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Integer Solutions to Some Diophantine Equations of Leech Type with Geometric Applications

Ralph Høibakk [1], Dag Lukkassen [1], Annette Meidell [1,*] and Lars-Erik Persson [1,2]

[1] Department of Computer Science and Computational Engineering, UiT The Arctic University of Norway, 8514 Narvik, Norway; hoibakk@yahoo.no (R.H.); dag.lukkassen@uit.no (D.L.); lpe090@post.uit.no (L.-E.P.)
[2] Department of Mathematics, Uppsala University, 753 10 Uppsala, Sweden
* Correspondence: annette.meidell@uit.no

Abstract

In this paper, we derive integer pseudo-parametric solutions to two sets of Diophantine equations. Moreover, we describe the so-called Double Crossed Ladder (DCL) and show how these results can be used to calculate an infinite number of integer solutions of its sides. In addition, we describe the fact that these results can be used to derive some corresponding sets of integer sides of more complex geometric structures.

Keywords: simultaneous Diophantine equations; parametric and pseudo-parametric solutions; Double Crossed Ladder problem; integer-sided geometric structures

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1. Introduction

The study of finding solutions to simultaneous Diophantine equations is old. Diophantus himself (∼300 AD) formulated the following problem: Find a right-angled triangle where the area added to the hypotenuse is a cube and the circumference is a square (see, e.g., [1]). This leads to the following three simultaneous equations:

$$\begin{aligned} x^2 + y^2 &= h^2, \\ \frac{xy}{2} + h &= K^3, \\ x + y + h &= Q^2. \end{aligned}$$

Without the modern use of coefficients and negative numbers, Diophantus arrived at a numerical solution:

$$(x, y, h, K, Q) = (\frac{24121185}{628864}, 2, \frac{24153953}{628864}, \frac{17}{4}, \frac{71}{8}).$$

Leonhard Euler (18th century) formulated two sets of three simultaneous quartic Diophantine equations (see, e.g., [2]):

$$\begin{aligned} x^2 + y^2 &= u^2, \\ x^2 + z^2 &= v^2, \\ y^2 + z^2 &= w^2, \end{aligned}$$

and

$$\begin{aligned} x^2 - y^2 &= u^2, \\ x^2 - z^2 &= v^2, \\ y^2 - z^2 &= w^2. \end{aligned}$$

He ventured to determine an infinite number of integer solutions in the form of a parametric representation in two parameters. For the first set, the solution is of degree 6 in the parameters, and of degree 10 in the second set. His solution to the last set is

$$\begin{aligned} x &= (r^2+s^2)(r^8+28r^6s^2+6r^4s^4+28r^2s^6+s^8), \\ y &= -(r^2+s^2)(r^8-4r^6s^2+70r^4s^4-4r^2s^6+s^8), \\ z &= (r^2-s^2)(r^8-20r^6s^2-26s^4s^4-20r^2s^6+s^8), \\ u &= 8rs(r^4-s^4)(r^4+6r^2s^2+s^4), \\ v &= 2rs(5r^8+12r^6s^2+30r^4s^4+12r^2s^6+5s^8), \\ w &= 2rs(3r^8-12r^6s^2-46r^4s^4-12r^2s^4+3s^8). \end{aligned}$$

With the study of cubic curves in the 19th and 20th century (see, e.g., [3–5]), new methods have been developed to solve more complex sets of simultaneous Diophantine equations (see, e.g., [6–22]).

Our main source of inspiration is that John Leech in 1981 wrote the article [23] with the remarkable title "Two Diophantine Birds With One Stone". Indeed, he discovered a surprising method to determine an infinite number of integer solutions to the set of equations

$$x_l^2(1,1,1,1) - 3(y_l^2, z_l^2, (y_l+z_l)^2, (y_l-z_l)^2) = (u_l^2, v_l^2, r_l^2, s_l^2). \tag{1}$$

and

$$x_1^2(1,1,1,1) + (y_1^2, z_1^2, (y_1+z_1)^2, (y_1-z_1)^2) = (u_1^2, v_1^2, r_1^2, s_1^2), \tag{2}$$

where $(x_i, y_i, z_i, u_i, v_i, r_i, s_i) \in \mathbb{Z}$.

Inspired by the papers by Leech, the main objectives of our paper are as follows:

(a) To determine a larger set of integer solutions to the set of equations

$$x^2(1,1,1,1) + N(y^2, z^2, (y+z)^2, (y-z)^2) = (u^2, v^2, r^2, s^2)$$

for $N = 3$, and secondly to develop a new direct method for obtaining solutions for $N = 1$.

(b) To determine the requirements for integer-sided Double Crossed Ladders (see Figure 1), and use the results from (a) to deduce the infinite number of integer-sided Double Crossed Ladders. Finally, we will use this on even more complex geometric structures.

