - \rho_f (\hat{u}_0 \cdot \nabla) \hat{u}_0) \cdot (\partial_t \hat{u})(0) \, d\mathbf{z} + \int_0^L (\beta \partial_x^2 h_0 - \alpha \partial_x^4 h_0) (\partial_t^2 h)(0) \, dx \\ & \quad - 2\mu \int_0^L \mathbf{e}_2 \cdot D(\hat{u}_0)(0, x, h_0) (-\partial_x h_0 \mathbf{e}_1 + \mathbf{e}_2) (\partial_t^2 h)(0) \, dx. \end{aligned}$$

This implies a uniform estimate, if $\hat{u}_0 \in H^2(\Omega_1)$ and $h_0 \in H^4(0, L)$. Moreover, it is linear in $\hat{u}_0 \in H^2(\Omega_1)$ and $h_0 \in H^4(0, L)$. Rigorously the initial data for the time-derivative will be established in Step 2 in the Galerkin construction below.

4. Construction of a Strong Solution

In order to make the estimates rigorous we have to derive the estimates on an approximate or mollified level. This is typical for higher-order in time estimates. We note that due to the (weak-strong)-uniqueness result shown in [44] constructing a smooth solution implies that it is unique.

The key that in order of being able to rigorously differentiating the equation in time is the correct treatment of the pressure. For that we introduce the following solenoidal substitute for the fluid velocity $\hat{v} = B_h^\top \hat{u}$, and in particular we note that

$$B_h^{-\top} \partial_t \hat{v} = \partial_t \hat{u} + B_h^{-\top} \partial_t B_h^\top \hat{u} = \partial_t \hat{u} + G,$$

where G is defined in (3.2).

We use a Galerkin approximation which allows us to rigorously perform the key estimate. For that the following steps will be performed:

- 4.1 We prepare the partial differential equation and its time-derivative such that it can be approximated by a Galerkin method. For that we introduce a weak equation in terms of solenoidal test functions and decouple the geometry from the equation.
- 4.2 We show the existence of a space-discrete and decoupled solution to the time-differentiated equation that solves an energy estimate. By a fixed-point argument we get the desired coupled approximation sequence.
- 4.3 We show that the a-priori estimates derived in (3.1) are valid on the discrete level.
- 4.4 We construct a strong solution. For that we pass to the limit with the approximation and show further spatial regularity using the properties of the steady Stokes operator. This establishes a strong solution.

Hence the strategy is to derive a discretized weak formulation for \hat{v} , which we will approximate by a solenoidal Galerkin basis. This next will be (formally) differentiated in time.

4.1. Preparation. Following the convention for Galerkin schemes we consider the following weak formulation for (\hat{u}, h) a solution to system (2.3)–(2.5). We introduce the space of L -periodic smooth functions

$$C_{\text{per}}^\infty(0, L) := \{f \in C^\infty(\mathbb{R}) \mid f(\cdot) \equiv f(\cdot + L)\},$$

and

$$C_{\text{per},0}^\infty(\Omega_1) := \{f \in C^\infty(\mathbb{R} \times [0, 1]) \mid f(\cdot, y) \in C_{\text{per}}^\infty(0, L) \text{ and } f(\cdot, 0) = 0\}.$$

We have for all time values t and $(\phi, \Phi) \in C_{\text{per}}^\infty(0, L) \times C_{\text{per},0}^\infty(\Omega_1)$, with $\text{div } B_h^T(t)\Phi = 0$ in Ω_1 , $\Phi(x, 1) = \phi(x) e_2$, $\Phi(x, 0) = 0$ and $\Phi(0, z) = \Phi(L, z)$, that

$$\begin{aligned} & \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot \Phi \, dz + \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \hat{u} \cdot \Phi \, dz - \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \Phi \cdot \hat{u} \, dz \\ & + \frac{1}{2} \int_0^L \rho_f |\partial_t h|^2 \phi \, dx - \int_{\Omega_1} \rho_f \partial_t \chi_h (B_h \nabla) \hat{u} \cdot \Phi \, dz + \mu \int_{\Omega_1} (A_h \nabla) \hat{u} : \nabla \Phi \, dz \\ & + \rho_s \int_0^L \partial_t^2 h \phi \, dx - \beta \int_0^L \partial_x^2 h \phi \, dx + \alpha \int_0^L \partial_x^4 h \phi \, dx = 0, \end{aligned} \tag{4.1}$$

with the relation $\hat{u}(t, x, 1) = \partial_t h(t, x) e_2$ for $(t, x) \in (0, T) \times (0, L)$.

The above weak formulation can be derived by multiplying the strong equation with ϕ and Φ respectively, integrating in time-space and performing integration by parts. In particular we used the following form for the convective term:

$$\begin{aligned} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \hat{u} \cdot \Phi \, dz &= \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \hat{u} \cdot \Phi \, dz \\ &- \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \Phi \cdot \hat{u} \, dz + \frac{1}{2} \int_0^L \rho_f |\partial_t h|^2 \phi \, dx, \end{aligned} \tag{4.2}$$

which plays an important role in the construction of the approximate solution and in particular preserves the structure of the energy estimate in the Galerkin procedure.

Following the weak formulation (4.1), we define

$$\Psi(x, z) = \begin{bmatrix} \Psi^1(x, z) \\ \Psi^2(x, z) \end{bmatrix} := B_h^\top(t, x, z)\Phi(x, z),$$

which gives that $\operatorname{div} \Psi = 0$ in Ω_1 . It is worthwhile noting that Ψ is independent of the time variable. Recalling the structure of the matrix B_h (please see (2.6)), with $\Phi = [\Phi^1 \ \Phi^2]^\top$ we have

$$\Psi = \begin{bmatrix} h & 0 \\ -z\partial_x h & 1 \end{bmatrix} \begin{bmatrix} \Phi^1 \\ \Phi^2 \end{bmatrix} = \begin{bmatrix} h\Phi^1 \\ -z\partial_x h\Phi^1 + \Phi^2 \end{bmatrix}.$$

Note that $\Phi^1(x, 1) = 0$, therefore we obtain that $\Psi(x, 1) = \Phi^2(x, 1)\mathbf{e}_2$.

This allows to introduce the corresponding weak formulation in terms of Ψ , which is now divergence free. Actually it suffices to consider steady functions $(\psi, \Psi) \in C_{\text{per}}^\infty(0, L) \times C^\infty(\Omega_1)$, with $\operatorname{div} \Psi = 0$ in Ω_1 , $\Psi(x, 1) = \psi(x)\mathbf{e}_2$, $\Psi(x, 0) = 0$ and $\Psi(0, z) = \Phi(L, z)$ only. Then we find for all fixed times that

$$\begin{aligned} & \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot B_h^{-\top} \Psi \, \mathbf{d}\mathbf{z} + \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \hat{u} \cdot B_h^{-\top} \Psi \, \mathbf{d}\mathbf{z} \\ & - \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) (B_h^{-\top} \Psi) \cdot \hat{u} \, \mathbf{d}\mathbf{z} \\ & + \frac{1}{2} \int_0^L \rho_f |\partial_t h|^2 \psi \, \mathbf{d}x - \int_{\Omega_1} \rho_f \partial_t \chi_h (B_h \nabla) \hat{u} \cdot B_h^{-\top} \Psi \, \mathbf{d}\mathbf{z} \quad (4.3) \\ & + \mu \int_{\Omega_1} (A_h \nabla) \hat{u} : \nabla (B_h^{-\top} \Psi) \, \mathbf{d}\mathbf{z} \\ & + \rho_s \int_0^L \partial_t^2 h \psi \, \mathbf{d}x - \beta \int_0^L \partial_x^2 h \psi \, \mathbf{d}x + \alpha \int_0^L \partial_x^4 h \psi \, \mathbf{d}x = 0, \end{aligned}$$

with the relation $\hat{u}(t, x, 1) = \partial_t h(t, x)\mathbf{e}_2$ for $(t, x) \in (0, T) \times (0, L)$.

Next we differentiate (4.3) in time. This implies for the same set of test functions and for every $t \in (0, T)$, which is given by¹

$$\begin{aligned} & \left\langle \rho_f \partial_t h \partial_t \hat{u}, B_h^{-\top} \Psi \right\rangle + \left\langle \rho_f h \partial_t^2 \hat{u}, B_h^{-\top} \Psi \right\rangle + \left\langle \rho_f h \partial_t \hat{u}, \partial_t B_h^{-\top} \Psi \right\rangle \\ & + \frac{1}{2} \left\langle \rho_f \partial_t \hat{u} (B_h \nabla) \hat{u}, B_h^{-\top} \Psi \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u} (\partial_t B_h \nabla) \hat{u}, B_h^{-\top} \Psi \right\rangle \\ & + \frac{1}{2} \left\langle \rho_f \hat{u} (B_h \nabla) \partial_t \hat{u}, B_h^{-\top} \Psi \right\rangle \\ & + \frac{1}{2} \left\langle \rho_f \hat{u} (B_h \nabla) \hat{u}, \partial_t B_h^{-\top} \Psi \right\rangle - \frac{1}{2} \left\langle \rho_f \partial_t \hat{u} (B_h \nabla) (B_h^{-\top} \Psi), \hat{u} \right\rangle \\ & - \frac{1}{2} \left\langle \rho_f \hat{u} (\partial_t B_h \nabla) (B_h^{-\top} \Psi), \hat{u} \right\rangle \\ & - \frac{1}{2} \left\langle \rho_f \hat{u} (B_h \nabla) (\partial_t B_h^{-\top} \Psi), \hat{u} \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u} (B_h \nabla) (B_h^{-\top} \Psi), \partial_t \hat{u} \right\rangle \end{aligned}$$

¹ From here on we use scalar products $\langle \cdot, \cdot \rangle$ for formal products and this would be justified in Galerkin procedure.

$$\begin{aligned}
 & + \left\langle \rho_f \partial_t^2 h \partial_t h, \psi \right\rangle \\
 & - \left\langle \rho_f \partial_t^2 \chi_h (B_h \nabla) \hat{u}, B_h^{-\top} \Psi \right\rangle - \left\langle \rho_f \partial_t \chi_h (\partial_t B_h \nabla) \hat{u}, B_h^{-\top} \Psi \right\rangle \\
 & - \left\langle \rho_f \partial_t \chi_h (B_h \nabla) \partial_t \hat{u}, B_h^{-\top} \Psi \right\rangle \\
 & - \left\langle \rho_f \partial_t \chi_h (B_h \nabla) \hat{u}, \partial_t B_h^{-\top} \Psi \right\rangle + \left\langle \mu (\partial_t A_h \nabla) \hat{u}, \nabla (B_h^{-\top} \Psi) \right\rangle \\
 & + \left\langle \mu (A_h \nabla) \partial_t \hat{u}, \nabla (B_h^{-\top} \Psi) \right\rangle \\
 & + \left\langle \mu (A_h \nabla) \hat{u}, \nabla (\partial_t B_h^{-\top} \Psi) \right\rangle + \left\langle \rho_s \partial_t^3 h, \psi \right\rangle - \left\langle \beta \partial_t \partial_x^2 h, \psi \right\rangle \\
 & + \left\langle \alpha \partial_t \partial_x^2 h, \partial_x^2 \psi \right\rangle = 0,
 \end{aligned}$$

and $\partial_t \hat{u}(t, x, 1) = \partial_t^2 h(t, x) e_2$ for all $(t, x) \in (0, T) \times (0, L)$.

