



Existence and uniqueness of the solutions to convection–diffusion equations

ALIREZA ATAEI

Abstract. In this work, we study convection–diffusion equations in the cases of bounded drifts and drifts induced by the gradient of a potential. We define a new notion of solution and prove its existence and uniqueness. Furthermore, we show the conservation of mass, the convergence to the initial data, and the strong maximum principle.

1. Introduction and the statements of the results

Convection–diffusion equations appear in many parts of science describing several phenomena, see [17, 20, 33, 36, 38, 39]. The general prototype of convection–diffusion equations is

$$\partial_t u - \operatorname{div} A \nabla u + \operatorname{div}(v u) = R,$$

with initial data g , where $g \in L^1(\mathbb{R}^n)$, $R \in L^1([0, T] \times \mathbb{R}^n)$, $T > 0$, $A = (a_{ij})_{i,j=1}^n$ is an elliptic matrix-valued function, see (2.2) for more detail, and v is either bounded or is of the form $v = -\nabla V$ where $e^V + e^{-V} \in L^1_{\text{loc}}(\mathbb{R}^n)$ and e^{-V} is a regular weight, which means that for every function $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{H^1_{e^{-V}}(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} (|f|^2 + |\nabla f|^2) e^{-V} \, dx < \infty,$$

there exists a sequence $f_i \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{i \rightarrow \infty} \|f_i - f\|_{H^1_{e^{-V}}} = 0.$$

The existence and uniqueness of solutions for such equations are not yet fully understood.

Mathematics Subject Classification: Primary: 35K10, 35K20; Secondary: 28A33

Keywords: Convection–diffusion equations, Regular and Muckenhoupt weights, Strong maximum principle, Conservation of mass.

There are several ways to define solutions if the drift v is not smooth, namely very weak solutions, weak solutions, entropy solutions, and renormalized solutions. The weak solution $u \in L^1([0, T]; \mathbf{W}_{\text{loc}}^{1,1}(\mathbb{R}^n))$ satisfies

$$\int_0^T \int_{\mathbb{R}^n} -u \partial_t \phi + A \nabla u \cdot \nabla \phi - u v \cdot \nabla \phi \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx, \quad (1.1)$$

for every $\phi \in C_0^\infty([0, T] \times \mathbb{R}^n)$ with $\phi(T, x) = 0$ for every $x \in \mathbb{R}^n$. Moreover, assuming that A satisfies

$$\sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_i} \in L^\infty([0, T] \times \mathbb{R}^n), \quad (1.2)$$

for all $1 \leq j \leq n$, then a very weak solution $u \in L^1([0, T]; L_{\text{loc}}^1(\mathbb{R}^n))$ is defined by the property

$$\int_0^T \int_{\mathbb{R}^n} u (-\partial_t \phi + \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi) \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx. \quad (1.3)$$

for every $\phi \in C_0^\infty([0, T] \times \mathbb{R}^n)$ with $\phi(T, x) = 0$ for every $x \in \mathbb{R}^n$. If the Aronson–Serrin condition holds, see [2], then one can use regularity estimates in [2] to derive the existence of weak solutions. In the case that v and $\operatorname{div} v$ are bounded, $R = 0$, and (1.2), see [19], it is proved that a unique very weak solution exists, see also [5, Thm. 6.6.2] for the case of replacing $\operatorname{div} A \nabla u$ with $\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (a_{ij} u)$ which does not require boundedness condition on $\operatorname{div} v$. In general case, however, there are some restrictions to the notion of weak and very weak solutions. First, it makes sense only if the product $u v$ is locally integrable, which is unclear if $v = -\nabla V$ where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ does not belong to a Sobolev space. The second and perhaps more crucial issue is uniqueness. If one removes the assumption of boundedness for v or (1.2), then there are counterexamples of uniqueness. For example, in [28, Thm. 1.2], it is proved that there are L^p -bounded drifts v satisfying $\operatorname{div} v = 0$ with infinitely many weak solutions satisfying (1.1) in the periodic case, replacing \mathbb{R}^n with \mathbb{T}^n . In the case of $R = 0$, $A = I$, and smooth drifts v , it is necessary to have a special growth bound on v , see [5, Ch. 9]; otherwise, there are infinitely many very weak solutions for any initial data which is a probability density function, see [6, Thm. 1]. Moreover, in Example 3.4, motivated by [32, 35], we prove the lack of uniqueness if (1.2) does not hold. To resolve the issues of existence and uniqueness, the notion of entropy solutions was introduced in [8, 32] for nonlinear parabolic equations, see also [3] for nonlinear elliptic equations. The advantages are that the entropy solutions exist even for square-integrable drifts, and whenever $v = 0$ they are unique. However, if the drift is nonzero, the uniqueness and the connection to weak solutions are not clear. Finally, the notion of the renormalized solutions was defined to address some of the previous

issues, see the original idea for transport equations in [12] and then for the Fokker–Planck equations in [23]. However, for the existence and uniqueness, one requires certain properties such as $\operatorname{div} v$ being bounded, see [23, 24]. We also mention that in [30] it is proven that if a weak solution u exists for $A = I$ and $|v|^2 |u|$ is integrable, then the weak solution is unique and it is a renormalized solution (although the proof is for bounded sets, one can generalize it to \mathbb{R}^n).

Motivated by [12, 30, 31], we define the following notion of solutions, see Definition 3.3 and Definition 4.8 for more details: A convection solution u of

$$\partial_t u - \operatorname{div} A \nabla u + \operatorname{div}(v u) = R,$$

in $[0, T] \times \mathbb{R}^n$ with initial data g , satisfies

$$\int_0^T \int_{\mathbb{R}^n} u f \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx \, dt, \tag{1.4}$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$ and ϕ satisfying the adjoint equation

$$-\partial_t \phi - \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi = f,$$

in $[0, T] \times \mathbb{R}^n$ with $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$, where A^* is the transpose of A . We prove, by Proposition 3.2 and Proposition 4.5, that the adjoint equation has a unique solution. The benefits of working with the adjoint equation $-\partial_t \phi - \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi = f$ are the maximum principle property and energy estimates, see Proposition 3.2 and Proposition 4.5, which implies the unique existence in a certain class of functions. The definition of convection solution is a generalization of very weak solutions by setting $-\partial_t \phi - \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi = f$ in (1.3). However, the uniqueness property and maximum principle are immediate from the definition of convection solutions, which is not the case for weak solutions.

In the first part of the paper, we focus on bounded drifts and prove the following: Let A be an elliptic measurable matrix-valued function, see (2.2). Then, the following holds.

Theorem 1.1. *Let $v \in L^\infty([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, and $R \in L^1([0, T] \times \mathbb{R}^n)$. Then, there exists a unique convection solution u , belonging to $L^1([0, T] \times \mathbb{R}^n) \cap L^p([0, T]; W_{\operatorname{loc}}^{1,p}(\mathbb{R}^n))$ for every $1 \leq p < \frac{n}{n-1}$, of*

$$\partial_t u - \operatorname{div} A \nabla u + \operatorname{div}(v u) = R,$$

in $[0, T] \times \mathbb{R}^n$ with initial data g . Moreover, u satisfies the following properties:

(i). Let φ_ϵ be a non-negative smooth approximation of the identity and $u_\epsilon \in L^2([0, T]; H^1(\mathbb{R}^n))$ be the smooth weak solution of

$$\partial_t u_\epsilon - \operatorname{div}(\varphi_\epsilon * A) \nabla u_\epsilon + \operatorname{div}((\varphi_\epsilon * v) u_\epsilon) = \varphi_\epsilon * R,$$

in $[0, T] \times \mathbb{R}^n$ with $u_\epsilon(0, x) = \varphi_\epsilon * g(x)$ for $x \in \mathbb{R}^n$. Then,

$$\lim_{\epsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^n} u_\epsilon \psi \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} u \psi \, dx \, dt,$$

for every $\psi \in C_0^\infty([0, T] \times \mathbb{R}^n)$.

(ii). There exists $F \subset [0, T]$ of measure zero such that

$$\lim_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n} u \psi \, dx \Big|_s = \int_{\mathbb{R}^n} g \psi \, dx,$$

for every bounded continuous function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$.

(iii). If R, g are non-negative, then u is non-negative. Furthermore, if R, g are non-negative and $(\ln(g))^- \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $u > 0$ a.e. in $[0, T] \times \mathbb{R}^n$, $\ln u \in L^2([0, T]; W^{1,2}_{\text{loc}}(\mathbb{R}^n))$, and

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} -\ln u \, \partial_t \varphi \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} A \nabla(\ln u) \cdot \nabla \varphi \, dx \, dt \\ & - \int_0^T \int_{\mathbb{R}^n} \varphi A \nabla(\ln u) \cdot \nabla(\ln u) \, dx \, dt \\ & - \int_0^T \int_{\mathbb{R}^n} v \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \varphi v \cdot \nabla(\ln u) \, dx \, dt \geq \int_0^T \int_{\mathbb{R}^n} \frac{R}{u} \varphi \, dx \, dt, \end{aligned} \tag{1.5}$$

for every non-negative $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^n)$.

(iv). The convection solution is a weak solution of $\partial_t u - \text{div} A \nabla u + \text{div}(v u) = R$ with initial data g .

(v). If (1.2) holds, then the convection solution satisfies the following conservation of mass:

$$\int_{\mathbb{R}^n} u \, dx \Big|_s = \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dx \, dt,$$

for a.e. $s \in [0, T]$.

To explain the theorem, property (i) shows that any sequence of solutions to the equations obtained by the regularization of data converges vaguely to the unique convection solution. In property (ii), the weak convergence to initial data g is proved, which is stronger than the vague convergence up to a subset depending on ψ that was the strongest result proven so far, see [5, Thm. 6.6.2]. Property (iii) derives both the weak and strong maximum principle for convection solutions. Finally, properties (iv) and (v) demonstrate that the convection solution is a generalization of the weak solution and satisfies the conservation of mass if (1.2) holds, respectively.

Due to a large number of research on convection–diffusion equations, it is impossible to mention all of them here. However, we briefly compare Theorem 1.1 with some highlights in the literature. The first case is when the Aronson–Serrin condition is

fulfilled, where $g \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ and $R \in L^p([0, T]; L^q(\mathbb{R}^n))$ for some $p, q > 1$, such that $\frac{n}{2p} + \frac{1}{q} < 1$. Then, see [2,5], Theorem 1.1 is well-known. In the case of (1.2), $R = 0$, $g \in L^2(\mathbb{R}^2)$, and $v = v_1 + v_2$ where $v_1 \in W^{1,1}_{\text{loc}}([0, T] \times \mathbb{R}^n)$ and $(\text{div } v_2)^- \in L^\infty(\mathbb{R}^2)$, then Theorem 1.1 is derived using a commutator estimate or stochastic differential equation approach; see [4,19,24] and references therein. Moreover, in [8,9,18], the existence of an entropy and weak solutions are derived if $R = 0$, where it is not clear if the solution is unique. On the other hand, for $R = 0$ and $g \geq 0$, the uniqueness of weak solutions is derived in [30], due to v^2u being integrable. Finally, we refer to [11] regarding boundary value problems with both Dirichlet and Neumann boundary conditions, assuming $R = 0$, $g \in L^2(\mathbb{R}^2)$ and using a heat kernel argument to derive Theorem 1.1. As we see, all of the mentioned results fail to give a unified approach to consider convection–diffusion equations with rough conditions R, g being L^1 -integrable which is resolved by our notion of convection solutions. Another advantage as we discussed is the immediate result of uniqueness of convection solutions in the space $L^1([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$ which fails for weak solutions; see Example 3.4.

The second part of this work focuses on the case of $A = I, v = \nabla V$, where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a measurable function satisfying

$$\begin{aligned} e^V + e^{-V} &\in L^1_{\text{loc}}(\mathbb{R}^n). \\ e^{-V} &\text{ is a regular weight.} \end{aligned} \tag{1.6}$$

Hence, we do not assume any integrability condition on ∇V or decay estimates on V at infinity since several interesting applications require considering such a large class of potentials. Some key examples are $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and $V(x) = C \ln |x|$ for $x \in \mathbb{R}^2$ where $-2 < C < 2$, see [13,36] and references therein. However, the literature on existence and uniqueness is restricted to potentials in Sobolev spaces, see [22] and [34, Thm. 2.2.29]. The idea in [22] is to use a minimization technique that requires regularity and decay assumptions on the potential. The other approach in [34, Thm. 2.2.29] is to show that $h = u e^V$ is the weak solution of $\partial_t h - \Delta h + \nabla V \cdot \nabla h = R e^V$ and then using regularity estimates to derive solutions. However, if V is not regular enough, then the transformation idea does not work. To define a well-defined notion of solution, we again define the adjoint convection solution ϕ for $-\partial_t \phi - \Delta \phi + \nabla V \cdot \nabla \phi = f$ in $[0, T] \times \mathbb{R}^n$ for every $f \in C^\infty_0([0, T] \times \mathbb{R}^n)$ with $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$. It is well-known that formally ϕ satisfies the weighted backward parabolic equation

$$-\partial_t \phi - e^V \text{div}(e^{-V} \nabla \phi) = f, \tag{1.7}$$

in $[0, T] \times \mathbb{R}^n$ with $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$. We prove in Proposition 4.5 that equation (1.7) has a unique solution. Now, we denote $\mathcal{M}([0, T] \times \mathbb{R}^n)$ as all the bounded signed Borel measures. For every $R \in L^1([0, T] \times \mathbb{R}^n), g \in L^1(\mathbb{R}^n)$, we define a convection solution $\mu \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ of

$$\partial_t \mu - \Delta \mu - \text{div}(\mu \nabla V) = R,$$

in $[0, T] \times \mathbb{R}^n$ with initial data g if

$$\int_0^T \int_{\mathbb{R}^n} f \, d\mu = \int_0^T \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx \, dt,$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$ and adjoint convection solution ϕ of $-\partial_t \phi - e^V \operatorname{div}(e^{-V} \nabla \phi) = f$ in $[0, T] \times \mathbb{R}^n$ with $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$.

Finally, we prove the following:

Theorem 1.2. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function, satisfying (1.6), $g \in L^1(\mathbb{R}^n)$, $R \in L^1([0, T] \times \mathbb{R}^n)$. Then, there exists a unique convection solution $\mu \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ of*

$$\partial_t \mu - \Delta \mu - \operatorname{div}(\mu \nabla V) = R,$$

in $[0, T] \times \mathbb{R}^n$ with initial data g , where

$$|\mu|([0, T] \times \mathbb{R}^n) \leq T \int_{\mathbb{R}^n} |g| \, dx + T \int_0^T \int_{\mathbb{R}^n} |R| \, dx \, dt.$$

Furthermore, μ satisfies the following:

(i). *Let φ_ϵ be a positive smooth approximation of the identity and $V_\epsilon \in C^{0,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be chosen such that $e^{V_\epsilon}, e^{-V_\epsilon}$ converge to e^V, e^{-V} in $L^1_{\text{loc}}(\mathbb{R}^n)$, respectively (see Lemma 4.4 for the existence of such a sequence). Assume that $\mu_\epsilon \in L^2([0, T]; H^1(\mathbb{R}^n))$ is the smooth weak solution of*

$$\partial_t \mu_\epsilon - \Delta \mu_\epsilon - \operatorname{div}(\mu_\epsilon \nabla V_\epsilon) = \varphi_\epsilon * R,$$

in $[0, T] \times \mathbb{R}^n$ with $\mu_\epsilon(0, x) = \varphi_\epsilon * g(x)$ for every $x \in \mathbb{R}^n$. Then,

$$\lim_{\epsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^n} \psi \mu_\epsilon \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} \psi \, d\mu,$$

for every $\psi \in C_0^\infty([0, T] \times \mathbb{R}^n)$.

(ii). *If R, g are non-negative, then μ is a positive measure and*

$$\liminf_{r \rightarrow 0} \frac{\mu((s-r, s+r) \times B(x, r))}{|(s-r, s+r) \times B(x, r)|} \geq (\inf g e^V + s \inf R e^V) e^{-V(x)},$$

for every $s \in (0, T)$ and every Lebesgue point $x \in \mathbb{R}^n$ of e^{-V} .

(iii). *If*

$$\int_{\mathbb{R}^n} |g|^q e^{\frac{q'}{q} V} \, dx + \int_0^T \left(\int_{\mathbb{R}^n} |R|^{p'} e^{\frac{p'}{p} V} \, dx \right)^{\frac{1}{p'}} dt < \infty,$$

for some $p > 1, q > 1$, then $\mu \in L^1([0, T] \times \mathbb{R}^n)$.

(iv). If

$$\|g e^V\|_{L^\infty} + \|R e^V\|_{L^\infty} < \infty, \tag{1.8}$$

then $\mu \in L^1([0, T] \times \mathbb{R}^n)$ and

$$\|\mu e^V\|_{L^\infty} \leq \|g e^V\|_{L^\infty} + T \|R e^V\|_{L^\infty}.$$

(v). If $e^{-V} \in L^1(\mathbb{R}^n)$, then the signed measure $\nu(I) := \mu(I \times \mathbb{R}^n)$, for Borel subsets $I \subset [0, T]$, is absolutely continuous with respect to Lebesgue’s measure \mathcal{L}^1 on $[0, T]$, and μ satisfies the following conservation of mass:

$$\frac{d\nu}{d\mathcal{L}^1}(s) = \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dx \, dt,$$

for a.e. $s \in [0, T]$.

(vi). For every $f \in C_0^\infty(\mathbb{R}^n)$, the following convergence to the initial data holds:

$$\lim_{(h,s) \rightarrow 0} \frac{1}{h} \int_s^{s+h} \int_{\mathbb{R}^n} f \, d\mu = \int_{\mathbb{R}^n} f \, g \, dx.$$

where in the limit we let (h, s) goes to zero in the domain $0 < h + s < T$.

To elaborate on the result, part (i) proves that any good regularization on the data, see Lemma 4.4, gives a sequence of solutions that converges vaguely to the convection solution. Parts (ii) and (iii) are a generalized form of the strong maximum principle and a sufficient condition for having L^1 -bounded solutions, respectively. In part (iv), we prove an upper bound estimate for the convection solution if (1.8) holds. Then, with the assumption $e^{-V} \in L^1(\mathbb{R}^n)$, we demonstrate the conservation of mass. Finally, the last part proves the convergence to the initial data in the vague sense.

