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RESEARCH ARTICLE

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Zeros of multiple orthogonal polynomials: location and interlacing

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Abstract

We prove a criterion for the possible locations of zeros of type I and type II multiple orthogonal polynomials in terms of normality of degree 1 Christoffel transforms. We provide another criterion in terms of degree 2 Christoffel transforms for establishing zero interlacing of the neighboring multiple orthogonal polynomials of type I and type II. We apply these criteria to establish zero location and interlacing of type I multiple orthogonal polynomials for Nikishin systems. Additionally, we recover the known results on zero location and interlacing for type I multiple orthogonal polynomials for Angelesco systems, as well as for type II multiple orthogonal polynomials for Angelesco and AT systems. Finally, we demonstrate that normality of the higher-order Christoffel transforms is naturally related to the zeros of the Wronskians of consecutive orthogonal polynomials.

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1 | INTRODUCTION

Given an infinite set Γ on the real line $\mathbb{R}$, let $\mathcal{M}(\Gamma)$ denote the set of all finite Borel measures μ of constant sign, with finite moments and infinite support $\operatorname{supp}\mu \subseteq \Gamma$. For any $\mu \in \mathcal{M}(\mathbb{R})$, its orthogonal polynomial of degree $n \in \mathbb{N} := \{0, 1, 2, \dots\}$ is defined to be the monic polynomial P_n of exact degree n satisfying

$$\int P_n(x)x^k d\mu(x) = 0, \qquad k = 0, 1, \dots, n-1. \tag{1.1}$$

It is easy to see that such P_n always exists and is unique. The two basic yet fundamental properties of the zeros of orthogonal polynomials are:

(i) All zeros of P_n are real, simple, and belong to the interior of the convex hull of supp μ;
(ii) Zeros of two consecutive polynomials P_n and P_{n+1} interlace, that is, between every two consecutive zeros of one of the polynomials there lies exactly one zero of the other polynomial.

For more details on the theory of orthogonal polynomials on the real line, see, for example, [11, 23, 40–42].

Let us now define *multiple orthogonal polynomials*. For an introduction to the theory, see [2, 23, 37, 39]. Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$, $r \geqslant 1$, be a system of (potentially complex) measures with all finite moments. We write $\boldsymbol{n}$ for multi-indices $(n_1, \dots, n_r) \in \mathbb{N}^r$, and $|\boldsymbol{n}| = n_1 + \dots + n_r$.

A type I multiple orthogonal polynomial at a location $\boldsymbol{n} \in \mathbb{N}^r \setminus \{\boldsymbol{0}\}$ is a vector of polynomials $\boldsymbol{A_n} = (A_{\boldsymbol{n}}^{(1)}, \dots, A_{\boldsymbol{n}}^{(r)})$ such that $\deg A_{\boldsymbol{n}}^{(j)} \leqslant n_j - 1$, $j = 1, \dots, r$, and

$$\sum_{j=1}^{r} \int A_{\boldsymbol{n}}^{(j)}(x)x^k d\mu_j(x) = \begin{cases} 0, & \text{for } k = 0, 1, \dots, |\boldsymbol{n}| - 2, \\ 1, & \text{for } k = |\boldsymbol{n}| - 1. \end{cases} \tag{1.2}$$

In cases where $n_j = 0$ for some $j = 1, \dots, r$, we interpret the degree condition $\deg A_{\boldsymbol{n}}^{(j)} \leqslant n_j - 1$ to mean that $A_{\boldsymbol{n}}^{(j)}(z) \equiv 0$. If $\boldsymbol{n} = \boldsymbol{0}$, we set $A_{\boldsymbol{0}}(z) \equiv \boldsymbol{0}$ by definition. For $r = 1$ and $n > 0$, the polynomial A_n coincides with P_{n-1} from (1.1), but with a nonmonic normalization.

A type II multiple orthogonal polynomial for $\boldsymbol{\mu}$ at a location $\boldsymbol{n} \in \mathbb{N}^r$ is a monic polynomial $P_{\boldsymbol{n}}(x)$ of exact degree $|\boldsymbol{n}|$ such that

$$\int P_{\boldsymbol{n}}(x)x^k d\mu_j(x) = 0, \qquad k = 0, 1, \dots, n_j - 1, \qquad j = 1, \dots, r \tag{1.3}$$

(if $r = 1$, then this reduces to P_n in (1.1)).

We say that an index $\boldsymbol{n}$ is normal with respect to $\boldsymbol{\mu}$ if $P_{\boldsymbol{n}}$ exists and is unique. It is easy to show that this is equivalent to the existence and uniqueness of $\boldsymbol{A_n}$ (see Section 3.1 for details). A system is called perfect if every $\boldsymbol{n} \in \mathbb{N}^r$ is normal.

If $r \geqslant 2$, it is not easy to identify perfect systems, but there are several wide classes of systems that are known to be perfect. These are Angelesco, AT, and Nikishin (see Sections 3.2, 3.3, and 3.4, for the definitions). In particular, all the known multiple orthogonality analogues of the classical orthogonal polynomials fall within these categories.

We are interested in studying properties of the zeros of type I and type II multiple orthogonal polynomials, specifically their possible locations and interlacing properties. The notion of the (multiple) Christoffel transform is crucial for our analysis.

Given a measure μ and a point $z_0 \in \mathbb{C}$, a one-step Christoffel transform of μ is the new measure $\widehat{\mu}$ defined by

$$\int f(x)d\widehat{\mu}(x) = \int f(x)(x - z_0)d\mu(x). \tag{1.4}$$

We denote such a measure by $(x - z_0)\mu$.

Our first main result is the criterion (Theorem 4.1) that says that z_0 is a zero of $P_{\boldsymbol{n}}$ if and only if the index $\boldsymbol{n}$ is not normal for the system

$$((x - z_0)\mu_1, \dots, (x - z_0)\mu_r). \tag{1.5}$$

Similarly, Theorem 4.8 says that z_0 is a zero of $A_{\boldsymbol{n}}^{(1)}$ if and only if the index $\boldsymbol{n} - \boldsymbol{e}_1$ (here $\boldsymbol{e}_\ell$ denotes the vector in $\mathbb{Z}^r$ with a 1 in the ℓ-th component and zeros elsewhere) is not normal for the system

$$((x - z_0)\mu_1, \mu_2 \dots, \mu_r) \tag{1.6}$$

(analogous statements hold for $A_{\boldsymbol{n}}^{(j)}$ with any other $j = 2, \dots, r$).

Normality of indices for the systems (1.5) and (1.6) are often easy to check, which provides the set free from zeros of the orthogonal polynomials. In particular, it can be applied to establish the possible locations of the zeros of Angelesco systems (for type I and type II polynomials), AT systems (for type II polynomials only, as for type I nothing general can be said), and Nikishin systems (for type I and type II polynomials). All of these results are well known, except for the case of type I polynomials for Nikishin systems (Theorem 4.13), which is new (see [33] for a somewhat related study on real-rootedness). The distinctive feature is that these zeros all appear on a set disjoint from the supports of the orthogonality measures, which is unlike any other case.

The second result of the paper is a criterion (Theorem 5.1) that says that, for each $\ell = 1, \dots, r$, two neighboring polynomials $P_{\boldsymbol{n}}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_\ell}$ have interlacing zeros if and only if the index $\boldsymbol{n}$ is normal for the system

$$\left((x - z_0)^2\mu_1, \dots, (x - z_0)^2\mu_r\right) \tag{1.7}$$

for every $z_0 \in \mathbb{R}$ (assuming real-rootedness of $P_{\boldsymbol{n}}$). One side of this equivalence was already observed by Haneczok and Van Assche [22], and our proof is in essence distilled from their paper.

A dual result to this is Theorem 5.6, which states that $A_{\boldsymbol{n}}^{(1)}$ and $A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)}$ have interlacing zeros if and only if $\boldsymbol{n} - 2\boldsymbol{e}_1$ is normal for the system

$$\left((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r\right) \tag{1.8}$$

for every $z_0 \in \mathbb{R}$ (assuming real-rootedness of $A_{\boldsymbol{n}}^{(1)}$), and similarly for $A_{\boldsymbol{n}}^{(j)}$ with any other $j = 2, \dots, r$.

We then apply these criteria to Angelesco and Nikishin systems to obtain interlacing for both type I and type II polynomials. Interlacing for type I neighboring polynomials for Nikishin systems (Theorem 5.9) is new.

The organization of the paper is as follows. Section 2 contains a collection of results that are needed for the proof. In Section 3, we remind the reader the definitions of Angelesco, AT, and Nikishin systems. In Section 4, we deal with location of zeros and in Section 5 with interlacing. Section 6 has a short discussion on the Christoffel transforms of higher order and their connection to Wronskians of consecutive orthogonal polynomials.

In order to keep the paper self-contained and friendly for the general audience, we present in Section 3, elementary proofs of perfectness for Angelesco and AT systems that do not involve zero counting, inspired by the arguments of Kuijlaars [31].

Interlacing property of the zeros of the neighboring type II polynomials for Angelesco systems was shown in [22] (and more generally for any systems with positive $a_{\boldsymbol{n},j}$-recurrence coefficients), see also [3]. Interlacing for type I was proved in [21], see also [13]. Interlacing of the zeros of the neighboring type II polynomials for Nikishin systems was shown in [16], and for AT systems in [16, 22, 26]. Some other types of interlacing properties of multiple orthogonal polynomials were investigated in [6, 7, 14, 20, 21, 28, 32, 35, 36]. Christoffel transforms of multiple orthogonal polynomials were the topic of [4, 10, 28, 29, 34].