Next this equation will be *decoupled*. Let us consider a geometry given by the function $\tilde{h} \in C^2([0, T] \times [0, L]; [h_{\min}, h_{\max}])$. Respectively we define the coefficients

$$\tilde{B} = B_{\tilde{h}}, \quad \tilde{A} = A_{\tilde{h}}, \quad \tilde{\chi} = \chi_{\tilde{h}},$$

which are now independent of the solution. Then we aim to solve the following decoupled system.

We look for the coupled solution (\hat{u}, g) that is periodic in $[0, L]$ satisfying $\operatorname{div} \tilde{B}^\top \hat{u} = 0$, $\hat{u}(t, x, 1) = \partial_t g(t, x) e_2$, $\hat{u}(t, x, 0) = 0$, and

$$\begin{aligned}
 & \left\langle \rho_f \partial_t \tilde{h} \partial_t \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle + \left\langle \rho_f \tilde{h} \partial_t^2 \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle + \left\langle \rho_f \tilde{h} \partial_t \hat{u}, \partial_t \tilde{B}^{-\top} \Psi \right\rangle \\
 & + \frac{1}{2} \left\langle \rho_f \partial_t \hat{u} (\tilde{B} \nabla) \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u} (\partial_t \tilde{B} \nabla) \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle \\
 & + \frac{1}{2} \left\langle \rho_f \hat{u} (\tilde{B} \nabla) \partial_t \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u} (\tilde{B} \nabla) \hat{u}, \partial_t \tilde{B}^{-\top} \Psi \right\rangle \\
 & - \frac{1}{2} \left\langle \rho_f \partial_t \hat{u} (\tilde{B} \nabla) (\tilde{B}^{-\top} \Psi), \hat{u} \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u} (\partial_t \tilde{B} \nabla) (\tilde{B}^{-\top} \Psi), \hat{u} \right\rangle \\
 & - \frac{1}{2} \left\langle \rho_f \hat{u} (\tilde{B} \nabla) (\partial_t \tilde{B}^{-\top} \Psi), \hat{u} \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u} (\tilde{B} \nabla) (\tilde{B}^{-\top} \Psi), \partial_t \hat{u} \right\rangle \\
 & + \frac{1}{2} \left\langle \rho_f \partial_t^2 \tilde{h} \partial_t g, \psi \right\rangle + \frac{1}{2} \left\langle \rho_f \partial_t \tilde{h} \partial_t^2 g, \psi \right\rangle \tag{4.4} \\
 & - \left\langle \rho_f \partial_t^2 \tilde{\chi} (\tilde{B} \nabla) \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle - \left\langle \rho_f \partial_t \tilde{\chi} (\partial_t \tilde{B} \nabla) \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle \\
 & - \left\langle \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \partial_t \hat{u}, \tilde{B}^{-\top} \Psi \right\rangle - \left\langle \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \hat{u}, \partial_t \tilde{B}^{-\top} \Psi \right\rangle \\
 & + \left\langle \mu (\partial_t \tilde{A} \nabla) \hat{u}, \nabla (\tilde{B}^{-\top} \Psi) \right\rangle + \left\langle \mu (\tilde{A} \nabla) \partial_t \hat{u}, \nabla (\tilde{B}^{-\top} \Psi) \right\rangle \\
 & + \left\langle \mu (\tilde{A} \nabla) \hat{u}, \nabla (\partial_t \tilde{B}^{-\top} \Psi) \right\rangle \\
 & + \left\langle \rho_s \partial_t^3 g, \psi \right\rangle - \left\langle \beta \partial_t \partial_x^2 g, \psi \right\rangle + \left\langle \alpha \partial_t \partial_x^2 g, \partial_x^2 \psi \right\rangle = 0,
 \end{aligned}$$

for all times and every $(\psi, \Psi) \in C_{\text{per}}^\infty(0, L) \times C^\infty(\Omega_1)$, with $\operatorname{div} \Psi = 0$ in Ω_1 , $\Psi(x, 1) = \psi(x) e_2$, $\Psi(x, 0) = 0$ and $\Psi(0, z) = \Phi(L, z)$. As the system is second

order/third order in time, strictly speaking the introduction of new variables would be necessary to make it a first order. But it eventually depends on (\hat{u}, g) and this is the unknown functions we are looking for.

Remark 4.1. We remark here that in (4.4), by fixing the geometry \tilde{h} and using g , we decouple $\langle \rho_f \partial_t^2 h \partial_t h, \psi \rangle$ into:

$$\frac{1}{2} \langle \rho_f \partial_t^2 \tilde{h} \partial_t g, \psi \rangle + \frac{1}{2} \langle \rho_f \partial_t \tilde{h} \partial_t^2 g, \psi \rangle.$$

This formulation plays an important role in the derivation of the energy estimate of (4.16) in Proposition 4.3.

4.2. Galerkin approximations. For every time-value we will seek a solution in the setting of the following Sobolev spaces. The periodic Sobolev spaces are introduced as

$$W_{\text{per}}^{k,p}(0, L) := \overline{C_{\text{per}}^\infty(0, L)}^{\|\cdot\|_{W^{k,p}(0,L)}},$$

for $1 \leq p < \infty$ and $k \in \mathbb{N}$.

We introduce the space

$$\dot{H}_{\text{per}}^2(0, L) := \left\{ \check{\psi} \in H_{\text{per}}^2(0, L) \mid \int_0^L \check{\psi} \, dx = 0 \right\},$$

which is a Hilbert space w.r.t. the scalar product $(\check{\psi}, \xi)_{\dot{H}^2(0,L)} := \int_0^L \partial_x^2 \check{\psi} \partial_x^2 \xi \, dx$ (this is indeed a Hilbert space as the affine functions in periodic spaces are constants). We take $(\check{\psi}_k)_{k \in \mathbb{N}}$ as a basis of $\dot{H}_{\text{per}}^2(0, L)$ that is orthonormal in $L^2(0, L)$.²

Further for the fluid velocities we consider

$$V_{\text{per},0}^1(\Omega_1) := \left\{ \hat{\Psi} \in H_{\text{per},0}^1(\Omega_1) \mid \text{div } \hat{\Psi} = 0 \right\},$$

where the subscript “0” has a similar meaning with the one introduced above (4.1), i.e. $\hat{\Psi}(\cdot, 0) = 0$. This Hilbert space $V_{\text{per},0}^1(\Omega_1)$ is w.r.t. the scalar product

$$(\hat{\Psi}, \hat{w})_{V_{\text{per},0}^1(\Omega_1)} = \int_{\Omega_1} \nabla \hat{\Psi} \cdot \nabla \hat{w} \, dx \, dy.$$

We will decompose this space into

$$V_{\text{per},0,0}^1(\Omega_1) := \left\{ \hat{\Psi} \in V_{\text{per},0}^1(\Omega_1) \mid \hat{\Psi}(\cdot, 1) = 0 \right\}$$

and its orthogonal complement $(V_{\text{per},0,0}^1(\Omega_1))^\perp$. We will construct bases for each part separately using the Stokes operator.

² These functions are precisely of the form $a_0 \frac{\sin(\frac{k\pi}{L}x)}{k^2}$ and $a_0 \frac{\cos(\frac{k\pi}{L}x)}{k^2}$, with a_0 depending on L .

First we collect all eigenfunctions of the following Stokes problem:

$$\begin{cases} -\Delta \hat{\Psi}_k + \nabla \hat{p}_k = \lambda_k \hat{\Psi}_k & \text{in } \Omega_1, \\ \operatorname{div} \hat{\Psi}_k = 0 & \text{in } \Omega_1, \\ \hat{\Psi}_k(\cdot, 0) = 0 = \hat{\Psi}_k(\cdot, 1) & \text{on } [0, L], \\ \hat{\Psi}_k(\cdot, z) = \hat{\Psi}_k(\cdot + L, z) & \text{for all } z \in [0, 1]. \end{cases}$$

where λ_k is the corresponding eigenvalue of the eigenfunction $\hat{\Psi}_k$ for every $k \in \mathbb{N}$. This provides a smooth basis of $V_{\text{per},0,0}^1(\Omega_1)$ which is orthonormal in $V_{\text{per},0}^1(\Omega_1)$ and orthogonal in $L^2(\Omega_1)$, respectively.

Next we construct $(\check{\Psi}_k)_{k \in \mathbb{N}}$ by extending $(\check{\psi}_k)_{k \in \mathbb{N}}$, the basis of $\dot{H}_{\text{per}}^2(0, L)$, through the system:

$$\begin{cases} -\Delta \check{\Psi}_k + \nabla \check{p}_k = 0 & \text{in } \Omega_1, \\ \operatorname{div} \check{\Psi}_k = 0 & \text{in } \Omega_1, \\ \check{\Psi}_k = \begin{cases} \check{\psi}_k(x) \mathbf{e}_2 & (0, L) \times \{z = 1\}, \\ 0 & (0, L) \times \{z = 0\}. \end{cases} \\ \check{\Psi}_k(\cdot, z) = \check{\Psi}_k(\cdot + L, z) & \text{for all } z \in [0, 1]. \end{cases}$$

These functions are smooth and linearly independent. Observe that for $\hat{w} \in V_{\text{per},0}^1(\Omega_1)$ we have

$$\begin{aligned} 0 &= \int_{\Omega_1} (-\Delta \check{\Psi}_k + \nabla \check{p}_k) \hat{w} \, dx \, dy = \int_{\Omega_1} \nabla \check{\Psi}_k \cdot \nabla \hat{w} \, dx \, dy \\ &\quad - \int_0^L (\partial_x \check{\Psi}_k^2)(\cdot, 1) \hat{w}^1(\cdot, 1) \, dx + \int_0^L (\check{p}_k - \partial_y \check{\Psi}_k^2)(\cdot, 1) \hat{w}^2(\cdot, 1) \, dx, \end{aligned}$$

which implies that $\check{\Psi}_k \in (V_{\text{per},0,0}^1(\Omega_1))^\perp$.