In Section 2 of this paper, we further develop the ideas of Leech and even point out a "hidden Diophantine bird", which can be of independent interest. Specifically, in this paper, we shortly reiterate Leech's solutions of (2) in order to use his method to derive pseudo-parametric integer solutions of the following set of simultaneous Diophantine equations:

$$x_3^2(1,1,1,1) + 3(y_3^2, z_3^2, (y_3+z_3)^2, (y_3-z_3)^2) = (u_3^2, v_3^2, r_3^2, s_3^2). \tag{3}$$

The main result of Section 2 is formulated in Theorem 1.

In Section 3, we describe a new method to derive the pseudo-parametric integer solutions of (2). The main result is presented in Theorem 2.

In Section 4, we present some new geometric applications of our results. We first introduce what we call Double Crossed Ladders (see Figure 1), then investigate the requirement for integer solutions of all sides of it, and finally present a set of solutions in Theorem 3. Moreover, we present some more geometric structures (see Figures 2–4) for which our methods can be used to derive requirements for an infinite number of solutions with integer values of each side.

The last section includes some final comments and concluding remarks, and we have also included some open questions for further research.

2. New Integer Solutions of (3), the Hidden Diophantine Bird

By developing Leech's basic idea further, we will deduce infinitely many integer solutions to (3). In order to perform this, we first need to show Leech's solutions to his set of equations in (1).

Leech starts from a parametric solution to a reduced set,

$$x_l^2(1,1,1) - 3(y_l^2, z_l^2, (y_l - z_l)^2) = (u_l^2, v_l^2, s_l^2),$$

namely

$$\begin{aligned}(x_l, y_l, z_l, (y_i - z_i), u_i, v_i) \quad = \quad & (2(3m^2+n^2), (3m-n)(m+n), (3m+n)(m-n), \\ & 4mn, \left(-6mn + 3m^2 - n^2\right), \left(6mn + 3m^2 - n^2\right))\end{aligned} \tag{4}$$

where $(m, n) \in \mathbb{Z}$.

In order to complete the solution to (1), Leech also needed to find a solution to $x_l^2 - 3(y_l + z_l)^2 = r_l^2$, which requires that

$$r_l^2 = (2(3m^2+n^2))^2 - 3(2\left(3m^2 - n^2\right))^2 = -8\left(-12m^2n^2 + 9m^4 + n^4\right) \tag{5}$$

i.e.,

$$r_l^2 = -8\left(-12m^2n^2 + 9m^4 + n^4\right). \tag{6}$$

Solutions to (6) can be solved by substituting

$$\begin{aligned} m &= a+b, \\ n &= a-b, \end{aligned} \tag{7}$$

where $(a, b) \in \mathbb{Z}$. This leads to the requirement

$$f^4 - 16f^3 - 42f^2 - 16f + 1^4 = Q^2, \tag{8}$$

where $Q \in \mathbb{Q}$ and $f = a/b$.

Equation (8) can be transformed into a cubic curve, which will give an infinite number of rational solutions for f of increasing order. Each solution for f will lead to corresponding numerical values for a ans b and further to m and n and then ultimately to r_l. The first-order solution is $f = -4/13$, giving $(a, b) = (4, -13)$. Substituting these values for (a, b) in (7) lead to $(m, n) = (-9, 17)$, and then by substituting (m, n) in (6) to calculate $r_l = \pm 1052$. This will lead to an integer solution to (1): $(x_l, y_l, z_l) = (266, 65, 88)$ after eliminating common factors.

To determine solutions to (2), Leech rewrote the basic requirement for a solution to (1), (5), i.e.,

$$r_l^2 = (2(3m^2+n^2))^2 - 3(2\left(3m^2 - n^2\right))^2$$

to read

$$(2(3m^2+n^2))^2 - r_l^2 = 3(2\left(3m^2-n^2\right))^2,$$

or factorized:

$$(2(3m^2+n^2)+r_l)(2(3m^2+n^2)-r_l)) = 3(2\left(3m^2-n^2\right))^2. \tag{9}$$

He then defined

$$\begin{aligned} h^2 &= 2(3m^2+n^2)+r_l, \\ 3k^2 &= 2(3m^2+n^2)-r_l, \end{aligned} \tag{10}$$

where $(h,k) \in \mathbb{Z}$, i.e.,

$$hk = 2\left(3m^2-n^2\right), \tag{11}$$

and

$$\begin{aligned} (h^2-3k^2) &= 2r_l, \\ (h^2+3k^2) &= 4(3m^2+n^2). \end{aligned} \tag{12}$$

From these calculations and the fact that the integer solution to (1) and to (2), found by computer search, exhibits the same numeric values to the variables $u_1 = u_l$ and $v_1 = v_l$, Leech deduced a pseudo-parametric solution to (2):

$$\begin{aligned} x_1 &= (r_l+hk) = h^2-4n^2 = 12m^2-3k^2, \\ y_1 &= 2(hn+3km), \\ z_1 &= 2(hn-3km). \end{aligned}$$

For $(m,n,h,k,r_l) = (-9,17,46,-2,1052)$, we arrive at $(x_1,y_1,z_1) = (120,209,182)$ after eliminating common factors.

Leech's mathematical "stone" hit "Two Diophantine Birds", (1) and (2). What he missed is that his "stone", i.e., his method, can be used to bring down even a third "Diophantine Bird", (3).