2 | PRELIMINARIES

2.1 | A generalization of the Andreief identity

In Section 3, we use the following generalization of the Andreief identity (which is the special case of (2.1) with $M = N$). It is particularly well suited for matrices with block structure which results in simple proofs of perfectness of Angelesco and AT systems. It can also be applied to Nikishin systems [12, 27, 31], as well as Angelesco, AT, and Nikishin systems on the unit circle [27, 30].

Proposition 2.1 [30]. *Let $(\phi_j)_{j=1}^N, (\psi_k)_{k=1}^N \in L^2(\mu)$ for some probability measure μ on a measure space (X, Σ, μ); $a_{j,k} \in \mathbb{C}$. Then we have the identity*

$$\int_{X^M} \det\begin{pmatrix} a_{1,1} & \cdots & a_{1,N} \\ \vdots & \ddots & \vdots \\ a_{N-M,1} & \cdots & a_{N-M,N} \\ \hdashline \psi_1(x_1) & \cdots & \psi_N(x_1) \\ \vdots & \ddots & \vdots \\ \psi_1(x_M) & \cdots & \psi_N(x_M) \end{pmatrix} \det\begin{pmatrix} \phi_1(x_1) & \dots & \phi_M(x_1) \\ \vdots & \ddots & \vdots \\ \phi_1(x_M) & \dots & \phi_M(x_M) \end{pmatrix} \prod_{j=1}^M d\mu(x_j)$$
$$= M!\det\begin{pmatrix} a_{1,1} & \cdots & a_{1,N} \\ \vdots & \ddots & \vdots \\ a_{N-M,1} & \cdots & a_{N-M,N} \\ \hdashline \int_X \phi_1(x)\psi_1(x)\,d\mu(x) & \cdots & \int_X \phi_1(x)\psi_N(x)\,d\mu(x) \\ \vdots & \ddots & \vdots \\ \int_X \phi_M(x)\psi_1(x)\,d\mu(x) & \cdots & \int_X \phi_M(x)\psi_N(x)\,d\mu(x) \end{pmatrix}. \tag{2.1}$$

2.2 | m-functions and Christoffel transforms

The m-function of $\mu \in \mathcal{M}(\mathbb{R})$ is defined by

$$m_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}, \qquad z \in \mathbb{C} \setminus \operatorname{supp}\mu. \tag{2.2}$$

Assuming additionally that $\mu \in \mathcal{M}(\mathbb{R})$ is a probability measure, m satisfies the following relation, see, for example, [41, Theorem 3.2.4],

$$\frac{1}{m_\mu(z)} = b_1 - z - a_1^2 m_{\mu^{(1)}}(z), \tag{2.3}$$

for some $a_1, b_1 \in \mathbb{R}$ with $a_1 > 0$, where

$$m_{\mu^{(1)}}(z) = \int_{\mathbb{R}} \frac{d\mu^{(1)}(x)}{x - z} \tag{2.4}$$

is the m-function of another probability measure $\mu^{(1)}$ (which is the spectral measure of the Jacobi operator of μ but with the first row and column stripped). Using (2.3), it is easy to show that if $\operatorname{supp}\mu$ belongs to some interval Γ, then $\operatorname{supp}\mu^{(1)}$ also belongs to Γ.

It is an easy exercise to see that the m-function (2.2) $m_{\widehat{\mu}}(z)$ of the Christoffel transform $\widehat{\mu} = (x - z_0)\mu$ (recall (1.4)) satisfies

$$m_{\widehat{\mu}}(z) = \int \frac{(x - z_0)d\mu(x)}{x - z} = \int d\mu(x) + (z - z_0)m_\mu(z). \tag{2.5}$$

2.3 | Zeros, real-rootedness, and interlacing

Given a polynomial $p(z)$, let $\mathcal{Z}[p]$ stand for the set of its zeros. We say that a polynomial is real-rooted if $\mathcal{Z}[p] \subset \mathbb{R}$.

We say that zeros of two real polynomials $p(x)$ and $q(x)$ interlace, and we write $p(x) \sim q(x)$, if all zeros of $p(x)$ and $q(x)$ are real and simple, and between every two consecutive zeros of (any) one of the polynomials there lies exactly one zero of the other (here we mean *strict* interlacing, i.e., $\mathcal{Z}[p] \cap \mathcal{Z}[q] = \emptyset$).

Recall that a Wronskian of ℓ polynomials $Q_1(x), \ldots, Q_\ell(x)$ is

$$W(Q_1, \ldots, Q_\ell; x) := \det \begin{pmatrix} Q_1(x) & Q_2(x) & \ldots & Q_\ell(x) \\ Q_1'(x) & Q_2'(x) & \ldots & Q_\ell'(x) \\ \vdots & \vdots & \ddots & \vdots \\ Q_1^{(\ell-1)}(x) & Q_2^{(\ell-1)}(x) & \ldots & Q_\ell^{(\ell-1)}(x) \end{pmatrix}. \tag{2.6}$$

To check interlacing we need to use the following standard result.

Lemma 2.2. *Let Q be a real-rooted polynomial and P be any real polynomial. Then $P \sim Q$ if and only if $W(P, Q; z_0) \neq 0$ for all $z_0 \in \mathbb{R}$ and $\deg P \leqslant \deg Q + 1$.*

Proof. Without loss of generality, we may assume P and Q to be monic. Suppose $W(P, Q; x) \neq 0$ on $\mathbb{R}$, with $\deg Q = N$, and $\deg P \leqslant N + 1$. Since $W(P, Q)$ is continuous, it keeps the same sign for all $x \in \mathbb{R}$. If x_j is a zero of Q, we have $W(P, Q; x_j) = P(x_j)Q'(x_j)$, so all zeros of Q are simple and different from those of P. We write $x_1 > x_2 > \ldots > x_N$ for the zeros of Q.

Since $Q'(x_j)$ alternates sign, we obtain that between each two consecutive zeros of Q there must be an odd number of zeros of P (counting multiplicities). Switching the roles of P and Q, we obtain that there is exactly one zero of P on each interval (x_{j+1}, x_j). This determines the location of $N - 1$ zeros of P and proves $P \sim Q$ unless $\deg P = N + 1$. If $\deg P = N + 1$, then we claim there is one zero of P on (x_1, ∞). If not then $P(x_1) > 0$ and therefore $W(P, Q; x_1) = P(x_1)Q'(x_1) > 0$. But this contradicts $W(P, Q; x) \sim x^{N+1}(Nx^{N-1}) - (N + 1)x^N x^N = -x^{2N}$ as $x \to \infty$. Similarly one shows that P must have a zero on $(-\infty, x_N)$. This completes the proof of $P \sim Q$.

For the converse, assume $\deg Q \geqslant \deg P$ and write

$$W(P,Q;x) = -Q(x)^2(P/Q)'(x) = Q(x)^2 \sum_{j=1}^{N} \frac{\lambda_j}{(x-x_j)^2}, \tag{2.7}$$

where λ_j is the residue of P/Q at x_j. Interlacing $P \sim Q$ implies that all λ_j's are of the same sign, and then the last equality proves that $W(P,Q;x)$ does not vanish on $\mathbb{R}$. The case $\deg Q \leqslant \deg P$ can be handled by swapping the roles of P and Q. □

3 | MULTIPLE ORTHOGONAL POLYNOMIALS

3.1 | Normality

Let $r \geqslant 1$ and consider a system of measures on the real line $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r) \in \mathcal{M}(\mathbb{R})^r$.

Recall the definition of the type I and type II multiple orthogonal polynomials (1.2) and (1.3).

It is not hard to verify that existence and uniqueness of either type I or type II polynomials is equivalent to the condition $\det H_{\boldsymbol{n}}[\boldsymbol{\mu}] \neq 0$, where

$$H_{\boldsymbol{n}}[\boldsymbol{\mu}] = \begin{pmatrix} c_0^{(1)} & c_1^{(1)} & \cdots & c_{|\boldsymbol{n}|-1}^{(1)} \\ c_1^{(1)} & c_2^{(1)} & \cdots & c_{|\boldsymbol{n}|}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n_1-1}^{(1)} & c_{n_1}^{(1)} & \cdots & c_{|\boldsymbol{n}|+n_1-2}^{(1)} \\ \hdashline & & \vdots & \\ \hdashline c_0^{(r)} & c_1^{(r)} & \cdots & c_{|\boldsymbol{n}|-1}^{(r)} \\ c_1^{(r)} & c_2^{(r)} & \cdots & c_{|\boldsymbol{n}|}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n_r-1}^{(r)} & c_{n_r}^{(r)} & \cdots & c_{|\boldsymbol{n}|+n_r-2}^{(r)} \end{pmatrix}$$

and $H_{\boldsymbol{0}}[\boldsymbol{\mu}] := 1$. Here $c_k^{(j)} = \int x^k d\mu_j(x)$. If this happens then we say that $\boldsymbol{n}$ is normal with respect to $\boldsymbol{\mu}$. $\boldsymbol{\mu}$ is called perfect if every $\boldsymbol{n} \in \mathbb{N}^r$ is normal.

