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# Christoffel transform and multiple orthogonal polynomials

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ABSTRACT

We investigate multiple orthogonal polynomials associated with the system of measures obtained by applying a Christoffel transform to each of the orthogonality measures. We present an algorithm for computing the transformed recurrence coefficients and determinantal formulas for the transformed multiple orthogonal polynomials of type I and type II.

We apply these results to show that zeros of multiple orthogonal polynomials of an Angelesco or an AT system interlace with the zeros of the polynomials corresponding to its one-step Christoffel transform. This allows us to prove a number of interlacing properties satisfied by the multiple orthogonality analogues of classical orthogonal polynomials. For the discrete polynomials, this also produces an estimate on the smallest distance between consecutive zeros.

We also identify a connection between the Christoffel transform of orthogonal polynomials and multiple orthogonality systems containing a finitely supported measure. In consequence, the compatibility relations for the nearest neighbour recurrence coefficients provide a new algorithm for the computation of the Jacobi coefficients of the one-step or multi-step Christoffel transforms.

## 1. Introduction

Let $\mu$ be a positive Borel measure on the real line with all the moments $c_j := \int x^j \, \mathrm{d}\mu(x)$ finite. Denote $P_n(x)$, $n \in \mathbb{N} := \{k \in \mathbb{Z} : k \geq 0\}$, to be the monic orthogonal polynomial of degree $n$ with respect to the inner product

$$\langle f(x), g(x) \rangle = \int_{\mathbb{R}} f(x)g(x) \, \mathrm{d}\mu(x). \tag{1}$$

These polynomials satisfy the famous three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x) \tag{2}$$

for some $a_n > 0$ and $b_n \in \mathbb{R}$, called the Jacobi coefficients of $\mu$.

Given a point $z_0 \in \mathbb{R}$, a one-step Christoffel transform of $\mu$ is a new (potentially signed) measure $\hat{\mu}$ defined by

$$\int f(x) \, \mathrm{d}\hat{\mu}(x) = \int f(x)(x - z_0) \, \mathrm{d}\mu(x). \tag{3}$$

More generally, given a polynomial $\Phi(x) = \prod_{j=1}^{m} (x - z_j)$ we define the multi-step Christoffel transform to be $\hat{\mu}$ given by

$$\int f(x) \, \mathrm{d}\hat{\mu}(x) = \int f(x)\Phi(x) \, \mathrm{d}\mu(x), \tag{4}$$

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which, of course, can be viewed as the one-step transform repeated $m$ times for each of the roots of $\Phi(x)$.

It is a natural question to understand the relationship between the orthogonal polynomials $(P_n(x))_{n=0}^\infty$ of $\mu$ and $(\widehat{P}_n(x))_{n=0}^\infty$ of $\widehat{\mu}$. In the one-step case (3) the following simple relation holds true:

$$\widehat{P}_n(x) = \frac{1}{x - z_0}\left(P_{n+1}(x) - \frac{P_{n+1}(z_0)}{P_n(z_0)} P_n(x)\right). \tag{5}$$

More generally, if $\deg \Phi = m$, then $\widehat{P}_n$ can be expressed in terms of $P_n, P_{n+1}, \dots, P_{n+m}$ using the Christoffel determinantal formula, shown in [1].

Another matter of interest here is the relationship between the Jacobi coefficients of $\widehat{\mu}$ and of $\mu$. For the one-step case $\deg \Phi = 1$ there are a number of closely related algorithms (among them are the $qd$ algorithm of Rutishauser [2], Galant's [3], and Gautschi's [4]) that allow to compute $\widehat{a}_n$'s and $\widehat{b}_n$'s recursively from $a_n$'s and $b_n$'s. There exist explicit algorithms for the quadratic factors, $\deg \Phi = 2$, see, e.g., [4, Sect 2.4.3]. For the general case $\deg \Phi = m$, one typically applies the one-step or two-step algorithms repeatedly.

The Christoffel transform, as well as the closely connected topic of the Darboux transformations, is a very well-studied topic both in pure and applied mathematics, see, e.g., [1,3–16].

Now let us introduce multiple orthogonal polynomials with respect to a system of two measures $(\mu_1, \mu_2)$ on $\mathbb{R}$ (we work in the more general setting of $r$ measures from Section 2 onwards). For $(n_1, n_2) \in \mathbb{N}^2$, let $P_{n_1,n_2}(x)$ be a non-zero monic polynomial of degree $n_1 + n_2$ satisfying

$$\int_{\mathbb{R}} P_{n_1,n_2}(x) x^p \, \mathrm{d}\mu_1(x) = 0, \qquad p = 0, 1, \dots, n_1 - 1, \tag{6}$$

$$\int_{\mathbb{R}} P_{n_1,n_2}(x) x^p \, \mathrm{d}\mu_2(x) = 0, \qquad p = 0, 1, \dots, n_2 - 1. \tag{7}$$

$P_{n_1,n_2}(x)$ is then called a multiple orthogonal polynomial at the multi-index $(n_1, n_2)$. We say that $(n_1, n_2)$ is normal for $(\mu_1, \mu_2)$ if such $P_{n_1,n_2}$ exists and is unique.

Assuming sufficiently many indices are normal, these polynomials satisfy [17,18] the nearest neighbour recurrence relations (compare with (2))

$$x P_{n_1,n_2}(x) = P_{n_1+1,n_2}(x) + b_{n_1,n_2;1} P_{n_1,n_2}(x) + a_{n_1,n_2;1} P_{n_1-1,n_2}(x) + a_{n_1,n_2;2} P_{n_1,n_2-1}(x), \tag{8}$$

$$x P_{n_1,n_2}(x) = P_{n_1,n_2+1}(x) + b_{n_1,n_2;2} P_{n_1,n_2}(x) + a_{n_1,n_2;1} P_{n_1-1,n_2}(x) + a_{n_1,n_2;2} P_{n_1,n_2-1}(x). \tag{9}$$

Subtracting (8) and (9) we can also obtain

$$P_{n_1+1,n_2}(x) - P_{n_1,n_2+1}(x) = (b_{n_1,n_2;2} - b_{n_1,n_2;1}) P_{n_1,n_2}(x). \tag{10}$$

The coefficients $\{a_{n_1,n_2;1}, a_{n_1,n_2;2}, b_{n_1,n_2;1}, b_{n_1,n_2;2}\}$ in (8) and (9) are called the nearest neighbour coefficients. These coefficients satisfy a set of partial difference equations [17, Eq. (3.6)–(3.8)] which we will call the compatibility conditions, or CC, for short. [19] showed that these equations provide an algorithm that allows to recursively compute all the nearest neighbour recurrence coefficients $\{a_{n_1,n_2;1}, a_{n_1,n_2;2}, b_{n_1,n_2;1}, b_{n_1,n_2;2}\}$ from the Jacobi coefficients of $\mu_1$ and $\mu_2$ (that is, from $\{a_{n_1,0;1}, a_{0,n_2;2}, b_{n_1,0;1}, b_{0,n_2;2}\}$).

The central idea of our paper is the simple observation that if $\mu_2$ is supported on $N$ distinct points $\{z_j\}_{j=1}^N$, then the multiple orthogonal polynomial $P_{n,N}(x)$ of the system $(\mu_1, \mu_2)$ coincides with $\widehat{P}_n(x)\Phi(x)$, where $\Phi(x) = \prod_{j=1}^N (x - z_j)$ and $\widehat{P}_n(x)$ is the $n$-th orthogonal polynomial of the Christoffel transform $\widehat{\mu}_1$ (4) of $\mu_1$ corresponding to the polynomial $\Phi(x)$. Indeed, $\widehat{P}_n(x)\Phi(x)$ is monic, has the right degree, trivially satisfies (6)–(7), and it only remains to resolve the issue of uniqueness, which we do in Theorem 3.1.

In particular, the nearest neighbour recurrence relation (8) along locations $\{(n, N) : n \in \mathbb{N}\}$ reduces to the three-term recurrence relation for $\widehat{\mu}_1$ (one should observe that $a_{n,N;2} = 0$ for all $n$, see Theorem 3.5), and the nearest neighbour coefficients along these locations coincide with the Jacobi coefficients of $\widehat{\mu}_1$.

Taking the simplest case $N = 1$, one realizes that Christoffel's formula (5) is just (10), while Gautschi's algorithm [4] for computing the Jacobi coefficients of the Christoffel transform is effectively the CC algorithm of [19] (after minor modifications related to restricting the coefficients to the strip $\mathbb{N} \times \{0, 1\}$, see Section 3.2). Furthermore, one can show that the well-known Gauss–Radau quadrature rule for $\mu_1$ is just the multiple Gauss quadrature rule for $(\mu_1, \mu_2)$ with $N = |\operatorname{supp} \mu_2| = 1$ (the Gauss–Lobatto rule corresponds to $N = |\operatorname{supp} \mu_2| = 2$).

For any $N$ the modified CC algorithm (see Section 3.2) therefore provides an algorithm for computation of the Jacobi coefficients of the multi-step Christoffel transform. It would be interesting to find out if there is any computational benefit of this algorithm compared to the repeated use of the one-step/two-step Gautschi/Galant algorithm. Such questions are important in numerical mathematics, see, e.g., [3,4,11–16] and references therein. We will not pursue this in this paper.

Our main focus is the study of the *multiple* Christoffel transform

$$(\widehat{\mu}_1, \dots, \widehat{\mu}_r) = (\Phi\widehat{\mu}_1, \dots, \Phi\widehat{\mu}_r)$$

of the multiple orthogonality system $(\mu_1, \dots, \mu_r)$ for $r \ge 2$. Such a transform appears naturally when one studies the multiple Gauss quadrature with fixed nodes at the zeros of $\Phi$.

We show how one can use the CC algorithm to compute the nearest neighbour recurrence coefficients of $(\widehat{\mu}_1, \dots, \widehat{\mu}_r)$ (Section 3.2) and establish the determinantal formula for the multiple orthogonal polynomials for $(\widehat{\mu}_1, \dots, \widehat{\mu}_r)$ (see Section 3.1 for type II, and Section 3.5 for type I).

For the special case $N = 1$ the determinantal formulas are known from the earlier literature: see [20, Prop 3.2] for type II, and [21] for type I for the case of two measures and multi-indices along the step-line. During the preparation of the manuscript there appeared [22] studying multiple Christoffel transforms using another approach (the Gauss–Borel factorization) for the step-line multi-indices.

In Section 3.7 we demonstrate that CC algorithm can be used to compute *repeated* Christoffel transforms, which is the natural setting for the *qd* algorithm of Rutishauser [2] and the discrete-time Toda lattices in one (see, e.g., [9]) and multiple dimensions, see [20,23] and references therein.

In Section 3.8 we classify all possible nearest neighbour recurrence coefficients $\{a_{\boldsymbol{n},j}, b_{\boldsymbol{n},j}\}$ that can occur for maximally-normal systems (such systems are called perfect). This was proved for systems with two positive infinitely-supported measures in [24]. We provide an alternative simple proof that allows either measures or linear moment functionals which may be finitely or infinitely supported. The main difference is that for finitely supported $\mu_j$'s the $a_{\boldsymbol{n},j}$-coefficients must be zero not only on the initial marginal indices (that is, with $n_j = 0$) but also on the final ones (with $n_j = |\operatorname{supp} \mu_j|$).

The next portion of the results (Section 3.9) concerns interlacing of the zeros of multiple orthogonal polynomials $P_{\boldsymbol{n}}$ for $(\mu_1, \dots, \mu_r)$ and $\widehat{P}_{\boldsymbol{n}}$ for $(\widehat{\mu}_1, \dots, \widehat{\mu}_r)$ when $\deg \Phi = 1$. We show that the zeros of $P_{\boldsymbol{n}}$ and $\widehat{P}_{\boldsymbol{n}}$ interlace for a wide class of systems including all Angelesco and AT systems, and the same result holds for type I polynomials for a class of measures containing all Angelesco systems. In particular, this applies to multiple Laguerre of the first and second kind, Jacobi–Piñeiro, Angelesco–Jacobi, Jacobi–Laguerre, Jacobi–Hermite, Charlier, Meixner of the first and second kind, Krawtchouk, and Hahn (Sections 3.10 and 3.11). This type of interlacing was shown very recently in [25,26] for Angelesco–Jacobi, Jacobi–Laguerre, Jacobi–Hermite systems for type II polynomials along the step-line multi-indices using much more involved arguments, see also [27] for related results which use the notion of free convolution.

The interlacing results for the discrete systems then produces the lower bound 1 for the distance between two consecutive zeros (see Section 3.12), a result that is well-known for the $r = 1$ case, see [28–30].

Note that the result of one-step or multi-step Christoffel transform is always a positive measure if all of the zeros of $\Phi(x)$ fall outside of the interior of the convex hull of $\operatorname{supp} \mu$ (or if there are an even number of them at each gap of the support). Otherwise however, $\widehat{\mu}$ is a signed/complex measure, which is convenient to view as a linear moment functional. It does not take too much extra effort to allow $\mu$ to be a linear moment functional from the beginning, which is what we do starting from Section 2 onwards.

In a companion paper [31] we obtain determinantal formulas for type I and type II multiple orthogonal polynomials for rational perturbations of measures, which includes general Geronimus [32] and Uvarov [33] transforms, as well as the Christoffel transforms with different polynomials $\Phi_j$ for each $\mu_j$.

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## 2. Preliminaries

### 2.1. Orthogonal polynomials with respect to moment functionals

We use the notation $\mathbb{N} := \{k \in \mathbb{Z} : k \geq 0\}$ and $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k > 0\}$. Let us assume that we are given an arbitrary sequence $\{c_n\}_{n=0}^{\infty}$ of complex numbers which will be referred to as the moment sequence. Define the corresponding moment functional $\mu$ to be the linear map on the space of all polynomials such that

$$\mu[x^n] = c_n, \qquad n \in \mathbb{N}. \tag{11}$$

Associated to $\mu$ we have the bilinear form

$$\left\langle P(x), Q(x) \right\rangle = \mu[P(x)Q(x)]. \tag{12}$$

In particular, note that

$$\left\langle P(x)R(x), Q(x) \right\rangle = \left\langle P(x), R(x)Q(x) \right\rangle \tag{13}$$

for any choice of polynomials $P$, $Q$, and $R$. Orthogonal polynomials with respect to $\mu$ are non-zero polynomials $P_n(x)$ such that $\deg P_n \leq n$ and

$$\left\langle P_n(x), x^p \right\rangle = 0, \qquad p = 0, 1, \dots, n-1. \tag{14}$$

Such polynomials always exist, since solving (14) for the first $n+1$ Maclaurin coefficients results in a homogeneous system of linear equations with more columns than rows. If we fix the coefficient at $x^n$ we get a linear system with coefficient matrix

$$M_n = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{pmatrix}. \tag{15}$$

Then we see that $\Delta_n = \det M_n \neq 0$ if and only if $P_n$ is unique up to multiplication by a constant and $\deg P_n = n$. In this case we always take $P_n$ to be monic.

Denote $\mathcal{L}_\infty$ to be the set of all *quasi-definite* moment functionals, which are those $\mu$ for which $\Delta_n \neq 0$ for all $n \in \mathbb{N}$. For such $\mu$ the monic orthogonal polynomial $P_n$ is unique for each $n \in \mathbb{N}$. The polynomials satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \qquad n \in \mathbb{N} \tag{16}$$

for some complex numbers $a_n$ and $b_n$, called the Jacobi coefficients of $\mu$ (in the case $n = 0$ we formally take $a_0 = 0$ and $P_{-1} = 0$).

It is well known that $a_n \neq 0$ for all $n > 0$. Conversely, Favard's Theorem states that any set of $a_n$ and $b_n$, with $a_0 = 0$ and $a_n \neq 0$ for $n > 0$, generates a sequence of polynomials from (16) that are the orthogonal polynomials with respect to some moment functional $\mu$.

The tridiagonal matrix

$$J = \begin{pmatrix} b_0 & 1 & 0 & \\ a_1 & b_1 & 1 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{17}$$

will be called the ("monic") Jacobi matrix associated with $\mu$. It is the matrix of the map $P(x) \mapsto xP(x)$ in the basis $\{P_n\}_{n=0}^\infty$, see (16).

Finally, define the Christoffel–Darboux kernel via

$$K_n(x, y) = \sum_{j=0}^{n-1} \frac{P_j(x)P_j(y)}{\langle P_j, P_j \rangle}. \tag{18}$$

Then the Christoffel–Darboux identity takes place:

$$K_n(x, y) = \frac{1}{\langle P_{n-1}, P_{n-1} \rangle} \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x - y}. \tag{19}$$

For more on the basics of orthogonal polynomials with respect to moment functionals, see for example [7].

Let $\mathcal{M}_\infty$ be the set of measures $\mu$ on $\mathbb{R}$ with infinite support and all the moments

$$c_n = \int x^n d\mu(x), \quad n \in \mathbb{N}, \tag{20}$$

finite. Such a measure generates a quasi-definite moment functional (11) which with a mild abuse of notation we also denote by $\mu$. In particular, (12) becomes the usual inner product in $L^2(\mu)$. With this convention, we can view $\mathcal{M}_\infty$ as a subset of $\mathcal{L}_\infty$. The setting $\mu \in \mathcal{M}_\infty$ corresponds to the standard theory of orthogonal polynomials with $a_n > 0$ for all $n \geq 1$ and $b_n \in \mathbb{R}$ for all $n \geq 0$.

### 2.2. $\mu$ associated with finite (complex) Jacobi matrices

If one takes a measure $\mu$ on $\mathbb{R}$ supported on exactly $N$ distinct points $\{z_j\}_{j=1}^N \subset \mathbb{R}$, $N \in \mathbb{Z}_+$, then it is known that $\Delta_n \neq 0$ for $0 \leq n \leq N$ and $\Delta_n = 0$ for $n > N$. Consequently, only $(P_n(x))_{n=0}^N$ are uniquely defined, with $P_N(x) = \prod_{j=1}^N (x - z_j)$. The three-term recurrence (16) holds for $0 \leq n \leq N-1$ with $a_n > 0$ for $1 \leq n \leq N-1$. The corresponding Jacobi matrix $J$ in (17) is finite of size $N \times N$ with $(b_n)_{n=0}^{N-1}$ on the diagonal and $(a_n)_{n=1}^{N-1}$ on the subdiagonal. Denote the set of such $N$-finitely supported measures by $\mathcal{M}_N$.

In what follows we want to allow *complex* finite Jacobi matrices and the associated linear functionals. Therefore we define $\mathcal{L}_N$, for each $N \in \mathbb{Z}_+$, to be the set of all the moment functionals $\mu$ for which $\Delta_n \neq 0$ for $1 \leq n \leq N$ and $\Delta_n = 0$ for $n > N$.

**Lemma 2.1.** *Suppose $\mu \in \mathcal{L}_N$ for some $N \in \mathbb{Z}_+$. Then*

$$\langle P_N(x), x^p \rangle = 0, \qquad p \in \mathbb{N}. \tag{21}$$

*Moreover, we have*

$$\langle P(x), x^p \rangle = 0, \qquad p = 0, 1, \ldots, N-1, \tag{22}$$

*if and only if $P(x)$ is divisible by $P_N(x)$.*

**Proof.** Since $\Delta_{N+1} = 0$, there has to exist some non-zero $P_{N+1}$ solving (14) (at $N+1$) for $n = N+1$ with degree $\deg P_{N+1} < N+1$. This is because $\Delta_{N+1} = 0$ implies the existence of two linearly independent monic solutions to (14) of degree $\leq N+1$, and if they both have degree $N+1$ then their difference is also non-zero and solves (14) but has degree $< N+1$. We can always scale $P_{N+1}$ to be monic.

Since $\Delta_N \neq 0$, we must have $P_{N+1} = P_N$, as there is only one monic solution at $N$. By the orthogonality relations of $P_{N+1}$ we then must have $\langle P_N(x), x^N \rangle = 0$. We proceed to prove by induction that $\langle P_N(x), x^p \rangle = 0$ for all $p \geq N$. By $\Delta_{n+1} = 0$ (assuming $n \geq N$), there is some monic $P_n$ (solving (14) at $n$) with $\deg P_n = n - k$, $0 \leq k \leq n - N$, such that $\langle P_n(x), x^n \rangle = 0$ (since there is some monic $P_{n+1}$ with $\deg P_{n+1} \leq n$). In particular, $\langle P_n(x), Q(x) \rangle = 0$ for every polynomial $Q$ with $\deg Q \leq n$, so we must have $\langle P_N(x), x^k P_n(x) \rangle = \langle P_n(x), x^k P_N(x) \rangle = 0$, by (13). Assuming $\langle P_N(x), x^p \rangle = 0$ is true for all $p < n$ (and here $n > N$), we then end up with $\langle P_N(x), x^n \rangle = \langle P_N(x), x^k P_n(x) \rangle = 0$, since $x^k P_n(x) = x^n + o(x^n)$, and $\langle P_N(x), o(x^n) \rangle = 0$. This proves (21).