We briefly discuss the new techniques in the proofs for Theorem 1.1 and Theorem 1.2. For the first one, we apply the argument in [30, Lem. 3.4], together with using regularization, to derive the unique adjoint solution for $-\partial_t \phi - \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi = f$ for bounded and square-integrable functions f in \mathbb{R}^n . We then prove the conservation of mass for regular solutions and use it to derive a sequence of functions converging vaguely to the solution u and prove continuous convergence to initial data. To prove the strong maximum principle, however, a nonlinear argument is applied. We first use the linearity and perturbation to consider the strictly positive approximating solutions u_i and then extract the properties of the equation for $\ln(u_i)$. This approach is new in the context of the strong maximum principle, but it has been used to prove the Harnack’s inequality; see [15, 25, 27]. For the second result, we need to use new arguments for all the steps. First, the notion of the adjoint convection solutions is defined on a suitable weighted Sobolev space. Then, uniqueness is followed by a test function argument, and existence is obtained by approximating solutions and their regularity estimates,

see Proposition 4.5. Then, again using the conservation of mass for the approximating smooth solutions, we derive the convection solution as a vague limit. For the upper and lower bounds, we need to use a weighted L^1 -estimate on the adjoint convection solution, see Lemma 4.7, together with the Radon–Nikodym theorem. In the next part, if $e^{-V} \in L^1(\mathbb{R}^n)$, then, by Prokhorov’s theorem, we derive the weak convergence of approximating solutions to convection solutions. This derives the conservation of mass for the convection solutions. Finally, we use properties of the time derivative of adjoint convection solutions, see Lemma 4.6, to obtain the continuous convergence to the initial data.

2. Preliminaries and notations

In the whole note, n is a positive integer number. The set $B(x, r)$, for $x \in \mathbb{R}^n, r > 0$, denotes the ball of radius r centered at x . We define the dilation $\epsilon B(x, r) = B(x, \epsilon r)$ for every $\epsilon > 0, r > 0, x \in \mathbb{R}^n$. The Hölder dual of a real $p > 1$ is denoted by $p' := \frac{p}{p-1}$.

2.1. Spaces and measures

Let $T > 0$ and $(\mathcal{B}, \|\cdot\|)$ be a normed vector space. For every $F \subset [0, T], t \in [0, T]$, and $f : [0, T] \rightarrow \mathbb{R}$, we say that

$$\limsup_{s \rightarrow t; s \in F} f(s) = g,$$

for $g \in \mathbb{R}$ if

$$\limsup_{n \rightarrow \infty} f(s_n) = g,$$

for every sequence $s_n \in F$ such that $\lim_{n \rightarrow \infty} s_n = t$. Likewise, we define $\liminf_{s \rightarrow t; s \in F} f$ and $\lim_{s \rightarrow t; s \in F} f$. A function $f : [0, T] \rightarrow \mathcal{B}$ is measurable if $f^{-1}(U)$ is Lebesgue measurable for every open subset $U \subset \mathcal{B}$. Let p, q be positive real numbers. Then, $L^p([0, T]; \mathcal{B})$ denotes the set of all the measurable functions $f : [0, T] \rightarrow \mathcal{B}$ such that

$$\|f\|_{L^p([0, T]; \mathcal{B})}^p := \int_0^T \|f(t)\|^p dt < \infty,$$

and $L^\infty([0, T]; \mathcal{B})$ includes all the measurable functions $g : [0, T] \rightarrow \mathcal{B}$ such that

$$\|g\|_{L^\infty([0, T]; \mathcal{B})} := \sup_{t \in [0, T]} \|g(t)\| < \infty.$$

For every $f : [0, T] \rightarrow \mathcal{B}$ and $h > 0$, we define the shift operator $\tau_h f : [-h, T-h] \rightarrow \mathcal{B}$ by $\tau_h f(t) = f(t+h)$ for every $t \in [-h, T-h]$. The space $L^q([0, T]; W_{loc}^{k,p}(\mathbb{R}^n))$ includes all the measurable real valued functions $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\int_0^T \left(\sum_{i=0}^k \int_B |\nabla^i f|^p \right)^{\frac{q}{p}} dx dt < \infty,$$

for every ball $B \subset \mathbb{R}^n$. We denote $C([0, T]; \mathcal{B})$ as the set of functions $f : [0, T] \rightarrow \mathcal{B}$ which are continuous, and $C([0, T]; L^p_{loc}(\mathbb{R}^n))$ is made of measurable functions $f : [0, T] \rightarrow L^p_{loc}(\mathbb{R}^n)$ such that $f \in C([0, T]; L^p(B))$ for every ball $B \subset \mathbb{R}^n$.

We define $A_2(\mathbb{R}^n)$ as all the positive functions $w \in L^1_{loc}(\mathbb{R}^n)$ such that for a constant $C > 0$ we have

$$\frac{\int_B w \, dx \int_B w^{-1} \, dx}{|B|^2} \leq C, \tag{2.1}$$

for every ball $B \subset \mathbb{R}^n$. The smallest constant C in (2.1) is denoted by $[w]_{A_2}$. Now, let $w, w^{-1} \in L^1_{loc}(\mathbb{R}^n)$. We define the measures $dw(x) := w(x) \, dx$, $dw^{-1}(x) := w^{-1}(x) \, dx$. For every Borel subset $B \subset \mathbb{R}^n$, define $w(B) := \int_B dw$, $w^{-1}(B) := \int_B dw^{-1}$. The space $L^2_w(\mathbb{R}^n)$ is defined by functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{L^2_w(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} f^2 \, dw < \infty,$$

and $L^2_{w,loc}(\mathbb{R}^n)$ includes all measurable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\int_B f^2 \, dw < \infty$ for every ball $B \subset \mathbb{R}^n$. The Sobolev space $H^1_w(\mathbb{R}^n)$ is defined by all the functions $f \in L^2_w(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |\nabla f|^2 \, dw < \infty$ and for a sequence $f_i \in C^\infty_0(\mathbb{R}^n)$, we have

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R}^n} |f_i - f|^2 \, dw + \int_{\mathbb{R}^n} |\nabla(f_i - f)|^2 \, dw = 0.$$

Note that by Cauchy–Schwarz inequality, we have

$$\left(\int_B |\nabla f| \, dx \right)^2 \leq w^{-1}(B) \int_B |\nabla f|^2 \, dw,$$

for every $f \in H^1_w(\mathbb{R}^n)$ and ball $B \subset \mathbb{R}^n$. Hence, $H^1_w(\mathbb{R}^n) \subset W^{1,1}_{loc}(\mathbb{R}^n)$. In the case that $w(x) = 1$, we denote $H^1_w(\mathbb{R}^n)$ by $H^1(\mathbb{R}^n)$. The norm $\|\cdot\|_{H^1_w(\mathbb{R}^n)}$ is defined by

$$\|f\|_{H^1_w(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} f^2 \, dw + \int_{\mathbb{R}^n} |\nabla f|^2 \, dw,$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$. The space $W^{1,2}_w(\mathbb{R}^n)$ includes functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\|f\|_{H^1_w(\mathbb{R}^n)} < \infty$. Note that, unlike the classical Sobolev space, it is possible for special weights $H^1_w(\mathbb{R}^n) \neq W^{1,2}_w(\mathbb{R}^n)$, see [10,40]. We say that w is a regular weight if $H^1_w(\mathbb{R}^n) = W^{1,2}_w(\mathbb{R}^n)$. Let $U \subset \mathbb{R}^m$ be a Lebesgue measurable subset for a positive integer m . We denote the set of all the bounded and continuous real-valued functions on U by $C_b(U)$. The space $C_0(U)$ includes all the functions in $C_b(U)$ with compact support. We represent the space of all bounded signed Borel measures on U by $\mathcal{M}(U)$. By the Jordan decomposition theorem, for every $\mu \in \mathcal{M}(U)$ there exist two unique bounded positive Borel measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$. We define the Borel measure $|\mu| := \mu^+ + \mu^-$. If $\mu, \nu \in \mathcal{M}(U)$ and μ is absolutely continuous

with respect to ν , then we denote $\frac{d\mu}{d\nu} : U \rightarrow \mathbb{R}$ as the bounded ν -measurable function satisfying

$$\mu(F) = \int_F \frac{d\mu}{d\nu} \, d\nu,$$

for every Borel subset $F \subset U$.

2.2. Coefficient

We assume that the matrix valued function $A = (a_{ij})_{i,j=1}^n$ is with real measurable entries $A_{ij} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ which, for some positive λ, Λ , satisfy the elliptic conditions

$$|A(t, x)\chi \cdot \xi| \leq \Lambda|\chi| |\xi|, \quad A(t, x)\chi \cdot \chi \geq \lambda|\chi|^2, \tag{2.2}$$

for all $\chi, \xi \in \mathbb{R}^n$ and a.e. $(t, x) \in [0, T] \times \mathbb{R}^n$. The matrix valued function A^* is the transpose of A .

2.3. Weak and vague convergence of measures

We apply two main convergences for finite signed Borel measures here. First, we say that $\mu_i \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ is weakly convergent to $\mu \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ if

$$\lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} f \, d\mu_i = \int_0^T \int_{\mathbb{R}^n} f \, d\mu,$$

for every $f \in C_b([0, T] \times \mathbb{R}^n)$. Second, we say that $\mu_i \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ converges vaguely to $\mu \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ if

$$\lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} f \, d\mu_i = \int_0^T \int_{\mathbb{R}^n} f \, d\mu,$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$. Note that every weak convergence is a vague convergence, but the reverse may not hold.

2.4. Truncation function

For every measurable function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ and $k > 0$, define

$$T_k(u)(x) := \begin{cases} u(x) & |u(x)| \leq k, \\ k \operatorname{sgn}(u(x)) & |u(x)| > k. \end{cases}$$

Then, by a standard result if $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$, for a $p \geq 1$, then $T_k(u) \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ and $\|\nabla T_k(u)\|_{L^p(U)} \leq \|\nabla u\|_{L^p(U)}$ for every $U \subset \mathbb{R}^n$. For $p > 1$, the space $\mathcal{T}^p(\mathbb{R}^n)$ includes all the measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $T_k f \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ for every $k > 0$. We refer to [3] for the properties of the spaces $\mathcal{T}^p(\mathbb{R}^n)$.

3. Existence for integrable drifts

In this section, we prove Theorem 1.1. We divide the proof into several steps. We assume that

$$T > 0, \quad v \in L^\infty([0, T] \times \mathbb{R}^n, \mathbb{R}^n). \tag{3.1}$$

To define the notion of a solution, we need the notion of the dual solution.

Definition 3.1. Let (3.1) and $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T] \times \mathbb{R}^n)$. We say that ϕ is an adjoint convection solution of

$$-\partial_t \phi - \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi = f, \tag{3.2}$$

in $[0, T] \times \mathbb{R}^n$ if $\phi \in L^2([0, T]; H^1(\mathbb{R}^n)) \cap C([0, T]; L^2(\mathbb{R}^n))$, $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$, and

$$\begin{aligned} & \int_{\mathbb{R}^n} \phi \psi \, dx \Big|_0^T + \int_0^T \int_{\mathbb{R}^n} \phi \partial_t \psi \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} A^* \nabla \phi \cdot \nabla \psi \, dx \, dt \\ & - \int_0^T \int_{\mathbb{R}^n} \psi v \cdot \nabla \phi \, dx \, dt = \int_{\mathbb{R}^n} f \phi \, dx \, dt, \end{aligned}$$

for every $\psi \in C_0^\infty([0, T] \times \mathbb{R}^n)$.

Proposition 3.2. If (3.1) holds, then, for every $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T] \times \mathbb{R}^n)$, there exists a unique adjoint convection solution ϕ for (3.2) which satisfies

$$\begin{aligned} (T - t) \inf f &\leq \phi(t, x) \leq (T - t) \sup f, \\ \|\phi\|_{L^2([0, T]; H^1(\mathbb{R}^n))} &\leq \frac{2}{\lambda} e^{(2 + \|v\|_{L^\infty}^2 / \lambda)T} \|f\|_{L^2([0, T] \times \mathbb{R}^n)}, \end{aligned}$$

for a.e. $(t, x) \in [0, T] \times \mathbb{R}^n$. Moreover, for every pointwise approximation of f , v , A by uniformly bounded sequences $f_i \in L^2([0, T] \times \mathbb{R}^n) \cap L^\infty([0, T] \times \mathbb{R}^n)$, $v_i \in L^\infty([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$, and $A_i \in L^\infty([0, T] \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$, where A_i satisfies (2.2) and

$$\lim_{i \rightarrow \infty} \|f_i - f\|_{L^2([0, T] \times \mathbb{R}^n)} = 0,$$

there exists a sequence of functions $\phi_i \in L^2([0, T]; H^1(\mathbb{R}^n)) \cap C([0, T]; L^2(\mathbb{R}^n))$ satisfying

$$-\partial_t \phi_i - \operatorname{div} A_i^* \nabla \phi_i - v_i \cdot \nabla \phi_i = f_i,$$

in $[0, T] \times \mathbb{R}^n$ with $\phi_i(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$, where ϕ_i converges to ϕ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$ and $L^2([0, T]; H^1_{\text{loc}}(\mathbb{R}^n))$.

Proof. To prove uniqueness, let $\phi_1, \phi_2 \in L^2([0, T]; H^1(\mathbb{R}^n)) \cap C([0, T]; L^2(\mathbb{R}^n))$ be two adjoint convection solutions of (3.2). Then, $\phi := \phi_1 - \phi_2 \in L^2([0, T]; H^1(\mathbb{R}^n)) \cap C([0, T]; L^2(\mathbb{R}^n))$ is an adjoint convection solution of

$$-\partial_t \phi - \operatorname{div} A^* \nabla \phi - v \cdot \nabla \phi = 0,$$

in $[0, T] \times \mathbb{R}^n$ with $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$. Let φ_ϵ be a non-negative approximation of identity in $[0, T] \times \mathbb{R}^n$. Then,

$$-\partial_t \varphi_\epsilon * \phi - \operatorname{div} \varphi_\epsilon * (A^* \nabla \phi) - \varphi_\epsilon * (v \cdot \nabla \phi) = 0,$$

weakly in $[\epsilon, T - \epsilon] \times \mathbb{R}^n$. Hence, by taking $\varphi_\epsilon * \phi$ as a test function and using Young’s inequality, together with (2.2), we obtain

$$\begin{aligned} & -\frac{1}{2} \partial_t \int_{\mathbb{R}^n} |\varphi_\epsilon * \phi|^2 \, dx \Big|_t + \int_{\mathbb{R}^n} \varphi_\epsilon * (A^* \nabla \phi) \cdot \varphi_\epsilon * \nabla \phi \, dx \Big|_t \\ & \leq \frac{\lambda}{2} \int_{\mathbb{R}^n} \varphi_\epsilon * |\nabla \phi|^2 \, dx \Big|_t + \frac{\|v\|_{L^\infty}^2}{2\lambda} \int_{\mathbb{R}^n} |\varphi_\epsilon * \phi|^2 \, dx \Big|_t \end{aligned}$$

for every $t \in [\epsilon, T - \epsilon]$. Now, by Grönwall’s inequality, we derive that

$$\begin{aligned} & \frac{1}{2} e^{\left(\frac{\|v\|_{L^\infty}^2}{\lambda} \epsilon\right)} \int_{\mathbb{R}^n} |\varphi_\epsilon * \phi|^2 \, dx \Big|_\epsilon + \int_\epsilon^{T-\epsilon} \int_{\mathbb{R}^n} e^{\left(\frac{\|v\|_{L^\infty}^2}{\lambda} t\right)} (\varphi_\epsilon * A^* \nabla \phi) \cdot \varphi_\epsilon * \nabla \phi \, dt \, dx \\ & \leq \frac{\lambda}{2} \int_\epsilon^{T-\epsilon} \int_{\mathbb{R}^n} e^{\left(\frac{\|v\|_{L^\infty}^2}{\lambda} t\right)} \varphi_\epsilon * |\nabla \phi|^2 \, dx \, dt + \frac{1}{2} e^{\left(\frac{\|v\|_{L^\infty}^2}{\lambda} (T-\epsilon)\right)} \int_{\mathbb{R}^n} |\varphi_\epsilon * \phi|^2 \, dx \Big|_{T-\epsilon}. \end{aligned}$$

By taking $\epsilon \rightarrow 0$ in the above inequality and using the elliptic condition (2.2) as well as $\phi \in L^2([0, T]; H^1(\mathbb{R}^n)) \cap C([0, T]; L^2(\mathbb{R}^n))$, it is implied that

$$\int_0^T \int_{\mathbb{R}^n} e^{\left(\frac{\|v\|_{L^\infty}^2}{\lambda} t\right)} |\nabla \phi|^2 \, dx \, dt = 0.$$

