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Topics in Operator Theory and Partial Differential Equations

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Abstract

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This thesis comprises a comprehensive summary and five research articles addressing analytical problems related to differential operators, with a focus on the interplay between operator theory and partial differential equations. The studies cover both linear and nonlinear problems, combining functional analytic techniques with methods from PDE analysis.

The first two papers investigate the Kato square root problem for parabolic operators with rough coefficients. Paper I considers operators that are elliptic with respect to A_2 Muckenhoupt weights, while Paper II focuses on coefficients in the space of functions with bounded mean oscillation. These works establish estimates for the square root of the operator, providing precise information about its domain and implications for the regularity of solutions to the associated evolution equations.

Paper III examines parabolic equations with time-dependent coefficients that are elliptic with respect to A_2 Muckenhoupt weights. The study proves the existence of fundamental solutions and derives Gaussian bounds, offering quantitative control over the propagation and decay of solutions.

The final two papers address nonlinear nonlocal equations involving fractional operators. Paper IV focuses on the linear fractional Laplacian, while Paper V investigates fractional p-Laplacian operators. By analyzing the boundary behavior of solutions, these works establish the isolation of the first eigenvalue of the associated fractional operators.

Together, the five articles demonstrate how operator-theoretic and PDE techniques can be combined to tackle differential operators in complex settings, including rough or time-dependent coefficients and nonlocal nonlinear phenomena. The results provide a deeper understanding of both linear and nonlinear operators, highlighting the rich interactions between operator theory and PDE analysis in contemporary mathematical research.

Keywords: Parabolic PDEs, Gaussian bounds, Kato square root problem, fractional p-Laplacian, Wiener criterion.

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Dedicated to my parents

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I The Kato square root problem for weighted parabolic operators

II The Kato square root problem for parabolic operators with an anti-symmetric part in BMO

III On fundamental solutions and Gaussian bounds for degenerate parabolic equations with time-dependent coefficients

IV A comparison method for the fractional Laplacian and applications

V Boundary behavior of solutions to fractional p-Laplacian equation

Contents

1. Introduction

This thesis is based on five articles that investigate the interplay between operator theory and partial differential equations (PDEs). Operator theory studies linear mappings and their spectral properties, which, in particular, provide insight into the frequencies and possible energy levels of atoms and molecules, while PDEs originated in physics as models describing phenomena such as heat flow, wave propagation, and potential fields.

Although these two areas may appear distinct, they are naturally connected: a linear PDE can be interpreted as a linear operator acting on a suitable function space. The comparison allows the techniques of operator theory to be applied to understand PDEs.

In the study of PDEs, particularly boundary value problems, it is often essential to understand the square root of the associated linear operator in order to establish existence and regularity results for solutions. This motivates the Kato square root problem, which seeks to obtain norm equivalences and bounds involving the square root of a linear operator. Importantly, the square root of a PDE operator does not generally correspond to another PDE operator, and its analysis therefore relies on abstract operator-theoretic techniques rather than classical PDE methods.

The first two papers of this thesis address the Kato square root problem for parabolic PDEs with rough coefficients. The distinction between the two lies in the assumptions imposed on the coefficients: the first paper considers operators that are elliptic with respect to an A_2 Muckenhoupt weight, while the second treats coefficients belonging to the space of functions with bounded mean oscillation (BMO).

The third paper focuses on the existence of fundamental solutions and the derivation of Gaussian bounds for parabolic equations with rough, time-dependent coefficients. Here again, linear operator methods play a central role in obtaining sharp estimates for solutions of parabolic PDEs.

The final two papers are of a different nature and approach the connection in the opposite direction: here, PDE techniques are used to solve a problem motivated by operator theory. Specifically, they focus on nonlocal (fractional) PDEs and analyze the boundary behavior of solutions. These results, in turn, allow us to prove the isolation of the first eigenvalue of the corresponding fractional operators. Unlike the first three papers, the PDEs studied in the last two works are nonlinear, so one cannot directly apply classical operator-theoretic methods.

Together, these works emphasize the deep and fruitful interplay between operator theory and PDEs, demonstrating how techniques from one area can illuminate challenging problems in the other.

2. Parabolic PDEs

The PDEs are conventionally divided into three types: elliptic, parabolic, and hyperbolic. In this section, we explain the notion of parabolic equations with rough coefficients. Then, we explore notions of solutions and Gaussian bounds.

2.1 Motivation

Before presenting the formalism, we start with a motivation to consider parabolic PDEs, which we keep in the back of our heads throughout the entire work. We begin with the simplest model: the heat equation. The fundamental question is how one can predict the temperature in a given region. This was systematically studied by Joseph Fourier, see [9], where he showed that the temperature $u(y,t)$ at time t and position y in a region $\Omega \subset \mathbb{R}^n$ satisfies the equation

$$\partial_t u - \Delta_y u = 0,$$

where Δ_y denotes the spatial Laplacian. Intuitively, the equation states that heat flows in different parts of a region so that at some level it becomes uniform inside the whole region. Now, a primary complication arises from the boundary $\partial\Omega$, which may be irregular or complex. To simplify the boundary, we use the transformation $y = \Phi(x)$, where x lies in a simpler domain, e.g., the upper half-space. In general, such a Φ may exist only locally, depending on the geometry of the boundary. However, to simplify our understanding, we focus on a simple model. The transformation Φ allows us to work with 'normal coordinates' and apply a simpler representation of the solutions. After change of the coordinates y to x, it is suitable to define