Now for the second part, we have $\langle P_N(x)Q(x), x^p\rangle = \langle P_N(x), x^pQ(x)\rangle = 0$ by (21). Conversely, suppose that $P$ satisfies (22) and write $P = P_N q + r$ with $\deg r < N$. Since $P_N$ is orthogonal to any polynomial, we get that $r$ also satisfies (22). This is only possible if $r \equiv 0$. □

**Theorem 2.2.** *If $\mu \in \mathcal{L}_N$, $N \le \infty$, then $(P_n(x))_{n=0}^N$ satisfy the three-term recurrence relation (16) for $0 \le n \le N-1$ with $a_n \in \mathbb{C}\setminus\{0\}$ for $1 \le n \le N-1$ and $b_n \in \mathbb{C}$ for $0 \le n \le N-1$. Conversely, for any non-zero $(a_n)_{n=1}^{N-1}$ and any $(b_n)_{n=0}^{N-1}$ there exists a functional $\mu \in \mathcal{L}_N$, unique up to multiplication by a non-zero constant, with exactly these Jacobi coefficients.*

**Remark 2.3.** See [35, Thm 4] for a more detailed description of $\mu$ from $\mathcal{L}_N$: if the $N$ roots of $P_N$ are all distinct then $\mu$ can be represented as integration against a (complex) measure supported on these roots. If some of the roots overlap then $\mu$ contains derivative(s) of the Dirac delta function at the corresponding root.

**Proof.** The first part of the statement follows from the same arguments as in [7, Ch. 1, Sect. 3–4]. The second part follows very similarly to the proof of [7, Ch.1, Thm 4.4]. First define $\mu[1] = c \neq 0$ and $\mu[P_n(x)] = 0$ for $n = 1, \dots, N-1$, where $P_n$ are generated by the recurrence coefficients through (16). If we also define $\mu[x^pP_N(x)] = 0$ for all $p \in \mathbb{N}$ (see Lemma 2.1) we can then extend $\mu$ uniquely to all polynomials. From the recurrence relation one can prove that $P_n$ are orthogonal polynomials with respect to $\mu$, for $n = 0, \dots, N-1$, and also $\langle P_n, x^n\rangle = a_n \cdots a_1 \neq 0$, which implies $\Delta_n \neq 0$ for $n = 1, \dots, N$. From $\langle P_N(x), x^{p+N}\rangle = 0$ we get $\Delta_n = 0$ for each $n > N$, so $\mu$ belongs to $\mathcal{L}_N$. Since these assumptions on $\mu$ were necessary for $(P_n)_{n=0}^N$ to be orthogonal with respect to some $\mu \in \mathcal{L}_N$, and the recurrence coefficients are uniquely determined by the recurrence, we get the full result. □

Finally, we define

$$\mathcal{L} = \left(\bigcup_{N=1}^{\infty} \mathcal{L}_N\right) \cup \mathcal{L}_\infty. \tag{23}$$

This is the set of moment functionals that we are working with throughout this paper. Note that $\mathcal{L}$ includes $\mathcal{M} = \left(\bigcup_{N=1}^{\infty} \mathcal{M}_N\right) \cup \mathcal{M}_\infty$, the set of all positive measures of finite or infinite support on $\mathbb{R}$ with finite moments.

### *2.3. Christoffel transform*

For $\mu \in \mathcal{L}$, a one-step Christoffel transform is a functional $\hat{\mu}$ given by

$$\hat{\mu}[x^n] = \mu[x^n(x - z_0)], \qquad n \in \mathbb{N}. \tag{24}$$

If $\mu \in \mathcal{M}$, then $\hat{\mu}$ becomes (3).

More generally, if $\mu \in \mathcal{L}$ and $\Phi(x) = \prod_{j=1}^m (x - z_j)$ is any polynomial, then we define the corresponding Christoffel transform

$$\hat{\mu}[x^n] = \mu[x^n\Phi(x)], \qquad n \in \mathbb{N}. \tag{25}$$

We will also occasionally employ the notation $\Phi\mu$ for $\hat{\mu}$ to make the dependence on $\Phi$ explicit. If $\mu \in \mathcal{M}$ and $\Phi$ has real coefficients and does not change sign on the convex hull of $\operatorname{supp}(\mu)$, then $\Phi\mu$ is also in $\mathcal{M}$ and is given as in (4).

Some authors choose to work with the normalized version of $\Phi\mu$ given by $\mu[\Phi(x)]^{-1}\mu[x^n\Phi(x)]$, under the assumption $\mu[\Phi(x)] \neq 0$. Note that the monic orthogonal polynomials and their recurrence coefficients for both versions of the Christoffel transform are the same, so this makes no significant difference.

Given any $\mu \in \mathcal{L}$ and its Christoffel transform $\hat{\mu} = \Phi\mu$, we write $\hat{P}_n$ for the orthogonal polynomials with respect to $\hat{\mu}$, and $\hat{a}_n$ and $\hat{b}_n$ for the Jacobi coefficients of $\hat{\mu}$. If $\mu, \hat{\mu} \in \mathcal{L}_\infty$ then the following Christoffel determinantal formula [1] holds:

$$\hat{P}_n(x) = \Phi(x)^{-1} D_n^{-1} \det \begin{pmatrix} P_{n+m}(x) & P_{n+m-1}(x) & \cdots & P_n(x) \\ P_{n+m}(z_1) & P_{n+m-1}(z_1) & \cdots & P_n(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n+m}(z_m) & P_{n+m-1}(z_m) & \cdots & P_n(z_m) \end{pmatrix}, \tag{26}$$

where $D_n$ is the normalizing constant

$$D_n = \det \begin{pmatrix} P_{n+m-1}(z_1) & P_{n+m-2}(z_1) & \cdots & P_n(z_1) \\ P_{n+m-1}(z_2) & P_{n+m-2}(z_2) & \cdots & P_n(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n+m-1}(z_m) & P_{n+m-2}(z_m) & \cdots & P_n(z_m) \end{pmatrix}. \tag{27}$$

In the one-step case $m = 1$ this becomes

$$\hat{P}_n(x) = \frac{1}{x - z_0}\left(P_{n+1}(x) - \frac{P_{n+1}(z_0)}{P_n(z_0)} P_n(x)\right). \tag{28}$$

By the Christoffel–Darboux formula (19) we get

$$\hat{P}_n(x) = \frac{\langle P_n, P_n\rangle}{P_n(z_0)} K_{n+1}(z_0, x). \tag{29}$$

Polynomials $K_n(z_0, z)$ are sometimes referred to as the kernel polynomials of $\mu$.

(28) together with the three-term recurrence (16) can be used to generate the recurrence equations (see [4])

$$\begin{aligned} \hat{b}_n - \delta_{n+1} &= b_{n+1} - \delta_n, \\ \hat{a}_n - \delta_n \hat{b}_n &= a_{n+1} - \delta_n b_n, \\ \delta_{n-1} \hat{a}_n &= \delta_n a_n, \end{aligned} \tag{30}$$

with initial conditions $\hat{a}_0 = 0$ and $\delta_0 = z_0 - b_0$. From this, it is possible to compute the Jacobi coefficients of $\hat{\mu}$ from the Jacobi coefficients of $\mu$, through the following algorithm,

$$\begin{aligned} &\delta_0 := z_0 - b_0; \\ &\hat{a}_0 = 0; \\ &\hat{b}_0 = b_0 - \frac{a_1}{\delta_0}; \\ &\text{for all } n \in \mathbb{Z}_+ : \\ &\quad \delta_n := \hat{b}_{n-1} - b_n + \delta_{n-1}; \\ &\quad \hat{a}_n = a_n \frac{\delta_n}{\delta_{n-1}}; \\ &\quad \hat{b}_n = b_n + \frac{\hat{a}_n - a_{n+1}}{\delta_n}. \end{aligned} \tag{31}$$

In Section 3.1 we show how this can be generalized to the multi-step Christoffel transform using multiple orthogonal polynomials.

### 2.4. Basics of multiple orthogonal polynomials on the real line (MOPRL)

Let $r \geq 1$ and consider a system of functionals $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r) \in \mathcal{L}^r$. Let us write $\left\langle P(x), Q(x) \right\rangle_j$ for $\mu_j[P(x)Q(x)]$.

**Definition 2.4.** Given a multi-index $\boldsymbol{n} \in \mathbb{N}^r$, a type II multiple orthogonal polynomial is a non-zero polynomial $P_{\boldsymbol{n}}(x)$ such that $\deg P_{\boldsymbol{n}} \leq |\boldsymbol{n}| := n_1 + \cdots + n_r$, and

$$\left\langle P_{\boldsymbol{n}}(x), x^p \right\rangle_j = 0, \qquad p = 0, 1, \dots, n_j - 1, \qquad j = 1, \dots, r. \tag{32}$$

**Definition 2.5.** A type I multiple orthogonal polynomial is a non-zero vector of polynomials $\boldsymbol{A}_{\boldsymbol{n}} = (A_{\boldsymbol{n}}^{(1)}, \dots, A_{\boldsymbol{n}}^{(r)})$ such that $\deg A_{\boldsymbol{n}}^{(j)} \leq n_j - 1$, $j = 1, \dots, r$, and

$$\sum_{j=1}^{r} \left\langle A_{\boldsymbol{n}}^{(j)}(x), x^p \right\rangle_j = 0, \qquad p = 0, 1, \dots, |\boldsymbol{n}| - 2. \tag{33}$$

Note that $A_{\boldsymbol{n}}^{(j)} = 0$ when $n_j = 0$ (we take the degree of 0 to be $-\infty$). Hence for $\boldsymbol{n} = \boldsymbol{0}$ there would be no non-zero solutions to (33). In this case we take $\boldsymbol{A}_{\boldsymbol{0}} = \boldsymbol{0}$ (the $r$-vector of zero polynomials) as the only type I polynomial.

It is easy to show that for any multi-index $\boldsymbol{n} \in \mathbb{N}^r \setminus \{\boldsymbol{0}\}$ the following statements are equivalent:

(i) There is a unique monic type II multiple orthogonal polynomial $P_{\boldsymbol{n}}$ such that $\deg P_{\boldsymbol{n}} = |\boldsymbol{n}|$;

(ii) There is a unique type I multiple orthogonal polynomial $(A_{\boldsymbol{n}}^{(1)}, \dots, A_{\boldsymbol{n}}^{(r)})$ such that

$$\sum_{j=1}^{r} \left\langle A_{\boldsymbol{n}}^{(j)}(x), x^{|\boldsymbol{n}|-1} \right\rangle_j = 1. \tag{34}$$

(iii) $\deg P_{\boldsymbol{n}} = |\boldsymbol{n}|$ for every non-zero solution of (32);

(iv) $\sum_{j=1}^{r} \left\langle A_{\boldsymbol{n}}^{(j)}(x), x^{|\boldsymbol{n}|-1} \right\rangle_j \neq 0$ for every non-zero solution of (33);

(v) $\det M_{\boldsymbol{n}} \neq 0$, where

$$M_{\boldsymbol{n}} = \left( \begin{array}{cccc} c_0^{(1)} & c_1^{(1)} & \cdots & c_{|\boldsymbol{n}|-1}^{(1)} \\ c_1^{(1)} & c_2^{(1)} & \cdots & c_{|\boldsymbol{n}|}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n_1-1}^{(1)} & c_{n_1}^{(1)} & \cdots & c_{|\boldsymbol{n}|+n_1-2}^{(1)} \\ \hline & & \vdots & \\ \hline c_0^{(r)} & c_1^{(r)} & \cdots & c_{|\boldsymbol{n}|-1}^{(r)} \\ c_1^{(r)} & c_2^{(r)} & \cdots & c_{|\boldsymbol{n}|}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n_r-1}^{(r)} & c_{n_r}^{(r)} & \cdots & c_{|\boldsymbol{n}|+n_r-2}^{(r)} \end{array} \right), \tag{35}$$

where $c_n^{(j)}$ are the moments

$$c_n^{(j)} = \mu_j[x^n].$$

**Definition 2.6.** If any and therefore each of the above conditions (i)–(v) are satisfied for an index $\boldsymbol{n} \neq \mathbf{0}$ then we say that $\boldsymbol{n}$ is normal. We always take the index $\mathbf{0}$ to be normal.

Whenever $\boldsymbol{n}$ is normal we will only work with polynomials satisfying (i) and (ii) above, that is, $P_{\boldsymbol{n}}$ will be monic and $\boldsymbol{A}_{\boldsymbol{n}}$ will satisfy (34).

**Proposition 2.7.** *If $\mu_j \in \mathcal{L}_{N_j}$ with $N_j < \infty$, then indices $\boldsymbol{n}$ with $n_j > N_j$ can never be normal.*

**Proof.** In the case $r = 1$ this follows from Lemma 2.1. In general, if $n_j > N_j$, we have

$$\left\langle P_{\boldsymbol{n}-\boldsymbol{e}_j}(x), x^{n_j-1} \right\rangle_j = \left\langle P_{\boldsymbol{n}-\boldsymbol{e}_j}(x), Q(x) \right\rangle_j$$

for any degree $n_j - 1$ monic polynomial $Q$, given any choice of type II polynomial $P_{\boldsymbol{n}-\boldsymbol{e}_j}$ (for the index $\boldsymbol{n} - \boldsymbol{e}_j$). If we choose $Q(x) = x^{n_j-1-N_j} P_{N_j}(x)$, where $P_{N_j}$ is the degree $N_j$ orthogonal polynomials with respect to $\mu_j$, then we get

$$\left\langle P_{\boldsymbol{n}-\boldsymbol{e}_j}(x), x^{n_j-1} \right\rangle_j = \left\langle P_{N_j}(x), P_{\boldsymbol{n}-\boldsymbol{e}_j}(x) x^{n_j-1-N_j} \right\rangle_j = 0$$

by Lemma 2.1. This means that $P_{\boldsymbol{n}-\boldsymbol{e}_j}$ satisfies all the orthogonality relations (32) for the index $\boldsymbol{n}$, but $\deg P_{\boldsymbol{n}-\boldsymbol{e}_j} < |\boldsymbol{n}|$, so $\boldsymbol{n}$ cannot be normal. □

A system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ in $\mathcal{L}^r$ or in $\mathcal{M}^r$ is usually said to be perfect if every $\mathbb{N}^r$-index is normal. We modify this notion to the case when $\mu_j \in \mathcal{L}_{N_j}$ where $N_j$ may be finite. We write $\boldsymbol{N} = (N_1, \ldots, N_r) \in (\mathbb{Z}_+ \cup \{+\infty\})^r$ and denote

$$\mathbb{N}^r_{\boldsymbol{N}} := \left\{ \boldsymbol{n} \in \mathbb{N}^r \mid 0 \le n_j \le N_j \right\}.$$

**Definition 2.8.** We say that $(\mu_1, \ldots, \mu_r)$ is a perfect system if all indices in $\mathbb{N}^r_{\boldsymbol{N}}$ are normal.

One important class of perfect systems is Angelesco systems. Usually they are defined for measures with infinite support, but we extend the notion of Angelesco system to allow measures of finite support. If $\boldsymbol{\mu} \in \mathcal{M}^r$ we write $\Delta_j$ for the convex hull of $\operatorname{supp}(\mu_j)$ and $\Delta_j^{\circ}$ for the interior of $\Delta_j$.

**Definition 2.9.** An Angelesco system is a $\mathcal{M}^r$-system such that $\Delta_k^{\circ} \cap \Delta_j^{\circ} = \emptyset$ and if $\mu_k$ and $\mu_j$ are both finitely supported then we also require $\Delta_k \cap \Delta_j = \emptyset$ $(k \neq j)$.

With this definition one can show that any Angelesco system is perfect in the sense of Definition 2.8. The proof goes along the same lines (counting real zeros of odd multiplicity of $P_{\boldsymbol{n}}$) as the standard argument for the case $\boldsymbol{N} = (\infty, \ldots, \infty)$, see e.g. [18, Thm 23.1.3].

### *2.5. MOPRL: nearest neighbour recurrence relations*

We remind the reader that whenever some index is normal then the type II multiple orthogonal polynomial at that location is always taken to be monic, and the type I polynomial is taken with the normalization (34).

It was shown by Van Assche [17,18] that multiple orthogonal polynomials of type II and type I satisfy the following set of equations, called the nearest neighbour recurrence relations (NNRR for short). Although it was originally stated in $\mathcal{M}^r_\infty$, the proof is no different in $\mathcal{L}^r$.

**Theorem 2.10** (*Van Assche, [17]*). *Let $\boldsymbol{\mu} \in \mathcal{L}^r$. If $\boldsymbol{n}$ and $\boldsymbol{n} + \boldsymbol{e}_j$ are normal, then*

$$x P_{\boldsymbol{n}}(x) = P_{\boldsymbol{n}+\boldsymbol{e}_j}(x) + b_{\boldsymbol{n},j} P_{\boldsymbol{n}}(x) + \sum_{i=1}^{r} a_{\boldsymbol{n},i} P_{\boldsymbol{n}-\boldsymbol{e}_i}(x) \tag{36}$$

*for some constants $a_{\boldsymbol{n},1}, \ldots, a_{\boldsymbol{n},r}$ and $b_{\boldsymbol{n},j}$. Here $a_{\boldsymbol{n},l} := 0$ whenever $n_l = 0$.*

**Remark 2.11.** The recurrence coefficient $b_{\boldsymbol{n},j}$ is defined when both $\boldsymbol{n}$ and $\boldsymbol{n} + \boldsymbol{e}_j$ are normal, and it is always unique, given our choice of normalization for the polynomials. The coefficient $a_{\boldsymbol{n},i}$ is defined when $\boldsymbol{n}$ is normal and $\boldsymbol{n} + \boldsymbol{e}_l$ is normal for some $l$. $a_{\boldsymbol{n},i}$ is unique if $\boldsymbol{n} - \boldsymbol{e}_i$ is also normal. Indeed, (36) implies

$$a_{\boldsymbol{n},i} = \frac{\left\langle P_{\boldsymbol{n}}(x), x^{n_i} \right\rangle_i}{\left\langle P_{\boldsymbol{n}-\boldsymbol{e}_i}(x), x^{n_i-1} \right\rangle_i} \tag{37}$$

if $n_i > 0$. Therefore the constants $a_{\boldsymbol{n},i}$ are independent of the choice of $j$ in (36) (i.e., if $\boldsymbol{n} + \boldsymbol{e}_j$ are normal for several choices of $j$), but may depend on the choice of $P_{\boldsymbol{n}-\boldsymbol{e}_1}, \ldots, P_{\boldsymbol{n}-\boldsymbol{e}_r}$ if any of the indices $\boldsymbol{n} - \boldsymbol{e}_1, \ldots, \boldsymbol{n} - \boldsymbol{e}_r$ are not normal.

**Remark 2.12.** In particular, for perfect systems $\boldsymbol{\mu} \in \mathcal{L}^r$ with $N_j \le \infty$, we have that $a_{\boldsymbol{n},j}$ is defined for all $1 \le j \le r$ and all $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}}$ except for $\boldsymbol{n} = \boldsymbol{N}$. This exception is only relevant if $N_j < \infty$ for all $j$, of course. Let us adopt the convention $a_{\boldsymbol{N},j} := 0$ for that case, for reasons that will become more clear later on in the paper. Assuming this, all perfect systems then have well-defined NNRR coefficients $\{a_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}}}$ and $\{b_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}}$ for each $j$.

It is well-known that the type I polynomials satisfy similar recurrence relations. Since the explicit proof is hard to locate we provide a proof in Appendix A.1, using minimal normality assumptions.

**Theorem 2.13.** *Let $\boldsymbol{\mu} \in \mathcal{L}^r$. If $\boldsymbol{n}$ and $\boldsymbol{n} - \boldsymbol{e}_j$ are normal, then*

$$x\boldsymbol{A}_{\boldsymbol{n}}(x) = \boldsymbol{A}_{\boldsymbol{n}-\boldsymbol{e}_j}(x) + b_{\boldsymbol{n}-\boldsymbol{e}_j,j}\boldsymbol{A}_{\boldsymbol{n}}(x) + \sum_{i=1}^{r} c_{\boldsymbol{n},i}\boldsymbol{A}_{\boldsymbol{n}+\boldsymbol{e}_i}(x), \tag{38}$$

*for some constants $c_{\boldsymbol{n},1}, \ldots, c_{\boldsymbol{n},r}$. If $\boldsymbol{n}+\boldsymbol{e}_i$ and $\boldsymbol{n}-\boldsymbol{e}_i$ are normal then $c_{\boldsymbol{n},i} = a_{\boldsymbol{n},i}$.*

**Remark 2.14.** As above, $c_{\boldsymbol{n},i}$ are independent of the choice of $j$, but may depend on the choice of the vectors $(A_{\boldsymbol{n}+\boldsymbol{e}_i,1}, \ldots, A_{\boldsymbol{n}+\boldsymbol{e}_i,r})$ when $\boldsymbol{n}+\boldsymbol{e}_i$ is not normal (for the details, see the Appendix).