Remark 3.1. Easy to see that $\boldsymbol{n} \in \mathbb{N}^r$ is normal if and only if there is no nonzero solution to (1.3) with $\deg P_{\boldsymbol{n}} < |\boldsymbol{n}|$. Similarly, $\boldsymbol{n}$ is normal if and only if there is no nonzero vector $\boldsymbol{A}_{\boldsymbol{n}}$ that satisfies $\deg A_{\boldsymbol{n}}^{(j)} \leqslant n_j - 1$ and

$$\sum_{j=1}^{r} \int A_{\boldsymbol{n}}^{(j)}(x) x^k d\mu_j(x) = 0, \qquad k = 0, \dots, |\boldsymbol{n}| - 1. \tag{3.1}$$

Note that (3.1) has one extra orthogonality condition compared with (1.2).

3.2 | Angelesco systems and their perfectness

Definition 3.2. For each $1 \leqslant j \leqslant r$, let $\mu_j \in \mathcal{M}(\mathbb{R})$. We call $(\mu_j)_{j=1}^r$ an Angelesco system [1] if there exist intervals Γ_j, $1 \leqslant j \leqslant r$, such that $\operatorname{supp} \mu_j \subseteq \Gamma_j$ and for each $j \neq k$, $\Gamma_j \cap \Gamma_k$ is either empty or consists of a single point.

Theorem 3.3. *Angelesco systems are perfect.*

Proof. The idea of the proof is borrowed from [31, Section 4.2]. The use of Proposition 2.1 slightly simplifies the argument. Write $H^{(j)}_{n_j,|\boldsymbol{n}|}$ for the j-th block in (3.1). Apply (2.1) r times to each of the $H^{(j)}_{n_j,|\boldsymbol{n}|}$ sequentially (use $N = |\boldsymbol{n}|$ and $M = n_j$) to get

$$\det H_{\boldsymbol{n}} = \frac{1}{n_1! \dots n_r!} \int_{\Gamma_1^{n_1}} \dots \int_{\Gamma_r^{n_r}} \Delta_{|\boldsymbol{n}|}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_r) \prod_{j=1}^{r} \Delta_{n_j}(\boldsymbol{x}_j)\, d^{n_j}\mu_j(\boldsymbol{x}_j),$$

where $\boldsymbol{x}_j = (x_{j,1}, \dots, x_{j,n_j})$, and $\Delta_k(x_1, \dots, x_k) = \prod_{i<j}(x_j - x_i)$ is the $k \times k$ Vandermonde determinant. We end up with the integral

$$\frac{1}{n_1! \dots n_r!} \int_{\Gamma_1^{n_1}} \dots \int_{\Gamma_r^{n_r}} \prod_{i<j} \Delta(\boldsymbol{x}_i; \boldsymbol{x}_j) \prod_{j=1}^{r} \Delta(\boldsymbol{x}_j)^2\, d^{n_j}\mu_j(\boldsymbol{x}_j), \tag{3.2}$$

where $\Delta(\boldsymbol{x}_i; \boldsymbol{x}_j) := \prod_{\alpha,\beta}(x_{j,\alpha} - x_{i,\beta})$. Clearly the integrand does not change sign, so cannot be zero, and we conclude that $\boldsymbol{n}$ is normal. ☐

Remark 3.4. Recall that the type II multiple orthogonal polynomials in a perfect system satisfy the nearest-neighbor recurrence relations [43]

$$xP_{\boldsymbol{n}}(x) = P_{\boldsymbol{n}+\boldsymbol{e}_k}(x) + b_{\boldsymbol{n},k} P_{\boldsymbol{n}}(x) + \sum_{\ell=1}^{r} a_{\boldsymbol{n},\ell} P_{\boldsymbol{n}-\boldsymbol{e}_\ell}(x).$$

Assuming intervals Γ_j are ordered according to $\Gamma_i \leqslant \Gamma_j$ for all $i < j$, the above proof also shows that $\det H_{\boldsymbol{n}} > 0$ for every $\boldsymbol{n} \in \mathbb{N}^r$. This fact can be used to show that all the nearest-neighbor recurrence coefficients $a_{\boldsymbol{n},\ell}$ for Angelesco systems are positive [3].

3.3 | AT systems and their perfectness

Let Γ be a closed interval of $\mathbb{R}$. A collection $(u_j(t))_{j=1}^n$ of continuous real-valued functions on Γ is called a Chebyshev system on Γ if the determinant

$$U_n(\boldsymbol{x}) := \det \begin{pmatrix} u_1(x_1) & u_1(x_2) & \cdots & u_1(x_n) \\ u_2(x_1) & u_2(x_2) & \cdots & u_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ u_n(x_1) & u_n(x_2) & \cdots & u_n(x_n) \end{pmatrix} \tag{3.3}$$

is nonzero and has a constant sign for any $x_1 < x_2 < \dots < x_n$ on Γ. See [25] to more details about the Chebyshev property and its equivalent form.

Definition 3.5. For each $1 \leqslant j \leqslant r$, let $d\mu_j(x) = w_j(x)d\mu(x)$ be an absolutely continuous measure with respect to some $\mu \in \mathcal{M}(\Gamma)$. We say that $(\mu_j)_{j=1}^r$ is an AT system on Γ for the index $\boldsymbol{n} = (n_1, \dots, n_r)$ if the functions

$$\left\{w_1, xw_1, \dots, x^{n_1-1}w_1, w_2, xw_2, \dots, x^{n_2-1}w_2, \dots, w_r, xw_r, \dots, x^{n_r-1}w_r\right\} \tag{3.4}$$

form a Chebyshev system on Γ.

Theorem 3.6. *If $(\mu_j)_{j=1}^r$ is an AT system for the index $\boldsymbol{n}$, then $\boldsymbol{n}$ is normal.*

Proof. The proof is borrowed from [31, Section 4.3]. Given $\boldsymbol{n} \in \mathbb{N}^r$, denote $U_{\boldsymbol{n}}(\boldsymbol{x})$ to be the determinant (3.3) for functions (3.4). Applying the Andreief identity ((2.1) with $M = N = |\boldsymbol{n}|$) and performing elementary row operations we arrive at

$$\det H_{\boldsymbol{n}}[\boldsymbol{\mu}] = \frac{1}{|\boldsymbol{n}|!} \int_{\Gamma^{|\boldsymbol{n}|}} U_{\boldsymbol{n}}(\boldsymbol{x}) \Delta_{|\boldsymbol{n}|}(\boldsymbol{x})\, d^{|\boldsymbol{n}|}\mu(\boldsymbol{x}), \tag{3.5}$$

where $\Delta_{|\boldsymbol{n}|}$ is again the Vandermonde determinant. Since both $U_{\boldsymbol{n}}$ and $\Delta_{|\boldsymbol{n}|}$ preserve sign on $x_1 < \dots < x_{|\boldsymbol{n}|}$ and simultaneously change sign when we permute the x_j's, we see that the integrand does not change sign. Hence $\det H_{\boldsymbol{n}} \neq 0$, so that $\boldsymbol{n}$ is normal. □

Remark 3.7. Similarly to Remark 3.4, we can see that $\operatorname{sgn} \det H_{\boldsymbol{n}} = \operatorname{sgn} U_{\boldsymbol{n}}$. The collection of signs of $U_{\boldsymbol{n}}$ therefore determines the signs of all the nearest-neighbor recurrence coefficients $a_{\boldsymbol{n},j}$ for AT systems.

3.4 | Nikishin systems

For any set $S \subseteq \mathbb{R}$, we denote $\mathring{S}$ to be the interior of S in the topology of $\mathbb{R}$. Given two closed intervals Γ_1 and Γ_2 with $\mathring{\Gamma}_1 \cap \mathring{\Gamma}_2 = \emptyset$, and two measures $\sigma_1 \in \mathcal{M}(\Gamma_1)$, $\sigma_2 \in \mathcal{M}(\Gamma_2)$, denote $\langle \sigma_1, \sigma_2 \rangle \in \mathcal{M}(\Gamma_1)$ to be the measure

$$d\langle \sigma_1, \sigma_2 \rangle(x) := m_{\sigma_2}(x) d\sigma_1(x), \tag{3.6}$$

where m_{σ_2} is the m-function of σ_2 (see Section 2.2). Note that the right-hand side of (3.6) always defines a finite sign-definite measure on Γ_1, assuming that Γ_1 and Γ_2 are disjoint. If Γ_1 and Γ_2 share a common endpoint, however, then by writing $\langle \sigma_1, \sigma_2 \rangle$, we implicitly *assume* that (3.6) defines a finite measure, as this is no longer automatic and depends on the behavior of σ_1 and σ_2 at that point.

Definition 3.8. We say that $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r) \in \mathcal{M}(\mathbb{R})^r$ forms a Nikishin system generated by $(\sigma_1, \dots, \sigma_r)$ (we then write $\boldsymbol{\mu} = \mathcal{N}(\sigma_1, \dots, \sigma_r)$), if there is a collection of closed intervals Γ_j, $j = 1, \dots, r$ such that

$$\mathring{\Gamma}_j \cap \mathring{\Gamma}_{j+1} = \emptyset, \qquad j = 1, \ldots, r-1, \tag{3.7}$$

and measures $\sigma_j \in \mathcal{M}(\Gamma_j)$, $j = 1, \ldots, r$, so that

$$\mu_1 = \sigma_1, \mu_2 = \langle \sigma_1, \sigma_2 \rangle, \mu_3 = \langle \sigma_1, \langle \sigma_2, \sigma_3 \rangle \rangle, \ldots, \mu_r = \langle \sigma_1, \langle \sigma_2, \langle \sigma_3, \ldots, \sigma_r \rangle \rangle \rangle.$$

It is well-known that Nikishin systems are perfect. The proof for $r = 2$ appeared in [15], and for any $r \geqslant 2$ this was shown in [18, 19]. See also [5, 9, 12, 16, 17, 27, 31, 32, 38] for related normality/perfectness results.