Now we define $(\Psi_k, \psi_k)_{k \in \mathbb{N}}$ in a way of enumeration by

$$(\Psi_k, \psi_k) := \begin{cases} \left(\check{\Psi}_{\frac{k+1}{2}}, \check{\psi}_{\frac{k+1}{2}} \right) & k \text{ odd} \\ \left(\hat{\Psi}_{\frac{k}{2}}, 0 \right) & k \text{ even} \end{cases} \quad \text{for all } k \in \mathbb{N}. \tag{4.5}$$

The span $\{(\Psi_k, \psi_k) \mid k \in \mathbb{N}\}$ consists of all couples $(\Psi, \psi) \in V_{\text{per},0}^1(\Omega_1) \times \dot{H}_{\text{per}}^2(0, L)$ with $\Psi(x, 1) = \psi \mathbf{e}_2$ for every $x \in (0, L)$. We observe further that the space span $\{(\tilde{B}^{-\top} \Psi_k, \psi_k) \mid k \in \mathbb{N}\}$ consists of the test function pairs:

$$(\Phi, \phi) \in \left\{ (\Phi, \phi) \in H_{\text{per},0}^1(\Omega_1) \times \dot{H}_{\text{per}}^2(0, L) \mid \operatorname{div}(\tilde{B}^\top \Phi) = 0 \text{ in } \Omega_1, \Phi(t, x, 1) = \phi \mathbf{e}_2 \right\},$$

which are C^2 in space-time since \tilde{B} is C^2 in space-time. On that space we introduce the projection \mathcal{P}_N , that is defined as

$$\begin{aligned} \mathcal{P}_N \phi &:= \sum_{k=1}^N P^k(\phi) \psi_k := \sum_{k=1}^N (\phi, \psi_k)_{H^2_{\text{per},0}(0,L)} \psi_k \\ \mathcal{P}_N \Phi &:= \tilde{B}^{-\top} \sum_{k=1}^N P^k(\Phi) \Psi_k = \tilde{B}^{-\top} \left(\sum_{k \leq N, k \text{ odd}} P^k(\phi) \Psi_k \right. \\ &\quad \left. + \sum_{k \leq N, k \text{ even}} (\Psi_k, \tilde{B}^\top \Phi)_{V^1_{\text{per},0}(\Omega_1)} \Psi_k \right). \end{aligned} \tag{4.6}$$

The following properties of the projection follow by standard Hilbert space theory and standard regularity theory for Stokes system:

- For any $m \in \mathbb{N}$, if (additionally) $\phi \in H^m, \|\mathcal{P}_N \phi\|_{H^m(0,L)} \leq c \|\phi\|_{H^m(0,L)}$ and $\mathcal{P}_N \phi \rightarrow \phi$ in $H^m(0,L)$.
- We can decompose $\tilde{B}^\top \Phi = \Phi_0 + \Phi_1 = (\tilde{B}^\top \Phi - \Phi_1) + \Phi_1$, where

$$\begin{cases} -\Delta \Phi_1 + \nabla p_1 = 0 & \text{in } \Omega_1, \\ \text{div } \Phi_1 = 0 & \text{in } \Omega_1, \\ \Phi_1 = \begin{cases} \phi e_2 & (0,L) \times \{z = 1\}, \\ 0 & (0,L) \times \{z = 0\}. \end{cases} \\ \Phi_1(\cdot, z) = \Phi_1(\cdot + L, z) & \text{for all } z \in [0, 1]. \end{cases}$$

- We find that Φ_1 is approximated by its boundary values. In particular, by Stokes theory and the properties of the eigenfunctions, we find that

$$\begin{aligned} &\| \sum_{k \leq N, k \text{ odd}} P^k(\phi) \Psi_k - \Phi_1 \|_{H^{\frac{5}{2}}(\Omega_1)} \\ &\leq c \| \sum_{k \leq N, k \text{ odd}} P^k(\phi) \psi_k - \phi \|_{H^2(0,L)} \rightarrow 0 \text{ with } N \rightarrow \infty. \end{aligned}$$

- If (additionally) $\Phi \in H^2$, we find that Φ_0 is also approximated in H^2 by the second projection part, i.e. for k even. For that observe that

$$(\Psi_k, \tilde{B}^\top \Phi)_{V^1_{\text{per},0}(\Omega_1)} = (\Psi_k, \Phi_0)_{V^1_{\text{per},0}(\Omega_1)} + \underbrace{(\Psi_k, \Phi_1)_{V^1_{\text{per},0}(\Omega_1)}}_{=0},$$

where we used the equation of Φ_1 and the fact that Ψ_k has zero boundary values in $\{z = 1\}$. Hence we get by the properties of the basis $\{\Psi_k\}_{k \text{ even}}$, that

$$\| \sum_{k \leq N, k \text{ even}} (\Psi_k, \tilde{B}^\top \Phi)_{V^1_{\text{per},0}(\Omega_1)} \Psi_k - \Phi_0 \|_{H^2(\Omega_1)} \rightarrow 0 \text{ with } N \rightarrow \infty.$$

- Combining the arguments above we find that

$$\| \mathcal{P}_N \Phi \|_{H^2(\Omega_1)} \leq C \| \Phi \|_{H^2(\Omega_1)} + c \| \phi \|_{H^2(0,L)}.$$

Moreover, $\mathcal{P}_N \Phi \rightarrow \Phi$ in $H^2(\Omega_1)$ as $N \rightarrow \infty$. Note that based on the similar but simpler analysis, the above estimate for $\mathcal{P}_N \Phi$ holds also for the H^1 and L^2 -norm.

We make the following ansatz, for every fixed $N \in \mathbb{N}$:

$$\begin{aligned} \hat{u}_N(t, x, z) &:= \tilde{B}^{-\top} \sum_{k=1}^N \alpha_N^k(t) \Psi_k(x, z) \quad \text{for all } t \in (0, T), \\ g_N(t, x) &:= \int_0^t \sum_{k=1}^N \alpha_N^k(\tau) \psi_k(x) d\tau + \mathcal{P}_N h_0 \quad \text{for all } t \in (0, T). \end{aligned} \tag{4.7}$$

From the construction (4.7), we see that $\hat{u}_N(t, x, 1) = \partial_t g_N e_2$ for $(t, x) \in (0, T) \times (0, L)$.

We construct now the solution (\hat{u}_N, g_N) in four steps.

Step 1: Existence of α_N for a given geometry. Assume that $\tilde{h} \in C^2([0, T] \times [0, L]; [h_{\min}, h_{\max}])$ is the given geometry and we still use the notation $\tilde{B} = B_{\tilde{h}}, \tilde{A} = A_{\tilde{h}}$ and $\tilde{\chi} = \chi_{\tilde{h}}$.

In what follows we seek for the a couple of discrete solutions (\hat{u}_N, g_N) of the form (4.7) with time-dependent coefficients $\alpha_N = (\alpha_N^k)_{k=1}^N$, which solve the following discrete equation, for $k = 1, \dots, N$:

$$\begin{aligned} &\left\langle \rho_f \partial_t \tilde{h} \partial_t \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle + \left\langle \rho_f \tilde{h} \partial_t^2 \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle + \left\langle \rho_f \tilde{h} \partial_t \hat{u}_N, \partial_t \tilde{B}^{-\top} \Psi_k \right\rangle \\ &+ \frac{1}{2} \left\langle \rho_f \partial_t \hat{u}_N (\tilde{B} \nabla) \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u}_N (\partial_t \tilde{B} \nabla) \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle \\ &+ \frac{1}{2} \left\langle \rho_f \hat{u}_N (\tilde{B} \nabla) \partial_t \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u}_N (\tilde{B} \nabla) \hat{u}_N, \partial_t \tilde{B}^{-\top} \Psi_k \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \partial_t \hat{u}_N (\tilde{B} \nabla) (\tilde{B}^{-\top} \Psi_k), \hat{u}_N \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u}_N (\partial_t \tilde{B} \nabla) (\tilde{B}^{-\top} \Psi_k), \hat{u}_N \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \hat{u}_N (\tilde{B} \nabla) (\partial_t \tilde{B}^{-\top} \Psi_k), \hat{u}_N \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u}_N (\tilde{B} \nabla) (\tilde{B}^{-\top} \Psi_k), \partial_t \hat{u}_N \right\rangle \\ &+ \frac{1}{2} \left\langle \rho_f \partial_t^2 \tilde{h} \partial_t g_N, \psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \partial_t \tilde{h} \partial_t^2 g_N, \psi_k \right\rangle \\ &- \left\langle \rho_f \partial_t^2 \tilde{\chi} (\tilde{B} \nabla) \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle - \left\langle \rho_f \partial_t \tilde{\chi} (\partial_t \tilde{B} \nabla) \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle \\ &- \left\langle \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \partial_t \hat{u}_N, \tilde{B}^{-\top} \Psi_k \right\rangle - \left\langle \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \hat{u}_N, \partial_t \tilde{B}^{-\top} \Psi_k \right\rangle \\ &+ \left\langle \mu (\partial_t \tilde{A} \nabla) \hat{u}_N, \nabla (\tilde{B}^{-\top} \Psi_k) \right\rangle + \left\langle \mu (\tilde{A} \nabla) \partial_t \hat{u}_N, \nabla (\tilde{B}^{-\top} \Psi_k) \right\rangle \\ &+ \left\langle \mu (\tilde{A} \nabla) \hat{u}_N, \nabla (\partial_t \tilde{B}^{-\top} \Psi_k) \right\rangle + \left\langle \rho_s \partial_t^3 g_N, \psi_k \right\rangle - \left\langle \beta \partial_t \partial_x^2 g_N, \psi_k \right\rangle \\ &+ \left\langle \alpha \partial_t \partial_x^2 g_N, \partial_x^2 \psi_k \right\rangle = 0, \end{aligned} \tag{4.8}$$

where (Ψ_k, ψ_k) are introduced in (4.5).

According to the definition of \hat{u}_N and g_N in (4.7), we have

$$\begin{aligned}\partial_t \hat{u}_N &= \sum_{j=1}^N \partial_t \tilde{B}^{-\top} \alpha_N^j(t) \Psi_j + \sum_{j=1}^N \tilde{B}^{-\top} \alpha_N^{j'}(t) \Psi_j, \\ \partial_t^2 \hat{u}_N &= \sum_{j=1}^N \partial_t^2 \tilde{B}^{-\top} \alpha_N^j(t) \Psi_j + \sum_{j=1}^N 2 \partial_t \tilde{B}^{-\top} \alpha_N^{j'}(t) \Psi_j + \sum_{j=1}^N \tilde{B}^{-\top} \alpha_N^{j''}(t) \Psi_j, \\ \nabla (\partial_t \hat{u}_N) &= \sum_{j=1}^N \alpha_N^{j'}(t) \nabla (\tilde{B}^{-\top} \Psi_j) + \sum_{j=1}^N \alpha_N^j(t) \nabla (\partial_t \tilde{B}^{-\top} \Psi_j), \\ \partial_t g_N &= \sum_{j=1}^N \alpha_N^j(t) \psi_j, \quad \partial_t^2 g_N = \sum_{j=1}^N \alpha_N^{j'}(t) \psi_j, \quad \partial_t^3 g_N = \sum_{j=1}^N \alpha_N^{j''}(t) \psi_j.\end{aligned}\tag{4.9}$$

