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Higher Auslander-Reiten Theory for Species with Relations

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Abstract
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One of the main topics in higher dimensional Auslander-Reiten theory is the study of so-called d-representation finite algebras. These are algebras of global dimension d that admit a d-cluster tilting module. In particular, 1-representation finite algebras are representation finite and hereditary. Over an algebraically closed field 1-representation finite algebras are therefore classified as path algebras of Dynkin quivers by Gabriel's Theorem. Over a general field (not necessarily algebraically closed) there is a similar classification result of 1-representation finite algebras due to Dlab and Ringel, in which Dynkin quivers are replaced by species of Dynkin type. In this thesis we study d-representation finite algebras over general fields and (in the spirit of the Dlab-Ringel Theorem) produce d-representation finite algebras given by species with relations.

The thesis consists of three papers. The first paper describes preprojective algebras of the representation finite species that appear in the classification of Dlab and Ringel. It is shown that they are almost Koszul. Moreover, taking tensor products of l-homogeneous representation finite species one obtains 2-representation finite algebras. Their 3-preprojective algebras are described using Segre products. The second paper uses species with potentials, which generalise quivers with potentials to describe the 3-preprojective algebras from the previous paper. Buan, Iyama, Reiten and Smith showed that there is a strong connection with mutation of quivers with potential and mutation in 2-Calabi-Yau categories. We generalise some of their results to species with potentials. This leads to new examples of self-injective species with potentials, which fit in the derived Auslander-Iyama correspondence due to Jasso and Muro. The third paper considers 2-APR tilting of 2-representation finite algebras whose 3-preprojective algebras are given by species with potentials. It is shown that under certain conditions 2-APR tilting is given by cut mutation (generalising a result by Herschend and Iyama for quivers with potentials). Moreover, a sufficient condition is given for transitivity of cut mutation showing that in these cases all 2-representation finite algebras that appear for the same species with potentials are derived equivalent.

Keywords: representation theory, higher representation finite algebras, higher Auslander-Reiten theory, higher preprojective algebra, tensor products, Nakayama automorphism, almost Koszul algebras, mutation of cluster tilting objects, species with relations, species with potentials, mutation of species with potentials, derived Auslander-Iyama correspondence, Jacobian algebras, truncated Jacobian algebras, APR tilting, cut-mutation

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Christoffer Söderberg. Preprojective Algebras of d-Representation Finite Species with Relations. *Journal of Pure and Applied Algebra*, 228(4):Paper No. 107520, 54, 2024.

II Christoffer Söderberg. Mutating Species with Potentials and Cluster Tilting Objects. 2025. arXiv:2509.24707

III Christoffer Söderberg. Jacobian Algebras of Species with Potentials and 2-Representation Finite Algebras. 2025. arXiv:2510.04865

Contents

1. Introduction

The main subject of this thesis is representation theory. Representation theory dates back to the end of 19th century when Frobenius [Fro96] introduced characters for finite groups and showed how one can use representation theory to solve various problems. Since then, representation theory has evolved beyond groups to other structures, where one of them being associative algebras. The main principle of representation theory is that we want to represent elements of abstract objects with linear transformations of vector spaces. Thus using the well understood theory of linear algebra we can extract information of the given object we want to study.

A quiver is a set of vertices together with a set of arrows between said vertices. Gabriel initiated the study of representations of quivers. He proved in [Gab73] the crucial result that a quiver is representation finite, i.e. there are finitely many non-isomorphic indecomposable representations, if and only if the diagram is a Dynkin diagram of type ADE. Dynkin diagrams were introduced by Dynkin in [Dyn50] in order to classify all semi-simple Lie algebras over algebraically closed fields. Gabriel also introduced the notion of species, which is the main protagonist of this thesis. Species is a generalisation of the notion of a quiver. Dlab and Ringel [DR75] proved that a species is representation finite if and only if its diagram is a Dynkin diagram of type ABCDEFG. This generalises Gabriels result to include all Dynkin diagrams.

To study representation finite algebras we use Auslander-Reiten theory. It uses concepts such as almost split sequences, Auslander-Reiten quivers and the Auslander-Reiten translation. This theory was introduced by Auslander and Reiten in [Aus74] and [AR74]. It serves as a vital tool when studying Artin algebras, see for example [ASS06] and [ARS95]. There has been extensive work done to understand representation finite quivers and representation finite species. By the result of Dlab and Ringel [DR75] we have a clear description of all almost split sequences and the Auslander-Reiten quiver in these cases.

In [Iya07b] Iyama introduced the higher dimensional Auslander-Reiten theory. He introduced a special type of category called the d-cluster tilting subcategory that serves as a natural domain for higher dimensional Auslander-Reiten theory. Iyama introduced d-almost split sequences and d-Auslander-Reiten translations that work on said categories. If the d-cluster tilting subcategory is generated by a single element then we say that Λ is d-representation finite. Even though the higher Auslander-Reiten theory is very natural in how it is defined, constructing examples is a challenge as the setting is highly restrictive.

There are many examples due to [Iya07a], [IO11], [HI11b]. Concretely these have lead to algebras given by quivers with relations. In the view of Gabriel's theorem compared to its generalisation by Dlab and Ringel, there should also be many interesting examples based on species with relations.

There are constructions using tensor product to generate d-representation finite algebras whenever the field is perfect [HI11a]. But even in these examples, they are not well understood. In this thesis we try to give insight of how the algebras constructed using tensor products of tensor algebras of representation finite species behave. The resulting algebra is an example of a 2-representation finite algebra. We provide various properties that are inherited from the given species. Furthermore, we provide the necessary tools to generate even more examples of 2-representation finite algebras using mutation.

The core of the thesis consists of three research papers devoted to the study of species and related structures. The summary is structured as follows. We begin by first presenting concepts without in depth details that are used throughout the thesis. Then we give a short summary of all articles to give a more overview how the thesis. Then we proceed it with a very brief summary in Swedish. Lastly, I devote the last section to express my gratitude towards everyone that has supported me throughout this journey.

2. Preliminaries

In this section we aim to provide the basic concepts that are present in all papers in this thesis. We include references, with more details for the interested reader.

2.1 Notation

We assume that $\mathbb{K}$ is a perfect field throughout this summary. An R-module is assumed to be a left R-module unless otherwise stated. Compositions of arrows are done from right to left.

Let Λ be a finite dimensional $\mathbb{K}$-algebra. We denote by modΛ the category of finitely generated Λ-modules. For a Λ-module $M \in \text{mod}\Lambda$ we define add(M) to be the full subcategory of modΛ closed under direct sums and direct summands. We say that a module M generates a subcategory $\mathscr{C}$ if add$(M) = \mathscr{C}$.

2.2 Auslander-Reiten theory

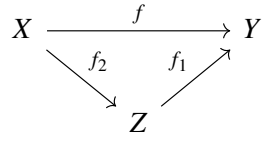
Auslander-Reiten theory provides a way to study properties the module category modΛ of a finite dimensional $\mathbb{K}$-algebra Λ. One can ask whether a Λ-module M can be decomposed into $M = X \oplus Y$ where X and Y are Λ-modules. If a module cannot be expressed as a direct sum of two Λ-modules we call it indecomposable. Indecomposable modules are the building blocks of modΛ. A natural question arises, what is the relation between all the indecomposable Λ-modules? The relations between Λ-modules can be described by morphisms between said Λ-modules. This is where Auslander-Reiten theory shows up. Auslander-Reiten theory was developed by Auslander and Reiten in the late 20th century. We refer the reader to [ARS95] for details.

2.2.1 Auslander-Reiten quiver and almost split sequences

For two Λ-modules X and Y Auslander-Reiten theory introduces irreducible morphism. A morphism $f : X \to Y$ is irreducible if

1. f is neither a section nor a retraction,

2. if $f = f_1 \circ f_2$, either f_1 is a retraction or f_2 is a section.



In other words, irreducible morphisms are morphisms that cannot be factored non-trivially. Irreducible morphisms give insight into how modules decompose and interact.

The Auslander-Reiten quiver of a algebra Λ is defined as the quiver given by the following.

1. The vertices are the isomorphism classes $[X]$ of indecomposable modules X.
2. There is an arrow $[X] \to [Y]$ whenever there is an irreducible map $X \to Y$.

For $X, Y \in \text{mod}\Lambda$ let $\text{rad}_\Lambda(X,Y)$ to be the set of radical morphisms from X to Y. We also let $\text{End}_\Lambda(X) = \text{Hom}_\Lambda(X,X)$. The Auslander-Reiten quiver describes $\text{mod}\Lambda$ to a certain extent. One of the main techniques of Auslander-Reiten theory is using almost split sequences. They were introduced in [Aa75] by Auslander and Reiten. To define almost split sequences we need the following definition.

Definition 2.2.1.

1. A morphism $f_0 \in \text{rad}_\Lambda(X, C_1)$ is called left almost split if $C_1 \in \text{mod}\Lambda$ and

$$\text{Hom}_\Lambda(C_1, Y) \xrightarrow{f_0} \text{rad}_\Lambda(X,Y) \to 0$$

   is exact for all $Y \in \text{mod}\Lambda$.
2. A morphism $f_0 : X \to C_1$ is called left minimal if every $h \in \text{End}_\Lambda(C_1)$ such that $h \circ f_0 = f_0$ is an isomorphism.
3. A left minimal and left almost split morphism is called a source morphism. A sink morphism is defined dually.

Now we can state the definition of an almost split sequence.

Definition 2.2.2. Let $X, C_1, Y \in \text{mod}\Lambda$. A sequence

$$0 \to X \xrightarrow{f_0} C_1 \xrightarrow{f_1} Y \to 0$$

is said to be an almost split sequence if f_0 is a source morphism and f_1 a sink morphism.