4 | ZERO LOCATION

4.1 | Zero location for type II multiple orthogonal polynomials

The following theorem provides a very simple yet useful tool for locating zeros of the type II multiple orthogonal polynomials. Recall that the notation $\mathcal{Z}[P]$ stands for the zero set of a polynomial P, and $(x - z_0)\mu$ stands for the Christoffel transform of μ (see Section 2.2). Let us also write $(x - z_0)\boldsymbol{\mu}$ for $((x - z_0)\mu_1, \ldots, (x - z_0)\mu_r)$.

Theorem 4.1. *Let $\boldsymbol{n} \in \mathbb{N}^r$ be normal with respect to $\boldsymbol{\mu}$. Then*

$$\mathcal{Z}[P_{\boldsymbol{n}}] = \left\{ z_0 \in \mathbb{C} : \boldsymbol{n} \text{ is not normal for } (x - z_0)\boldsymbol{\mu} \right\}. \tag{4.1}$$

Remark 4.2. The same proof shows that z_0 being a zero of $P_{\boldsymbol{n}}$ of multiplicity $\geqslant 2$ implies that $\boldsymbol{n}$ is not normal for $(x - z_0)^2\boldsymbol{\mu}$.

Remark 4.3. Suppose each $\mu_j \in \mathcal{M}(\mathbb{R})$ and let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then the same proof shows that $P_{\boldsymbol{n}}(z_0) = 0$ implies that $\boldsymbol{n}$ is not normal for $|x - z_0|^2\boldsymbol{\mu}$.

Proof. If $P_{\boldsymbol{n}}(z_0) = 0$, then $Q(x) = P_{\boldsymbol{n}}(x)/(x - z_0)$ satisfies every orthogonality relation for the index $\boldsymbol{n}$ with respect to $(x - z_0)\boldsymbol{\mu}$. Since $\deg Q < |\boldsymbol{n}|$, we obtain that $\boldsymbol{n}$ is not normal for $(x - z_0)\boldsymbol{\mu}$ by Remark 3.1.

Conversely, if $\boldsymbol{n}$ is not normal for $(x - z_0)\boldsymbol{\mu}$, then by Remark 3.1 there is some monic polynomial $Q \neq 0$ with $\deg Q < |\boldsymbol{n}|$ and

$$\int Q(x) x^k (x - z_0) d\mu_j(x) = 0, \qquad k = 0, \ldots, n_j - 1. \tag{4.2}$$

Since $\boldsymbol{n}$ is normal, we then have $P_{\boldsymbol{n}}(x) = Q(x)(x - z_0)$, so $P_{\boldsymbol{n}}(z_0) = 0$. □

We now illustrate how this approach leads to zero location for type II polynomials for AT, Nikishin, and Angelesco systems. These results are, of course, well known.

Corollary 4.4. *Let $\boldsymbol{\mu}$ be an AT system on a closed interval Γ for an index $\boldsymbol{n} \in \mathbb{N}^r$. Then $\mathcal{Z}[P_{\boldsymbol{n}}] \subset \mathring{\Gamma}$, and each zero is simple.*

Proof. Let $z_0 \in \mathbb{R} \setminus \mathring{\Gamma}$. $\boldsymbol{\mu}$ being an AT system on Γ at $\boldsymbol{n}$ implies that so is $(x - z_0)\boldsymbol{\mu}$: just replace the reference measure μ in Definition 3.5 with $(x - z_0)\mu$. By Theorems 3.6 and 4.1, we get that z_0 is not a zero of $P_{\boldsymbol{n}}$.

Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. It is easy to see that $|x - z_0|^2\boldsymbol{\mu}$ is an AT system on Γ for $\boldsymbol{n}$. By Theorem 3.6 and Remark 4.3, we get that z_0 is not a zero of $P_{\boldsymbol{n}}$.

That each zero must be simple follows from Remark 4.2 and AT property of $(x - z_0)^2\boldsymbol{\mu}$ on Γ for $\boldsymbol{n}$ for any $z_0 \in \mathbb{R}$. □

Corollary 4.5. *Let $\boldsymbol{\mu}$ be a Nikishin system. Then $\mathcal{Z}[P_{\boldsymbol{n}}] \subset \mathring{\Gamma}_1$, and each zero is simple.*

Proof. If $z_0 \notin \mathring{\Gamma}_1$, then $(x - z_0)\boldsymbol{\mu}$ is a Nikishin system which is perfect. □

If $\boldsymbol{\mu}$ is an Angelesco system, then both $(x - z_0)\boldsymbol{\mu}$ (for $z_0 \notin \mathbb{R} \setminus \bigcup_{j=1}^{r} \mathring{\Gamma}_j$) and $|x - z_0|^2\boldsymbol{\mu}$ (for $z_0 \in \mathbb{C} \setminus \mathbb{R}$) are also Angelesco systems. Therefore, by applying the exact same argument as in the case of AT systems, we obtain $\mathcal{Z}[P_{\boldsymbol{n}}] \subset \bigcup_{j=1}^{r} \mathring{\Gamma}_j$, and all zeros are simple. This statement can be sharpened using the following well-known result. We give a slight variation of the standard proof, based on the notion of the Christoffel transform.

Proposition 4.6. *Let Γ be an interval in $\mathbb{R}$ and $\mu \in \mathcal{M}(\Gamma)$. If a polynomial $P(x)$ satisfies*

$$\int P(x)x^k d\mu(x) = 0, \qquad k = 0, \dots, n-1, \tag{4.3}$$

then it has at least n distinct zeros on $\mathring{\Gamma}$.

Proof. Let $Q(x)$ be the monic polynomial whose zeros are precisely the zeros of P of odd multiplicity that lie in $\mathring{\Gamma}$, each counted only once. Then the Christoffel transform $\widehat{\mu} := \frac{P(x)}{Q(x)}\mu$ belongs to $\mathcal{M}(\Gamma)$, since all zeros of P/Q in $\mathring{\Gamma}$ occur with even multiplicity. Consequently, $\widehat{\mu}$ admits a unique orthogonal polynomial of degree n. But (4.3) implies

$$\int Q(x)x^k d\widehat{\mu}(x) = 0, \qquad k = 0, \dots, n-1. \tag{4.4}$$

The n-th orthogonal polynomial of $\widehat{\mu}$ cannot have degree $< n$, which implies $\deg Q \geqslant n$. □

Corollary 4.7. *Let $\boldsymbol{\mu}$ be an Angelesco system and $\boldsymbol{n} \in \mathbb{N}^r$. Then $P_{\boldsymbol{n}}$ has exactly n_j zeros on each $\mathring{\Gamma}_j$, and each zero is simple.*

Proof. Since $\boldsymbol{n}$ is normal, we have $\deg P_{\boldsymbol{n}} = |\boldsymbol{n}|$. By Proposition 4.6, $P_{\boldsymbol{n}}$ has at least n_j distinct zeros on each $\mathring{\Gamma}_j$. Since $\mathring{\Gamma}_j$'s are pairwise disjoint, we obtain the statement. □

4.2 | Zero location for type I multiple orthogonal polynomials

The main result of this section provides a criterion for the zero locations of the type I polynomials $A_{\boldsymbol{n}}^{(m)}$. For simplicity, we state it for $m = 1$ only, though it holds for any $m = 1, \dots, r$. The general case of Theorem 4.8 and Remarks 4.9 and 4.10 is obtained by replacing $A_{\boldsymbol{n}}^{(1)}$ with $A_{\boldsymbol{n}}^{(m)}$, $\boldsymbol{e}_1$ with

$\boldsymbol{e}_m$, and applying the Christoffel transforms to the m-th component of $\boldsymbol{\mu}$ instead of the first ($m = 1, \dots, r$).

Theorem 4.8. *Let $\boldsymbol{n} \in \mathbb{N}^r$ with $n_1 > 0$ be normal with respect to $\boldsymbol{\mu}$. Then*

$$\mathcal{Z}[A_{\boldsymbol{n}}^{(1)}] = \{z_0 \in \mathbb{C} : \boldsymbol{n} - \boldsymbol{e}_1 \text{ is not normal for } ((x - z_0)\mu_1, \mu_2, \dots, \mu_r)\}.$$

Remark 4.9. The same proof shows that if z_0 is a zero of $A_{\boldsymbol{n}}^{(1)}$ of multiplicity $\geqslant 2$, then $\boldsymbol{n} - 2\boldsymbol{e}_1$ and $\boldsymbol{n} - \boldsymbol{e}_1$ are not normal for $((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)$.

Remark 4.10. Suppose each $\mu_j \in \mathcal{M}(\mathbb{R})$ and let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then the same proof shows that $A_{\boldsymbol{n}}^{(1)}(z_0) = 0$ implies that $\boldsymbol{n} - 2\boldsymbol{e}_1$ and $\boldsymbol{n} - \boldsymbol{e}_1$ are not normal for $(|x - z_0|^2\mu_1, \mu_2, \dots, \mu_r)$.