$$v(x,t) := u(\Phi(x),t).$$

Let $J := D\Phi(x)$ be the Jacobian of the transformation, and define

$$w(x) := \det D\Phi(x).$$

Assuming that Φ preserves orientation, w represents the volume distortion induced by Φ. Then, by straightforward computation, v satisfies

$$\partial_t v - \frac{1}{w}\operatorname{div}_x A\nabla_x v = 0, \tag{2.1.1}$$

where

$$A(x) := w(x)\, D\Phi^{-1}(x)(D\Phi^{-1}(x))^T,$$

and $(D\Phi^{-1}(x))^T$ is the transpose of $D\Phi^{-1}(x)$. The equation (2.1.1), with often $w = 1$ in the literature, is referred to as a second-order parabolic PDEs in divergence form. In the rest of the work, we focus exclusively on (2.1.1) and ignore Φ. Moreover, we generalize the formulation to allow A to depend on both x and t.

2.2 Coefficients with A_2 weights and bounded mean oscillation

In this subsection, two different scenarios are considered for the coefficients. First, we assume that w belongs to the Muckenhoupt class $A_2(\mathbb{R}^n)$ and $\frac{A}{w}$ is uniformly elliptic in (2.1.1). To be more precise, w satisfies

$$[w]_{A_2} := \sup_Q \left(\fint_Q w \, \mathrm{d}x \right) \left(\fint_Q w^{-1} \, \mathrm{d}x \right) < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and $A(x,t)$ defined on $\mathbb{R}^{n+1}$ is assumed to have complex measurable entries satisfying

$$c_1|\xi|^2 w(x) \leq \mathrm{Re}(A(x,t)\xi \cdot \overline{\xi}), \qquad |A(x,t)\xi \cdot \zeta| \leq c_2 w(x)|\xi||\zeta|, \tag{2.2.1}$$

for some $c_1, c_2 \in (0,\infty)$ and for all $\xi, \zeta \in \mathbb{C}^n$, $(x,t) \in \mathbb{R}^{n+1}$. To define the second type of assumption on coefficients, we use the space $\mathrm{BMO}(\mathbb{R}^n)$ which is the space of real-valued functions with bounded mean oscillation, i.e., $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if

$$||f||_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \fint_Q \left| f - \fint_Q f \right| < \infty,$$

where the supremum is taken with respect to all cubes $Q \subset \mathbb{R}^n$. We generalize the definition to $\mathrm{BMO}(\mathbb{R}^n;\mathbb{C}^k)$ by defining it to be the space of all the vector valued functions $(f_1, \cdots, f_k) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^k)$, such that $\mathrm{Re} f_i, \mathrm{Im} f_i \in \mathrm{BMO}(\mathbb{R}^n)$ for all $1 \leq i \leq k$. Now, the second type of assumption is to take $w = 1$ and $A(x,t)$ to have complex measurable entries, which can be decomposed as in $A(x,t) = S(x,t) + D(x,t)$. The complex $n \times n$-dimensional matrix-valued function $S(x,t)$ is assumed to satisfy the uniform ellipticity condition

$$c_1|\xi|^2 \leq \mathrm{Re}(S(x,t)\xi \cdot \overline{\xi}), \qquad |S(x,t)\xi \cdot \zeta| \leq c_2|\xi||\zeta|, \tag{2.2.2}$$

for some $c_1, c_2 \in (0,\infty)$ and for all $\xi, \zeta \in \mathbb{C}^n$, $(x,t) \in \mathbb{R}^{n+1}$, and $D(x,t)$ is assumed to be a real, measurable, $n \times n$-dimensional, and anti-symmetric matrix-valued function satisfying

$$||D||_{\mathrm{L}^\infty(\mathbb{R},\mathrm{BMO}(\mathbb{R}^n,\mathrm{d}x))} \leq c_3, \tag{2.2.3}$$

for some $c_3 \in (0,\infty)$.

2.3 Energy space and a concept of solutions

Here, we present a well-defined space of functions to consider solutions to the parabolic equations, which is convenient for operator theory. We start with the issue that the operator

$$\mathscr{H}u := \partial_t u - \frac{1}{w}\operatorname{div}_x A(x,t)\nabla_x u \tag{2.3.1}$$

is not really well-defined even if u is smooth, since A is not necessarily regular here to define derivatives. The trick to get rid of this issue is to use the integration by parts and the notion of duality. We first need some definitions.

Definition 2.3.1. We define the Hilbert transform H_t and half order time derivative $D_t^{1/2}$ through the Fourier transform as follows:

$$\begin{aligned} H_t f &:= \mathscr{F}^{-1}(\mathrm{i}\,\mathrm{sgn}(\tau)\mathscr{F}f), \\ D_t^{1/2} f &:= \mathscr{F}^{-1}(|\tau|^{1/2}\mathscr{F}f), \end{aligned}$$

where $\mathscr{F}^{-1}$ is the inverse Fourier transform, and τ is the Fourier variable.