2.2.2 Auslander-Reiten translation

The endpoints of almost split sequences are related by the Auslander-Reiten translation. We first consider the functor

$$\mathrm{Hom}_\Lambda(-,\Lambda) : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda^{\mathrm{op}}.$$

Starting with a Λ-module M, we can write down its minimal projective presentation

$$P_1 \xrightarrow{p_1} P_0 \to M \to 0.$$

By applying $\mathrm{Hom}_\Lambda(-,\Lambda)$ we obtain the sequence

$$\mathrm{Hom}_\Lambda(P_0,\Lambda) \to \mathrm{Hom}_\Lambda(P_1,\Lambda) \to \mathrm{coker}(\mathrm{Hom}_\Lambda(p_1,\Lambda)) \to 0.$$

The Auslander-Bridger transpose is defined as $\mathrm{Tr}(M) = \mathrm{coker}(\mathrm{Hom}_\Lambda(p_1,\Lambda))$. The Auslander-Reiten translation functors are now defined by composing the $\mathbb{K}$-dual $D = \mathrm{Hom}_\mathbb{K}(-,\mathbb{K})$ with Tr

$$\tau = D \circ \mathrm{Tr}, \qquad \tau^- = \mathrm{Tr} \circ D.$$

Note that $\tau(P) = 0$ and $\tau^-(I) = 0$, where P and I are projective and injective modules respectively. By restricting ourselves to the stable and the costable Λ-module categories $\underline{\mathrm{mod}}\Lambda$ and $\overline{\mathrm{mod}}\Lambda$ we get the following.

Proposition 2.2.3. *The Auslander-Reiten translations τ and τ^- induce mutually inverse equivalences*

$$\underline{\mathrm{mod}}\Lambda \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \overline{\mathrm{mod}}\Lambda.$$

Using the Auslander-Reiten translation we can construct almost split sequences. For every non-projective indecomposable Λ-module M there is an almost split sequence

$$0 \to \tau(M) \to N \to M \to 0.$$

The existence of such almost split sequences can be proven using the following theorem.

Theorem 2.2.4. *(the Auslander-Reiten formulas) Let M and N be two Λ-modules. There exist isomorphisms*

$$\mathrm{Ext}^1_\Lambda(M,N) \cong D\underline{\mathrm{Hom}}_\Lambda(\tau^-(N),M) \cong D\overline{\mathrm{Hom}}_\Lambda(N,\tau(M))$$

that are functorial in both variables.

Corollary 2.2.5.

1. *For any indecomposable non-projective Λ-module M, there exists an almost split sequence* $0 \to \tau(M) \to E \to M \to 0$ *in* $\mathrm{mod}\Lambda$.
2. *For any indecomposable non-injective Λ-module N, there exists an almost split sequence* $0 \to N \to F \to \tau^-(N) \to 0$ *in* $\mathrm{mod}\Lambda$.

Definition 2.2.6. An algebra is hereditary if $\mathrm{gldim}(\Lambda) \leq 1$.

If Λ is a hereditary algebra, then we define the preprojective component of the Auslander-Reiten quiver as the full subquiver consisting of modules in

$$\mathrm{add}\{\tau^{-i}(\Lambda) \mid i \geq 0\}.$$

In this case, indecomposable modules in the preprojective component are called preprojective modules.

2.2.3 Representation finite hereditary algebras and preprojective algebras

Definition 2.2.7. A finite dimensional $\mathbb{K}$-algebra is said to be representation finite if it has finitely many finitely generated non-isomorphic indecomposable modules.

If a finite dimensional $\mathbb{K}$-algebra is representation finite, then its Auslander-Reiten quiver only consist of one component. We refer [ARS95] for more details. If we also assume that the algebra is hereditary then the preprojective component coincides with the whole Auslander-Reiten quiver. By the discussion above, in this case every indecomposable module is preprojective and thus

$$\mathrm{mod}\Lambda = \mathrm{add}\{\tau^{-i}(\Lambda) \mid i \geq 0\}.$$

One way to describe the preprojective component is to use the preprojective algebra.

Definition 2.2.8. Let Λ be a hereditary (finite dimensional) $\mathbb{K}$-algebra. The preprojective algebra $\Pi(\Lambda)$ is defined as

$$\Pi(\Lambda) = \bigoplus_{i \geq 0} \mathrm{Hom}_\Lambda(\Lambda, \tau^{-i}(\Lambda)).$$

Understanding the preprojective algebra yields a lot of useful information about the algebra, in particular when the algebra is representation finite. For a representation finite path algebra it was proven in [BBK02], although it was folklore before, that its preprojective algebra is finite dimensional and self-injective, thus Frobenius. Later, Iyama and Oppermann proved that if an algebra Λ is representation finite then its preprojective algebra $\Pi(\Lambda)$ is a finite

dimensional self-injective algebra [IO13], and thus enjoys a Frobenius structure. Hence there exists a unique automorphism $\gamma : \Pi(\Lambda) \to \Pi(\Lambda)$ such that $\Pi(\Lambda) \cong D\Pi(\Lambda)_\gamma$ called the Nakayama automorphism. The Nakayama automorphism is vital when trying to understand $\Pi(\Lambda)$ since it describes the Frobenius structure of $\Pi(\Lambda)$.

2.3 Higher dimensional Auslander-Reiten theory

A higher analogue of the Auslander-Reiten theory which is called the higher dimensional Auslander-Reiten theory was introduced by Iyama in [Iya07b]. Iyama fixed an integer $d \geq 1$ and defined a subcategory which enjoys similarly properties as the module category of an algebra Λ in the usual Auslander-Reiten theory. Such a subcategory is called a d-cluster tilting subcategory. If the d-cluster tilting subcategory is generated by a single module, then it has finitely many non-isomorphic indecomposables. Thus giving us a natural way to define the concept of d-representation finite. Setting $d = 1$ we recover the classical Auslander-Reiten theory. For an introduction for higher dimensional Auslander-Reiten theory we refer to [JK19].

The higher dimensional Auslander-Reiten theory is more restrictive than the classical Auslander-Reiten theory. Thus making it harder to construct examples. Some known examples are due to [Iya07a], [IO11], [HI11b].

2.3.1 d-cluster tilting subcategories and d-almost split sequences

The higher analogues of representation finite hereditary algebras care called d-representation finite algebras. They are defined using the notion of d-cluster tilting subcategories and d-cluster tilting modules.

Definition 2.3.1.

1. A functorially finite subcategory $\mathscr{C} \subseteq \mathrm{mod}\Lambda$ is said to be a d-cluster tilting subcategory if

$$\begin{aligned}\mathscr{C} &= \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^i_\Lambda(X,M) = 0, \text{ for all } M \in \mathscr{C}, i = 1,\dots,d-1\} = \\ &= \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^i_\Lambda(M,X) = 0, \text{ for all } M \in \mathscr{C}, i = 1,\dots,d-1\}.\end{aligned}$$

2. We say that a Λ-module M is called a d-cluster tilting Λ-module if $\mathrm{add}(M)$ is a d-cluster tilting subcategory in $\mathrm{mod}\Lambda$.
3. An algebra Λ is called d-representation finite if $\mathrm{gldim}(\Lambda) \leq d$ and there exists a d-cluster tilting module.

Remark 2.3.2.

1. The subcategory $\mathrm{add}(M)$ is always functorially finite. Moreover, note that $\Lambda, D\Lambda \in \mathrm{add}(M)$.

2. The word finite in d-representation finite refers to the fact that add(M) has finitely many indecomposable objects up to isomorphism.

Remark 2.3.3. Note that for the case when $d = 1$ the d-cluster tilting subcategory becomes modΛ. So 1-representation finite coincides with the usual definition of being representation finite and hereditary.

In complete analogy of the classical Auslander-Reiten theory we recall the following definitions and results. We start with d-almost split sequences. For this we need to introduce source and sink sequences.

Definition 2.3.4. Let Λ be an algebra. We call a complex

$$X \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} C_d$$

a source sequence of X if the following conditions are satisfied.

1. $C_i \in \text{mod}\Lambda$ and $f_i \in \text{rad}_\Lambda(C_i, C_{i+1})$ for any $1 \leq i \leq d-1$.
2. we have the following exact sequence for all $Y \in \text{mod}\Lambda$.

$$\begin{aligned}\text{Hom}_\Lambda(C_d, Y) \xrightarrow{f_{d-1}} \cdots \xrightarrow{f_2} \text{Hom}_\Lambda(C_2, Y) \xrightarrow{f_1} \\ \xrightarrow{f_1} \text{Hom}_\Lambda(C_1, Y) \xrightarrow{f_0} \text{rad}_\Lambda(X, Y) \to 0\end{aligned}$$

A sink sequence is defined dually.

With this we can define d-almost split sequences.

Definition 2.3.5. We call an exact sequence

$$0 \to X \to C_1 \to C_2 \to \cdots C_d \to Y \to 0$$

is a d-almost split sequence if it is a source sequence of $X \in \text{mod}\Lambda$ an a sink sequence of $Y \in \text{mod}\Lambda$.

To define the higher Auslander-Reiten translation, or the d-Auslander-Reiten translation, we compose the regular Auslander-Reiten translation with the $(d-1)$th syzygy.

Definition 2.3.6. The d-Auslander-Reiten translation is defined as

$$\begin{aligned}\tau_d &= D\text{Tr}\Omega^{d-1} : \underline{\text{mod}}\Lambda \to \overline{\text{mod}}\Lambda, \\ \tau_d^- &= \text{Tr}D\Omega^{-(d-1)} : \overline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda.\end{aligned}$$

Similarly to the $d = 1$ case we have the following result.

Proposition 2.3.7. *Let $\mathscr{C}$ be a d-cluster tilting subcategory of* modΛ*. Then*

$$\tau_d : \underline{\mathscr{C}} \to \overline{\mathscr{C}}, \qquad \tau_d^- : \overline{\mathscr{C}} \to \underline{\mathscr{C}}$$

are mutually inverse equivalences.

The relation between the d-Auslander-Reiten translation and the d-almost split sequences are similar to as in the $d = 1$ case.