In conclusion, ϕ only depends on time and $-\partial_t \phi = 0$ weakly in $[0, T] \times \mathbb{R}^n$. This together with $\phi_i(T, x) = 0$ a.e. in \mathbb{R}^n implies that $\phi = 0$ a.e. in $[0, T] \times \mathbb{R}^n$. For the existence part, we follow the argument in [30, Lem. 3.4]. Let $f_i \in C_0^\infty([0, T] \times \mathbb{R}^n)$, $v_i \in C^\infty([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$, and $A_i \in C^\infty([0, T] \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$ be uniformly bounded sequence of functions converging to f, v, A pointwise, where A_i satisfies (2.2) and $\lim_{i \rightarrow \infty} \|f_i - f\|_{L^2([0, T] \times \mathbb{R}^n)} = 0$. The construction of such sequences is derived by a standard mollification argument. By a standard result on parabolic equations, see [1], there exist smooth functions $\phi_i \in L^2([0, T]; H^1(\mathbb{R}^n))$ satisfying

$$-\partial_t \phi_i - \operatorname{div} A_i^* \nabla \phi_i - v_i \cdot \nabla \phi_i = f_i,$$

weakly in $[0, T] \times \mathbb{R}^n$ with $\phi(T, x) = 0$ for every $x \in \mathbb{R}^n$. Hence,

$$\begin{aligned} & -\partial_t (\phi_i - (T - t) \sup f_i) - \operatorname{div} A_i^* \nabla (\phi_i - (T - t) \sup f_i) \\ & - v_i \cdot \nabla (\phi_i - (T - t) \sup f_i) \leq 0, \end{aligned}$$

weakly in $[0, T] \times \mathbb{R}^n$. By the maximum principle, see [1], $\phi_i - (T - t) \sup f_i \leq 0$. Likewise, one can prove that $\phi_i - (T - t) \inf f_i \geq 0$. Now, by taking ϕ_i as a test function and using (2.2), we derive that

$$-\frac{1}{2} \partial_t \|\phi_i(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|\nabla \phi_i(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} \phi_i v_i \cdot \nabla \phi_i \, dx + \int_{\mathbb{R}^n} f_i \phi_i \, dx.$$

for every $0 \leq t \leq T$. Hence, by Young and Grönwall’s inequalities, we derive that

$$\sup_{0 \leq t \leq T} \frac{1}{2} \|\phi_i(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + \frac{\lambda}{2} \|\nabla \phi_i(t, \cdot)\|_{L^2([0, T] \times \mathbb{R}^n)}^2 \leq e^{\left(1 + \frac{\|v_i\|_{L^\infty}^2}{\lambda}\right) T} \|f_i\|_{L^2([0, T] \times \mathbb{R}^n)}^2. \tag{3.3}$$

Then, by using the equation, it is obtained that ϕ_i is bounded uniformly in $L^2([0, T]; H^1(\mathbb{R}^n))$ and $\partial_t \phi_i$ is bounded uniformly in $L^2([0, T]; H^{-1}(\mathbb{R}^n)) + L^1([0, T]; L^1_{loc}(\mathbb{R}^n))$. Hence, by the classical result in [37, Thm. 6], ϕ_i converges to $\phi \in L^2([0, T]; H^1(\mathbb{R}^n))$ pointwise a.e. in $[0, T] \times \mathbb{R}^n$ and strongly in $L^1([0, T]; L^1_{loc}(\mathbb{R}^n))$, up to a subsequence. Moreover, ϕ satisfies (3.2) weakly in $[0, T] \times \mathbb{R}^n$ and

$$(T - t) \inf f \leq \phi(t, x) \leq (T - t) \sup f. \tag{3.4}$$

Since $\phi \in L^2([0, T]; H^1(\mathbb{R}^n))$ and $\partial_t \phi \in L^2([0, T]; H^{-1}(\mathbb{R}^n)) + L^1([0, T]; L^1_{loc}(\mathbb{R}^n))$, by [29, Thm. 1], we deduce that $\phi \in C([0, T]; L^1_{loc}(\mathbb{R}^n))$. Then, by (3.4), we get

$$\int_B |\phi(t, x) - \phi(s, x)|^2 \, dx \leq 2T \|f\|_{L^\infty} \int_B |\phi(t, x) - \phi(s, x)| \, dx,$$

for every $0 \leq s \leq t \leq T$ and ball $B \subset \mathbb{R}^n$. Hence, $\phi \in C([0, T]; L^2_{loc}(\mathbb{R}^n))$.

Now, we demonstrate that ϕ_i converges to ϕ in the space $C([0, T]; L^1_{loc}(\mathbb{R}^n))$. Let us fix $\varphi \in C^\infty_0([0, T] \times \mathbb{R}^n)$. By taking the differences of the equations for ϕ , ϕ_i and using $(\phi - \phi_i) \varphi^2$ as a test function and using (2.2), we obtain

$$\begin{aligned} & \lambda \int_0^T \int_{\mathbb{R}^n} |\nabla(\phi - \phi_i)|^2 \varphi^2 \, dx \, dt \\ & \leq \int_0^T \int_{\mathbb{R}^n} (A_i^* - A^*) \nabla \phi \cdot (\varphi^2 \nabla(\phi - \phi_i) + 2(\phi - \phi_i) \varphi \nabla \varphi) \, dx \, dt \\ & \quad - \int_0^T \int_{\mathbb{R}^n} \varphi^2 (\phi_i - \phi) v_i \cdot \nabla(\phi_i - \phi) \, dx \, dt \\ & \quad - \int_0^T \int_{\mathbb{R}^n} \varphi^2 (\phi_i - \phi) (v_i - v) \cdot \nabla \phi \, dx \, dt \\ & \quad + \int_0^T \int_{\mathbb{R}^n} \varphi^2 (f_i - f) (\phi - \phi_i) \, dx \, dt \end{aligned}$$

for $t \in [0, T]$. In conclusion, by using strong convergence of ϕ_i to ϕ in $L^2([0, T]; L^2_{\text{loc}}(\mathbb{R}^n))$, (3.3), and Lebesgue’s dominated convergence, we obtain ϕ_i converges to ϕ strongly in the space $L^2([0, T]; H^1_{\text{loc}}(\mathbb{R}^n))$. Then, by using the equations for ϕ, ϕ_i , we derive that $\partial_t \phi_i$ converges to $\partial_t \phi$ in $L^2([0, T]; L^2(B)) + L^2([0, T]; H^{-1}(B))$ for every ball $B \subset \mathbb{R}^n$. Hence, by [29, Thm. 1], ϕ_i converges to ϕ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$. Note that the argument above works also for any pointwise approximation of f, v, A by uniformly bounded sequences $f_i \in L^2([0, T] \times \mathbb{R}^n) \cap L^\infty([0, T] \times \mathbb{R}^n)$, $v_i \in L^\infty([0, T]; L^\infty(\mathbb{R}^n \times \mathbb{R}^n))$, $A_i \in L^\infty([0, T] \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$, where A_i satisfies (2.2) and $\lim_{i \rightarrow \infty} \|f_i - f\|_{L^2([0, T] \times \mathbb{R}^n)} = 0$. Finally, we prove that in fact $\phi \in C([0, T]; L^2(\mathbb{R}^n))$. To elaborate the proof, let $r > 0$ and $\psi \in C^\infty(\mathbb{R}^n)$ be a function which satisfies $0 \leq \psi \leq 1$ in \mathbb{R}^n , $\psi = 0$ in $B(0, r/2)$, $\psi = 1$ in $\mathbb{R}^n \setminus B(0, r)$, and $|\nabla \psi| \leq \frac{C}{r}$ in \mathbb{R}^n for a constant $C > 0$. By using the equation for ϕ_i , we take $\psi \phi_i$ as a test function to arrive at

$$-\frac{1}{2} \partial_t \int_{\mathbb{R}^n} \phi_i^2 \psi^2 \, dx \Big|_t + \int_{\mathbb{R}^n} A_i^* \nabla \phi_i \cdot \nabla (\phi_i \psi^2) \, dx \Big|_t \leq \int_{\mathbb{R}^n} \|v_i\|_{L^\infty} |\nabla \phi_i| |\phi_i| \psi^2 \, dx \Big|_t + \int_{\mathbb{R}^n} |f_i| |\phi_i| \psi^2 \, dx \Big|_t.$$

Then, by using Young’s inequality, Grönwall’s inequality, and (2.2), we obtain

$$\sup_{0 \leq t \leq T} \frac{1}{2} \int_{\mathbb{R}^n} \phi_i^2 \psi^2 \, dx \Big|_t \leq e^{\left(1 + \frac{\|v_i\|_{L^\infty}^2}{\lambda}\right) T} \left(\int_0^T \int_{\mathbb{R}^n} \frac{2\Lambda^2}{\lambda} \phi_i^2 |\nabla \psi|^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} f_i^2 \psi^2 \, dx \, dt \right).$$

Hence, by strong convergence of ϕ_i to ϕ in both $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$ and $L^2([0, T]; L^2_{\text{loc}}(\mathbb{R}^n))$, Fatou’s lemma, and Lebesgue’s dominated convergence, we obtain

$$\sup_{0 \leq t \leq T} \frac{1}{2} \int_{\mathbb{R}^n} \phi^2 \psi^2 \, dx \Big|_t \leq e^{\left(1 + \frac{\|v\|_{L^\infty}^2}{\lambda}\right) T} \left(\frac{2(C\Lambda)^2}{r^2 \lambda} \int_0^T \int_{B(0,r)} \phi^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^n \setminus B(0,r/2)} f^2 \, dx \, dt \right).$$

Thus, we obtain

$$\lim_{r \rightarrow \infty} \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n \setminus B(0,r)} \phi^2 \, dx \Big|_t = 0.$$

Since $\phi \in C([0, T]; L^2_{\text{loc}}(\mathbb{R}^n))$, for every $t \in [0, T]$, $\epsilon > 0$, ball $B \subset \mathbb{R}^n$, there exists a constant $\delta(t, \epsilon, B) > 0$ depending on t, ϵ, B , such that

$$\int_B |\phi(t, x) - \phi(s, x)|^2 \, dx < \epsilon,$$

for all $s \in [0, T]$ satisfying $|s - t| < \delta(t, \epsilon, B)$. Let $\epsilon > 0$ and $r > 0$ satisfies

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n \setminus B(0,r)} \phi^2 \, dx \Big|_t < \frac{\epsilon}{8}.$$

Then, for every $t > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\phi(t, x) - \phi(s, x)|^2 \, dx &= \int_{B(0,r)} |\phi(t, x) - \phi(s, x)|^2 \, dx \\ &+ \int_{\mathbb{R}^n \setminus B(0,r)} |\phi(t, x) - \phi(s, x)|^2 \, dx \\ &< \frac{\epsilon}{2} + 2 \int_{\mathbb{R}^n \setminus B(0,r)} \phi^2 \, dx \Big|_t + 2 \int_{\mathbb{R}^n \setminus B(0,r)} \phi^2 \, dx \Big|_s < \epsilon, \end{aligned}$$

for every $s \in [0, T]$ satisfying $|s - t| \leq \delta(t, \epsilon/2, B(0, r))$. In conclusion, $\phi \in C([0, T]; L^2(\mathbb{R}^n))$. This completes the proof. \square

Now, we define a notion of solution for convection–diffusion equations.

Definition 3.3. Let (3.1), $g \in L^1(\mathbb{R}^n)$, and $R \in L^1([0, T] \times \mathbb{R}^n)$. Then, we say that u is a convection solution for

$$\partial_t u - \operatorname{div} A \nabla u + \operatorname{div}(v u) = R,$$

in $[0, T] \times \mathbb{R}^n$ with initial value g if $u \in L^1([0, T] \times \mathbb{R}^n)$ and

$$\int_0^T \int_{\mathbb{R}^n} u f \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx \, dt,$$

for every $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T] \times \mathbb{R}^n)$ and dual solution ϕ satisfying (3.2). We say that a convection solution u for

$$\partial_t u - \operatorname{div} A \nabla u + \operatorname{div}(v u) = R,$$

in $[0, T] \times \mathbb{R}^n$ with the initial value g satisfies the conservation of mass if

$$\int_{\mathbb{R}^n} u \, dx \Big|_s = \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dx \, dt,$$

for a.e. $s \in [0, T]$.

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. The proof is divided into several steps.

1. First, we prove the uniqueness of convection solutions. If $u_1, u_2 \in L^1([0, T] \times \mathbb{R}^n)$ are convection solutions of (3.2) with the initial value g , then

$$\int_0^T \int_{\mathbb{R}^n} (u_1 - u_2) f \, dx \, dt = 0$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$. Hence, $u_1 = u_2$ a.e. in $[0, T] \times \mathbb{R}^n$.

2. In this step, we prove some properties of approximate solutions. For now, we assume that R, g are non-negative and $(\ln g)^- \in L^1_{\text{loc}}(\mathbb{R}^n)$. We remove the extra assumptions later in Step 6. Let φ_ϵ be a positive approximation of identity. We define $\tilde{R}_i = \varphi_{1/i} * R, \tilde{g}_i := \varphi_{1/i} * g, A_i = \varphi_{1/i} * A$, and $v_i := \varphi_{1/i} * v$. Then, $R_i = \tilde{R}_i + \frac{1}{i}e^{-x^2}, g_i = \tilde{g}_i + \frac{1}{i}e^{-x^2}$ are positive smooth approximations of R, g in $L^1([0, T]; L^1(\mathbb{R}^n))$, respectively. Note that, by Young’s convolution inequality, $R_i \in L^p([0, T] \times \mathbb{R}^n), g_i \in L^p(\mathbb{R}^n)$ for every $p \geq 1$ and i . Hence, by a standard theorem on parabolic equations, see [1], there exists a smooth function $u_i \in L^2([0, T]; H^1(\mathbb{R}^n))$ such that $u_i(0, x) = g_i(x)$ for every $x \in \mathbb{R}^n$ and

$$\partial_t u_i - \operatorname{div} A_i \nabla u_i + \operatorname{div}(v_i u_i) = R_i, \tag{3.5}$$

in $[0, T] \times \mathbb{R}^n$. By the strong maximum principle, see [1], $u_i > 0$ in $[0, T] \times \mathbb{R}^n$. Then, taking u_i^α as a test function for $\alpha > 0$, we obtain

$$\begin{aligned} & \frac{1}{\alpha + 1} \partial_t \int_{\mathbb{R}^n} u_i^{\alpha+1} \, dx \Big|_t + \alpha \int_{\mathbb{R}^n} u_i^{\alpha-1} A_i \nabla u_i \cdot \nabla u_i \, dx \Big|_t \\ & - \alpha \int_{\mathbb{R}^n} u_i^\alpha v_i \cdot \nabla u_i \, dx \Big|_t = \int_{\mathbb{R}^n} R_i u_i^\alpha \, dx \Big|_t, \end{aligned}$$

for every $t \in [0, T]$. Then, by Young’s inequality and (2.2), we obtain

$$\begin{aligned} & \frac{1}{\alpha + 1} \partial_t \int_{\mathbb{R}^n} u_i^{\alpha+1} \, dx \Big|_t + \frac{\alpha \lambda}{2} \int_{\mathbb{R}^n} |\nabla u_i|^2 u_i^{\alpha-1} \, dx \Big|_t \\ & \leq \left(\frac{\alpha \|v_i\|_{L^\infty}^2}{2\lambda} + \frac{\alpha}{\alpha + 1} \right) \int_{\mathbb{R}^n} u_i^{\alpha+1} \, dx + \frac{1}{\alpha} \int_{\mathbb{R}^n} R_i^\alpha \, dx \Big|_t. \end{aligned}$$

for every $t \in [0, T]$. Hence, by Grönwall’s inequality, we have

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} u_i^{\alpha+1} \, dx \, dt \leq e^{\left(\frac{(\alpha+1)\|v_i\|_{L^\infty}^2}{2\lambda} + \alpha\right)T} \left(\frac{\alpha + 1}{\alpha} \int_0^T \int_{\mathbb{R}^n} R_i^\alpha \, dx \, dt + \int_{\mathbb{R}^n} g_i^{\alpha+1} \, dx \right). \tag{3.6}$$

Now, take $s \in [0, T]$ and a sequence functions $\psi_j \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\begin{aligned} & 1_{B(0,j)} \leq \psi_j \leq 1_{B(0,2j)}, \\ & \psi_j \leq \psi_{j+1}, \\ & |\nabla^k \psi_j| \leq \frac{C_k}{j^k}, \end{aligned} \tag{3.7}$$

for all non-negative integers j, k , where C_k is a constant depending on n, k . Then,

$$\begin{aligned} \int_{\mathbb{R}^n} u_i \psi_j \, dx \Big|_s &= \int_{\mathbb{R}^n} g_i \psi_j \, dx \Big|_0 + \int_0^s \int_{\mathbb{R}^n} u_i \operatorname{div} A_i \nabla \psi_j \, dx \, dt \\ & + \int_0^s \int_{\mathbb{R}^n} u_i v_i \cdot \nabla \psi_j \, dx \, dt + \int_0^s \int_{\mathbb{R}^n} R_i \psi_j \, dx \, dt, \end{aligned} \tag{3.8}$$

for every $0 \leq s \leq T$. Hence, using (3.6) and Hölder’s inequality, we deduce

$$\begin{aligned} & \left| \int_0^s \int_{\mathbb{R}^n} u_i \operatorname{div} A_i \nabla \psi_j \, dx \, dt + \int_0^s \int_{\mathbb{R}^n} u_i v_i \cdot \nabla \psi_j \, dx \, dt \right| \\ & \leq (\|A_i\|_{L^\infty} + \|v_i\|_{L^\infty} + \|\nabla A_i\|_{L^\infty} + 1) \frac{C_1 + C_2 + 1}{j} \int_0^s \int_{B(0,2j)} u_i \, dx \, dt \\ & \leq (\|A_i\|_{L^\infty} + \|v_i\|_{L^\infty} + \|\nabla A_i\|_{L^\infty} + 1) (|B(0, 1)|(2j)^n)^{\frac{\alpha}{\alpha+1}} \frac{C_1 + C_2 + 1}{j} \\ & \quad \left(\int_0^s \int_{B(0,2j)} u_i^{\alpha+1} \, dx \, dt \right)^{\frac{1}{\alpha+1}} \\ & \leq C(2j)^{\frac{n\alpha}{\alpha+1}-1} T^{\frac{1}{\alpha+1}} e^{\left(\frac{\|v_i\|_{L^\infty}^2}{2\lambda} + \frac{\alpha}{\alpha+1}\right)T} \left(\frac{\alpha + 1}{\alpha} \int_0^T \int_{\mathbb{R}^n} R_i^\alpha \, dx \, dt + \int_{\mathbb{R}^n} g_i^{\alpha+1} \, dx \right)^{\frac{1}{\alpha+1}}, \end{aligned}$$

for $\alpha > 0$, where C is a constant depending on n and $\|A_i\|_{L^\infty} + \|v_i\|_{L^\infty} + \|\nabla A_i\|_{L^\infty}$. Taking α small enough ($\alpha < \frac{1}{n-1}$), letting $j \rightarrow \infty$ in (3.8), and using monotone convergence theorem, we arrive at

$$\int_{\mathbb{R}^n} u_i \, dx \Big|_s = \int_{\mathbb{R}^n} g_i \, dx + \int_0^s \int_{\mathbb{R}^n} R_i \, dx \, dt. \tag{3.9}$$