Proof. Let us denote $((x - z_0)\mu_1, \mu_2, \dots, \mu_r)$ by $\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_1, \dots, \widehat{\mu}_r)$. Since $\boldsymbol{n}$ is $\boldsymbol{\mu}$-normal, we have

$$\sum_{j=1}^{r} \int A_{\boldsymbol{n}}^{(j)}(x)x^k d\mu_j(x) = 0, \qquad k = 0, \dots, |\boldsymbol{n}| - 2. \tag{4.5}$$

If $A_{\boldsymbol{n}}^{(1)}(z_0) = 0$, then let $\boldsymbol{B}(x)$ be $(A_{\boldsymbol{n}}^{(1)}(x)/(x - z_0), A_{\boldsymbol{n}}^{(2)}(x), \dots, A_{\boldsymbol{n}}^{(r)}(x))$. By (4.5),

$$\sum_{j=1}^{r} \int B^{(j)}(x)x^k d\widehat{\mu}_j(x) = 0, \qquad k = 0, \dots, |\boldsymbol{n}| - 2. \tag{4.6}$$

Since $\deg B^{(1)} \leqslant n_1 - 2$ and $\deg B^{(j)} \leqslant n_j - 1$ for $j = 2, \dots, r$, Remark 3.1 shows that $\boldsymbol{n} - \boldsymbol{e}_1$ is not $\widehat{\boldsymbol{\mu}}$-normal.

Conversely, if $\boldsymbol{n} - \boldsymbol{e}_1$ is not normal for $\widehat{\boldsymbol{\mu}}$, then by Remark 3.1 there is a vector $\boldsymbol{B} = (B^{(1)}, \dots, B^{(r)}) \neq \boldsymbol{0}$ with $\deg B^{(1)} \leqslant n_1 - 2$ and $\deg B^{(j)} \leqslant n_j - 1$ for $j = 2, \dots, r$, that satisfies (4.6). Then $\boldsymbol{A}_{\boldsymbol{n}}(x)$ must be equal to $(B^{(1)}(x)(x - z_0), B^{(2)}(x), \dots, B^{(r)}(x))$, up to a multiplicative normalization, so $A_{\boldsymbol{n}}^{(1)}(z_0) = 0$.

As for the first remark, assume that $A_{\boldsymbol{n}}^{(1)}(z_0) = 0$ of multiplicity $\geqslant 2$. Denote $((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)$ by $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_r)$ and let $\boldsymbol{B}(x)$ be the system $(A_{\boldsymbol{n}}^{(1)}(x)/(x - z_0)^2, A_{\boldsymbol{n}}^{(2)}(x), \dots, A_{\boldsymbol{n}}^{(r)}(x))$. By (4.5),

$$\sum_{j=1}^{r} \int B^{(j)}(x)x^k d\tilde{\mu}_j(x) = 0, \qquad k = 0, \dots, |n| - 2. \tag{4.7}$$

Since $\deg B^{(1)} \leqslant n_1 - 3$ and $\deg B^{(j)} \leqslant n_j - 1$ for $j = 2, \dots, r$, Remark 3.1 shows that both $\boldsymbol{n} - 2\boldsymbol{e}_1$ and $\boldsymbol{n} - \boldsymbol{e}_1$ are not $\tilde{\boldsymbol{\mu}}$-normal. The same argument applies for the second remark, just with $(x - z_0)^2\mu_1$ replaced by $(x - z_0)(x - \bar{z}_0)\mu_1 = |x - z_0|^2\mu_1$. □

We now apply this criterion to the Angelesco systems (this result is well known). Application to the Nikishin systems is in the next section.

Corollary 4.11. *Let $\boldsymbol{\mu}$ be an Angelesco system. Then $\mathcal{Z}[A_{\boldsymbol{n}}^{(j)}] \subset \mathring{\Gamma}_j$ for any $j = 1, \dots, r$ and any $\boldsymbol{n} \in \mathbb{N}^r$ with $n_j > 0$, and each zero is simple.*

Proof. For any j and any $z_0 \in \mathbb{R} \setminus \mathring{\Gamma}_j$, replacing μ_j with $(x - z_0)\mu_j$ leads to another Angelesco system. By Theorems 3.3 and 4.8, we get $A_{\boldsymbol{n}}^{(j)}(z_0) \neq 0$.

Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$ be arbitrary. Then Remark 4.10 together with the Angelesco property of the system $(|x - z_0|^2\mu_1, \mu_2, \dots, \mu_r)$ shows that $A_{\boldsymbol{n}}^{(j)}(z_0) \neq 0$.

Finally, simplicity of zeros follows from Remark 4.9 and the Angelesco property of the system $((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)$ for $z_0 \in \mathbb{R}$. □

4.3 | Zero location for type I multiple orthogonal polynomials in Nikishin systems

Note that if $\boldsymbol{\mu}$ is an AT system on Γ, then replacing μ_1 with $(x - z_0)\mu_1$ with $z_0 \in \mathbb{R} \setminus \Gamma$ may destroy the AT property. Therefore, one should not expect general results about the zeros of type I multiple polynomials for AT systems. This should not be surprising, since such polynomials need not even be real-rooted in general. We can say something about zeros of type I polynomials for Nikishin system, however. For simplicity, we consider the case $r = 2$ (see Remark 4.16 below). We employ the following simple result.

Lemma 4.12. *Let* $\boldsymbol{\mu} := (\mu_1, \mu_2)$ *and* $\widetilde{\boldsymbol{\mu}} := (\mu_1, \widetilde{\mu}_2)$, *where*

$$d\widetilde{\mu}_2(x) = \alpha\, d\mu_2(x) + Q(x)d\mu_1(x), \quad Q(x) = \sum_{j=0}^{s} k_j x^j, \quad \alpha \neq 0. \tag{4.8}$$

Then for any $\boldsymbol{n} = (n_1, n_2) \in \mathbb{N}^2$ *with* $n_2 \leqslant n_1 - s$, $\boldsymbol{n}$ *is normal with respect to* $\boldsymbol{\mu}$ *if and only if it is normal with respect to* $\widetilde{\boldsymbol{\mu}}$. *In this case,*

$$\widetilde{P}_{\boldsymbol{n}}(x) = P_{\boldsymbol{n}}(x), \tag{4.9}$$

$$\widetilde{A}_{\boldsymbol{n}}^{(1)}(x) = A_{\boldsymbol{n}}^{(1)}(x) - \tfrac{1}{\alpha}Q(x)A_{\boldsymbol{n}}^{(2)}(x), \tag{4.10}$$

$$\widetilde{A}_{\boldsymbol{n}}^{(2)}(x) = \tfrac{1}{\alpha}A_{\boldsymbol{n}}^{(2)}(x), \tag{4.11}$$

where $\widetilde{P}_{\boldsymbol{n}}$ *are the type II and* $\widetilde{\boldsymbol{A}}_{\boldsymbol{n}}$ *are the type I polynomials of* $\widetilde{\boldsymbol{\mu}}$.

Proof. Observe that $H^{(1)}_{n_1,|\boldsymbol{n}|}$ block is common for both $H_{\boldsymbol{n}}[\boldsymbol{\mu}]$ and $H_{\boldsymbol{n}}[\widetilde{\boldsymbol{\mu}}]$. Now note that the k-th row of $\widetilde{H}^{(2)}_{n_2,|\boldsymbol{n}|}$ $(k = 1, \dots, n_2)$ is

$$\left(\widetilde{c}^{(2)}_{k-1}, \widetilde{c}^{(2)}_{k}, \dots, \widetilde{c}^{(2)}_{k-2+|\boldsymbol{n}|}\right) = \alpha\left(c^{(2)}_{k-1}, c^{(2)}_{k}, \dots, c^{(2)}_{k-2+|\boldsymbol{n}|}\right) + \sum_{j=0}^{s} k_j\left(c^{(1)}_{k+j-1}, c^{(1)}_{k+j}, \dots, c^{(1)}_{k+j-2+|\boldsymbol{n}|}\right). \tag{4.12}$$

If $n_2 \leqslant n_1 - s$, each row of moments in the last sum can be canceled in $\det H_{\boldsymbol{n}}[\widetilde{\boldsymbol{\mu}}]$ by subtracting a multiple of the corresponding row of $H^{(1)}_{n_1,|\boldsymbol{n}|}$ (use $n_2 \leqslant n_1 - s$). This shows that $\det H_{\boldsymbol{n}}[\widetilde{\boldsymbol{\mu}}]$ is equal to $\alpha^{n_2} \det H_{\boldsymbol{n}}[\boldsymbol{\mu}]$, which proves the first statement.

