

# On the regularity of systems of dispersive partial differential equations

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**Abstract** We extend and improve a regularity result that shows polynomial growth in time of solutions of certain systems of partial differential equations that model dispersive equations interacting with a general class of multilinear operators. This is done by a general multilinearisation method that extends estimates on linear oscillatory integral operators to the case of multilinear operators.

## 1. Introduction

We consider the multilinear operator  $T_\sigma$  defined by

$$T_\sigma(f_1, \dots, f_N)(x) = \int_{\mathbb{R}^{Nn}} \sigma(\Xi) \prod_{j=1}^N e^{ix \cdot \xi_j} \widehat{f}_j(\xi_j) \, d\Xi,$$

where  $\Xi = (\xi_1, \dots, \xi_N)$  and  $\sigma$  is a smooth multivariate symbol on  $\mathbb{R}^{Nn}$ , belonging for example to some  $S_{1,0}^m(n, N)$ -class. This is thus a pseudoproduct, which includes the pointwise product,  $f_1 f_2$  (where  $m=0$ ), product with derivatives  $\partial^\alpha f_1 \partial^\beta f_2$  (where  $m = |\alpha + \beta|$ ), and many others.

Consider now the equations

$$(1.1) \quad \begin{cases} i\partial_t u - \rho_0(D)u = T_\sigma(v_1, \dots, v_N) \\ i\partial_t v_j - \rho_j(D)v_j = 0, \quad j=1, \dots, N, \end{cases}$$

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where  $\rho_j$  are homogeneous of degree  $\lambda > 0$ , with Cauchy data

$$\begin{cases} u(0, x) = 0 \\ v_j(0, x) = f_j(x), \quad j = 1, \dots, N. \end{cases}$$

The functions  $u$ ,  $v_j$  and  $f_j$  map  $\mathbb{R}^n$  to  $\mathbb{C}$ , and  $\rho_j(D)$  is the Fourier multiplier

$$\rho_j(D)f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \rho_j(\xi) \hat{f}(\xi) \, d\xi,$$

with  $d\xi = (2\pi)^{-n} d\xi$ . Depending on the parameter  $\lambda$ , the above system is used in order to study the nonlinear interaction of various free solutions, such as quantum mechanical waves ( $\lambda = 2$ ), as a first step towards understanding a nonlinear problem of the type  $i\partial_t u - \rho(D)u = F(u)$ , with a suitable nonlinearity  $F$ .

Here the main question is: if the data  $f_j$  belong to certain Banach spaces, then what is the regularity of the solution  $u$ ? This system has been studied in the case  $N = 2$  and  $n = 1, 2, 3$  by F. Bernicot and P. Germain in [4], [5], see also [3]. They used an amplitude  $\sigma$  with compact support and were able to produce global-in-time estimates of  $u$ .

In [1], A. Bergfeldt, S. Rodríguez-López, D. Rule and W. Staubach studied this system for general  $\sigma \in S_{1,0}^{m_\sigma}(n, N)$ ,  $m_\sigma \in \mathbb{R}$  (this class is specified in Section 4), and showed that the solution  $u$  obeys

$$\|u\|_{L^q([0, T], H^{s_0, p_0}(\mathbb{R}^n))} \leq C_T \prod_{j=1}^N \|f_j\|_{H^{s_j, p_j}(\mathbb{R}^n)},$$

where  $q \in [1, \infty]$  and  $p_0, \dots, p_N \in (1, \infty)$  with  $1/p_0 = \sum_{j=1}^N 1/p_j$  and

$$0 \leq s_0 \leq -m_\sigma + \sum_{j=0}^N m_c(p_j, \lambda) + \min_{1 \leq j \leq N} s_j,$$

where

$$(1.2) \quad m_c(p, \lambda) = \begin{cases} -n\lambda \left| \frac{1}{p} - \frac{1}{2} \right| & \text{when } \lambda \neq 1 \\ -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right| & \text{when } \lambda = 1. \end{cases}$$

The method used in [1] relies on proving boundedness results for multilinear oscillatory integral operators, and is in turn based on extensions of the Coifman–Meyer methods in [7] to the case of multilinear operators with nonlinear phase functions.

In the present paper, we are utilising a different method to produce the required multilinear Sobolev space estimates. This allows us to improve this estimate with respect to the range of  $s_0$  above, and we also cover all end-point cases where  $p=1$  or  $p=\infty$ . We have the following theorem.

**Theorem 1.1.** *Let  $p_0, \dots, p_N \in [1, \infty]$  with  $1/p_0 = \sum_{j=1}^N 1/p_j$  and let  $s_0, \dots, s_N \in \mathbb{R}$  satisfy, for some  $\varepsilon > 0$ ,*

$$0 \leq s_0 \leq -m_\sigma + m_c(p_0, \lambda) + \min_{1 \leq j \leq N} \left[ s_j + m_c(p_j, \lambda) + \sum_{\substack{1 \leq \ell \leq N \\ \ell \neq j}} \min(s_\ell + m_c(p_\ell, \lambda), -\varepsilon \gamma_j) \right],$$

where  $m_c$  is as in equation (1.2) and  $\gamma_j = 0$  if  $p_j = 2$  and otherwise equal to 1. Then the solution  $u$  of system (1.1) satisfies the following bound uniformly in  $f_1 \in X^{p_1}, \dots, f_N \in X^{p_N}$ :

$$\|\langle D \rangle^{s_0} u(t, \cdot)\|_{X^{p_0}(\mathbb{R}^n)} \leq P(t) \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}(\mathbb{R}^n)},$$

where  $P$  is a polynomial. The spaces  $X^p$  are defined as

$$X^p = \begin{cases} h^1 & \text{if } p=1 \\ L^p & \text{if } 1 < p < \infty \\ \text{bmo} & \text{if } p=\infty. \end{cases}$$

Here,  $h^1$  is the local Hardy space of power 1 and  $\text{bmo}$  is the local variant of the space of functions of bounded mean oscillation. With  $\langle D \rangle^s$  we mean the Bessel potential of order  $s$ , that is  $\langle D \rangle^s f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \, d\xi$ . This means in particular that  $\|\langle D \rangle^s f\|_{X^p} = \|f\|_{H^{s,p}}$  when  $1 < p < \infty$ , where  $H^{s,p}$  is the  $L^p$ -based Sobolev space, and in general for  $p \in [1, \infty]$  that  $\|\langle D \rangle^s f\|_{X^p} = \|f\|_{F_{p,2}^s}$  where  $F_{p,q}^s$  are the Triebel–Lizorkin spaces (see [13, Theorem 1.22]).

**Remark 1.1.** The condition on  $s_0$  in Theorem 1.1 can be written

$$0 \leq s_0 \leq -m_\sigma + \sum_{j=0}^N m_c(p_j, \lambda) + \min_{1 \leq j \leq N} \left[ s_j + \sum_{\substack{1 \leq \ell \leq N \\ \ell \neq j}} \min(s_\ell, -m_c(p_\ell, \lambda) - \varepsilon \gamma_j) \right].$$

When not in the situation  $N=2$  and  $p_1=p_2=2$ , the quantity

$$\sum_{\substack{1 \leq \ell \leq N \\ \ell \neq j}} \min(s_\ell, -m_c(p_\ell, \lambda) - \varepsilon \gamma_j)$$

is always positive for some  $j$  and sufficiently small  $\varepsilon$ , and hence this is indeed an improvement of previous results also when  $p_j \in (1, \infty)$ .

In order to achieve this result, we extend the regularity results about multilinear oscillatory integral operators in [1] and [2]. The work in [1] treats operators connected to a wide class of differential equations, in the absence of space dependent potentials. The typical examples are phases of the form  $\varphi(x, \xi) = x \cdot \xi + |\xi|^\lambda$  for  $\lambda > 0$ . This theory was then extended to Sobolev spaces and applied to systems of partial differential equations.

The results in [2] are concerned with operators that are at most quadratic in frequency in the phase, but allow for a nontrivial space dependency. The typical example is the Schrödinger equation harmonic oscillator phase  $\varphi(x, \xi) = x \cdot \xi + |\xi|^2 + |x|^2$ . The techniques used in these two works are however different, and the origin of the present work was the observation that the methods of [2] are more flexible in how growth and decay in frequency can be distributed between the functions that the multilinear operator acts on.

As an example of this flexibility, let us consider the Kato–Ponce inequality

$$\|\langle D \rangle^s (fg)\|_{L^{p_0}} \leq \|\langle D \rangle^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|\langle D \rangle^s g\|_{L^{p_2}} \lesssim \|\langle D \rangle^s f\|_{L^{p_1}} \|\langle D \rangle^s g\|_{L^{p_2}},$$

with for example  $p \in (1, \infty)$  and  $1/p_0 = 1/p_1 + 1/p_2$ . Notice that for the combined estimate, we have to have  $s$  extra derivatives (i. e. Bessel potentials) on the right hand side in this multilinear estimate. Recall also (see e. g. [6]) that there is a nonpositive number  $m_c(p)$  that counts the number of derivatives that are lost for the linear oscillatory integral operator. That is, if  $\sigma \in S_{1,0}^{m_\sigma}$ , then

$$\|\langle D \rangle^{s_0} T_\sigma^\varphi f\|_{L^p} \lesssim \|\langle D \rangle^{s_1} f\|_{L^p} \quad \text{if } s_0 + m_\sigma \leq s_1 + m_c(p).$$

Then the question is, what shall we expect for the multilinear operator? In [1], the best that could be shown was that, similar to the Kato–Ponce inequality, we will have an estimate of the form

$$\begin{aligned} \|\langle D \rangle^{s_0} T_\sigma^\Phi(f, g)\|_{L^p} &\lesssim \|\langle D \rangle^{s_1} f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|\langle D \rangle^{s_1} f\|_{L^{p_2}} \\ &\lesssim \|\langle D \rangle^{s_1} f\|_{L^{p_1}} \|\langle D \rangle^{s_1} g\|_{L^{p_2}} \end{aligned}$$

if  $s_0 + m_\sigma \leq m_c(p_1) + m_c(p_2) + s_1$ . In the present work, we show that if the orders  $s_1, s_2$  are smaller than  $|m_c(p_1)|, |m_c(p_2)|$  then we have in fact the improved bound

$$\|\langle D \rangle^{s_0} T_\sigma^\Phi(f, g)\|_{L^{p_0}} \lesssim \|\langle D \rangle^{s_1} f\|_{L^{p_1}} \|\langle D \rangle^{s_2} g\|_{L^{p_2}}$$

if  $s_0 + m_\sigma \leq m_c(p_1) + m_c(p_2) + s_1 + s_2$ . In this case, we therefore do not need extra derivatives on the right hand side.

Building upon the framework developed in [2], this property follows directly by distributing the growth or decay in frequency to the corresponding amplitudes, as seen under *Treatment of  $\Sigma_a$*  in Section 5. The presentation there in turn hinges on

the particular frequency decomposition into high frequency and reducible regions that was developed for [2] and differs from the more standard one used in [1]. In particular, one can use simpler amplitudes independent of the frequency parameter  $k$  and general results such as estimate (2.5) in Lemma 2.1. These techniques do however rely on duality arguments and the properties of  $h^1$  and  $\text{bmo}$ , and therefore, they cannot be extended to the Hardy spaces of power smaller than 1 that are studied in [1].

A novelty of the present work is the use of a general multilinearisation theorem to find multilinear estimates from linear ones. In particular, we produce the corollaries 4.1 and 4.2, which extend the main results of [2] and [1] to  $F_{p,2}^s$  with  $s \neq 0$ .