3. In this step, we prove logarithm-type estimates. Since $u_i > 0$, we can define the smooth function $S_i := \ln u_i$. Hence, by (3.5), it is implied that

$$\partial_t S_i - \operatorname{div} A_i \nabla S_i - A_i \nabla S_i \cdot \nabla S_i + \operatorname{div} v_i + v_i \cdot \nabla S_i = R_i e^{-S_i}, \tag{3.10}$$

weakly in $[0, T] \times \mathbb{R}^n$. Note that by (3.9), we obtain

$$\int_{\mathbb{R}^n} S_i^+ \, dx \Big|_s \leq \int_{\mathbb{R}^n} e^{S_i} \, dx \Big|_s = \int_{\mathbb{R}^n} g_i \, dx + \int_0^s \int_{\mathbb{R}^n} R_i \, dx \, dt. \tag{3.11}$$

for every $s \in [0, T]$. We need to prove estimates for S_i^- . By using φ^2 , where $\varphi \in C_0^\infty(\mathbb{R}^n)$, as a test function and (3.10), we get

$$\begin{aligned} & \int_0^s \int_{\mathbb{R}^n} \varphi^2 A \nabla S_i \cdot \nabla S_i \, dx \, dt + \int_0^s \int_{\mathbb{R}^n} R_i e^{-S_i} \varphi^2 + \int_{\mathbb{R}^n} S_i \varphi^2 \Big|_0 = \int_{\mathbb{R}^n} S_i \varphi^2 \Big|_s \\ & + \int_0^s \int_{\mathbb{R}^n} 2\varphi A \nabla S_i \cdot \nabla \varphi \, dx \, dt - \int_0^s \int_{\mathbb{R}^n} 2\varphi v_i \cdot \nabla \varphi \, dx \, dt + \int_0^s \int_{\mathbb{R}^n} \varphi^2 v_i \cdot \nabla S_i \, dx \, dt, \end{aligned} \tag{3.12}$$

for every $s \in [0, T]$. Then, by using Young’s inequality and (2.2) again, together with (3.11), we obtain

$$\begin{aligned} & \frac{\lambda}{2} \int_0^T \int_{\mathbb{R}^n} |\nabla S_i|^2 \varphi^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} R_i e^{-S_i} \varphi^2 \, dx \, dt + \sup_{0 \leq s \leq T} \int_{\mathbb{R}^n} S_i^- \varphi^2 \, dx \Big|_s \\ & \leq 2 \int_{\mathbb{R}^n} S_i^- \varphi^2 \, dx \Big|_0 + \|\varphi\|_{L^\infty}^2 \int_{\mathbb{R}^n} g_i \, dx + \|\varphi\|_{L^\infty}^2 \int_0^T \int_{\mathbb{R}^n} R_i \, dx \, dt \\ & + \left(\frac{1}{\lambda} + \frac{\lambda}{\Lambda^2} \right) \int_0^T \int_{\mathbb{R}^n} v_i^2 \varphi^2 \, dx \, dt + \frac{5\Lambda^2}{\lambda} \int_0^T \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx \, dt. \end{aligned} \tag{3.13}$$

4. Now, we use the estimates to get to a limit. By (3.10) and (3.13), for every ball $B \subset \mathbb{R}^n$, $\|S_i\|_{L^2([0, T]; H^1(B))}$, $\|R_i e^{-S_i}\|_{L^1([0, T] \times B)}$ is uniformly bounded and $\partial_t S_i$ has uniformly bounded norm in $L^2([0, T]; H^{-1}(B)) + L^1([0, T]; L^1(B))$. Hence, by [37, Thm. 6], S_i converges to $S \in L^2([0, T]; H^1_{loc}(\mathbb{R}^n))$ strongly in $L^1([0, T]; L^1_{loc}(\mathbb{R}^n))$ and weakly in $L^2([0, T]; H^1_{loc}(\mathbb{R}^n))$, up to a subsequence. In particular, up to a subsequence, S_i converges pointwise to S a.e. in $[0, T] \times \mathbb{R}^n$. In conclusion, by Fatou’s lemma and (3.9), we derive

$$\int_0^T \int_{\mathbb{R}^n} e^S \, dx \, dt \leq \liminf_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} e^{S_i} \, dx \, dt = T \int_{\mathbb{R}^n} g \, dx + T \int_0^T \int_{\mathbb{R}^n} R \, dx \, dt.$$

Also, by letting $i \rightarrow \infty$ in (3.12) and using [7, Thm. 4.3], together with Fatou’s lemma, we conclude (1.5). Now, we prove that $u := e^S$ is the desired convection solution. Let us fix a function $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T] \times \mathbb{R}^n)$ and assume that $f_i \in C^\infty_0([0, T] \times \mathbb{R}^n)$ converges to f in $L^\infty([0, T] \times \mathbb{R}^n)$ -weak* and strongly in $L^2([0, T] \times \mathbb{R}^n)$. We define the sequence $f_i^k := f_i 1_{\{u_i \leq k\}}$. Since u_i, v_i are smooth and satisfy (3.5) weakly, we obtain

$$\int_0^T \int_{\mathbb{R}^n} u_i f_i^k \, dx \, dt = \int_{\mathbb{R}^n} g_i \phi_i^k \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R_i \phi_i^k \, dx \, dt, \tag{3.14}$$

for every dual solution ϕ_i^k satisfying

$$-\partial_t \phi_i^k - \operatorname{div} A_i^* \nabla \phi_i^k - v_i \cdot \nabla \phi_i^k = f_i^k,$$

weakly in $[0, T] \times \mathbb{R}^n$. By Proposition 3.2, ϕ_i^k are uniformly bounded and $\phi_i^k, \phi_i^k(0, \cdot)$ converge pointwise to $\phi^k, \phi^k(0, \cdot)$, respectively. Moreover, ϕ^k satisfies

$$-\partial_t \phi^k - \operatorname{div} A^* \nabla \phi^k - v \cdot \nabla \phi^k = f^k, \tag{3.15}$$

weakly in $[0, T] \times \mathbb{R}^n$ with $\phi^k(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$. Since $|u_i f_i^k| \leq k|f|$ and, up to a subsequence, u_i converges pointwise to u , by Lebesgue’s dominated convergence and taking $i \rightarrow \infty$ in (3.14), it is implied that

$$\int_0^T \int_{\mathbb{R}^n} u f 1_{\{u \leq k\}} \, dx \, dt = \int_{\mathbb{R}^n} g \phi^k \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi^k \, dx \, dt.$$

By the same argument as above, up to a subsequence, ϕ^k and $\phi^k(0, \cdot)$ are uniformly bounded and converge pointwise to ϕ and $\phi(0, \cdot)$, respectively, where ϕ satisfies (3.2). Hence, by the dominated convergence, we obtain

$$\int_0^T \int_{\mathbb{R}^n} u f \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx \, dt.$$

Thus, u is the desired convection solution. Since $u_i \in L^2([0, T]; H^1(\mathbb{R}^n))$ is the smooth solution of (3.5) with $u_i(0, x) = g_i(x)$ for $x \in \mathbb{R}^n$, by using the test function $T_k u_i$ for the equation of u_i and (2.2), we obtain that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} \frac{1}{k} |T_k(u_i)|^2 \, dx \Big|_t + \frac{\lambda}{2k} \int_0^T \int_{\mathbb{R}^n} |\nabla T_k u_i|^2 \, dx \, dt \\ & \leq e^{\left(1 + \frac{\|v_i\|_{L^\infty}^2}{\lambda}\right)T} \left(\int_{\mathbb{R}^n} g_i \, dx + \int_0^T \int_{\mathbb{R}^n} R_i \, dx \, dt \right). \end{aligned}$$

Hence, by Fatou’s lemma and weak convergence of $\nabla T_k u_i$ to $\nabla T_k u$ in $L^2([0, T] \times \mathbb{R}^n)$, it is implied that $u \in T^2([0, T] \times \mathbb{R}^n)$ and

$$\begin{aligned} & \sup_{0 \leq t \leq T} \frac{1}{k} \int_{\mathbb{R}^n} |T_k(u)|^2 \, dx + \frac{\lambda}{2k} \int_0^T \int_{\mathbb{R}^n} |\nabla T_k u|^2 \, dx \, dt \\ & \leq e^{\left(1 + \frac{\|v\|_{L^\infty}^2}{\lambda}\right)T} \left(\int_{\mathbb{R}^n} g \, dx + \int_0^T \int_{\mathbb{R}^n} R \, dx \, dt \right). \end{aligned}$$

By a simple application of the layer cake representation, for every bounded subset $\Omega \subset \mathbb{R}^n$ and $1 \leq p < \frac{n}{n-1}$, we have

$$\begin{aligned} \int_{\Omega} |Du|^p & \leq |\Omega| + p \int_1^\infty \lambda^{p-1} |\{x \in \Omega : |Du| > \lambda\}| \, d\lambda \\ & \lesssim |\Omega| + p \int_1^\infty \lambda^{p-1-\frac{n}{n-1}} \, d\lambda = |\Omega| + \frac{p}{\frac{n}{n-1} - p}. \end{aligned}$$

for $n > 2$, where we used [3, Lem. 4.2] on the last inequality. For $n \leq 2$, one can simply use the same argument by applying Hölder’s inequality and q instead of $\frac{n}{n-1}$, which satisfies $p < q < \frac{n}{n-1}$. Hence, $u \in L^2([0, T]; W_{loc}^{1,p}(\mathbb{R}^n))$ for all $1 \leq p < \frac{n}{n-1}$ and u_i converges to u weakly in $L^2([0, T]; L_{loc}^p(\mathbb{R}^n))$, up to subsequence depending on p . Moreover,

$$\int_0^T \int_{\mathbb{R}^n} u_i f \, dx \, dt = \int_{\mathbb{R}^n} g_i \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R_i \phi \, dx \, dt,$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$ and i . Hence,

$$\lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} u_i f \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} u f \, dx \, dt.$$

In conclusion, u_i converges to u vaguely in $[0, T] \times \mathbb{R}^n$ without any need to pass to a subsequence.

5. In this step, we prove that there exists a set $F \subset [0, T]$ of measure zero such that

$$\lim_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n} u \psi \, dx \Big|_s = \int_{\mathbb{R}^n} g \psi \, dx \tag{3.16}$$

for every $\psi \in C_b(\mathbb{R}^n)$. By the Stone–Weierstrass theorem and σ -compactness of \mathbb{R}^n , there exists a countable dense subset $D \subset C_0^\infty(\mathbb{R}^n)$. Let us fix an element $\varphi \in D$ and

$0 \leq t_1 < t_2 \leq T$. Then, by using (2.2), the equation of u_i , and test function φ , we have

$$\left| \int_{\mathbb{R}^n} u_i \varphi \, dx \Big|_{t_1} - \int_{\mathbb{R}^n} u_i \varphi \, dx \Big|_{t_2} \right| \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Lambda |\nabla u_i| |\nabla \varphi| \, dx \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |v_i| |u_i| |\nabla \varphi| \, dx \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} R_i \varphi \, dx \, dt.$$

Hence, by Hölder’s inequality, we derive that

$$\limsup_{i \rightarrow \infty} \left| \int_{\mathbb{R}^n} u_i \varphi \, dx \Big|_{t_1} - \int_{\mathbb{R}^n} u_i \varphi \, dx \Big|_{t_2} \right| \leq \sqrt{t_2 - t_1} |\text{supp } \varphi|^{\frac{1}{p'}} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} (\Lambda + \|v\|_{L^\infty}) \limsup_{i \rightarrow \infty} \|u_i + |\nabla u_i|\|_{L^2([0, T]; L^p(\text{supp } \varphi))} + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} R \varphi \, dx \, dt,$$

for every $p < \frac{n}{n-1}$. Moreover, by (3.9), it is obtained that

$$\left| \int_{\mathbb{R}^n} u_i \varphi \, dx \Big|_s \right| \leq \|\varphi\|_{L^\infty} \left| \int_{\mathbb{R}^n} u_i \, dx \Big|_s \leq \|\varphi\|_{L^\infty} \left(\int_{\mathbb{R}^n} g_i + \int_0^s \int_{\mathbb{R}^n} R_i \, dx \, dt \right),$$

for every $0 \leq s \leq T$. In conclusion, by the Arzelà-Ascoli theorem, the sequence $\int_{\mathbb{R}^n} u_i \varphi \, dx : [0, T] \rightarrow \mathbb{R}$ converges uniformly to a continuous function $f_\varphi : [0, T] \rightarrow \mathbb{R}$, up to a subsequence. Since u_i converges weakly to u in $L^1([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$, up to a subsequence, $f_\varphi = \int_{\mathbb{R}^n} u \varphi \, dx$ a.e. in $[0, T]$. By combining the previous results, $\int_{\mathbb{R}^n} u \varphi \, dx$ is continuous in $[0, T]$ outside a set F_φ of measure zero, which depends on φ . We define $F := \cup_{\varphi \in D} F_\varphi \cup G$, where $[0, T] \setminus G$ is the set of points $s \in [0, T]$ such that $\int_{\mathbb{R}^n} u \, dx \Big|_s$ is finite and s is Lebesgue point for the function $\int_{\mathbb{R}^n} u \, dx$. By Fubini’s theorem and Lebesgue’s differentiation theorem, the subset F is of Lebesgue measure zero. Let $s \in (0, T) \setminus F$ and $0 < \epsilon < \min(|T - s|, s)$. Then, by (3.9) and Fatou’s lemma,

$$\begin{aligned} \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} \int_{\mathbb{R}^n} u \, dx \, dt &\leq \frac{1}{2\epsilon} \liminf_{i \rightarrow \infty} \int_{s-\epsilon}^{s+\epsilon} \int_{\mathbb{R}^n} u_i \, dx \, dt \\ &= \liminf_{i \rightarrow \infty} \int_{\mathbb{R}^n} g_i \, dx + \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} \int_0^t \int_{\mathbb{R}^n} R_i \, dx \, dr \, dt \\ &= \int_{\mathbb{R}^n} g \, dx + \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} \int_0^t \int_{\mathbb{R}^n} R \, dx \, dr \, dt. \end{aligned}$$

Now, by letting $\epsilon \rightarrow 0$ in the above inequality and using Lebesgue’s differentiation theorem, we arrive at

$$\int_{\mathbb{R}^n} u \, dx \Big|_s \leq \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dx \, dt. \tag{3.17}$$

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_i \in D$ be a sequence of functions converging to ψ in $C_b(\mathbb{R}^n)$. Hence, by (3.17), for every $s \in [0, T] \setminus F$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u \psi \, dx \Big|_s - \int_{\mathbb{R}^n} g \psi \, dx \right| &\leq \left| \int_{\mathbb{R}^n} u \varphi_i \, dx \Big|_s - \int_{\mathbb{R}^n} g \varphi_i \, dx \right| \\ &\quad + \|\psi - \varphi_i\|_{L^\infty(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} u \, dx \Big|_s + \int_{\mathbb{R}^n} g \, dx \right) \\ &\leq \left| \int_{\mathbb{R}^n} u \varphi_i \, dx \Big|_s - \int_{\mathbb{R}^n} g \varphi_i \, dx \right| \\ &\quad + \|\psi - \varphi_i\|_{L^\infty(\mathbb{R}^n)} \left(2 \int_{\mathbb{R}^n} g \, dx + \int_0^T \int_{\mathbb{R}^n} R \, dx \, dt \right). \end{aligned}$$

By letting $s \rightarrow 0$, we derive

$$\begin{aligned} \limsup_{s \rightarrow 0; s \in [0, T] \setminus F} \left| \int_{\mathbb{R}^n} u \psi \, dx \Big|_s - \int_{\mathbb{R}^n} g \psi \, dx \right| \\ \leq \|\psi - \varphi_i\|_{L^\infty(\mathbb{R}^n)} \left(2 \int_{\mathbb{R}^n} g \, dx + \int_0^T \int_{\mathbb{R}^n} R \, dx \, dt \right). \end{aligned}$$

Since i is arbitrary, it is implied that

$$\lim_{s \rightarrow 0; [0, T] \setminus F} \int_{\mathbb{R}^n} u \psi \, dx \Big|_s = \int_{\mathbb{R}^n} g \psi \, dx. \tag{3.18}$$

Now, we prove that in fact the convergence in (3.18) holds for every $\psi \in C_b(\mathbb{R}^n)$. First, by (3.17), we obtain

$$\limsup_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n} u \, dx \Big|_s \leq \int_{\mathbb{R}^n} g \, dx.$$

Then, by (3.18), we arrive at

$$\int_{\mathbb{R}^n} g \psi_i \, dx = \lim_{s \rightarrow 0; [0, T] \setminus F} \int_{\mathbb{R}^n} u \psi_i \, dx \Big|_s \leq \liminf_{s \rightarrow 0; [0, T] \setminus F} \int_{\mathbb{R}^n} u \, dx \Big|_s,$$

where ψ_i are the increasing sequence of functions in (3.7). Thus, by combining the previous inequalities, we conclude

$$\lim_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n} u \, dx \Big|_s = \int_{\mathbb{R}^n} g \, dx.$$

Moreover,

$$\begin{aligned} \lim_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n} u (1 - \psi_i) \, dx \Big|_s &= \lim_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n} u \, dx \Big|_s - \int_{\mathbb{R}^n} u \psi_i \, dx \Big|_s \\ &= \int_{\mathbb{R}^n} g (1 - \psi_i) \, dx. \end{aligned}$$

Hence,

$$\lim_{i \rightarrow \infty} \limsup_{s \rightarrow 0; s \in [0, T] \setminus F} \int_{\mathbb{R}^n \setminus B(0, 2i)} u \, dx \Big|_s = 0.$$

In conclusion, by Prokhorov’s theorem, we imply (3.16) for every $\psi \in C_b(\mathbb{R}^n)$.