By comparing (36) for two different $j$, and doing the same for (38), we obtain the following.

**Corollary 2.15.** *Assume $\boldsymbol{n}$, $\boldsymbol{n}+\boldsymbol{e}_j$ and $\boldsymbol{n}+\boldsymbol{e}_l$ are normal. Then*

$$P_{\boldsymbol{n}+\boldsymbol{e}_l} - P_{\boldsymbol{n}+\boldsymbol{e}_j} = (b_{\boldsymbol{n},j} - b_{\boldsymbol{n},l})P_{\boldsymbol{n}}. \tag{39}$$

*Similarly, if $\boldsymbol{n}$, $\boldsymbol{n}-\boldsymbol{e}_j$ and $\boldsymbol{n}-\boldsymbol{e}_l$ are normal, then*

$$\boldsymbol{A}_{\boldsymbol{n}-\boldsymbol{e}_l} - \boldsymbol{A}_{\boldsymbol{n}-\boldsymbol{e}_j} = (b_{\boldsymbol{n}-\boldsymbol{e}_j,j} - b_{\boldsymbol{n}-\boldsymbol{e}_l,l})\boldsymbol{A}_{\boldsymbol{n}} \tag{40}$$

Van Assche showed in [17] that (assuming sufficient normality conditions) the recurrence coefficients satisfy the partial difference equations in the theorem below. Their proof works for $\boldsymbol{\mu} \in \mathcal{L}^r$ without any changes. When we write $\boldsymbol{m} \le \boldsymbol{n}$ we mean $m_j \le n_j$ for each $j = 1, \ldots, r$.

**Theorem 2.16** (*Van Assche, [17]*). *Let $\boldsymbol{\mu} \in \mathcal{L}^r$. Assume all indices $\boldsymbol{m} \le \boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_l$ is normal. If $j \ne l$ then*

$$b_{\boldsymbol{n}+\boldsymbol{e}_l,j} - b_{\boldsymbol{n}+\boldsymbol{e}_j,l} = b_{\boldsymbol{n},j} - b_{\boldsymbol{n},l}, \tag{41}$$

$$b_{\boldsymbol{n},l}b_{\boldsymbol{n}+\boldsymbol{e}_l,j} - b_{\boldsymbol{n},j}b_{\boldsymbol{n}+\boldsymbol{e}_j,l} = \sum_{i=1}^{r} a_{\boldsymbol{n}+\boldsymbol{e}_l,i} - \sum_{i=1}^{r} a_{\boldsymbol{n}+\boldsymbol{e}_j,i}, \tag{42}$$

$$a_{\boldsymbol{n}+\boldsymbol{e}_l,j}\left(b_{\boldsymbol{n}-\boldsymbol{e}_j,j} - b_{\boldsymbol{n}-\boldsymbol{e}_j,l}\right) = a_{\boldsymbol{n},j}\left(b_{\boldsymbol{n},j} - b_{\boldsymbol{n},l}\right). \tag{43}$$

*If $n_j = 0$ or $n_l = 0$ then we only get the first two equations.*

We can use (41) to rewrite the left hand side of (42) as

$$b_{\boldsymbol{n},l}b_{\boldsymbol{n}+\boldsymbol{e}_l,j} - b_{\boldsymbol{n},j}b_{\boldsymbol{n}+\boldsymbol{e}_j,l} = b_{\boldsymbol{n},l}b_{\boldsymbol{n}+\boldsymbol{e}_l,j} - b_{\boldsymbol{n},j}(b_{\boldsymbol{n}+\boldsymbol{e}_l,j} + b_{\boldsymbol{n},l} - b_{\boldsymbol{n},j}) = (b_{\boldsymbol{n},l} - b_{\boldsymbol{n},j})b_{\boldsymbol{n}+\boldsymbol{e}_l,j} - (b_{\boldsymbol{n},l} - b_{\boldsymbol{n},j})b_{\boldsymbol{n},j}.$$

We write $\delta_{\boldsymbol{n},j,l}$ for $b_{\boldsymbol{n},l} - b_{\boldsymbol{n},j}$. Now we can rewrite (41)–(43) in the alternative form,

$$b_{\boldsymbol{n}+\boldsymbol{e}_l,j} - \delta_{\boldsymbol{n}+\boldsymbol{e}_j,j,l} = b_{\boldsymbol{n}+\boldsymbol{e}_j,j} - \delta_{\boldsymbol{n},j,l}, \tag{44}$$

$$\sum_{i=1}^{r} a_{\boldsymbol{n}+\boldsymbol{e}_l,i} - \delta_{\boldsymbol{n},j,l}b_{\boldsymbol{n}+\boldsymbol{e}_l,j} = \sum_{i=1}^{r} a_{\boldsymbol{n}+\boldsymbol{e}_j,i} - \delta_{\boldsymbol{n},j,l}b_{\boldsymbol{n},j}, \tag{45}$$

$$\delta_{\boldsymbol{n}-\boldsymbol{e}_j,j,l}a_{\boldsymbol{n}+\boldsymbol{e}_l,j} = \delta_{\boldsymbol{n},j,l}a_{\boldsymbol{n},j}, \tag{46}$$

which will be better adapted for our purposes.

Consider a sequence of normal indices $(\boldsymbol{n}_k)_{k=0}^{\infty}$ that forms a path starting at the origin, i.e., $\boldsymbol{n}_0 = \boldsymbol{0}$ and $\boldsymbol{n}_{k+1} = \boldsymbol{n}_k + \boldsymbol{e}_{j_k}$ for some $j_k \in \{1, \ldots, r\}$, $k \in \mathbb{N}$. Define the generalized Christoffel–Darboux kernel $\boldsymbol{K}_{\boldsymbol{n}}(x,y) = (K_{\boldsymbol{n}}^{(1)}(x,y), \ldots, K_{\boldsymbol{n}}^{(r)}(x,y))$ via

$$\boldsymbol{K}_{\boldsymbol{n}}(x,y) = \sum_{k=0}^{N-1} P_{\boldsymbol{n}_k}(x)\boldsymbol{A}_{\boldsymbol{n}_{k+1}}(y). \tag{47}$$

Using the NNRR one can show [17] that the following Christoffel–Darboux formula holds

$$(x-y)\boldsymbol{K}_{\boldsymbol{n}}(x,y) = P_{\boldsymbol{n}_N}(x)\boldsymbol{A}_{\boldsymbol{n}_N}(y) - \sum_{j=1}^{r} a_{\boldsymbol{n},j}P_{\boldsymbol{n}_N-\boldsymbol{e}_j}(x)\boldsymbol{A}_{\boldsymbol{n}_N+\boldsymbol{e}_j}(y), \tag{48}$$

originally due to [36].

### 2.6. Normality criteria

Here we collect a series of results that allow to establish normality of indices for certain multiple orthogonality systems $\boldsymbol{\mu} \in \mathcal{L}^r$. While basic, these results may be of interest on their own. We will use these results in Section 3.1 to find necessary and sufficient conditions for $(\widehat{\mu}_1, \dots, \widehat{\mu}_r)$ to be perfect.

We start with the following simple lemma. One direction is standard, see, e.g., [18, Cor 23.1.1–23.1.2], while the other has appeared in [37], as well as [38] (for the unit circle), see also [39].

**Lemma 2.17.** *$\boldsymbol{n} + \boldsymbol{e}_j$ is normal if and only if*

$$\left\langle P_{\boldsymbol{n}}(x), x^{n_j} \right\rangle_j \neq 0 \tag{49}$$

*for every type II multiple orthogonal polynomial $P_{\boldsymbol{n}}$ corresponding to the index $\boldsymbol{n}$.*

*Similarly, $\boldsymbol{n} - \boldsymbol{e}_j$ is normal if and only if $A_{\boldsymbol{n}}^{(j)}$ has degree $n_j - 1$ for every vector of type I multiple orthogonal polynomials $(A_{\boldsymbol{n}}^{(1)}, \dots, A_{\boldsymbol{n}}^{(r)})$.*

**Proof.** If $\left\langle P_{\boldsymbol{n}}(x), x^{n_j} \right\rangle_j$ is zero for some multiple orthogonal polynomial $P_{\boldsymbol{n}}$, then $P_{\boldsymbol{n}}$ satisfies every orthogonality condition for the index $\boldsymbol{n} + \boldsymbol{e}_j$, but $P_{\boldsymbol{n}}$ has degree $\le |\boldsymbol{n}|$, so $\boldsymbol{n} + \boldsymbol{e}_j$ cannot be normal. Conversely, if $\boldsymbol{n} + \boldsymbol{e}_j$ is not normal then (32) has a solution $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ of degree $\le |\boldsymbol{n}|$. $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ satisfies the orthogonality conditions (32) for the index $\boldsymbol{n}$, so it is a multiple orthogonal polynomial for the index $\boldsymbol{n}$. In other words, $P_{\boldsymbol{n}+\boldsymbol{e}_j} = P_{\boldsymbol{n}}$ for some choice of $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ and $P_{\boldsymbol{n}}$, but then (32) for the index $\boldsymbol{n} + \boldsymbol{e}_j$ implies $\left\langle P_{\boldsymbol{n}}(x), x^{n_j} \right\rangle_j = 0$.

If there is some $A_{\boldsymbol{n}}^{(j)}$ of degree $< n_j - 1$ then we have a non-zero solution to the system (33) for the index $\boldsymbol{n} - \boldsymbol{e}_j$, so $\boldsymbol{n} - \boldsymbol{e}_j$ is not normal since normality condition (iv) is not satisfied. Conversely, if $\boldsymbol{n} - \boldsymbol{e}_j$ is not normal then there is a non-zero solution $(A_{\boldsymbol{n}-\boldsymbol{e}_j}^{(1)}, \dots, A_{\boldsymbol{n}-\boldsymbol{e}_j}^{(r)})$ to

$$\sum_{i=1}^{r} \left\langle A_{\boldsymbol{n}-\boldsymbol{e}_j}^{(i)}(x), x^p \right\rangle_i = 0, \qquad p = 0, 1, \dots, |\boldsymbol{n}| - 2,$$

but then this is also a solution for the index $\boldsymbol{n}$, which means that there is an $A_{\boldsymbol{n},j}$ of degree $< n_j - 1$. □

In order to define the recurrence coefficients we required some indices to be normal. In turn, the recurrence coefficients can give us information about the normality of neighbouring indices, as we show next.

**Lemma 2.18.** *The following holds true for the recurrence coefficients of a system $\mu \in \mathcal{L}^r$.*

*(a) Let $\boldsymbol{n}$ and $\boldsymbol{n} + \boldsymbol{e}_i$ be normal for some $i$ and $n_j > 0$. Then $a_{\boldsymbol{n},j} \neq 0$ if and only if $\boldsymbol{n} + \boldsymbol{e}_j$ is normal.*
*(b) Let $\boldsymbol{n}$, $\boldsymbol{n} + \boldsymbol{e}_j$ and $\boldsymbol{n} + \boldsymbol{e}_l$ be normal for some $j \neq l$. Then $b_{\boldsymbol{n},j} \neq b_{\boldsymbol{n},l}$ if and only if $\boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_l$ is normal.*

**Proof.** *(a)* follows by Lemma 2.17 and (37). To show *(b)*, take the $j$th inner product with respect to $x^{n_j}$ in (39) to get

$$b_{\boldsymbol{n},j} - b_{\boldsymbol{n},l} = \frac{\left\langle P_{\boldsymbol{n}+\boldsymbol{e}_l}(x), x^{n_j} \right\rangle_j}{\left\langle P_{\boldsymbol{n}}(x), x^{n_j} \right\rangle_j}. \tag{50}$$

Then *(b)* follows from Lemma 2.17. □

To check perfectness of certain systems the following two results may be useful. Note that direct application of Lemma 2.17 would require to check the condition (49) for *every* orthogonal polynomial at every index $\boldsymbol{n}$. We show that it is enough to check it for only *one* choice of orthogonal polynomial at every index $\boldsymbol{n}$. This is useful since occasionally finding one such choice is easy from a Rodrigues-type formula or other arguments. One such application can be found in Section 3.1 below. The first of the two results appeared in [37, Lemma 3.4].

**Theorem 2.19** (*[37]*). *$\boldsymbol{\mu} \in \mathcal{L}^r$ is a perfect system if and only if there is a choice of type II multiple orthogonal polynomials $P_{\boldsymbol{n}}$ for each $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}}$, such that*

$$\left\langle P_{\boldsymbol{n}}(x), x^{n_j} \right\rangle_j \neq 0 \tag{51}$$

*whenever $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}$.*

**Proof.** Necessity of (51) is trivial from Lemma 2.17. Let us now show sufficiency. Note that (51) is only imposed on one choice of $P_{\boldsymbol{n}}$ so this direction is not immediate from Lemma 2.17. We use induction on $N = |\boldsymbol{n}|$ to prove that $\boldsymbol{n}$ is normal whenever $0 \le n_j \le N_j$. The case $N = 0$ is obvious. If $N > 0$, assume $\boldsymbol{n}$ is normal whenever $|\boldsymbol{n}| < N$ and take $\boldsymbol{m} = (m_1, \dots, m_r)$ such that $|\boldsymbol{m}| = N$. There is some $j$ such that $\boldsymbol{m} = \boldsymbol{n} + \boldsymbol{e}_j$. $\boldsymbol{n}$ is normal by assumption, so the monic $P_{\boldsymbol{n}}$ is unique, and

$$\left\langle P_{\boldsymbol{n}}(x), x^{n_j} \right\rangle_j \neq 0,$$

so $\boldsymbol{n} + \boldsymbol{e}_j$ is normal by Lemma 2.17. □

**Remark 2.20.** In fact the same proof shows that all indices along any path $(\boldsymbol{n}_l)_{l=0}^m$ (where $m$ may be infinite) with $\boldsymbol{n}_0 = \boldsymbol{0}$ and $\boldsymbol{n}_{l+1} = \boldsymbol{n}_l + \boldsymbol{e}_{j_l}$, where $1 \le j_l \le r$, are normal if and only if

$$\left\langle P_{\boldsymbol{n}_l}(x), x^{(\boldsymbol{n}_l)_{j_l}} \right\rangle_{j_l} \neq 0, \qquad l = 0, \dots, m-1, \tag{52}$$

for a choice of type II multiple orthogonal polynomials $P_{\boldsymbol{n}_l}$.

**Theorem 2.21.** *$\boldsymbol{\mu} \in \mathcal{L}^r$ is a perfect system if and only if there is a choice of type II multiple orthogonal polynomials $P_{\boldsymbol{n}}$ for each $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}}$, such that for each $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j-\boldsymbol{e}_l}$*

$$P_{\boldsymbol{n}+\boldsymbol{e}_l} - P_{\boldsymbol{n}+\boldsymbol{e}_j} = c_{\boldsymbol{n},j,l} P_{\boldsymbol{n}} \tag{53}$$

*for some constants $c_{\boldsymbol{n},j,l} \neq 0$ ($j \neq l$).*

**Proof.** We prove that $\boldsymbol{n}$ is normal by induction on $M = |\boldsymbol{n}|$. The cases $M = 0$ and $M = 1$ are obvious. For $M > 1$, assume $\boldsymbol{n}$ is normal whenever $|\boldsymbol{n}| < M$ and take $\boldsymbol{m} = (m_1, \dots, m_r)$ such that $|\boldsymbol{m}| = M$. Note that we know the normality in the case $\boldsymbol{m} = m\boldsymbol{e}_i$, so assume $m_j > 0$ and $m_l > 0$ for some $j$ and $k$ with $j \neq l$. Then $\boldsymbol{n} = \boldsymbol{m} - \boldsymbol{e}_j - \boldsymbol{e}_l$ is normal, as well as $\boldsymbol{n} + \boldsymbol{e}_j = \boldsymbol{m} - \boldsymbol{e}_l$ and $\boldsymbol{n} + \boldsymbol{e}_l = \boldsymbol{m} - \boldsymbol{e}_j$. Hence by Corollary 2.15 we get

$$(b_{\boldsymbol{n},j} - b_{\boldsymbol{n},l}) P_{\boldsymbol{n}} = P_{\boldsymbol{n}+\boldsymbol{e}_l} - P_{\boldsymbol{n}+\boldsymbol{e}_j} = c_{\boldsymbol{n},j,l} P_{\boldsymbol{n}},$$

so we must have $b_{\boldsymbol{n},j} - b_{\boldsymbol{n},l} = c_{\boldsymbol{n},j,l} \neq 0$, which implies that $\boldsymbol{m} = \boldsymbol{n} + \boldsymbol{e}_j + \boldsymbol{e}_l$ is normal, by Lemma 2.18 (b). □

A related result to Theorem 2.21 can be found in [40, Prop. 3].

## 3. Main results

Let us setup the following notation for the remainder of this paper. We write

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_r) \in \mathcal{L}^r,$$
$$\boldsymbol{\nu} = (\mu_1, \dots, \mu_{r-1}) \in \mathcal{L}^{r-1},$$

that is, $\boldsymbol{\nu}$ is $\boldsymbol{\mu}$ with $\mu_r$ removed, where $r \ge 2$.

As usual we allow $\mu_j \in \mathcal{L}_{N_j}$ with $N_j \le \infty$, and denote $\boldsymbol{N} = (N_1, \dots, N_r)$. We use $\boldsymbol{n}$ for a multi-index in $\mathbb{N}^r_{\boldsymbol{N}}$, and $\boldsymbol{k} = (k_1, \dots, k_{r-1})$ for a multi-index in $\mathbb{N}^{r-1}_{\boldsymbol{K}}$, where we denote $\boldsymbol{K} = (N_1, \dots, N_{r-1})$.

Finally, it is clear that the type II multiple orthogonal polynomials for $\boldsymbol{\nu}$ at a location $\boldsymbol{k}$ coincide with type II multiple orthogonal polynomials for $\boldsymbol{\mu}$ at the location $(\boldsymbol{k}, 0)$. So we use the same label $P$ for type II polynomials for $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ interchangeably, as it cannot lead to any ambiguity: $P_{(\boldsymbol{k},0)} = P_{\boldsymbol{k}}$, and similarly for type I polynomials $A^{(j)}_{(\boldsymbol{k},0)} = A^{(j)}_{\boldsymbol{k}}$ for $1 \le j \le r-1$.

### 3.1. *Christoffel transforms of type II polynomials*

Let $\mu_r \in \mathcal{L}_m$ with $m = N_r < \infty$, and write $\Phi(x) = \prod_{j=1}^m (x - z_j)$ for the unique monic orthogonal polynomial of degree $m$ with respect to $\mu_r$. For example, if all $z_j$'s are distinct (which is true if $\mu_r \in \mathcal{M}_m \subset \mathcal{L}_m$), then $\mu_r$ is of the form

$$\sum_{j=1}^m w_j \delta_{z_j} \tag{54}$$

where $w_j \in \mathbb{C} \setminus \{0\}$ for each $j = 1, \dots, m$. But in general $z_j$'s may overlap leading to more complicated functionals, see Remark 2.3.

Consider the Christoffel transforms $\widehat{\mu}_j = \Phi \mu_j$ for $1 \le j \le r-1$, as in (25). We refer to

$$\widehat{\boldsymbol{\nu}} = \Phi \boldsymbol{\nu} = (\widehat{\mu}_1, \dots, \widehat{\mu}_{r-1})$$

as the Christoffel transform of $\boldsymbol{\nu} = (\mu_1, \dots, \mu_{r-1})$. We write $P_{\boldsymbol{k}}$ for the type II polynomials with respect to $\boldsymbol{\nu}$ and $\widehat{P}_{\boldsymbol{k}}$ for the type II polynomials with respect to $\widehat{\boldsymbol{\nu}}$, $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$. Let $\{a_{\boldsymbol{k},j}, b_{\boldsymbol{k},j}\}$ denote the recurrence coefficients of $\boldsymbol{\nu}$, and $\{\widehat{a}_{\boldsymbol{k},j}, \widehat{b}_{\boldsymbol{k},j}\}$ denote the recurrence coefficients of $\widehat{\boldsymbol{\nu}}$. The connection between the systems $\widehat{\boldsymbol{\nu}}$ and $\boldsymbol{\mu} = (\boldsymbol{\nu}, \mu_r)$ is summarized by the following result.