Proposition 2.3.8. *If Λ is a d-representation finite algebra with d-cluster tilting subcategory $\mathscr{C}$, then for every non-projective indecomposable Λ-module $X \in \mathscr{C}$ there is a d-almost split sequence*

$$0 \to \tau_d(X) \to C_1 \to C_2 \to \cdots \to C_d \to X \to 0,$$

where $C_i \in \mathscr{C}$ for $1 \leq i \leq d$. Moreover, in this case the d-cluster tilting subcategory is given as

$$\mathscr{C} = \{\tau_d^{-i}(\Lambda) \mid i \geq 0\}$$

and is unique.

Therefore we can view $\mathscr{C}$ as the "d-preprojective component", i.e. that for all indecomposable modules $X \in \mathscr{C}$ there exists an integer $n \geq 0$ such that $\tau_d^n(X)$ is projective, in particular when X is an injective indecomposable Λ-module. Note that this coincides with the usual preprojective component whenever $d = 1$.

2.3.2 Higher preprojective algebras and d-representation finite algebras

The preprojective algebra is define by considering all morphism in the preprojective component from the projective modules. Since we are considering $\mathscr{C}$ instead of the whole module category, it is natural to define the higher analogue of the preprojective algebra as all morphisms from the preprojective modules to objects in $\mathscr{C}$.

Iyama and Oppermann in [IO13] introduced the $(d+1)$-preprojective algebra of Λ in the following way.

Definition 2.3.9. The $(d+1)$-preprojective algebra $\Pi_{d+1}(\Lambda)$ is defined as

$$\Pi_{d+1}(\Lambda) = T_\Lambda(E),$$

where $E = \mathrm{Ext}^d_\Lambda(D\Lambda, \Lambda)$.

Similarly as in the $d = 1$ case, we can identify the $(d+1)$-preprojective algebra of Λ with

$$\Pi_{d+1}(\Lambda) = \bigoplus_{i \geq 0} \mathrm{Hom}_\Lambda(\Lambda, \tau_d^{-i}(\Lambda)),$$

(see [GI20]). Moreover, we have the following useful result.

Theorem 2.3.10. *If Λ is a d-representation finite algebra, then $\Pi_{d+1}(\Lambda)$ is a finite dimensional self-injective algebra.*

In this case, $\Pi_{d+1}(\Lambda)$ is a Frobenius algebra and therefore there exists a Nakayama automorphism. As in the classical case, describing the Nakayama automorphism yields crucial information about the $(d+1)$-preprojective algebra.

2.3.3 Tensor products of d-representation finite algebras

One way to construct examples of d-representation finite algebra is to use tensor products. This is based upon the work of Herschend and Iyama in [HI11a].

Before we present the crucial result we need the following. Let Λ be a basic d-representation finite $\mathbb{K}$-algebra. Fix a set of complete set of primitive orthogonal idempotents $\{e_i\}_{i\in I}$ for some index set I. Since Λ is d-representation finite, the d-cluster tilting subcategory $\mathscr{C}$ consists of finitely many indecomposable modules. Hence we can deduce that

$$\mathscr{C} = \operatorname{add}\{\tau_d^{-j}(\Lambda e_i) \mid i \in I, j \in \mathbb{Z}\} = \operatorname{add}\{\tau_d^{j}(D(e_i\Lambda)) \mid i \in I, j \in \mathbb{Z}\}.$$

Therefore there exists integers l_i such that $\tau_d^{-l_i}(D(e_i\Lambda))$ is projective. We define the permutation $\sigma : I \to I$ such that

$$\Lambda e_{\sigma(i)} = \tau_d^{l_i-1} D(e_i\Lambda)$$

for all $i \in I$. The permutation σ is called the Nakayama permutation for Λ, which coincides with the Nakayama permutation of $\Pi_{d+1}(\Lambda)$ induced by the Nakayama automorphism.

Definition 2.3.11. [HI11a, Definition 1.2] The algebra Λ is said to be l-homogeneous if $l_i = l$ for all $i \in I$.

Herschend and Iyama showed the following useful result.

Theorem 2.3.12. *[HI11a, Corollary 1.5] Let $\mathbb{K}$ be a perfect field and l a positive integer. If Λ_i is l-homogeneous d_i-representation finite for each $i \in \{1,2\}$, then $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$ is an l-homogeneous (d_1+d_2)-representation-finite algebra with a (d_1+d_2)-cluster tilting module $\bigoplus_{i=0}^{l-1}(\tau_{d_1}^{-i}\Lambda_1 \otimes_{\mathbb{K}} \tau_{d_2}^{-i}\Lambda_2)$.*

Since there are not any known classification of d-representation finite $\mathbb{K}$-algebras, Theorem 2.3.12 gives us a tool to construct many new examples of

d-representation finite $\mathbb{K}$-algebras. Also note that the resulting algebra is l-homogeneous, meaning that we can iterate the process to get more examples. In this thesis we explore the properties of $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$ whenever Λ_i is a tensor algebra of some representation finite species for $i \in \{1,2\}$.

2.4 Quivers and Species

In this section we will recall the definition of the main protagonist of this thesis, species. We refer the reader to [Gab73] and [Ber11] for more details. Species play an important role in representation theory due to the fact that many algebras can be described using a species and some relations. We will begin by first recalling the definition of path algebras of quivers and then move on to defining tensor algebras of species.

2.4.1 Quivers

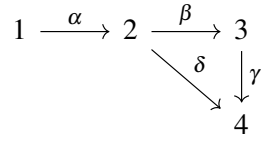
An important type of a $\mathbb{K}$-algebra is the path algebra over a quiver. A quiver is a tuple (Q_0, Q_1, s, t), where Q_0 is the set of vertices, Q_1 is the set of arrows and

$$s,t : Q_1 \to Q_0$$

are maps such that $s(\alpha)$ and $t(\alpha)$ are the source and target of an arrow $\alpha \in Q_1$. In other words, for an arrow $\alpha \in Q_1$ we can write

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Usually we only write $Q = (Q_0, Q_1)$ as a pair and omit writing out s and t. We illustrate quivers using diagrams such as



for example. The path algebra over a quiver Q is denoted by $\mathbb{K}Q$ and is defined as the $\mathbb{K}$-linear span of all paths in the quiver Q, where the multiplication is given by following. If p and q are two paths in $\mathbb{K}Q$, then $p \cdot q$ is defined as the concatenation if possible and otherwise $p \cdot q = 0$.

An ideal of a path algebra is called admissible if it is generated by paths of length at least 2 and all paths of length m of some integer m lies inside the ideal. The following theorem is a well known fact and motivates the study of path algebras.

Theorem 2.4.1. *[ASS06, Theorem 3.7] Let $\mathbb{K}$ be an algebraically closed field. Every finite dimensional $\mathbb{K}$-algebra is Morita equivalent to $\mathbb{K}Q/I$, where $\mathbb{K}Q/I$ is a path algebra over a quiver Q quotient by an admissible ideal I.*

The ideal I in Theorem 2.4.1 is zero if and only if $\mathbb{K}Q/I$ is hereditary, so studying hereditary algebras over algebraically closed fields is reduced to studying $\mathbb{K}Q$. By applying Gabriel's theorem we arrive at the following corollary.

Corollary 2.4.2. *Let $\mathbb{K}$ be an algebraically closed field. Every finite dimensional hereditary $\mathbb{K}$-algebra is Morita equivalent to $\mathbb{K}Q$, where Q is a disjoint union of quivers of Dynkin type ADE.*

Note that Theorem 2.4.1 is no longer true if we do not assume that $\mathbb{K}$ is algebraically closed. We can for example choose $\mathbb{K} = \mathbb{R}$ and study the $\mathbb{R}$-subalgebra

$$\begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix} \tag{4.2.1}$$

of complex 2×2-matrices. Note that for path algebras $\mathbb{K}Q/I$ the endomorphism algebra of any simple $\mathbb{K}Q/I$-module is isomorphic to $\mathbb{K}$. This is not true for the algebra (4.2.1). Take for example the simple modules

$$S_1 = \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} \mathbb{C} \\ \mathbb{R} \end{bmatrix} \Big/ \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}.$$

Then $\mathrm{End}_\Lambda(S_1) \cong \mathbb{C}$ and $\mathrm{End}_\Lambda(S_2) \cong \mathbb{R}$. Therefore it cannot be Morita equivalent to some $\mathbb{K}Q/I$. It can still be described as a path algebra over some ideal, but the ideal would no longer be admissible. However, it can be described as a tensor algebra of a species.

2.4.2 Species

Species are used to generalise the concept of path algebras over some quiver. As said earlier, species are the main protagonists of this thesis and play an important role of classifying representation finite algebras. Species were introduced by Gabriel in [Gab73]. For details about result in this section we refer the reader to [Ber11].

We define species in the following way.

Definition 2.4.3. Let Q be a quiver.

1. A species $S = (D_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a collection where each D_i is a semisimple $\mathbb{K}$-algebra and $M_\alpha \in D_j$-D_i-mod, where $\alpha : i \to j$, such that

   $$\mathrm{Hom}_{D_i^{\mathrm{op}}}(M_\alpha, D_i) \cong \mathrm{Hom}_{D_j}(M_\alpha, D_j)$$

   as D_i-D_j-modules, such that $\mathbb{K}$ is central in D_i and M_α for all $i \in Q_0$ and $\alpha \in Q_1$.
2. A finite dimensional species S is a species S such that $\dim_{\mathbb{K}} D_i < \infty$ and $\dim_{\mathbb{K}} M_\alpha < \infty$ for all $i \in Q_0$ and $\alpha \in Q_1$.

3. We say that a finite dimensional species S is a $\mathbb{K}$-species if all D_i are division $\mathbb{K}$-algebras over a common central subfield $\mathbb{K}$.
4. We call a species S acyclic if Q is acyclic.