6. In this step, we remove the extra assumptions in Step 2 to prove the existence and vague convergence to the initial data. Notice that, by the definition, the convection solutions depend linearly on R, g . Hence, it is enough to find a convection solution for R^+, g^+ . By Step 5, for the initial data $R = 0, g = e^{-|x|^2}$ there exist a convection solution u_e . Since $\ln(g^+ + e^{-x^2}) \geq -|x|^2$, there exists a convection u_{eg} solution with initial data $R^+, g^+ + e^{-x^2}$. Taking the difference $u_{eg} - u_e$ gives us the desired result. Also, by Step 5, there exist sets $F_1, F_2 \subset [0, T]$ of Lebesgue measure zero such that

$$\begin{aligned} \lim_{s \rightarrow 0; s \in [0, T] \setminus F_1} \int_{\mathbb{R}^n} u_e \psi \, dx \Big|_s &= \int_{\mathbb{R}^n} e^{-x^2} \psi \, dx, \\ \lim_{s \rightarrow 0; s \in [0, T] \setminus F_2} \int_{\mathbb{R}^n} u_{eg} \psi \, dx \Big|_s &= \int_{\mathbb{R}^n} (g^+ + e^{-x^2}) \psi \, dx, \end{aligned}$$

for every $\psi \in C_b(\mathbb{R}^n)$. Hence,

$$\lim_{s \rightarrow 0; s \in [0, T] \setminus (F_1 \cup F_2)} \int_{\mathbb{R}^n} (u_{eg} - u_e) \psi \, dx \Big|_s = \int_{\mathbb{R}^n} g^+ \psi \, dx,$$

for every $\psi \in C_b(\mathbb{R}^n)$.

7. Finally, we prove that the convection solutions are weak solutions and they satisfy the conservation of mass. Let $\phi \in C_0^\infty([0, T] \times \mathbb{R}^n)$ such that $\phi(T, x) = 0$ for every $x \in \mathbb{R}^n$. Then,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} -u_i \partial_t \phi + A_i \nabla u_i \cdot \nabla \phi - u_i v_i \cdot \nabla \phi \, dx \, dt \\ &= \int_{\mathbb{R}^n} g_i \phi \, dx + \int_0^T \int_{\mathbb{R}^n} R_i \phi \, dx \, dt. \end{aligned}$$

Since u_i converges to u pointwise a.e. in $[0, T] \times \mathbb{R}^n$ and weakly in $L^2([0, T], W_{\text{loc}}^{1,p}(\mathbb{R}^n))$ for $1 < p < \frac{n}{n-1}$, up to a subsequence depending on p , we derive that

$$\int_0^T \int_{\mathbb{R}^n} -u \partial_t \phi + A \nabla u \cdot \nabla \phi - u v \cdot \nabla \phi \, dx \, dt = \int_{\mathbb{R}^n} g \phi \, dx + \int_0^T \int_{\mathbb{R}^n} R \phi \, dx \, dt.$$

Thus, u is a weak solution. Finally, we prove that u satisfies the conservation of mass if (1.2) holds. Let take $F \subset [0, T]$ defined in Step 6 and $s \in [0, T] \setminus F$. Assume that $\psi_i \in C_0^\infty(\mathbb{R}^n)$ is the sequence of functions in (3.7).

Define

$$\theta_i(r) := \frac{i}{2} \int_r^T 1_{[s-\frac{1}{i}, s+\frac{1}{i}]}(t) \, dt,$$

for every $r \in [0, T]$. Then, by (1.2), we have

$$f_i := -\partial_t(\theta_i \psi_i) - \operatorname{div} A^* \nabla(\theta_i \psi_i) - v \cdot \nabla(\theta_i \psi_i) \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T] \times \mathbb{R}^n),$$

weakly in $[0, T] \times \mathbb{R}^n$ and $\theta_i \psi_i(T, x) = 0$ for every $x \in \mathbb{R}^n$. Hence,

$$\int_0^T \int_{\mathbb{R}^n} f_i u \, dx \, dt = \int_{\mathbb{R}^n} \theta_i \psi_i g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \theta_i \psi_i R \, dx \, dt.$$

Taking $i \rightarrow \infty$ above and using Lebesgue’s dominated convergence and Lebesgue’s density theorems, we derive that

$$\int_{\mathbb{R}^n} u \, dx \Big|_s = \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dt \, dx,$$

which completes the proof. □

Example 3.4. We use the idea in [32, 35] to demonstrate the non-uniqueness of weak solutions for coefficients that do not satisfy (1.2). Assume that $a_{ij}(t, x) = \delta_{ij} + (a - 1) \frac{x_i x_j}{|x|^2}$ for $(t, x) \in [0, T] \times \mathbb{R}^n$ and $n = 2, 3$, where the positive constant a is to be determined later. Then, by a classical result of Serrin, see [35], the function $u(x) = x_1 |x|^{1-n-\epsilon}$ is a weak solution for $-\operatorname{div} A \nabla u = 0$ in \mathbb{R}^n , where $0 < \epsilon < \frac{1}{2}$ and $a = \frac{n-1}{\epsilon(\epsilon+n-2)}$. We note that $\nabla u \notin W_{loc}^{1,2}(\mathbb{R}^n)$ and $\nabla u \in W_{loc}^{1,p}(\mathbb{R}^n)$ for every $p < \frac{n}{n+\epsilon-1}$. Now, let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\varphi = 1$ in the unit ball $B(0, 1)$ and define $R := -\operatorname{div} A \nabla(u \varphi)$. Since $-\operatorname{div} A \nabla u = 0$ in \mathbb{R}^n weakly and A, u are smooth in $\mathbb{R}^n \setminus \{0\}$, we obtain $R \in C_0^\infty(\mathbb{R}^n)$. If we define $g := u \varphi$, then $u \varphi$ is the weak solution of $\partial_t(u \varphi) - \operatorname{div} A \nabla(u \varphi) = R$ in $[0, T] \times \mathbb{R}^n$ with initial data g , and $u \varphi$ does not belong to $L^2([0, T]; W_{loc}^{1,2}(\mathbb{R}^n))$. Since $g \in L^2(\mathbb{R}^n)$, $R \in L^2([0, T] \times \mathbb{R}^n)$, by variational method in [26] or Galerkin method in [14], there exists a solution $\tilde{u}_i \in L^2([0, T]; W_0^{1,2}(B(0, i)))$ which satisfies $\partial_t \tilde{u}_i - \operatorname{div} A \nabla \tilde{u}_i = R$ weakly in $[0, T] \times B(0, i)$ with initial data g . By standard regularity estimates, we derive

$$a \|\tilde{u}_i\|_{L^2([0, T]; W^{1,2}(B(0, i)))}^2 \leq \|R\|_{L^2([0, T] \times B(0, i))}^2 + \|g\|_{L^2(B(0, i))}^2.$$

Hence, up to a subsequence, \tilde{u}_i converges weakly to \tilde{u} in $L^2([0, T]; W^{1,2}(\mathbb{R}^n))$ which satisfies $\partial_t \tilde{u} - \operatorname{div} A \nabla \tilde{u} = R$ weakly in $[0, T] \times \mathbb{R}^n$ with initial data g . Moreover, $w := u - \tilde{u}$ belongs to $L^1([0, T]; W_{loc}^{1,1}(\mathbb{R}^n))$ and satisfies $\partial_t w - \operatorname{div} A \nabla w = 0$ weakly in $[0, T] \times \mathbb{R}^n$ with the initial data 0 but $w \neq 0$.

4. Existence for drifts induced by potentials

In this section, we assume that $T > 0$ and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a measurable function, which satisfies (1.6). The following examples are the main motivations:

Example 4.1. If $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, then $e^V + e^{-V} \in L^1_{\text{loc}}(\mathbb{R}^n)$. To prove that e^{-V} is a regular weight consider $f \in H^1_{e^{-V}}(\mathbb{R}^n)$ and $\psi_i \in C^\infty_0(\mathbb{R}^n)$ constructed in (3.7). Then,

$$\|(1 - \psi_i)f\|_{H^1_{e^{-V}}(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} \left((1 - \psi_i)^2 + \frac{C_1^2}{i^2} \right) f^2 + (1 - \psi_i)^2 |\nabla f|^2 \, de^{-V}.$$

Hence, by Lebesgue’s dominated convergence $\lim_{i \rightarrow \infty} \|(1 - \psi_i)f\|_{H^1_{e^{-V}}(\mathbb{R}^n)} = 0$.

Moreover, since $\psi_i f$ is compactly supported and e^{-V} is comparable with Lebesgue measure on compact subsets of \mathbb{R}^n , there exists $f_i \in C^\infty_0(\mathbb{R}^n)$ such that $\|\psi_i f - f_i f\|_{H^1_{e^{-V}}(\mathbb{R}^n)} \leq \frac{1}{i}$. In conclusion, by triangle inequality, we derive that $\lim_{i \rightarrow \infty} \|f_i - f\|_{H^1_{e^{-V}}(\mathbb{R}^n)} = 0$.

Example 4.2. Let $V(x) = C \ln|x|$ for $x \in \mathbb{R}^n$ and a constant $-n < C < n$. Then, e^{-V} belongs to Muckenhoupt weight $A_2(\mathbb{R}^n)$, see [21, Ch. 15]. Hence, $e^V + e^{-V} \in L^1_{\text{loc}}(\mathbb{R}^n)$ and, by [21, Thm. 15.21], e^{-V} is a regular weight.

We need the following concept of an adjoint convection solution.

Definition 4.3. Let $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2_{e^{-V}}([0, T] \times \mathbb{R}^n)$. We define the adjoint convection solution $\phi \in L^2([0, T]; H^1_{e^{-V}}(\mathbb{R}^n)) \cap C([0, T]; L^2_{e^{-V}}(\mathbb{R}^n))$ of

$$-\partial_t \phi - e^V \operatorname{div}(e^{-V} \nabla \phi) = f, \tag{4.1}$$

in $[0, T] \times \mathbb{R}^n$ if $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$ and

$$\begin{aligned} & \int_{\mathbb{R}^n} \phi \psi \, de^{-V} \Big|_0^T + \int_0^T \int_{\mathbb{R}^n} \phi \partial_t \psi \, de^{-V} \, dt \\ & + \int_0^T \int_{\mathbb{R}^n} \nabla \phi \cdot \nabla \psi \, de^{-V} \, dt = \int_0^T \int_{\mathbb{R}^n} f \psi \, de^{-V} \, dt, \end{aligned}$$

for every $\psi \in C^\infty_0([0, T] \times \mathbb{R}^n)$.

We need the following auxiliary lemma.

Lemma 4.4. *There exists a sequence $V_i \in C^{0,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that $\inf_{\mathbb{R}^n} V_i > -\infty$, V_i converges pointwise to V , and e^{V_i}, e^{-V_i} converge to e^V, e^{-V} in $L^1(B)$, respectively, for every ball $B \subset \mathbb{R}^n$. Moreover, if $e^{-V} \in L^1(\mathbb{R}^n)$, then e^{-V_i} converges to e^{-V} in $L^1(\mathbb{R}^n)$.*

Proof. Let φ_ϵ be a positive approximation of identity, where ϵ is a positive constant. Define $V_{\epsilon, \theta, \delta}(x) := \varphi_\epsilon * (V \mathbf{1}_{B(0, \frac{1}{\theta})})(x) + \delta \sqrt{|x|^2 + 1} \in C^{0,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ for every $x \in \mathbb{R}^n$, where θ, δ are positive constants. Then, $\inf_{\mathbb{R}^n} V_{\epsilon, \theta, \delta} > -\infty$. Since $e^V + e^{-V} \in L^1_{\text{loc}}(\mathbb{R}^n)$, we obtain that $e^{|V|} \in L^1_{\text{loc}}(\mathbb{R}^n)$. Hence, by $\frac{|V|^n}{n!} \leq e^{|V|}$ for every positive integer n , we derive that $V \in L^p_{\text{loc}}(\mathbb{R}^n)$ for every $p > 0$. Now, we prove that

$$\lim_{(\epsilon, \theta, \delta) \rightarrow 0} \|e^{V_{\epsilon, \theta, \delta}} - e^V\|_{L^1(B)} + \|e^{-V_{\epsilon, \theta, \delta}} - e^{-V}\|_{L^1(B)} = 0, \tag{4.2}$$

for every ball $B \subset \mathbb{R}^n$. Let us fix a ball $B \subset \mathbb{R}^n$. Since B is bounded,

$$e^{V_{\epsilon,\theta,\delta}} + e^{-V_{\epsilon,k,\delta}} \Big|_B \leq C_B e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})} \Big|_B,$$

for constant C_B depending on B . By a variant of Lebesgue’s dominated convergence, see [16, Thm. 1.20], it is enough to prove that

$$\lim_{(\epsilon,\theta) \rightarrow 0} \|e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})} - e^{|V|}\|_{L^1(B)} = 0. \tag{4.3}$$

By Young’s convolution inequality, we obtain

$$\int_B |\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})|^p dx \leq \int_{\mathbb{R}^n} |\varphi_\epsilon * (|V|1_{(1+\epsilon)B})|^p dx \leq \int_{(1+\epsilon)B} |V|^p dx.$$

for every $p \geq 1$. Hence, by the monotone convergence theorem, it is obtained that

$$\begin{aligned} \int_B e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})} dx &= \sum_{j=1}^\infty \int_B \frac{|\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})|^j}{j!} dx \\ &\leq \sum_{j=1}^\infty \int_{(1+\epsilon)B} \frac{|V|^j}{j!} dx = \int_{(1+\epsilon)B} e^{|V|} < \infty. \end{aligned}$$

Also, by Fatou’s lemma, we have

$$\int_B e^{|V|} dx \leq \liminf_{(\epsilon,\theta) \rightarrow 0} \int_B e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})} dx.$$

Hence, $\lim_{(\epsilon,\theta) \rightarrow 0} \int_B e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})} dx = \int_B e^{|V|} dx$. Then, since $e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})}$ converges pointwise to $e^{|V|}$, by Riesz–Scheffé–Brezis–Lieb lemma we obtain $e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})}$ converges to $e^{|V|}$ in $L^1(B)$ as (ϵ, θ) goes to 0. This completes the proof of (4.2). Now, if $e^{-V} \in L^1(\mathbb{R}^n)$, we construct a sequence of $\epsilon_i, \theta_i, \delta_i$ converging to zero such that

$$\lim_{i \rightarrow \infty} \|e^{-V_{\epsilon_i,\theta_i,\delta_i}} - e^{-V}\|_{L^1(\mathbb{R}^n)} = 0 \tag{4.4}$$

Since $e^{-V_{\epsilon,\theta,\delta}} \leq 1_{B(0,\frac{1+\epsilon}{\theta})} e^{\varphi_\epsilon * (|V|1_{B(0,\frac{1}{\theta})})} + e^{-\delta\sqrt{|x|^2+1}}$ and $\int_{\mathbb{R}^n} e^{-\delta\sqrt{|x|^2+1}} < \infty$, by (4.3) and a variant of Lebesgue’s dominated convergence, see [16, Thm. 1.20], we have

$$\lim_{\epsilon \rightarrow 0} \|e^{-V_{\epsilon,\theta,\delta}} - e^{-V1_{B(0,\frac{1}{\theta})} - \delta\sqrt{|x|^2+1}}\|_{L^1(\mathbb{R}^n)} = 0. \tag{4.5}$$

Also, by Lebesgue’s dominated convergence and $e^{-V1_{B(0,\frac{1}{\theta})} - \delta\sqrt{|x|^2+1}} \leq e^{-V} + e^{-\delta\sqrt{|x|^2+1}}$, we deduce

$$\begin{aligned} \lim_{\theta \rightarrow 0} \|e^{-V1_{B(0,\frac{1}{\theta})} - \delta\sqrt{|x|^2+1}} - e^{-V - \delta\sqrt{|x|^2+1}}\|_{L^1(\mathbb{R}^n)} &= 0, \\ \lim_{\delta \rightarrow 0} \|e^{-V - \delta\sqrt{|x|^2+1}} - e^{-V}\|_{L^1(\mathbb{R}^n)} &= 0. \end{aligned} \tag{4.6}$$

In conclusion, by triangle inequality, (4.5), and (4.6), there exists a sequence $(\epsilon_i, \theta_i, \delta_i)$ converging to 0 such that (4.4) holds. \square

For simplicity, define the weights

$$w_i := e^{-V_i}, \quad w := e^{-V},$$

where V_i is the sequence in Lemma 4.4. We now construct an approximating sequence of weak solutions for (4.1).

Proposition 4.5. *For every $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T]; L^2_w(\mathbb{R}^n))$, there exists a unique adjoint convection solution $\phi \in L^2([0, T]; H^1_w(\mathbb{R}^n)) \cap C([0, T]; L^2_w(\mathbb{R}^n))$ of (4.1) in $[0, T] \times \mathbb{R}^n$. Also, ϕ satisfies the following:*

(i). *For every uniformly bounded sequence $f_i \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T]; L^2_w(\mathbb{R}^n))$ which converges to f in $L^2_w([0, T] \times \mathbb{R}^n)$, there exists a sequence ϕ_i belonging to $L^2([0, T]; H^1_w(\mathbb{R}^n)) \cap C([0, T]; L^2_w(\mathbb{R}^n))$ which are adjoint convection solutions of*

$$-\partial_t \phi_i - e^{V_i} \operatorname{div}(e^{-V_i} \nabla \phi_i) = f_i, \tag{4.7}$$

in $[0, T] \times \mathbb{R}^n$, and ϕ_i converges to ϕ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$.

(ii).

$$(T - t) \inf f \leq \phi(t, \cdot) \leq (T - t) \sup f,$$

$$\begin{aligned} \sup_{0 \leq t \leq T} \frac{1}{p} \|\phi(t, \cdot)\|_{L^p_w(\mathbb{R}^n)}^p + \|\phi\|_{L^2_w([0, T] \times \mathbb{R}^n)}^2 &\leq \frac{e^{(p-1)T}}{p} \|f\|_{L^p_w([0, T] \times \mathbb{R}^n)}^p, \\ \sup_{0 \leq t \leq T-h} \|\tau_h \phi(t, \cdot) - \phi(t, \cdot)\|_{L^2_w(\mathbb{R}^n)}^2 &\leq e^T \left(\|\tau_h f - f\|_{L^2_w([0, T-h] \times \mathbb{R}^n)}^2 + \int_{T-h}^T \int_{\mathbb{R}^n} |f|^2 \, dw \right), \end{aligned}$$

for every $p > 1$, where $f \in L^p_w(\mathbb{R}^n)$, and $0 < h < T$.