**Theorem 3.1.** *An index $(\boldsymbol{k}, m)$ is normal for the system $\boldsymbol{\mu} = (\boldsymbol{\nu}, \mu_r)$ if and only if $\boldsymbol{k}$ is normal for the system $\widehat{\boldsymbol{\nu}}$. In that case we have $P_{(\boldsymbol{k},m)}(x) = \Phi(x) \widehat{P}_{\boldsymbol{k}}(x)$.*

**Proof.** Suppose $(\boldsymbol{k}, m)$ is normal. By the orthogonality relations

$$\left\langle P_{(\boldsymbol{k},m)}(x), x^p \right\rangle_r = 0, \qquad p = 0, 1, \dots, m-1, \tag{55}$$

we must have $P_{(\boldsymbol{k},m)}(x) = \Phi(x) Q(x)$ for some polynomial $Q$ with $\deg Q \le |\boldsymbol{k}|$, by Lemma 2.1. For the other inner products we then have

$$\left\langle \Phi(x) Q(x), x^p \right\rangle_j = 0, \qquad p = 0, 1, \dots, k_j - 1, \qquad j = 1, \dots, r-1. \tag{56}$$

So $Q$ must be a multiple orthogonal polynomial for to $\hat{\nu}$. Conversely, any multiple orthogonal polynomial $Q = \hat{P}_{\boldsymbol{k}}$ with respect to $\hat{\nu}$ satisfies (56) by definition. (55) is then satisfied with $P_{(\boldsymbol{k},m)}(x)$ replaced by $\Phi(x)\hat{P}_{\boldsymbol{k}}(x)$, by Lemma 2.1. Now, $(\boldsymbol{k}, m)$ is normal for $(\boldsymbol{\nu}, \mu_r)$ if and only if every $P_{(\boldsymbol{k},m)}$ has degree $|\boldsymbol{k}| + m$, if and only if every $\hat{P}_{\boldsymbol{k}}$ has degree $|\boldsymbol{k}|$, if and only if $\boldsymbol{k}$ is normal for $\hat{\nu}$. □

Now we find the determinantal formula, together with a necessary and sufficient condition on the perfectness of $\hat{\nu}$.

**Theorem 3.2.** *Suppose that $\nu \in \mathcal{L}_\infty^{r-1}$ is a perfect system, and all $\{z_j\}_{j=1}^m$ are distinct.*

*(i) If $(\boldsymbol{k}, m)$ is normal, then for any sequence of $\mathbb{N}^{r-1}$-indices $(\boldsymbol{s}_j)_{j=0}^m$ with $\boldsymbol{s}_0 = 0$ and $|\boldsymbol{s}_j| = j$, we have $D_{\boldsymbol{k}} = \det(P_{\boldsymbol{k}+\boldsymbol{s}_{m-i}}(z_j))_{i,j=1}^m \neq 0$ and the determinantal formula*

$$\hat{P}_{\boldsymbol{k}}(x) = \Phi(x)^{-1} D_{\boldsymbol{k}}^{-1} \det \begin{pmatrix} P_{\boldsymbol{k}+\boldsymbol{s}_m}(x) & P_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(x) & \cdots & P_{\boldsymbol{k}}(x) \\ P_{\boldsymbol{k}+\boldsymbol{s}_m}(z_1) & P_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(z_1) & \cdots & P_{\boldsymbol{k}}(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ P_{\boldsymbol{k}+\boldsymbol{s}_m}(z_m) & P_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(z_m) & \cdots & P_{\boldsymbol{k}}(z_m) \end{pmatrix}, \qquad \boldsymbol{k} \in \mathbb{N}_{\boldsymbol{K}}^{r-1}. \tag{57}$$

*(ii) If $D_{\boldsymbol{k}} = \det(P_{\boldsymbol{k}+\boldsymbol{s}_{m-i}}(z_j))_{i,j=1}^m \neq 0$ for any sequence of $\mathbb{N}^{r-1}$-indices $(\boldsymbol{s}_j)_{j=0}^m$ with $\boldsymbol{s}_0 = 0$ and $|\boldsymbol{s}_j| = j$, then $\hat{\nu}$ is perfect.*

**Proof.** (i) If $D_{\boldsymbol{n}} = 0$ then the columns of $(P_{\boldsymbol{k}+\boldsymbol{s}_{m-i}}(z_j))_{i,j=1}^m$ are linearly dependent. This would imply

$$P(x) = \sum_{j=0}^{m-1} c_j P_{\boldsymbol{k}+\boldsymbol{s}_j}(x) = 0, \qquad x = z_1, \ldots, z_r,$$

for some $(c_1, \ldots, c_r) \neq (0, \ldots, 0)$. However, since we have a linear combination of polynomials with different degrees, $P$ cannot be identically 0. $P$ satisfies the orthogonality conditions at the index $(\boldsymbol{k}, m)$ with respect to $\mu_j$ with $1 \le j \le r-1$ since $\boldsymbol{k} \le \boldsymbol{k} + \boldsymbol{s}_i$ for every $i = 0, \ldots, m-1$, and with respect to $\mu_r$ by Lemma 2.1 since $P$ is divisible by $\Phi$. Since $\deg P \le |\boldsymbol{k}| + m - 1$ this would contradict the normality of $(\boldsymbol{k}, m)$.

The determinant in (57) is a polynomial of degree $|\boldsymbol{k}| + m$. It is a linear combination of $P_{\boldsymbol{k}}, \ldots, P_{\boldsymbol{k}+\boldsymbol{s}_m}$, and $\boldsymbol{k} \le \boldsymbol{k} + \boldsymbol{s}_i$, so it is orthogonal to $x^p$ with respect to $\mu_j$ for $p = 0, \ldots, n_j - 1$, $j = 1, \ldots, r-1$. Since it vanishes at $z_1, \ldots, z_m$ it is orthogonal to everything with respect to $\mu_r$, by Lemma 2.1. Hence the normality of $(\boldsymbol{k}, m)$ implies that $P_{(\boldsymbol{k},m)}$ coincides with the determinant in (57) divided by the normalization constant $D_{\boldsymbol{n}}$. Finally, Theorem 3.1 implies (57).

(ii) Conversely, assume $\boldsymbol{\nu}$ is perfect and $D_{\boldsymbol{k}} \neq 0$ for any choice of path and polynomials for all $\boldsymbol{k} \in \mathbb{N}_{\boldsymbol{K}}^{r-1}$. Fix any path $(\boldsymbol{k}_l)_{k=0}^M$ of multi-indices in $\mathbb{N}_{\boldsymbol{K}}^{r-1}$ with $\boldsymbol{k}_0 = \boldsymbol{0}$ and $\boldsymbol{k}_{l+1} = \boldsymbol{k}_l + \boldsymbol{e}_{j_l}$. For each $\boldsymbol{k}_l \in \mathbb{N}_{\boldsymbol{K}}^{r-1}$, denote $Q_{\boldsymbol{k}_l}$ to be the polynomial on the right-hand side of (57), where we choose $\boldsymbol{s}_j$ in such a way that $\boldsymbol{k} + \boldsymbol{s}_j$ for each $j$ belongs to the chosen path $(\boldsymbol{k}_l)_{l=0}^M$. It is clear that $Q_{\boldsymbol{k}_l}$ is monic of degree $|\boldsymbol{k}_l|$ and satisfies all the type II orthogonality conditions for $\hat{\nu}$. Let us show that $Q_{\boldsymbol{k}_l}$ satisfy the conditions of Remark 2.20 with respect to the system $\hat{\nu}$. Indeed, by (57),

$$\left\langle \Phi(x) Q_{\boldsymbol{k}_l}(x), x^{(\boldsymbol{k}_l)_{j_l}} \right\rangle_{j_l} = \frac{(-1)^m D_{\boldsymbol{k}_{l+1}}}{D_{\boldsymbol{k}_l}} \left\langle P_{\boldsymbol{k}_l}(x), x^{(\boldsymbol{k}_l)_{j_l}} \right\rangle_{j_l}, \tag{58}$$

which is non-zero by the assumptions and Lemma 2.17 applied to $\boldsymbol{\nu}$. Hence by Remark 2.20 every index along the path $(\boldsymbol{k}_l)_{l=0}^M$ is normal for $\hat{\nu}$, and by taking different paths that cover all $\mathbb{N}_{\boldsymbol{K}}^{r-1}$-indices, we see that $\hat{\nu}$ is perfect. □

**Remark 3.3.** If we remove the assumption $\boldsymbol{\nu} \in \mathcal{L}_\infty^{r-1}$ then we get complications when $N_j < \infty$ for all $j = 1, \ldots, r-1$, since a sequence of normal indices $(\boldsymbol{k} + \boldsymbol{s}_j)_{j=0}^m$ will not exist for every $\boldsymbol{k} \in \mathbb{N}_{\boldsymbol{K}}^{r-1}$. Indeed, if $|\boldsymbol{k}| > |\boldsymbol{K}| - m$, then $\boldsymbol{k} + \boldsymbol{s}_j$ will have to be outside of $\mathbb{N}_{\boldsymbol{K}}^{r-1}$ for $j > |\boldsymbol{K}| - |\boldsymbol{k}|$ and therefore will not be normal. Theorem 3.2 still works if one replaces $P_{\boldsymbol{k}+\boldsymbol{s}_j}(x)$ for those $j$'s with, for example, $x^{|\boldsymbol{k}|-|\boldsymbol{K}|+j} P_{\boldsymbol{K}}(x)$. The above proof works without any significant change. More generally, as long as $\deg P_{\boldsymbol{k}+\boldsymbol{s}_j} = |\boldsymbol{k}| + j$ for each $j = 1, \ldots, m$, the proof still holds even if these indices are not normal.

**Remark 3.4.** If we remove the assumption that $\{z_j\}_{j=1}^m$ are distinct, then the theorem still holds but the matrix in (57) should be modified as follows. If $z_j$ is a root of $\Phi$ of multiplicity $l$, then instead of $l$ copies of the row $\begin{pmatrix} P_{\boldsymbol{k}+\boldsymbol{s}_m}(z_j) & P_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(z_j) & \cdots & P_{\boldsymbol{k}}(z_j) \end{pmatrix}$ we have the rows

$$\begin{pmatrix} P_{\boldsymbol{k}+\boldsymbol{s}_m}(z_j) & P_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(z_j) & \cdots & P_{\boldsymbol{k}}(z_j) \\ P'_{\boldsymbol{k}+\boldsymbol{s}_m}(z_j) & P'_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(z_j) & \cdots & P'_{\boldsymbol{k}}(z_j) \\ \vdots & \vdots & \ddots & \vdots \\ P^{(l-1)}_{\boldsymbol{k}+\boldsymbol{s}_m}(z_j) & P^{(l-1)}_{\boldsymbol{k}+\boldsymbol{s}_{m-1}}(z_j) & \cdots & P^{(l-1)}_{\boldsymbol{k}}(z_j) \end{pmatrix},$$

and the corresponding modifications should be done in the determinant $D_{\boldsymbol{k}}$ to make $\hat{P}_{\boldsymbol{k}}$ monic. This change ensures that $z_j$ is a root of multiplicity $l$ of the right-hand side of (57). The previous remark still holds.

### *3.2. Nearest neighbour recurrence coefficients for the Christoffel transforms*

In Theorem 3.1 we stated the relationship between the type II polynomials of $\hat{\nu}$ and $\boldsymbol{\mu}$ (for type I polynomials, see Theorem 3.14 below). It will come as no surprise that the nearest neighbour recurrence coefficients of $\hat{\nu}$ and $\boldsymbol{\mu}$ are also related.

**Theorem 3.5.** *Suppose $\boldsymbol{\mu} \in \mathcal{L}^r$ with $\mu_r \in \mathcal{L}_m$, where $m < \infty$. Then:*

*(i)* $a_{(\boldsymbol{k},m),r} = 0$ *for all* $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$.
*(ii) Additionally, suppose* $\boldsymbol{\nu}$ *and* $\hat{\boldsymbol{\nu}}$ *are perfect. Then*

$$a_{(\boldsymbol{k},m),j} = \widehat{a}_{\boldsymbol{k},j}, \qquad \boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}, \qquad j = 1, \dots, r-1, \tag{59}$$

$$b_{(\boldsymbol{k},m),j} = \widehat{b}_{\boldsymbol{k},j}, \qquad \boldsymbol{k} \in \mathbb{N}^{r}_{\boldsymbol{K}-\boldsymbol{e}_j}, \qquad j = 1, \dots, r-1. \tag{60}$$

**Proof.** (i) This follows immediately from Lemma 2.18 (a) along with Proposition 2.7 (except in the case $\boldsymbol{k} = \boldsymbol{K}$, in which case it follows by definition through Remark 2.12, for the sake of convenience). *(ii)* This is clear from Theorem 3.1 and (i). □

Now let $\boldsymbol{\mu} \in \mathcal{L}^r$ be perfect in the extended sense of Definition 2.8. Suppose that the Jacobi coefficients of each $\mu_j \in \mathcal{L}$ is given, i.e., $(a_{n\boldsymbol{e}_j,j})_{n=1}^{N_j}$ and $(b_{n\boldsymbol{e}_j,j})_{n=0}^{N_j-1}$ for each $1 \le j \le r$ (here $a_{n\boldsymbol{e}_j,N_j} = 0$ if $N_j < \infty$). It is clear that this information is sufficient to determine all the NNRR coefficients $\{a_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}}}$ and $\{b_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}}$ for each $1 \le j \le r$. Filipuk, Haneczok, and Van Assche in [19] put forward a recursive algorithm for this, based on the compatibility conditions (41)–(43). In the next result, we state their algorithm in the more general setting $\boldsymbol{\mu} \in \mathcal{L}^r$, where we allow $N_j$'s to be finite. The proof, which is a straightforward application of Theorem 2.16, is following the same lines as the proof of [19, Thm 3.3] except that extra care needs to be taken with the indices $\boldsymbol{n}$ with $n_j = N_j$ if $N_j < \infty$. Indeed, the $a_{\boldsymbol{n},j}$ coefficients vanish there, as we just showed in Theorem 3.5(i).

**Theorem 3.6** (*The CC Algorithm, [19]*). *Suppose $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$ is perfect. Given the Jacobi coefficients of each $\mu_j$, $j = 1, \dots, r$, the following algorithm produces all the NNRR coefficients:*

*for all* $1 \le j, k \le r$ *with* $j \ne k$ :

$a_{\boldsymbol{n},j} := 0$ *for all* $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}}$ *with* $n_j = 0$;

$\delta_{\boldsymbol{0},j,k} := b_{\boldsymbol{0},k} - b_{\boldsymbol{0},j}$;

*for all* $d \in \mathbb{N}$ :

*for all* $1 \le j, k \le r$ *with* $j \ne k$ :

*for all* $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}}$ *with* $n_k > 0, n_j > 0, |\boldsymbol{n}| = d$ :

$$a_{\boldsymbol{n},j} = \begin{cases} 0 & \text{if } a_{\boldsymbol{n}-\boldsymbol{e}_k,j} = 0, \\ a_{\boldsymbol{n}-\boldsymbol{e}_k,j} \dfrac{\delta_{\boldsymbol{n}-\boldsymbol{e}_k,k,j}}{\delta_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_j,k,j}} & \text{otherwise}; \end{cases}$$

*for all* $1 \le j, k \le r$ *with* $j \ne k$ :

*for all* $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}$ *with* $n_k > 0, |\boldsymbol{n}| = d$ :

$$b_{\boldsymbol{n},j} = b_{\boldsymbol{n}-\boldsymbol{e}_k,j} + \frac{\sum_{i=1}^r a_{\boldsymbol{n},i} - \sum_{i=1}^r a_{\boldsymbol{n}+\boldsymbol{e}_j-\boldsymbol{e}_k,i}}{\delta_{\boldsymbol{n}-\boldsymbol{e}_k,k,j}};$$

*for all* $1 \le j, k \le r$ *with* $j \ne k$ :

*for all* $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j-\boldsymbol{e}_k}, |\boldsymbol{n}| = d$ :

$$\delta_{\boldsymbol{n},j,k} := b_{\boldsymbol{n},k} - b_{\boldsymbol{n},j}.$$

**Proof.** Assume all coefficients of indices with size $< d$ are computed and consider indices $\boldsymbol{n}$ with $|\boldsymbol{n}| = d$. Using (46) we can compute the coefficients $a_{\boldsymbol{n},j}$ when $0 < n_j < N_j$ by

$$a_{\boldsymbol{n},j} = a_{\boldsymbol{n}-\boldsymbol{e}_k,j} \frac{\delta_{\boldsymbol{n}-\boldsymbol{e}_k,k,j}}{\delta_{\boldsymbol{n}-\boldsymbol{e}_k-\boldsymbol{e}_j,k,j}},$$

where $k \ne j$ and $n_k > 0$ (if such a $k$ does not exist then $\boldsymbol{n} = d\boldsymbol{e}_j$, in which case $a_{\boldsymbol{n},j}$ is already given). If $n_j = 0$ then we have $a_{\boldsymbol{n},j} = 0$ by definition, and if $n_j = N_j$ then we have $a_{\boldsymbol{n},j} = 0$ by Theorem 3.5, which agrees with the algorithm.

With every $a_{\boldsymbol{n},j}$ computed, we can then use (45) to compute the coefficients $b_{\boldsymbol{n},j}$ by

$$b_{\boldsymbol{n},j} = b_{\boldsymbol{n}-\boldsymbol{e}_k,j} + \frac{\sum_{i=1}^r a_{\boldsymbol{n},i} - \sum_{i=1}^r a_{\boldsymbol{n}+\boldsymbol{e}_j-\boldsymbol{e}_k,i}}{\delta_{\boldsymbol{n}-\boldsymbol{e}_k,k,j}},$$

with $k$ chosen such that $k \ne j$ and $n_k > 0$. Hence all the NNRR coefficients can be computed, by induction on $|\boldsymbol{n}|$ (in the case $|\boldsymbol{n}| = 0$ there is nothing to compute). □

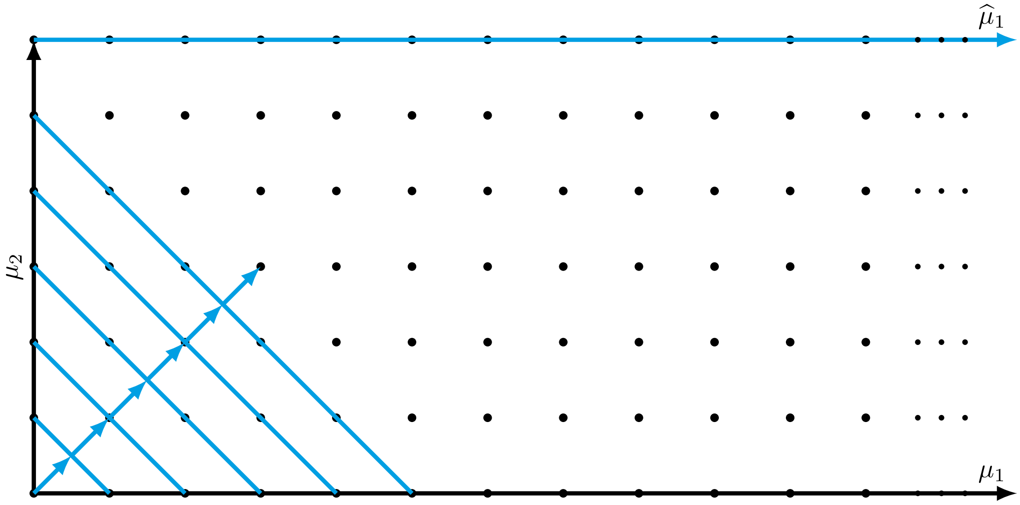


**Fig. 1.** The recurrence coefficients of $(\mu_1, \mu_2)$ can be computed via the CC algorithm (Theorem 3.6). On the upper boundary we find the recurrence coefficients of $\widehat{\mu}_1$.

**Remark 3.7.** Note that the algorithm never breaks down for perfect systems since $\delta_{\boldsymbol{n},j,k} \neq 0$ for all $\boldsymbol{n} \in \mathbb{N}^r_{\boldsymbol{N}-e_j-e_k}$, by Lemma 2.18(b).

**Remark 3.8.** The algorithm in the way that is written here is not optimized since most of the coefficients are computed more than once (generically $r$ times, of course all leading to the same answer); if one cares about the computational efficiency then one would need to be more careful with the loops to avoid repetitions.