Definition 2.3.2. Let $\mathrm{d}w = w(x)\,\mathrm{d}x$ and $\mu(x,t) := \mathrm{d}w\,\mathrm{d}t$ be measures on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively. Then, the energy space E_μ is defined as the measurable functions $f : \mathbb{R}^{n+1} \to \mathrm{C}$ satisfying

$$\|f\|_{\mathrm{E}_\mu}^2 = \iint_{\mathbb{R}^{n+1}} |f|^2 + |D_t^{1/2} f|^2 + |\nabla_x f|^2 \,\mathrm{d}\mu < \infty.$$

Note that H_t is an isometry from E_μ to E_μ with respect to the Hilbert norm induced by $\| \|_{\mathrm{E}_\mu}$.

Now, we are able to define $\mathscr{H}$.

Definition 2.3.3. Let $(\mathrm{E}_\mu)^*$ denote the dual space of E_μ. Then, the operator $\mathscr{H} : \mathrm{E}_\mu \to (\mathrm{E}_\mu)^*$ is defined by

$$(\mathscr{H}u)(v) := \iint_{\mathbb{R}^{n+1}} w^{-1} A\nabla_x u \cdot \overline{\nabla_x v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v}\,\mathrm{d}\mu \quad (u,v \in \mathrm{E}_\mu).$$

In the above definition, we used the observations that

$$\begin{aligned} \partial_t f = D_t^{1/2} H_t D_t^{1/2} f &\in (\mathrm{E}_\mu)^* \quad (f \in \mathrm{E}_\mu), \\ (w^{-1}\operatorname{div}_x w)(w^{-1} f) = w^{-1}\operatorname{div}_x f &\in \mathrm{L}^2_\mu \quad (f \in \mathrm{E}_\mu). \end{aligned}$$

For the case of A satisfying (2.2.1), it is not hard to show that $\mathscr{H}$ is well-defined by applying the Cauchy-Schwarz inequality. However, the case of $A = S + D$, where S, D satisfy (2.2.2) and (2.2.3), is a bit subtle. First, we need the notion of Hardy spaces. Consider a fixed non-negative radial function $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x = 1$ and define

$$\phi_\varepsilon(x) := \frac{\phi\left(\frac{x}{\varepsilon}\right)}{\varepsilon^n}, \quad x \in \mathbb{R}^n.$$

Then, the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$||f||_{\mathscr{H}^1(\mathbb{R}^n)} := \left\|\sup_{\varepsilon>0} |\phi_\varepsilon * f|\right\|_{\mathrm{L}^1(\mathbb{R}^n)} < \infty.$$

Note that $\mathscr{H}^1(\mathbb{R}^n)$ is independent of the choice of ϕ, and the norms $||\,||_{\mathscr{H}^1(\mathbb{R}^n)}$ are equivalent for different choices of ϕ. Therefore, by abuse of notation, we will denote all of them by $\mathscr{H}^1(\mathbb{R}^n)$ having the norm $||\,||_{\mathscr{H}^1(\mathbb{R}^n)}$. We refer to Chapter III in [16] for more details on Hardy spaces. The notation $A \lesssim B$ means, unless otherwise stated, that A/B is bounded from above by a positive constant depending at most on the constants in (2.2.2) and (2.2.3). Now, we need to use the following compensated compactness lemma; see [6].

Lemma 2.3.4. *Let $n \geq 2$ and let $f, g : \mathbb{R}^n \to \mathbb{R}$ be in $\mathrm{H}^1(\mathbb{R}^n)$. Then,*

$$||\partial_{x_i} f\, \partial_{x_j} g - \partial_{x_i} g\, \partial_{x_j} f||_{\mathscr{H}^1(\mathbb{R}^n)} \lesssim ||\nabla_x f||_{\mathrm{L}^2} ||\nabla_x g||_{\mathrm{L}^2},$$

for all $i, j \in \{1, ..., n\}$.

Finally, combining Lemma 2.3.4, (2.2.3), the duality between $\mathrm{BMO}(\mathbb{R}^n)$ and the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$; see Chapter IV in [16], and Cauchy-Schwarz inequality, we derive

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^{n+1}} D(x,t)\nabla_x u \cdot \overline{\nabla_x v}\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\lesssim \int_{\mathbb{R}} \|\partial_{x_i} u(\cdot,t)\, \overline{\partial_{x_j} v(\cdot,t)} - \partial_{x_j} u(\cdot,t)\, \overline{\partial_{x_i} v(\cdot,t)}\|_{\mathscr{H}^1(\mathbb{R}^n,\mathrm{d}x)}\,\mathrm{d}t \\
&\lesssim \int_{\mathbb{R}} \|\nabla_x u(\cdot,t)\|_{\mathrm{L}^2} \|\nabla_x v(\cdot,t)\|_{\mathrm{L}^2}\,\mathrm{d}t \lesssim \|\nabla_x u\|_{\mathrm{L}^2} \|\nabla_x v\|_{\mathrm{L}^2}.
\end{aligned}$$

In the last part, we define the concept of energy solution as follows:

Definition 2.3.5. Let $f \in (\mathrm{E}_\mu)^*$. Then, we say that $u \in \mathrm{E}_\mu$ is a solution to $\mathscr{H} u = f$ if

$$(\mathscr{H} u)(v) = \iint_{\mathbb{R}^{n+1}} f\, v\,\mathrm{d}\mu,$$

for every $v \in \mathrm{E}_\mu$.