Computer search shows that integer solutions to (3) also exhibits the same numerical values to the variables u_3 and v_3 as in both (1) and (2). This allows us to also deduce a new pseudo-parametric solution to (3).

From (9) and (10), we can formulate the following two identities:

$$(h^2-3(2m)^2)^2 + 3(4hm)^2 = \left(h^2+3(2m)^2\right)^2,$$

and

$$((2n)^2-3k^2)^2 + 3(4nk)^2 = \left((2n)^2+3k^2\right)^2.$$

Comparing with the numerical values found by computer search and the numerical values determined for (m,n,h,k), we see that

$$\left|(h^2-3(2m)^2\right| \text{ and } \left|(2n)^2-3k^2\right| \text{ are both equal to the calculated } |x_3|, \tag{13}$$

$$(|4hm|, |4nk|) \text{ are equal to the calculated } (|y+z|_3, |y-z|_3),$$

and

$$(\left|h^2+3(2m)^2\right|, \left|(2n)^2+3k^2\right|) \text{ are equal to the calculated } (|r_3|, |s_3|),$$

where $|\ |$ denotes the absolute value. From (13), we can deduce that

$$\begin{aligned} y_3 &= 2(hm + kn) \\ &\text{and} \\ z_3 &= 2(hm - kn). \end{aligned} \tag{14}$$

This assumption proves right for the calculated numerical values of (m, n, h, k). For generality we need to show that the symbolic expressions found for (x_3, y_3, z_3) in (13) and (14) lead to corresponding symbolic values for u_3 and v_3. Since the assumption is that $(u_3, v_3) = (u_l, v_l)$, we must require that

$$\begin{aligned} u_3^2 &= x_3^2 + 3y_3^2 = \left(-6mn + 3m^2 - n^2\right)^2, \\ &\text{and} \\ v_3^2 &= x_3^2 + 3z_3^2 = \left(6mn + 3m^2 - n^2\right)^2, \end{aligned} \tag{15}$$

where (u_l, v_l) has been derived from (4). By using the symbolic values for (x_3, y_3, z_3) from (13), (14), and also the relations (10)–(12), the requirements (15) are confirmed. We then have proven the following:

Theorem 1. *The following pseudo-parametric representation gives integer solutions to (3):*

$$\begin{aligned} x_3 &= \frac{1}{8}(r_l - hk) = \frac{1}{8}\left(h^2 - 3(2m)^2\right) = \frac{1}{8}\left((2n)^2 - 3k^2\right), \\ y_3 &= \frac{1}{4}(hm + kn), \\ z_3 &= \frac{1}{4}(hm - kn). \\ u_3 &= \frac{1}{4}(6mn + 3m^2 - n^2), \\ u_4 &= \frac{1}{4}(-6mn + 3m^2 - n^2), \\ r_3 &= \frac{1}{8}\left(h^2 + 3(2m)^2\right), \\ s_3 &= \frac{1}{8}\left((2n)^2 + 3k^2\right), \end{aligned}$$

where (m, n, r_l) satisfy (6) and (h, k) satisfy (10).

Example 1. *Below we have tabulated corresponding values for $(m, n, r_l, h, k$ and $x_3, y_3, z_3)$ of increasing order:*

m	n	r_l	h	k
-9	17	± 1052	± 46	∓ 2
-937	3041	$\pm 6307\,492$	± 4178	∓ 3166
$-1807\,937$	4816 657	$\pm 46\,954\,543\,851\,076$	$\pm 4365\,506$	$\mp 6136\,414$

and

x_3	y_3	z_3
143	95	112
865 007	$-1428\,255$	3385 648
2520 748 964 449	5416 110 406 719	9362 390 317 280

Note: In the calculated numerical solutions to (2) and (1) shown above, we used absolute values for (x_i, y_i, z_i).

3. A New Method for Solutions to Equation (2)

A new more direct method than the one described in Section 1 is demonstrated in the following:

The four equations in (2) are

$$\begin{aligned} x_1^2 + y_1^2 &= u_1^2, \\ x_1^2 + z_1^2 &= v_1^2, \end{aligned} \tag{16}$$

and

$$\begin{aligned} x_1^2 + (y_1 + z_1)^2 &= r_1^2, \\ x_1^2 + (y_1 - z_1)^2 &= s_1^2. \end{aligned} \tag{17}$$

From Equation (16), we have that

$$y_1^2 - z_1^2 = u_1^2 - v_1^2,$$

i.e.,

$$y_1^2 + v_1^2 = z_1^2 + u_1^2.$$

The complete parametric integer solution to solve this equation is given by

$$(y_1, v_1, z_1, u_1) = (mq - np, mp + nq, mp - nq, np + mq),$$

where $(m, n, p, q) \in \mathbb{Z}$ (see [24]), leading to

$$x_1^2 = u_1^2 - y_1^2 = v_1^2 - z_1^2 = 4mnpq.$$

For $x_1^2 = 4mnpq$ to be a complete quadratic expression, we must have that

$$(m, n, p, q) = (ka^2, lb^2, kc^2, ld^2),$$

where $(k, l) \in \mathbb{Z}$, leading to

$$\begin{aligned} x_1 &= 2klabcd, \\ y_1 &= kl(ad - bc)(ad + bc), \\ z_1 &= (ack - bdl)(ack + bdl). \end{aligned} \tag{18}$$