We substitute (4.9) in the system (4.8) and obtain the second-order nonlinear ODE system for $\alpha_N = (\alpha_N^k)_{k=1}^N$ as follows:

$$\begin{aligned}\mathcal{A}(t) \alpha_N''(t) + \mathcal{B}(t) \alpha_N'(t) + \mathcal{C}(t) \alpha_N(t) + (\mathcal{D}(t) \cdot \alpha_N'(t)) \alpha_N(t) \\ + (\mathcal{E}(t) \cdot \alpha_N(t)) \alpha_N(t) = 0,\end{aligned}\tag{4.10}$$

where the coefficients matrices $\mathcal{A} = (A_{j,k})_{j,k=1}^N$, $\mathcal{B} = (B_{j,k})_{j,k=1}^N$, $\mathcal{C} = (C_{j,k})_{j,k=1}^N$, $\mathcal{D} = (D_{j,k}^l)_{j,k,l=1}^N$ and $\mathcal{E} = (E_{j,k}^l)_{j,k,l=1}^N$ are

$$\begin{aligned}A_{j,k} &= \langle \rho_f \tilde{h} \tilde{B}^{-\top} \Psi_j, \tilde{B}^{-\top} \Psi_k \rangle + \langle \rho_s \psi_j, \psi_k \rangle, \\ B_{j,k} &= \langle \rho_f \partial_t \tilde{h} \tilde{B}^{-\top} \Psi_j, \tilde{B}^{-\top} \Psi_k \rangle + 2 \langle \rho_f \tilde{h} \partial_t \tilde{B}^{-\top} \Psi_j, \tilde{B}^{-\top} \Psi_k \rangle \\ &\quad + \langle \rho_f \tilde{h} \tilde{B}^{-\top} \Psi_j, \partial_t \tilde{B}^{-\top} \Psi_k \rangle - \langle \rho_f \partial_t \tilde{\chi} \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \rangle \\ &\quad + \mu \langle \tilde{A} \nabla (\tilde{B}^{-\top} \Psi_j), \nabla (\tilde{B}^{-\top} \Psi_k) \rangle + \frac{1}{2} \langle \rho_f \partial_t \tilde{h} \psi_j, \psi_k \rangle, \\ C_{j,k} &= \langle \rho_f \partial_t \tilde{h} \partial_t \tilde{B}^{-\top} \Psi_j, \tilde{B}^{-\top} \Psi_k \rangle + \langle \rho_f \tilde{h} \partial_t^2 \tilde{B}^{-\top} \Psi_j, \tilde{B}^{-\top} \Psi_k \rangle \\ &\quad + \langle \rho_f \tilde{h} \partial_t \tilde{B}^{-\top} \Psi_j, \partial_t \tilde{B}^{-\top} \Psi_k \rangle - \langle \rho_f \partial_t^2 \tilde{\chi} \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \rangle \\ &\quad - \langle \rho_f \partial_t \tilde{\chi} \partial_t \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \rangle - \langle \rho_f \partial_t \tilde{\chi} \tilde{B} \nabla (\partial_t \tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \rangle \\ &\quad - \langle \rho_f \partial_t \tilde{\chi} \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \partial_t \tilde{B}^{-\top} \Psi_k \rangle + \langle \mu \partial_t \tilde{A} \nabla (\tilde{B}^{-\top} \Psi_j), \nabla (\tilde{B}^{-\top} \Psi_k) \rangle \\ &\quad + \langle \mu \tilde{A} \nabla (\partial_t \tilde{B}^{-\top} \Psi_j), \nabla (\tilde{B}^{-\top} \Psi_k) \rangle + \langle \mu \tilde{A} \nabla (\tilde{B}^{-\top} \Psi_j), \nabla (\partial_t \tilde{B}^{-\top} \Psi_k) \rangle \\ &\quad + \langle \beta \partial_x \psi_j, \partial_x \psi_k \rangle + \langle \alpha \partial_x^2 \psi_j, \partial_x^2 \psi_k \rangle + \frac{1}{2} \langle \rho_f \partial_t^2 \tilde{h} \psi_j, \psi_k \rangle, \\ D_{j,k}^l &= \frac{1}{2} \langle \rho_f \tilde{B}^{-\top} \Psi_l \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \rangle + \frac{1}{2} \langle \rho_f \tilde{B}^{-\top} \Psi_j \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_l), \tilde{B}^{-\top} \Psi_k \rangle \\ &\quad - \frac{1}{2} \langle \rho_f \tilde{B}^{-\top} \Psi_l \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \rangle - \frac{1}{2} \langle \rho_f \tilde{B}^{-\top} \Psi_j \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_k), \tilde{B}^{-\top} \Psi_l \rangle,\end{aligned}$$

$$\begin{aligned} \mathcal{E}_{j,k}^l &= \frac{1}{2} \left\langle \rho_f \partial_t \tilde{B}^{-\top} \Psi_l \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \tilde{B}^{-\top} \Psi_j \partial_t \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_l), \tilde{B}^{-\top} \Psi_k \right\rangle \\ &+ \frac{1}{2} \left\langle \rho_f \tilde{B}^{-\top} \Psi_j \tilde{B} \nabla (\partial_t \tilde{B}^{-\top} \Psi_l), \tilde{B}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \tilde{B}^{-\top} \Psi_j \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_l), \partial_t \tilde{B}^{-\top} \Psi_k \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \partial_t \tilde{B}^{-\top} \Psi_l \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_j), \tilde{B}^{-\top} \Psi_k \right\rangle - \frac{1}{2} \left\langle \rho_f \tilde{B}^{-\top} \Psi_j \partial_t \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_l), \tilde{B}^{-\top} \Psi_k \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \tilde{B}^{-\top} \Psi_j \tilde{B} \nabla (\partial_t \tilde{B}^{-\top} \Psi_l), \tilde{B}^{-\top} \Psi_k \right\rangle - \frac{1}{2} \left\langle \rho_f \tilde{B}^{-\top} \Psi_j \tilde{B} \nabla (\tilde{B}^{-\top} \Psi_k), \partial_t \tilde{B}^{-\top} \Psi_l \right\rangle. \end{aligned}$$

We shall show in what follows that for given initial data $\alpha_N(0)$ and $\alpha_N'(0)$, the differential equation (4.10) allows a unique solution α_N . The choice of the initial value $\alpha_N'(0)$ will be discussed in Step 2 below. We see that the matrix \mathcal{A} is obviously symmetric and for every $\xi \in \mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned} \mathcal{A}\xi \cdot \xi &= \rho_f \tilde{h} \sum_{j,k=1}^N \xi_j \xi_k \left\langle \tilde{B}^{-\top} \Psi_j, \tilde{B}^{-\top} \Psi_k \right\rangle + \rho_s \sum_{j,k=1}^N \xi_j \xi_k \langle \psi_j, \psi_k \rangle \\ &\geq C \left\langle \tilde{B}^{-\top} \sum_{j=1}^N \xi_j \Psi_j, \tilde{B}^{-\top} \sum_{k=1}^N \xi_k \Psi_k \right\rangle + c \sum_{k=1}^N \|\xi_k \psi_k\|^2 \\ &> 0. \end{aligned} \tag{4.11}$$

Recalling that in the given geometry $\tilde{h} \in C^2([0, T] \times [0, L]; [h_{\min}, h_{\max}])$, all the coefficients in (4.10) are continuous in t and all the non-linear quantities in (4.10) are locally Lipschitz continuous in α_N and α_N' . According to the Picard-Lindelöf theorem, there is a unique solution of (4.10) in short time. Therefore, we obtain a solution (\hat{u}_N, g_N) of (4.8) which is given in form of (4.7).

Step 2: Choosing the initial values for the time-derivative. For the initial data, we take

$$\hat{u}_N(0, x, z) = \mathcal{P}_N \hat{u}_0, \quad g_N(0, x) = \mathcal{P}_N h_0, \quad (\partial_t g_N)(0, x) = \mathcal{P}_N h_1, \tag{4.12}$$

where the projection \mathcal{P}_N has been defined in (4.6). From the definition of \hat{u}_N in (4.7), we note that actually $\alpha_N(0)$ is determined by \hat{u}_0 and h_0 , while $\alpha_N'(0)$ is free. With the assumption (4.12) we choose the initial value $\alpha_N'(0)$ by considering the following equality:

$$\begin{aligned} &\left\langle \rho_f h_0 (\partial_t \hat{u}_N)(0), B_{h_0}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \mathcal{P}_N \hat{u}_0 (B_{h_0} \nabla) \mathcal{P}_N \hat{u}_0, B_{h_0}^{-\top} \Psi_k \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \mathcal{P}_N \hat{u}_0 (B_{h_0} \nabla) (B_{h_0}^{-\top} \Psi_k), \mathcal{P}_N \hat{u}_0 \right\rangle + \frac{1}{2} \left\langle \rho_f h_1 \mathcal{P}_N h_1, \psi_k \right\rangle \\ &- \left\langle \rho_f \partial_t \chi_{h_0} (B_{h_0} \nabla) \mathcal{P}_N \hat{u}_0, B_{h_0}^{-\top} \Psi_k \right\rangle + \mu \left\langle (A_{h_0} \nabla) \mathcal{P}_N \hat{u}_0, \nabla (B_{h_0}^{-\top} \Psi_k) \right\rangle \\ &+ \rho_s \left\langle (\partial_t^2 g_N)(0), \psi_k \right\rangle - \beta \left\langle \partial_x^2 \mathcal{P}_N h_0, \psi_k \right\rangle + \alpha \left\langle \partial_x^4 \mathcal{P}_N h_0, \psi_k \right\rangle = 0, \end{aligned} \tag{4.13}$$

where the projection operator \mathcal{P}_N has been introduced in (4.6). Using the expression in (4.9) for $(\partial_t \hat{u}_N)(0)$ and $(\partial_t^2 g_N)(0)$, we immediately obtain from (4.13) the equation for $\alpha_N'(0)$:

$$M \alpha_N'(0) = F(\alpha_N(0), \hat{u}_0, h_0, h_1), \tag{4.14}$$

where the coefficient matrix $M = (M_{j,k})_{j,k=1}^N$ is given by

$$M_{j,k} = \rho_f h_0 \left\langle B_{h_0}^{-\top} \Psi_j, B_{h_0}^{-\top} \Psi_k \right\rangle + \rho_s \langle \psi_j, \psi_k \rangle.$$

By a similar analysis argument of \mathcal{A} in (4.11), we know that the matrix M is invertible. Therefore, the initial value $\alpha_N'(0)$ is uniquely determined by the equality (4.14).

Now we consider obtaining the initial value $(\partial_t \hat{u}_N)(0)$ and $(\partial_t^2 g_N)(0)$ from the assumption of the regularity of \hat{u}_0 , h_0 and h_1 . For this, we have the following proposition.

Proposition 4.2. *With the assumptions in (4.12), let the initial data \hat{u}_0 , h_0 and h_1 satisfy*

$$\hat{u}_0 \in H^2(\Omega_1), \quad h_0 \in H^4(0, L), \quad h_1 \in H^2(0, L), \quad (4.15)$$

then $(\partial_t \hat{u}_N)(0)$ and $(\partial_t^2 g_N)(0)$ is uniformly bounded in $L^2(\Omega_1)$ and $L^2(0, L)$, respectively.