Proof. For the uniqueness, it is enough to prove that 0 is the only adjoint convection solution of (4.1) if $f = 0$. Let $\phi \in L^2([0, T]; H^1_w(\mathbb{R}^n)) \cap C([0, T]; L^2_w(\mathbb{R}^n))$ be an adjoint convection solution of

$$-\partial_t \phi - e^V \operatorname{div}(e^{-V} \nabla \phi) = 0, \tag{4.8}$$

in $[0, T] \times \mathbb{R}^n$. We assume that $0 < \epsilon < T$, $\eta_\epsilon \in C^\infty_0(\mathbb{R})$ is an approximation of identity, and $\beta \in C^\infty_0((\epsilon, T - \epsilon))$. Then, for every $\psi \in C^\infty_0([0, T] \times \mathbb{R}^n)$ we have

$$\int_0^T \int_{\mathbb{R}^n} \phi \partial_t (\eta_\epsilon * (\beta \psi)) \, dw \, dt + \int_0^T \int_{\mathbb{R}^n} \nabla \phi \cdot \nabla (\eta_\epsilon * (\beta \psi)) \, dw \, dt = 0,$$

where we used $\eta_\epsilon * (\beta \psi)(t, x) = 0$ if $t = 0, T$ and $x \in \mathbb{R}^n$. Hence,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} -\beta \psi \partial_t (\eta_\epsilon * \phi) \beta \psi \, dw \, dt \\ &+ \int_0^T \int_{\mathbb{R}^n} \beta \nabla (\eta_\epsilon * \phi) \cdot \nabla \psi \, dw \, dt = 0 \end{aligned}$$

for every $\psi \in C_0^\infty([0, T] \times \mathbb{R}^n)$. Since w is a regular weight and $\phi \in L^2([0, T]; H_w^1(\mathbb{R}^n))$, we can take an approximation of $\eta_\epsilon * \phi$ by elements in $C_0^\infty([\epsilon, T - \epsilon] \times \mathbb{R}^n)$ in $L^2([\epsilon, T - \epsilon]; H_w^1(\mathbb{R}^n))$ to obtain

$$\int_0^T \int_{\mathbb{R}^n} -\beta \partial_t(\eta_\epsilon * \phi) \eta_\epsilon * \phi \, dw \, dt + \int_0^T \int_{\mathbb{R}^n} \beta \nabla \eta_\epsilon * \phi \cdot \nabla \eta_\epsilon * \phi \, dw \, dt = 0.$$

Thus,

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^n} (\eta_\epsilon * \phi)^2 \partial_t \beta \, dw \, dt + \int_0^T \int_{\mathbb{R}^n} \beta \nabla \eta_\epsilon * \phi \cdot \nabla \eta_\epsilon * \phi \, dw \, dt = 0.$$

Letting $\epsilon \rightarrow 0$ in the above derives that

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \phi^2 \partial_t \beta \, dw \, dt + \int_0^T \int_{\mathbb{R}^n} \beta |\nabla \phi|^2 \, dw \, dt = 0 \tag{4.9}$$

for every $\beta \in C_0^\infty((0, T))$. By approximation and the assumption $\phi \in C([0, T]; L_w^2(\mathbb{R}^n)) \cap L^2([0, T]; H_w^1(\mathbb{R}^n))$, we can assume that $\beta \in C^{0,1}([0, T])$ and $\beta(0) = \beta(T) = 0$ in (4.9). Define

$$\beta_i(t) := \begin{cases} 0 & 0 \leq t \leq \frac{1}{i}, \\ it - 1 & \frac{1}{i} \leq t \leq \frac{2}{i}, \\ 1 & \frac{2}{i} \leq t \leq 1 - \frac{2}{i}, \\ i(T - t) - 1 & T - \frac{2}{i} \leq t \leq T - \frac{1}{i}, \\ 0 & T - \frac{1}{i} \leq t \leq T, \end{cases}$$

for $i > \frac{1}{T} + 1$. Then, by definition $\beta_i \in C^{0,1}([0, T])$ and $\beta_i(0) = \beta_i(T) = 0$. Hence, by using $\phi \in C([0, T]; L_w^2(\mathbb{R}^n)) \cap L^2([0, T]; H_w^1(\mathbb{R}^n))$ and replacing β by β_i in (4.9), we deduce

$$\int_{\mathbb{R}^n} \phi^2 \, dw \Big|_0 - \int_{\mathbb{R}^n} \phi^2 \, dw \Big|_T + \int_0^T \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dw \, dt = \lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} \phi^2 \partial_t \beta_i \, dw \, dt + \int_0^T \int_{\mathbb{R}^n} \beta_i |\nabla \phi|^2 \, dw \, dt = 0.$$

Hence, $\nabla \phi = 0$ and ϕ is a function of time. By (4.8), we imply that $-\partial_t \phi = 0$ in $[0, T] \times \mathbb{R}^n$. Since $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$, we conclude that $\phi = 0$. Now, to prove the existence part, let $f_i \in C_0^\infty([0, T] \times \mathbb{R}^n)$ where f_i is a uniformly bounded sequence which converges to f in $L^2([0, T]; L_w^2(\mathbb{R}^n))$. By the standard theory of parabolic equations, see [1], there exists a smooth weak solution $\phi_i \in L^2([0, T]; H^1(\mathbb{R}^n))$ satisfying (4.7) and $\phi_i(T, x) = 0$ for every $x \in \mathbb{R}^n$. Then, by Proposition 3.2, we have

$$(T - t) \inf f_i \leq \phi_i(t, \cdot) \leq (T - t) \sup f_i, \tag{4.10}$$

for every $0 \leq t \leq T$. Let $0 \leq T' \leq T$. Using $\phi_i |\phi_i|^{p-2} w_i$, for $p > 1$, as a test function gives

$$\partial_t \left| \frac{1}{p} \int_{\mathbb{R}^n} |\phi_i|^p dw_i \right|_t + p \int_{\mathbb{R}^n} |\phi_i|^{p-2} |\nabla \phi_i|^2 dw_i dt \Big|_t \leq \int_{\mathbb{R}^n} |f_i| |\phi_i|^{p-1} dw_i \Big|_t.$$

Hence, by Young and Grönwall’s inequality, we obtain

$$\begin{aligned} & \sup_{T' \leq t \leq T} \left| \frac{1}{p} \int_{\mathbb{R}^n} |\phi_i|^p dw_i \right|_t + p e^{-(p-1)T} \int_{T'}^T \int_{\mathbb{R}^n} |\phi_i|^{p-2} |\nabla \phi_i|^2 dw_i dt \\ & \leq \frac{e^{(p-1)T}}{p} \int_{T'}^T \int_{\mathbb{R}^n} |f_i|^p dw_i dt, \end{aligned} \tag{4.11}$$

for every $p > 1$. Now, by definition $\Phi_i := \tau_h \phi_i - \phi_i$ is the weak solution of

$$-\partial_t \Phi_i - \Delta \Phi_i + \nabla V_i \cdot \nabla \Phi_i = \tau_h f_i - f_i,$$

in $[0, T - h] \times \mathbb{R}^n$ with $\Phi_i(T - h, x) = -\phi_i(T - h, x)$ for $x \in \mathbb{R}^n$, where the shift operator τ_h is defined in Sect. 2.1. Taking $\Phi_i w_i$ as a test function and using Young’s inequality as above, we obtain

$$\sup_{0 \leq t \leq T-h} \int_{\mathbb{R}^n} \Phi_i^2 dw_i \Big|_t \leq e^T \int_0^{T-h} \int_{\mathbb{R}^n} |\tau_h f_i - f_i|^2 dw_i dt + \int_{\mathbb{R}^n} \Phi_i^2(T - h, x) dw_i. \tag{4.12}$$

Note that by integration by part the term $\nabla V_i \cdot \nabla \phi_i$ disappears. Indeed, we used

$$\begin{aligned} & \int_0^{T-h} \int_{\mathbb{R}^n} (-\Delta \Phi_i + \nabla V_i \cdot \nabla \Phi_i) \Phi_i w_i dx dt \\ & = \int_0^{T-h} \int_{\mathbb{R}^n} |\nabla \Phi_i|^2 dw_i dt + \int_0^{T-h} \int_{\mathbb{R}^n} \Phi_i \nabla \Phi_i \cdot \nabla w_i dx dt \\ & \quad + \int_0^{T-h} \int_{\mathbb{R}^n} \Phi_i w_i \nabla V_i \cdot \nabla \Phi_i \\ & = \int_0^{T-h} \int_{\mathbb{R}^n} |\nabla \phi_i|^2 dw_i dt \geq 0, \end{aligned}$$

where we noted that $\nabla w_i = \nabla e^{-V_i} = -w_i \nabla V_i$. Let $B \subset \mathbb{R}^n$ be a fixed ball and $0 \leq t_1 < t_2 \leq T$. Then, by Cauchy–Schwarz inequality and (4.11), it is implied that

$$\begin{aligned} & \left(\sup_{0 \leq t \leq T-h} \int_B |\tau_h \phi_i - \phi_i| dx \Big|_t \right)^2 \leq w_i^{-1}(B) \sup_{0 \leq t \leq T-h} \int_B |\tau_h \phi_i - \phi_i|^2 dw_i \Big|_t \\ & \leq e^T w_i^{-1}(B) \left(\int_0^{T-h} \int_{\mathbb{R}^n} |\tau_h f_i - f_i|^2 dw_i dt + \int_{T-h}^T \int_{\mathbb{R}^n} |f_i|^2 dw_i dt \right). \end{aligned}$$

In conclusion, $\lim_{h \rightarrow 0} \sup_i \|\tau_h \phi_i - \phi_i\|_{L^1([0, T-h] \times B)} = 0$. Moreover, by (4.11) and Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left(\int_B \left| \nabla \int_{t_1}^{t_2} \phi_i \, dt \right| dx \right)^2 &\leq (t_2 - t_1) w_i^{-1}(B) \int_B \int_{t_1}^{t_2} |\nabla \phi_i|^2 \, dt \, dw_i \\ &\leq T e^T w_i^{-1}(B) \int_B \int_{t_1}^{t_2} f_i^2 \, dt \, dw_i \end{aligned}$$

Thus, by Sobolev embedding, $\int_{t_1}^{t_2} \phi_i \, dt$ is relatively compact in $L^1(B)$. Now, by the classical result in [37, Thm. 1], ϕ_i is relatively compact in $C([0, T]; L^1(B))$. Hence, up to a subsequence, ϕ_i converges to $\phi \in L^\infty([0, T] \times \mathbb{R}^n) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$ and $L^\infty([0, T] \times \mathbb{R}^n)$ -weak*, where $\phi(T, x) = 0$ for a.e. $x \in \mathbb{R}^n$. By Lemma 4.4 and (4.11), for every ball $B \subset \mathbb{R}^n$, we have $w_i^{\frac{1}{2}} \nabla \phi_i$ is uniformly bounded in $L^2([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$ and $w_i^{\frac{1}{2}}, w_i^{-\frac{1}{2}}$ converges to $w^{\frac{1}{2}}, w^{-\frac{1}{2}}$, respectively, in $L^2_{\text{loc}}(\mathbb{R}^n)$. Now, we will use ϕ_i converges to ϕ in $L^\infty([0, T] \times \mathbb{R}^n)$ -weak* to show that $w_i^{\frac{1}{2}} \nabla \phi_i$ converges to $w^{\frac{1}{2}} \nabla \phi$ weakly in $L^2([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$. In fact, if $w_i^{\frac{1}{2}} \nabla \phi_i$ converges to $F = (F_1, \dots, F_n)$ weakly in $L^2([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$, then

$$\lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} \Phi w_i^{\frac{1}{2}} \partial_{x_j} \phi_i \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} \Phi F_j \, dx \, dt,$$

for every $1 \leq j \leq n$ and $\Phi \in L^2([0, T] \times \mathbb{R}^n)$. Now, by Hölder’s inequality and the convergence of $w_i^{-\frac{1}{2}}$ to $w^{-\frac{1}{2}}$ in $L^2(\mathbb{R}^n)$, we obtain

$$\lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} w_i^{-\frac{1}{2}} \varphi \cdot w_i^{\frac{1}{2}} \partial_{x_j} \phi_i \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} \varphi F_j w^{-\frac{1}{2}} \, dx \, dt,$$

for every $1 \leq j \leq n$ and $\varphi \in C^\infty_0([0, T] \times \mathbb{R}^n)$. Then, by the convergence of ϕ_i to ϕ in $L^\infty([0, T] \times \mathbb{R}^n)$ -weak*, we arrive at

$$\int_0^T \int_{\mathbb{R}^n} \varphi F_j w^{-\frac{1}{2}} \, dx \, dt = \lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} \varphi \partial_{x_j} \phi_i \, dx \, dt = - \int_0^T \int_{\mathbb{R}^n} \phi \partial_{x_j} \varphi \, dx \, dt.$$

In conclusion, $F = w^{\frac{1}{2}} \nabla \phi$. Moreover, using (4.11) and Lemma 4.4 again, we derive that $w_i^{\frac{1}{2}} \phi_i$ converges to $w^{\frac{1}{2}} \phi$ weakly in $L^2([0, T] \times \mathbb{R}^n)$. Now, we prove that $\phi \in C([0, T]; L^2_w(\mathbb{R}^n))$. By Fatou’s lemma, (4.11), and (4.12), we obtain

$$\begin{aligned} &\sup_{0 \leq t \leq T-h} \int_{\mathbb{R}^n} |\tau_h \phi - \phi|^2 \, dw \Big|_t \\ &\leq e^T \left(\int_0^{T-h} \int_{\mathbb{R}^n} |\tau_h f - f|^2 \, dw \, dt + \int_{T-h}^T \int_{\mathbb{R}^n} f^2 \, dw \, dt \right), \end{aligned}$$

which proves that $\phi \in C([0, T]; L^2_w(\mathbb{R}^n))$. Then, using (4.7), (4.10), and the weak convergence of $w_i^{\frac{1}{2}} \nabla \phi_i, w_i^{\frac{1}{2}} \phi_i$ to $w^{\frac{1}{2}} \nabla \phi, w^{\frac{1}{2}} \phi$, respectively, we imply that ϕ is a weak

solution of (4.1) with $\phi(T, x) = 0$ a.e. in \mathbb{R}^n . Since ϕ is unique, we derive that ϕ_i converges to ϕ in $C([0, T]; L^1_{loc}(\mathbb{R}^n))$ without any need to pass to a subsequence. To complete the proof of part (i), we note that the proof above applies to any $f_i \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2_w([0, T] \times \mathbb{R}^n)$ converging to f in $L^\infty([0, T] \times \mathbb{R}^n)$ -weak* and $L^2_w([0, T] \times \mathbb{R}^n)$. Finally, recall that, ϕ_i converges to ϕ in $C([0, T]; L^1_{loc}(\mathbb{R}^n))$. Hence, part (ii) is obtained by applying Fatou’s lemma in (4.10), (4.11), and in (4.12). \square

The following lemma derives some properties of the adjoint convection solution if the right-hand side is time-independent.

Lemma 4.6. *Let $f \in C^\infty_0(\mathbb{R}^n)$ and ϕ be the adjoint convection solution of (4.1) in the set $[0, T] \times \mathbb{R}^n$. Then, for every $0 \leq s \leq T$, $\phi^s(t, x) := \phi(t + T - s, x)$, for $0 \leq t \leq s$, $x \in \mathbb{R}^n$, is the adjoint convection solution for*

$$-\partial_t \phi^s - e^V \operatorname{div}(e^{-V} \nabla \phi^s) = f,$$

in $[0, s] \times \mathbb{R}^n$. Moreover, for every weak solution $\phi_i \in C^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T]; H^1(\mathbb{R}^n))$ of $-\partial_t \phi_i - \Delta \phi_i + \nabla V_i \cdot \nabla \phi_i = 0$ with $\phi_i(T, \cdot) = 0$, we have $\partial_t \phi_i$ converges to $\partial_t \phi$ in $C([0, T]; L^1_{loc}(\mathbb{R}^n))$, $\partial_t \phi(T, x) = -f(x)$ for a.e. $x \in \mathbb{R}^n$, and

$$\|\partial_t \phi\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

Proof. The proof uses the argument in Proposition 4.5. Let V_i be as in Lemma 4.4 and $\phi_i \in L^2([0, T]; H^1(\mathbb{R}^n))$ be the smooth weak solution for

$$-\partial_t \phi_i - \Delta \phi_i + \nabla V_i \cdot \nabla \phi_i = f,$$

in $[0, T] \times \mathbb{R}^n$ with $\phi_i(T, x) = 0$ for every $x \in \mathbb{R}^n$. Then, $\phi_i^s(t, x) := \phi_i(t + T - s, x)$ for $0 \leq t \leq s$, $x \in \mathbb{R}^n$ is the adjoint convection solution of

$$-\partial_t \phi_i^s - e^{V_i} \operatorname{div}(e^{-V_i} \nabla \phi_i^s) = f,$$

in $[0, s] \times \mathbb{R}^n$. Since ϕ_i^s converges pointwise to ϕ^s , by Proposition 4.5, ϕ^s is the adjoint convection solution of

$$-\partial_t \phi^s - e^V \operatorname{div}(e^{-V} \nabla \phi^s) = f,$$

in $[0, s] \times \mathbb{R}^n$. For the second part, we have

$$-\partial_t(\partial_t \phi_i) - \Delta(\partial_t \phi_i) + \nabla V_i \cdot \nabla(\partial_t \phi_i) = 0, \tag{4.13}$$

weakly in $[0, T] \times \mathbb{R}^n$ and

$$\partial_t \phi_i(T, x) = -\Delta \phi_i(T, x) + \nabla V_i(x) \cdot \nabla \phi_i(T, x) - f(x) = -f(x) \tag{4.14}$$

for every $x \in \mathbb{R}^n$. Hence, by the maximum principle, see [1], we arrive at

$$\|\partial_t \phi_i\|_{L^\infty} \leq \|f\|_{L^\infty}. \tag{4.15}$$

Define $\Phi_i := \partial_t \phi_i - f$. Then, Φ_i is the weak solution of

$$-\partial_t \Phi_i - \Delta \Phi_i + \nabla V_i \cdot \nabla \Phi_i = \Delta f - \nabla V_i \cdot \nabla f, \tag{4.16}$$

in $[0, T] \times \mathbb{R}^n$ with $\Phi_i(T, x) = 0$ for every $x \in \mathbb{R}^n$. Hence, by taking $\Phi_i w_i$ as a test function for (4.16), we obtain that

$$-\frac{1}{2} \partial_t \int_{\mathbb{R}^n} \Phi_i^2 dw_i \Big|_t + \int_{\mathbb{R}^n} |\nabla \Phi_i|^2 dw_i = - \int_{\mathbb{R}^n} \nabla f \cdot \nabla \Phi_i dw_i.$$