The alternative expression $(m, n, p, q) = (ka^2, kb^2, lc^2, ld^2)$ implies that kl will be a common factor for (x_1, y_1, z_1), which then can be eliminated without loss of generality. From (17) and (18), it follows that

$$\begin{aligned} r_1^2 &= x_1^2 + (y + z)^2 = a^4c^4k^4 + b^4d^4l^4 + a^4d^4k^2l^2 + b^4c^4k^2l^2 \\ &\quad -2a^2b^2c^4k^3l - 2a^2b^2d^4kl^3 + 2a^4c^2d^2k^3l + 2b^4c^2d^2kl^3, \\ s_1^2 &= x_1^2 + (y_1 - z_1)^2 = a^4c^4k^4 + b^4d^4l^4 + a^4d^4k^2l^2 + b^4c^4k^2l^2 \\ &\quad +2a^2b^2c^4k^3l + 2a^2b^2d^4kl^3 - 2a^4c^2d^2k^3l - 2b^4c^2d^2kl^3. \end{aligned} \tag{19}$$

We can then deduce

$$r_1^2 - s_1^2 = (r_1 + s_1)(r_1 - s_1) = 4kl(ad - bc)(ad + bc)(ack - bdl)(ack + bdl).$$

From this equation we can arrive at representations for $(r_1 + s_1)$ and $(r_1 - s_1)$. To separate these representations from the still undefined r_1 and s_1, we have in the following chosen to give them separate indices, r_{11} and s_{11}. For generality $(r_{11} + s_{11})$ and $(r_{11} - s_{11})$ must be of the same degree in the variables (k, l, a, b, c, d). The alternatives $(r_1 + s_1) = 2kl(ad - bc)(ad + bc)$ and $(r_1 - s_1) = (ack - bdl)(ack + bdl)$ lead to a trivial solution. The other permutations of the variables all lead to the same end result.

In the following we use the fact that

$$\begin{aligned} r_{11} + s_{11} &= 2k(ad + bc)(ack + bdl), \\ r_{11} - s_{11} &= 2l(ad - bc)(ack - bdl), \end{aligned}$$

leading to

$$\begin{aligned} r_{11} &= abc^2k^2 - abd^2l^2 + a^2cdk^2 + b^2cdl^2 - abc^2kl + abd^2kl + a^2cdkl + b^2cdkl, \\ s_{11} &= abc^2k^2 + abd^2l^2 + a^2cdk^2 - b^2cdl^2 + abc^2kl + abd^2kl - a^2cdkl + b^2cdkl. \end{aligned} \tag{20}$$

We must require that

$$r_{11}^2 - r_1^2 = 0 \quad \text{and} \quad s_{11}^2 - s_1^2 = 0$$

from (19) and (20). These two equations turn out to be identical:

$$\begin{gathered} a^4c^4k^4 + b^4d^4l^4 - a^2b^2c^4k^4 - a^2b^2d^4l^4 - a^4c^2d^2k^4 - b^4c^2d^2l^4 + a^4d^4k^2l^2 + b^4c^4k^2l^2 \\ -a^2b^2c^4k^2l^2 - a^2b^2d^4k^2l^2 - a^4c^2d^2k^2l^2 - b^4c^2d^2k^2l^2 - 2a^3bc^3dk^4 + 2ab^3cd^3l^4 - 4a^2b^2c^2d^2kl^3 \\ -4a^2b^2c^2d^2k^3l + 2ab^3c^3dkl^3 - 2ab^3c^3dk^3l + 2a^3bcd^3kl^3 - 2a^3bcd^3k^3l - 2ab^3cd^3k^2l^2 + 2a^3bc^3dk^2l^2 \\ = 0. \end{gathered}$$

For $k = -2$, $l = 1$, we arrive at

$$\begin{aligned} &\left(4a^2c^2 - 2a^2d^2 - 2b^2c^2 + b^2d^2 - 2abc^2 + abd^2 + 2a^2cd - b^2cd - 4abcd\right) \\ &\left(4a^2c^2 - 2a^2d^2 - 2b^2c^2 + b^2d^2 + 2abc^2 - abd^2 - 2a^2cd + b^2cd - 4abcd\right) \\ = {} & 0. \end{aligned} \tag{21}$$

The two second-degree polynomes both lead to the same end result when they are set equal to 0. Note that

$$\frac{k}{l} = -2 \quad \text{or} \quad \frac{k}{l} = -\frac{1}{2}$$

will lead to the same end result.