For a species S we define the tensor algebra $T(S)$ to be the tensor algebra $T(D,M)$. More explicitly,

$$T(S) = T(D,M) = \bigoplus_{k\geq 0} M^k = \bigoplus_{k\geq 0} M^{\otimes_D k}, \quad M^0 = D.$$

A species over a quiver Q can be presented using a diagram of shape Q where the vertices are given by D_i and the arrows are decorated with M_α. For example, the algebra in (4.2.1) can be described as the tensor algebra of the species

$$\mathbb{C} \xrightarrow{\mathbb{C}} \mathbb{R}.$$

For any quiver Q the species S over Q defined by setting all $D_i = M_\alpha = \mathbb{K}$ for all $i \in Q_0$ and $\alpha \in Q_1$ we recover the path algebra $T(S) = \mathbb{K}Q$.

Our definition of species does not coincide with the definition given in for example [Gab73] and [Ber11]. In those articles assume that D_i is a division $\mathbb{K}$-algebra for all $i \in Q_0$. Usually we generally only care about $\mathbb{K}$-species where our definitions match. We relaxed our definition of species due to technical reasons, being that we are considering tensor products of tensor algebra of species. Taking tensor products of tensor algebras of species cannot necessarily be described in the obvious way using a species with relations. To illustrate this we can consider the $\mathbb{R}$-species S given by $\mathbb{C}$, i.e. the species with $Q = \bullet$. Its tensor algebra is $T(S) = \mathbb{C}$. Taking the tensor product over $\mathbb{R}$ we can show that

$$T(S) \otimes_{\mathbb{R}} T(S) = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}.$$

Glossing over the details and refer the reader to Paper II and Paper III for more details, we expect the new species that describes the tensor product to be a species over the same quiver Q. But since $\mathbb{C} \oplus \mathbb{C}$ is not a division $\mathbb{R}$-algebra it cannot possibly be a species. However, we show in Paper II that $T(S_1) \otimes_{\mathbb{K}} T(S_2)$ is Morita equivalent to a $\mathbb{K}$-species if S_1 and S_2 are $\mathbb{K}$-species. In the above example it is Morita equivalent to the species given by

$$\mathbb{C} \quad \mathbb{C}.$$

In this case the quiver consists of two isolated points.

Now we can state the more general version of Theorem 2.4.1 using species.

Theorem 2.4.4. *[Ber11, Corollary 4.3] Let $\mathbb{K}$ be a perfect field. Every finite dimensional $\mathbb{K}$-algebra is Morita equivalent to $T(S)/I$ for some $\mathbb{K}$-species S and some admissible ideal I.*

Similarly to the quiver case, if $I = 0$ then $T(S)/I$ is hereditary. Thus for studying hereditary algebras over perfect fields it is enough to study $T(S)$. We can replace the assumption that $\mathbb{K}$ is perfect for a more general, but more complicated, assumption. We refer the reader to [Ber11] for the details.

2.5 Representation finiteness of species

Gabriel's theorem classifying representation finite quivers was generalised to $\mathbb{K}$-species by Dlab and Ringel in [DR75].

Theorem 2.5.1. *[DR75] A $\mathbb{K}$-species is representation finite if and only if it is isomorphic to a disjoint union of species of Dynkin type ABCDEFG.*

A species is of Dynkin type if its diagram is given by a Dynkin diagram. Note that here we include all Dynkin diagrams rather than just the Dynkin diagrams of type ADE in Gabriel's theorem. Dlab and Ringel showed that such a Dynkin species is isomorphic to a species where $D_i, M_\alpha \in \{F, G\}$ for all $i \in Q_0$ and $\alpha \in Q_1$ and F and G are two division $\mathbb{K}$-algebras such that $F \subseteq G$. They are represented in Figure 2.1, with their corresponding diagram in Figure 2.2. The decorated edge $i \overset{(k)}{\Longrightarrow} j$, where $k = \dim_{\mathbb{K}} G / \dim_{\mathbb{K}} F$, means that $D_i = G$ and $D_j = F$. Such a decorated edge only exists in the non-simply laced cases, i.e. the Dynkin diagrams of type BCGF. Note that the valuation (k) in their diagram does not correspond to the orientation. The orientations of the $\mathbb{K}$-species in Figure 2.1 are omitted since all possible orientations yields a representation finite $\mathbb{K}$-species.

2.5.1 Preprojective algebras of species

In the later article [DR80] Dlab and Ringel provided a description of the preprojective algebra of a species. Before we present the construction of the preprojective algebra we need to define Casimir elements. Let S be a $\mathbb{K}$-species and let $\alpha \in Q_1$. There exist $x_1, \ldots, x_n \in M_\alpha$ and $f_1, \ldots, f_n \in \operatorname{Hom}_{D_{s(\alpha)}^{\mathrm{op}}}(M_\alpha, D_{s(\alpha)})$ such that for every $x \in M_\alpha$ we have

$$x = \sum_{i=1}^{n} f_i(x) x_i.$$

Definition 2.5.2. Let x_i and f_i be as above. Then, the element

$$c_\alpha = \sum_{i=1}^{n} x_i \otimes_{D_{s(\alpha)}} f_i \in M_\alpha \otimes_{D_{s(\alpha)}} \operatorname{Hom}_{D_{s(\alpha)}^{\mathrm{op}}}(M_\alpha, D_{s(\alpha)})$$

is called the Casimir element of $M_\alpha \otimes_{D_{s(\alpha)}} \operatorname{Hom}_{D_{s(\alpha)}^{\mathrm{op}}}(M_\alpha, D_{s(\alpha)})$.

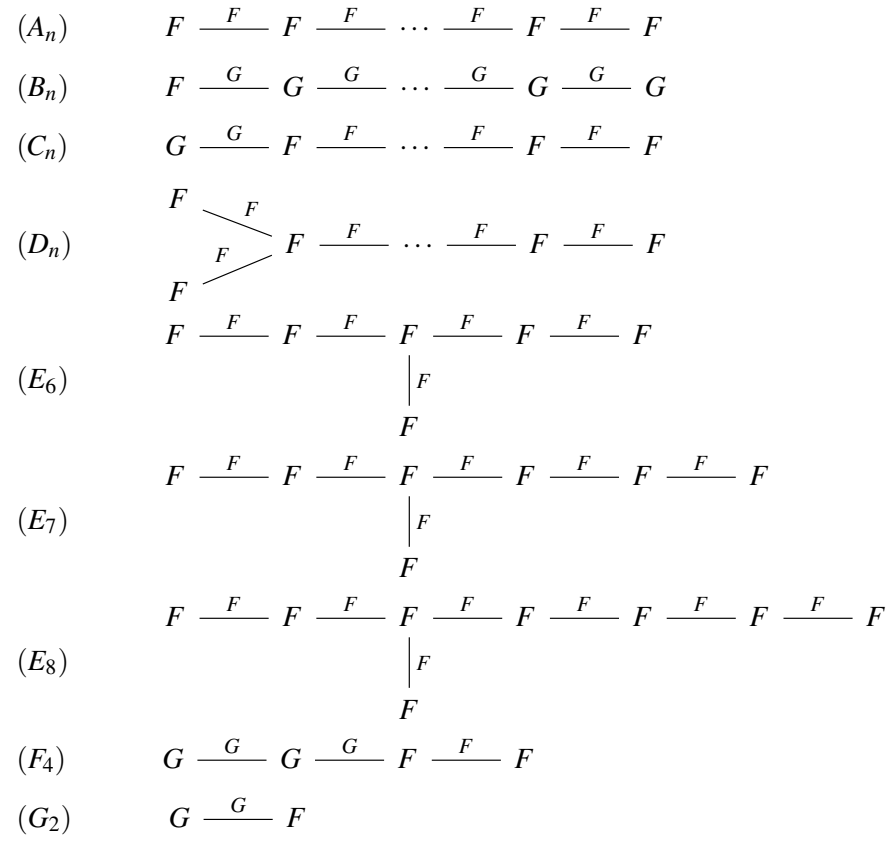


Figure 2.1. Description of species over Dynkin diagrams

The Casimir does not depend on the choice of bases $\{x_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$, therefore Casimir elements are uniquely determined. Using Casimir elements we can define the preprojective algebra.

Definition 2.5.3. [DR80] Let S be a representation finite $\mathbb{K}$-species.

1. The double quiver $\overline{Q}$ is defined to be $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \cup Q_1^*$ where

$$Q_1^* = \{\alpha^* : j \to i \mid \alpha : i \to j \in Q_1\}.$$

2. Let $\overline{S}$ be the species over $\overline{Q}$ where $\overline{D_i} = D_i$ for all $i \in \overline{Q}_0$ and $\overline{M}_\alpha = M_\alpha$ when $\alpha \in Q_1$ and $\overline{M}_{\alpha^*} = \mathrm{Hom}_{D_{s(\alpha)}^{\mathrm{op}}}(M_\alpha, D_{s(\alpha)})$.
3. For each $\alpha \in \overline{Q}_1$ let c_α be the Casimir element of $\overline{M}_\alpha \otimes_{D_{s(\alpha)}} \overline{M}_{\alpha^*}$. Define