In conclusion, by Young’s inequality, we derive that

$$\sup_{T' \leq t \leq T} \int_{\mathbb{R}^n} |\Phi_i|^2 dw_i \Big|_t + \int_{T'}^T \int_{\mathbb{R}^n} |\nabla \Phi_i|^2 dw_i dt \leq \int_{T'}^T \int_{\mathbb{R}^n} |\nabla f|^2 dw_i dt, \tag{4.17}$$

for every $0 \leq T' \leq T$. Let $0 < h < T$ and $\Psi_i^h := \tau_h \Phi_i - \Phi_i$. Then, $\Psi_i^h(T - h, x) = -\Phi_i(T - h, x)$ for every $x \in \mathbb{R}^n$ and

$$-\partial_t \Psi_i^h - \Delta \Psi_i^h + \nabla V_i \cdot \nabla \Psi_i^h = 0, \tag{4.18}$$

in $[0, T - h] \times \mathbb{R}^n$. By taking the test function $\Psi_i^h w_i$ for (4.18) and using (4.17), it is obtained that

$$\begin{aligned} \sup_{0 \leq t \leq T-h} \int_{\mathbb{R}^n} |\Psi_i^h|^2 dw_i \Big|_t + \int_{T'}^T \int_{\mathbb{R}^n} |\nabla \Psi_i^h|^2 dw_i dt &\leq \int_{\mathbb{R}^n} |\Phi_i|^2 dw_i \Big|_{T-h} \\ &\leq \int_{T-h}^T \int_{\mathbb{R}^n} |\nabla f|^2 dw_i dt. \end{aligned}$$

Hence, by Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left| \int_0^{T-h} \int_B |\Psi_i^h| dx dt \right|^2 &\leq T w_i^{-1}(B) \int_0^{T-h} \int_B |\Psi_i^h|^2 dw_i dt \\ &\leq T^2 w_i^{-1}(B) \int_{T-h}^T \int_{\mathbb{R}^n} |\nabla f|^2 dw_i dt, \end{aligned}$$

for every ball $B \subset \mathbb{R}^n$. Therefore,

$$\lim_{h \rightarrow 0} \limsup_{i \rightarrow \infty} \|\tau_h \Phi_i - \Phi_i\|_{L^1([0, T-h] \times B)} = 0,$$

for every ball $B \subset \mathbb{R}^n$. Moreover, by using Cauchy–Schwarz inequality and (4.17), we deduce that

$$\begin{aligned} \left(\int_B \left| \nabla \int_{t_1}^{t_2} \Phi_i dt \right| dx \right)^2 &\leq (t_2 - t_1) w_i^{-1}(B) \int_B \int_{t_1}^{t_2} |\nabla \Phi_i|^2 dt dw_i \\ &\leq T w_i^{-1}(B) \int_B \int_{t_1}^{t_2} |\nabla f|^2 dt dw_i \end{aligned}$$

for every $0 \leq t_1 \leq t_2 \leq T$ and ball $B \subset \mathbb{R}^n$. Then, for every $0 \leq t_1 \leq t_2 \leq T$, by Sobolev embedding, $\int_{t_1}^{t_2} \Phi_i \, dt$ is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^n)$. Hence, by the classical result in [37, Thm. 1], $\partial_t \phi_i = \Phi_i + f$ is relatively compact in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$. Then, since ϕ_i converges to ϕ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$ by Proposition 4.5, we obtain that $\partial_t \phi_i$ converges to $\partial_t \phi$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$. This together with (4.14) and (4.15) completes the proof. \square

We also need the following L^1 -estimate lemma.

Lemma 4.7. *Let $f \in C_0([0, T] \times \mathbb{R}^n)$ and ϕ be the adjoint convection solution of (4.1) in $[0, T] \times \mathbb{R}^n$. Then,*

$$\sup_{0 \leq t \leq T} \left| \int_{\mathbb{R}^n} |\phi| \, dw \right|_t \leq \int_0^T \int_{\mathbb{R}^n} |f| \, dw \, dt.$$

Moreover, if $w \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \phi \, dw \Big|_s = \int_s^T \int_{\mathbb{R}^n} f \, dw \, dt,$$

for every $0 \leq s \leq T$.

Proof. Without loss of generality, by decomposing $f = f^+ - f^-$, we can assume that f is non-negative. Let ϕ_i be the approximating sequence in Lemma 4.4 and consider the test functions $\psi_j w_i$ where ψ_j is defined in (3.7). Then, by Lemma 4.4, $0 \leq \phi_i \leq T \|f\|_{L^\infty([0, T] \times \mathbb{R}^n)}$ and

$$\int_{\mathbb{R}^n} \phi_i \psi_j \, dw_i \Big|_t + \int_t^T \int_{\mathbb{R}^n} -\phi_i \operatorname{div}(w_i \nabla \psi_j) \, dx \, dt = \int_t^T \int_{\mathbb{R}^n} f \psi_j \, dw_i \, dt.$$

Now, by Hölder’s inequality and Lemma 4.5, we obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left| \int_t^T \int_{\mathbb{R}^n} -\phi_i \operatorname{div}(w_i \nabla \psi_j) \, dx \, dt \right| \\ & \leq \|w_i \phi_i\|_{L^p([0, T] \times \mathbb{R}^n)} \|w_i^{-1} \operatorname{div}(w_i \nabla \psi_j)\|_{L^{p'}([0, T] \times \mathbb{R}^n)} \\ & \leq e^T \sup_{\mathbb{R}^n} w_i^{\frac{p-1}{p}} \|f\|_{L^p_{w_i}([0, T] \times \mathbb{R}^n)} \|w_i^{-1} \operatorname{div}(w_i \nabla \psi_j)\|_{L^{p'}([0, T] \times \mathbb{R}^n)}, \end{aligned}$$

for every $p > 1$. Letting $p < \frac{n}{n-1}$ and taking $j \rightarrow \infty$ in the inequality above, we arrive at

$$\int_{\mathbb{R}^n} \phi_i \, dw_i \Big|_s = \int_s^T \int_{\mathbb{R}^n} f \, dw_i \, dt,$$

for every $s \in [0, T]$, where we used Lemma 4.4 to derive

$$\begin{aligned} \sup_{\mathbb{R}^n} w_i &= e^{-\inf_{\mathbb{R}^n} V_i} < \infty, \\ \sup_{\mathbb{R}^n} w_i^{-1} |\nabla w_i| &\leq \|V_i\|_{C^{0,1}(\mathbb{R}^n)} < \infty. \end{aligned}$$

Now, using Lemma 4.4 and Proposition 4.5, $\phi_i(s, \cdot)$ converges pointwise to $\phi(s, \cdot)$ for every $s \in [0, T]$, up to a subsequence depending on s , and w_i converges to w pointwise and in $L^1_{loc}(\mathbb{R}^n)$. Hence, by Fatou’s lemma and Lebesgue’s dominated convergence, it is obtained that

$$\int_{\mathbb{R}^n} \phi \, dw \Big|_s \leq \int_s^T \int_{\mathbb{R}^n} f \, dw \, dt.$$

for every $0 \leq s \leq T$. Moreover, by Lemma 4.4, Proposition 4.5, and Lebesgue’s dominated convergence, the inequality above is equality if $w \in L^1(\mathbb{R}^n)$. \square

Now, we define the notion of convection solutions.

Definition 4.8. We say that $\mu \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ is a convection solution for

$$\partial_t \mu - \Delta \mu - \operatorname{div}(\mu \nabla V) = R, \tag{4.19}$$

with initial value g if

$$\int_0^T \int_{\mathbb{R}^n} f \, d\mu = \int_{\mathbb{R}^n} \phi g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \phi R \, dx \, dt,$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$ and weak solution ϕ of (4.1). We say that a convection solution μ for

$$\partial_t \mu - \Delta \mu - \operatorname{div}(\mu \nabla V) = R,$$

with the initial value g satisfies the conservation of mass if the measure $\nu(I) := \mu(I \times \mathbb{R}^n)$, for Borel subsets $I \subset [0, T]$, is absolutely continuous with respect to Lebesgue’s measure and

$$\frac{d\nu}{d\mathcal{L}^1}(s) = \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dx \, dt,$$

for a.e. $s \in [0, T]$.

Proof of Theorem 1.2. We prove the Theorem in several steps.

1. Similar to Theorem 1.1, the uniqueness follows directly by the definition. To prove the existence, by the linearity of the equation, we can assume that g, R are non-negative. Let g_i, R_i be the non-negative smooth approximation of g, R in $L^1(\mathbb{R}^n), L^1([0, T] \times \mathbb{R}^n)$ constructed in Theorem 1.1. By a standard theory of parabolic equations, see [1], there exists a non-negative weak solution $\mu_i \in C^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T]; H^1(\mathbb{R}^n))$ of

$$\partial_t \mu_i - \Delta \mu_i - \operatorname{div}(\mu_i \nabla V_i) = R_i.$$

with $\mu_i(0, x) = g_i(x)$ for every $x \in \mathbb{R}^n$. By Theorem 1.1, we have

$$\mu_i([0, T] \times \mathbb{R}^n) \leq T \int_{\mathbb{R}^n} g_i \, dx + T \int_0^T \int_{\mathbb{R}^n} R_i \, dx \, dt,$$

and

$$\int_0^T \int_{\mathbb{R}^n} f \, d\mu_i = \int_{\mathbb{R}^n} \phi_i g_i \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \phi_i R_i \, dx \, dt,$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$, where $\phi_i \in C^\infty([0, T] \times \mathbb{R}^n) \cap L^2([0, T]; H^1(\mathbb{R}^n))$ is the weak solution of

$$-\partial_t \phi_i - \Delta \phi_i + \nabla \phi_i \cdot \nabla V_i = f,$$

in $[0, T] \times \mathbb{R}^n$ with $\phi(T, x) = 0$ for $x \in \mathbb{R}^n$. Hence, up to a subsequence, μ_i converges to $\mu \in \mathcal{M}([0, T] \times \mathbb{R}^n)$ in the vague topology, see [16, Thm. 1.41]. Moreover, by Lemma 4.4, ϕ_i is uniformly bounded and converges to $\phi \in L_w^2([0, T]; H_w^1(\mathbb{R}^n)) \cap C([0, T]; L_w^2(\mathbb{R}^n))$ in $C([0, T]; L_{loc}^1(\mathbb{R}^n))$, where ϕ is the unique adjoint convection solution of (4.1). In conclusion,

$$\int_0^T \int_{\mathbb{R}^n} f \, d\mu = \int_{\mathbb{R}^n} \phi g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \phi R \, dx \, dt, \tag{4.20}$$

for every $f \in C_0^\infty([0, T] \times \mathbb{R}^n)$, where ϕ is the adjoint convection solution of (4.1), and

$$|\mu|([0, T] \times \mathbb{R}^n) \leq T \int_{\mathbb{R}^n} g \, dx \Big|_0 + T \int_0^T \int_{\mathbb{R}^n} R \, dx \, dt. \tag{4.21}$$

Moreover,

$$\lim_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} f \, d\mu_i = \int_{\mathbb{R}^n} \phi g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \phi_i R \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} f \, d\mu.$$

Thus, μ_i converges vaguely to μ without any need to pass to a subsequence.

2. Assume that

$$\int_{\mathbb{R}^n} |g|^{q'} w^{-\frac{q'}{q}} \, dx + \int_0^T \left(\int_{\mathbb{R}^n} |R|^{p'} w^{-\frac{p'}{p}} \, dx \right)^{\frac{1}{p'}} dt < \infty,$$

for some $p > 1, q > 1$. To show that $\mu \in L^1([0, T] \times \mathbb{R}^n)$, by Radon–Nikodym, it is enough to prove that $\mu(E) = 0$ for every subset $E \subset [0, T] \times \mathbb{R}^n$ of Lebesgue measure zero. Consider a subset $E \subset [0, T] \times \mathbb{R}^n$ of Lebesgue measure zero and functions $f_i \in C_0^\infty([0, T] \times \mathbb{R}^n)$ approximating 1_E in $L^\infty([0, T] \times \mathbb{R}^n)$ -weak*. By Lemma 4.4, there exists an adjoint convection solution $\gamma_i \in L^2([0, T]; H_w^1(\mathbb{R}^n)) \cap C([0, T]; L_w^2(\mathbb{R}^n))$ of

$$-\partial_t \gamma_i - e^V \operatorname{div}(e^{-V} \nabla \gamma_i) = f_i,$$

in $[0, T] \times \mathbb{R}^n$. Hence,

$$\int_0^T \int_{\mathbb{R}^n} f_i \, d\mu = \int_{\mathbb{R}^n} \gamma_i g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \gamma_i R \, dx \, dt.$$

Then, by Lemma 4.4 and Hölder’s inequality, we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^n} f_i \, d\mu \right| &\leq \|\gamma_i(0, \cdot)\|_{L_w^q(\mathbb{R}^n)} \|g w^{-1/q}\|_{L^{q'}(\mathbb{R}^n)} \\ &\quad + T^{1/p} \sup_{0 \leq t \leq T} \|\gamma_i(t, \cdot)\|_{L_w^p(\mathbb{R}^n)} \|R w^{-1/p}\|_{L^1([0, T]; L^{p'}(\mathbb{R}^n))} \\ &\leq e^T \|f_i\|_{L_w^q([0, T] \times \mathbb{R}^n)} \|g w^{-1/q}\|_{L^{q'}(\mathbb{R}^n)} \\ &\quad + T^{1/p} e^T \|f_i\|_{L_w^p([0, T] \times \mathbb{R}^n)} \|R w^{-1/p}\|_{L^1([0, T]; L^{p'}(\mathbb{R}^n))}. \end{aligned}$$

Taking $i \rightarrow \infty$ implies that $\mu(E) = 0$.

3. In this part, we derive an upper bound estimate for the convection solution. Let R, g satisfy (1.8). Without loss of generality, again we assume that g, R are non-negative and μ is the convection solution of (4.19) with initial data g . Let $s \in (0, T)$ and $x \in \mathbb{R}^n$. We consider a sequence of function $f_i \in C_0^\infty((1 + \frac{1}{i})I \times (1 + \frac{1}{i})B)$ satisfying $f_i = 1$ in $I \times B$, where $I \subset [0, T], B \subset \mathbb{R}^n$ are open balls satisfying $s \in I, x \in B$, and i is large enough such that $(1 + \frac{1}{i})I \subset [0, T]$. Then,

$$\int_0^T \int_{\mathbb{R}^n} \frac{1}{|I \times B|} f_i \, d\mu = \int_{\mathbb{R}^n} \psi_i g \, dx + \int_0^T \int_{\mathbb{R}^n} \psi_i R \, dx \, dt,$$

where $\psi_i \in L_w^2([0, T]; H_w^1(\mathbb{R}^n)) \cap C([0, T]; L_w^2(\mathbb{R}^n))$ is the non-negative adjoint convection solution of

$$-\partial_t \psi_i - e^V \operatorname{div}(e^{-V} \nabla \psi_i) = \frac{1}{|I \times B|} f_i,$$

in $[0, T] \times \mathbb{R}^n$. Hence, by Lemma 4.7, it is concluded that

$$\begin{aligned} \frac{\mu(I \times B)}{|I \times B|} &\leq \frac{1}{|I \times B|} \int_0^T \int_{\mathbb{R}^n} f_i \, d\mu = \frac{1}{|I \times B|} \left(\int_{\mathbb{R}^n} \psi_i g \, dx + \int_0^T \int_{\mathbb{R}^n} \psi_i R \, dx \, dt \right) \\ &\leq \frac{1}{|I \times B|} \|g w^{-1}\|_{L^\infty} \int_0^T \int_{\mathbb{R}^n} f_i \, dw \, dt \\ &\quad + \frac{T}{|I \times B|} \|R w^{-1}\|_{L^\infty} \int_0^T \int_{\mathbb{R}^n} f_i \, dw \, dt. \end{aligned}$$

Hence, by [16, Lemma 1.2], Radon–Nikodym theorem, and Step 1, it is concluded that $\mu \in L^1([0, T] \times \mathbb{R}^n)$ and

$$\| \mu w^{-1} \|_{L^\infty} \leq \|g w^{-1}\|_{L^\infty} + T \|R w^{-1}\|_{L^\infty}.$$

To prove the strong maximum principle, without loss of generality, let R, g be non-negative and $\inf g w^{-1} + \inf R w^{-1} > 0$. Then, $w \in L^1(\mathbb{R}^n)$ and, by Lemma 4.7, we

have

$$\begin{aligned} & \frac{\mu\left(\left(1 + \frac{1}{i}\right)I \times \left(1 + \frac{1}{i}\right)B\right)}{\left|\left(1 + \frac{1}{i}\right)I \times \left(1 + \frac{1}{i}\right)B\right|} \geq \frac{1}{\left|\left(1 + \frac{1}{i}\right)I \times \left(1 + \frac{1}{i}\right)B\right|} \int_0^T \int_{\mathbb{R}^n} f_i \, d\mu \\ & = \frac{1}{|I \times B|} \left(\int_{\mathbb{R}^n} \psi_i g \, dx + \int_0^T \int_{\mathbb{R}^n} \psi_i R \, dx \, dt \right) \\ & \geq \frac{\inf g w^{-1}}{\left|\left(1 + \frac{1}{i}\right)I \times \left(1 + \frac{1}{i}\right)B\right|} \int_0^T \int_{\mathbb{R}^n} f_i \, dw \, dt \\ & + \frac{\inf R w^{-1}}{\left|\left(1 + \frac{1}{i}\right)I \times \left(1 + \frac{1}{i}\right)B\right|} \int_0^T \int_t^T \int_{\mathbb{R}^n} f_i \, dw \, dr \, dt. \end{aligned}$$

This concludes that

$$\liminf_{r \rightarrow 0} \frac{\mu((s - r, s + r) \times B(x, r))}{|(s - r, s + r) \times B(x, r)|} \geq \inf(g w^{-1} + s \inf R w^{-1}) w(x),$$

for every $0 < s < T$ and Lebesgue point $x \in \mathbb{R}^n$ of w .