Remark 1. *The only value for $\frac{k}{l}$ that we have found that leads to integer solutions for a, b, c, and d are $\frac{k}{l} = (-2)$ and $\frac{k}{l} = (-\frac{1}{2})$.*

For further calculation, we use the following requirement from (21):

$$\left(4a^2c^2 - 2a^2d^2 - 2b^2c^2 + b^2d^2 + 2abc^2 - abd^2 - 2a^2cd + b^2cd - 4abcd\right) = 0. \tag{22}$$

Solving for a and b, we find that

$$\begin{aligned} a &= 2c^2 - d^2 + \sqrt{24cd^3 - 48c^3d - 12c^2d^2 + 36c^4 + 9d^4} - 4cd, \\ b &= 4cd - 8c^2 + 4d^2. \end{aligned} \tag{23}$$

The root in Equation (23) requires that

$$24cd^3 - 48c^3d - 12c^2d^2 + 36c^4 + 9d^4 = R^2$$

where $R \in \mathbb{Q}$ has an infinite number of integer solutions for c and d that can be found by using the tools for determining rational solutions to cubic curves (see [22]). We can then calculate the corresponding values for (a, b) by inserting (c, d) in (23). We have then proven the following.

Theorem 2. *The following pseudo-parametric representation gives integer solutions to (2):*

$$\begin{aligned} x_1 &= 2abcd \\ y_1 &= a^2d^2 - b^2c^2 \\ z_1 &= \tfrac{1}{2}(4a^2c^2 - b^2d^2) \\ u_1 &= a^2d^2 + b^2c^2 \\ v_1 &= \tfrac{1}{2}(4a^2c^2 + b^2d^2) \\ r_1 &= \tfrac{1}{2}(-6a^2cd - 2abc^2 + abd^2 + 3b^2cd) \\ s_1 &= \tfrac{1}{2}(-6abc^2 + 3abd^2 - 2a^2cd + b^2cd) \end{aligned} \tag{24}$$

where the corresponding values for $a, b, c,$ and d can be calculated from (22).

Example 2. *Below we have tabulated corresponding values of (a, b, c, d) and (x_1, y_1, z_1) of increasing order:*

a	b	c	d	x_1	y_1	z_1
1	5	3	4	120	209	182
35	19	27	−68	2441 880	5401 231	951 418
1379	1273	935	−2808	9217886 998 320	13577473696799	3063901180078

(25)

Remark 2. *From (22) and (24), we can deduce the following linear requirement between the solutions to (2):*

$$-x_1 - (u_1 - v_1) + \frac{1}{2}(r_1 - s_1) = 0.$$

We observe the striking observation that there exists such a linear connection between the solutions to (2).

From the solutions to (1) and (3) shown in Section 2 and to (2), we can deduce solutions to (1) and to (3) expressed with the parameters $a, b, c,$ and d. We arrive at

$$\begin{aligned} h &= 2(2a^2 - b^2), \\ k &= \left(2c^2 - d^2\right), \\ m &= -\frac{1}{3}\left(2a^2 + b^2\right), \\ n &= \frac{1}{2}\left(2c^2 + d^2\right). \end{aligned}$$

Then we find for (1),

$$
\begin{aligned}
x_l &= \frac{1}{2}(3m^2+n^2) = -(2ac-bd)(ad+bc),\\
y_l &= \frac{1}{4}(3m-n)(m+n) = \frac{1}{6}\left(4a^2c^2-4a^2d^2-4b^2c^2+b^2d^2\right),\\
z_l &= \frac{1}{4}(3m+n)(m-n) = \frac{1}{3}\left(4a^2c^2-a^2d^2-b^2c^2+b^2d^2\right),\\
u_l &= \frac{1}{4}\left(-6mn+3m^2-n^2\right) = \frac{1}{2}\left(4a^2c^2+b^2d^2\right),\\
v_l &= \frac{1}{4}\left(6mn+3m^2-n^2\right) = -\left(a^2d^2+b^2c^2\right),\\
r_l &= \pm\frac{1}{4}(h^2-2(3m^2+n^2)) = \pm\frac{1}{2}(2abc^2-abd^2-2a^2cd+b^2cd+4abcd),\\
s_l &= \frac{1}{2}\left(3m^2-n^2\right) = \frac{1}{2}\left(2c^2-d^2\right)\left(2a^2-b^2\right),
\end{aligned}
$$

and for (3),

$$
\begin{aligned}
x_3 &= \frac{1}{8}(r_l-hk) = \frac{1}{8}(h^2-3(2m)^2) = \frac{1}{8}((2n)^2-3k^2) = -\frac{1}{4}(4c^4-8c^2d^2+d^4),\\
y_3 &= \frac{1}{4}(hm+kn) = -\frac{1}{24}\left(16a^4-4b^4-12c^4+3d^4\right),\\
z_3 &= \frac{1}{4}(hm-kn) = -\frac{1}{24}\left(16a^4-4b^4+12c^4-3d^4\right),\\
u_3 &= \frac{1}{4}(6mn+3m^2-n^2) = -\left(a^2d^2+b^2c^2\right),\\
v_3 &= \frac{1}{4}(-6mn+3m^2-n^2) = \frac{1}{2}\left(4a^2c^2+b^2d^2\right),\\
r_3 &= \frac{1}{8}(h^2+3(2m)^2) = \frac{2}{3}\left(-2a^2b^2+4a^4+b^4\right),\\
s_3 &= \frac{1}{8}((2n)^2+3k^2) = \frac{1}{2}\left(-2c^2d^2+4c^4+d^4\right).
\end{aligned}
$$