**Corollary 3.9.** *Suppose $\boldsymbol{\mu}$ is perfect and $\mu_r \in \mathcal{L}_m$. Then the recurrence coefficients of $\hat{\boldsymbol{v}}$ can be computed from the recurrence coefficients of $\boldsymbol{v}$ using the CC algorithm (see Theorem 3.6) applied to $\boldsymbol{\mu}$ (see Fig. 1).*

**Proof.** This is immediate from Theorem 3.5 and Theorem 3.6. □

### 3.3. Two examples

When $m = 1$ in Theorem 3.2 we get the one-step Christoffel transform $\hat{\boldsymbol{v}}$ of a multiple orthogonal system $\boldsymbol{v}$. This corresponds to taking $\mu_r$ to be the Dirac delta measure $\delta_{z_0}$ at $z_0$ for some $z_0 \in \mathbb{C}$ (i.e., $\delta_{z_0}[x^n] := z_0^n$) and $\Phi(x) = x - z_0$. Here, the determinantal condition is particularly interesting. The formula (61) has appeared earlier in [20]. In the case $r-1 = 1$ this is a result Gautschi [13].

**Theorem 3.10.** *Suppose $\boldsymbol{v} \in \mathcal{L}^{r-1}$ is perfect. Then $\hat{\boldsymbol{v}}$ is perfect if and only if $z_0$ is not a root of any $P_{\boldsymbol{k}}(x)$, $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$, and then the type II polynomials with respect to $\boldsymbol{v}$ are given by*

$$\widehat{P}_{\boldsymbol{k}}(x) = \frac{1}{x - z_0}\left(P_{\boldsymbol{k}+e_j}(x) - \frac{P_{\boldsymbol{k}+e_j}(z_0)}{P_{\boldsymbol{k}}(z_0)} P_{\boldsymbol{k}}(x)\right), \qquad \boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}-e_j}, \qquad j = 1, \dots, r-1. \tag{61}$$

**Proof.** $D_{\boldsymbol{k}} = P_{\boldsymbol{k}}(z_0)$. □

The compatibility conditions/CC algorithm now starts to resemble the original computation algorithm (30). If we write $\delta_{\boldsymbol{k},j}$ for $\delta_{(\boldsymbol{k},0),j,r}$, $j = 1, \dots, r-1$, the Eqs. (44)–(46) for the system $\boldsymbol{\mu} = (\boldsymbol{v}, \delta_{z_0})$ with $k = r$ then turn into

$$\widehat{b}_{\boldsymbol{k},j} - \delta_{\boldsymbol{k}+e_j,j} = b_{\boldsymbol{k}+e_j,j} - \delta_{\boldsymbol{k},j}, \tag{62}$$

$$\sum_{i=1}^{r-1} \widehat{a}_{\boldsymbol{k},i} - \delta_{\boldsymbol{k},j}\widehat{b}_{\boldsymbol{k},j} = \sum_{i=1}^{r-1} a_{\boldsymbol{k}+e_j,i} - \delta_{\boldsymbol{k},j} b_{\boldsymbol{k},j}, \tag{63}$$

$$\delta_{\boldsymbol{k}-e_j,j}\widehat{a}_{\boldsymbol{k},j} = \delta_{\boldsymbol{k},j} a_{\boldsymbol{k},j}, \tag{64}$$

with the initial conditions $\delta_{\boldsymbol{0},j} = z_0 - b_{\boldsymbol{0},j}$ and $\widehat{a}_{\boldsymbol{k},j} = 0$ when $k_j = 0$. In the case $r = 2$ we end up exactly with Gautschi's Eqs. (30). We can now generalize the computation algorithm to one-step Christoffel transforms for multiple orthogonal polynomials.

**Theorem 3.11.** *Suppose $\boldsymbol{v}$ is perfect and $z_0$ is not a root of any $P_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$. Given the NNRR coefficients of $\boldsymbol{v}$, the following algorithm produces all the NNRR coefficients of $\hat{\boldsymbol{v}} := (z - z_0)\boldsymbol{v}$:*

*for all* $1 \le j \le r-1$ :

$\delta_{\boldsymbol{0},j} := z_0 - b_{\boldsymbol{0},j}$;

$\hat{a}_{\boldsymbol{k},j} := 0$ *for all* $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$ *with* $k_j = 0$;

$\hat{b}_{\boldsymbol{0},j} := b_{\boldsymbol{0},j} - \dfrac{a_{e_j,j}}{\delta_{\boldsymbol{0},j}}$;

*for all* $d \in \mathbb{N}$ :

*for all* $1 \le j \le r-1$ :

*for all* $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}-e_j}$ *with* $|\boldsymbol{k}| = d$ :

$\delta_{\boldsymbol{k},j} := \hat{b}_{\boldsymbol{k}-e_j,j} - b_{\boldsymbol{k},j} + \delta_{\boldsymbol{k}-e_j,j}$;

*for all* $1 \le j \le r-1$ :

*for all* $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$ *with* $k_j > 0, |\boldsymbol{k}| = d$ :

$$\hat{a}_{\boldsymbol{k},j} = \begin{cases} a_{\boldsymbol{k},j}\frac{\delta_{\boldsymbol{k},j}}{\delta_{\boldsymbol{k}-e_j,j}} & \text{if } k_j < N_j, \\ 0 & \text{if } k_j = N_j; \end{cases}$$

*for all* $1 \le j \le r-1$ :

*for all* $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}-e_j}$ *with* $|\boldsymbol{k}| = d$ :

$$\hat{b}_{\boldsymbol{k},j} = b_{\boldsymbol{k},j} + \frac{\sum_{i=1}^{r-1} \hat{a}_{\boldsymbol{k},i} - \sum_{i=1}^{r-1} a_{\boldsymbol{k}+e_j,i}}{\delta_{\boldsymbol{k},j}}.$$

Another example one often wants to consider is the two-step Christoffel transform of $v \in \mathcal{M}$ with $\Phi(z) = (x - z_0)(x - \bar{z}_0)$ where $z_0 \in \mathbb{C}\setminus\mathbb{R}$. The most obvious choice of the multiple orthogonality system $(v, \frac{1}{2}\delta_{z_0} + \frac{1}{2}\delta_{\bar{z}_0})$ might not be perfect however. For example, if $v$ is symmetric with respect to $\operatorname{Re} z_0$ and $n$ is odd, then $P_n(x)$, the degree $n$ orthogonal polynomial of $v$, satisfies all the orthogonality conditions for $(v, \frac{1}{2}\delta_{z_0} + \frac{1}{2}\delta_{\bar{z}_0})$ at the location $(n, 1)$, which shows that $(n, 1)$ is not normal in that case. Instead, let us consider the multiple orthogonality system $(v, \omega)$, where $\omega \in \mathcal{L}_2$ given by

$$\omega = w_0\delta_{z_0} + (1 - w_0)\delta_{\bar{z}_0} \tag{65}$$

with $w_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, 1\}$. Indices $(n, 0)$ and $(n, 2)$ are normal for all $n$ since $v$ and its transform are both in $\mathcal{M}$. Suppose that some index $(n, 1)$ is not normal. Then $P_n(x)$ must be one of the type II polynomials at index $(n, 1)$, implying

$$0 = \omega[P_n(x)] = w_0 P_n(z_0) + (1 - w_0)\overline{P_n(z_0)}. \tag{66}$$

The latter equality implies $\operatorname{Re} P_n(z_0) = 0$ and $(2w_0 - 1)\operatorname{Im} P_n(z_0) = 0$. Hence $P_n(z_0) = P_n(\bar{z}_0) = 0$, but then $P_n$ solves all orthogonality relations with respect to the index $(n, 2)$, which contradicts the normality of $(n, 2)$. This proves that $(n, 1)$ is normal and therefore $(v, \omega)$ is perfect for any $w_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, 1\}$. In order to run the CC algorithm with $(v, \omega)$ for computing the Jacobi coefficients of $\hat{v}$, one needs to know the Jacobi coefficients (17) of $\omega$. Elementary calculations show that these are

$$b_0 = x_0 - iy_0(1 - 2w_0), \tag{67}$$

$$b_1 = x_0 + iy_0(1 - 2w_0), \tag{68}$$

$$a_1 = -4w_0(1 - w_0)y_0^2, \tag{69}$$

where $x_0 = \operatorname{Re} z_0$, $y_0 = \operatorname{Im} z_0$. Note that all the arguments used here work just as well in the more general case $\boldsymbol{v} \in \mathcal{M}^{r-1}$ with $r-1 > 2$ except that the perfectness of $v$ and of $\hat{v}$ is no longer given for free (but this easily holds if we assume that $\boldsymbol{v}$ is an Angelesco or an AT system, for example).

### *3.4. An example on the step-line*

In this section we work with a perfect system of two functionals $\boldsymbol{v} = (\mu_1, \mu_2)$ and write $p_n$ for the polynomials defined by $p_{2n} = P_{n,n}$ and $p_{2n+1} = P_{n+1,n}$ for $n \in \mathbb{N}$. $(p_n)_{n=0}^{\infty}$ are called the multiple orthogonal polynomials on the step-line. In the previous sections we have worked with recurrence coefficients from the nearest neighbour recurrence relation, but another recurrence relation restricted to the step-line is also commonly used.

We assume indices on the step-line $\{(n, n)\}_{n\in\mathbb{N}} \cup \{(n + 1, n)\}_{n\in\mathbb{N}}$ are normal. The step-line recurrence relation is then given by

$$xp_n(x) = p_{n+1}(x) + c_n p_n(x) + b_n p_{n-1}(x) + a_n p_{n-2}(x). \tag{70}$$

If we put $a_0 = a_1 = 0$ and $b_1 = 0$ then the recurrence relation holds for any $n \in \mathbb{N}$ (with any choice of $p_{-1}$ and $p_{-2}$). To see that (70) holds, simply choose $b_n$ and $c_n$ such that the polynomial $xp_n(x) - p_{n+1}(x) - c_n p_n(x) - b_n p_{n-1}(x)$ is of minimal degree, and then verify that it satisfies all the orthogonality conditions for the polynomial $p_{n-2}$.

We put $\Phi(x) = x - z_0$ and assume that $p_n(z_0) \neq 0$ for all $n \in \mathbb{N}$. A slight modification of Theorem 3.10 shows that $\hat{\nu} = \Phi\nu$ has all indices normal on the step-line. The polynomials $\hat{p}_n$ on the step-line then satisfy the recurrence relation

$$x\hat{p}_n(x) = \hat{p}_{n+1}(x) + \hat{c}_n\hat{p}_n(x) + \hat{b}_n\hat{p}_{n-1}(x) + \hat{a}_n\hat{p}_{n-2}(x). \tag{71}$$

Let us show how to compute the recurrence coefficients $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$, assuming we are given $a_n$, $b_n$, and $c_n$ for each $n \in \mathbb{N}$. Note that [19] describes how to compute the step-line recurrence coefficients from the nearest neighbour recurrence coefficients (see [41, Sect 2.2] for the same problem for an arbitrary increasing path of indices). Here, however, we would like to go directly from the step-line coefficients of $\boldsymbol{\nu}$ to the step-line coefficients of $\hat{\boldsymbol{\nu}}$.

For the system $\boldsymbol{\mu} = (\mu_1, \mu_2, \delta_{z_0})$ we have $P_{n,m,0}(x) = P_{n,m}(x)$ and $P_{n,m,1}(x) = (x - z_0)\hat{P}_{n,m}(x)$. When $(n, m)$ is on the step-line we have the recurrence relations (70) and (71). We also have the recurrence relation (39), which turns into

$$p_{n+1}(x) - (x - z_0)\hat{p}_n(x) = \delta_n p_n(x). \tag{72}$$

We proceed by computing $(x - z_0)\hat{p}_{n+1}(x)$ using (70), (71), and (72), in two different ways (similarly to the proof of Theorem 2.16 in [17]). First, we have

$$(x - z_0)\hat{p}_{n+1} = p_{n+2} - \delta_{n+1}p_{n+1} = xp_{n+1} - (c_{n+1} + \delta_{n+1})p_{n+1} - b_{n+1}p_n - a_{n+1}p_{n-1}.$$

On the other hand, we have

$$\begin{aligned}(x - z_0)\hat{p}_{n+1} &= x(x - z_0)\hat{p}_n - \hat{c}_n(x - z_0)\hat{p}_n - \hat{b}_n(x - z_0)\hat{p}_{n-1} - \hat{a}_n(x - z_0)\hat{p}_{n-2}\\ &= x(p_{n+1} - \delta_n p_n) - \hat{c}_n(p_{n+1} - \delta_n p_n) - \hat{b}_n(p_n - \delta_{n-1}p_{n-1}) - \hat{a}_n(p_{n-1} - \delta_{n-2}p_{n-2})\\ &= xp_{n+1} - \delta_n(p_{n+1} + c_np_n + b_np_{n-1} + a_np_{n-2}) - \hat{c}_np_{n+1} - (\hat{b}_n - \delta_n\hat{c}_n)p_n\\ &\qquad - (\hat{a}_n - \delta_{n-1}\hat{b}_n)p_{n-1} + \delta_{n-2}\hat{a}_np_{n-2}\\ &= xp_{n+1} - (\delta_n + \hat{c}_n)p_{n+1} - (\delta_nc_n + \hat{b}_n - \delta_n\hat{c}_n)p_n - (\delta_nb_n + \hat{a}_n - \delta_{n-1}\hat{b}_n)p_{n-1}\\ &\qquad - (\delta_na_n - \delta_{n-2}\hat{a}_n)p_{n-2}.\end{aligned}$$

By comparing the two results, using liner independence, we get the following result.

**Theorem 3.12.** *If all indices on the step-line are normal for $\boldsymbol{\nu}$ and $\hat{\boldsymbol{\nu}}$, then*

$$\hat{c}_n - \delta_{n+1} = c_{n+1} - \delta_n, \tag{73}$$

$$\hat{b}_n - \delta_n\hat{c}_n = b_{n+1} - \delta_nc_n, \tag{74}$$

$$\hat{a}_n - \delta_{n-1}\hat{b}_n = a_{n+1} - \delta_nb_n, \tag{75}$$

$$\delta_{n-2}\hat{a}_n = \delta_na_n. \tag{76}$$

*If $n = 0$ we only get the first two equations, and if $n = 1$ we only get the first three equations.*

We now get the following computation algorithm.

**Theorem 3.13.** *Suppose all indices on the step-line are normal for $\boldsymbol{\nu} = (\mu_1, \mu_2)$ and $\hat{\boldsymbol{\nu}} = (x - z_0)\boldsymbol{\nu}$. Given the step-line coefficients of $\boldsymbol{\nu}$, the following algorithm produces all step-line coefficients of $\hat{\boldsymbol{\nu}}$:*

$\delta_0 := z_0 - c_0;$
$\hat{a}_0 = 0;$
$\hat{b}_0 = 0;$
$\hat{c}_0 = c_0 - \dfrac{b_1}{\delta_0};$
$\delta_1 = \hat{c}_0 - c_1 + \delta_0;$
$\hat{a}_1 = 0;$
$\hat{b}_1 = \dfrac{\delta_1b_1 - a_2}{\delta_0};$
$\hat{c}_1 = c_1 + \dfrac{\hat{b}_1 - b_2}{\delta_1};$
*for all* $n \geq 2$ :
  $\delta_n := \hat{c}_{n-1} - c_n + \delta_{n-1};$
  $\hat{a}_n = a_n\dfrac{\delta_n}{\delta_{n-2}};$
  $\hat{b}_n = \dfrac{\delta_nb_n + \hat{a}_n - a_{n+1}}{\delta_{n-1}};$
  $\hat{c}_n = c_n + \dfrac{\hat{b}_n - b_{n+1}}{\delta_n}.$

**Proof.** for $n = 0$, (72) turns into $(x - c_0) - (x - z_0) = \delta_0$, which allows us to compute $\delta_0$. The rest clearly follows from Theorem 3.12. We can divide by $\delta_n$ since $\delta_n \neq 0$ by Lemma 2.18 (b). □

### 3.5. Christoffel transforms of type I polynomials

We write $\boldsymbol{A}_{\boldsymbol{k}} = (A^{(1)}_{\boldsymbol{k}}, \dots, A^{(r-1)}_{\boldsymbol{k}})$ and $\widehat{\boldsymbol{A}}_{\boldsymbol{k}} = (\widehat{A}^{(1)}_{\boldsymbol{k}}, \dots, \widehat{A}^{(r-1)}_{\boldsymbol{k}})$ for the type I polynomials with respect to $\boldsymbol{v}$ and $\widehat{\boldsymbol{v}}$, respectively. We get similar results to the previous two sections here. However, in this case the matrix is larger and the sequence of indices requires the extra constraints $\boldsymbol{s}_j \leq m\mathbf{1}$ where $m\mathbf{1} = (m, \dots, m) \in \mathbb{N}^{r-1}$. For simplicity we assume that $\boldsymbol{v} \in \mathcal{L}^{r-1}_{\infty}$ and all zeros of $\Phi$ are simple, ignoring the details of the special cases discussed in Remark 3.3–3.4.

**Theorem 3.14.** *Let $\boldsymbol{v}$ be a perfect system and $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$. Let $(\boldsymbol{k} + \boldsymbol{s}_j)_{j=0}^{(r-1)m}$ be a sequence of $\mathbb{N}^{r-1}$-indices where $|\boldsymbol{s}_j| = j$ and $\boldsymbol{s}_j \in \mathbb{N}^{r-1}_{m\mathbf{1}}$ for each $j = 0, 1, \dots, (r-1)m$. Consider the determinant*

$$D_{\boldsymbol{k}} = \det \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_1}(z_1) & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_2}(z_1) & \cdots & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_{(r-1)m}}(z_1) \\ \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_1}(z_2) & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_2}(z_2) & \cdots & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_{(r-1)m}}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_1}(z_m) & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_2}(z_m) & \cdots & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_{(r-1)m}}(z_m) \end{pmatrix}, \tag{77}$$

*where $\boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_j}(z_k)$ denotes the column vector with elements $A^{(1)}_{\boldsymbol{k}+\boldsymbol{s}_j}(z_k), \dots, A^{(r-1)}_{\boldsymbol{k}+\boldsymbol{s}_j}(z_k)$.*

*If $(\boldsymbol{k}, m)$ is normal then $D_{\boldsymbol{k}} \neq 0$ and the following determinantal formula holds*

$$A^{(j)}_{(\boldsymbol{k},m)}(x) = \widehat{A}^{(j)}_{\boldsymbol{k}}(x) = \Phi(x)^{-1} D_{\boldsymbol{k}}^{-1} \det \begin{pmatrix} A^{(j)}_{\boldsymbol{k}}(x) & A^{(j)}_{\boldsymbol{k}+\boldsymbol{s}_1}(x) & \cdots & A^{(j)}_{\boldsymbol{k}+\boldsymbol{s}_{(r-1)m}}(x) \\ \boldsymbol{A}_{\boldsymbol{k}}(z_1) & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_1}(z_1) & \cdots & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_{(r-1)m}}(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_{\boldsymbol{k}}(z_m) & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_1}(z_m) & \cdots & \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_{(r-1)m}}(z_m) \end{pmatrix}, \qquad 1 \leq j \leq r-1. \tag{78}$$

**Proof.** In the proof we use the notation $\langle \boldsymbol{A}(x), P(x) \rangle$, for polynomials $P(x)$ and vectors of polynomials $(A^{(1)}(x), \dots, A^{(r-1)}(x))$, to represent $\sum_{j=1}^{r-1} \langle A^{(j)}(x), P(x) \rangle_j$. Note that this is clearly bilinear. Then the orthogonality relations for $\boldsymbol{A}_{\boldsymbol{k}}(x)$ can be rephrased as

$$\langle \boldsymbol{A}_{\boldsymbol{k}}(x), x^p \rangle = 0, \qquad p = 0, \dots, |\boldsymbol{k}| - 2. \tag{79}$$

We first prove that $D_{\boldsymbol{k}} \neq 0$. $D_{\boldsymbol{k}} = 0$ would imply that the columns are linearly dependent. In particular this means that

$$\sum_{j=1}^{(r-1)m} c_j \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_j}(x) = \mathbf{0}, \qquad x = z_1, \dots, z_r,$$

for some $(c_1, \dots, c_{(r-1)m}) \neq (0, \dots, 0)$. On the other hand, we note that the sum cannot vanish for every $x$, since in that case the smallest $l$ such that $c_l \neq 0$ would satisfy

$$\left\langle \sum_{j=1}^{(r-1)m} c_j \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_j}(x), x^{|\boldsymbol{k}|-1+l} \right\rangle = c_l \left\langle \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_l}(x), x^{|\boldsymbol{k}|-1+l} \right\rangle = 0.$$

Normality of $\boldsymbol{k} + \boldsymbol{s}_l$ would then imply $c_l = 0$. Hence we have a non-zero vector $\sum_{j=1}^{(r-1)m} c_j \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_j}$ which is also divisible by $\Phi$. Write $\boldsymbol{A} = (A^{(1)}, \dots, A^{(r-1)})$ for $\Phi(x)^{-1} \sum_{j=1}^{(r-1)m} c_j \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_j}$. Since $\boldsymbol{s}_j \in \mathbb{N}_{m\mathbf{1}}$ for each $j = 0, 1, \dots, (r-1)m$ we have $\deg A^{(i)} \leq k_i - 1$. This vector also satisfies

$$\langle \Phi(x) \boldsymbol{A}(x), x^p \rangle = 0, \qquad p = 0, \dots, |\boldsymbol{k}| - 1,$$

which implies that $\boldsymbol{k}$ is not $\widehat{\boldsymbol{v}}$-normal. This contradicts the normality of $(\boldsymbol{k}, m)$ by Theorem 3.1. Hence we must have $D_{\boldsymbol{k}} \neq 0$.