Proof. Still using the structure in (4.9) and recalling the projection \mathcal{P}_N defined in (4.6), we first take an integration by parts with respect to space for the third and the sixth terms in (4.13):

$$\begin{aligned} & - \left\langle \mathcal{P}_N \hat{u}_0 (B_{h_0} \nabla) (B_{h_0}^{-\top} \Psi_k), \mathcal{P}_N \hat{u}_0 \right\rangle \\ &= - \int_{\Omega_1} \operatorname{div} \left((B_{h_0}^{\top} \mathcal{P}_N \hat{u}_0) \otimes B_{h_0}^{-\top} \Psi_k \right) \cdot \mathcal{P}_N \hat{u}_0 \, d\mathbf{z} \\ &= - \int_0^L |\mathcal{P}_N h_1|^2 \psi_k \, dx + \int_{\Omega_1} B_{h_0}^{-\top} \Psi_k \cdot \mathcal{P}_N \hat{u}_0 (B_{h_0} \nabla) \mathcal{P}_N \hat{u}_0 \, d\mathbf{z}, \end{aligned}$$

and

$$\begin{aligned} & \left\langle (A_{h_0} \nabla) \hat{u}_0 : \nabla (B_{h_0}^{-\top} \Psi_k) \right\rangle \\ &= \int_0^L \left[A_{h_0} \nabla \hat{u}_0 B_{h_0}^{-\top} \Psi_k \right] (z=1) \cdot \mathbf{e}_2 \, dx - \int_{\Omega_1} \operatorname{div} (A_{h_0} \nabla \hat{u}_0) \cdot B_{h_0}^{-\top} \Psi_k \, d\mathbf{z}. \end{aligned}$$

Multiply the equality (4.13) by $(\alpha_N^k)'(0)$ and sum over $k = 1, \dots, N$. Note that we have

$$\begin{aligned} & \sum_{k=1}^N \alpha_N^k'(0) \psi_k = (\partial_t^2 g_N)(0), \\ & \sum_{k=1}^N B_{h_0}^{-\top} \alpha_N^k'(0) \Psi_k = B_{h_0}^{-\top} \partial_t \left(B_{h_0}^{\top} \hat{u}_N(0) \right) = B_{h_0}^{-\top} \partial_t B_{h_0}^{\top} \hat{u}_N(0) + (\partial_t \hat{u}_N)(0). \end{aligned}$$

Under the assumption (4.15), with (4.12) and (4.9), we derive that

$$\begin{aligned} & \|(\partial_t \hat{u}_N)(0)\|_{L^2(\Omega_1)}^2 + \|(\partial_t^2 g_N)(0)\|_{L^2(0,L)}^2 \\ & \leq c(C_0) \left(\|\hat{u}_0\|_{H^2(\Omega_1)}^2 + \|h_0\|_{H^4(0,L)}^2 + \|h_1\|_{H^2(0,L)}^2 \right), \end{aligned}$$

which ends the proof. In the above estimate, we used the properties of the projection discussed below (4.6) for initial data. \square

Step 3: Energy estimate for the discrete decoupled solution. Now we derive the energy estimate to extend the existence interval of α_N . Taking integration of (4.8) with respect to time variable and recalling the equation (4.13), we thereby obtain

$$\begin{aligned}
 & \int_{\Omega_1} \rho_f \tilde{h} \partial_t \hat{u}_N \cdot \tilde{B}^{-\top} \Psi_k \, d\mathbf{z} + \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u}_N (\tilde{B} \nabla) \hat{u}_N \cdot \tilde{B}^{-\top} \Psi_k \, d\mathbf{z} \\
 & - \frac{1}{2} \int_{\Omega_1} \rho_f \hat{u}_N (\tilde{B} \nabla) \left(\tilde{B}^{-\top} \Psi_k \right) \cdot \hat{u}_N \, d\mathbf{z} + \frac{1}{2} \int_0^L \rho_f \partial_t \tilde{h} \partial_t g_N \psi_k \, dx \\
 & - \int_{\Omega_1} \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \hat{u}_N \cdot \tilde{B}^{-\top} \Psi_k \, d\mathbf{z} + \mu \int_{\Omega_1} (\tilde{A} \nabla) \hat{u}_N : \nabla \left(\tilde{B}^{-\top} \Psi_k \right) \, d\mathbf{z} \\
 & + \rho_s \int_0^L \partial_t^2 g_N \psi_k \, dx - \beta \int_0^L \partial_x^2 g_N \psi_k \, dx + \alpha \int_0^L \partial_x^2 g_N \partial_x^2 \psi_k \, dx = 0.
 \end{aligned} \tag{4.16}$$

Proposition 4.3. *For the solution to the ODE we have the following energy estimates:*

$$\begin{aligned}
 & h_{\min} \rho_f \|\hat{u}_N\|_{L^\infty(0,T;L^2(\Omega_1))}^2 + \mu c(\tilde{h}) \|\nabla \hat{u}_N\|_{L^2(0,T;L^2(\Omega_1))}^2 \\
 & + \rho_s \|\partial_t g_N\|_{L^\infty(0,T;L^2(0,L))}^2 + \beta \|\partial_x g_N\|_{L^\infty(0,T;L^2(0,L))}^2 + \alpha \|\partial_x^2 g_N\|_{L^\infty(0,T;L^2(0,L))}^2 \\
 & \leq C \left(\|\hat{u}_0\|_{L^2(\Omega_1)} + \|h_1\|_{L^2(0,L)}^2 + \|h_0\|_{H^2(0,L)}^2 \right) =: \tilde{C}.
 \end{aligned} \tag{4.17}$$

The constant \tilde{C} above does not depend on \tilde{h} , h_{\min} , h_{\max} .

Proof. Multiply (4.16) by $\alpha_N^k(t)$ and sum over $k = 1, \dots, N$. Note that

$$\partial_t \tilde{h} = \operatorname{div} (\tilde{B}^\top \partial_t \tilde{\chi}),$$

by using integration by parts we have

$$\begin{aligned}
 \frac{1}{2} \int_{\Omega_1} \rho_f \partial_t \tilde{h} |\hat{u}_N|^2 \, d\mathbf{z} &= \frac{1}{2} \int_{\Omega_1} \rho_f \operatorname{div} (\tilde{B}^\top \partial_t \tilde{\chi}) |\hat{u}_N|^2 \, d\mathbf{z} \\
 &= \frac{1}{2} \int_0^L \rho_f \partial_t \tilde{h} |\partial_t g_N|^2 \, dx - \int_{\Omega_1} \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \hat{u}_N \cdot \hat{u}_N \, d\mathbf{z},
 \end{aligned}$$

which further implies that

$$\begin{aligned}
 & \int_{\Omega_1} \rho_f \tilde{h} \hat{u}_N \cdot \partial_t \hat{u}_N \, d\mathbf{z} + \frac{1}{2} \int_0^L \rho_f \partial_t \tilde{h} |\partial_t g_N|^2 \, dx - \int_{\Omega_1} \rho_f \partial_t \tilde{\chi} (\tilde{B} \nabla) \hat{u}_N \cdot \hat{u}_N \, d\mathbf{z} \\
 & = \frac{1}{2} \frac{d}{dt} \int_{\Omega_1} \rho_f \tilde{h} |\hat{u}_N|^2 \, d\mathbf{z}.
 \end{aligned}$$

Recalling the structure in (4.7) and integrating the equation with respect to time, we immediately obtain the energy balance (4.17). \square

Step 4: Performing the fixed-point theorem. We consider the following map for a fixed point. Let h_{\min} , h_{\max} and T be fixed by the initial conditions (See Remark 2.3).

Similar with α_N introduced in (4.10), here we use the notation $\beta_N := (\beta_N^k)_{k=1}^N$. We take K_N as the set of all vector-valued functions β_N satisfying the following conditions:

$$K_N := \left\{ \begin{array}{l} \beta_N \in C^1([0, T]; \mathbb{R}^N) \left| h_\beta(t, x) := \int_0^t \sum_{k=1}^N \beta_N^k(\tau) \psi_k(x) d\tau + \mathcal{P}_N h_0 \right. \\ \left. \text{such that } h_{\min} \leq h_\beta \leq h_{\max}, \quad \rho_s \|\partial_t h_\beta\|_{L^\infty(0, T; L^2(0, L))}^2 \leq \tilde{C}. \right. \end{array} \right\},$$

where the projection \mathcal{P}_N has been introduced in (4.6). Taking such a function h_β as the geometry, for N given we consider the following map F_N :

$$\begin{aligned} F_N : K_N &\rightarrow C^2([0, T]; \mathbb{R}^N) \\ \beta_N &\mapsto F_N(\beta_N) = \alpha_N, \end{aligned}$$

where α_N is the unique solution to the ODE (4.10). We shall use the Theorem of Schauder for fixed-point argument. For that we have to check the following points:

- The set K_N is convex and closed, as can be seen from its definition.
- The mapping is continuous and compact, as follows directly from classical ODE theory as the system is continuous and solutions are in $C^2[0, T]$ which is a compact subset of $C^1[0, T]$.
- The mapping is onto K_N , because of Remark 2.3 and Proposition 4.3. Indeed the energy estimate allows bounds on $\partial_t g_N$ and $\partial_x^2 g_N$ independent of \tilde{h} . This implies in particular that for T_0 fixed by Remark 2.3 and according choices of h_{\min} and h_{\max} that any function with bounded energy and with given initial values stays in the interval $[h_{\min}, h_{\max}]$.

Hence the map F_N possesses a fixed point in the set K_N . This means we find the existence of a coupled solution (\hat{u}_N, g_N) to the system:

$$\begin{aligned} &\left\langle \rho_f \partial_t g_N \partial_t \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle + \left\langle \rho_f g_N \partial_t^2 \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle + \left\langle \rho_f g_N \partial_t \hat{u}_N, \partial_t B_{g_N}^{-\top} \Psi_k \right\rangle \\ &+ \frac{1}{2} \left\langle \rho_f \partial_t \hat{u}_N (B_{g_N} \nabla) \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u}_N (\partial_t B_{g_N} \nabla) \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle \\ &+ \frac{1}{2} \left\langle \rho_f \hat{u}_N (B_{g_N} \nabla) \partial_t \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle + \frac{1}{2} \left\langle \rho_f \hat{u}_N (B_{g_N} \nabla) \hat{u}_N, \partial_t B_{g_N}^{-\top} \Psi_k \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \partial_t \hat{u}_N (B_{g_N} \nabla) (B_{g_N}^{-\top} \Psi_k), \hat{u}_N \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u}_N (\partial_t B_{g_N} \nabla) (B_{g_N}^{-\top} \Psi_k), \hat{u}_N \right\rangle \\ &- \frac{1}{2} \left\langle \rho_f \hat{u}_N (B_{g_N} \nabla) (\partial_t B_{g_N}^{-\top} \Psi_k), \hat{u}_N \right\rangle - \frac{1}{2} \left\langle \rho_f \hat{u}_N (B_{g_N} \nabla) (B_{g_N}^{-\top} \Psi_k), \partial_t \hat{u}_N \right\rangle \\ &+ \left\langle \rho_f \partial_t^2 g_N \partial_t g_N, \psi_k \right\rangle \\ &- \left\langle \rho_f \partial_t^2 \chi_{g_N} (B_{g_N} \nabla) \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle - \left\langle \rho_f \partial_t \chi_{g_N} (\partial_t B_{g_N} \nabla) \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle \\ &- \left\langle \rho_f \partial_t \chi_{g_N} (B_{g_N} \nabla) \partial_t \hat{u}_N, B_{g_N}^{-\top} \Psi_k \right\rangle \\ &- \left\langle \rho_f \partial_t \chi_{g_N} (B_{g_N} \nabla) \hat{u}_N, \partial_t B_{g_N}^{-\top} \Psi_k \right\rangle \\ &+ \left\langle \mu (\partial_t A_{g_N} \nabla) \hat{u}_N, \nabla (B_{g_N}^{-\top} \Psi_k) \right\rangle \end{aligned}$$

$$\begin{aligned}
 & + \left\langle \mu(A_{g_N} \nabla) \partial_t \hat{u}_N, \nabla(B_{g_N}^{-\top} \Psi_k) \right\rangle \\
 & + \left\langle \mu(A_{g_N} \nabla) \hat{u}_N, \nabla(\partial_t B_{g_N}^{-\top} \Psi_k) \right\rangle + \left\langle \rho_s \partial_t^3 g_N, \psi_k \right\rangle - \left\langle \beta \partial_t \partial_x^2 g_N, \psi_k \right\rangle \\
 & + \left\langle \alpha \partial_t \partial_x^2 g_N, \partial_x^2 \psi_k \right\rangle = 0,
 \end{aligned} \tag{4.18}$$

where $\hat{u}_N(t, x, 1) = \partial_t g_N e_2$ for all $(t, x) \in (0, T) \times (0, L)$.