4. Applications to Double Crossed Ladders and Some Related Constructions

Some sets of Diophantine equations relate to defined geometrical structures where solutions to the equations also lead to integer-sided geometries. Cases in point are the Single Ladder Problem, the Open and Closed Ladder Problem, and the Common Hypotenuse Problem; see, e.g., [14]. In the following, we have taken this a step further and defined more complex integer-sided geometrical structures.

In this paper we use a variation of what is called the Crossed Ladders Problem (CLP), a geometrical/mathematical problem of unknown origin; see, e.g., [15,16]. The original description is as follows: Two ladders of length u and v are leaning against opposite walls of an alley of width x. The distance from the alley floor to where the ladders cross is c. Determine the alley width, x, and the heights, y and z, where the ladders meet the walls when u, v, and c are known.

In this paper we consider a new geometric structure, namely what we have chosen to call the Double Crossed Ladder (see Figure 1). Our objective is to determine the requirements for the sides $(x, y, z, u, v, (r_1+r_2), s)$ to be integer-valued.

Two of the requirements for integer-valued sides are integer solutions to the equations

$$
\begin{aligned}
x_1^2+y^2 &= r_1^2,\\
x_2^2+z^2 &= r_2^2.
\end{aligned}
\tag{26}
$$

From the geometry in Figure 1, we can deduce that

$$x_1 = \frac{xy}{y+z}, \quad x_2 = \frac{xz}{y+z}, \quad r_1 = \frac{(r_1+r_2)y}{y+z}, \quad r_2 = \frac{(r_1+r_2)z}{y+z}.$$

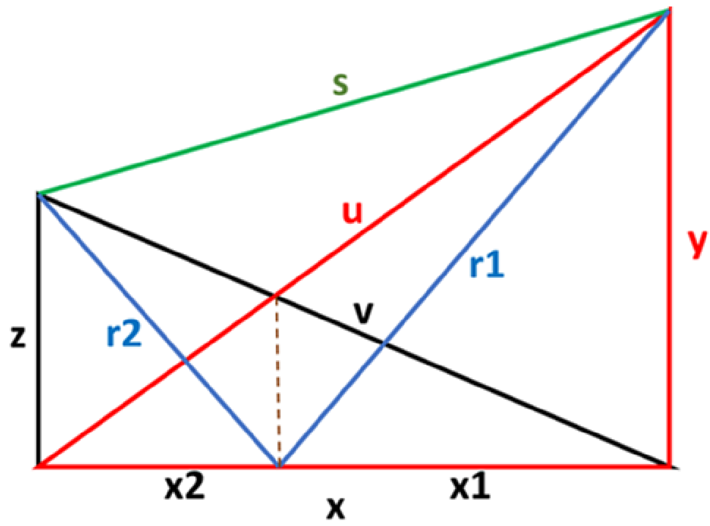

Figure 1. Double Crossed Ladders.

When we substitute these values into Formula (26), they transform to a single requirement:

$$x^2 + (y+z)^2 = (r_1+r_2)^2.$$

Hence, we prove the following:

Theorem 3. *The complete set of requirements for integer-sided Double Crossed Ladder is then comprised of integer solutions to the following set of equations:*

$$\begin{aligned} x^2 + y^2 &= u^2 \\ x^2 + z^2 &= v^2 \\ x^2 + (y+z)^2 &= (r_1+r_2)^2 \\ x^2 + (y-z)^2 &= s^2. \end{aligned}$$

We recognize the set of equations for (2). Using the demonstrated solution in (2), in particular, we can derive an infinite number of integer-sided Double Crossed Ladder. In (25), we have calculated the integer solution to (2). We arrive at the first-order integer-sided solution to the Double Crossed Ladders:

$$(x, y, z, u, v, (r_1+r_2), s) = (120, 209, 182, 241, 218, 409, 123).$$

We may refine our geometric structure in several ways; see, e.g., Figure 2.
From this geometry, we can deduce

$$\begin{aligned} c &= \frac{yz}{y+z}, \\ x_1 &= \frac{xy}{y+z}, \quad x_2 = \frac{xz}{y+z}, \end{aligned}$$

$$\begin{aligned} y_1 &= \frac{y^2}{y+z}, \quad y_2 = \frac{yz}{y+z}, \\ z_1 &= \frac{z^2}{y+z}, \quad z_2 = \frac{yz}{y+z}, \\ u_1 &= \frac{yu}{y+z}, \quad u_2 = \frac{uz}{y+z}, \\ v_1 &= \frac{vy}{y+z}, \quad v_2 = \frac{vz}{y+z}, \\ r_1 &= \frac{ry}{y+z}, \quad r_2 = \frac{rz}{y+z}, \\ s_1 &= \frac{sy}{y+z}, \quad s_2 = \frac{sz}{y+z}. \end{aligned}$$