$$c = \sum_{\alpha \in \overline{Q}_1} \mathrm{sgn}(\alpha) c_\alpha,$$

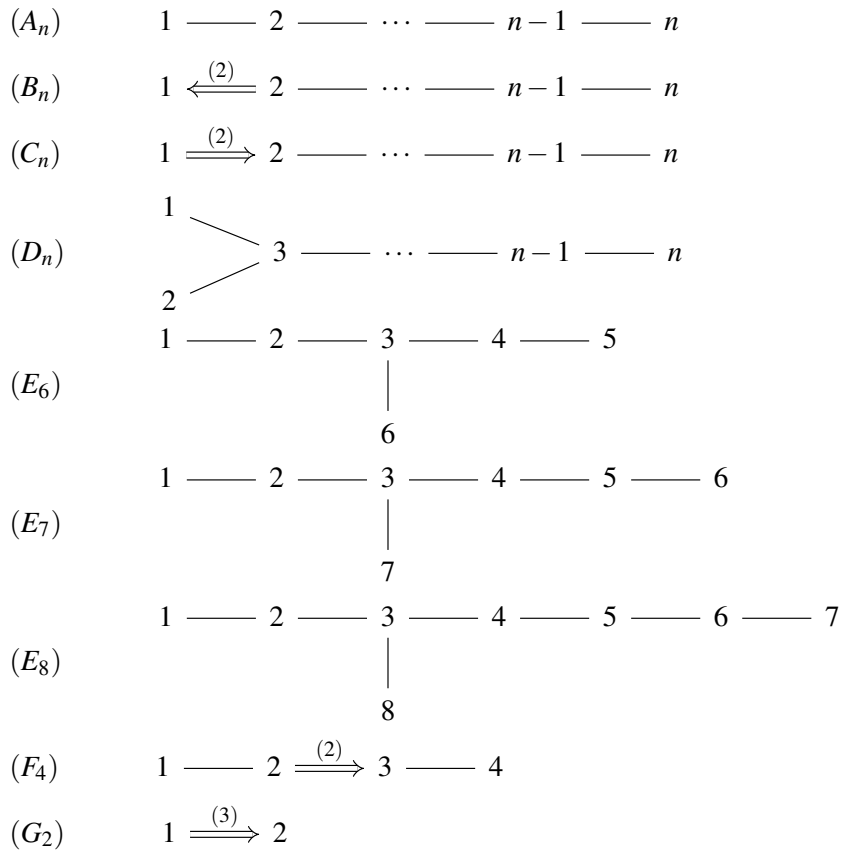


Figure 2.2. Dynkin Diagrams

where

$$\operatorname{sgn}(\alpha) = \begin{cases} 1, & \alpha \in Q_1 \\ -1, & \text{else.} \end{cases}$$

We define the preprojective algebra of S as $\Pi(S) = T(\overline{S})/\langle c \rangle$.

We have the following expected theorem.

Theorem 2.5.4. *There exists an isomorphism $\Pi(S) \cong \Pi(T(S))$, where $\Pi(T(S))$ is the usual definition of the preprojective algebra $\Pi(T(S))$ given in Definition 2.2.8.*

Remark 2.5.5. Theorem 2.5.4 was proven by Ringel in [Rin98] for the case when S is a Dynkin species of type ADE. It is also proven for a generalisation

of species called phylum in [GKKP22]. In Paper I we provide a proof for all Dynkin species.

2.6 Cluster categories and quivers with potential

The notions of d-cluster tilting objects and d-cluster tilting subcategories can be generalised to any triangulated category $\mathscr{T}$. This is of particular interest if $\mathscr{T}$ is d-Calabi–Yau, meaning $\mathscr{T}$ is $\mathbb{K}$-linear with finite dimensional hom spaces such that there is a natural isomorphism

$$\mathscr{T}(X,Y) \cong D\mathscr{T}(Y,X[d]),$$

i.e. the d-fold suspension $[d]$ is a Serre functor. Cluster categories provide a useful source of d-Calabi–Yau categories. In this section we recall various definitions an results related to these.

2.6.1 Cluster categories of algebras with global dimension d

Let $\mathbb{K}$ be a perfect field and assume that Λ is a finite dimensional $\mathbb{K}$-algebra with $\mathrm{gldim}(\Lambda) \leq d$. Following [Ami08, Ami09] we construct the Amiot d-cluster category for Λ. Let $\Gamma = \Lambda \oplus D\Lambda[-d-1]$ be a DG algebra with zero differential. The we have a natural restriction functor

$$\mathscr{D}^b_\Lambda \to \mathscr{D}^b_\Gamma \tag{6.0.1}$$

where $\mathscr{D}^b_\Lambda$ and $\mathscr{D}^b_\Gamma$ are the bounded derived categories of Λ and Γ respectively. The Amiot d-cluster category $\mathscr{C}^d_\Lambda$ is defined as

$$\mathscr{C}^d_\Lambda := \mathrm{thick}_{\mathscr{D}^b_\Gamma}(\Lambda)/\mathrm{perf}(\Gamma),$$

where $\mathrm{thick}_{\mathscr{D}^b_\Gamma}(\Lambda)$ is the smallest thick subcategory of $\mathscr{D}^b_\Gamma$ containing the image of Λ through (6.0.1) and $\mathrm{perf}(\Gamma)$ is the category containing all of the perfect complexes over Γ.

Lemma 2.6.1. *[Ami08, Ami09] The restriction functor induces a functor $\pi : \mathscr{D}^b_\Lambda \to \mathscr{C}_\Lambda$ which commutes with the Serre functors of the two categories.*

Theorem 2.6.2. *[Ami08, Ami09] If Λ is τ_d-finite, i.e. $\tau_d^{-i}\Lambda = 0$ for sufficiently large i, then:*

1. *The Amiot d-cluster category $\mathscr{C}^d_\Lambda$ is a d-Calabi–Yau triangulated category.*
2. *$\pi\Lambda$ is a d-cluster tilting object in the Amiot d-cluster category $\mathscr{C}^d_\Lambda$. Moreover, the endomorphism algebra $\mathrm{End}_{\mathscr{C}^d_\Lambda}(\pi\Lambda)^{op}$ is isomorphic to $\Pi_{d+1}(\Lambda)$.*

In [JKM23] Jasso and Muro proved a derived version of the Auslander-Iyama correspondence.

Theorem 2.6.3. *[JKM23, Theorem A] Suppose that $\mathbb{K}$ is a perfect field and $d \geq 1$ an integer. There are bijective correspondences between the following:*

1. *Equivalence classes of pairs $(\mathscr{C}, T)$ consisting of,*
   a) *an algebraic triangulated category $\mathscr{C}$ with finite-dimensional morphism spaces and split idempotents and*
   b) *a basic $d\mathbb{Z}$-cluster tilting object $T \in \mathscr{C}$, that is a d-cluster tilting object such that $\mathscr{C}(T, T[i]) = 0$ for all $i \notin d\mathbb{Z}$.*
2. *Equivalence classes of pairs (Π, I) consisting of*
   a) *a basic finite-dimensional self-injective algebra Π that is twisted $(d+2)$-periodic and*
   b) *an invertible Π-bimodule I such that $\Omega^{d+2}_{\Pi^e}(\Pi) \cong I$ in the stable category of Π-bimodules.*

Remark 2.6.4. Note that if Λ is a d-representation finite algebra then it is τ_d-finite. Moreover, its preprojective algebra is $\Pi_{d+1}(\Lambda)$ is self-injective, and thus by [IO13, Proposition 3.6]

$$(\pi\Lambda)[d] \cong \pi\Lambda.$$

Hence $\pi\Lambda \in \mathscr{C}^d_\Lambda$ is a $d\mathbb{Z}$-cluster tilting object which fits into Theorem 2.6.3.

Thus by Remark 2.6.4 we can provide examples for Theorem 2.6.3 by applying Theorem 2.3.12.

2.6.2 Quivers and potentials and the 3-preprojective algebra

The case $d = 2$ is of particular importance. In fact d-cluster tilting is a generalisation of 2-cluster tilting which is usually referred to simply as cluster tilting. Moreover, in this case $\mathscr{C}^2_\Lambda$ and $\Pi_3(\Lambda)$ can be understood using quivers and potentials, in case Λ itself is given by a quiver with admissible relations. In this section we recall some of the relevant constructions.

We follow [BIRS11] for the definition of quivers with potentials. Let Q be a quiver. Consider the complete path algebra

$$\widehat{\mathbb{K}Q} = \prod_{i \geq 0} \mathbb{K}Q_i$$

where $\mathbb{K}Q_i$ is the $\mathbb{K}$-vector space with basis Q_i consisting of path of length i. A quiver with potential (Q, W) consists of a quiver Q and a potential $W \in \prod_{i \geq 0} \mathbb{K}Q_{i,cyc}$, where $\mathbb{K}Q_{i,cyc}$ is the $\mathbb{K}$-vector space with a basis $Q_{i,cyc}$ consisting of cycles in Q_i. For a given element $a \in Q_1$, we define the cyclic derivative

$\partial_a(W)$ by

$$\partial_a(a_1a_2\cdots a_n) = \sum_{a_i=a} a_{i+1}a_{i+2}\cdots a_na_1\cdots a_{i-1}$$

and extended linearly and continuously. For a given quiver with potential (Q,W) we define its Jacobian algebra

$$\mathscr{P}(Q,W) = \widehat{\mathbb{K}Q}/\mathscr{J}(Q,W)$$

where $\mathscr{J}(Q,W) = \overline{\langle \partial_a(W) \mid a \in Q_1\rangle}$ is the closure of $\langle \partial_a(W) \mid a \in Q_1\rangle$ with respect to the $\widehat{\mathbb{K}Q}_{\geq 1}$-adic topology.

Definition 2.6.5. [Kel11] Let Λ be a finite dimensional $\mathbb{K}$-algebra of global dimension of at most two. For a presentation $\Lambda = \widehat{\mathbb{K}Q}/\overline{\langle r_1, r_\cdots, r_l\rangle}$ by a quiver Q and minimal set of relations $\{r_1, r_2, \ldots, r_l\}$, we define (Q_Λ, W_Λ) by

$$\begin{aligned} Q_{\Lambda,0} &= Q_0, \\ Q_{\Lambda,1} &= Q_0 \cup \{p_i : t(r_i) \to s(r_i) \mid 1 \leq i \leq l\}, \\ W_\Lambda &= \sum_{i=1}^{l} p_i r_l. \end{aligned}$$

The connection between Jacobian algebras and preprojective algebras are due to [Kel11].