Although it is not immediately possible to show that a solution to $\mathscr{H}u = f$ exists, by perturbing $\mathscr{H}$ to $\sigma + \mathscr{H}$ for $\sigma \in \mathrm{C}$ satisfying $\operatorname{Re}\sigma > 0$, we can prove the existence of solutions for $(\sigma + H)u = f$; see [2, Lem. 4.1] and [4, Lem. 4.1].

2.4 Fundamental solutions and Gaussian bounds

In this part, we explain the notion of fundamental solutions and Gaussian bounds for parabolic equations with rough coefficients. Our focus would be on equation (2.1.1) where w belongs to the Muckenhoupt $A_2(\mathbb{R}^n)$ class and A satisfies (2.2.1). It is aimed to solve the Cauchy problem

$$\begin{aligned}
&\text{(i) } \mathscr{H}u = \partial_t u - w^{-1}\operatorname{div}_x(A(x,t)\nabla_x u) = 0 \text{ in } \mathbb{R}^n \times (0,T),\\
&\text{(ii) } \lim_{t\to 0} u(x,t) = f(x).
\end{aligned} \tag{2.4.1}$$

Now, we again get into the trouble of defining the exact meaning of solutions to (2.4.1). Thus, the dual notion is used again to make sense of the equation.

Definition 2.4.1. Define the weighted Sobolev space $\mathrm{H}^1_w(\mathbb{R}^n)$ including $u \in \mathrm{L}^2_w(\mathbb{R}^n)$ for which the distributional gradient ∇u belongs to $\mathrm{L}^2_w(\mathbb{R}^n;\mathbb{R}^n)$. A weak solution to (2.4.1) is defined by the following properties:
$u \in \mathrm{L}^\infty([0,T],\mathrm{L}^2_w(\mathbb{R}^n)) \cap \mathrm{L}^2((0,T],\mathrm{H}^1_w(\mathbb{R}^n))$ and
(i).

$$\int_0^T \int_{\mathbb{R}^n} u(x,t)\partial_t\phi(x,t)\,\mathrm{d}w\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n} A(x,t)\nabla_x u(x,t)\cdot\nabla_x\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\phi \in C_0^\infty(\mathbb{R}^n \times (0,T))$.
(ii).

$$u(\cdot,t) \to f(\cdot) \text{ in } \mathrm{L}^2_w(\mathbb{R}^n) \text{ as } t \to 0^+.$$

As defined above, there is a linear connection between f and u. Hence, one way to think of solutions is to find a non-negative Kernel $K_t(x,y)$ which satisfies

$$u(x,t) = \int_{\mathbb{R}^n} K_t(x,y)f(y)\,w(y)\,\mathrm{d}y, \tag{2.4.2}$$

for all $(x,t) \in \mathbb{R}^n \times (0,T)$. Then, K_t is called the fundamental solution for $\mathscr{H}$. Finally, it remains to define the notion of Gaussian bounds. We motivate the reader with the Gaussian process, where

$$G_t(x,y) = \frac{1}{(4\pi\kappa t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t\kappa}}, \quad x,y \in \mathbb{R}^n,\, t,\kappa > 0.$$

This is the fundamental solution to (2.1.1) for $w = 1$ and $A = \kappa I$. The natural question is how one can compare the properties of $K_t(x,y)$ with $G_t(x,y)$. Let $B_t(x) \subset \mathbb{R}^n$ denote the ball of radius t and center x, where $t > 0, x \in \mathbb{R}^n$. Since we have $|B_{\sqrt{t}}(x)| \sim t^{n/2}$, it is natural in the weighted setting to replace $t^{n/2}$ by the weighted volume of the ball. Therefore, we define

$$w_t(x) := \int_{B_t(x)} w(y)\, \mathrm{d}y,$$

for every $x \in \mathbb{R}^n, t > 0$. Now, the following definition answers our question.

Definition 2.4.2. We say that $K_t(x,y)$ satisfies the Gaussian bounds if there exist constants $c > 1, \nu > 0$ such that

$$K_t(x,y) \leq \frac{c}{\sqrt{w_t(x)w_t(y)}} e^{-\frac{|x-y|^2}{ct}},$$

for all $t > 0, x, y \in \mathbb{R}^n$, and

$$|K_t(x+h,y) - K_t(x,y)| \leq \frac{c}{\sqrt{w_t(x)w_t(y)}} \left(\frac{|h|}{t^{1/2} + |x-y|} \right)^{\nu} e^{-\frac{|x-y|^2}{ct}},$$
$$|K_t(x,y+h) - K_t(x,y)| \leq \frac{c}{\sqrt{w_t(x)w_t(y)}} \left(\frac{|h|}{t^{1/2} + |x-y|} \right)^{\nu} e^{-\frac{|x-y|^2}{ct}},$$

for all $t > 0, x, y, h \in \mathbb{R}^n$, satisfying $2|h| \leq t^{1/2} + |x-y|$.

3. Kato square root problem

In this section, we introduce the Kato square root problem for parabolic operators and develop the functional analytic tools required for its solution. The problem concerns the identification of the domain of the square root of a divergence-form operator and its equivalence with a natural energy space.