## Overview of the proof of Theorem 1.1

The proof of Theorem 1.1 starts by using Duhamel's formula to present the solution of system (1.1), and from this one can identify the linear and multilinear oscillatory integral operators that occur in the analysis. Lemma 4.3 is then needed, since it gives time-dependent estimates from time-independent ones, and finally we need estimates of linear and multilinear operators connected to the system of equations. Estimates of the multilinear operator are provided by Theorem 4.1, which creates multilinear estimates from the linear ones shown in Theorem 3.1 and earlier work. Lastly, Theorem 4.1 is proved in large by following the steps of Theorem 1.1 in [2] and replacing key lemmas and estimates therein with new ones adapted to include Bessel potentials.

The material is organised as follows. We start by recalling key definitions and properties in Section 2. The new linear estimates that we need are provided in Section 3. In Section 4 we state the multilinearisation theorem, and as corollaries we show improved estimates of multilinear oscillatory operators. These then lead to the proof of Theorem 1.1. Finally, the proof of Theorem 4.1 is presented in Section 5.

## Notations

Constants that can be easily estimated by given parameters are all denoted by  $C$ , even though the precise values will vary from line to line. We also use the notation  $A \lesssim B$ , if there exists a constant  $C$  (independent of  $B$ ) such that  $A \leq CB$ .  $A \approx B$  means  $A \lesssim B$  and  $B \lesssim A$ . For clarity, we sometimes indicate the parameters on which a constant depends as subscripts. Most constants will depend on the underlying dimensions  $n, N$ , and this will not be indicated. When we say that an estimate is valid uniformly for  $f \in \mathcal{S}$  then we mean that the constant on the

estimate does not depend on  $f \in \mathcal{S}$ . This terminology is also used in the multilinear setting.

We will also drop the domain  $\mathbb{R}^n$  in the notation of all function spaces used in this paper, i. e. we will write  $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ ,  $L^p = L^p(\mathbb{R}^n)$ ,  $\text{bmo} = \text{bmo}(\mathbb{R}^n)$ , and so on. We will write pseudodifferential operators as  $a(x, D)f(x) = \int e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi$  and use the shorthand notation  $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ . Finally the domains of integration are often left out if they are  $\mathbb{R}^n$  or  $\mathbb{R}^{nN}$ .

## 2. Definitions and tools

The operators that we shall be interested in have their amplitudes in some of the multilinear Hörmander classes  $S_{1,0}^m(n, N)$ . These consist of all smooth  $\sigma: \mathbb{R}^n \times \mathbb{R}^{Nn} \rightarrow \mathbb{C}$  such that for all multi-indices  $\alpha, \beta$ ,

$$|\partial_x^\alpha \partial_\Xi^\beta \sigma(x, \Xi)| \leq C_{\alpha\beta} \langle \Xi \rangle^{m - |\beta|}, \quad \text{for all } x, \Xi.$$

For the linear amplitudes, we use the shorthand  $S_{1,0}^m = S_{1,0}^m(n, 1)$ , and  $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S_{1,0}^m$ .

We also recall the definition and some properties of the function spaces that will be of interest in this paper.

Let  $p > 0$  and let  $\varphi \in \mathcal{S}$  be such that  $\int \varphi(x) \, dx \neq 0$ . The Hardy space  $\mathcal{H}^p$  consists of all  $f \in \mathcal{S}'$  such that

$$\|f\|_{\mathcal{H}^p} := \left\| \sup_{t>0} |\varphi_t * f| \right\|_{L^p} < \infty,$$

where  $\varphi_t(x) = t^{-n} \varphi(x/t)$  and  $\mathcal{H}^p$  does not depend on the choice of the function  $\varphi$ . A local version of the Hardy space was defined by D. Goldberg [8]. The local Hardy space  $h^p$ , also denoted the inhomogeneous Hardy space, consists of all  $f \in \mathcal{S}'$  such that

$$\|f\|_{h^p} := \left\| \sup_{0 < t < 1} |\varphi_t * f| \right\|_{L^p} < \infty.$$

There are continuous inclusions  $\mathcal{H}^p \rightarrow h^p$  and  $h^p \rightarrow L^p$ . A key feature of the local Hardy spaces is that pseudodifferential operators of order 0 are bounded on these spaces,

$$(2.1) \quad \|a(\cdot, D)f\|_{h^p} \lesssim \|f\|_{h^p},$$

whenever  $a \in S_{1,0}^0$ . We also recall that  $h^p = L^p$  when  $1 < p < \infty$ .

The dual of  $\mathcal{H}^1$  is John–Nirenberg’s space of functions of bounded mean oscillations denoted by BMO, which consists of all functions  $f \in L_{\text{loc}}^1$  such that

$$\|f\|_{\text{BMO}} := \sup_B \frac{1}{|B|} \int_B |f(x) - \text{avg}_B f| \, dx < \infty,$$

where  $\text{avg}_B f = |B|^{-1} \int_B f(x) dx$ , and  $B$  ranges over the balls in  $\mathbb{R}^n$ . The dual of  $h^1$  is the local BMO-space, denoted by  $\text{bmo}$ , which consists of all functions  $f \in L^1_{\text{loc}}$  such that

$$\|f\|_{\text{bmo}} := \sup_{|B| \leq 1} \frac{1}{|B|} \int_B |f(x) - \text{avg}_B f| dx + \sup_{|B| > 1} \frac{1}{|B|} \int_B |f(x)| dx < \infty.$$

By duality, any Fourier multiplier  $a(D)$  with  $a \in S^0_{1,0}$  is also bounded  $\text{bmo} \rightarrow \text{bmo}$ , and there is a continuous inclusion  $L^\infty \rightarrow \text{bmo}$  as well as  $\text{bmo} \rightarrow \text{BMO}$ .

A straight forward use of the triangle inequality, Fubini's theorem and Lebesgue's dominated convergence theorem shows that if  $I$  is a bounded interval, then

$$(2.2) \quad \begin{aligned} \left\| \int_I f(\cdot, r) dr \right\|_{h^1} &\leq \int_I \|f(\cdot, r)\|_{h^1} dr && \text{if } f \in h^1(\mathbb{R}^n \times I) \\ \left\| \int_I f(\cdot, r) dr \right\|_{\text{bmo}} &\leq \int_I \|f(\cdot, r)\|_{\text{bmo}} dr && \text{if } f \in \text{bmo}(\mathbb{R}^n \times I). \end{aligned}$$

The following Fourier multiplier regularity result was shown in Lemmas 2.5 and 2.6 in [2].

**Lemma 2.1.** *Let  $\theta, \psi \in \mathcal{C}^\infty_c$  with  $\psi$  supported in an annulus, let  $\mu \in \mathcal{C}^\infty$  vanish in a neighbourhood of the origin and be equal to 1 outside of compact set, and let  $\delta > 0$ . Then the following bounds are uniform in  $k \in \mathbb{R}$  and  $p \in [1, \infty]$ :*

$$(2.3) \quad \|\psi(2^{-k}D)f\|_{Y^p} \lesssim \|f\|_{Z^p}$$

$$(2.4) \quad \|2^{-\delta k} \psi(2^{-k}D) \langle D \rangle^\delta f\|_{L^p} \lesssim \|f\|_{L^p}, \quad k \geq 0$$

$$(2.5) \quad \|2^{-\delta k} \mu(D) \theta(2^{-k}D) |D|^\delta f\|_{Y^p} \lesssim \|f\|_{Z^p},$$

where  $Y^p = Z^p = L^p$  when  $p \in (1, \infty)$ ,  $Y^1 = \mathcal{H}^1$ ,  $Z^1 = L^1$ ,  $Y^\infty = L^\infty$  and  $Z^\infty = \text{BMO}$ .

### 3. Linear estimates

We will here consider oscillatory integral operators, defined initially on  $\mathcal{S}$  by

$$T_\sigma^\varphi f(x) = \int e^{i\varphi(x, \xi)} \sigma(x, \xi) d\xi,$$

where  $\varphi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is called the phase of the operator, and  $\sigma: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$  is the amplitude. We will deal both with general phases  $\varphi$  depending on  $x$  and  $\xi$  as above, and the special ones of the form  $\varphi(x, \xi) = x \cdot \xi + \rho(\xi)$ . In the special case the operators are commonly referred to as  $T_\rho^p$  rather than  $T_\sigma^\varphi$ .

**Definition 3.1.** Given  $\lambda > 0$  we define the *phases of order  $\lambda$*  as the set of functions  $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$  that satisfy the condition

$$(3.1) \quad \sup_{\xi \neq 0} |\xi|^{|\alpha| - \lambda} |\partial_\xi^\alpha \rho(\xi)| \leq C_\alpha, \quad |\alpha| \geq 0.$$

**Theorem 3.1.** *If  $\rho$  is a phase of order  $\lambda$ ,  $1 \leq p \leq \infty$ ,  $s \in \mathbb{R}$  and  $\sigma \in S_{1,0}^{m_\sigma}$  with  $m_\sigma \leq -n\lambda|1/p - 1/2|$ , we have for all  $f \in \mathcal{S}$  that*

$$(3.2) \quad \|\langle D \rangle^s T_\sigma^\rho f\|_{X^p} \lesssim \|\langle D \rangle^s f\|_{X^p},$$

and when  $m_\sigma < -n\lambda/2$ ,

$$(3.3) \quad \|T_\sigma^\rho f\|_{L^\infty} \lesssim \|f\|_{\text{bmo}}.$$

Furthermore, if  $\rho$  is positively homogeneous of degree 1, then estimate (3.2) is valid provided  $m_\sigma \leq -(n-1)|1/p - 1/2|$  and estimate (3.3) is valid provided that  $m_\sigma < -(n-1)/2$ .

Therefore there is a certain order  $m_c$ , which we shall refer to as the critical order, for which operators with amplitude of order  $m_\sigma \leq m_c$  are bounded. As the above theorem shows, operators of subcritical order, that is  $m_\sigma < m_c$ , are even better behaved. Before we prove this theorem, we recall a composition theorem which is a special case of Theorem 3.11 in [6].

**Theorem 3.2.** *Let  $m, s, \lambda \in \mathbb{R}$  and suppose that  $a \in S_{1,0}^m$ ,  $b \in S_{1,0}^s$  and  $\rho \in C_c^\infty$ . Let  $\sigma$  be the amplitude of the composition operator  $T_\sigma^\rho := b(\cdot, D)T_a^\rho$  given by*

$$\sigma(x, \xi) = \iint a(y, \xi) b(x, \eta) e^{i(x-y) \cdot (\eta - \xi)} \mathfrak{d}\eta \, dy.$$

Then one can write  $\sigma$  as

$$\sigma(x, \xi) = b(x, \xi) a(x, \xi) + r(x, \xi),$$

where  $r \in S_{1,0}^{s+m-\varepsilon}$  for some  $\varepsilon > 0$ .

*Proof of Theorem 3.1.* Using this result one has that the principal part (leading term) of the amplitude of the operator  $\langle D \rangle^{s_0} T_\sigma^\rho \langle D \rangle^{-s_1}$  is  $\langle \xi \rangle^{s_0 - s_1} \sigma(x, \xi)$  which belongs to  $S^{m+s_0-s_1}$ . The second term in the expansion has greater decay, and therefore Theorem 3.5 in [6] yields estimate (3.2).