4. Now, we prove the conservation of mass. Assume that $w \in L^1(\mathbb{R}^n)$ and R, g are non-negative functions. We take the same sequences of functions R_i, g_i, V_i, μ_i as in Step 1. Define $f = 1_{\mathbb{R}^n \setminus B}$ for a ball $B \subset \mathbb{R}^n$. Then, by Lemma 4.4, w_i converges to w in $L^1(\mathbb{R}^n)$ and $f \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^2_{w_i}([0, T] \times \mathbb{R}^n)$ for every i . Hence, by Proposition 4.5 and Lemma 4.7, there exists an adjoint convection solution $\phi_i^B \in C([0, T]; L^2_w(\mathbb{R}^n)) \cap L^2([0, T]; H^1_w(\mathbb{R}^n))$ of

$$-\partial_t \phi_i^B - e^{V_i} \operatorname{div}(e^{-V_i} \nabla \phi_i^B) = f,$$

in $[0, T] \times \mathbb{R}^n$, where

$$\|\phi_i^B\|_{L^\infty(\mathbb{R}^n)} \leq T, \quad \sup_{0 \leq t \leq T} \|\phi_i^B(t, \cdot)\|_{L^1_{w_i}(\mathbb{R}^n)} \leq w_i(\mathbb{R}^n \setminus B).$$

Moreover, as $i \rightarrow \infty$, ϕ_i^B converges to ϕ^B in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$. Hence, by Fatou's lemma and Lebesgue's dominated convergence, we derive

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\phi^B(t, \cdot)\|_{L^1_w(\mathbb{R}^n)} \leq w(\mathbb{R}^n \setminus B), \\ & \limsup_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n \setminus B} d\mu_i \leq \int_{\mathbb{R}^n} \phi^B g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \phi^B R \, dx \, dt. \end{aligned}$$

In conclusion, by replacing B by $B(0, j)$ and using again Proposition 4.5, we derive that, $\phi^{B(0, j)}$ is uniformly bounded and converges to zero in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n))$. Then,

$$\lim_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n \setminus B(0, j)} d\mu_i = 0,$$

where we used Lebesgue's dominated convergence above. Hence, μ_i is a tight sequence, and, by Prokhorov's theorem and Step 1, it converges weakly to μ . Let

$\Phi \in C^1([0, T])$ be a fixed function where $\Phi(T) = 0$. Then, $-\partial_t \Phi - \Delta \Phi + \nabla \Phi \cdot \nabla V_i = -\partial_t \Phi$ weakly in $[0, T] \times \mathbb{R}^n$ and

$$\int_0^T \int_{\mathbb{R}^n} -\partial_t \Phi \mu_i \, dt \, dx = \int_{\mathbb{R}^n} \Phi g_i \, dx + \int_0^T \int_{\mathbb{R}^n} \Phi R_i \, dx \, dt.$$

Taking $i \rightarrow \infty$ and using the weak convergence of μ_i to μ , we arrive at

$$\int_0^T \int_{\mathbb{R}^n} -\partial_t \Phi \, d\mu = \int_{\mathbb{R}^n} \Phi g \, dx + \int_0^T \int_{\mathbb{R}^n} \Phi R \, dx \, dt.$$

Let $s \in (0, T)$ be a fixed point, $i > \max(\frac{1}{T-s}, \frac{1}{s})$, and φ_ϵ is an approximation of the identity. Setting $\Phi_\epsilon(t) = \frac{i}{2} \int_t^T \varphi_\epsilon * 1_{[s-\frac{1}{i}, s+\frac{1}{i}]}$ for every $t \in [0, T]$, we arrive at

$$\int_0^T \int_{\mathbb{R}^n} -\partial_t \Phi_\epsilon \, d\mu = \int_{\mathbb{R}^n} \Phi_\epsilon g \, dx \Big|_0 + \int_0^T \int_{\mathbb{R}^n} \Phi_\epsilon R \, dx \, dt.$$

Taking $\epsilon \rightarrow 0$ implies that

$$\begin{aligned} \frac{i}{2} \int_{s-\frac{1}{i}}^{s+\frac{1}{i}} \int_{\mathbb{R}^n} \, d\mu &= \int_{\mathbb{R}^n} g \, dx + \int_0^{s-\frac{1}{i}} \int_{\mathbb{R}^n} R \, dt \, dx \\ &+ \frac{i}{2} \int_{s-\frac{1}{i}}^{s+\frac{1}{i}} \int_{\mathbb{R}^n} \left(\frac{is + 1 - it}{2} \right) R \, dr \, dt \, dx. \end{aligned}$$

Hence, by [16, Lem. 1.2], Radon–Nikodym theorem, and letting $i \rightarrow \infty$, we arrive at the signed measure $\nu(I) := \mu(I \times \mathbb{R}^n)$, for Borel subsets $I \subset [0, T]$, is absolutely continuous with respect to Lebesgue’s measure and

$$\frac{d\nu}{d\mathcal{L}^1}(s) = \int_{\mathbb{R}^n} g \, dx + \int_0^s \int_{\mathbb{R}^n} R \, dx \, dt$$

for a.e. $s \in [0, T]$.

5. Finally, we prove the continuous to the initial data. Without loss of generality, we assume that R, g are non-negative functions. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $\phi_i \in L^2([0, T]; H^1(\mathbb{R}^n))$ be the smooth weak solution of

$$-\partial_t \phi_i - \Delta \phi_i + \nabla V_i \cdot \nabla \phi_i = f,$$

in $[0, T] \times \mathbb{R}^n$ with $\phi_i(T, x) = 0$ for every $x \in \mathbb{R}^n$, and g_i, R_i, μ_i be the sequences defined in Step 1. Then, by Lemma 4.6, we have

$$\int_0^s \int_{\mathbb{R}^n} f \mu_i \, dx \, dt = \int_{\mathbb{R}^n} \phi_i^s g_i \, dx \Big|_0 + \int_0^s \int_{\mathbb{R}^n} \phi_i^s R_i \, dx \, dt,$$

where $s \in (0, T)$, $\phi_i^s(t, x) := \phi_i(t + T - s, x)$ for every $0 \leq t \leq s, x \in \mathbb{R}^n$. Let $0 < s < T$ be fixed. Now, using $\phi_i^s(s, x) = \phi_i(T, x) = 0$ for every $x \in \mathbb{R}^n$ and taking the derivative of both sides of above equality with respect to s , we obtain

$$\int_{\mathbb{R}^n} f \mu_i \, dx \Big|_s = \int_{\mathbb{R}^n} -\partial_t \phi_i g_i \, dx \Big|_{T-s} + \int_0^s \int_{\mathbb{R}^n} -\partial_t \phi_i(t + T - s, \cdot) R_i \, dx \, dt.$$

Hence, by Lemma 4.6, it is concluded that

$$\left| \int_{\mathbb{R}^n} f \mu_i \, dx \right|_s - \int_{\mathbb{R}^n} f g \, dx \leq \left| \int_{\mathbb{R}^n} -\partial_t \phi_i g_i \, dx \right|_{T-s} - \int_{\mathbb{R}^n} f g \, dx + \|f\|_{L^\infty} \int_0^s \int_{\mathbb{R}^n} R_i \, dx \, dt.$$

Then, by vague convergence of μ_i to μ proved in Step 1, Lemma 4.6, and Lebesgue’s dominated convergence, we have

$$\begin{aligned} \left| \frac{1}{h} \int_s^{s+h} \int_{\mathbb{R}^n} f \, d\mu - \int_{\mathbb{R}^n} f g \, dx \right| &= \lim_{i \rightarrow \infty} \left| \frac{1}{h} \int_s^{s+h} \int_{\mathbb{R}^n} f \mu_i \, dx \, dt - \int_{\mathbb{R}^n} f g \, dx \right| \\ &\leq \frac{1}{h} \int_s^{s+h} \left| \int_{\mathbb{R}^n} -\partial_t \phi g \, dx \right|_{T-t} - \int_{\mathbb{R}^n} f g \, dx \Big| dt \\ &\quad + \|f\|_{L^\infty} \frac{1}{h} \int_s^{s+h} \int_0^t \int_{\mathbb{R}^n} R \, dx \, dr \, dt, \end{aligned}$$

for every $0 < h < T - s$. In conclusion, by using Lemma 4.6 and Lebesgue’s dominated convergence again, we arrive at

$$\lim_{(h,s) \rightarrow 0} \left| \frac{1}{h} \int_s^{s+h} \int_{\mathbb{R}^n} f \, d\mu - \int_{\mathbb{R}^n} f g \, dx \right| = 0,$$

where the limit is taken in the domain $0 < h + s < T$. □

Example 4.9. If $R = 0$, $g = e^{-V}$, then by Theorem 1.2, the convection solution μ of

$$\partial_t \mu - \Delta \mu - \operatorname{div}(\mu \nabla V) = 0, \tag{4.22}$$

in $[0, T] \times \mathbb{R}^n$ with initial data g satisfies $\mu \in L^1([0, T] \times \mathbb{R}^n)$ and

$$1 \leq \mu e^V \leq 1$$

for a.e. $(t, x) \in [0, T] \times \mathbb{R}^n$. Hence, $\mu = e^{-V}$ which is the stable solution of (4.22).

Remark 4.10. We remark that if $\nabla V \in L^\infty(\mathbb{R}^n)$, equivalently V is globally Lipschitz, then, by Proposition 3.2 and Proposition 4.5, Definition 3.1 and Definition 4.3 coincide. In conclusion, by Theorem 1.1 and Theorem 1.2, the two definitions of convection solutions, namely Definition 3.3 and Definition 4.8, are equivalent whenever $\nabla V \in L^\infty(\mathbb{R}^n)$. Furthermore, with a slight change in the arguments, we can generalize the definition of convection–diffusion solutions and Theorem 1.2 to $\partial_t \mu - \operatorname{div} A \nabla \mu - \operatorname{div}(\mu \nabla V)$, where A is a positive definitive matrix with constant coefficients.

Acknowledgements

I want to thank Kaj Nyström for his encouragement for this work. Moreover, I thank Benny Avelin for motivating discussions and comments on this paper. The author also

acknowledges a communication with Pierre-Louis Lions and appreciates his comments.

Funding Open access funding provided by Uppsala University. Does not apply since no fund was applied on this research.

Data Availability Statement I approve the availability of my data for the public.

Declarations

Conflict of interest Does not apply since there is no Conflict of interest.

Ethical approval I declare and approve all the ethical rules in this journal are applied to my manuscript.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

REFERENCES

- [1] D. G. ARONSON. (1968), *Non-negative solutions of linear parabolic equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** 607-694.
- [2] D. G. ARONSON and J. SERRIN. *Local behavior of solutions of quasilinear parabolic equations*. Arch. Rational Mech. Anal. **25** (1967), 81-122.
- [3] P. BÉNILAN, L. BOCCARDO, T. GALLOUËT, R. GARIEPY, M. PIERRE, and J. VÁZQUEZ. (1995) *An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** 241-273.
- [4] P. BONICATTO, G. CIAMPA, and G. CRIPPA. *A regularity result for the Fokker-Planck equation with non-smooth drift and diffusion*. Nonlinear Anal. **254** (2025), 113748
- [5] V. I. BOGACHEV, N. KRYLOV, M. RÖCKNER, and S. SHAPOSHNIKOV. *Fokker-Planck-Kolmogorov equations*. Math. Surveys Monogr., 207 American Mathematical Society, Providence, RI, (2015).
- [6] V. I. BOGACHEV, T. I. KRASOVITSKII, and S. V. SHAPOSHNIKOV. *On nonuniqueness of probability solutions to the Cauchy problem for the Fokker-Planck-Kolmogorov equation*. (English summary) Translation of Dokl. Akad. Nauk **498** (2021), 16-20 Dokl. Math. **103** (2021), no.3, 108-112.
- [7] L. BOCCARDO and F. MURAT. *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*. Nonlinear Anal. **19** (1992), no.6, 581-597.
- [8] L. BOCCARDO, L. ORSINA, and A. PORRETTA. (2003) *Some noncoercive parabolic equations with lower order terms in divergence form*. J. Evol. Equ. **3**, no.3, 407-418.
- [9] L. BOCCARDO, L. ORSINA, and M.M. PORZIO. *Regularity results and asymptotic behavior for a noncoercive parabolic problem*. J. Evol. Equ. **21** (2021), 2195–2211.

- [10] V. CHIADÒ PIAT and F. SERRA CASSANO. (1994) *Some remarks about the density of smooth functions in weighted Sobolev spaces*. J. Convex Anal. **1**, no.2, 135-142.
- [11] D. DANERS. *Heat kernel estimates for operators with boundary conditions*, Math. Nachr., 217 (2000), 13–41.
- [12] R. J. DI PERNA and P.-L. LIONS. *Ordinary differential equations, transport theory and Sobolev spaces*. Invent. Math. **98** (1989), no.3, 511-547.
- [13] A. DECHANT, E. LUTZ, E. BARKAI, and D. A. KESSLER. *Solution of the Fokker-Planck equation with a logarithmic potential*. J. Stat. Phys. **145** (2011), no.6, 1524-1545.
- [14] L. EVANS. *Partial differential equations*. Second edition Grad. Stud. Math., 19 American Mathematical Society, Providence, RI, 2010, 378-379
- [15] L. EVANS. 2004 *A survey of entropy methods for partial differential equations*. Bull. Amer. Math. Soc. **41**(4): 409–438,
- [16] L. EVANS and F. GARIEPY. *Measure theory and fine properties of functions*. CRC Press, Boca Raton, FL, 2015.
- [17] P. FABRIE and T. GALLOUËT. *Modelling wells in porous media flows*. Math. Models Methods Appl. Sci. **10** (2000), 673-709.
- [18] F. FARRONI and G. MOSCARIELLO. *A nonlinear parabolic equation with drift term*. Nonlinear Anal. **177** (2018), 397-412.
- [19] A. FIGALLI. *Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients*. J. Funct. Anal. **254** (2008), no.1, 109-153.
- [20] C. W. GARDINER, *Handbook of stochastic methods*, 2nd ed., Springer-Verlag, Berlin, Heidelberg, 1985.
- [21] J. HEINONEN, T. KILPELÄINEN, and O. MARTIO. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications Inc., Mineola, NY, (2006).
- [22] R. JORDAN, D. KINDERLEHRER, and F. OTTO. *The variational formulation of the Fokker-Planck equation*. SIAM J. Math. Anal. **29** (1998), no. 1, 1-17.
- [23] C. LE BRIS and P.-L. LIONS. *Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients*. Comm. Partial Differential Equations **33** (2008), no.7-9, 1272-1317.
- [24] C. LE BRIS and P.-L. LIONS. *Parabolic equations with irregular data and related issues-applications to stochastic differential equations*. De Gruyter, Berlin, (2019).
- [25] P. LI and S.-T. YAU. *On the parabolic kernel of the Schrödinger operator*. Acta Math. **156**, 153201 (1986).
- [26] J.-L. LIONS and E. MAGENES. *Problèmes aux limites non homogènes et applications*. Dunod, 1968.
- [27] G. METAFUNE, D. PALLARA, and A. RHANDI. *Global properties of invariant measures*. J. Funct. Anal., **223** (2):396–424, 2005
- [28] S. MODENA and L. SZÉKELYHIDI. *Non-uniqueness for the Transport Equation with Sobolev Vector Fields*. Ann. PDE **4** (2018), no.2, Paper No. 18, 38 pp.
- [29] A. PORRETTA. (1999) *Existence results for nonlinear parabolic equations via strong convergence of truncations*. Ann. Mat. Pura Appl. **177**(4), 143-172.
- [30] A. PORRETTA. *Weak solutions to Fokker-Planck equations and mean field games*. Arch. Ration. Mech. Anal. **216** (2015), no.1, 1-62.
- [31] A. PORRETTA. *Decay rates of convergence for Fokker-Planck equations with confining drift*. Adv. Math. **436** (2024), Paper No. 109393, 57 pp.
- [32] A. PRIGNET. (1997) *Existence and uniqueness of entropy solutions of parabolic problems with L data*. Nonlinear Anal. **28** no. 12, 1943-1954.
- [33] H. RISKEN. *The Fokker-Planck equation, Methods of solution and applications*. Springer-Verlag, Berlin, 1989
- [34] G. ROYER. *An initiation to logarithmic Sobolev inequalities*. Translated from the 1999 French original by Donald Babbitt SMF/AMS Texts Monogr., 14 American Mathematical Society, Providence, RISociété Mathématique de France, Paris, 2007.
- [35] J. SERRIN. (1964) *Pathological solutions of elliptic differential equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **18** 385-387.
- [36] Z. SCHUSS. *Singular perturbation methods in stochastic differential equations of mathematical physics*. SIAM Rev., **22** (1980), pp. 119-155.

- [37] J. SIMON. *Compact sets in $L^p(0, T; B)$* . Ann. Mat. Pura Appl. 146(4), 65-96 (1987).
- [38] T. STOCKER. *Introduction to Climate Modelling*. Springer, Berlin, Heidelberg, 2011, pp. 55-57.
- [39] S.A. SOCOLOFSKY and G.H. JIRKA. *Special Topics in Mixing and Transport Processes in the Environment: Engineering - Lectures*. 2005. pp. 29-31.
- [40] V. V. ZHIKOV. *On weighted Sobolev spaces*. (Russian. Russian summary) Mat. Sb. **189** (1998), no.8, 27-58; translation in Sb. Math. **189** (1998), no.7-8, 1139-1170.

Alireza Ataei
Department of Mathematics
Uppsala University
S-751 06 Uppsala
Sweden
E-mail: alireza.ataei@math.uu.se

Received: 28 January 2025

Accepted: 8 January 2026