Next, we note that the determinant in (78) vanishes at $z_1 \dots, z_r$, so it is divisible by $\Phi(x)$ (for each $j = 0, \dots, r-1$). Denote the rightmost expression of (78) by $B^{(j)}_{\boldsymbol{k}}$ and $\boldsymbol{B}_{\boldsymbol{k}} = (B^{(1)}_{\boldsymbol{k}}, \dots, B^{(r-1)}_{\boldsymbol{k}})$. Clearly $\boldsymbol{B}_{\boldsymbol{k}}$ is then orthogonal to $x^p$ for $p = 0, \dots, |\boldsymbol{k}| - 2$ with respect to $\widehat{\boldsymbol{v}}$ in the sense of (79). Now note that

$$\langle \boldsymbol{B}_{\boldsymbol{k}}(x), x^{|\boldsymbol{k}|-1} \rangle_{\widehat{\boldsymbol{v}}} = \langle D_{\boldsymbol{k}}^{-1} D_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}}(x), x^{|\boldsymbol{k}|-1} \rangle_{\boldsymbol{v}} = 1. \tag{80}$$

This shows that $\widehat{\boldsymbol{A}}_{\boldsymbol{k}} = \boldsymbol{B}_{\boldsymbol{k}}$. Finally,

$$\langle \boldsymbol{A}_{(\boldsymbol{k},m)}, x^{p+m} \rangle_{\boldsymbol{\mu}} = \langle \boldsymbol{A}_{(\boldsymbol{k},m)}, x^p \Phi(x) \rangle_{\boldsymbol{\mu}} = \sum_{j=1}^{r-1} \langle \Phi(x) A^{(j)}_{(\boldsymbol{k},m)}, x^p \rangle_j = \begin{cases} 0, & p = 0, \dots, |\boldsymbol{k}| - 2, \\ 1, & p = |\boldsymbol{k}| - 1, \end{cases}$$

by Lemma 2.1. Also $\deg A_{(\boldsymbol{k},m)} \leq k_j - 1$, $j = 1, \dots, r-1$. Since $\boldsymbol{k}$ is normal for $\widehat{\boldsymbol{v}}$, we conclude with $\widehat{A}^{(j)}_{\boldsymbol{k}} = A^{(j)}_{(\boldsymbol{k},m)}$. □

### 3.6. Determinantal formula for type I polynomials: one-step case

For the case $m = 1$, the determinantal formula in Theorem 3.14 is not as simple as in Theorem 3.10. Rather, we get a formula in terms of $\boldsymbol{A}_{\boldsymbol{k}}, \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_1}, \dots, \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{s}_{r-1}}$. However, we can get new formulas by deforming the sequence $(\boldsymbol{k}+\boldsymbol{s}_j)_{j=0}^{r-1}$ using (40). Perhaps the nicest choice is writing the columns in terms of the nearest neighbours, which the following theorem shows is possible, even with very relaxed normality assumptions.

**Theorem 3.15.** *Suppose $\boldsymbol{k}$ is normal for both $\boldsymbol{v}$ and $\hat{\boldsymbol{v}} = (x - z_0)\boldsymbol{v}$. Consider the matrix*

$$D_{\boldsymbol{k}} = \det \begin{pmatrix} A^{(1)}_{\boldsymbol{k}+\boldsymbol{e}_1}(z_0) & A^{(1)}_{\boldsymbol{k}+\boldsymbol{e}_2}(z_0) & \cdots & A^{(1)}_{\boldsymbol{k}+\boldsymbol{e}_{r-1}}(z_0) \\ A^{(2)}_{\boldsymbol{k}+\boldsymbol{e}_1}(z_0) & A^{(2)}_{\boldsymbol{k}+\boldsymbol{e}_2}(z_0) & \cdots & A^{(2)}_{\boldsymbol{k}+\boldsymbol{e}_{r-1}}(z_0) \\ \vdots & \vdots & \ddots & \vdots \\ A^{(r-1)}_{\boldsymbol{k}+\boldsymbol{e}_1}(z_0) & A^{(r-1)}_{\boldsymbol{k}+\boldsymbol{e}_2}(z_0) & \cdots & A^{(r-1)}_{\boldsymbol{k}+\boldsymbol{e}_{r-1}}(z_0) \end{pmatrix}. \tag{81}$$

*Then $D_{\boldsymbol{k}} \neq 0$ and the following determinantal formula holds*

$$\widehat{A}^{(j)}_{\boldsymbol{k}}(x) = (x - z_0)^{-1} D_{\boldsymbol{k}}^{-1} \det \begin{pmatrix} A^{(j)}_{\boldsymbol{k}}(x) & A^{(j)}_{\boldsymbol{k}+\boldsymbol{e}_1}(x) & \cdots & A^{(j)}_{\boldsymbol{k}+\boldsymbol{e}_{r-1}}(x) \\ A^{(1)}_{\boldsymbol{k}}(z_0) & A^{(1)}_{\boldsymbol{k}+\boldsymbol{e}_1}(z_0) & \cdots & A^{(1)}_{\boldsymbol{k}+\boldsymbol{e}_{r-1}}(z_0) \\ \vdots & \vdots & \ddots & \vdots \\ A^{(r-1)}_{\boldsymbol{k}}(z_0) & A^{(r-1)}_{\boldsymbol{k}+\boldsymbol{e}_1}(z_0) & \cdots & A^{(r-1)}_{\boldsymbol{k}+\boldsymbol{e}_{r-1}}(z_0) \end{pmatrix}, \qquad 1 \le j \le r-1. \tag{82}$$

**Proof.** If $D_{\boldsymbol{k}} = 0$ then for some non-trivial linear combination,

$$\sum_{j=1}^{r-1} c_k \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{e}_j}(z_0) = 0, \qquad j = 1, \dots, r-1.$$

Then define the vector $\boldsymbol{A} = (A^{(1)}, \dots, A^{(r-1)})$ by

$$\boldsymbol{A}(x) = (x - z_0)^{-1} \sum_{j=1}^{r-1} c_j \boldsymbol{A}_{\boldsymbol{k}+\boldsymbol{e}_j}(x).$$

Clearly $\deg A^{(i)} \le k_i - 1$, $i = 1, \dots, r-1$, and $\boldsymbol{A}$ satisfies the orthogonality conditions

$$\sum_{i=1}^{r-1} \left\langle (x - z_0) A^{(i)}(x), x^p \right\rangle_i = 0, \qquad p = 0, \dots, |\boldsymbol{k}| - 1.$$

Since $\boldsymbol{A} \neq \boldsymbol{0}$ by Lemma 2.22 we get a contradiction with the normality of the index $\boldsymbol{k}$ for the system $\hat{\boldsymbol{v}}$. We conclude that $D_{\boldsymbol{k}} \neq 0$, and (82) now follows from similar arguments as in the proof of Theorem 3.14. □

**Remark 3.16.** We can get a generalized version of Theorem 3.15 for arbitrary $m$, by considering a determinantal formula in terms of $A^{(j)}_{\boldsymbol{k}}$ and $A^{(j)}_{\boldsymbol{k}+l\boldsymbol{e}_i}$ for each $l = 1, \dots, m$ and $i = 1, \dots, r-1$ (although here we will need to assume more indices than just $\boldsymbol{k}$ to be normal). This can also be seen through repeated elementary row operations in the determinant in (78) combined with the relations in (40).

We can also relate the one-step Christoffel transform to the kernel polynomials (47), similarly to (29), by the following result.

**Theorem 3.17.** *Suppose $\boldsymbol{k}$ is normal for both $\boldsymbol{v}$ and $\hat{\boldsymbol{v}} = (x - z_0)\boldsymbol{v}$. Then $P_{\boldsymbol{k}}(z_0) \neq 0$ and*

$$\boldsymbol{K}_{\boldsymbol{k}}(z_0, x) = -P_{\boldsymbol{k}}(z_0)\widehat{\boldsymbol{A}}_{\boldsymbol{k}}(x). \tag{83}$$

**Proof.** If $P_{\boldsymbol{k}}(z_0) = 0$ then $P_{\boldsymbol{k}}$ satisfies all the orthogonality conditions for the index $(\boldsymbol{k}, 1)$ with respect to the system $(\boldsymbol{v}, \delta_{z_0})$, which is impossible by Theorem 3.1, so $P_{\boldsymbol{k}}(z_0) \neq 0$. By (47) we have

$$\sum_{j=1}^{r} \left\langle (x - z_0) K^{(j)}_{\boldsymbol{k}}(z_0, x), x^p \right\rangle_j = -P_{\boldsymbol{k}}(z_0) \sum_{j=1}^{r} \left\langle A^{(j)}_{\boldsymbol{k}}(x), x^p \right\rangle_j, \qquad p = 0, \dots, |\boldsymbol{k}| - 1,$$

so $\boldsymbol{K}_{\boldsymbol{k}}(z_0, x)$ satisfies the same orthogonality relations as $-P_{\boldsymbol{k}}(z_0)\widehat{\boldsymbol{A}}_{\boldsymbol{k}}(x)$. Clearly we also have $\deg K^{(j)}_{\boldsymbol{k}}(z_0, x) \le k_j - 1$, so we get (83). □

**Remark 3.18.** For the type II polynomials we see that (61) does not resemble $\boldsymbol{K}_{\boldsymbol{k}}(x, z_0)$. We can however get a similar result to (83) by considering linear combinations of $\widehat{P}_{\boldsymbol{k}-\boldsymbol{e}_1}, \dots, \widehat{P}_{\boldsymbol{k}-\boldsymbol{e}_r}$. A quick check using (48) and (61) shows that

$$P_{\boldsymbol{k}}(z_0) K^{(i)}_{\boldsymbol{k}}(z_0, x) = \sum_{j=1}^{r} a_{\boldsymbol{k},j} P_{\boldsymbol{k}-\boldsymbol{e}_j}(z_0) A^{(i)}_{\boldsymbol{k}+\boldsymbol{e}_j}(z_0) \widehat{P}_{\boldsymbol{k}-\boldsymbol{e}_j}(x). \tag{84}$$

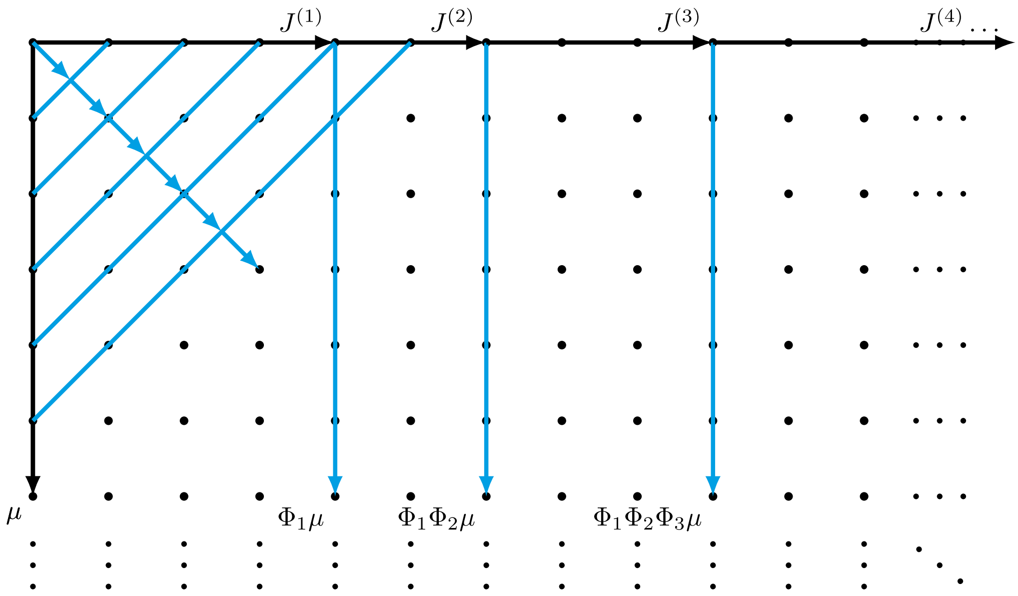


**Fig. 2.** Along the blue vertical arrows we find the Christoffel transforms $\Phi_1\mu, (\Phi_1\Phi_2)\mu, (\Phi_1\Phi_2\Phi_3)\mu, \dots$. The Jacobi coefficients of each transform can be computed via the NNCC algorithm.

### 3.7. Repeated Christoffel transform

In the MOPRL construction, even for the discrete case $\mu_l \in \mathcal{L}_{N_l}$, we always have $a_{je_l,l} \neq 0$ for $1 \le j \le N_l - 1$, and we terminate $a$'s as soon as we reach $a_{N_l e_l,l} = 0$. If one thinks in terms of the Jacobi matrices rather than in terms of the orthogonality measures then it is natural to allow some of the $a_{je_l,l}$'s to be zero. Let us do so for the last measure $\mu_r$ only.

To this end, for each $l = 1, \dots, r$, let $J_l$ be an $N_l \times N_l$ Jacobi matrix (with $N_l$ finite or infinite) with Jacobi coefficients $\{a_{je_l,l}\}_{j=1}^{N_l-1}$ and $\{b_{je_l,l}\}_{j=0}^{N_l-1}$. We assume that $J_1, \dots, J_{r-1}$ are proper, that is, $a_{je_l,l} \neq 0$ for each $1 \le j \le N_l - 1$ and $1 \le l \le r-1$. Suppose $J_r$ is the direct sum $J_r = \bigoplus_{j=1}^{\infty} J_r^{(j)}$ (which is improper), where $J_r^{(j)}$ is a proper Jacobi matrix of size $m_j \times m_j$, with $m_j < \infty$. Denote $\det(z - J_r^{(j)}) = \Phi_j(z)$. We then have $\deg \Phi_j = m_j$.

Now observe that we can still run the CC algorithm without changes. We then treat $N_r$ as $\infty$, and *not* as $m_1$, as we would normally. The algorithm will be able to compute all the recurrence coefficients $\{a_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{(\boldsymbol{K},\infty)}}$ and $\{b_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{(\boldsymbol{K},\infty)-e_j}}$ for each $1 \le j \le r$, as long as all the $\delta_{\boldsymbol{n},j,i}$ coefficients are nonzero. All of these coefficients satisfy the compatibility Eqs. (41)–(43) on the extended lattice $\mathbb{N}^r_{(\boldsymbol{K},\infty)}$ (here $\boldsymbol{K} = (N_1, \dots, N_{r-1})$ as usual), and therefore one can uniquely define the polynomials $P_{\boldsymbol{n}}$ and $A_{\boldsymbol{n}}$ using (36) and (38) for all $\boldsymbol{n} \in \mathbb{N}^r_{(\boldsymbol{K},\infty)}$.

By the results of the previous section it is clear that if $\boldsymbol{n} \in \mathbb{N}^r_{(\boldsymbol{K},\infty)}$ has the form $\boldsymbol{n} = (\boldsymbol{k}, m_1 + \dots + m_s)$ with $\boldsymbol{k} \in \mathbb{N}^{r-1}_{\boldsymbol{K}}$ for some $s \in \mathbb{N}$, then:

*(i)* $a_{\boldsymbol{n},r} = 0$;
*(ii)* $a_{\boldsymbol{n},j}$ and $b_{\boldsymbol{n},j}$ for $1 \le j \le r-1$ are the NNRR coefficients of $(\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}$, the Christoffel transform of the system $\boldsymbol{\nu} = (\mu_1, \dots, \mu_{r-1})$ corresponding to the polynomial $\Phi_1(x) \cdots \Phi_s(x)$;
*(iii)* $P_{\boldsymbol{n}}(x)$ is equal to $\Phi_1(x) \cdots \Phi_s(x)$ times the type II multiple orthogonal polynomial with respect to the system $(\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}$.
*(iv)* $A_{\boldsymbol{n}}^{(j)}(x)$ for $1 \le j \le r-1$ is the type I multiple orthogonal polynomial with respect to the system $(\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}$.

See Fig. 2. Furthermore, in the region $\boldsymbol{n} \in \mathbb{N}^r_{(\boldsymbol{K},\infty)}$ with $m_1 + \dots + m_s \le n_r < m_1 + \dots + m_{s+1}$ for some $s \in \mathbb{N}$, we have:

*(v)* $a_{\boldsymbol{n},j}$ and $b_{\boldsymbol{n},j}$ for $1 \le j \le r$ are the NNRR coefficients of the system $((\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}, \omega_s)$, where $\omega_s$ is the spectral measure/moment functional corresponding to the Jacobi submatrix $J_r^{(s)}$.
*(vi)* $P_{\boldsymbol{n}}(x)$ is equal to $\Phi_1(x) \cdots \Phi_{s-1}(x)$ times the type II multiple orthogonal polynomial with respect to the system $((\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}, \omega_s)$.
*(vii)* $A_{\boldsymbol{n}}^{(j)}(x)$ is equal to the type I multiple orthogonal polynomial with respect to the system $((\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}, \omega_s)$.

The condition $\delta_{\boldsymbol{n},j,l} \neq 0$ that ensures that the CC algorithm in Theorem 3.6 does not break down is then equivalent to the perfectness of all the systems $((\Phi_1 \cdots \Phi_s)\boldsymbol{\nu}, \omega_s)$ for all $s \ge 1$. For example, this is easily seen to be true if $(\mu_1, \dots, \mu_{r-1}) \in \mathcal{M}^{r-1}$ is Angelesco and $\sigma(J_r)$ is disjoint from each $\Delta_j$ (the convex hull of $\mathrm{supp}(\mu_j)$). The same holds true if $(\mu_1, \dots, \mu_{r-1}) \in \mathcal{M}^{r-1}$ is an AT system (see, e.g., [18, Sect 23.1.2] for the definition) on an interval $I$, and $\sigma(J_r)$ is disjoint from $I$.

The simplest example of this construction is to take $r = 2$ with an arbitrary $J_1$, the Jacobi matrix of some $\mu_1 \in \mathcal{M}$ with $\operatorname{supp}(\mu_1) \subseteq \mathbb{R}_+$, and $J_2$ to be the zero matrix, i.e., $m_j = 1$ with $J_r^{(j)}$ being the $1 \times 1$ zero matrices. Then for each $k \in \mathbb{N}$, $(a_{(k,s),1})_{k=1}^{\infty}$ and $(b_{(k,s),1})_{k=0}^{\infty}$ are the Jacobi coefficients of the measure $x^s d\mu_1(x)$. The corresponding polynomials are often called the *associated polynomials*. The CC algorithm from above has a close connection to the $qd$-algorithm of Rutishauser [2] and the Toda lattice with discrete time.

Similarly, the special case of the given construction with $r > 2$ and $m_j = 1$ for all $j$ (so that $J_r^{(j)}$ is the diagonal matrix with $(\lambda_1, \lambda_2, \ldots)$ on the diagonal) leads to the multi-dimensional Toda lattice with discrete time ("dm-Toda lattice"), studied in [20]. The integrable system requires the perfectness of each $\boldsymbol{v}, (x-\lambda_1)\boldsymbol{v}, (x-\lambda_1)(x-\lambda_2)\boldsymbol{v}, \ldots$, see [20, Remark 3.5]. As was discussed in the current section, perfectness holds true if $\boldsymbol{v}$ is any Angelesco or any AT system, and each $\lambda_j$ lies outside of the interior of the convex hull of each $\operatorname{supp} \mu_l$. This provides a wealth of examples of well-defined integrable systems of [20] corresponding to the repeated one-step Christoffel transforms. The same holds true for repeated one-step Geronimus transform if $\boldsymbol{v}$ is any Angelesco or any AT system, and each $\lambda_j$ lies outside of the convex hull of each $\operatorname{supp} \mu_l$.

### 3.8. Recurrence coefficients and normality

As discussed in Section 2.5, every perfect system (in the extended sense of Definition 2.8) $\boldsymbol{\mu} \in \mathcal{L}^r$ produces the nearest neighbour recurrence coefficients $\{a_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}}}$ and $\{b_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}}$ for each $1 \le j \le r$ that satisfy the compatibility conditions (41), (42), (43). In this section we identify what conditions are necessary and sufficient in order for coefficients $\{a_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}}}$ and $\{b_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}}$ to be the nearest neighbour recurrence coefficients of some perfect system.