Now if  $m_\sigma < -n\lambda/2$ , we will first show that an operator  $T_a^\rho$  with amplitude  $a \in S_{1,0}^{m_\sigma}$  maps  $L^\infty$  to itself. Here we will assume that the amplitude of  $a$  vanishes for small  $|\xi|$ . Indeed, the low frequency portion of the operator could be handled

using Lemma 2.7 in [1]. To show the desired boundedness for the high-frequency portion of  $T_a^\rho$ , we make the following decomposition of the integral kernel

$$K(x, y) = \int_{\mathbb{R}^n} a(x, \xi) e^{i(x-y) \cdot xi + i\varphi(\xi)} \, d\xi$$

of the operator  $T_a^\rho$ .

Start with Littlewood–Paley shells  $\{\xi; 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$  and their associated partition of unity given by functions  $\psi_j$ ,  $j=1, 2, \dots$  and let  $\psi_0 = 1 - \sum_{j=1}^\infty \psi_j$ . For each  $j$ , fix  $O(2^{n\lambda j/2})$  vectors  $\xi_j^\nu$ ,  $\nu=1, \dots, O(2^{n\lambda j/2})$ , distributed evenly in  $\text{supp } \psi_j$ . Let  $\{\chi_j^\nu\}_\nu$  be a family of smooth functions, where  $\text{supp } \chi_j^\nu$  is a ball of radius  $2^{(1-\lambda/2)j}$  centred at  $\xi_j^\nu$ , chosen in such a way that the supports of  $\{\chi_j^\nu\}_\nu$  cover  $\text{supp } \psi_j$ . One may for example take a smooth bump function  $\beta$  supported in a ball of radius 1 about the origin and from this form  $\chi_j^\nu(\xi) = \beta(2^{(\lambda/2-1)j}(\xi - \xi_j^\nu)) / \sum_\nu \beta(2^{(\lambda/2-1)j}(\xi - \xi_j^\nu))$ . This of course yields that  $\sum_\nu \chi_j^\nu = 1$ .

We also have that the  $\chi_j^\nu$ 's satisfy

$$|\partial^\alpha \chi_j^\nu(\xi)| \leq C_\alpha 2^{|\alpha|(\lambda/2-1)j}.$$

Using  $\chi_j^\nu$  and  $\psi_j$ , we obtain a partition of unity

$$\psi_0(\xi) + \sum_{j=1}^\infty \sum_\nu \chi_j^\nu(\xi) \psi_j(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

With this partition of unity, we may therefore write the integral kernel of  $T_a^\rho \psi_j(D)$  as  $K_j(x, y) = \sum_\nu K_j^\nu(x, y)$ , with

$$K_j^\nu(x, y) = \int a(x, \xi) \chi_j^\nu(\xi) \psi_j(\xi) e^{i(x-y) \cdot \xi + i\rho(\xi)} \, d\xi.$$

In order to get the desired estimates for the kernel, we rewrite the phase of this integral as

$$(3.4) \quad \begin{aligned} (x-y) \cdot \xi + \rho(\xi) &= (x-y + \nabla \rho(\xi_j^\nu)) \cdot \xi + h_j^\nu(\xi), \\ \text{with } h_j^\nu(\xi) &= \rho(\xi) - \nabla \rho(\xi_j^\nu) \cdot \xi, \end{aligned}$$

which in turn yields

$$K_j^\nu(x, y) = \int b_j^\nu(x, \xi) e^{i(x-y + \nabla \rho(\xi_j^\nu)) \cdot \xi} \, d\xi,$$

where  $b_j^\nu(x, \xi) = a(x, \xi) \chi_j^\nu(\xi) \psi_j(\xi) e^{ih_j^\nu(\xi)}$ . The mean-value theorem then yields that  $\partial_i h_j^\nu(\xi) = \nabla \partial_i \rho(\eta) \cdot (\xi - \xi_j^\nu)$  for some  $\eta$  on the line segment between  $\xi$  and  $\xi_j^\nu$ . On supp  $\psi_j \chi_j^\nu$ , we therefore have from condition (3.1) that

$$|\partial^\alpha h_j^\nu(\xi)| \lesssim \begin{cases} 2^{(\lambda/2-1)j} & |\alpha|=1 \\ 2^{(\lambda-|\alpha|)j} & |\alpha|>1. \end{cases}$$

Thus  $\partial_\xi^\alpha e^{ih_j^\nu(\xi)}$  is bounded by a constant times  $2^{(\lambda/2-1)|\alpha|j}$ . With these estimates at hand, we find that on the support of  $\psi_j \chi_j^\nu$ ,

$$\begin{aligned} |\partial_\xi^\alpha b_j^\nu(x, \xi)| &\leq C_\alpha \sum_{\sum \alpha_\ell = \alpha} |\partial^{\alpha_1} a(x, \xi) \partial^{\alpha_2} \chi_j^\nu(\xi) \partial^{\alpha_3} \psi_j(\xi) \partial_\xi^{\alpha_4} (e^{ih_j^\nu(\xi)})| \\ &\leq C_\alpha \sum_{\sum \alpha_\ell = \alpha} 2^{(m_\sigma - |\alpha_1 + \alpha_3| + (\lambda/2-1)|\alpha_2| + (\lambda/2-1)|\alpha_4|)j} \leq C_\alpha 2^{(m_\sigma + (\lambda/2-1)|\alpha|)j}. \end{aligned}$$

Now observe that the size of the  $\xi$ -support of  $b_j^\nu$  is  $O(2^{jn(1-\frac{\lambda}{2})})$  uniformly in  $\nu$  and  $j$ , and therefore one trivially has

$$|K_j^\nu(x, y)| \lesssim 2^{jm_\sigma} 2^{jn(1-\frac{\lambda}{2})}.$$

Moreover, integrating by parts and using the estimate for the derivatives of  $b_j^\nu$  above, one also has

$$|K_j^\nu(x, y)| \lesssim \frac{2^{jm_\sigma - j(1-\lambda/2)M} 2^{jn(1-\frac{\lambda}{2})}}{|\nabla_\xi \rho(\xi_j^\nu) + x - y|^M},$$

for an arbitrary positive integer  $M$ . Combining the two kernel estimates above, one obtains

$$|K_j^\nu(x, y)| \lesssim \frac{2^{jm_\sigma} 2^{jn(1-\frac{\lambda}{2})}}{(2^{j(1-\lambda/2)} (\nabla_\xi \rho(\xi_j^\nu) + x - y))^M},$$

for all  $j \geq 1$  and  $M \geq 0$ .

Thus summing in  $j$  and recalling that there are  $O(2^{nj\lambda/2})$  terms involved, one obtains

$$|K_j(x, y)| \lesssim \frac{2^{jm_\sigma} 2^{nj\lambda/2} 2^{jn(1-\frac{\lambda}{2})}}{(2^{j(1-\lambda/2)} (\nabla_\xi \rho(\xi_j^\nu) + x - y))^M},$$

and therefore, if we take large enough  $M$ , one readily sees that if  $m_\sigma < -n\lambda/2$  then the integral kernel of  $T_a^\rho$  is integrable. This yields that  $T_a^\rho$  maps  $L^\infty$  continuously to itself.

Now returning to our original problem, since the operator  $\langle D \rangle^{s_0} T_\sigma^\rho \langle D \rangle^{-s_1} r_j(D)$  (where  $r_j(D)$ ,  $j=1, \dots, n$ , are the local Riesz transforms with smooth symbols in  $S_{1,0}^0$ ), has an amplitude whose principal part belongs to  $S_{1,0}^{m+s_0-s_1}$ , and since  $m+s_0-$

$s_1$  was assumed to be strictly less than  $-n\lambda/2$ , the discussion above and Lemma 2.1 in [2] shows that estimate (3.3) is also valid.

Now, in the case of  $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$  being positively homogeneous of degree 1, estimate (3.2) was shown in [10]. Therefore, we confine ourselves to the proof of estimate (3.3). To this end, the Littlewood–Paley shell  $\{\xi: 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$  is partitioned into  $O(2^{j(n-1)/2})$  truncated cones of thickness  $2^{j/2}$ , where each such cone is a rectangle whose major axis has length of the size  $2^j$ , while all the other sides have length  $2^{j/2}$ .

This time, we choose for each  $j \in \mathbb{N}$  a collection of *unit* vectors  $\{\xi_j^\nu\}_\nu$  such that

- $|\xi_j^\nu - \xi_j^{\nu'}| \geq 2^{-\frac{j}{2}}$  for  $\nu \neq \nu'$ , and
- for each  $\xi \in \mathbb{S}^{n-1}$ , there exists a  $\xi_j^\nu$  such that  $|\xi - \xi_j^\nu| < 2^{-j/2}$ ,

which is maximal with respect to the first property. It follows that it contains at most  $O(2^{j(n-1)/2})$  elements. Associated to each  $\xi_j^\nu$  is a cone

$$\Gamma_j^\nu := \left\{ \xi \in \mathbb{R}^n : \left| \frac{\xi}{|\xi|} - \xi_j^\nu \right| \leq 2 \cdot 2^{-\frac{j}{2}} \right\},$$

whose central axis lies along  $\xi_j^\nu$ .

One can construct a partition of unity  $\sum_\nu \chi_j^\nu = 1$  of  $\mathbb{R}^n \setminus \{0\}$  subordinate to  $\{\Gamma_j^\nu\}_{j,\nu}$  which satisfies the estimates

$$(3.5) \quad \left| \partial_\xi^\alpha \chi_j^\nu(\xi) \right| \leq C_\alpha 2^{j \frac{|\alpha|}{2}} |\xi|^{-|\alpha|}$$

for all multi-indices  $\alpha$ . If we choose the coordinate axes in the  $\xi$ -space so that  $\xi_1$  is in the direction of  $\xi_j^\nu$  and  $\xi' = (\xi_2, \dots, \xi_n)$  is perpendicular to  $\xi_j^\nu$ , we also have

$$(3.6) \quad \left| \partial_{\xi_1}^N \chi_j^\nu(\xi) \right| \leq C_N |\xi|^{-N},$$

for  $N \geq 1$ . We omit the details of this standard argument, which can be found in e. g. [12, Section IX.4]. Now with  $\psi_j$  as in the proof of the first part of the theorem we have

$$\psi_0(\xi) + \sum_{j=1}^{\infty} \sum_{\nu} \chi_j^\nu(\xi) \psi_j(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

With this partition of unity, the integral kernel of the high frequency portion of the Littlewood–Paley pieces of  $T_a^\rho$  is given by  $K_j^\nu(x, y) = \sum_\nu K_j^\nu(x, y)$ , with

$$K_j^\nu(x, y) = \int a(x, \xi) \chi_j^\nu(\xi) \psi_j(\xi) e^{i(x-y) \cdot \xi + i\rho(\xi)} d\xi.$$

We then again define  $h_j^\nu$  as in equation (3.4), and it can be shown that when  $\rho$  is homogeneous of degree 1, each  $h_j^\nu$  satisfies estimates akin to those for  $\chi_j^\nu$  in estimates (3.5) and (3.6). This gives us the form

$$K_j^\nu(x, y) = \int c_j^\nu(x, \xi) e^{i(x-y+\nabla\rho(\xi_j^\nu))\cdot\xi} d\xi,$$

where  $c_j^\nu(x, \xi) = a(x, \xi)\chi_j^\nu(\xi)\psi_j(\xi)e^{ih_j^\nu(\xi)}$ . Using the differential operator

$$L = I - 2^{2j}\partial_{\xi_1}^2 - 2^j\nabla_{\xi'}$$

and estimates (3.5) and (3.6), we therefore have for any positive integer  $N$ ,

$$|L^N c_j^\nu(x, \xi)| \lesssim 2^{m_\sigma j}.$$

This estimate, integration by parts and the size of  $\text{supp } \chi_j^\nu$  yield that

$$|K_j^\nu(x, y)| \lesssim \frac{2^{(m_\sigma + \frac{n+1}{2})j}}{\left(1 + |2^j(x_1 - y_1 + \partial_{\xi_1}\rho(\xi_j^\nu))|^2\right)^N \left(1 + |2^{\frac{j}{2}}(x' - y' + \nabla_{\xi'}\rho(\xi_j^\nu))|^2\right)^N}.$$