Theorem 2.6.6. *[Kel11, Theorem 6.10] The $\mathbb{K}$-algebras $\Pi_3(\Lambda)$ and $\mathscr{P}(Q_\Lambda, W_\Lambda)$ are isomorphic.*

Let Λ_1 and Λ_2 be two l-homogeneous representation finite algebras. Then by Theorem 2.3.12 $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$ is a 2-representation finite algebra and thus fits into Theorem 2.6.3 due to Remark 2.6.4. Hence, if Λ_1 and Λ_2 are path algebras over Dynkin quivers, then Theorem 2.6.6 yields an explicit description of the 3-preprojective algebra $\Pi_3(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2)$ in this case. This is demonstrated in [HI11b, Section 5.2].

3. Summary of papers

As discussed in Chapter 2, the Auslander-Reiten theory of a representation finite hereditary algebra Λ is encoded in its preprojective algebra $\Pi(\Lambda)$. If Λ is given as a path algebra of a quiver, or more generally, as a tensor algebra of a species, then $\Pi(\Lambda)$ can be explicitly described.

In higher dimensional Auslander-Reiten theory one instead considers a d-representation finite algebra Λ and its $(d+1)$-preprojective algebra $\Pi_{d+1}(\Lambda)$. Explicit descriptions of $\Pi_{d+1}(\Lambda)$ were previously only known when Λ is given by a quiver with admissible relations. For instance quivers with potentials can be used for $d = 2$.

The aim of this thesis is to expand such explicit descriptions to include algebras Λ given by species with admissible relations. Along the way, some results for classical preprojective algebras are also extended to the species case. In particular, the thesis presents results regarding the Nakayama automorphism for preprojective algebras of Dynkin species, almost Koszul resolutions, preprojective algebras of tensor products, mutation theory for species with potentials, self-injective species with potentials, cuts for species with potentials, the relationship between cut-mutation and 2-APR tilting, and more.

The results presented in the thesis yields a rich source of tools on how to generate more examples of 2-representation finite algebras given by species with relations. We also illustrate our results by computing examples.

3.1 Paper I

In [BBK02] Brenner, Butler and King gave a complete description of the Nakayama automorphism for the preprojective algebras of Dynkin type ADE. They also introduced the notion of almost Koszul algebras and show that these preprojective algebras are almost Koszul. Several homological properties can be described using the almost Koszul property, and it allows for a more general theory of periodic algebras. In the first part of Paper I we extend the main result of [BBK02] to preprojective algebras of Dynkin type ABCDEFG. The first main theorem is as follows.

Theorem 3.1.1. *Let S be a Dynkin species over division algebras $F \subseteq G$. There is a Nakayama automorphism γ of $\Pi(S)$ given by*

$$\gamma(y^i_\alpha) = \begin{cases} y^i_{\sigma(\alpha)}, & \text{if } \alpha \in Q_1 \\ \operatorname{sgn}(\sigma(\alpha)) y^i_{\sigma(\alpha)}, & \text{if } \alpha \notin Q_1 \end{cases},$$

and $\gamma(e_i) = e_{\sigma(i)}$ *for all* $i \in Q_0$.

Here we have chosen the elements $y_\alpha^1, \ldots, y_\alpha^{\max(\dim_{D_{s(\alpha)}} \overline{M}_\alpha, \dim_{D_{t(\alpha)}} \overline{M}_\alpha)} \in \overline{M}_\alpha$ such that $\{y_\alpha^1, \ldots, y_\alpha^{\dim_{D_{s(\alpha)}} \overline{M}_\alpha}\}$ is a basis for $\overline{M}_\alpha$ when $\overline{M}_\alpha$ is viewed as a $D_{s(\alpha)}$-module and $\{y_\alpha^1, \ldots, y_\alpha^{\dim_{D_{t(\alpha)}} \overline{M}_\alpha}\}$ is a basis for $\overline{M}_\alpha$ when $\overline{M}_\alpha$ is viewed as a $D_{t(\alpha)}$-module.

Note that for the case when the tensor algebra is isomorphic to a path algebra of a quiver, $\max(\dim_{D_{s(\alpha)}} \overline{M}_\alpha, \dim_{D_{t(\alpha)}} \overline{M}_\alpha) = 1$ and thus we recover [BBK02, Corollary 4.7].

We also provide a complete list of l-homogeneous finite representation species, and compute the specific value of l in each case. By using Theorem 2.3.12 we get a lot of examples of 2-representation finite algebras which are obtained by taking tensor products of tensor algebras of Dynkin species. Moreover, we show that the resulting 2-representation finite algebra as well as its 3-preprojective algebra can be described using species with relations. In Paper II we improve on this result by showing that the 3-preprojective algebra can be described using a species with potential.

In the second part of the article we study preprojective algebras of acyclic d-representation finite Koszul algebras. All tensor algebras of Dynkin species fall under the case when $d = 1$. In [Pas17] Pasquali proved that for a d-representation finite algebra Λ a d-almost split sequence starting at an indecomposable non-injective module in $\operatorname{add}(\tau_d^{-i}\Lambda)$ is given by a mapping cone of a certain morphism. Since Koszul resolutions for simple $\Pi(\Lambda)$-modules are computed using d-almost split sequences, we arrive at the following result.

Theorem 3.1.2. *Let* Λ *be an acyclic* d*-representation finite Koszul algebra and let* S *be a graded simple* $\Pi(\Lambda)$*-module, then there exist graded* $\Pi(\Lambda)$*-module complexes* $R_\bullet, Q_\bullet$ *of length* d *and an almost quasi-isomorphism* $\varphi : Q_\bullet \to R_\bullet$ *such that* $\deg^\star(\varphi) = 1$ *and* $\operatorname{Cone}(\varphi)$ *is the almost Koszul complex for* S.

Recall from Theorem 2.3.12 that if Λ_i is an acyclic l-homogeneous d_i-representation finite algebra over a perfect field $\mathbb{K}$ for $i \in \{1,2\}$, then $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$ is (d_1+d_2)-representation finite. A natural question arises, can we describe the Koszul resolutions of $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$ given the same data for Λ_1 and Λ_2? The answer is yes, but we need to introduce an operation called the Segre product. For graded algebras $\Gamma_i = \bigoplus_{k\in\mathbb{Z}} \Gamma_{i,k}$, where $\Gamma_{i,k}$ is the space of elements of degree k in Γ_i and $i \in \{1,2\}$. We define the Segre product of Γ_1 and Γ_2 as

$$\Gamma_1 \mathbin{\mathbb{\triangle}_{\mathbb{K}}} \Gamma_2 = \bigoplus_{k\in\mathbb{Z}} \Gamma_{1,k} \otimes_{\mathbb{K}} \Gamma_{2,k}.$$

We show in the above setting that

$$\Pi_{d_1+d_2+1}(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2) \cong \Pi_{d_1+1}(\Lambda_1) \mathbin{\mathbb{\triangle}_{\mathbb{K}}} \Pi_{d_2+1}(\Lambda_2).$$

The Segre product can be considered as a bi-functor on graded algebras and analogously on complexes. Now we can state our second main theorem.

Theorem 3.1.3. *Let $\mathbb{K}$ be a perfect field. Let Λ_i be an acyclic d_i-representation finite l-homogeneous Koszul algebra for each $i \in \{1,2\}$. Let S^{Λ_i} be a graded simple $\Pi(\Lambda_i)$-module and let $\varphi^{\Lambda_i} : Q_\bullet^{\Lambda_i} \to R_\bullet^{\Lambda_i}$ be an almost quasi-isomorphism as in Theorem 3.1.2 such that* $\mathrm{Cone}(\varphi^{\Lambda_i})$ *is the almost Koszul complex for S^{Λ_i}. The complex*

$$\mathrm{Cone}(\mathrm{Tot}(\varphi^{\Lambda_1} \barwedge \varphi^{\Lambda_2}) : \mathrm{Tot}(Q_\bullet^{\Lambda_1} \barwedge_{\mathbb{K}} Q_\bullet^{\Lambda_2}) \to \mathrm{Tot}(R_\bullet^{\Lambda_1} \barwedge_{\mathbb{K}} R_\bullet^{\Lambda_2}))$$

is the almost Koszul complex for $S^{\Lambda_1} \otimes_{\mathbb{K}} S^{\Lambda_2} \in \Pi(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2)-\mathrm{mod}$.

We provide the description of φ^{Λ_i} when $\Lambda_i = T(S^i)$ and S^i a Dynkin species. Thus using the above theorem, we automatically have total control of

$$\mathrm{Cone}(\mathrm{Tot}(\varphi^{\Lambda_1} \barwedge \varphi^{\Lambda_2}),$$

i.e. all almost Koszul resolutions of simple modules. Moreover, in this case we describe the Nakayama automorphism of $\Pi_{d_1+d_2+1}(T(S^1) \otimes_{\mathbb{K}} T(S^2))$ as $\gamma_1 \barwedge \gamma_2$, where γ_1 and γ_2 are the Nakayama automorphisms for S^1 and S^2.

3.2 Paper II

The theory of cluster algebras was introduced by Fomin and Zelevinsky in [FZ02]. They introduced mutation for quivers and play a central role in the study of cluster algebras. Motivated by the mutation for quivers by Fomin and Zelevinsky, a mutation theory on cluster tilting objects in 2-Calabi-Yau categories was developed in [BMR$^+$06] and [IY08]. Afterwards came a theory of mutation of quiver with potentials introduced by Derksen, Weyman and Zelevinsky in [DWZ07]. A relationship between these types of mutation is established by Buan, Iyama, Reiten and Smith in [BIRS11]. They proved that mutating on cluster tilting objects corresponds to mutation of certain Jacobian algebras of quivers with potentials under certain conditions.