This corresponds to a result central to the paper [24], solved for $\boldsymbol{\mu} \in \mathcal{L}^r_\infty$, see also [40, Prop. 3]. Here we give an alternative simple proof, based on the lemmas in Section 2.6, along with the CC algorithm. The special feature that appears when some of the measures/functionals are finitely supported is property *(b)* below.

**Theorem 3.19.** *Let $\boldsymbol{N} = (N_j)_{j=1}^r$, where $N_j \le \infty$. Suppose we are given the sets of complex coefficients $\{a_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}}}$ and $\{b_{\boldsymbol{n},j}\}_{\boldsymbol{n}\in\mathbb{N}^r_{\boldsymbol{N}-\boldsymbol{e}_j}}$ for each $1 \le j \le r$. Assume they satisfy the partial difference equations (41), (42), (43) along with the boundary conditions*

- *(a)* $a_{\boldsymbol{n},j} = 0$ *when* $n_j = 0$,
- *(b)* $a_{\boldsymbol{n},j} = 0$ *when* $n_j = N_j$.

*and the normality conditions*

- *(c)* $a_{\boldsymbol{n},j} \ne 0$ *when* $0 < n_j < N_j$,
- *(d)* $b_{\boldsymbol{n},j} - b_{\boldsymbol{n},k} \ne 0$ *when* $j \ne k$.

*Then there is a unique perfect system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in \mathcal{L}^r$ such that $a_{\boldsymbol{n},i}$ and $b_{\boldsymbol{n},i}$ are the nearest neighbour recurrence coefficients of $\boldsymbol{\mu}$ with $\mu_j \in \mathcal{L}_{N_j}$ for each $j = 1, \ldots, r$.*

**Remark 3.20.** By induction using (43), we see that (d) can be replaced with $b_{\boldsymbol{0},j} - b_{\boldsymbol{0},k} \ne 0$ when $j \ne k$. Alternatively, (c) can be replaced with $a_{n\boldsymbol{e}_j,j} \ne 0$ for each $0 < n < N_j$ when $1 \le j \le r$.

**Remark 3.21.** If we also assume $a_{n\boldsymbol{e}_j,j} > 0$ for each $0 < n < N_j$ and some $1 \le j \le r$, then additionally $\mu_j \in \mathcal{M}_{N_j}$.

**Proof.** By the Favard Theorem 2.2, we can get functionals $\mu_1, \ldots, \mu_r$ such that $a_{n\boldsymbol{e}_j,j}$ and $b_{n\boldsymbol{e}_j,j}$ are the Jacobi coefficients of $\mu_j \in \mathcal{L}_{N_j}$ for each $j = 1, \ldots, r$, by *(c)*. We now want to prove that $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ is perfect and that $a_{\boldsymbol{n},j}$ and $b_{\boldsymbol{n},j}$ are the recurrence coefficients of $\boldsymbol{\mu}$. We will use induction, to prove that for each $M \in \mathbb{N}$,

1. Every index $\boldsymbol{n} \le \boldsymbol{N}$ such that $|\boldsymbol{n}| \le M$ is normal.
2. The nearest neighbour recurrence coefficients of $\boldsymbol{\mu}$, for every index $\boldsymbol{n} \le \boldsymbol{N}$ such that $|\boldsymbol{n}| \le M - 1$ are exactly the recurrence coefficients $\{a_{\boldsymbol{n},i}, b_{\boldsymbol{n},i}\}_{i=1}^r$.

This is true when $M = 0$ and $M = 1$, since then we already know which indices of the form $\boldsymbol{n} = n_j\boldsymbol{e}_j$ are normal, and the nearest neighbour recurrence coefficients of $\boldsymbol{n} = \boldsymbol{0}$ are $a_{\boldsymbol{0},j}$ and $b_{\boldsymbol{0},j}$. Now assume (1) and (2) hold for $1, \ldots, M$ and suppose $\boldsymbol{n} \le \boldsymbol{N}$ and $|\boldsymbol{n}| = M + 1$. If $\boldsymbol{n} = n_j\boldsymbol{e}_j$ then we already know that (1) holds. Otherwise we can write $\boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_j + \boldsymbol{e}_k$ for some index $\boldsymbol{m}$ and $k \ne j$. Since $|\boldsymbol{m}| = M - 1$ we know that the nearest neighbour recurrence coefficients for $\boldsymbol{m}$ are $a_{\boldsymbol{m},i}$ and $b_{\boldsymbol{m},i}$. In particular $b_{\boldsymbol{m},j} \ne b_{\boldsymbol{m},k}$, so that $\boldsymbol{n} = \boldsymbol{m}+\boldsymbol{e}_j+\boldsymbol{e}_k$ is normal by Corollary 2.15. Now the conditions of Theorem 2.16 are satisfied and we can apply CC algorithm to compute the nearest neighbour recurrence coefficients of the system $\boldsymbol{\mu}$ for every $\boldsymbol{n}$ with $|\boldsymbol{n}| = M$. Since our recurrence coefficients given in the Theorem are computed we get (2), and the perfectness of $\boldsymbol{\mu}$ follows. □

### 3.9. Zero interlacing for Christoffel transforms: general results

In this section we prove some new interlacing results for multiple orthogonal polynomials of type I and type II. Given two real polynomials $p(x)$ and $q(x)$ we write $p(x) \sim q(x)$ and say that the zeros of polynomials $p(x)$ and $q(x)$ interlace if all the zeros of $p(x)$

and $q(x)$ are pairwise distinct, real, simple, and between every two consecutive zeros of one of the polynomials there lies exactly one zero of the other polynomial.

Let us call a polynomial real-rooted if all of its zeros are real. Recall (see, e.g., [18]) that type II and type I polynomials for any Angelesco system are real-rooted. Type II polynomials for any AT systems are also real-rooted.

The following result is well-known (this is part of the Hermite–Kakeya–Obreschkoff theorem).

**Lemma 3.22.** *Suppose $p$ and $q$ are two polynomials with interlacing zeros $p(x) \sim q(x)$. Then*

$$p(x) + \alpha q(x) \sim q(x), \text{ for any } \alpha \in \mathbb{R}, \tag{85}$$

$$p(x) + \alpha q(x) \sim p(x), \text{ for any } \alpha \neq 0. \tag{86}$$

**Corollary 3.23.** *Suppose $\boldsymbol{v}$ and $\hat{\boldsymbol{v}} = (x - z_0)\boldsymbol{v}$ are perfect for some $z_0 \in \mathbb{R}$. If $P_{\boldsymbol{k}+\boldsymbol{e}_j}(x) \sim P_{\boldsymbol{k}}(x)$, then $\widehat{P}_{\boldsymbol{k}}$ is real-rooted. Moreover,*

$$(x - z_0)\widehat{P}_{\boldsymbol{k}}(x) \sim P_{\boldsymbol{k}}(x), \tag{87}$$

$$(x - z_0)\widehat{P}_{\boldsymbol{k}}(x) \sim P_{\boldsymbol{k}+\boldsymbol{e}_j}(x). \tag{88}$$

**Proof.** Immediate from (61) and Lemma 3.22. □

**Remark 3.24.** Let $\boldsymbol{v}$ be any Angelesco or AT system and suppose $z_0$ is not a zero of any $P_{\boldsymbol{k}}(x)$. By Theorem 3.10, $\hat{\boldsymbol{v}}$ is perfect. Since interlacing for $\boldsymbol{v}$ holds at every multi-index [18,42,43], Corollary 3.23 applies for any $z_0 \in \mathbb{R}$, and we obtain that every $\widehat{P}_{\boldsymbol{k}}$ is real-rooted and interlacings (87)–(88) hold true for all such systems.

In particular, if $z_0$ belongs to the support of one of the measures, then $\hat{\boldsymbol{v}}$ is no longer a system of positive measures, so its perfectness is not trivial in that case.

**Corollary 3.25.** *Suppose $\boldsymbol{v}$ and $\hat{\boldsymbol{v}} = (x - z_0)\boldsymbol{v}$ are perfect for some $z_0 \in \mathbb{R}$. If $\widehat{A}^{(j)}_{\boldsymbol{k}-\boldsymbol{e}_j}(x) \sim \widehat{A}^{(j)}_{\boldsymbol{k}}(x)$, then $A^{(j)}_{\boldsymbol{k}}(x)$ is real-rooted. Moreover,*

$$A^{(j)}_{\boldsymbol{k}}(x) \sim \widehat{A}^{(j)}_{\boldsymbol{k}}(x), \tag{89}$$

$$A^{(j)}_{\boldsymbol{k}}(x) \sim \widehat{A}^{(j)}_{\boldsymbol{k}-\boldsymbol{e}_j}(x). \tag{90}$$

**Proof.** Use (40) and the leftmost equality in (78) to get

$$\boldsymbol{A}_{\boldsymbol{k}} - \widehat{\boldsymbol{A}}_{\boldsymbol{k}-\boldsymbol{e}_j} = \delta \widehat{\boldsymbol{A}}_{\boldsymbol{k}}, \tag{91}$$

where $\delta = \delta_{(\boldsymbol{k}-\boldsymbol{e}_j,0),m,j}$ is non-zero by Lemma 2.18 *(b)*, and (41). If $z_0 \in \mathbb{R}$ then $\delta \in \mathbb{R}$, and Lemma 3.22 completes the proof. □

**Remark 3.26.** If $\boldsymbol{v}$ is Angelesco and $z_0$ is outside of the convex hull of each $\mu_j$, then $\hat{\boldsymbol{v}}$ is also Angelesco. For these systems, the zeros of $\widehat{A}^{(j)}_{\boldsymbol{k}-\boldsymbol{e}_j}(x)$ and $\widehat{A}^{(j)}_{\boldsymbol{k}}(x)$ interlace for any index $\boldsymbol{k}$ and any $j$ (see [44,45]), so Corollary 3.25 applies. Similarly, if $\boldsymbol{v}$ is Nikishin, then so is $\hat{\boldsymbol{v}}$. $\widehat{A}^{(j)}_{\boldsymbol{k}-\boldsymbol{e}_j}(x)$ and $\widehat{A}^{(j)}_{\boldsymbol{k}}(x)$ do interlace but not for all the indices $\boldsymbol{n}$ (see [46] for more details), so Corollary 3.25 applies for such indices.

### *3.10. Zero interlacing: continuous examples*

While Corollaries 3.23 and 3.25 might seem tame and almost obvious, they allow to obtain interesting interlacing results when applied to the multiple analogues of the classical orthogonal polynomials. We collect them in Sections 3.10 and 3.11.

We remark that recurrences (61) and (91) (as well as the NNRR for type II and type I polynomials) provide not only interlacing properties but also various relations that the classical MOPRL (as well as classical OPRL, as a special case $r = 1$) must satisfy. It is highly likely that these have appeared in the literature in various forms. We focus on interlacing properties only.

#### *3.10.1. The multiple Laguerre polynomials of the first kind*

Let $P^{\boldsymbol{\alpha}}_{\boldsymbol{n}}$ be the type II multiple orthogonal polynomials corresponding to the system $\boldsymbol{\mu}^{\boldsymbol{\alpha}} \in \mathcal{M}^r$ given by

$$d\mu^{\boldsymbol{\alpha}}_j = x^{\alpha_j} e^{-x} \chi_{[0,\infty)}(x) dx,$$

where $\alpha_j > -1$ and $\alpha_j - \alpha_k \notin \mathbb{Z}$ for $j \neq k$. Here and everywhere below $\chi_S(x)$ is the characteristic function of a set $S$.

Since $\boldsymbol{\mu}^{\boldsymbol{\alpha}}$ is an AT system and $x\boldsymbol{\mu}^{\boldsymbol{\alpha}} = \boldsymbol{\mu}^{\boldsymbol{\alpha}+\boldsymbol{1}}$ (where $\boldsymbol{1}$ is the vector of 1's), Corollary 3.23 gives

$$P^{\boldsymbol{\alpha}+\boldsymbol{1}}_{\boldsymbol{n}}(x) \sim P^{\boldsymbol{\alpha}}_{\boldsymbol{n}}(x) \quad \text{and} \quad P^{\boldsymbol{\alpha}+\boldsymbol{1}}_{\boldsymbol{n}}(x) \sim P^{\boldsymbol{\alpha}}_{\boldsymbol{n}+\boldsymbol{e}_j}(x)$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ and any $1 \leq j \leq r$.

*3.10.2. The multiple Laguerre polynomials of the second kind*

Let $P_{\boldsymbol{n}}^{c;\alpha}$ be the type II multiple orthogonal polynomials corresponding to the system $\boldsymbol{\mu}^{c;\alpha} \in \mathcal{M}^r$ given by

$$d\mu_j^{c;\alpha} = x^\alpha e^{-c_j x} \chi_{[0,\infty)}(x) dx,$$

where $\alpha > -1$, $c_j > 0$, and $c_j \neq c_k$ for $j \neq k$.

Since $\boldsymbol{\mu}^{c;\alpha}$ is an AT system and $x\boldsymbol{\mu}^{c;\alpha} = \boldsymbol{\mu}^{c;\alpha+1}$, Corollary 3.23 gives

$$P_{\boldsymbol{n}}^{c;\alpha+1}(x) \sim P_{\boldsymbol{n}}^{c;\alpha}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{c;\alpha+1}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{c;\alpha}(x)$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ and any $1 \le j \le r$.

*3.10.3. The Jacobi–Piñeiro polynomials*

Let $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta}$ be the type II multiple orthogonal polynomials corresponding to the system $\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta} \in \mathcal{M}^r$ given by

$$d\mu_j^{\boldsymbol{\alpha};\beta} = x^{\alpha_j}(1-x)^\beta \chi_{[0,1]}(x) dx,$$

where $\beta > -1$, $\alpha_j > -1$, and $\alpha_j - \alpha_k \notin \mathbb{Z}$ for $j \neq k$.

Since $\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta}$ is an AT system and $x\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta} = \boldsymbol{\mu}^{\boldsymbol{\alpha}+\boldsymbol{1};\beta}$ and $(1-x)\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta} = \boldsymbol{\mu}^{\boldsymbol{\alpha};\beta+1}$ Corollary 3.23 gives

*(i)* $P_{\boldsymbol{n}}^{\boldsymbol{\alpha}+\boldsymbol{1};\beta}(x) \sim P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta}(x)$ and $P_{\boldsymbol{n}}^{\boldsymbol{\alpha}+\boldsymbol{1};\beta}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta}(x)$;
*(ii)* $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta+1}(x) \sim P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta}(x)$ and $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta+1}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta}(x)$,

for any $\boldsymbol{n} \in \mathbb{N}^r$ and any $1 \le j \le r$.

*3.10.4. The Angelesco–Jacobi polynomials*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{\alpha,\beta,\gamma}$ corresponding to the system $\boldsymbol{\mu}^{\alpha,\beta,\gamma} \in \mathcal{M}^2$ where

$$d\mu_1^{\alpha,\beta,\gamma} = (1-x)^\alpha (x-a)^\beta |x|^\gamma \chi_{[a,0]}(x)\, dx, \tag{92}$$
$$d\mu_2^{\alpha,\beta,\gamma} = (1-x)^\alpha (x-a)^\beta |x|^\gamma \chi_{[0,1]}(x)\, dx, \tag{93}$$

where $\alpha, \beta, \gamma > -1$, $a < 0$. Denote $\boldsymbol{A}_{\boldsymbol{n}}^{\alpha,\beta,\gamma} = (A_{\boldsymbol{n},1}^{\alpha,\beta,\gamma}, A_{\boldsymbol{n},2}^{\alpha,\beta,\gamma})$ to be the corresponding type I polynomials.

Since $\boldsymbol{\mu}^{\alpha,\beta,\gamma} \in \mathcal{M}^2$ is an Angelesco system, Corollary 3.23 and Corollary 3.25 can be applied with any choice of $z_0$ away from the zeros of type II multiple orthogonal polynomials which are known to all belong to $(a,0) \cup (0,1)$. Note that in our notation, $(x-1)\boldsymbol{\mu}^{\alpha,\beta,\gamma} = \boldsymbol{\mu}^{\alpha+1,\beta,\gamma}$, $(x-a)\boldsymbol{\mu}^{\alpha,\beta,\gamma} = \boldsymbol{\mu}^{\alpha,\beta+1,\gamma}$, and $x\boldsymbol{\mu}^{\alpha,\beta,\gamma} = \boldsymbol{\mu}^{\alpha,\beta,\gamma+1}$ up to a trivial multiplicative normalization. Then

*(i)* $P_{\boldsymbol{n}}^{\alpha+1,\beta,\gamma}(x) \sim P_{\boldsymbol{n}}^{\alpha,\beta,\gamma}(x)$ and $P_{\boldsymbol{n}}^{\alpha+1,\beta,\gamma}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\alpha,\beta,\gamma}(x)$;
*(ii)* $P_{\boldsymbol{n}}^{\alpha,\beta+1,\gamma}(x) \sim P_{\boldsymbol{n}}^{\alpha,\beta,\gamma}(x)$ and $P_{\boldsymbol{n}}^{\alpha,\beta+1,\gamma}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\alpha,\beta,\gamma}(x)$;
*(iii)* $xP_{\boldsymbol{n}}^{\alpha,\beta,\gamma+1}(x) \sim P_{\boldsymbol{n}}^{\alpha,\beta,\gamma}(x)$ and $xP_{\boldsymbol{n}}^{\alpha,\beta,\gamma+1}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\alpha,\beta,\gamma}(x)$;
*(iv)* $A_{\boldsymbol{n},j}^{\alpha+1,\beta,\gamma}(x) \sim A_{\boldsymbol{n},j}^{\alpha,\beta,\gamma}(x)$ and $A_{\boldsymbol{n},j}^{\alpha+1,\beta,\gamma}(x) \sim A_{\boldsymbol{n}+\boldsymbol{e}_j,j}^{\alpha,\beta,\gamma}(x)$;
*(v)* $A_{\boldsymbol{n},j}^{\alpha,\beta+1,\gamma}(x) \sim A_{\boldsymbol{n},j}^{\alpha,\beta,\gamma}(x)$ and $A_{\boldsymbol{n},j}^{\alpha,\beta+1,\gamma}(x) \sim A_{\boldsymbol{n}+\boldsymbol{e}_j,j}^{\alpha,\beta,\gamma}(x)$;
*(vi)* $A_{\boldsymbol{n},j}^{\alpha,\beta,\gamma+1}(x) \sim A_{\boldsymbol{n},j}^{\alpha,\beta,\gamma}(x)$ and $A_{\boldsymbol{n},j}^{\alpha,\beta,\gamma+1}(x) \sim A_{\boldsymbol{n}+\boldsymbol{e}_j,j}^{\alpha,\beta,\gamma}(x)$,

for any $\boldsymbol{n} \in \mathbb{N}^2$ and any $j = 1, 2$.

Note that (i)–(iii) follows from Corollary 3.23. For (i) and (ii) the extra zero at $a$ or at 1 does not matter since all the zeros of type II polynomials belong to $(a,1)$. (iv)–(vi) is immediate from Corollary 3.25.

*3.10.5. The Jacobi–Laguerre polynomials*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{\beta,\gamma}$ corresponding to the system $\boldsymbol{\mu}^{\beta,\gamma} \in \mathcal{M}^2$ where

$$d\mu_1^{\beta,\gamma} = (x-a)^\beta |x|^\gamma e^{-x} \chi_{[a,0]}(x)\, dx, \tag{94}$$
$$d\mu_2^{\beta,\gamma} = (x-a)^\beta |x|^\gamma e^{-x} \chi_{[0,\infty)}(x)\, dx, \tag{95}$$

where $\beta, \gamma > -1$, $a < 0$. Denote $\boldsymbol{A}_{\boldsymbol{n}}^{\beta,\gamma} = (A_{\boldsymbol{n},1}^{\beta,\gamma}, A_{\boldsymbol{n},2}^{\beta,\gamma})$ to be the corresponding type I polynomials.

Since $\boldsymbol{\mu}^{\beta,\gamma} \in \mathcal{M}^2$ is an Angelesco system, and $(x-a)\boldsymbol{\mu}^{\beta,\gamma} = \boldsymbol{\mu}^{\beta+1,\gamma}$, $x\boldsymbol{\mu}^{\beta,\gamma} = \boldsymbol{\mu}^{\beta,\gamma+1}$ up to a trivial multiplicative normalization, we get as in the previous section:

*(i)* $P_{\boldsymbol{n}}^{\beta+1,\gamma}(x) \sim P_{\boldsymbol{n}}^{\beta,\gamma}(x)$ and $P_{\boldsymbol{n}}^{\beta+1,\gamma}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\beta,\gamma}(x)$;
*(ii)* $xP_{\boldsymbol{n}}^{\beta,\gamma+1}(x) \sim P_{\boldsymbol{n}}^{\beta,\gamma}(x)$ and $xP_{\boldsymbol{n}}^{\beta,\gamma+1}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\beta,\gamma}(x)$;
*(iii)* $A_{\boldsymbol{n},j}^{\beta+1,\gamma}(x) \sim A_{\boldsymbol{n},j}^{\beta,\gamma}(x)$ and $A_{\boldsymbol{n},j}^{\beta+1,\gamma}(x) \sim A_{\boldsymbol{n}+\boldsymbol{e}_j,j}^{\beta,\gamma}(x)$;
*(iv)* $A_{\boldsymbol{n},j}^{\beta,\gamma+1}(x) \sim A_{\boldsymbol{n},j}^{\beta,\gamma}(x)$ and $A_{\boldsymbol{n},j}^{\beta,\gamma+1}(x) \sim A_{\boldsymbol{n}+\boldsymbol{e}_j,j}^{\beta,\gamma}(x)$.

for any $\boldsymbol{n} \in \mathbb{N}^2$ and any $j = 1, 2$.