Thus summing in  $j$  and recalling that there are  $O(2^{(n-1)j/2})$  terms involved, one finally obtains

$$|K_j(x, y)| \lesssim \frac{2^{(m_\sigma + 1)j}}{\left(1 + |2^j(x_1 - y_1 + \partial_{\xi_1}\rho(\xi_j^\nu))|^2\right)^N \left(1 + |2^{\frac{j}{2}}(x' - y' + \nabla_{\xi'}\rho(\xi_j^\nu))|^2\right)^N}.$$

Therefore, once again Schur's lemma yields that if  $m_\sigma < -(n-1)/2$  then the integral kernel of  $T_a^\rho$  is integrable, provided that  $N > n$ . This yields that  $T_a^\rho$  maps  $L^\infty$  continuously to itself. The rest of the argument is similar to the inhomogeneous case and is hence omitted.  $\square$

#### 4. Multilinear estimates

This section deals with multilinear oscillatory integral operators, which for  $N \geq 1$  are initially defined on  $\mathcal{S}^N$  as

$$T_\sigma^\Phi(f_1, \dots, f_N) = \int e^{i\Phi(x, \Xi)} \sigma(x, \Xi) \prod_{j=1}^N \widehat{f}_j(\xi_j) d\Xi,$$

where  $\Xi = (\xi_1, \dots, \xi_N)$ , each  $\xi_j \in \mathbb{R}^n$  and  $\Phi: \mathbb{R}^n \times \mathbb{R}^{Nn} \rightarrow \mathbb{R}$ . In the present setting, the phase will be a sum  $\Phi(x, \Xi) = \sum_{j=1}^N \varphi_j(x, \xi_j)$ .

A large part of the work in proving Theorem 1.1 is done by the following multilinearisation theorem. Given estimates (4.5) and (4.6) below on a linear operator, it ensures estimates (4.7) and (4.8) of the corresponding multilinear operator. The proof of this result is presented in Section 5.

Some comments are in order about the requirements of this theorem. We say that a (phase) function  $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  is *strongly non-degenerate* (SND) if there is a  $C > 0$  such that  $|\det \partial_{x,\xi}^2 \varphi(x, \xi)| \geq C$  for all  $x, \xi$ . Together with condition (4.4) below, this will ensure that when  $|\xi| \gtrsim 1$ , we have the key property

$$(4.1) \quad |\nabla_x \varphi_j(x, \xi)| \approx |\xi| \quad \text{for all } j.$$

See [2, p. 399] for details on this, and why condition (4.4) can be dropped in some cases.

**Theorem 4.1.** *Let  $T_\sigma^\Phi$  be a multilinear oscillatory operator with phase  $\Phi(x, \Xi) = \sum_{j=1}^N \varphi_j(x, \xi_j)$ , where each  $\varphi_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  is SND and satisfies for all  $x, \xi \in \mathbb{R}^n$  and  $\Xi \in \mathbb{R}^{Nn}$  that*

$$(4.2) \quad |\partial_x^\alpha \varphi_j(x, \xi)| \leq C_{\alpha,j} \quad \text{for all } |\alpha| \geq 2$$

$$(4.3) \quad |\partial_x^\alpha \partial_\Xi^\beta \Phi(x, \Xi)| \leq C_{\alpha\beta} \langle \Xi \rangle^{1-|\beta|} \quad \text{for all } \alpha, \beta \text{ with } |\alpha|, |\beta| > 0$$

$$(4.4) \quad |\nabla_x \Phi(x, 0)| \leq C.$$

Assume also that there is a function  $m_c: [1, \infty] \rightarrow (-\infty, 0]$  with  $m_c(2) = 0$  and otherwise  $< 0$ , such that  $m_c(1/\cdot)$  is linear on  $[0, 1/2]$  and  $[1/2, 1]$ , with the property that any linear operator  $T_\sigma^\varphi$  with SND phase  $\varphi$  and amplitude  $\sigma \in S_{1,0}^{m_\sigma}$  that fulfils estimates (4.2)–(4.4), also satisfies

$$(4.5) \quad \begin{aligned} \|\langle D \rangle^s T_\sigma^\varphi f\|_{h^p} &\lesssim \|\langle D \rangle^s f\|_{h^p}, \quad 1 \leq p < \infty, \quad s \in \mathbb{R} \\ \|T_\sigma^\varphi f\|_{\text{bmo}} &\lesssim \|f\|_{\text{bmo}}, \quad (p = \infty), \end{aligned}$$

under the assumption that  $m_\sigma \leq m_c(p)$ , and the estimate

$$(4.6) \quad \|T_\sigma^\varphi f\|_{L^\infty} \lesssim \|f\|_{\text{bmo}},$$

under the assumption that  $m_\sigma < m_c(\infty)$ . Here we assume that the estimates above are satisfied uniformly for  $f \in \mathcal{S}$ .

Now let  $m_\sigma \in \mathbb{R}$ , and  $p_0, \dots, p_N \in [1, \infty]$  with  $1/p_0 = \sum_{j=1}^N 1/p_j$  and  $s_1, \dots, s_N \in \mathbb{R}$  satisfying, for some  $\varepsilon > 0$ ,

$$m_\sigma \leq \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\substack{1 \leq \ell \leq N \\ \ell \neq j}} \min(m_c(p_\ell) + s_\ell, -\varepsilon \gamma_j) \right],$$

where  $\gamma_\ell = 0$  when  $p_\ell = 2$  and is otherwise 1.

If the estimates (4.5) and (4.6) hold, then for any  $\sigma \in S_{1,0}^{m_\sigma}(n, N)$ , and any  $f_1, \dots, f_N \in \mathcal{S}$ , one has the bound

$$(4.7) \quad \|T_\sigma^\Phi(f_1, \dots, f_N)\|_{X^{p_0}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}}.$$

Condition (4.4) can be dropped when the target space is  $L^{p_0}$ ,  $p_0 \in [1, \infty)$ .

Furthermore, when the phases are of the form  $\varphi_j(x, \xi) = x \cdot \xi + \rho_j(\xi)$ , which in particular are always SND and satisfy conditions (4.2)–(4.4), and if the amplitude  $\sigma$  is independent of the  $x$  variable and  $s_0, \dots, s_N \in \mathbb{R}$  satisfy, for some  $\varepsilon > 0$ ,

$$0 \leq s_0 \leq -m_\sigma + \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\substack{1 \leq \ell \leq N \\ \ell \neq j}} \min(m_c(p_\ell) + s_\ell, -\varepsilon \gamma_j) \right],$$

then we also have

$$(4.8) \quad \|\langle D \rangle^{s_0} T_\sigma^\Phi(f_1, \dots, f_N)\|_{X^{p_0}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}}.$$

Recall that a *multilinear Schrödinger integral operator* was defined in [2] as a multilinear oscillatory operator that satisfies conditions (4.2) and (4.3) as well as

$$|\partial_x^\alpha \varphi_j(x, \xi)| \leq C_{\alpha,j} \quad \text{for all } |\alpha| \geq 2, \text{ all } j \text{ and all } x, \xi \in \mathbb{R}^n.$$

Now Theorem 4.1 immediately yields a generalisation of Theorem 1.1 in [2], using the linear estimates as stated in that paper. Namely we have

**Corollary 4.1.** *If a multilinear Schrödinger integral operator has SND phases and satisfies condition (4.4), then estimates (4.7) and (4.8) hold true for that operator with  $m_c(p) = -2n|1/p - 1/2|$ .*

Using Theorem 4.1, the linear estimates in Theorem 3.1 lead us to deduce the following result, which through estimate (4.7) extends Theorem 1.3 and Theorem 1.4 in [1] in the case  $\varphi_0 = 0$  with Banach source and target spaces. Moreover estimate (4.8) improves the estimate on  $T_\zeta^{(1)}$  on p. 7600 in [1] and extends it to source and target spaces  $h^1$  and  $\text{bmo}$ .

**Corollary 4.2.** *Let  $\varphi_j(x, \xi) = x \cdot \xi + \rho_j(\xi)$ ,  $j = 1, \dots, N$ , where  $\rho_j$  are phases of order  $\lambda > 0$ . Then estimates (4.7) and (4.8) hold with  $m_c(p) = -n\lambda|1/p - 1/2|$ .*

*Furthermore, if  $\rho$  is homogeneous of order 1, then estimates (4.7) and (4.8) hold with  $m_c(p) = -(n-1)|1/p - 1/2|$ .*

Incidentally, we can use this corollary to study pointwise products of functions, by taking  $\sigma=1$  and all  $\rho_j=0$ , with  $\lambda$  arbitrary. This extends the Kato–Ponce inequality to the  $X^p$ -spaces, up to an arbitrarily small loss in derivatives. It is clear from the proof of Theorem 4.1 that the outer minima in the conditions on  $s_j$  stem from a sum of  $N$  terms with different possible choices of  $s_j$  on the right hand side. In particular, we can look at the case with  $s_0=0$  and find that for any  $\varepsilon>0$ ,

$$\|fg\|_{\text{bmo}} \leq C_\varepsilon (\| \langle D \rangle^\varepsilon f \|_{\text{bmo}} \|g\|_{\text{bmo}} + \|f\|_{\text{bmo}} \| \langle D \rangle^\varepsilon g \|_{\text{bmo}}),$$

which is optimal with respect to  $\varepsilon$ , in light of the well-known examples of  $f, g \in \text{bmo}$  such that  $fg \notin \text{bmo}$ .

Before we take on the last proof, we need to establish the following multilinear result, which is a variation of Lemma 8.2 in [1] that in particular extends the source and target spaces to  $X^p$  with  $p \in [1, \infty]$ .