3.1 Motivation

To motivate the problem, consider the boundary value problem in the upper half-space

$$\begin{aligned} &\partial_t v - \frac{1}{w}\operatorname{div}_{\lambda,x} B(x,t)\nabla_{\lambda,x} v = 0, \quad \lambda > 0 \\ &v(0,x,t) = u(x,t). \end{aligned}$$

Assume that the weight w is independent of λ and t, and that

$$B = \begin{pmatrix} w & 0 \\ 0 & A(x,t) \end{pmatrix},$$

where $A(x,t)$ is an $n \times n$ matrix with complex coefficients. With this block structure, the equation reduces to

$$\partial_\lambda^2 v = \mathscr{H} v, \tag{3.1.1}$$

where $\mathscr{H}$ is as (2.3.1). Formally, this is analogous to solving the abstract second-order ODE (3.1.1) whose solution can be written as

$$v(\lambda) = e^{-\lambda\sqrt{\mathscr{H}}} u.$$

Making this rigorous requires a functional calculus for $\mathscr{H}$. Thus, it remains to make sense of $\sqrt{\mathscr{H}}, e^{-\lambda\sqrt{\mathscr{H}}}$, and understand the largest space of functions that one can define $\sqrt{\mathscr{H}}$, i.e.,

$$\mathsf{D}(\sqrt{\mathscr{H}}) := \{u \in \mathrm{E}_\mu(\mathbb{R}^{n+1}) : \sqrt{\mathscr{H}} u \in \mathrm{L}^2_\mu(\mathbb{R}^{n+1})\}.$$

If $A = I$ and the operator is purely elliptic (i.e., without the time derivative), then $\mathscr{H} = -\Delta$ is self-adjoint and non-negative. Hence,

$$\|\sqrt{\mathscr{H}} u\|_{\mathrm{L}^2}^2 = \langle \mathscr{H} u, u \rangle = \|\nabla u\|_{\mathrm{L}^2}^2.$$

In conclusion, $\mathsf{D}(\sqrt{\mathscr{H}}) = H^1(\mathbb{R}^n)$ and

$$\|\sqrt{\mathscr{H}}u\|_{\mathrm{L}^2} = \|\nabla u\|_{\mathrm{L}^2}.$$

For general parabolic operators with rough coefficients, a natural generalization is

$$\mathsf{D}(\sqrt{\mathscr{H}}) = \mathrm{E}_\mu(\mathbb{R}^{n+1}),$$

with the norm equivalence

$$\|\sqrt{\mathscr{H}}u\|_{\mathrm{L}^2_\mu} \simeq \|\nabla_x u\|_{\mathrm{L}^2_\mu} + \|D_t^{1/2}u\|_{\mathrm{L}^2_\mu}.$$

This type of identification is known as the Kato square root problem.

3.2 Definition of square root

We now set aside the PDE interpretation of $\mathscr{H}$ and focus on the abstract definition of $\sqrt{L}$ and $e^{-\lambda L}$ for a special class of linear operators L via functional calculus. The idea, which goes back to Riesz and Dunford, see [11, Ch.1 and Ch. 2], is to see them as functions of operators, namely $f(z) = z^{\frac{1}{2}}, f(z) = e^{-\lambda z}$, and use the Cauchy formula, i.e.,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi,$$

where Γ is a contour enclosing z. To generalize this for operators, we need the notion of spectrum.

Definition 3.2.1. Let $L : \mathsf{D}(L) \subset H \to H$ be a linear operator on a Hilbert space H. Then, the spectrum of L is defined as

$$\sigma(L) = \mathrm{C} \setminus \left\{\xi \in \mathrm{C} : \xi - L \text{ is invertible and } (\xi - L)^{-1} \text{ is bounded}\right\}.$$

Definition 3.2.2. For $\omega \in [0, \pi]$, define

$$S_\omega = \begin{cases} \{z \in \mathrm{C} : z \neq 0, |\arg z| \leq \omega\}, & \omega \in (0, \pi], \\ (0, \infty), & \omega = 0. \end{cases}$$

An operator $L : \mathsf{D}(L) \subset H \to H$ is called of type $\omega \in [0, \pi)$ if it is closed and densely defined in H, and, for every $\omega' \in (\omega, \pi)$, we have
(i) $\sigma(L) \subset \overline{S_\omega}$.
(ii) $\sup\left\{\|\xi(\xi - L)^{-1}\| : \xi \in \mathrm{C}\setminus\overline{S_{\omega'}}\right\} < \infty$.

Definition 3.2.3. For $\omega \in (0, \pi)$, we introduce three classes of functions on S_ω:

(i)

$$H_\infty(S_\omega) = \{f : S_\omega \to \mathrm{C} : f \text{ is analytic and } \|f\|_{\mathrm{L}^\infty} < \infty\}.$$

(ii)

$$\Psi(S_\omega) = \Bigg\{ f \in H_\infty(S_\omega) : \text{ for some } s > 0, c \geq 0, \\ |f(z)| \leq \frac{c\,|z|^s}{1+|z|^{2s}} \text{ for every } z \in S_\omega \Bigg\}.$$

(iii)

$$\mathscr{F}(S_\omega) = \Bigg\{ f : S_\omega \to \mathrm{C} : f \text{ is analytic and for some } s > 0, c \geq 0, \\ |f(z)| \leq c(|z|^s + |z|^{-s}) \text{ for every } z \in S_\omega \Bigg\}.$$