Now we write

$$\int \widetilde{A}_{\boldsymbol{n}}^{(1)}(x)x^p d\mu_1(x) + \int \widetilde{A}_{\boldsymbol{n}}^{(2)}(x)x^p d\widetilde{\mu}_2(x) \tag{4.13}$$

$$= \int \left(\widetilde{A}_{\boldsymbol{n}}^{(1)}(x) + Q(x)\widetilde{A}_{\boldsymbol{n}}^{(2)}(x)\right)x^p d\mu_1(x) + \int \alpha\widetilde{A}_{\boldsymbol{n}}^{(2)}(x)x^p d\mu_2(x). \tag{4.14}$$

If $n_2 \leqslant n_1 - s$, then $\deg QA_{\boldsymbol{n}}^{(2)} \leqslant n_1 - 1$. Hence, by normality of $\boldsymbol{n}$ for $\boldsymbol{\mu}$, we see that $(A_{\boldsymbol{n}}^{(1)}(x), A_{\boldsymbol{n}}^{(2)}(x)) = (\widetilde{A}_{\boldsymbol{n}}^{(1)}(x) + Q(x)\widetilde{A}_{\boldsymbol{n}}^{(2)}(x), \alpha\widetilde{A}_{\boldsymbol{n}}^{(2)}(x))$ which proves (4.10) and (4.11). The proof of (4.9) is similar. □

Theorem 4.13. *Let $(\mu_1, \mu_2) = \mathcal{N}(\sigma_1, \sigma_2)$ be a Nikishin system.*

(i) *Let $\boldsymbol{n} \in \mathbb{N}^2$ with $n_1 + 1 \leqslant n_2$, $n_1 \neq 0$. Then $A_{\boldsymbol{n}}^{(1)}$ is real-rooted and*

$$\mathcal{Z}[A_{\boldsymbol{n}}^{(1)}] \subset \mathring{\Gamma}_2, \tag{4.15}$$

and each zero is simple.

(ii) *Let $\boldsymbol{n} \in \mathbb{N}^2$ with $n_1 + 1 \geqslant n_2$, $n_2 \neq 0$. Then $A_{\boldsymbol{n}}^{(2)}$ is real-rooted and*

$$\mathcal{Z}[A_{\boldsymbol{n}}^{(2)}] \subset \mathring{\Gamma}_2, \tag{4.16}$$

and each zero is simple.

Remark 4.14. Our simulations clearly show that the above statement is optimal in the sense that (4.15) does not have to hold if $n_1 + 1 > n_2$ and (4.16) does not have to hold if $n_1 + 1 < n_2$. However, real-rootedness for $A_{\boldsymbol{n}}^{(1)}$ also holds if $n_1 = n_2$, and for $A_{\boldsymbol{n}}^{(2)}$ if $n_1 = n_2 - 2$, see Remark 5.10. Beyond these locations, one typically should expect an appearance of complex zeros.

Remark 4.15. Theorem 4.13 and Remark 5.10 show that all polynomials $A_{\boldsymbol{n}}^{(j)}$ are real-rooted along the step-line indices (n, n) and $(n, n+1)$. Note, however, that this is not necessarily the case for indices $(n+1, n)$, which are also commonly referred to as step-line indices.

Proof. (ii) Let $z_0 \in \mathbb{R} \setminus \mathring{\Gamma}_2$. Note that

$$(\mu_1, (x - z_0)\mu_2) = (\mu_1, (x - z_0)m_{\sigma_2}(x)\mu_1). \tag{4.17}$$

By (2.5), $(x - z_0)m_{\sigma_2}(x) = m_{\widehat{\sigma}_2}(x) + c$, where $\widehat{\sigma}_2 = (x - z_0)\sigma_2$ and $c = -\int d\sigma_2(t) \in \mathbb{R}$, which means that (4.17) can be represented as

$$(\mu_1, m_{\widehat{\sigma}_2}(x)\mu_1 + c\mu_1). \tag{4.18}$$

Note that $\operatorname{supp}\widehat{\sigma}_2 \subseteq \operatorname{supp}\sigma_2 \subseteq \Gamma_2$, so $(\mu_1, m_{\widehat{\sigma}_2}(x)\mu_1)$ is a Nikishin system, and thus it has every index normal. Applying Lemma 4.12 with $s = 0$, we obtain that (4.18) has every (n_1, n_2) with $n_2 \leqslant n_1$ normal. Theorem 4.8 then proves that z_0 is not a zero of $A_{\boldsymbol{n}}^{(2)}$ for any (n_1, n_2) with $n_2 - 1 \leqslant n_1$.

Now let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Consider $\widetilde{\mu}_2 = |x - z_0|^2\mu_2 = (x - z_0)(x - \bar{z}_0)m_{\sigma_2}(x)\mu_1$. Denote $\widetilde{\sigma}_2 = (x - z_0)(x - \bar{z}_0)\sigma_2$. Apply (2.5) twice to get

$$(x - z_0)(x - \bar{z}_0)m_{\sigma_2}(x) = m_{\widetilde{\sigma}_2}(x) + cx + d,$$

with $c = -\int d\sigma_2(t) \in \mathbb{R}$ and $d = -2c\,\mathrm{Re}(z_0) - \int t d\sigma_2(t) \in \mathbb{R}$. Therefore,

$$(\mu_1, \widetilde{\mu}_2) = (\mu_1, m_{\widetilde{\sigma}_2}(x)\mu_1 + (cx+d)\mu_1). \tag{4.19}$$

Since $(\mu_1, \widetilde{\mu}_2)$ is a Nikishin system, it is perfect. Applying Lemma 4.8 (with $s = 1$), we obtain that (4.19) has every (n_1, n_2) with $n_2 \leqslant n_1 - 1$ normal. Then Remark 4.10 shows that z_0 is not a zero of $A_{\boldsymbol{n}}^{(2)}$ for any (n_1, n_2) with $n_2 - 2 \leqslant n_1 - 1$. This completes the proof of (4.16). To show simplicity of zeros, apply the exact same argument but with Remark 4.9 instead of Remark 4.10.

(i) We use the well-known reversal trick here. Let $\widetilde{\mu}_1 = m_{\sigma_2}(x)\mu_1$ so that $\boldsymbol{\mu}$ can be rewritten as $(m_{\sigma_2}(x)^{-1}\widetilde{\mu}_1, \widetilde{\mu}_1)$. Define the reversed system $\boldsymbol{\nu} = (\widetilde{\mu}_1, m_{\sigma_2}(x)^{-1}\widetilde{\mu}_1)$. Its type I polynomials $B^{(1)}_{n_1,n_2}$, $B^{(2)}_{n_1,n_2}$ are then given by $A^{(2)}_{n_2,n_1}$, $A^{(1)}_{n_2,n_1}$, respectively. Applying the relation (2.3), we see that

$$\boldsymbol{\nu} = (\widetilde{\mu}_1, (b-x)\widetilde{\mu}_1 - a^2 m_{\sigma_2^{(1)}}\widetilde{\mu}_1). \tag{4.20}$$

for some $a \neq 0$. Finally, let $\boldsymbol{\eta} = (\widetilde{\mu}_1, m_{\sigma_2^{(1)}}\widetilde{\mu}_1)$ and $C^{(1)}_{n_1,n_2}$, $C^{(2)}_{n_1,n_2}$ be its type I polynomials. By Lemma 4.12, $C^{(2)}_{n_1,n_2} = -\frac{1}{a^2}B^{(2)}_{n_1,n_2}$ for any $n_2 \leqslant n_1 - 1$. Since $\boldsymbol{\eta}$ is a Nikishin system ($\operatorname{supp}\sigma_2^{(1)} \subseteq \Gamma_2$ since Γ_2 is an interval), we obtain $\mathcal{Z}[C^{(2)}_{n_1,n_2}] \subset \mathring{\Gamma}_2$ for every $n_1 + 1 \geqslant n_2$ (with all zeros simple) by (ii). Combining this all together produces $\mathcal{Z}[A^{(1)}_{n_2,n_1}] \subset \mathring{\Gamma}_2$ whenever $n_2 \leqslant n_1 - 1$ which is (i). □

Remark 4.16. It is a natural question whether the conclusions of Theorem 4.13 (as well as those of Theorem 5.9 below) remain valid for Nikishin systems $(\mu_1, \dots, \mu_r)$ with $r \geqslant 3$. The current proof of part (ii) extends with only minor modifications, yielding that $A^{(r)}_{\boldsymbol{n}}$ is real-rooted with $\mathcal{Z}[A^{(r)}_{\boldsymbol{n}}] \subset \mathring{\Gamma}_r$ whenever $\boldsymbol{n} \in \mathbb{N}^r$ satisfies $n_r \leqslant \min_j\{n_j\} + 1$. It is reasonable to expect that the reversal trick used in the proof of part (i) can be adapted to obtain analogous statement for the zeros of $A^{(j)}_{\boldsymbol{n}}$, $j = 1, \dots, r-1$. Indeed, this strategy was successfully implemented by Fidalgo and López [18] in their proof of the perfectness of Nikishin systems for any r, suggesting that the same idea can be useful here as well.

5 | ZERO INTERLACING

5.1 | Zero interlacing for type II multiple orthogonal polynomials

Let $(x - z_0)^2\boldsymbol{\mu}$ stand for the system $((x-z_0)^2\mu_1, \dots, (x-z_0)^2\mu_r)$, where $(x-z_0)^2\mu_j$ is the double Christoffel transform of μ_j, see Section 2.2.

Theorem 5.1. *Assume $\boldsymbol{n}$ and $\boldsymbol{n} + \boldsymbol{e}_j$ are normal for $\boldsymbol{\mu}$. Then*

$$\mathcal{Z}[W(P_{\boldsymbol{n}+\boldsymbol{e}_j}, P_{\boldsymbol{n}})] = \left\{z_0 \in \mathbb{C} : \boldsymbol{n} \textit{ is not normal for } (x-z_0)^2\boldsymbol{\mu}\right\}. \tag{5.1}$$

In particular, if $P_{\boldsymbol{n}}$ is real-rooted, then $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}}$ if and only if $\boldsymbol{n}$ is normal with respect to $(x-z_0)^2\boldsymbol{\mu}$ for every $z_0 \in \mathbb{R}$.