**Lemma 4.3.** *Let  $\rho_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$ ,  $j=1, \dots, N$ , be homogeneous of degree  $\lambda > 0$  and let  $\sigma \in S_{1,0}^m(n, N)$  with  $m \in \mathbb{R}$ . Define the multilinear operator  $T_\sigma^{(t)}$  by*

$$T_\sigma^{(t)}(f_1, \dots, f_N) = \int_{\mathbb{R}^{Nn}} \sigma(\Xi) \prod_{j=1}^N \hat{f}_j(\xi_j) e^{ix \cdot \xi_j + it\rho_j(\xi_j)} d\Xi.$$

Assume that for some  $p_0, \dots, p_N \in [1, \infty]$  and  $s_1, \dots, s_N \in \mathbb{R}$  the estimate

$$\| \langle D \rangle^{-s_0} T_\sigma^{(1)}(f_1, \dots, f_N) \|_{X^p} \lesssim \prod_{j=1}^N \| \langle D \rangle^{s_j} f_j \|_{X^{p_j}}$$

holds uniformly in  $f_1, \dots, f_N \in \mathcal{S}$ , where  $X^p$  is as in Theorem 1.1 and the hidden constant in this estimate depends only on a finite number of seminorms of  $\sigma$  and upper bounds of the size of a finite number of derivatives of  $\rho_j$ . Then it follows that for all  $t \geq 0$ ,

$$\| \langle D \rangle^{-s_0} T_\sigma^{(t)}(f_1, \dots, f_N) \|_{X^{p_0}} \leq C \langle t \rangle^{(1+\max(-m,0)+\sum_{j=0}^N \max(s_j,0))/\lambda} \prod_{j=1}^N \| \langle D \rangle^{s_j} f_j \|_{X^{p_j}}.$$

*Proof.* Here, the difference from Lemma 8.2 in [1] is that we allow the endpoint source and target spaces  $h^1$  and  $\text{bmo}$ , and in order to accommodate this change we have to add  $1/\lambda$  to the exponent of  $\langle t \rangle$  of the resulting estimate. To see how this works, notice first that when  $t \leq 1$ , the upper bounds of the derivatives of  $t\rho_j$  will be satisfied uniformly in  $t$ , and therefore this boundedness is trivial. We therefore assume that  $t > 1$ , and following [1] we use the representation

$$\langle D \rangle^{-s_0} T_\sigma^{(t)}(f_1, \dots, f_N)(x)$$

$$\begin{aligned}
&= t^{(\max(-m,0) + \sum_{j=0}^N \max(s_j,0))/\lambda} \\
&\quad \times \langle D \rangle^{-s_0} T_{\sigma_t}^{(1)} (\langle D \rangle^{-s_1} S_1 g_1(t^{1/\lambda} \cdot), \dots, \langle D \rangle^{-s_N} S_N g_N(t^{1/\lambda} \cdot)) (t^{-1/\lambda} x),
\end{aligned}$$

where  $g_j = \langle D \rangle^{s_j} f_j$  and

$$\begin{aligned}
\sigma_t(\Xi) &= t^{\min(m,0)/\lambda} \sigma(t^{-1/\lambda} \Xi) \\
S_j &= t^{-\max(s_j,0)/\lambda} \langle t^{-1/\lambda} D \rangle^{-s_j} \langle D \rangle^{s_j}.
\end{aligned}$$

Now, the operators  $S_j$  are bounded  $X^p \rightarrow X^p$  for  $p \in [1, \infty]$ , as follows from Theorem 4 in [8]. Thus it remains to handle the scalings in the source and target variables. For the spaces  $L^p$ , these scalings cancel out. To treat the endpoint cases, we use that for any  $a > 0$ ,

$$\begin{aligned}
\|f(a \cdot)\|_{\text{bmo}} &= \sup_{|B| \leq 1} \frac{1}{|B|} \int_B \left| f(ax) - \text{avg}_B f(a \cdot) \right| dx + \sup_{|B| > 1} \frac{1}{|B|} \int_B |f(ax)| dx \\
&= \sup_{|B| \leq 1} \frac{1}{a^n |B|} \int_{aB} \left| f(x) - \text{avg}_{aB} f \right| dx + \sup_{|B| > 1} \frac{1}{a^n |B|} \int_{aB} |f(x)| dx \\
&= \sup_{|B| \leq a^n} \frac{1}{|B|} \int_B \left| f(x) - \text{avg}_B f \right| dx + \sup_{|B| > a^n} \frac{1}{|B|} \int_B |f(x)| dx \\
&=: |f(a \cdot)|_1 + |f(a \cdot)|_2,
\end{aligned}$$

where  $aB$  denotes the ball  $B$  with radius scaled with  $a$ . If  $a \leq 1$ , we deduce from the third row that  $|f(a \cdot)|_1 \leq |f|_1$ , and we read from the second row that  $|f(a \cdot)|_2 \leq a^{-n} |f|_2$ . Now if  $a > 1$ , then

$$\sup_{|B| > a^n} \frac{1}{|B|} \int_B |f(x)| dx \leq \sup_{|B| > 1} \frac{1}{|B|} \int_B |f(x)| dx \leq \|f\|_{\text{bmo}}.$$

Therefore, since

$$\sup_{|B| \leq a^n} \frac{1}{|B|} \int_B |f(x) - \text{avg}_B f| dx \leq \|f\|_{\text{BMO}} \leq \|f\|_{\text{bmo}},$$

we conclude that  $\|f(a \cdot)\|_{\text{bmo}} \leq 2 \|f\|_{\text{bmo}}$ . Using duality, we therefore have for all  $a > 0$  that

$$\begin{aligned}
\|f(a \cdot)\|_{\text{bmo}} &\leq 2 \max(1, a^{-n}) \|f\|_{\text{bmo}}, \\
\|f(a \cdot)\|_{h^1} &\leq 2 a^{-n} \max(1, a^{-n}) \|f\|_{h^1}.
\end{aligned}$$

Coming back to the estimate of  $T_{\sigma}^{(t)}$  and the scalings  $t^{1/\lambda}$  and  $t^{-1/\lambda}$  with  $t > 1$  in the representation above, we see that these yield an extra factor  $t^{1/\lambda}$  in the estimate when the target space is  $\text{bmo}$  or if one of the source spaces is  $h^1$ . Since any of these can only happen once, it is enough to add the factor  $t^{1/\lambda}$ .  $\square$

With this at hand, it is straightforward to show the main theorem of the paper.

*Proof of Theorem 1.1.* The solution  $u$  of the system (1.1) is given by Duhamel's formula as

$$\begin{aligned} u(t, x) &= \int_0^t \int_{\mathbb{R}^{Nn}} e^{i(t-r)\rho_0(\xi_1 + \dots + \xi_N)} \sigma(\Xi) \prod_{j=1}^N e^{ix \cdot \xi_j + ir\rho_j(\xi_j)} \hat{f}_j(\xi_j) d\Xi dr \\ &= \int_0^t e^{i(t-r)\rho_0(D)} \langle D \rangle^{m_c(p_0)} \langle D \rangle^{-m_c(p_0)} T_\sigma^{(r)}(f_1, \dots, f_N)(x) dr. \end{aligned}$$

Now using Minkowski's inequality together with estimate (2.2), followed by Lemma 4.3 together with Corollary 4.2 on the linear operator  $e^{(t-r)\varphi_0(D)} \langle D \rangle^{m_c(p_j)}$  and the  $N$ -linear operator  $T_\sigma^{(r)}$ , we have that

$$\begin{aligned} &\| \langle D \rangle^{s_0} u(t, \cdot) \|_{X^{p_0}} \\ &\leq \int_0^t \| e^{i(t-r)\rho_0(D)} \langle D \rangle^{m_c(p_0)} \langle D \rangle^{s_0 - m_c(p_0)} T_\sigma^{(r)}(f_1, \dots, f_N) \|_{X^p} dr \\ &\lesssim \int_0^t \langle t-r \rangle^{(1-m_c(p_0))/\lambda} \| \langle D \rangle^{s_0 - m_c(p_0)} T_\sigma^{(r)}(f_1, \dots, f_N) \|_{X^p} dr \\ &\lesssim \int_0^t \langle t-r \rangle^{(1-m_c(p_0))/\lambda} \langle r \rangle^{(1+\max(-m_\sigma, 0) + \max(m_c(p_0) - s_0, 0) + \sum_{j=1}^N \max(s_j, 0))/\lambda} dr \\ &\quad \times \prod_{j=1}^N \| \langle D \rangle^{s_j} f_j \|_{X^{p_j}}. \quad \square \end{aligned}$$

## 5. Proof of Theorem 4.1

This result follows from a number of additions and alterations to the proof of Theorem 1.1 in [2], in the following way. Firstly, the claim with  $s_0 = s_1 = \dots = s_N = m_\sigma - \sum_{j=1}^N m_c(p_j) = 0$  is a direct consequence of that theorem, since the method relies solely on conditions (4.2) and (4.3) and the SND property, along with the properties of the linear operator.

However, in order to show the estimates with  $s_j \neq 0$ , we need to make a few additions to the proof. Some of the lemmas that were used in that proof will need to be replaced with more general ones. Two of them are the following lemmas, which will be used in the high frequency regime and will take the role of Corollary 3.1 in [2]. The first lemma is a straightforward generalisation of that corollary to the operators in Theorem 4.1, and the proof follows the same steps.

**Lemma 5.1.** *Let  $p_j \in [1, \infty]$ ,  $j=1, \dots, N$ , and let  $T_{jk}$ ,  $k \geq 0$ , be a family of operators that are of either of the following forms:*

$$T_{jk} = A_j = T_a^\varphi, \text{ an oscillatory integral operator as in Theorem 4.1} \\ \text{with } a \in S_{1,0}^{m_j}, \quad m_j < m_c(p_j),$$

$$T_{jk} = B_k A_j, \text{ with } A_j \text{ as above and} \\ B_k = \psi(2^{-k}D) \text{ or } B_k = 2^{-\varepsilon k} \mu(D) |D|^\varepsilon \theta(2^{-k}D), \quad \varepsilon > 0,$$

where  $\mu, \theta, \psi$  are as in Lemma 2.1, and let  $\lambda_k \in C^\infty$  satisfy  $\sup_k |\partial^\alpha \lambda_k| \leq C_\alpha$  for all  $\alpha$ . Then for  $\delta > 0$  and all  $f_j \in \mathcal{S}$

$$\left\| \sum_{k=1}^{\infty} 2^{-\delta k} \lambda_k \prod_{j=1}^N T_{jk} f_j \right\|_{Y^{p_0}} \lesssim \prod_{j=1}^N \|f_j\|_{X^{p_j}}, \quad p_j \in [1, \infty],$$

where  $1/p_0 = \sum_{j=1}^N 1/p_j$ ,  $Y^p = L^p$  for  $p > 1$  and  $Y^1 = h^1$  and  $X^p$  are as in Theorem 1.1. Furthermore, we have

$$\left\| \lambda_k \prod_{j=1}^N A_j f_j \right\|_{Y^{p_0}} \lesssim \prod_{j=1}^N \|f_j\|_{X^{p_j}}, \quad p_j \in [1, \infty].$$

The second lemma shows similar estimates with Bessel potentials, and requires an arbitrary decay in the amplitudes of the oscillatory integral operators.

**Lemma 5.2.** *Let  $p_j \in [1, \infty]$ ,  $j=1, \dots, N$ , and let  $T_{jk}$ ,  $k \geq 0$ , be a family of operators that are of either of the following forms:*

$$T_{jk} = A_j = T_a^\varphi, \text{ an oscillatory integral operator as in Theorem 4.1} \\ \text{with } a \in S_{1,0}^{m_j}, \quad m_j \in \mathbb{R},$$

$$T_{jk} = B_k A_j, \text{ with } A_j \text{ as above and} \\ B_k = \psi(2^{-k}D) \text{ or } B_k = 2^{-\varepsilon k} \mu(D) |D|^\varepsilon \theta(2^{-k}D), \quad \varepsilon > 0,$$

where  $\mu, \theta, \psi$  are as in Lemma 2.1, and let  $\lambda_k \in C^\infty$  satisfy  $\sup_k |\partial^\alpha \lambda_k| \leq C_\alpha$  for all  $\alpha$ . Then for all  $s_0 \geq 0$ ,  $s_1, \dots, s_N \in \mathbb{R}$ , there is an  $L \in \mathbb{R}$  such that if  $m_1, \dots, m_N \leq -L$  then for all  $f_j \in \mathcal{S}$ ,

$$\sup_{k \geq 0} \left\| \langle D \rangle^{s_0} \left[ \lambda_k \prod_{j=1}^N T_{jk} f_j \right] \right\|_{Y^{p_0}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}}, \quad p_j \in [1, \infty],$$

where  $1/q = \sum_{j=1}^N 1/p_j$  and  $Y^p$  are as in Lemma 5.1 and  $X^p$  are as in Theorem 1.1.