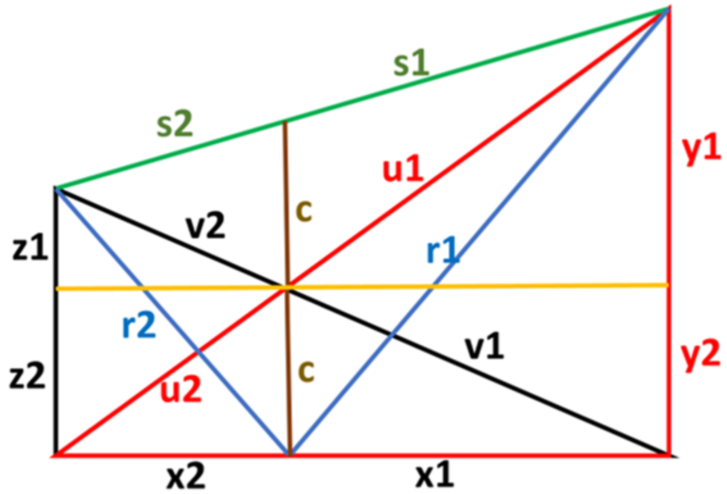

Figure 2. The refined geometric structure; the detailed Double Crossed Ladder.

By using the determined values for (x, y, z, u, v, r, s) and scaling the calculated values in Figure 2 with a common factor

$$S_1 = \frac{(y+z)}{\gcd(x, y, z)},$$

we can determine integer solutions to all sides in Figure 2.

Example 3. *The first-order integer solution to the detailed Double Crossed Ladders is*

$$\begin{aligned} c &= 38038, \\ x_1 &= 25080, \quad x_2 = 21840, \\ y_1 &= 43681, \quad y_2 = 38038, \\ z_1 &= 33124, \quad z_2 = 38038, \\ u_1 &= 50369, \quad u_2 = 43862, \\ v_1 &= 45562, \quad v_2 = 39676, \\ r_1 &= 85481, \quad r_2 = 74438, \\ s_1 &= 25707, \quad s_2 = 23386. \end{aligned}$$

We may refine our geometrical structure even further to what we call Iterative Double Crossed Ladders; see Figure 3.

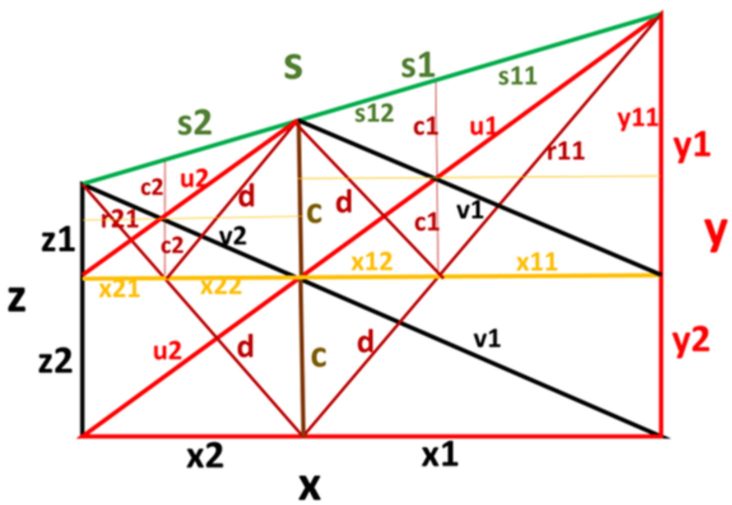

Figure 3. The Iterative Double Crossed Ladders.

We observe that the figures of the two smaller Crossed Ladders in the upper right and upper left corners of the structures are miniaturized versions of the original Crossed Ladders. The individual sections of the Crossed Ladders in the right upper and left corners of Figure 3, as indicated by x_{11}, x_{12}, x_{21}, and x_{22}, can all be determined to be of the type

$$\frac{f(x,y,z,u,v,r,s)}{(y+z)^2},$$

where the nominator is of order 3. We deduce that

$$\begin{aligned}
x_{11} &= \frac{xy^2}{(y+z)^2}, \quad x_{12} = x_{22} = \frac{xyz}{(y+z)^2}, \quad x_{21} = \frac{xz^2}{(y+z)^2}, \\
y_{11} &= \frac{y^3}{(y+z)^2}, \quad y_{12} = c_1 = \frac{y^2 z}{(y+z)^2}, \\
z_{11} &= \frac{z^3}{(y+z)^2}, \quad z_{12} = c_2 = \frac{yz^2}{(y+z)^2}, \\
u_{11} &= \frac{y^2 u}{(y+z)^2}, \quad u_{12} = \frac{yzu}{(y+z)^2}, \\
v_{21} &= \frac{vz^2}{(y+z)^2}, \quad v_{22} = \frac{yzv}{(y+z)^2}, \\
r_{11} &= \frac{yr^2}{(y+z)^2}, \quad r_{12} = r_{22} = d = \frac{yzr}{(y+z)^2}, \quad r_{21} = \frac{z^2 r}{(y+z)^2}, \\
s_{11} &= \frac{y^2 s}{(y+z)^2}, \quad s_{12} = s_{22} = \frac{yzs}{(y+z)^2}, \quad s_{21} = \frac{z^2 s}{(y+z)^2}.
\end{aligned}$$

For the sake of clarity, we have avoided marking all the sections with symbols in Figure 3.