Remark 5.2. The same proof shows that if $\boldsymbol{n}+\boldsymbol{e}_j$, $\boldsymbol{n}+\boldsymbol{e}_k$ and $\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k$ are normal for $\boldsymbol{\mu}$, $j \neq k$, then

$$\mathcal{Z}[W(P_{\boldsymbol{n}+\boldsymbol{e}_j}, P_{\boldsymbol{n}+\boldsymbol{e}_k})] = \{z_0 \in \mathbb{C} : \boldsymbol{n} \text{ is not normal for } (x-z_0)^2\boldsymbol{\mu}\}. \tag{5.2}$$

In particular, if $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ is real-rooted, then $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}+\boldsymbol{e}_k}$ if and only if $\boldsymbol{n}$ is normal with respect to $(x-z_0)^2\boldsymbol{\mu}$ for every $z_0 \in \mathbb{R}$.

Proof. Note that $W(P_{\boldsymbol{n}+\boldsymbol{e}_j}, P_{\boldsymbol{n}}; z_0) = 0$ if and only if the system of equations

$$\begin{cases} aP_{\boldsymbol{n}+\boldsymbol{e}_j}(z_0) + bP_{\boldsymbol{n}}(z_0) = 0 \\ aP'_{\boldsymbol{n}+\boldsymbol{e}_j}(z_0) + bP'_{\boldsymbol{n}}(z_0) = 0 \end{cases} \tag{5.3}$$

holds for some $(a,b) \neq (0,0)$. This is equivalent to $aP_{\boldsymbol{n}+\boldsymbol{e}_j}(x) + bP_{\boldsymbol{n}}(x) = (x-z_0)^2 Q(x)$ for some polynomial $Q \neq 0$. Such a Q would satisfy

$$\int Q(x)x^p(x-z_0)^2 d\mu_j(x) = 0, \qquad p = 0, \dots, n_j - 1, \tag{5.4}$$

but $\deg Q < |\boldsymbol{n}|$, which contradicts to the normality of $\boldsymbol{n}$ for $(x-z_0)^2\boldsymbol{\mu}$.

Conversely, if $\boldsymbol{n}$ is not normal for $(x-z_0)^2\boldsymbol{\mu}$, then there is some $Q \neq 0$ with $\deg Q < |\boldsymbol{n}|$ satisfying (5.4). Pick a such that $(x-z_0)^2Q(x) - aP_{\boldsymbol{n}+\boldsymbol{e}_j}$ has at most degree $|\boldsymbol{n}|$. By comparing orthogonality relations, we find that $(x-z_0)^2Q(x) - aP_{\boldsymbol{n}+\boldsymbol{e}_j} = bP_{\boldsymbol{n}}$, due to normality of $\boldsymbol{n}$ with respect to $\boldsymbol{\mu}$. Since $Q \neq 0$, we must have $(a,b) \neq (0,0)$, so $W(P_{\boldsymbol{n}+\boldsymbol{e}_j}, P_{\boldsymbol{n}}; z_0) = 0$. This concludes the proof, if we note that the interlacing follows from Lemma 2.2.

For Remark 5.2, the same proof works if we note that $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_k}$ are linearly independent when $\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k$ is normal (the converse also holds). To see this, assume $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ is a nonzero multiple of $P_{\boldsymbol{n}+\boldsymbol{e}_k}$ or vice versa. Then $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_k}$ satisfy all the orthogonality relations for the index $\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k$, but $\deg P_{\boldsymbol{n}+\boldsymbol{e}_j} < |\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k|$, so $\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k$ cannot be normal. Note that the linear independence of $P_{\boldsymbol{n}}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ was immediate since they have different degrees. □

We now apply this criterion to provide a streamlined approach to type II interlacing for Angelesco systems (originally due to [3, 22]), AT systems ([16, 22, 26]), and Nikishin systems ([16]).

Corollary 5.3. *Let $\boldsymbol{\mu}$ be an Angelesco system. Then $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}+\boldsymbol{e}_k}$ for any $\boldsymbol{n} \in \mathbb{N}^r$, any $j = 1, \dots, r$, and any $k \neq j$.*

Proof. By Corollary 4.7, all polynomials $P_{\boldsymbol{n}}$ are real-rooted. $(x-z_0)^2\boldsymbol{\mu}$ is an Angelesco system for any $z_0 \in \mathbb{R}$, so Theorem 5.1 applies, along with Remark 5.2. □

Corollary 5.4. *Let $\boldsymbol{\mu}$ be an AT system on Γ for $\boldsymbol{n}$ and $\boldsymbol{n}+\boldsymbol{e}_j$. Then zeros of $P_{\boldsymbol{n}}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_j}$ interlace. If $\boldsymbol{\mu}$ is AT on Γ for $\boldsymbol{n}+\boldsymbol{e}_j$, $\boldsymbol{n}+\boldsymbol{e}_k$ and $\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k$, then $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}+\boldsymbol{e}_k}$ for any $j \neq k$.*

Proof. By Corollary 4.4, all polynomials $P_{\boldsymbol{n}}$ are real-rooted. If $\boldsymbol{\mu}$ is an AT system for $\boldsymbol{k}$, then so is $(x-z_0)^2\boldsymbol{\mu}$ for any $z_0 \in \mathbb{R}$ (just replace the reference measure μ, see Definition 3.5, with $(x-z_0)^2\mu$), so Theorem 5.1 applies. □

Corollary 5.5. *Let $\boldsymbol{\mu}$ be a Nikishin system. Then $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_j} \sim P_{\boldsymbol{n}+\boldsymbol{e}_k}$ for any $\boldsymbol{n} \in \mathbb{N}^r$, any $j = 1, \dots, r$, and any $k \neq j$.*

Proof. By Corollary 4.5, all polynomials $P_{\boldsymbol{n}}$ are real-rooted. If $\boldsymbol{\mu}$ is a Nikishin system, then so is $(x - z_0)^2\boldsymbol{\mu}$ for any $z_0 \in \mathbb{R}$, so Theorem 5.1 applies. □

5.2 | Zero interlacing for type I multiple orthogonal polynomials

The main result of this section provides a criterion for the zero interlacing of neighboring type I polynomials $A_{\boldsymbol{n}}^{(m)}$. Just like in Section 4.2, we state it for $m = 1$ only, though it holds for any $m = 1, \dots, r$. The general case of Theorem 5.6 and Remark 5.7 is obtained by replacing $A_{\boldsymbol{n}}^{(1)}$ with $A_{\boldsymbol{n}}^{(m)}$, $\boldsymbol{e}_1$ with $\boldsymbol{e}_m$, and applying the Christoffel transform to the m-th component of $\boldsymbol{\mu}$ instead of the first ($m = 1, \dots, r$).

Theorem 5.6. *Assume that $\boldsymbol{n}$, $\boldsymbol{n} - \boldsymbol{e}_1$ are normal for $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_r)$. For any $\ell \in \{1, \dots, r\}$ such that $\boldsymbol{n} - \boldsymbol{e}_\ell$ is normal, we have*

$$\mathcal{Z}[W(A_{\boldsymbol{n}}^{(1)}, A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)})] = \left\{z_0 \in \mathbb{C} : \boldsymbol{n} - 2\boldsymbol{e}_1 \text{ is not normal for } ((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)\right\}. \tag{5.5}$$

If $A_{\boldsymbol{n}}^{(1)}$ is real-rooted, then $A_{\boldsymbol{n}}^{(1)} \sim A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)}$ if and only if $\boldsymbol{n} - 2\boldsymbol{e}_1$ is normal for the system $((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)$ for every $z_0 \in \mathbb{R}$.

Remark 5.7. Assuming additionally the $\boldsymbol{\mu}$-normality of $\boldsymbol{n} - \boldsymbol{e}_k$ and $\boldsymbol{n} - \boldsymbol{e}_k - \boldsymbol{e}_\ell$, with $k \neq \ell$, then $A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)} \sim A_{\boldsymbol{n}-\boldsymbol{e}_k}^{(1)}$ if and only if $\boldsymbol{n} - 2\boldsymbol{e}_1$ is normal for $((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)$ for every $z_0 \in \mathbb{R}$. The proof is almost identical.