*Proof.* Take first  $q > 1$ . Then one can use an appropriate Kato–Ponce inequality (see e. g. [11]) that yields for  $s_0 > 0$  that

$$\left\| \langle D \rangle^{s_0} \left[ \lambda_k \prod_{j=1}^N T_{jk} f_j \right] \right\|_{L^q} \lesssim \sum_{\ell=0}^N \|\langle D \rangle^{s_0 \delta_{0\ell}} \lambda_k\|_{L^\infty} \prod_{j=1}^N \|\langle D \rangle^{s_0 \delta_{j\ell}} T_{jk} f_j\|_{L^{p_j}},$$

where  $\delta$  here is the Kronecker delta. The first factor on the right hand side can be bounded by a standard kernel estimate. Estimates (4.5) and (4.6) then show  $\|\langle D \rangle^{s_0 \delta_{j\ell}} A_j f\|_{L^{p_j}} \lesssim \|\langle D \rangle^{s_j} f\|_{X^{p_j}}$  for sufficiently small  $m_j$  (we need  $m_j < m_c(p_j) + s_j - s_0$ ), while the case  $T_{jk} = B_k A_j$  is treated by first bounding the action of  $B_k$  using estimates (2.3) and (2.5).

To deal with the case  $q = 1$ , take  $\theta \in C_c^\infty$  equal to 1 at the origin and  $\psi := \theta - \theta(2 \cdot)$  so that  $1 = \theta + \sum_{k=1}^\infty \psi(2^{-k} \cdot)$ . Then for any constant  $\varepsilon > 0$  the following estimates are true uniformly for all locally integrable functions  $f$ :

$$\begin{aligned} \|\langle D \rangle^{s_0} f\|_{h^1} &\leq \|\theta(D) \langle D \rangle^{s_0} f\|_{h^1} + \sum_{k=1}^\infty \|\psi(2^{-k} D) \langle D \rangle^{s_0} f\|_{h^1} \\ &= \|\theta(D) \langle D \rangle^{s_0} f\|_{h^1} + \sum_{k=1}^\infty 2^{-\varepsilon k} \|\psi_1(2^{-k} D) |D|^\varepsilon \langle D \rangle^{s_0} f\|_{h^1} \\ &\lesssim \|\theta_1(D) f\|_{h^1} + \sum_{k=1}^\infty 2^{-\varepsilon k} \|\psi_1(2^{-k} D) \langle D \rangle^{s_0 + \varepsilon} f\|_{h^1} \\ &\lesssim \|\theta_1(D) f\|_{h^1} + \|\langle D \rangle^{s_0 + \varepsilon} f\|_{L^1}, \end{aligned}$$

where  $\psi_1 = \psi / |\cdot|^\varepsilon$  and in the penultimate estimate we used Theorem 4 in [8] and in the last one estimate (2.3). Applying this reasoning to the product in the statement of the lemma, we will hence find two terms, where the second can be dealt with using the Kato–Ponce inequality as in the  $q > 1$  case above to yield the result.

To deal with the first term, we use that for any locally integrable functions  $f, g$  (see Lemma 2.2 in [2]),

$$\|\langle D \rangle^{s_0} \theta(D)(fg)\|_{h^1} \lesssim \|fg\|_{h^1} \lesssim \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_{L^\infty} \|g\|_{h^1}.$$

Repeated use of this estimate yields that for any  $f_0, \dots, f_N$ , we have

$$\left\| \prod_{j=0}^N f_j \right\|_{h^1} = \left\| \left( \prod_{p_j = \infty} f_j \right) \prod_{p_\ell \neq \infty} f_\ell \right\|_{h^1} \lesssim \left( \prod_{p_j = \infty} \sum_{|\alpha| \leq 1} \|\partial^\alpha f_j\|_{L^\infty} \right) \left\| \prod_{p_\ell \neq \infty} f_\ell \right\|_{h^1}.$$

If the index set  $I = \{\ell; p_\ell \neq \infty\}$  is a singleton, we are done, since we can apply this reasoning to the product in the statement of the lemma, and then choose  $m_j$

sufficiently small as in the  $q > 1$  case. Note that  $\partial^\alpha = \partial^\alpha \langle D \rangle^{-n-2} \langle D \rangle^{n+2}$  and a simple kernel estimate shows that  $\partial^\alpha \langle D \rangle^{-n-2}$  is bounded  $L^\infty \rightarrow L^\infty$ .

If  $I$  is not a singleton, we can use Lemma 2.6 in [9], to show that for any  $p \in (1, \infty)$ , and  $\varepsilon > 0$ ,

$$\|fg\|_{h^1} \lesssim \|\langle D \rangle^\varepsilon (fg)\|_{h^1} \lesssim \|\langle D \rangle^\varepsilon (fg)\|_{\mathcal{H}^1} \lesssim \|\langle D \rangle^\varepsilon f\|_{L^p} \|g\|_{L^{p'}} + \|f\|_{L^p} \|\langle D \rangle^\varepsilon g\|_{L^{p'}},$$

where  $1/p' = 1 - 1/p$ . One can then again use the reasoning from the case  $q > 1$  to complete the argument.  $\square$

We will also make use of the following result. It follows directly from the proof of Theorem 5.1 in [2].

**Theorem 5.1.** *Let  $T_{ga}^\Phi$  be an  $N$ -linear operator as in Theorem 4.1 with amplitude*

$$g(2^{-k}P(x, \Xi))a(2^{-k}, x, \Xi),$$

where  $g \in \mathcal{S}$ ,  $k \geq 0$ ,  $a \in S_{1,0}^m(n, N)$  uniformly in  $k$ , and either of

$$P(x, \Xi) = \nabla_x \Phi(x, \Xi)$$

$$P(x, \Xi) = \nabla_x \Phi(x, \Xi) - 2\nabla_x \varphi_1(x, \xi_1) \text{ with } \langle \xi_1 \rangle, |P(x, \Xi)| \gtrsim \langle \Xi \rangle$$

holds. Then

$$T_{ga}^\Phi = g(2^{-k}D)T_a^\Phi + T_r^\Phi$$

with

$$|\partial_x^\alpha \partial_\Xi^\beta r(k, x, \Xi)| \leq C_{\alpha\beta} \min(2^k, \langle \Xi \rangle)^{m-|\beta|-1},$$

where the min is replaced with max if the exponent is positive. Furthermore, the support of  $r$  is contained in that of  $a$ .

Lastly, we shall also need this result about subcritical operators. It follows in a straightforward manner from the proof of Corollary 2.16 in [2].

**Lemma 5.3.** *Let  $T_\sigma^\varphi$  be an oscillatory integral operator that satisfies the first row of estimate (4.5). Then for all  $p \in [1, \infty)$  there is a bound such that for all  $f \in \text{bmo}$  and  $g \in \mathcal{S}$ ,*

$$\|fT_\sigma^\varphi g\|_{L^p} \lesssim \|f\|_{\text{bmo}} \|g\|_{h^p}.$$

With those preliminaries set, we are ready to start showing Theorem 4.1. The proof first splits the operator  $T_\sigma^\Phi$  into a high frequency part and a reducible part, as explained in section 4 of [2]. In short, the amplitude is divided into  $\sigma = \sum_{j=1}^N \sigma_j^H + \sum_{j=1}^N \sigma_j^R$ , and for the high frequency parts it will suffice to treat  $\sigma_1^H$ , while for the reducible part it will be convenient to look at  $\sigma_N^R$ .

### The high frequency part

For this part we can show both results of theorem at once, that is we show estimate (4.8) for the general  $x$ -dependent phases. The amplitude  $\sigma_1^H$  of the high frequency part of the operator is further split into the pieces  $d_+$  and  $a_+$ , as defined in equation (5.5) in [2], together with  $d_-$  and  $a_-$  which are treated similarly. In what follows, we have avoided giving the explicit expressions for the amplitudes,  $a_\pm$  and  $d_\pm$  (which are rather involved), as well as repeating the long arguments in Section 5 of [2], which are the ones that are followed below, albeit with some significant modifications that are displayed here. This decision was made for the sake of brevity, even though it might cause some inconvenience for the reader.

Following the exposition in [2], one can reduce  $T_{d_+}^\Phi$  to

$$T_{d_+}^\Phi(f_1, \dots, f_N) = \int_{\mathbb{R}^{Nn}} \lambda(\cdot, U) \prod_{j=1}^N T_{d_j}^{\varphi_j} f_j \frac{dU}{(1+|U|^2)^M},$$

for an arbitrary positive integer  $M$ , where  $\sup_{x,U} |\partial_x^\alpha \lambda(x, U)| \leq C_\alpha$  for all  $\alpha$  and  $d_j \in S^{-\infty}$  for all  $j$ . It therefore suffices to treat the integrand in this expression, and this is done by Lemma 5.2.

As for  $a_+$ , this amplitude is of the form

$$a_+(x, \Xi) = \sum_{k=1}^{\infty} \hat{a}_k(x, \Xi) = \sum_{k=0}^{\infty} \psi \left( 2^{-k} \sum_{j=1}^N \nabla_j \varphi_j(x, \xi_j) \right) \hat{a}_k(x, \Xi),$$

where  $\psi$  is smooth and annulus-supported and  $|\sum_{j=1}^N \nabla_j \varphi_j(x, \xi_j)| \approx \langle \Xi \rangle$  on the support of  $\hat{a}_k$  ( $\nabla_j$  denotes  $\nabla_{\xi_j}$  with  $\xi_j \in \mathbb{R}^n$ ). We may, without changing the value of the expression, multiply this with

$$\psi_1 \left( 2^{-k} \sum_{j=1}^N \nabla_j \varphi_j(x, \xi_j) \right),$$

for a smooth annulus-supported  $\psi_1$  that equals 1 on the support of  $\psi$ . We can then apply Theorem 5.1 to deduce that

$$T_{\hat{a}_k}^\Phi = \psi_1(2^{-k} D) T_{\hat{a}_k}^\Phi + T_{r_k}^\Phi = \psi_1(2^{-k} D) T_{\hat{a}_k}^\Phi + 2^{-k/(N+1)} T_{r_{1,k}}^\Phi,$$

where  $r_{1,k}(x, \Xi) = 2^{k/(N+1)} r_k(x, \Xi)$ , so that  $r_{1,k} \in S_{1,0}^{m_\sigma - N/(N+1)}(n, N)$ . This last fact follows from the support property of  $\hat{a}_k$ , mentioned above, which  $r_k$  inherits. Since the amplitude  $r_{1,k}$  has its support within that of  $\hat{a}_k$ , we may also multiply  $r_{1,k}$  with

the  $\psi_1$ -expression above without changing its value. Therefore this procedure can be repeated on  $r_{1,k}$ . We do this  $L(N+1)$  times, and thus find that for any integer  $L$ ,

$$\begin{aligned} T_{\hat{a}_k}^\Phi &= \psi_1(2^{-k}D)T_{\hat{a}_k}^\Phi + 2^{-k/(N+1)}\psi_1(2^{-k}D)T_{r_{1,k}}^\Phi + \dots \\ &\quad + 2^{-(L(N+1)-1)k/(N+1)}\psi_1(2^{-k}D)T_{r_{L(N+1)-1,k}}^\Phi + 2^{-Lk}T_{R_k}^\Phi \\ &=: \psi_1(2^{-k}D)T_{a_k}^\Phi + 2^{-Lk}T_{R_k}^\Phi, \end{aligned}$$

where  $a_k \in S_{1,0}^{m_\sigma}(n, N)$  and  $R_k \in S_{1,0}^{m_\sigma - LN}(n, N)$ . In line with the notation of [2], we define

$$\Sigma_a = \sum_{k=1}^{\infty} \psi_1(2^{-k}D)T_{a_k}^\Phi, \quad \Sigma_r = \sum_{k=1}^{\infty} 2^{-Lk}T_{R_k}^\Phi.$$