In [HI11b] Herschend and Iyama studied self-injective quivers with potentials. A quiver with potential (Q,W) is said to be self-injective if the associated Jacobian algebra $\mathscr{P}(Q,W)$ is finite dimensional and self-injective, hence $\mathscr{P}(Q,W)$ is a Frobenius algebra with a Nakayama permutation σ. Using the main result in [BIRS11], Herschend and Iyama showed that mutating a self-injective quiver with potential (Q,W) along a Nakayama orbit of a vertex yields another self-injective quiver with potential. They also provided a structure theorem for 2-representation finite algebras over an algebraically closed field. It says that every basic 2-representation finite algebra over an algebraically closed field is isomorphic to some degree zero part of the Jacobian

algebra of some self-injective quiver with potential, where the grading is such that W is homogeneous of degree 1. They also proved the converse, that every degree zero part of the Jacobian algebra is a 2-representation finite algebra, if the grading is such that W is homogeneous of degree 1.

In this paper we assume that our field $\mathbb{K}$ is perfect. We define a species with potential (S,W) as a pair of a species S and a potential W consisting of cycles in $T(S)$. Similarly as in the quiver case, we define the Jacobian algebra of a species with potential (S,W) which we denote by $\mathscr{P}(S,W)$ as $T(S)$ quotient with the ideal generated by derivations of W. The construction of $\mathscr{P}(S,W)$ is derived from the more general theory by Nguefack [Ngu12]. The first part of our main result is a generalisation of [BIRS11, Theorem 5.2] to the species case. We prove, under certain assumptions, that mutating cluster tilting objects corresponds to mutation of species with potentials. As in [HI11b], we use this new result to show that mutating along the Nakayama orbit of a self-injective species with potential yields a self-injective species with potential. This is formulated as the second part of our main theorem. Thus the main theorem is as follows.

Theorem 3.2.1. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category and $T = \bigoplus_{i=1}^{n} T_i \in \mathscr{C}$ a basic cluster tilting object with indecomposable summands T_i, that satisfies the vanishing condition at i for all $i \in Q_0$ and $k \in Q_0$ is mutable.*

1. *If $\mathrm{End}_{\mathscr{C}}(T)^{op} \cong \mathscr{P}(S,W)$ for a reduced species with potential (S,W), then $\mathrm{End}_{\mathscr{C}}(\mu_{T_k}(T))^{op} \cong \mathscr{P}(\mu_k(S,W))$.*
2. *Assume furthermore, that $\mathscr{P}(S,W)$ is self-injective with Nakayama permutation σ and Nakayama automorphism γ satisfying conditions*
   (A) $\gamma|_M : M \to M$ where $S = (D,M)$,
   (B) $\gamma(W) = W$.
   If the Nakayama orbit $(k) = \{\sigma^i(k) \mid i \geq 0\}$ is sparse, then $\mathscr{P}(\mu_{(k)}(S,W))$ is self-injective with Nakayama automorphism $\mu_{(k)}(\gamma)$.

The vanishing condition for all vertices and that $k \in Q_0$ is mutable are two assumptions that are needed for technical reasons. The vanishing condition at every vertex is automatically satisfied whenever $\mathrm{End}_{\mathscr{C}}(T)^{op}$ is a self-injective algebra. The conditions (A) and (B) are needed to define the Nakayama automorphism $\mu_{(k)}(\gamma)$. A Nakayama orbit (k) is said to be sparse if there are no arrows between the vertices, and if for each mutation step yields a reduced species with potential. This is needed in order to apply the first part of Theorem 3.2.1 repeatedly.

As an application consider $\Lambda = \Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$, where Λ_i is l-homogeneous and representation finite, specifically $\Lambda_i = T(S_i)$ when S_i is a Dynkin species, for $i \in \{1,2\}$. Similarly to Theorem 2.6.6, we show that $\Pi_3(\Lambda) \cong \mathscr{P}(S,W)$ for a certain species with potential. Using that $\mathrm{End}_{\mathscr{C}}(\pi\Lambda)^{op} = \Pi_3(\Lambda)$ we can set $T = \pi\Lambda$, which induces an isomorphism $\mathscr{P}(S,W) \cong \mathrm{End}_{\mathscr{C}}(T)^{op}$. We prove in this setting that T satisfies the vanishing condition at i for all $i \in Q_0$ and $k \in Q_0$

is mutable, where the former follows from the fact that Λ is 2-representation finite. Now if (k) is a sparse orbit, then we can apply Theorem 3.2.1 to get that $\mathscr{P}(\mu_{(k)}(S,W))$ is a self-injective algebra with Nakayama automorphism $\mu_{(k)}(\gamma)$. The resulting algebra $\mathscr{P}(\mu_{(k)}(S,W))$ will satisfy the conditions in Theorem 2.6.3, and therefore expanding the collection of examples for Theorem 2.6.3.

3.3 Paper III

In this article we generalise results of [HI11b] regarding the truncated Jacobian algebras. Their structure theorem, mentioned above in the summary of Paper II, says that every basic 2-representation finite $\mathbb{K}$-algebra appears as a truncated Jacobian algebra $\mathscr{P}(Q,W,C)$ of a self-injective quiver with potential (Q,W) by a cut C. A cut is a subset $C \subseteq Q_1$ such that every cycle in W contains exactly one arrow in C. It defines a grading on the arrows of Q such that W is homogeneous of degree 1 and thus induces a grading on $\mathscr{P}(Q,W)$. The truncated Jacobian algebra $\mathscr{P}(Q,W,C)$ is the degree zero part of $\mathscr{P}(Q,W)$ with respect to this grading. Thus as a consequence, every cut of a self-injective quiver with potential (Q,W) corresponds to a 2-representation finite algebra.

In the paper [IO11], Iyama and Oppermann introduced an operation called cut-mutation. For a quiver with potential (Q,W) and a cut C, cut-mutation is defined at strict sources and strict sinks. A vertex x is a strict source if all arrows ending at x belong to C and all arrows starting at x does not belong in C. Dually we define strict sinks. Cut-mutation at a strict source x is defined as removing all arrows in Q ending at x from C and add all arrows in Q starting at x to C. Dually we define cut-mutation for strict sinks. Herschend and Iyama proved in [HI11b] that cut-mutation of (Q,W,C) corresponds to 2-APR tilting of $\mathscr{P}(Q,W,C)$, which is a consequence from the combinatorics presented in [IO11]. An important remark is that 2-APR tilting preserves 2-representation finiteness, which was proven in [IO11]. Therefore cut-mutation provides us with a new way of constructing new 2-representation finite algebra. Herschend and Iyama also show that under certain assumptions all cuts are transitive under successive cut-mutation.

In this paper we define cuts similarly as in [HI11b] and introduce quivers with cycles inspired by the combinatorics of [HI11b, Section 8]. A quiver with cycles is a triple (Q_0,Q_1,Q_2), where (Q_0,Q_1) is a quiver and Q_2 is a set of distinguished cycles for (Q_0,Q_1). For every species with potential (S,W) we can associate it to a quiver with cycles, where the distinguished cycles are defined as the support of W. A cut for a quiver with cycles is a subset $C \subseteq Q_1$ such that all elements in Q_2 contains exactly one element in C. We prove the analogues statement presented in [HI11b] that under certain assumptions all cuts are transitive under successive cut-mutation. This is formulated as the following.

Theorem 3.3.1. *Let Q be a quiver with cycles which is fully compatible, covered and has enough cuts, then the set of all cuts of Q is transitive under successive cut-mutations.*

The various assumptions in Theorem 3.3.1 are technical and can be hard to check in practice, especially the fully compatible condition. To address this we define the canvas of a quiver with cycles as in [HI11b]. The canvas of a quiver with cycles is a CW-complex where vertices are 0-cells, arrows are 1-cells and the set of distinguished cycles give rise to 2-cells. Canvases are topological spaces and hence we can study their topological features. A quiver with cycles is said to be simply connected if its canvas is simply connected. We prove the following.

Theorem 3.3.2. *Let Q be a simply connected quiver with cycles, which is covered and has enough cuts. Then the set of all cuts of Q is transitive under successive cut-mutations.*

We now shift our focus back to species with potential. For a cut C, the truncated Jacobian algebra $\mathscr{P}(S,W,C)$ is defined as the degree zero part of $\mathscr{P}(S,W)$ with respect to the grading induced by C. We say that a cut C is preprojective if $\operatorname{gldim}\mathscr{P}(S,W,C)\leq 2$ and there is an isomorphism of complete graded algebras $\varphi:\Pi_3(\mathscr{P}(S,W,C))\cong\mathscr{P}(S,W)$ with $\mathscr{P}(S,W)$ graded via C such that φ is the identity on the degree 0 part. We show that 2-APR tilting of $\mathscr{P}(S,W,C)$ corresponds to cut-mutation on C, and with this we state one of our main theorems, which generalisation of the main result in [HI11b, Theorem 7.8].

Theorem 3.3.3. *Let (S,W) be a self-injective simply connected species with potential with enough cuts. Assume that (S,W) has a preprojective cut. Then all cuts C are preprojective and the corresponding truncated Jacobian algebras $\mathscr{P}(S,W,C)$ are iterated 2-APR tilts of each other. In particular, each $\mathscr{P}(S,W,C)$ is 2-representation finite.*

In the quiver case, all cuts of self-injective quiver with potentials are preprojective due to Theorem 2.6.6.

If S^1 and S^2 are two l-homogeneous Dynkin species, then $\Lambda=T(S^1)\otimes_{\mathbb{K}}T(S^2)$ is a 2-representation finite algebra and $\Pi_3(\Lambda)=\mathscr{P}(S,W)$ for a species with potential (S,W) described in Paper II. The species S is unfortunately not always a $\mathbb{K}$-species when S^1 and S^2 are $\mathbb{K}$-species, but we prove in Paper II that there exists a Morita equivalent $\mathbb{K}$-species with potential $(\tilde{S},\tilde{W})$ that we calculate in this case.

Proposition 3.3.4. *Let S^1 and S^2 be two l-homogeneous Dynkin species over division algebras $F=\mathbb{K}\subseteq G$. Moreover, if*

1. there are $i \in Q_0^1$ and $j \in Q_0^2$ such that $D_i^1 = D_j^2 = \mathbb{K}$ or,

2. $D_i^1 = D_j^2 = G$ for all $i \in Q_0^1$ and $j \in Q_0^2$,

then all truncated Jacobian algebras of $(\tilde{S}, \tilde{W})$ are iterated 2-APR tilts of each other, where $(\tilde{S}, \tilde{W})$ is the Morita equivalent $\mathbb{K}$-species with potential.