4.3. The higher order in time-estimate. It can be seen from the definition of the ODE, that multiplying (4.18) with α_N is precisely testing the time-derivative equation with the coupled test-function $(\partial_t^2 g_N, \partial_t \hat{u}_N + G_N)$ with G_N similarly defined as in (3.2), i.e. $G_N = B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \hat{u}_N$. Hence the formal estimate of Sect. 3 can be performed rigorously on the discrete level. Therefore, we have the following theorem. We observe that together with Proposition 4.2 this implies uniform higher order estimates.

Theorem 4.4. *Under the conditions (4.15) there exists a discrete solution to (4.18), that satisfies the first-order equation (4.16) (for the coupled equation with \tilde{h} replaced by g_N), the energy inequality (4.17) and the additional time-derivative estimate:*

$$\begin{aligned}
 & \rho_f \|\partial_t \hat{u}_N\|_{L^\infty(0,T;L^2(\Omega_1))}^2 + \mu \|\nabla \partial_t \hat{u}_N\|_{L^2(0,T;L^2(\Omega_1))}^2 + \rho_s \|\partial_t^2 g_N\|_{L^\infty(0,T;L^2(0,L))} \\
 & + \beta \|\partial_t \partial_x g_N\|_{L^\infty(0,T;L^2(0,L))} + \alpha \|\partial_t \partial_x^2 g_N\|_{L^\infty(0,T;L^2(0,L))} \\
 & \leq C(h_{\min}, \|\hat{u}_0\|_{L^2(\Omega_1)}, \|h_0\|_{H^2(0,L)}, \|h_1\|_{L^2(0,L)}) \\
 & \left(\|(\partial_t \hat{u})(0)\|_{L^2(\Omega_1)}^2 + \|(\partial_t^2 h)(0)\|_{L^2(0,L)}^2 + \|h_1\|_{H^2(0,L)}^2 \right),
 \end{aligned} \tag{4.19}$$

which holds in particular independent of N .

Proof. The existence follows by the fixed point performed in the last subsection. We are left to show the time-derivative estimate of \hat{u}_N , i.e. (4.19). To do this, we first derive the following identities:

$$\begin{aligned}
 & \sum_{k=1}^N \alpha_N^{k'}(t) \psi_k = \partial_t^2 g_N, \\
 & \sum_{k=1}^N B_{g_N}^{-\top} \alpha_N^{k'}(t) \Psi_k = B_{g_N}^{-\top} \partial_t \left(B_{g_N}^{\top} \hat{u}_N \right) = B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \hat{u}_N + \partial_t \hat{u}_N, \\
 & \sum_{k=1}^N \partial_t B_{g_N}^{-\top} \alpha_N^{k'}(t) \Psi_k = \partial_t B_{g_N}^{-\top} \partial_t \left(B_{g_N}^{\top} \hat{u}_N \right) = \partial_t B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \hat{u}_N + \partial_t B_{g_N}^{-\top} B_{g_N}^{\top} \partial_t \hat{u}_N.
 \end{aligned}$$

By using the fact that $\nabla(Av) = (\nabla A)v + A(\nabla v)$, where A is a 2×2 matrix and v is a 2×1 matrix, we also derive that

$$\begin{aligned}
 & \sum_{k=1}^N \alpha_N^{k'}(t) \nabla \left(B_{g_N}^{-\top} \Psi_k \right) = \nabla \left(B_{g_N}^{-\top} \partial_t \left(B_{g_N}^{\top} \hat{u}_N \right) \right) \\
 & = \nabla B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \hat{u}_N + \nabla B_{g_N}^{-\top} B_{g_N}^{\top} \partial_t \hat{u}_N + B_{g_N}^{-\top} \nabla \partial_t B_{g_N}^{\top} \hat{u}_N
 \end{aligned}$$

$$\begin{aligned}
& + B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \nabla \hat{u}_N + B_{g_N}^{-\top} \nabla B_{g_N}^{\top} \partial_t \hat{u}_N + \nabla \partial_t \hat{u}_N \\
\text{and } & \sum_{k=1}^N \alpha_N^{k'}(t) \nabla \left(\partial_t B_{g_N}^{-\top} \Psi_k \right) = \nabla \left(\partial_t B_{g_N}^{-\top} \partial_t \left(B_{g_N}^{\top} \hat{u}_N \right) \right) \\
& = \nabla \partial_t B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \hat{u}_N + \nabla \partial_t B_{g_N}^{-\top} B_{g_N}^{\top} \partial_t \hat{u}_N + \partial_t B_{g_N}^{-\top} \nabla \partial_t B_{g_N}^{\top} \hat{u}_N \\
& \quad + \partial_t B_{g_N}^{-\top} \partial_t B_{g_N}^{\top} \nabla \hat{u}_N + \partial_t B_{g_N}^{-\top} \nabla B_{g_N}^{\top} \partial_t \hat{u}_N + \partial_t B_{g_N}^{-\top} B_{g_N}^{\top} \nabla \partial_t \hat{u}_N.
\end{aligned}$$

Now multiply (4.18) by $\alpha_N^{k'}(t)$, sum over $k = 1, \dots, N$ and substitute the above sum expressions in the resulting equation, which allows to reconstruct (3.7).

First taking the integration with time on $(0, t)$ for $t \in (0, T)$, we note that the terms kept on the left side are

$$\begin{aligned}
& \rho_f \|\partial_t \hat{u}_N\|_{L^\infty(0, T; L^2(\Omega_1))}^2 + \mu \|\nabla \partial_t \hat{u}_N\|_{L^2(0, T; L^2(\Omega_1))}^2 + \rho_s \|\partial_t^2 g_N\|_{L^\infty(0, T; L^2(0, L))} \\
& \quad + \beta \|\partial_t \partial_x g_N\|_{L^\infty(0, T; L^2(0, L))} + \alpha \|\partial_t \partial_x^2 g_N\|_{L^\infty(0, T; L^2(\Omega_1))}^2.
\end{aligned}$$

which precisely corresponds to the left hand side of (3.7). We show in what follows that, multiplying (4.18) by $\alpha_N^{k'}(t)$, the resulting discrete equation also has the same right hand side as (3.7) in the continuous level. For that we first note from the above series structure that $\sum_{k=1}^N B_{g_N}^{-\top} \alpha_N^{k'}(t) \Psi_k$ represents $\partial_t \hat{u}_N + G_N$. With the definition of φ in (3.3), we derive from the formula of G in (3.2) that

$$\varphi = B_h^{-\top} \partial_t B_h^{\top} (\partial_t \hat{u} + B_h^{-\top} \partial_t B_h^{\top} \hat{u}) = -\partial_t B_h^{-\top} \partial_t B_h^{\top} \hat{u} - \partial_t B_h^{-\top} B_h^{\top} \partial_t \hat{u},$$

which corresponds to $\sum_{k=1}^N \partial_t B_{g_N}^{-\top} \alpha_N^{k'}(t) \Psi_k$ in the discrete level. We used in the above the fact that $B_h^{-\top} \partial_t B_h^{\top} = -\partial_t B_h^{-\top} B_h^{\top}$. Hence, the series $\sum_{k=1}^N \alpha_N^{k'}(t) \nabla \left(\partial_t B_{g_N}^{-\top} \Psi_k \right)$ thereby corresponds to $\nabla \varphi$. From here, we realized that the terms depending on φ on the right hand side in (3.7) appear here due to the time derivative of $\partial_t B_{g_N}^{-\top}$, which is precisely related to the pressure.

Comparing the right hand sides of the continuous equation (3.7) with the discrete equation, we realize that, after doing the integration by parts for the solid part, only the convective terms need to be clarified. For that we rewrite the convective terms as it appears in the continuous equation in (3.7), such that it connects to (4.18) multiplied by $\alpha_N^{k'}(t)$. Using the structure (4.2) we derive that

$$\begin{aligned}
& -\frac{1}{2} \int_0^t \int_{\Omega_1} \rho_f \partial_t \hat{u} (B_h \nabla) \hat{u} \cdot (\partial_t \hat{u} + G) \, d\mathbf{z} \\
& = \frac{1}{2} \int_0^t \int_{\Omega_1} \rho_f \partial_t \hat{u} (B_h \nabla) (\partial_t \hat{u} + G) \cdot \hat{u} \, d\mathbf{z} - \frac{1}{2} \int_0^t \rho_f |\partial_t^2 h|^2 \partial_t h \, dx,
\end{aligned}$$

and

$$\begin{aligned}
& -\frac{1}{2} \int_0^t \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \partial_t \hat{u} \cdot (\partial_t \hat{u} + G) \, d\mathbf{z} \\
& = \frac{1}{2} \int_0^t \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) (\partial_t \hat{u} + G) \cdot \partial_t \hat{u} \, d\mathbf{z} - \frac{1}{2} \int_0^t \rho_f |\partial_t^2 h|^2 \partial_t h \, dx,
\end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \int_0^t \int_{\Omega_1} \rho_f \hat{u} (\partial_t B_h \nabla) \hat{u} \cdot (\partial_t \hat{u} + G) \, dz \\
 & = \frac{1}{2} \int_0^t \int_{\Omega_1} \rho_f \hat{u} (\partial_t B_h \nabla) (\partial_t \hat{u} + G) \cdot \hat{u} \, dz,
 \end{aligned}$$

where we observe that no boundary term in the last equality since $\partial_t B_h^\top \hat{u} \otimes \hat{u} (\partial_t \hat{u} + G)$ vanishes at $z = 1$.

Based on the above analysis, we obtain that the resulting equation by multiplying (4.18) by $\alpha_N^{k'}(t)$ is exactly corresponds to the structure (3.7) in the continuous sense. Moreover, (\hat{u}_N, g_N) preserves the energy estimate (4.17). Therefore, the proof follows line by line using the estimates of (3.7). This finishes the proof. \square

4.4. Existence of a strong solution. By Theorem 4.4 together with Proposition 4.2 we got the time-derivative estimate on the discrete level. This allows us to reconstruct a weak solution, with additional regularity properties. Together this implies the existence of a strong solution. With the help of (4.17) and (4.19), we are able to deal with the limit procedure.

Limit passage for \hat{u} and h .

Based on the uniform estimates of Proposition 4.3 and Theorem 4.4, sending $N \rightarrow \infty$, we obtain the following convergence, up to a subsequence, for every $T > 0$,

$$\begin{aligned}
 \hat{u}_N & \rightharpoonup \hat{u} \quad \text{weak-}^* \text{ in } W^{1,\infty}(0, T; L^2(\Omega_1)), \\
 \hat{u}_N & \rightharpoonup \hat{u} \quad \text{weakly in } H^1(0, T; H^1(\Omega_1)), \\
 g_N & \rightharpoonup h \quad \text{weak-}^* \text{ in } W^{1,\infty}(0, T; H^2(0, L)), \\
 \partial_t^2 g_N & \rightharpoonup \partial_t^2 h \quad \text{weak-}^* \text{ in } L^\infty(0, T; L^2(0, L)).
 \end{aligned} \tag{4.20}$$

This implies by compactness that up to a subsequence, we have using the analysis (2.9) that

$$\begin{aligned}
 \hat{u}_N & \rightarrow \hat{u} \quad \text{strongly in } L^2((0, T) \times \Omega_1), \\
 \partial_t g_N & \rightarrow \partial_t h \quad \text{strongly in } C^{0,\alpha}([0, T] \times [0, L]), \\
 g_N & \rightarrow h \quad \text{strongly in } C^1([0, T] \times [0, L]),
 \end{aligned}$$

for $0 < \alpha < \frac{2}{3}$. This allows us to pass to the limit with (4.16) (the version by replacing \tilde{h} by g_N due to the fixed point procedure) and we thereby obtain

$$\begin{aligned}
 & \int_{\Omega_1} \rho_f h \partial_t \hat{u} \cdot B_h^{-\top} \Psi \, dz + \int_{\Omega_1} \rho_f \hat{u} (B_h \nabla) \hat{u} \cdot B_h^{-\top} \Psi \, dz \\
 & - \int_{\Omega_1} \rho_f \partial_t \chi_h (B_h \nabla) \hat{u} \cdot B_h^{-\top} \Psi \, dz + \mu \int_{\Omega_1} (A_h \nabla) \hat{u} : \nabla (B_h^{-\top} \Psi) \, dz \\
 & + \rho_s \int_0^L \partial_t^2 h \psi \, dx - \beta \int_0^L \partial_x^2 h \psi \, dx + \alpha \int_0^L \partial_x^2 h \partial_x^2 \psi \, dx = 0,
 \end{aligned}$$

for all sufficiently smooth $\psi(t, x) = \Psi(t, x, 1)$ with $\text{div } \Psi = 0$.