### Treatment of $\Sigma_a$

Following the steps in paper [2], we find that (see below estimate (5.10) in that paper)

$$(5.1) \quad a_k(x, \Xi) = 2^{-s_0 k} \int \lambda_k(U, x) \pi_k(x, \xi_1) \prod_{j=2}^N \tau_{k,j}(x, \xi_j) \frac{dU}{(1+|U|^2)^M}$$

with  $U = (u_1, \dots, u_N)$ ,  $M$  an arbitrary positive integer and  $|\partial_x^\alpha \lambda_k(x, U)| \leq C_\alpha$  for all multi-indices  $\alpha$ . The amplitudes  $\pi_k$  and  $\tau_{k,j}$  are defined by

$$\begin{aligned} \pi_k(x, \xi_1) &= b_1(x, \xi_1) \Psi(2^{-k} \nabla_x \varphi_1(x, \xi_1)) \\ \tau_{k,j}(x, \xi_j) &= b_j(x, \xi_j) \Theta_j(2^{-k} \nabla_x \varphi_j(x, \xi_j)) \mu(\xi_j) \\ b_1(x, \xi_1) &= |\nabla_x \varphi_1(x, \xi_1)|^{m_1} \mu(\xi_1)^2 \\ b_j(x, \xi_j) &= |\nabla_x \varphi_j(x, \xi_j)|^{m_j} \mu(\xi_j), \quad j \neq 1 \\ \Psi(\xi) &= |\xi|^{m_\sigma - m_1} \psi_1(\xi) e^{-iu_1 \cdot \xi} \\ \Theta_j(\xi) &= |\xi|^{-m_j} \theta_1(\xi) e^{-iu_j \cdot \xi}, \end{aligned}$$

where  $\mu$  is a smooth radial function that vanishes in a neighbourhood of the origin and equals 1 outside of a compact set, and the parameters  $m_j$  are given by

$$\begin{aligned} m_1 &= m_c(p_1) + s_1 \\ m_j &= \min(m_c(p_j) + s_j, -\varepsilon \gamma_j), \quad j \neq 1 \end{aligned}$$

for an arbitrary  $\varepsilon > 0$ , where  $\gamma_j = 0$  if  $p_j = 2$  and otherwise 1, and

$$s_0 = -m_\sigma + \sum_{j=1}^N m_j = -m_\sigma + m_c(p_1) + s_1 + \sum_{j \neq 1} \min(m_c(p_j) + s_j, -\varepsilon \gamma_j).$$

Here we thus made a change in these variables, which resulted in the overall factor  $2^{-s_0k}$ . Notice that the orders  $m_j$  for  $j \neq 1$  have been chosen such that estimate (2.5) will hold for the resulting operator  $\Theta_j$ . When  $p_j=2$  estimate (2.5) is not used, and therefore this order need not be negative in that case.

We therefore have

$$\begin{aligned} \Sigma_a(f_1, \dots, f_N)(x) &= \sum_{k=1}^{\infty} \psi(2^{-k}D) T_{a_k}^{\Phi}(f_1, \dots, f_N)(x) \\ &= \int \sum_{k=1}^{\infty} 2^{-s_0k} \psi(2^{-k}D) \left[ \lambda_k T_{\pi_k}^{\varphi_1} f_1 \prod_{j \neq 1} T_{\tau_{k,j}}^{\varphi_j} f_j \right] (x) \frac{dU}{(1+|U|^2)^M}. \end{aligned}$$

Now, the property (4.1) and the fact that  $k \geq 1$  allow us to write

$$T_{\pi_k}^{\varphi_1} f_1 = T_{\pi_k}^{\varphi_1} |D|^{-s_1} \mu_1(D) |D|^{s_1} \langle D \rangle^{-s_1} \langle D \rangle^{s_1} f_1 =: T_{\pi_k}^{\varphi_1} |D|^{-s_1} S_1 \langle D \rangle^{s_1} f_1,$$

for a suitable  $\mu_1$  that has the properties of  $\mu$ , and similarly for  $T_{\tau_{k,j}}^{\varphi_j} f_j$ . We can then utilise that the operators  $S_j = \mu_j(D) |D|^{s_j} \langle D \rangle^{-s_j}$  are bounded  $X^p \rightarrow X^p$  for all  $p \in [0, \infty]$ , because of estimate (2.1). Namely, if we could show that

$$(5.2) \quad \|\langle D \rangle^{s_0} \Sigma_a(|D|^{-s_1} f_1, \dots, |D|^{-s_N} f_N)\|_{X^{p_0}} \lesssim \prod_{j=1}^N \|f_j\|_{X^{p_j}},$$

we would have

$$\begin{aligned} \|\langle D \rangle^{s_0} \Sigma_a(f_1, \dots, f_N)\|_{X^{p_0}} &= \|\langle D \rangle^{s_0} \Sigma_a(|D|^{-s_1} S_1 \langle D \rangle^{s_1} f_1, \dots, |D|^{-s_N} S_N \langle D \rangle^{s_N} f_N)\|_{X^{p_0}} \\ &\lesssim \prod_{j=1}^N \|S_j \langle D \rangle^{s_j} f_j\|_{X^{p_j}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}}, \end{aligned}$$

which is what we want to prove. In order to show estimate (5.2), we include the  $|D|^{-s_j}$  operators into the multilinear operator, by changing the amplitudes  $b_1$  and  $b_j$  into

$$\begin{aligned} \tilde{b}_1(x, \xi_1) &= b(x, \xi_1) |\xi_1|^{-s_1} = |\nabla_x \varphi_1(x, \xi_1)|^{m_1} \mu(\xi_1)^2 |\xi_1|^{-s_1} \\ \tilde{b}_j(x, \xi_j) &= b(x, \xi_j) |\xi_j|^{-s_j} = |\nabla_x \varphi_1(x, \xi_j)|^{m_j} \mu(\xi_j) |\xi_j|^{-s_j}. \end{aligned}$$

It is clear that  $\tilde{b}_j \in S_{1,0}^{m_c(p_j)}$ ,  $j=1, \dots, N$ , and therefore these amplitudes are of the form used in the proof in [2]. We can also make the rewrite

$$\begin{aligned} 2^{-s_0k} \psi_1(2^{-k}D) &= |D|^{-s_0k} \mu_1(D) (|\cdot|^{s_0} \psi_1)(2^{-k}D) =: |D|^{-s_0k} \mu_1(D) \psi_2(2^{-k}D) \\ &= \langle D \rangle^{-s_0} \langle D \rangle^{s_0} \mu_1(D) |D|^{-s_0} \psi_2(2^{-k}D) =: \langle D \rangle^{-s_0} \tilde{S}_0 \psi_2(2^{-k}D), \end{aligned}$$

where  $\psi_2$  is smooth and supported in an annulus and  $\tilde{S}_0$  is bounded  $X^p \rightarrow X^p$  for  $p \in [0, \infty]$ . Now, if we let  $\tilde{\Sigma}_a$  denote the operator  $\Sigma_a$  with  $b_j$  replaced with  $\tilde{b}_j$ ,  $j=1, \dots, N$ , without the overall factor  $2^{-s_0 k}$  and with  $\psi_1$  replaced with  $\psi_2$ , this multilinear operator is precisely of the form used in the proof of Theorem 1.1 in [2], and therefore

$$\|\tilde{\Sigma}_a(f_1, \dots, f_N)\|_{X^{p_0}} \lesssim \prod_{j=1}^N \|f_j\|_{X^{p_j}}.$$

Hence, we have

$$\begin{aligned} \|\langle D \rangle^{s_0} \Sigma_a(|D|^{-s_1} f_1, \dots, |D|^{-s_N} f_N)\|_{X^{p_0}} &= \|\tilde{S}_0 \tilde{\Sigma}_a(f_1, \dots, f_N)\|_{X^{p_0}} \\ &\lesssim \|\tilde{\Sigma}_a(f_1, \dots, f_N)\|_{X^{p_0}} \lesssim \prod_{j=1}^N \|f_j\|_{X^{p_j}}, \end{aligned}$$

as requested.

### Treatment of $\Sigma_r$

This part has the form

$$\Sigma_r = \sum_{k=1}^{\infty} 2^{-Lk} T_{R_k}^{\Phi},$$

where  $R_k \in S_{1,0}^{m_\sigma - LN}$  for a large positive integer  $L$ . We can again make the representation as in equation (5.1), but this time with

$$m_j = m_c(p_j) - L$$

and the overall factor is  $2^{m_0 k}$  with

$$m_0 = \sum_{j=1}^N m_c(p_j) - m_\sigma - L.$$

Therefore, we find that

$$\Sigma_r(f_1, \dots, f_N)(x) = \int \sum_{k=1}^{\infty} 2^{m_0 k} \lambda_k T_{\pi_k}^{\varphi_1} f_1(x) \prod_{j \neq 1} T_{\tau_{k,j}}^{\varphi_j} f_j(x) \frac{dU}{(1+|U|^2)^M},$$

where  $\pi_k \in S_{1,0}^{m_1}$  and  $\tau_k \in S_{1,0}^{m_j}$ . Since  $L$  was arbitrary, Lemma 5.2 can then be applied to show the requested boundedness of this term.

We have therefore shown

**Proposition 5.4.** *If  $\text{supp } \sigma \subset \mathbb{R}^n \times \{\xi \in \mathbb{R}^n; |\xi| \gtrsim 1\}^N$ , then Theorem 4.1 holds.*

Before we move on to the reducible part, we must show the following result, which shows boundedness  $X^{p_1} \times \dots \times X^{p_N} \rightarrow L^{p_0}$  of subcritical operators. This is akin to Proposition 6.1 in [2].

**Proposition 5.5.** *Let  $T_\sigma^\Phi$  be an  $N$ -linear oscillatory operator as in Theorem 4.1, for which the linear operator satisfies estimates (4.5) and (4.6). Then if  $\sigma \in S_{1,0}^{m_\sigma}(n, N)$  with*

$$m_\sigma < \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\ell \neq j} \min(m_c(p_\ell) + s_\ell, -\varepsilon \gamma_j) \right],$$

for  $m_c$  as in Theorem 4.1, and for some  $p_0, \dots, p_N \in [0, \infty]$  with  $1/p_0 = \sum_{j=1}^N 1/p_j$ ,  $s_1, \dots, s_N \in \mathbb{R}$  and  $\varepsilon > 0$ . Then the bound

$$(5.3) \quad \|T_\sigma^\Phi(f_1, \dots, f_N)\|_{L^{p_0}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}}$$

is uniform in  $f_1, \dots, f_N \in \mathcal{S}$ . Furthermore, if  $s_0 \in \mathbb{R}$  and

$$0 \leq s_0 < -m_\sigma + \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\ell \neq j} \min(m_c(p_\ell) + s_\ell, -\varepsilon \gamma_\ell) \right],$$

then

$$(5.4) \quad \|\langle D \rangle^{s_0} T_\sigma^\Phi(f_1, \dots, f_N)\|_{L^{p_0}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}}.$$

*Proof.* We divide the operator into high frequency and reducible parts. The high frequency parts are dealt with by Proposition 5.4 for all cases except  $\text{bmo} \times \dots \times \text{bmo} \rightarrow L^\infty$ , so this must be shown. Following the reasoning from above, the part from the amplitude  $d_+$  maps into  $L^\infty$  (using Lemma 5.2), and this is also true for the part  $\Sigma_r$ . Only the part  $\Sigma_a$  hence remains to treat.