Definition 3.2.4. Let $L : \mathsf{D}(L) \to H$ be a linear operator of type ω, where $0 < \omega < \mu < \pi$. Then,

$$\psi(L) := \frac{1}{2\pi i} \int_\gamma (\chi - L)^{-1} \psi(\chi)\, \mathrm{d}\chi,$$

for every $\psi \in \Psi(S_\omega)$, where γ be the contour defined by

$$\gamma = \begin{cases} te^{i\theta}, & -\infty < t \leq 0, \\ te^{-i\theta}, & 0 \leq t < \infty, \end{cases}$$

for $\omega < \theta < \mu$. This is called the Dunford-Riesz formula. Moreover, for every $f \in \mathscr{F}(S_\omega)$, it is defined that

$$f(L) := \psi(L)^{-1}(f \cdot \psi)(L),$$

for

$$\psi(z) := \frac{z^{k+1}}{(1+z^2)^{k+1}}, \quad z \in \mathrm{C},$$

where k is a large enough natural number, such that, for some $c > 0$, we have

$$|f(z)| \leq c(|z|^k + |z|^{-k}), \quad z \in S_\omega.$$

Now, we summarize important properties of the Dunford-Riesz formula; see [15].

Proposition 3.2.5. *Let $L : \mathsf{D}(L) \to H, \omega, \mu$ be as Definition 3.2.4. Then,*

(i) *Definition 3.2.4 is independent of θ and ψ, and $f(L)$ is closed and densely defined operator.*

(ii) *For* $f,g \in \mathscr{F}(S_\omega), \alpha \in \mathrm{C}$, *we have*

$$\alpha f(L) + g(L) = (\alpha f + g)(L)\Big|_{\mathsf{D}(f(L))},$$
$$g(L)f(L) = (g \cdot f)(L)\Big|_{\mathsf{D}(f(L))},$$
$$(f(L))^* = \bar{f}(L^*).$$

Hence, by taking $f(\chi) = \chi^{\frac{1}{2}}$, we can define

$$\sqrt{L} := f(L),$$

which satisfies $\sqrt{L}^2 = L$ on $\mathsf{D}(L)$. Likewise, one can define $e^{-\lambda L}$, satisfying $\partial_\lambda e^{-\lambda L} u = -L e^{-\lambda L} u$ on a dense subset of H. Here, ∂_λ is defined as

$$\lim_{\delta \to 0} \frac{e^{-(\lambda+\delta)L} u - e^{-\lambda L} u}{\delta},$$

where the limit is taken with respect to the norm in H. A fundamental result due to McIntosh establishes quadratic estimates that characterize the functional calculus for sectorial operators, see [11, Thm. 5.2.6] and [15] for the proof. We remind the reader that

$$\mathscr{R}(L) := \{Lu : u \in \mathsf{D}(L)\},$$

for a linear operator $L : \mathsf{D}(L) \to H$.

Theorem 3.2.6. *Let H be a Hilbert space, $L : \mathsf{D}(L) \to H$ be a linear operator of type ω, and $\psi \in H_\infty(S_\omega)$ such that $\int_0^\infty \psi(t)\, \frac{\mathrm{d}t}{t} = 1$. Then,*

$$\int_0^\infty \psi(\lambda L) u \frac{\mathrm{d}\lambda}{\lambda} = u, \quad \text{for all } u \in \overline{\mathsf{D}(L) \cap \mathscr{R}(L)}$$
$$\int_0^\infty \|\psi(\lambda L) u\|^2 \frac{\mathrm{d}\lambda}{\lambda} \leq \|u\|^2, \quad \text{for all } u \in H.$$

By an approximation argument and applying Theorem 3.2.6 on function $\psi(t) = \frac{16}{\pi} \frac{t^3}{(1+t^2)^3}$, operator $\sqrt{L}$, and function $\sqrt{L}u$ instead of u, we get

Corollary 3.2.7. *Let L, H, ω be as in Theorem 3.2.6. Then,*

$$\sqrt{L}u = \frac{16}{\pi} \int_0^\infty \lambda^3 L^2 (1 + \lambda^2 L)^{-3} u \frac{\mathrm{d}\lambda}{\lambda},$$

for every $u \in \mathsf{D}(\sqrt{L})$.

This corollary gives us an equivalent definition for the square root of an operator, which is applied to prove the Kato square root problem.

In the last part, we briefly mention *maximal accretive*, which is most applicable in the PDE theory. An operator L on a Hilbert space H is called maximal accretive if it is closed, densely defined, and satisfies the resolvent estimate

$$\|(\sigma+L)^{-1}\| \leq (\operatorname{Re}\sigma)^{-1}, \quad \operatorname{Re}\sigma > 0.$$

Every maximal accretive operator is sectorial of type $\pi/2$. In particular, we use that $\mathscr{H}$ defined in (2.3.1) is a maximal accretive operator; see [2, Prop. 4.2] and [4, Lem. 4.1], which allows us to define $\sqrt{\mathscr{H}}, e^{-\lambda\sqrt{\mathscr{H}}}$ for $\lambda > 0$ on a dense subset of E_μ.

4. Fractional p-Laplacian equations

In this section, we give preliminary ideas on nonlocal Laplacian-type equations. Then, we discuss the Wiener criterion and an eigenvalue problem for the fractional p-Laplacian.