Regularity of u , p and h .

Here we rely on the properties of the steady Stokes equation. Let us consider the steady Stokes system

$$\begin{cases} \Delta u - \nabla \pi = f, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega_h} = u_\partial, \end{cases} \tag{4.21}$$

in a domain $\Omega_h \subset \mathbb{R}^2$ with unit normal \mathbf{n} . The result given in the following theorem is a maximal regularity estimate for the solution of (4.21) in terms of the right-hand side. We quote here the Stokes estimate from [11, Theorem 3.1] and [12, Theorem 2.8] for the system (4.21).

Theorem 4.5. *Let $q \in (1, \infty)$, $s \geq 1 + \frac{1}{q}$ and*

$$\varrho \geq q \text{ if } q(s - 1) \geq 2; \quad \varrho \geq \frac{2q}{q(s-1)-1} \text{ if } q(s - 1) < 2,$$

such that $2(\frac{1}{q} - \frac{1}{2}) + 1 \leq s$. Suppose that Ω_h is a $B_{\varrho,q}^\theta$ -domain³ and $h \in L^\infty(\bar{I}; B_{\varrho,q}^\theta \cap C^1[0, L])$ for some $\theta > s - 1/q$ with locally small Lipschitz constant, $f \in W^{s-2,q}(\Omega_h)$ and $u_\partial \in W^{s-1/q,q}(\partial\Omega_h)$ with $\int_{\partial\Omega_h} u_\partial \cdot \mathbf{n} dS = 0$. Then there is a unique solution to (4.21) with $\int_{\Omega_h} \pi dx = 0$ that satisfies

$$\|u\|_{W^{s,q}(\Omega_h)} + \|\pi\|_{W^{s-1,q}(\Omega_h)} \lesssim \|f\|_{W^{s-2,q}(\Omega_h)} + \|u_\partial\|_{W^{s-1/q,q}(\partial\Omega_h)}. \tag{4.22}$$

Remark 4.6. We remark here that the above result in [11] and [12] is stated in a general $B_{\varrho,q}^\theta$ -domain \mathcal{O} . It is also available for the moving domain Ω_h with $h \in L^\infty(\bar{I}; B_{\varrho,q}^\theta \cap C^1[0, L])$, and it suffices to verify that $\partial\Omega_h \in B_{\varrho,q}^\theta$. This has been explained in [11, Remark 3.4] and [12, Remark 2.9] in a detailed way.

By the weak lower semi-continuity, we derive from (4.20) that (\hat{u}, h) also satisfy the energy balance (4.17) and the time-derivative estimate (4.19). We will explain in the following that this, together with Lemma 4.5, implies more regularity for \hat{u} . Moreover, with the help of Proposition 2.1, we could derive more regularity directly for u .

Theorem 4.7. *Assume that the initial data (u_0, h_0, h_1) satisfy $h_0 \in H^4(0, L)$, $h_1 \in H^2(0, L)$, $u_0 \in H^2(\Omega_{h_0})$ and $\min_{x \in (0, L)} h_0 = \delta > 0$, then the system (1.1)–(1.7) admits a unique solution (u, p, h) , for $T > 0$, satisfying*

$$\begin{aligned} h &\in W^{1,\infty}(0, T; H^2(0, L)) \cap W^{2,\infty}(0, T; L^2(0, L)) \cap L^\infty(0, T; H^4(0, L)), \\ u &\in L^\infty(0, T; H^2(\Omega_h)) \cap L^4(0, T; H^{\frac{5}{2}}(\Omega_h)), \\ p &\in L^\infty(0, T; H^1(\Omega_h)) \cap L^4(0, T; H^{\frac{3}{2}}(\Omega_h)). \end{aligned} \tag{4.23}$$

This solution exists until the self-intersection of Ω_h .

Proof. The regularity of h can be obtained directly from the estimate (4.17) and (4.19) that

$$h \in W^{1,\infty}(0, T; H^2(0, L)) \cap W^{2,\infty}(0, T; L^2(0, L)).$$

For the initial data, recalling that $\hat{u}_0(x, z) = u_0(x, h_0 z) = u_0(x, y)$, we notice from Proposition 2.1 that $\|\hat{u}_0\|_{H^2(\Omega_1)}$ is equivalent to $\|u_0\|_{H^2(\Omega_{h_0})}$. This implies that the initial

³ For bounded domain \mathcal{O} , the Besov spaces are defined as $B_{\varrho,q}^\theta(\mathcal{O}) := \{f|_{\mathcal{O}} : f \in B_{\varrho,q}^\theta(\mathbb{R}^n)\}$ with the norm $\|g\|_{B_{\varrho,q}^\theta(\mathcal{O})} = \inf\{\|f\|_{B_{\varrho,q}^\theta(\mathbb{R}^n)} : f|_{\mathcal{O}} = g\}$.

conditions (4.15) is satisfied. The pressure is constructed in two parts. For that we decompose it into

$$p(t, x) = p_0(t, x) + p_1(t) \text{ such that } \int_{\Omega_{h(t)}} p_0(t, x) \, dx = 0. \tag{4.24}$$

Now p_0 is directly constructed via Stokes equation, while p_1 is constant in space.

To present the proof clearly, we divide it in the following steps.

Step 1: We show that $u \in L^\infty(0, T; W^{2,r}(\Omega_h))$ for every $1 < r < 2$. Note that we have the regularity for \hat{u} :

$$\hat{u} \in W^{1,\infty}(0, T; L^2(\Omega_1)) \cap H^1(0, T; H^1(\Omega_1))$$

from the weak convergence (4.20). This gives us that $\nabla \hat{u} \in L^\infty(0, T; L^2(\Omega_1))$. From Proposition 2.1 we obtain that $\nabla u \in L^\infty(0, T; L^2(\Omega_h))$. Using Hölder’s inequality and Sobolev embedding $H^1(\Omega_h) \hookrightarrow L^{\frac{2r}{2-r}}(\Omega_h)$, for every $1 < r < 2$ we have

$$\|u \cdot \nabla u\|_{L^r(\Omega_h)} \lesssim \|u\|_{L^{\frac{2r}{2-r}}(\Omega_h)} \|\nabla u\|_{L^2(\Omega_h)} \lesssim \|\nabla u\|_{L^2(\Omega_h)}^2.$$

Hence we obtain that $u \cdot \nabla u \in L^\infty(0, T; L^r(\Omega_h))$ for every $1 < r < 2$.

Now we apply the Stokes estimate (4.22) for $s = 2, q = r \in (1, 2), f = \partial_t u + u \cdot \nabla u$ and $u_\partial = \partial_t h$. Thereby we have

$$\begin{aligned} \|u\|_{W^{2,r}(\Omega_h)} + \|p_0\|_{W^{1,r}(\Omega_h)} &\lesssim \|\partial_t u + u \cdot \nabla u\|_{L^r(\Omega_h)} + \|\partial_t h\|_{W^{2-\frac{1}{r},r}(0,L)} \\ &\lesssim \|\partial_t u\|_{L^r(\Omega_h)} + \|u \cdot \nabla u\|_{L^r(\Omega_h)} + \|\partial_t h\|_{W^{2-\frac{1}{r},r}(0,L)}. \end{aligned} \tag{4.25}$$

Note that $\partial_t \hat{u} \in L^\infty(0, T; L^2(\Omega_1))$, then through Proposition 2.1 we get that $\partial_t u \in L^\infty(0, T; L^2(\Omega_h))$. Moreover, recall that $\partial_t h \in L^\infty(0, T; H^2(0, L))$, using the continuous embedding $H^2(0, L) \hookrightarrow W^{2-\frac{1}{r},r}(0, L)$, we derive from (4.25) that $u \in L^\infty(0, T; W^{2,r}(\Omega_h))$ for $1 < r < 2$.

Step 2: We prove qualitatively that $u \in L^\infty(0, T; H^2(\Omega_h))$ and $p \in L^\infty(0, T; H^1(\Omega_h))$. Using the regularity $u \in L^\infty(0, T; W^{2,r}(\Omega_h))$ and the continuous embedding $W^{2,r}(\Omega_h) \hookrightarrow L^\infty(\Omega_h)$ for $1 < r < 2$, we have $u \in L^\infty((0, T) \times \Omega_h)$. Moreover, we note that

$$\|u \cdot \nabla u\|_{L^2(\Omega_h)} \lesssim \|u\|_{L^\infty(\Omega_h)} \|\nabla u\|_{L^2(\Omega_h)}.$$

This, together with $\nabla u \in L^\infty(0, T; L^2(\Omega_h))$, gives us that $u \cdot \nabla u \in L^\infty(0, T; L^2(\Omega_h))$. At this stage, we apply the Stokes estimate (4.22) again for $s = 2 = q$ and obtain

$$\begin{aligned} \|u\|_{H^2(\Omega_h)} + \|p_0\|_{H^1(\Omega_h)} &\lesssim \|\partial_t u + u \cdot \nabla u\|_{L^2(\Omega_h)} + \|\partial_t h\|_{H^{\frac{3}{2}}(0,L)} \\ &\lesssim \|\partial_t u\|_{L^2(\Omega_h)} + \|u \cdot \nabla u\|_{L^2(\Omega_h)} + \|\partial_t h\|_{H^{\frac{3}{2}}(0,L)}. \end{aligned} \tag{4.26}$$

Since we already notice that $\partial_t u \in L^\infty(0, T; L^2(\Omega_h))$, then (4.26) shows that $u \in L^\infty(0, T; H^2(\Omega_h))$ and $p_0 \in L^\infty(0, T; H^1(\Omega_h))$.

Based on the decomposition (4.24), now we define p_1 in such a way that (1.2) is satisfied.

Recall that we have the zero mean condition (2.1) for the source term $\phi(u, p, h)$ defined in (1.4). This, together with (4.24), gives us that $\sigma(u, p) = \sigma(u, p_0) - p_1 \mathbb{1}_{2 \times 2}$ and thereby we have

$$Lp_1(t) = \int_0^L e_2 \cdot \sigma(u, p_0)(t, x, h(t, x))(-\partial_x h e_1 + e_2) dx \in L^\infty(0, T).$$

Hence, with the current regularity of u and p , we derive that the Cauchy stress tensor $\sigma(u, p) \in L^\infty(0, T; H^1(\Omega_h))$. Thereby the source term of the structure $\phi(u, p, h)$ belongs to $L^\infty(0, T; H^{\frac{1}{2}}(0, L))$. Based on the beam equation (1.2) and the regularity of h , we finally have $\partial_x^4 h \in L^\infty(0, T; L^2(0, L))$. Therefore, the strong solution (u, p, h) of the system (1.1)–(1.7) is established in both space and time.