*3.10.6. The Jacobi–Hermite polynomials*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{\gamma}$ corresponding to the system $\boldsymbol{\mu}^{\gamma} \in \mathcal{M}^2$ given by

$$d\mu_1^{\gamma} = |x|^{\gamma} e^{-x^2} \chi_{(-\infty,0]}(x)\, dx, \tag{96}$$

$$d\mu_2^{\gamma} = x^{\gamma} e^{-x^2/2} \chi_{[0,\infty)}(x)\, dx, \tag{97}$$

where $\gamma > -1$. Since $\boldsymbol{\mu}^{\gamma} \in \mathcal{M}^2$ is an Angelesco system, and $x\boldsymbol{\mu}^{\gamma} = \boldsymbol{\mu}^{\gamma+1}$ up to a trivial multiplicative normalization, we get:

*(i)* $xP_{\boldsymbol{n}}^{\gamma+1}(x) \sim P_{\boldsymbol{n}}^{\gamma}(x)$ and $xP_{\boldsymbol{n}}^{\gamma+1}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\gamma}(x)$;

*(ii)* $A_{\boldsymbol{n},j}^{\gamma+1}(x) \sim A_{\boldsymbol{n},j}^{\gamma}(x)$ and $A_{\boldsymbol{n},j}^{\gamma+1}(x) \sim A_{\boldsymbol{n}+\boldsymbol{e}_j,j}^{\gamma}(x)$.

for any $\boldsymbol{n} \in \mathbb{N}^2$ and any $j = 1, 2$.

Zero interlacing for type II Angelesco–Jacobi, Jacobi–Laguerre, and Jacobi–Hermite polynomials were proved by Martínez-Finkelshtein–Morales in their recent [25, Thm 2.2] with more involved arguments and for $\boldsymbol{n}$ along the stepline (see also [26, Lem 2.1]).

*3.11. Zero interlacing: discrete examples*

*3.11.1. The multiple Charlier polynomials*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{\boldsymbol{a}}$ corresponding to the discrete system $\boldsymbol{\mu}^{\boldsymbol{a}} \in \mathcal{M}^r$ given by

$$\mu_j^{\boldsymbol{a}} = \sum_{k=0}^{\infty} \frac{a_j^k}{k!} \delta_k,$$

where $a_j > 0$, and $\alpha_j \neq \alpha_s$ for $j \neq s$.

Recall that $\boldsymbol{\mu}^{\alpha}$ is an AT system. Now note that $x\boldsymbol{\mu}^{\boldsymbol{a}}$ is supported on $\{k \in \mathbb{Z} : k \geq 1\}$ and its weight at $\{k\}$ is $\frac{a_j^k}{k!} k = a_j \frac{a_j^{k-1}}{(k-1)!}$. This means that up to an inconsequential multiplicative normalization, $x\boldsymbol{\mu}^{\boldsymbol{a}}$ shifted by 1 to the left coincides with $\boldsymbol{\mu}^{\boldsymbol{a}}$ itself. Corollary 3.23 then gives

$$P_{\boldsymbol{n}}^{\boldsymbol{a}}(x-1) \sim P_{\boldsymbol{n}}^{\boldsymbol{a}}(x) \quad \text{and} \quad xP_{\boldsymbol{n}}^{\boldsymbol{a}}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{a}}(x) \tag{98}$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ and any $1 \leq j \leq r$.

The self-interlacing property $xP_{\boldsymbol{n}}^{\boldsymbol{a}}(x-1) \sim P_{\boldsymbol{n}}^{\boldsymbol{a}}(x)$ implies a number of interesting properties, see [47, Sect 8.7]. In particular, it is easy to see that (98) implies that the distance between any two roots of $P_{\boldsymbol{n}}^{\boldsymbol{a}}$ is at least 1. In Theorem 3.27 below we provide a more general statement.

*3.11.2. The multiple Meixner polynomials of the first kind*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{\boldsymbol{c};\beta}$ corresponding to the discrete system $\boldsymbol{\mu}^{\boldsymbol{c};\beta} \in \mathcal{M}^r$ given by

$$\mu_j^{\boldsymbol{c};\beta} = \sum_{k=0}^{\infty} \frac{(\beta)_k c_j^k}{k!} \delta_k,$$

where $\beta > 0$, $0 < c_j < 1$, and $c_j \neq c_s$ for $j \neq s$. Here $(x)_0 = 1$, $(x)_k = x(x+1)\ldots(x+k-1)$ is the Pochhammer symbol.

Again, $\boldsymbol{\mu}^{\boldsymbol{c};\beta}$ is an AT system, and $x\boldsymbol{\mu}^{\boldsymbol{c};\beta}$ is supported on $\{k \in \mathbb{Z} : k \geq 1\}$ with the weight at $\{k\}$ $(k \geq 1)$ being $\frac{(\beta)_k c_j^k}{k!} k = \beta c_j \frac{(\beta+1)_{k-1} c_j^{k-1}}{(k-1)!}$. This means that up to an inconsequential multiplicative normalization, $x\boldsymbol{\mu}^{\boldsymbol{c};\beta}$ shifted by 1 to the left coincides with $\boldsymbol{\mu}^{\boldsymbol{c};\beta+1}$. Corollary 3.23 then gives

$$P_{\boldsymbol{n}}^{\boldsymbol{c};\beta+1}(x-1) \sim P_{\boldsymbol{n}}^{\boldsymbol{c};\beta}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{\boldsymbol{c};\beta+1}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{c};\beta}(x)$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ and any $1 \leq j \leq r$.

*3.11.3. The multiple Meixner polynomials of the second kind*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{c;\boldsymbol{\beta}}$ corresponding to the discrete system $\boldsymbol{\mu}^{c;\boldsymbol{\beta}} \in \mathcal{M}^r$ given by

$$\mu_j^{c;\boldsymbol{\beta}} = \sum_{k=0}^{\infty} \frac{(\beta_j)_k c^k}{k!} \delta_k,$$

where $\beta_j > 0$, $0 < c < 1$, and $\beta_j - \beta_s \notin \mathbb{Z}$ for $j \neq s$.

$\boldsymbol{\mu}^{c;\boldsymbol{\beta}}$ is an AT system, and up to an inconsequential multiplicative normalization, $x\boldsymbol{\mu}^{c;\boldsymbol{\beta}}$ shifted by 1 to the left coincides with $\boldsymbol{\mu}^{c;\boldsymbol{\beta}+\mathbf{1}}$. Corollary 3.23 then gives

$$P_{\boldsymbol{n}}^{c;\boldsymbol{\beta}+\mathbf{1}}(x-1) \sim P_{\boldsymbol{n}}^{c;\boldsymbol{\beta}}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{c;\boldsymbol{\beta}+\mathbf{1}}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{c;\boldsymbol{\beta}}(x)$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ and any $1 \leq j \leq r$.

*3.11.4. The multiple Krawtchouk polynomials*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{N;\boldsymbol{p}}$ corresponding to the discrete finite system $\boldsymbol{\mu}^{N;\boldsymbol{p}} \in \mathcal{M}^r$ given by

$$\mu_j^{N;p} = \sum_{k=0}^{N} \binom{N}{k} p_j^k (1-p_j)^{N-k} \delta_k,$$

where $N \in \mathbb{Z}_+$, $0 < p_j < 1$, and $p_j - p_s \neq 0$ for $j \neq s$.

$\boldsymbol{\mu}^{N;\boldsymbol{p}}$ is a discrete AT system (see [48]), and up to an inconsequential multiplicative normalization, $x\boldsymbol{\mu}^{N;\boldsymbol{p}}$ shifted by 1 to the left coincides with $\boldsymbol{\mu}^{N-1;\boldsymbol{p}}$. Corollary 3.23 then gives

$$P_{\boldsymbol{n}}^{N-1;\boldsymbol{p}}(x-1) \sim P_{\boldsymbol{n}}^{N;\boldsymbol{p}}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{N-1;\boldsymbol{p}}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{N;\boldsymbol{p}}(x)$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ with $|\boldsymbol{n}| < N$ and any $1 \le j \le r$.

Similarly, up to an inconsequential multiplicative normalization, $(N-x)\boldsymbol{\mu}^{N;\boldsymbol{p}}$ coincides with $\boldsymbol{\mu}^{N-1;\boldsymbol{p}}$. Corollary 3.23 then gives

$$P_{\boldsymbol{n}}^{N-1;\boldsymbol{p}}(x) \sim P_{\boldsymbol{n}}^{N;\boldsymbol{p}}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{N-1;\boldsymbol{p}}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{N;\boldsymbol{p}}(x)$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ with $|\boldsymbol{n}| < N$ and any $1 \le j \le r$.

*3.11.5. The multiple Hahn polynomials*

These are the type II multiple orthogonal polynomials $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N}$ corresponding to the discrete finite system $\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta;N} \in \mathcal{M}^r$ given by

$$\mu_j^{\alpha;\beta;N} = \sum_{k=0}^{N} \frac{(\alpha_j+1)_k(\beta+1)_{N-k}}{k!(N-k)!} \delta_k$$

where $N \in \mathbb{Z}_+$, $\beta > -1$, $\alpha_j > -1$, and $\alpha_j - \alpha_s \neq 0$ for $j \neq s$.

Again, $\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta;N}$ is a discrete AT system (see [48]), and up to an inconsequential multiplicative normalization, $x\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta;N}$ shifted by 1 to the left coincides with $\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta;N-1}$, and $(N-x)\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta;N}$ coincides with $\boldsymbol{\mu}^{\boldsymbol{\alpha};\beta;N-1}$. Corollary 3.23 then gives

$$P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x-1) \sim P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x) \tag{99}$$

and

$$P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x) \sim P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N}(x) \quad \text{and} \quad P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x) \tag{100}$$

for any $\boldsymbol{n} \in \mathbb{N}^r$ with $|\boldsymbol{n}| < N$ and any $1 \le j \le r$.

### 3.12. Minimal distance between roots

In Section 3.11.1 we observed that the minimal distance between consecutive roots (sometimes called the *mesh*) of any type II multiple Charlier polynomial is larger than 1. In the next theorem we generalize this to all the other discrete multiple orthogonal polynomials discussed here. This also serves as a new elementary proof for the case of one measure $r = 1$, for which this property has been well known, see [28–30].

**Theorem 3.27.** *The minimal distance between roots of any type II multiple Hahn (for $|\boldsymbol{n}| < N$), multiple Krawtchouk (for $|\boldsymbol{n}| < N$), or multiple Charlier polynomial is larger than 1. The minimal distance between roots of any type II multiple Meixner polynomial of the first or second kind is not smaller than 1.*

**Proof.** For the Charlier case this is clear from (98).

Consider the multiple Hahn case. Since all the zeros of type II polynomials belong to the interval $[0, N]$ (AT systems), then interlacing in (99) and (100) can be strengthened to $xP_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x)$ and $(N-x)P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x-1) \sim P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x)$.

This implies that below the first zero of $P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x)$ there is no zero of either $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x-1)$ nor $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x)$. Then between the first and the second zero of $P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x)$ there is exactly one zero of $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x-1)$ and of $P_{\boldsymbol{n}}^{\boldsymbol{\alpha};\beta;N-1}(x)$. This proves that the distance between the first two zeros of $P_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha};\beta;N}(x)$ is larger than one. An easy inductive argument can be used to complete the proof.

The proof for the multiple Krawtchouk polynomials is identical.

For the Meixner of the first and second kind, recall that they can be obtained from the Hahn polynomials by taking a certain limit with respect to the coefficients $\boldsymbol{\alpha}, \beta, N$, see [37]. Since the minimal distance for the Hahn polynomials is $> 1$, we get $\ge 1$ in the limit. □

## Appendix A

### A.1. Proof of Theorem 2.13

**Lemma 2.22.** *Suppose $\mathbf{n}$ is normal. Then the vectors $\left\{(A^{(1)}_{\mathbf{n}+e_i},\dots,A^{(r)}_{\mathbf{n}+e_i})\right\}_{i=1}^{r}$ are linearly independent.*

**Proof.** Consider the equation

$$\sum_{i=1}^{r} c_i(A^{(1)}_{\mathbf{n}+e_i},\dots,A^{(r)}_{\mathbf{n}+e_i}) = 0.$$

Since $\mathbf{n}$ is normal $A^{(j)}_{\mathbf{n}+e_j}$ has degree $n_j$, by Lemma 2.17, but $A^{(j)}_{\mathbf{n}+e_i}$ has degree $< n_j$ when $i \neq j$, so $c_j = 0$, which proves linear independence. □

**Remark 2.23.** The type II equivalent of Lemma 2.22 is the linear independence of $P_{\mathbf{n}-e_1},\dots,P_{\mathbf{n}-e_r}$, which also follows from Lemma 2.17 (see [18,36]).

**Proof of Theorem 2.13.** We have

$$\sum_{k=1}^{r}\left\langle xA^{(k)}_{\mathbf{n}}(x) - A^{(k)}_{\mathbf{n}-e_j}, x^p\right\rangle_k = 0, \qquad p = 0,1,\dots,|\mathbf{n}|-2.$$

Now choose $d_{\mathbf{n},j}$ such that

$$\sum_{k=1}^{r}\left\langle xA^{(k)}_{\mathbf{n}}(x) - A^{(k)}_{\mathbf{n}-e_j} - d_{\mathbf{n},j}A^{(k)}_{\mathbf{n}}, x^p\right\rangle_k = 0, \qquad p = 0,1,\dots,|\mathbf{n}|-1.$$

The polynomial $B^{(k)}_{\mathbf{n}} = xA^{(k)}_{\mathbf{n}}(x) - A^{(k)}_{\mathbf{n}-e_j} - d_{\mathbf{n},j}A^{(k)}_{\mathbf{n}}$ has degree $n_k$. Hence the above system of orthogonality relations is homogeneous with matrix $M^t_{\mathbf{n}}$ with $r$ columns added, where $M_{\mathbf{n}}$ is given by (15). This matrix has nullity $r$, since $M^t_{\mathbf{n}}$ has nullity 0, by normality. Hence the solution space is of dimension $r$, and $(B^{(1)}_{\mathbf{n}},\dots,B^{(r)}_{\mathbf{n}}) = (A^{(1)}_{\mathbf{n}+e_i},\dots,A^{(r)}_{\mathbf{n}+e_i})$ are solutions for each $i = 1,\dots,r$. By Lemma 2.22 every solution is thus on the form

$$B^{(k)}_{\mathbf{n}} = \sum_{i=1}^{r} c_{\mathbf{n},i}A^{(k)}_{\mathbf{n}+e_i}.$$

In other words we have the recurrence relation

$$xA^{(k)}_{\mathbf{n}}(x) = A^{(k)}_{\mathbf{n}-e_j} + d_{\mathbf{n},j}A^{(k)}_{\mathbf{n}} + \sum_{i=1}^{r} c_{\mathbf{n},i}A^{(k)}_{\mathbf{n}+e_i}.$$

To show that $d_{\mathbf{n},j} = b_{\mathbf{n}-e_j,j}$, we compare with the recurrence relation

$$xP_{\mathbf{n}-e_j}(x) = P_{\mathbf{n}}(x) + b_{\mathbf{n}-e_j,j}P_{\mathbf{n}-e_j}(x) + \sum_{i=1}^{r} a_{\mathbf{n}-e_j,i}P_{\mathbf{n}-e_j-e_i}(x).$$

From here we can write

$$\begin{aligned}\sum_{k=1}^{r}\left\langle xA^{(k)}_{\mathbf{n}}(x), P_{\mathbf{n}-e_j}(x)\right\rangle = {}& \sum_{k=1}^{r}\left\langle A^{(k)}_{\mathbf{n}}(x), P_{\mathbf{n}}(x)\right\rangle \\ & + b_{\mathbf{n}-e_j,j}\sum_{k=1}^{r}\left\langle A^{(k)}_{\mathbf{n}}(x), P_{\mathbf{n}-e_j}(x)\right\rangle \\ & \sum_{i=1}^{r} a_{\mathbf{n}-e_j,i}\sum_{k=1}^{r}\left\langle A^{(k)}_{\mathbf{n}}(x), P_{\mathbf{n}-e_j-e_i}(x)\right\rangle.\end{aligned}$$

The first term vanishes by the orthogonality relations of $P_{\mathbf{n}}$, and the last term vanishes by the orthogonality relations of $\mathbf{A}_{\mathbf{n}}$. The second factor in the middle term is 1, so we have

$$\sum_{k=1}^{r}\left\langle xA^{(k)}_{\mathbf{n}}(x), P_{\mathbf{n}-e_j}(x)\right\rangle_k = b_{\mathbf{n}-e_j,j}.$$

On the other hand, if we instead apply a similar argument using the type I recurrence relation we end up with

$$\sum_{k=1}^{r}\left\langle xA^{(k)}_{\mathbf{n}}(x), P_{\mathbf{n}-e_j}(x)\right\rangle_k = d_{\mathbf{n},j},$$

so $d_{\mathbf{n},j} = b_{\mathbf{n}-e_j,j}$.

Write $\kappa_{\boldsymbol{n},j}$ for the degree $n_j - 1$ coefficient of $A_{\boldsymbol{n}}^{(j)}$, so that $A_{\boldsymbol{n}}^{(j)}(x) = \kappa_{\boldsymbol{n},j} x^{n_j-1} + o(x^{n_j-1})$. By comparing the degree $n_i$ coefficients in (38) we see that $\kappa_{\boldsymbol{n},i} = c_{\boldsymbol{n},i}\kappa_{\boldsymbol{n}+\boldsymbol{e}_i,i}$. By Lemma 2.17 we know that $\kappa_{\boldsymbol{n}+\boldsymbol{e}_i,i} \neq 0$, so we get

$$c_{\boldsymbol{n},i} = \frac{\kappa_{\boldsymbol{n},i}}{\kappa_{\boldsymbol{n}+\boldsymbol{e}_i,i}},$$

which shows independence of $j$.

What remains is to show that $c_{\boldsymbol{n},i} = a_{\boldsymbol{n},i}$. Note that

$$\sum_{k=1}^{r} \left\langle A_{\boldsymbol{n}}^{(k)}(x), P_{\boldsymbol{n}-\boldsymbol{e}_i}(x) \right\rangle_k = 1,$$

by the orthogonality relations of $\boldsymbol{A}_{\boldsymbol{n}}$, assuming $\boldsymbol{n} - \boldsymbol{e}_i$ is normal. By the orthogonality relations of $P_{\boldsymbol{n}-\boldsymbol{e}_i}$ we instead get

$$\begin{aligned}\sum_{k=1}^{r} \left\langle A_{\boldsymbol{n}}^{(k)}(x), P_{\boldsymbol{n}-\boldsymbol{e}_i}(x) \right\rangle_k &= \left\langle A_{\boldsymbol{n}}^{(i)}(x), P_{\boldsymbol{n}-\boldsymbol{e}_i}(x) \right\rangle_i \\ &= \kappa_{\boldsymbol{n},i}\left\langle x^{n_i-1}, P_{\boldsymbol{n}-\boldsymbol{e}_i}(x) \right\rangle_i.\end{aligned}$$

Similarly, if $\boldsymbol{n} + \boldsymbol{e}_i$ is normal then

$$\kappa_{\boldsymbol{n}+\boldsymbol{e}_i,i}\left\langle x^{n_i}, P_{\boldsymbol{n}}(x) \right\rangle = 1,$$

so we get

$$a_{\boldsymbol{n},i} = \frac{\left\langle x^{n_i}, P_{\boldsymbol{n}}(x) \right\rangle}{\left\langle x^{n_i-1}, P_{\boldsymbol{n}-\boldsymbol{e}_i}(x) \right\rangle} = \frac{\kappa_{\boldsymbol{n},i}}{\kappa_{\boldsymbol{n}+\boldsymbol{e}_i,i}} = c_{\boldsymbol{n},i},$$

which completes the proof. □

## Data availability

No data was used for the research described in the article.

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