This implies that all the individual sections in Figure 3 can be made an integer by scaling with a factor

$$S_2 = \frac{(y+z)^2}{\gcd(x,y,z)}.$$

From this, we can conclude that the iterative process of sectioning of the sides can be continued ad infinitum, and that by scaling with appropriate factors, all structures will obtain integer sides.

We may even expand our geometry to the following larger structure, which we call the Symmetric Double Crossed Ladders (Figure 4).

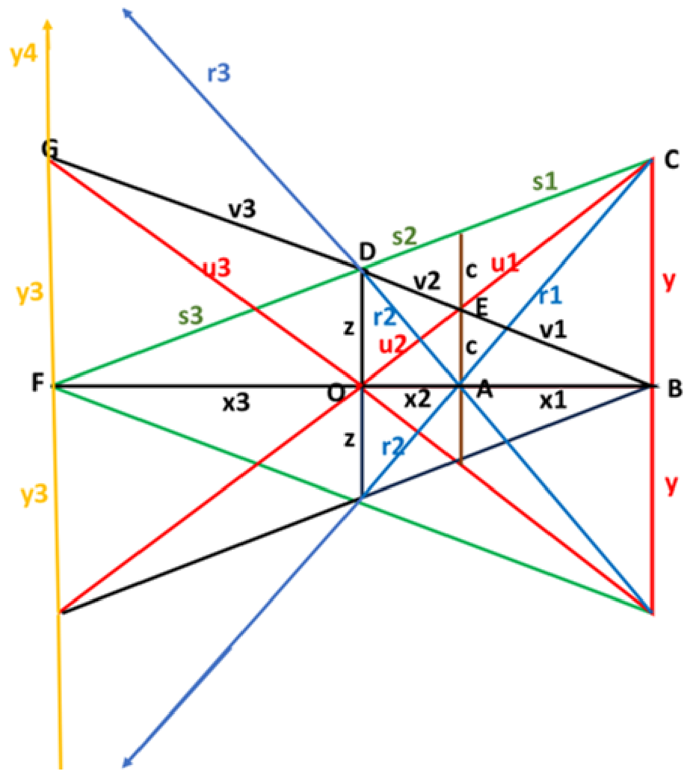

Figure 4. The Symmetric Double Crossed Ladders.

We have omitted the horizontal line through point **E** and other lines that would obscure visual clarity in Figure 4.

From this structure, we can deduce that

$$\begin{aligned} x_3 &= \frac{xz}{y-z}, \\ y_3 &= y_4 = \frac{yz}{y-z}, \\ u_3 &= \frac{uz}{y-z}, \\ v_3 &= \frac{vz}{y-z}, \\ r_3 &= \frac{rz}{y-z}, \\ s_3 &= \frac{sz}{y-z}. \end{aligned}$$

By scaling with

$$S_3 = \frac{(y+z)^2(y-z)}{\gcd(x,y,z)},$$

we can obtain integer solutions to all sides and sections in Figures 3 and 4.

Remark 3. *It is possible to derive an infinite number of integer solutions on the pseudo-parametric form of all sides in all of these figures, but we leave this for the reader to formulate the corresponding theorems in each case.*

Remark 4. *For some basic ideas in this paper, we refer to the Ph.D. thesis of R. Høibakk [13].*

5. Some Final Comments and Concluding Remarks

The objective of this paper was to determine integer solutions to the set of equations

$$x_N^2(1,1,1,1) + N(y_N^2, z_N^2, (y_N + z_N)^2, (y_N - z_N)^2) = (u_N^2, v_N^2, r_N^2, s_N^2)$$

for $N = 3$ and $N = 1$.

The results of our work are as follows: We have introduced the concept of *integer pseudo-parametric solutions* for $N = 3$, and we have derived a new method for finding pseudo-parametric solutions for $N = 1$. We have also used these solutions to obtain an infinite number of integer-sided Double Crossed Ladders and to even more complex geometries.

We believe that there are challenging areas for future research in the study of the chosen simultaneous Diophantine equations. In particular, we conclude with the following questions interesting:

(i) For which $N \in \mathbb{Z}$ will the set of equations have integer solutions? How many? Can they be represented by parameters?
(ii) Will the pseudo-parametric solutions to $N = \{-3, 1, 3\}$ give all integer solutions in each of the sets of equations?
(iii) Will a similar linear connection between the variables that was observed in the set of equations for $N = 1$,
$$-x_1 - (u_1 - v_1) + \frac{1}{2}(r_1 - s_1) = 0,$$
also appear for other values of N?

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