Step 3: We have further regularity: $u \in L^4(0, T; H^{\frac{5}{2}}(\Omega_h))$ and $p \in L^4(0, T; H^{\frac{3}{2}}(\Omega_h))$. Recalling the regularity of \hat{u} , we have $\partial_t \hat{u} \in L^\infty(0, T; L^2(\Omega_1)) \cap L^2(0, T; H^1(\Omega_1))$. With the result in Step 2, we know that $u \in L^\infty(0, T; H^2(\Omega_h))$ and then by Proposition 2.1 we get $\partial_t u \in L^\infty(0, T; L^2(\Omega_h)) \cap L^2(0, T; H^1(\Omega_h))$ as well. By the interpolation inequality:

$$\|\partial_t u\|_{L^4(0, T; H^{\frac{1}{2}}(\Omega_h))} \lesssim \|\partial_t u\|_{L^\infty(0, T; L^2(\Omega_h))}^{\frac{1}{2}} \|\partial_t u\|_{L^2(0, T; H^1(\Omega_h))}^{\frac{1}{2}}, \tag{4.27}$$

we obtain further that $\partial_t u \in L^4(0, T; H^{\frac{1}{2}}(\Omega_h))$. Now we use the Stokes estimate (4.22) for $s = \frac{5}{2}$ and $q = 2$ once more and have

$$\begin{aligned} \|u\|_{H^{\frac{5}{2}}(\Omega_h)} + \|p_0\|_{H^{\frac{3}{2}}(\Omega_h)} &\lesssim \|\partial_t u + u \cdot \nabla u\|_{H^{\frac{1}{2}}(\Omega_h)} + \|\partial_t h\|_{H^2(0, L)} \\ &\lesssim \|\partial_t u\|_{H^{\frac{1}{2}}(\Omega_h)} + \|u \cdot \nabla u\|_{H^{\frac{1}{2}}(\Omega_h)} + \|\partial_t h\|_{H^2(0, L)} \\ &\lesssim \|\partial_t u\|_{H^{\frac{1}{2}}(\Omega_h)} + \|u\|_{L^\infty(\Omega_h)} \|\nabla u\|_{H^{\frac{1}{2}}(\Omega_h)} + \|\partial_t h\|_{H^2(0, L)}. \end{aligned} \tag{4.28}$$

Combining with the above analysis, we obtain from (4.28) that

$$\begin{aligned} &\|u\|_{L^4(0, T; H^{\frac{5}{2}}(\Omega_h))} + \|p_0\|_{L^4(0, T; H^{\frac{3}{2}}(\Omega_h))} \\ &\lesssim \|\partial_t u\|_{L^4(0, T; H^{\frac{1}{2}}(\Omega_h))} + C(T) \left(\|u\|_{L^\infty(0, T; H^2(\Omega_h))} + \|\partial_t h\|_{L^\infty(0, T; H^2(0, L))} \right). \end{aligned}$$

Using the decomposition argument again, we obtain the regularity of p .

Putting the above results together, we conclude that the regularity in (4.23) holds, which ends the proof. \square

4.5. Proof of Theorem 1.1. Now we are in a position to finish the proof of Theorem 1.1. The existence of a strong solution was obtained in Theorem 4.7. Hence we are left to show *quantitatively* that the estimate is linear with respect to the higher order norms of the initial conditions.

Beginning from Step 2 in the proof of Theorem 4.7, we derive the following estimate of u in the spaces $L^\infty(0, T; H^2(\Omega_h))$ and $L^4(0, T; H^{\frac{5}{2}}(\Omega_h))$, which is linear with respect to the higher order norms of the initial conditions. Based on the inequality (4.26), we have

$$\|u\|_{L^\infty(0, T; H^2(\Omega_h))}^2 + \|p_0\|_{L^\infty(0, T; H^1(\Omega_h))}^2$$

$$\lesssim \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \|u \cdot \nabla u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \|\partial_t h\|_{L^\infty(0,T;H^{\frac{3}{2}}(0,L))}^2.$$

Note that for the convective term above, we estimate by using Ladyzhenskaya inequality that

$$\begin{aligned} \|u \cdot \nabla u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 &\leq \|u\|_{L^\infty(0,T;L^4(\Omega_h))}^2 \|\nabla u\|_{L^\infty(0,T;L^4(\Omega_h))}^2 \\ &\lesssim \|u\|_{L^\infty(0,T;L^2(\Omega_h))} \|\nabla u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 \|\nabla^2 u\|_{L^\infty(0,T;L^2(\Omega_h))} \\ &\leq \varepsilon \|\nabla^2 u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 \\ &\quad + C(\varepsilon) \|u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 \|\nabla u\|_{L^\infty(0,T;L^2(\Omega_h))}^4. \end{aligned}$$

According to Proposition 2.1, we know that $\|\nabla u\|_{L^\infty(0,T;L^2(\Omega_h))} \sim \|\nabla \hat{u}\|_{L^\infty(0,T;L^2(\Omega_1))}$. Then we consider the following interpolation inequality:

$$\|\nabla \hat{u}\|_{L^\infty(0,T;L^2(\Omega_1))} \lesssim \|\nabla \hat{u}\|_{L^2(0,T;L^2(\Omega_1))}^{\frac{1}{2}} \|\nabla \hat{u}\|_{H^1(0,T;L^2(\Omega_1))}^{\frac{1}{2}} \lesssim C_0^{\frac{1}{4}} \|\nabla \hat{u}\|_{H^1(0,T;L^2(\Omega_1))}^{\frac{1}{2}},$$

This gives us that

$$\|u \cdot \nabla u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 \leq \varepsilon \|\nabla^2 u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + C(\varepsilon) C_0^2 \|\nabla \hat{u}\|_{H^1(0,T;L^2(\Omega_1))}^2, \tag{4.29}$$

where we used the energy balance (2.8). Combining with the above estimate, we derive from Proposition 4.2 and Theorem 4.4 that

$$\begin{aligned} &\|u\|_{L^\infty(0,T;H^2(\Omega_h))}^2 + \|p_0\|_{L^\infty(0,T;H^1(\Omega_h))}^2 \\ &\leq c(C_0) \left(\|\hat{u}_0\|_{H^2(\Omega_1)}^2 + \|h_0\|_{H^4(0,L)}^2 + \|h_1\|_{H^2(0,L)}^2 + 1 \right). \end{aligned}$$

Similarly, we can derive a ‘linear estimate’ from (4.28). We consider

$$\begin{aligned} &\|u\|_{L^4(0,T;H^{\frac{5}{2}}(\Omega_h))}^4 + \|p_0\|_{L^4(0,T;H^{\frac{3}{2}}(\Omega_h))}^4 \\ &\lesssim \|\partial_t u\|_{L^4(0,T;H^{\frac{1}{2}}(\Omega_h))}^4 + \|u \cdot \nabla u\|_{L^4(0,T;H^{\frac{1}{2}}(\Omega_h))}^4 + \|\partial_t h\|_{L^2(0,T;H^2(0,L))}^4. \end{aligned} \tag{4.30}$$

By using the interpolation (4.27), we deal with the convective term again as follows:

$$\|u \cdot \nabla u\|_{L^4(0,T;H^{\frac{1}{2}}(\Omega_h))}^4 \lesssim \|u \cdot \nabla u\|_{L^\infty(0,T;L^2(\Omega_h))}^2 \|u \cdot \nabla u\|_{L^2(0,T;H^1(\Omega_h))}^2. \tag{4.31}$$

Observe that

$$\|u \cdot \nabla u\|_{L^2(0,T;H^1(\Omega_h))}^2 \lesssim \int_0^T \left(\|\nabla u\|_{L^2(\Omega_h)}^4 + \|u \cdot \nabla^2 u\|_{L^2(\Omega_h)}^2 \right) dt := \text{I} + \text{II}.$$

Now we estimate I and II, respectively. We first have

$$\begin{aligned} \text{I} &\lesssim \int_0^T \|u\|_{L^2(\Omega_h)}^{\frac{12}{5}} \|\nabla u\|_{H^{\frac{3}{2}}(\Omega_h)}^{\frac{8}{5}} dt \\ &\lesssim \left(\int_0^T \|u\|_{L^2(\Omega_h)}^4 dt \right)^{\frac{3}{5}} \left(\int_0^T \|\nabla u\|_{H^{\frac{3}{2}}(\Omega_h)}^4 dt \right)^{\frac{2}{5}}, \end{aligned}$$

where we used Hölder’s inequality and the interpolation inequality:

$$\|\nabla u\|_{L^2(\Omega_h)} \lesssim \|\nabla u\|_{H^{-1}(\Omega_h)}^{\frac{3}{5}} \|\nabla u\|_{H^{\frac{3}{2}}(\Omega_h)}^{\frac{2}{5}}.$$

For II, we also have

$$\begin{aligned} \text{II} &\leq \int_0^T \|u\|_{L^4(\Omega_h)}^2 \|\nabla^2 u\|_{L^4(\Omega_h)}^2 dt \\ &\lesssim \int_0^T \|u\|_{L^2(\Omega_h)} \|\nabla u\|_{L^2(\Omega_h)} \|\nabla^2 u\|_{H^{\frac{1}{2}}(\Omega_h)}^2 dt \\ &\leq \left(\int_0^T \|\nabla^2 u\|_{H^{\frac{1}{2}}(\Omega_h)}^4 dt \right)^{\frac{1}{2}} \left(\int_0^T \|\nabla u\|_{L^2(\Omega_h)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

This implies using (4.29) and (4.31) that

$$\begin{aligned} &\|u \cdot \nabla u\|_{L^4(0,T;H^{\frac{1}{2}}(\Omega_h))}^4 \\ &\lesssim \left(\left(\int_0^T \|\nabla^2 u\|_{H^{\frac{1}{2}}(\Omega_h)}^4 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|\nabla u\|_{H^{\frac{3}{2}}(\Omega_h)}^4 dt \right)^{\frac{2}{5}} \right) \\ &\quad \times \|\nabla^2 u\|_{L^\infty(0,T;L^2(\Omega_h))} \|\nabla \hat{u}\|_{H^1(0,T;L^2(\Omega_1))} \\ &\leq \varepsilon \int_0^T \|u\|_{H^{\frac{5}{2}}(\Omega_h)}^4 dt + \left(\|\hat{u}_0\|_{H^2(\Omega_1)} + \|h_0\|_{H^4(0,L)} + \|h_1\|_{H^2(0,L)} + 1 \right)^4. \end{aligned}$$

Hence, we conclude from (4.30) and the above that

$$\begin{aligned} &\|u\|_{L^4(0,T;H^{\frac{5}{2}}(\Omega_h))}^4 + \|p_0\|_{L^4(0,T;H^{\frac{3}{2}}(\Omega_h))}^4 \\ &\leq c(C_0) \left(\|\hat{u}_0\|_{H^2(\Omega_1)}^4 + \|h_0\|_{H^4(0,L)}^4 + \|h_1\|_{H^2(0,L)}^4 + 1 \right). \end{aligned}$$

This finishes the proof of Theorem 1.1.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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