Since 2-APR tilts of a 2-representation finite algebra is 2-representation finite, we get that all truncated Jacobian algebras in the above proposition are 2-representation finite, thus expanding our collection of 2-representation finite algebras.

4. Summary in Swedish (svensk sammanfattning)

Denna avhandling består av tre artiklar inom representationsteori. Vi kommer att kort gå igenom bakgrunden till avhandlingen och sedan sammanfatta resultatet i artiklarna.

4.1 Bakgrund

Ett känt resultat är att om vi har en ändligt dimensionell algebra över en algebraisk sluten grundkropp så är den Moritaekvivalent med en kogeralgebra kvotat med ett tillåtligt ideal. Vi kan lätta på kraven till sin grundkropp genom att anta att den är perfekt istället för algebraisk sluten. I det här fallet gäller inte det nämnda resultatet utan vi måste introducera det mer allmänna konceptet arter. Alltså om grundkroppen är perfekt, så är varje ändligt dimensionell algebra Moritaekvivalent med en tensoralgebra av någon art kvotat med ett tillåtligt ideal. Detta innebär att om vi vill undersöka ändligt dimensionella algebror över en perfekt grundkropp så räcker det att studera arter.

För att förstå modulkategorin av en given algebra använder vi oftast Auslander-Reitenteorin. Auslander-Reitenteorin är ett verktyg för att skapa en bild av hela modulkategorin med hjälp av koncept som nästan kluvna följder, Auslander-Reitentransformationen och Auslander-Reitenkoger. Nästan kluvna följder och Auslander-Reitentransformationen används för att bygga upp Auslander-Reitenkogret av en algebra, vilket är kogret som beskriver modulkategorin. Den är definierad så att punkterna korresponderar till icke-isomorfa ouppdelbara moduler och det existerar en pil mellan två punkter ifall det existerar en irreducibel morfism mellan de korresponderade modulerna för punkterna. Denna teori utvecklades på sjuttiotalet av Auslander och Reiten [Aa75]. Det finns en speciell mängd av algebror där deras Auslander-Reitenkoger är ändligt, med andra ord har de ändligt många icke-isomorfa ouppdelbara moduler i sin modulkategori. Dessa algebror kallas för representationsändliga algebror. Gabriel visade i [Gab73] att en kogeralgebra över en algebraisk sluten kropp är representationsändlig ifall diagrammet av kogret är ett Dynkin diagram av typ ADE. Dlab och Ringel generaliserade detta resultat genom att visa att en tensoralgebra av en art över en perfekt kropp är representationsändlig om dess diagram är ett Dynkin diagram av typ ABCDEFG. I dessa fall är Auslander-Reitenkogren kända.

För att beskriva mer komplicerade algebror som inte är representationsändliga använder vi oss utav en mer allmän teori kallad för högre Auslander-Reitenteori. Den högre Auslander-Reitenteorin introducerar koncept som är mer allmänna och analoga till nästan kluvna följder, Auslander-Reitentransformationen och Auslander-Reitenkoger. Istället för att studera hela modulkategorin så väljer vi ut en speciell delkategori, som kallas för en d-kluster tilting delkategori, där d är ett positivt heltal. I den här delkategorin kan vi beskriva d-nästan kluvna följder och d-Auslander-Reitentransformationen. Om vi sätter $d = 1$ så återfår vi den vanliga Auslander-Reitenteorin där 1-kluster tilting delkategorin är hela modulkategorin. Vi säger att en algebra är d-representationsändlig ifall d-kluster tilting delkategorin har ändligt många objekt, vilket visar sig vara ekvivalent till att d-kluster tilting delkategorin är genererad av en modul som kallas för en d-kluster tilting modul.

En speciell algebra som används för att beskriva d-kluster tilting delkategorin är den $(d+1)$-preprojektiva algebran. Den definieras som mängden av morfismer från de projektiva objekten i modulkategorin till alla transformerade objekten via d-Auslander-Reitentransformationen. För $d = 1$ har Dlab och Ringel i [DR80] gett en beskrivning av preprojektiva algebran av representationsändliga arter. Om en algebra är d-representationsändlig så är den $(d+1)$-preprojektiva algebran ändligt dimensionell och självinjektiv. I det här fallet är den en så kallad Frobeniusalgebra, vilket betyder att algebran har en speciell typ av automorfi som kallas för Nakayamaautomorfin. Genom att beskriva Nakayamaautomorfin utvinner vi mycket information av $(d+1)$-preprojektiva algebran.

Det finns ingen klassifikation för d-representationsändliga algebror, inte ens när $d = 2$. En anledning är att det är svårt att hitta exempel på såna algebror. Dock har Herschend och Iyama [HI11a] visat att ifall vi tar tensorprodukten av två l-homogena d_1- respektive d_2-representationsändliga algebror så får vi en l-homogen $(d_1 + d_2)$-representationsändlig algebra när grundkroppen är perfekt. En algebra är l-homogen ifall varje Auslander-Reitentransformationsbana av de projektiva modulerna har längd l, där l är något positivt heltal. Detta ger oss ett sätt att generera exempel på d-representationsändliga algebror.

4.2 Sammanfattning av artiklar

4.2.1 Artikel I

Artikel I består av två delar. Den första delen handlar om hur vi beskriver Nakayamaautomorfin för representationsändliga arter. Detta generaliserar ett resultat från [BBK02] där de ger en beskrivning av Nakayamaautomorfin för fallet av kogeralgebror. Vi ger även en komplett beskrivning av alla l-homogena arter, som i sin tur används senare för att skapa d-representationsändliga arter

med hjälp av tensorprodukter. Vi ger även en beskrivning av Koszulupplösningar av de representationsändliga arterna.

I den andra delen av artikeln fokuserar vi på d-representationsändliga algebror. Vi beskriver Koszulupplösningar för tensorprodukter av l-homogena d_1-respektive d_2-representationsändliga algebror i termer av Koszulupplösningar för algebrorna i tensorprodukten. Vi introducerar Segreprodukten för graderade algebror. Genom att använda Segreprodukten och att den högre preprojektiva algebran är naturligt graderad ger vi en beskrivning av den högre preprojektiva algebra för tensorprodukten av algebror. Eftersom den resulterande algebran av tensorprodukten av två l-homogena algebror är l-homogen, så har vi att processen ovan är iterativ.

Vi ger också en beskrivning av tensorprodukten av två l-homogena representationsändliga arter, genom att förse en konstruktion på en beskrivningen av tensorprodukten som en tensoralgebran av en art kvotat med ett tillåtligt ideal. Detta gör att vi får en mängd av exempel på 2-representationsändliga algebror där vi kan applicera resultaten i den andra delen av artikeln.

4.2.2 Artikel II

En potential i ett koger är en linjärkombination av cykler i vägalgebran av kogret. Det finns en mutationsteori för koger med potential som introducerades av Derksen, Weyman och Zelevinsky i [DWZ07]. För ett koger med potential kan vi definiera Jacobianalgebran som vägalgebran av kogret kvotat med idealet som genereras av alla deriveringar av potentialen. I artikeln [BIRS11] visar de att mutationer av vissa koger med potential korresponderar till mutation av kluster tilting objekt. Vårt första huvudresultat i den här artikeln är att vi generaliserar resultatet från [BIRS11] till arter med potential.

En art med potential är självinjektiv ifall dess Jacobianalgebra är självinjektiv och ändligt dimensionell. I det här fallet kommer Jacobianalgebran att vara Frobenius, vilket i sin tur betyder att det existerar en Nakayamapermutation för punkterna kogret. Vårt andra huvudresultat visar vi att om vi muterar en självinjektiv art med potential längs en Nakayamapermutationsbana ger det en ny självinjektiv art med potential. Detta generaliserar det analoga resultatet för koger med potential i [HI11b], och ger oss ett nytt verktyg för att skapa fler självinjektiva arter med potential.

Det visar sig att algebror som beskrivs av tensorprodukter av två l-homogena representationsändliga arter är exempel på algebror som passar in i den deriverade Auslander-Iyama korrespondensen som är etablerad av [JKM23]. Genom att applicera vårt andra huvudresultat får vi fler exempel på algebror som passar in i den deriverade Auslander-Iyama korrespondensen.

4.2.3 Artikel III

I den här artikeln studerar vi trunkerade Jacobianalgebror. Ett snitt av en art med potential är en mängd med pilar i kogret för arten så att potentialen är homogen av grad ett med avseende på graderingen inducerad av snittet. För en art med en potential och ett snitt kan vi definiera den trunkerade Jacobianalgebran som tensoralgebran av arten kvotat med idealet som genereras av alla deriveringar av potential med avseende på alla pilar i snittet. Vi kan definiera snittmutation på liknande sätt som i [IO11]. Vårt första resultat är att, givet några antaganden, alla snitt är transitivt under successiv snittmutation. Vi visar även att dessa antaganden är uppfyllda ifall vår art med potential är självinjektiv, enkelt sammanhängande och har nog med snitt.

För en art med potential och ett snitt säger vi att snittet är preprojektivt ifall den trunkerade Jacobianalgebran är av global dimension 2 och att Jacobianalgebran är dens preprojektiva algebra. Vårt andra huvudresultat är följande. Fixera en självinjektiv enkelt sammanhängande art med potential som har nog med snitt. Om det finns ett snitt som är preprojektivt, så är alla snitt preprojektiva och deras trunkerade Jacobianalgebror är 2-representationsändliga. Exempel på arter med potential som uppfyller kraven kommer från Artikel I, där vi tar tensorprodukter av två l-homogena representationsändliga arter. Detta expanderar vår samling av exempel på 2-representationsändliga algebror.

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