4.1 Motivation

Up to now, we have worked with parabolic PDEs with rough coefficients. One of the important features is that the information of the solution is local, which means that to understand the behavior of solutions at a particular point, it is enough to know their behavior in a small neighborhood. However, in real life, particles may interact with each other over a long range, breaking the local nature of the problem. To model the problem, we consider the density of particles as a function $u : \mathbb{R}^n \to \mathbb{R}_+$, where $u = 0$ outside an open bounded subset $\Omega \subset \mathbb{R}^n$, which minimizes the energy

$$\mathscr{E}_{s,p}[u] = \frac{1}{p} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda}{q} \int_{\mathbb{R}^n} |u(x)|^q \,\mathrm{d}x,$$

for some $p, q > 1, 0 < s < 1, \lambda > 0$. Increasing p imposes a higher penalty on large differences in particle density, while s controls the interaction range. The parameters q and λ determine the strength of aggregation relative to spreading. Computing the first variation of $\mathscr{E}_{s,p}[u]$ yields

$$(-\Delta_p)^s u(x) = \lambda |u(x)|^{q-2} u(x), \quad \text{for } x \in \Omega, \tag{4.1.1}$$

where the s-fractional p-Laplacian $(-\Delta_p)^s$ is defined by

$$(-\Delta_p)^s u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+ps}} \,\mathrm{d}y, \quad \text{for } x \in \mathbb{R}^n.$$

In the case of $p = 2$, we obtain the linear s-fractional Laplacian $(-\Delta)^s$, whose Fourier symbol is $|\xi|^{2s}$, up to a multiplicative constant. This clarifies how $(-\Delta)^s$ is s-fractional power of $-\Delta$. To characterize the natural energy space in which $\mathscr{E}_{s,p}[u]$ is finite, we introduce the following function space: for the boundary value of solutions, we consider the space of functions

$$V^{s,p}(\Omega|\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \;:\; u|_\Omega \in \mathrm{L}^p(\Omega)\,,\; \frac{u(x) - u(y)}{|x-y|^{\frac{n}{p}+s}} \in \mathrm{L}^p(\Omega \times \mathbb{R}^n) \right\},$$

with the norm

$$\|u\|^p_{V^{s,p}(\Omega|\mathbb{R}^n)} := \int_\Omega |u(x)|^p \, \mathrm{d}x + \int_{\Omega\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} \, \mathrm{d}x\,\mathrm{d}y.$$

For every measurable function $u : \mathbb{R}^n \to \mathbb{R}$, define the semi-norm

$$[u]_{V^{s,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}.$$

The completion of the space $C_0^\infty(\Omega)$ in $V^{s,p}(\Omega|\mathbb{R}^n)$ with respect to $[]_{V^{s,p}(\mathbb{R}^n)}$ is denoted by $V_0^{s,p}(\Omega|\mathbb{R}^n)$, and the space of functions with boundary $g \in V^{s,p}(\Omega|\mathbb{R}^n)$ is defined by

$$V_g^{s,p}(\Omega|\mathbb{R}^n) := \left\{ u \in V^{s,p}(\Omega|\mathbb{R}^n) : u - g \in V_0^{s,p}(\Omega|\mathbb{R}^n) \right\}.$$

4.2 Wiener Criterion

The notion of Wiener regular boundary is used to find a solution to a boundary value PDE problem which converges continuously to the boundary value, see [17]. We focus on a simple example of the local Laplacian, and then we generalize it to the s-fractional p-Laplacian. Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset. We consider the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{in } \partial\Omega, \end{aligned} \tag{4.2.1}$$

for $f \in C(\Omega), g \in C(\overline{\Omega})$. The question is if there is a solution $u \in C(\overline{\Omega})$ to (4.2.1). Unfortunately, such solutions do not exist for arbitrary boundaries $\partial\Omega$. We need the boundary to be sufficiently thick in the sense of capacity. To be more precise, we require the notion of capacity. Let $K \subset \mathbb{R}^n$ be a compact subset and $n \geq 3$. The capacity of K is defined by

$$\mathrm{cap}(K) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \ : \ u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } K \right\}.$$

For $n = 2$, one uses the logarithmic capacity:

$$\mathrm{cap}(K) = \exp\left(-\inf_\mu \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \, d\mu(x) d\mu(y) \right),$$

where μ runs over all probability measures supported on K. Now, let $\Omega \subset \mathbb{R}^n$ be a domain and $\xi_0 \in \partial\Omega$. Then, we say that the point ξ_0 satisfies the Wiener criterion for the local Laplacian if

$$\int_0^1 \frac{\mathrm{cap}(\overline{B_r(\xi_0)} \setminus \Omega)}{r^{n-2}} \, \frac{\mathrm{d}r}{r} = \infty.$$

If the integral above converges, then ξ_0 is called an irregular boundary point, and the solution of Laplace's equation in (4.2.1) may fail to attain the boundary value at ξ_0:

$$\lim_{x\to\xi_0} u(x) \neq g(\xi_0).$$

Now, the concept of Wiener criterion can be generalized for fractional p-Laplacian as follows; see [1, 13, 14] :