Proof. Fix any $z_0 \in \mathbb{C}$ and denote $\widetilde{\boldsymbol{\mu}}$ to be $((x - z_0)^2\mu_1, \mu_2, \dots, \mu_r)$. Note that $W(A_{\boldsymbol{n}}^{(1)}, A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)}; z_0) = 0$ if and only if $aA_{\boldsymbol{n}}^{(1)}(x) + bA_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)}(x) = (x - z_0)^2B(x)$ for some $(a, b) \neq (0, 0)$ and some polynomial B with $\deg B \leqslant n_1 - 3$. Note that $\boldsymbol{A}_{\boldsymbol{n}}$ and $\boldsymbol{A}_{\boldsymbol{n}-\boldsymbol{e}_\ell}$ are linearly independent, since

$$\sum_{j=1}^{r} \int A_{\boldsymbol{n}}^{(j)}(x)x^{|\boldsymbol{n}|-2}d\mu_j(x) = 0 \neq \sum_{j=1}^{r} \int A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(j)}(x)x^{|\boldsymbol{n}|-2}d\mu_j(x). \tag{5.6}$$

Then the vector $(B, aA_{\boldsymbol{n}}^{(2)} + bA_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(2)}, \dots, aA_{\boldsymbol{n}}^{(r)} + bA_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(r)})$ is nonzero and satisfies all the degree and orthogonality conditions for the index $\boldsymbol{n} - 2\boldsymbol{e}_1$ with respect to $\widetilde{\boldsymbol{\mu}}$. Since we also have

$$\sum_{j=1}^{r} \int (aA_{\boldsymbol{n}}^{(j)}(x) + bA_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(j)}(x))x^{|\boldsymbol{n}|-3}d\mu_j(x) = 0, \tag{5.7}$$

we see that $\boldsymbol{n} - 2\boldsymbol{e}_1$ is not normal for $\widetilde{\boldsymbol{\mu}}$ by Remark 3.1.

Conversely, if $\boldsymbol{n} - 2\boldsymbol{e}_1$ is not normal for $\widetilde{\boldsymbol{\mu}}$, then there is some vector $\boldsymbol{B} = (B^{(1)}, B^{(2)}, \dots, B^{(r)}) \neq \boldsymbol{0}$ with $\deg B^{(1)} \leqslant n_1 - 3$ and $\deg B^{(j)} \leqslant n_j - 1$ for $j = 2, \dots, r$, such that

$$\sum_{j=1}^{r} \int B^{(j)}(x)x^p d\widetilde{\mu}_j(x) = 0, \qquad p = 0, \dots, |\boldsymbol{n}| - 3. \tag{5.8}$$

Pick a such that $((x-z_0)^2B^{(1)}(x), B^{(2)}(x)\dots, B^{(r)}(x)) - a\boldsymbol{A}_{\boldsymbol{n}-\boldsymbol{e}_\ell}(x)$ satisfies the same degree and orthogonality conditions as $\boldsymbol{A}_{\boldsymbol{n}}$ (with the correct choice of a we get the one missing orthogonality condition). This shows that $aA_{\boldsymbol{n}}^{(1)}(x) + bA_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)}(x) = (x-z_0)^2B^{(1)}(x)$ for some $(a,b)\neq(0,0)$ and completes the proof of (5.5).

The statement on interlacing then follows from Lemma 2.2 since normality of $\boldsymbol{n}-\boldsymbol{e}_1$ gives us $\deg A_{\boldsymbol{n}}^{(1)} = n_1 - 1$ (use [23, Corollary 23.1.1]), and $\deg A_{\boldsymbol{n}-\boldsymbol{e}_\ell}^{(1)} \leqslant n_1 - 1 \leqslant \deg A_{\boldsymbol{n}}^{(1)} + 1$. □

Corollary 5.8 [13, 21]. *Let $\boldsymbol{\mu}$ be an Angelesco system. Then $A_{\boldsymbol{n}}^{(i)} \sim A_{\boldsymbol{n}+\boldsymbol{e}_j}^{(i)}$ and $A_{\boldsymbol{n}+\boldsymbol{e}_j}^{(i)} \sim A_{\boldsymbol{n}+\boldsymbol{e}_k}^{(i)}$, assuming $n_i \geqslant 1$ and $j \neq k$.*

Proof. By Corollary 4.11, all polynomials $A_{\boldsymbol{n}}^{(i)}$ are real-rooted. For any $z_0 \in \mathbb{R}$, $((x-z_0)^2\mu_1, \mu_2, \dots, \mu_r)$ is an Angelesco system, so Theorem 5.6 and Remark 5.7 apply. □

For AT systems, real-rootedness of type I polynomials does not have to hold, so one cannot expect interlacing, of course. For Nikishin systems, Theorem 5.6 can be used, however.

Theorem 5.9. *Let $(\mu_1, \mu_2) = \mathcal{N}(\sigma_1, \sigma_2)$ be a Nikishin system. Then*

(i) *$A_{\boldsymbol{n}}^{(1)}$, $A_{\boldsymbol{n}-\boldsymbol{e}_1}^{(1)}$, and $A_{\boldsymbol{n}-\boldsymbol{e}_2}^{(1)}$ are pairwise interlacing if $n_1 + 1 \leqslant n_2$.*
(ii) *$A_{\boldsymbol{n}}^{(2)}$, $A_{\boldsymbol{n}-\boldsymbol{e}_1}^{(2)}$, and $A_{\boldsymbol{n}-\boldsymbol{e}_2}^{(2)}$ are pairwise interlacing if $n_1 + 1 \geqslant n_2$.*

Remark 5.10. In particular, this shows that the polynomials $A_{\boldsymbol{n}}^{(1)}$ with $n_1 = n_2$ and $A_{\boldsymbol{n}}^{(2)}$ with $n_1 + 2 = n_2$ are real-rooted, which improves the statement from Theorem 4.13 (see Remark 4.14). Interlacing also shows that all, except potentially one, of the real roots of each of these polynomials belong to $\mathring{\Gamma}_2$.

Proof. Recall that $A_{\boldsymbol{n}}^{(2)}$ is real-rooted for any $\boldsymbol{n} \in \mathbb{N}^2$ with $n_1 + 1 \geqslant n_2 > 0$ by Theorem 4.13(ii). As in the proof of that theorem, for any $z_0 \in \mathbb{R}$ we get

$$(\mu_1, (x-z_0)^2\mu_2) = (\mu_1, m_{\widetilde{\sigma}_2}(x)\mu_1 + Q(x)\mu_1), \tag{5.9}$$

with $\operatorname{supp}\widetilde{\sigma}_2 \subseteq \operatorname{supp}\sigma_2$, $\deg Q \leqslant 1$. Lemma 4.12 implies normality of this system for indices with $n_2 \leqslant n_1 - 1$. Then $\boldsymbol{n} - 2\boldsymbol{e}_2$ is normal if $n_2 - 2 \leqslant n_1 - 1$. This together with Theorem 5.6 and Remark 5.7 proves (ii).

(i) follows from the same argument as in the proof of Theorem 4.13(i). □

6 | WRONSKIANS OF HIGHER ORDER

It should come as no surprise that normality of higher-degree Christoffel transforms is linked to the zeros of higher-order Wronskians. Fix an "increasing path" of indices $\{\boldsymbol{n}_s\}_{s=1}^{\ell}$ with $\ell \in \{1, 2, \dots\}$, $\boldsymbol{n}_s \in \mathbb{N}^r$, so that

$$\boldsymbol{n}_{s+1} = \boldsymbol{n}_s + \boldsymbol{e}_{j_s}$$

for every $s \in \{1, 2, \dots, \ell - 1\}$, and for some $j_s \in \{1, \dots, r\}$.

Theorem 6.1. *Suppose each multi-index in* $\{\boldsymbol{n}_s\}_{s=1}^{\ell}$ *is* $\boldsymbol{\mu}$*-normal. Then*

$$\mathcal{Z}[W(P_{\boldsymbol{n}_1},\dots,P_{\boldsymbol{n}_\ell};x)] = \left\{z_0 \in \mathbb{C} : \boldsymbol{n}_1 \text{ is not normal for } (x-z_0)^{\ell}\boldsymbol{\mu}\right\}.$$

We skip the proof since it follows the same steps as the proof of Theorem 5.1. This result immediately gives us the following corollary.

Corollary 6.2. *Let* $\boldsymbol{\mu}$ *be an Angelesco, AT, or Nikishin system, and* ℓ *be even. Then* $W(P_{\boldsymbol{n}_1},\dots,P_{\boldsymbol{n}_\ell};x)$ *has no real zeros.*

For AT systems this was shown by Zhang and Filipuk in [44], and for $r=1$ this goes back to the classical result of Karlin and Szegő [24, Theorem 1].

A similar result can be stated for type I polynomials as follows.

Theorem 6.3. *Let each index in* $\{\boldsymbol{n}_s\}_{s=1}^{\ell}$ *be* $\boldsymbol{\mu}$*-normal,* $(\boldsymbol{n}_1)_1 \geqslant 2$*. Then*

$$\mathcal{Z}[W(A^{(1)}_{\boldsymbol{n}_1},\dots,A^{(1)}_{\boldsymbol{n}_\ell};x)] = \left\{z_0 \in \mathbb{C} : \boldsymbol{n}_1 - \boldsymbol{e}_1 \text{ is not normal for } ((x-z_0)^{\ell}\mu_1,\mu_2,\dots,\mu_r)\right\}. \tag{6.1}$$

Analogous statements hold for $A^{(j)}_{\boldsymbol{k}}$ for $j=2,\dots,r$, if we instead transform the j-th measure and replace $\boldsymbol{e}_1$ with $\boldsymbol{e}_j$.

Corollary 6.4. *Let* $\boldsymbol{\mu}$ *be an Angelesco system, and* ℓ *be even. Then, for any* $j=1,\dots,r$*, the Wronskian* $W(A^{(j)}_{\boldsymbol{n}_1},\dots,A^{(j)}_{\boldsymbol{n}_\ell};x)$ *has no real zeros.*

One can show a similar statement for Nikishin systems for indices in certain cones, similarly as in Theorem 5.9.

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