Note first that since the amplitude is subcritical, we have

$$(5.5) \quad \begin{aligned} m_\sigma &= -s_0 + \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\ell \neq j} \min(m_c(p_\ell) + s_\ell, -\varepsilon \gamma_j) \right] - \delta \\ &\leq -s_0 + m_c(p_1) + s_1 + \sum_{\ell \neq 1} \min(m_c(p_\ell) + s_\ell, -\varepsilon \gamma_1) - \delta, \end{aligned}$$

for some  $\delta > 0$ . For simplicity, we take  $m_\sigma$  to be the value given by the second row of (5.5). Then the triangle inequality and estimate (2.4) yield that

$$\left\| \langle D \rangle^{s_0} \sum_{k=1}^{\infty} \psi_1(2^{-k} D) T_{a_k}^\Phi(f_1, \dots, f_N) \right\|_{L^\infty}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-\delta k/(N+1)} \|T_{2^{(s_0+\delta/(N+1))k} a_k}^{\Phi}(f_1, \dots, f_N)\|_{L^\infty}.$$

Now we let

$$\begin{aligned} m_1 &= m_c(p_1) + s_1 - \delta/(N+1) \\ m_j &= \min(m_c(p_j) + s_j, -\varepsilon\gamma_1) - \delta/(N+1), \quad j \neq 1 \end{aligned}$$

so that

$$-m_\sigma + \sum_{j=1}^N m_j = s_0 + \delta/(N+1).$$

Therefore, following the reasoning of representation (5.1), we have

$$a_k(x, \Xi) = 2^{-(s_0+\delta/(N+1))k} \int \lambda_k(U, x) \pi_k(x, \xi_1) \prod_{j=2}^N \tau_{k,j}(x, \xi_j) \frac{dU}{(1+|U|^2)^M}.$$

Following a similar line of reasoning as the one regarding estimate (5.2), it is enough to show the bound

$$\sup_{k \geq 1} \|T_{2^{(s_0+\delta/(N+1))k} a_k}^{\Phi}(|D|^{s_1} f_1, \dots, |D|^{s_N} f_N)\|_{L^\infty} \lesssim \prod_{j=1}^N \|f_j\|_{X^{p_j}}.$$

Here, the amplitudes  $b_j$  have therefore been changed into  $\tilde{b}$  with  $\tilde{b}_j(x, \xi) = b_j(x, \xi)|\xi|^{-s_j}$ , which are in particular of subcritical order  $m_c(p_j) - \delta/(N+1)$ . We can then use Theorem 5.1 to write

$$\begin{aligned} T_{\pi_k}^{\varphi_1} |D|^{-s_1} &= \Psi(2^{-k} D) T_{\tilde{b}_1}^{\varphi_1} + 2^{-k/2} T_{r_{1,k}}^{\varphi_1} \\ T_{\tau_{k,j}}^{\varphi_j} |D|^{-s_j} &= \mu(D) \Theta_j(2^{-k} D) T_{\tilde{b}_j}^{\varphi_j} + 2^{-k/2} T_{r_{j,k}}^{\varphi_j}, \end{aligned}$$

where the amplitudes  $r_{j,k} \in S_{1,0}^{m_c(p_j) - \delta/(N+1)}$  for  $1 \leq j \leq N$ . Hence we have

$$\begin{aligned} &2^{s_0+\delta/(N+1)k} T_{a_k}^{\Phi}(|D|^{-s_1} f_1, \dots, |D|^{-s_N} f_N) \\ &= \int \lambda_k(U, \cdot) \left( \Psi(2^{-k} D) T_{\tilde{b}_1}^{\varphi_1} + 2^{-k/2} T_{r_{1,k}}^{\varphi_1} \right) f_1 \\ &\quad \times \prod_{j \neq 1} \left( \mu(D) \Theta_j(2^{-k} D) T_{\tilde{b}_j}^{\varphi_j} + 2^{-k/2} T_{r_{j,k}}^{\varphi_j} \right) f_j \frac{dU}{(1+|U|^2)^M}. \end{aligned}$$

Then boundedness of this product follows from Lemma 5.1.

It remains to show boundedness of the reducible part, and it suffices to treat  $T_{\sigma_N^{\Phi}}^{\Phi}$ . We start by showing estimate (5.3). From presentation 6.2 in [2], it follows that it is enough to show that

$$\sup_k \|T_{c_k}^{\Phi_{N-1}}(f_1, \dots, f_{N-1})T_{\zeta}^{\varphi_N} f_N\|_{L^{p_0}} \lesssim \prod_{j=1}^N \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}},$$

where  $\Phi_{N-1}(x, \xi_1, \dots, \xi_{N-1}) = \varphi_1(x, \xi_1) + \dots + \varphi_{N-1}(x, \xi_{N-1})$ ,  $c_k \in S_{1,0}^{m_{\sigma}}(n, N-1)$  uniformly in  $k \in \mathbb{Z}^n$  and  $\zeta \in \mathcal{C}_c^{\infty}$ . Hölder's inequality yields that

$$\|T_{c_k}^{\Phi_{N-1}}(f_1, \dots, f_{N-1})T_{\zeta}^{\varphi_N} f_N\|_{L^{p_0}} \lesssim \|T_{c_k}^{\Phi_{N-1}}(f_1, \dots, f_{N-1})\|_{L^{p^*}} \|T_{\zeta}^{\varphi_N} f_N\|_{L^{p_N}},$$

where  $1/p^* = 1/p_0 - 1/p_N = \sum_{j=1}^{N-1} 1/p_j$ . Now, since estimates (4.5) and (4.6) ensure that estimate (5.3) is true in the case  $N=1$ , and we have shown boundedness for the high frequency part, we can by induction assume that any  $(N-1)$ -linear operator that satisfies the conditions of Proposition 5.5 will satisfy estimate (5.3). These conditions are all met, since in particular

$$\begin{aligned} m_{\sigma} &= \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\substack{1 \leq \ell \leq N \\ \ell \neq j}} \min(m_c(p_{\ell}) + s_{\ell}, -\varepsilon\gamma_j) \right] - \delta \\ &\leq \min_{1 \leq j \leq N-1} \left[ m_c(p_j) + s_j + \sum_{\substack{1 \leq \ell \leq N-1 \\ \ell \neq j}} \min(m_c(p_{\ell}) + s_{\ell}, -\varepsilon\gamma_j) \right] - \delta. \end{aligned}$$

Therefore, we have by induction that

$$\|T_{c_k}^{\Phi_{N-1}}(f_1, \dots, f_{N-1})\|_{L^{p^*}} \lesssim \prod_{j=1}^{N-1} \|\langle D \rangle^{s_j} f_j\|_{X^{p_j}},$$

while estimates (4.5) and (4.6) show that

$$\|T_{\zeta}^{\varphi_N} f_N\|_{L^{p_N}} = \|T_{\zeta \langle \cdot \rangle^{-s_N}}^{\varphi_N} \langle D \rangle^{s_N} f_N\|_{L^{p_N}} \lesssim \|\langle D \rangle^{s_N} f_N\|_{X^{p_N}}.$$

Finally, to show estimate (5.4), we start by writing  $\langle D \rangle^{s_0} T_{\sigma_N^{\Phi}}^{\Phi} = T_{\hat{\sigma}}^{\Phi}$ , so that the amplitude  $\hat{\sigma}$  is given by

$$\hat{\sigma}(\Xi) = \left\langle \sum_{j=1}^N \xi_j \right\rangle^{s_0} \sigma_N^{\mathbb{R}}(\Xi) = \left\langle \sum_{j=1}^{N-1} \xi_j \right\rangle^{s_0} \frac{\left\langle \sum_{j=1}^N \xi_j \right\rangle^{s_0} \sigma_N^{\mathbb{R}}(\Xi)}{\left\langle \sum_{j=1}^{N-1} \xi_j \right\rangle^{s_0}} =: \left\langle \sum_{j=1}^{N-1} \xi_j \right\rangle^{s_0} \check{\sigma}(\Xi),$$

where  $\bar{\sigma} \in S_{1,0}^{m_\sigma}(n, N)$  if  $|\xi_N| \leq 1/2$ . We can choose the low frequency cut-off function  $\mu$  in our construction such that this holds. With an amplitude of this form, it is, following the construction of  $c_k$  above equation (6.2) in [2], enough to show the estimate

$$(5.6) \quad \begin{aligned} & \sup_k \|T_{c_k}^{\Phi_{N-1}} \langle \xi_1 + \dots + \xi_{N-1} \rangle^{s_0} (f_1, \dots, f_{N-1}) T_\zeta^{\varphi_N} f_N\|_{L^{p_0}} \\ &= \sup_k \| \langle D \rangle^{s_0} T_{c_k}^{\Phi_{N-1}} (f_1, \dots, f_{N-1}) T_\zeta^{\varphi_N} f_N\|_{L^{p_0}} \lesssim \prod_{j=1}^N \| \langle D \rangle^{s_j} f_j\|_{X^{p_j}}. \end{aligned}$$

The rest of the proof then follows the same lines as those of the proof of estimate (5.3).  $\square$

We are now ready to show boundedness of the reducible part of the operator with critical amplitude.

### Treatment of the reducible part

We show first estimate (4.7). Just as in the proof of Proposition 5.5, it suffices to control  $T_{c_k}^{\Phi_{N-1}}(f_1, \dots, f_{N-1})T_\zeta^{\varphi_N} f_N$  uniformly in  $k$ . We will once again use an induction argument, and therefore note that

$$\begin{aligned} m_\sigma &\leq \min_{1 \leq j \leq N} \left[ m_c(p_j) + s_j + \sum_{\ell \neq j} \min(m_c(p_\ell) + s_\ell, -\varepsilon\gamma_j) \right] \\ &\leq \min_{1 \leq j \leq N-1} \left[ m_c(p_j) + s_j + \sum_{\substack{1 \leq \ell \leq N-1 \\ \ell \neq j}} \min(m_c(p_\ell) + s_\ell, -\varepsilon\gamma_j) \right] \\ &\quad + \min(m_c(p_N) + s_N, -\varepsilon\gamma_N). \end{aligned}$$

Therefore, except when  $p_N=2$ , we have  $\min(m_c(p_N) + s_N, -\varepsilon\gamma_N) \leq -\varepsilon < 0$ , and hence  $m_\sigma$  is subcritical when regarded as an order of an  $(N-1)$ -linear operator. Hence, Proposition 5.5 is applicable to  $T_{c_k}^{\Phi_{N-1}}$  in these cases.

The rest of the proof then follows the same lines as the treatment of the reducible part in [2], with the notable alteration that our Lemma 5.3 plays the role of Corollary 2.16 in [2].

To show estimate (4.8), it is, just as in the proof of Proposition 5.5, enough to show estimate (5.6). One can then use the same line of reasoning as for the proof of estimate (4.7).

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