Definition 4.2.1. Define

$$\operatorname{cap}_{s,p}(\overline{B_r(\xi_0)} \setminus \Omega, B_{2r}(\xi_0)) := \inf_v \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\, \mathrm{d}x\, \mathrm{d}y,$$

where the infimum is taken over all $v \in C_0^\infty(B_{2r}(\xi_0))$ such that $v \geq 1$ on the set $\overline{B_r(\xi_0)} \setminus \Omega$. This quantity measures the minimal 'energy' of functions that are ≥ 1 on the complement of Ω near ξ_0. We say that a point $\xi_0 \in \partial\Omega$ satisfies the Wiener criterion for the s-fractional p-Laplacian if

$$\int_0^1 \left(\frac{\operatorname{cap}_{s,p}(\overline{B_r(\xi_0)} \setminus \Omega, B_{2r}(\xi_0))}{r^{n-ps}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r} = \infty.$$

In the next step, we define the notion of regular boundaries.

Definition 4.2.2. A point $\xi_0 \in \partial\Omega$ is regular for the s-fractional p-Laplacian if for every $f \in \mathrm{L}^\infty(\Omega), g \in C(\mathbb{R}^n) \cap V^{s,p}(\Omega|\mathbb{R}^n), u \in V_g^{s,p}(\Omega|\mathbb{R}^n)$, satisfying $(-\Delta_p)^s u = f$ weakly in Ω, we have $\lim_{\xi\to\xi_0} u(\xi) = g(\xi_0)$. We say that Ω has Wiener regular boundary for the s-fractional p-Laplacian if all the points on $\partial\Omega$ are regular for the s-fractional p-Laplacian.

It turns out that these two notions are equivalent; we refer to [1, Prop. 3.3] for the proof.

Proposition 4.2.3. *A point $\xi_0 \in \partial\Omega$ is regular for the s-fractional p-Laplacian if and only if it satisfies the Wiener criterion for the s-fractional p-Laplacian.*

4.3 An eigenvalue problem

The eigenvalues of a linear operator $L : \mathsf{D}(H) \to H$ for a Hilbert space H are the values λ such that $Lu = \lambda u$ for some $u \in H \setminus 0$. This notion helps us to understand the spectral properties of a linear operator. Although traditionally the eigenvalues are defined for linear problems, one can generalize them to nonlinear cases. We consider a special nonlinear equation, motivated by

(4.1.1). Define (s,p)-eigenvalue $\lambda > 0$ and an associated (s,p)-eigenfunction $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ as a solution of

$$\begin{aligned} &(-\Delta_p)^s u = \lambda |u|^{q-2}u, \quad \text{in } \Omega, \\ &\|u\|_{\mathrm{L}^q(\Omega)} = 1. \end{aligned} \tag{4.3.1}$$

By normalization, we can write (4.1.1) as an (s,p)-eigenvalue problem with a change of λ. Indeed, by considering $v = \frac{u}{\|u\|_{\mathrm{L}^q(\Omega)}}$, (4.1.1) becomes

$$(-\Delta_p)^s v = \lambda \|u\|^{q-p}_{\mathrm{L}^q(\Omega)} |v|^{q-2}v, \quad \text{in } \Omega.$$

Now, assume that $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ is a solution to (4.3.1). Then, by testing the weak formulation of (4.3.1) with $\varphi = u$, it is obtained that

$$2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\, u(x)\,\mathrm{d}y\,\mathrm{d}x = \lambda \int_{\mathbb{R}^n} |u(x)|^q\,\mathrm{d}x. \tag{4.3.2}$$

By symmetry of the kernel and exchanging x and y, we obtain

$$\begin{aligned} &2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\, u(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy. \end{aligned}$$

Therefore,

$$\lambda = \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.3.3}$$

Considering the smallest λ above, minimizing the energy over all admissible functions, motivates us to define

$$\Lambda_{p,q} := \inf_{\phi \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y : \|\phi\|_{\mathrm{L}^q(\Omega)} = 1 \right\}, \tag{4.3.4}$$

for $1 < q < p_s^*$, where

$$p_s^* := \begin{cases} \frac{pn}{n-ps}, & \text{if } ps < n, \\ \infty, & \text{if } ps \geq n. \end{cases}$$

Here, p_s^* is the critical exponent in the fractional Sobolev embedding

$$V_0^{s,p}(\Omega|\mathbb{R}^n) \hookrightarrow \mathrm{L}^{p_s^*}(\Omega),$$

for $sp < n$, see [1, Thm 2.1]. For $s = 1$, p_1^* reduces to the classical Sobolev critical exponent.

With the first eigenvalue $\Lambda_{p,q}$ defined, a natural question is whether it is simple and isolated from higher eigenvalues. This is addressed positively in

[1, Thm. 1.5] for the case of $1 < q \le p$ under the condition of Wiener regular boundary.

We briefly discuss the case of $p < q < p_s^*$, where the issue of simplicity arises. Here, it is difficult to prove the first eigenvalue $\Lambda_{p,q}$ is simple, and it is often required to have more assumptions; see [5, Example. 4.7]. However, if we know the simplicity of the first eigenvalue $\Lambda_{p,q}$, then there is a well-known argument; see [5], which implies the isolation of the first eigenvalue. We summarize the key steps in the ideas here.

Let u be the first eigenfunction associated with $\Lambda_{p,q}$. It is known that u does not change sign, so we assume that it is non-negative. Now, assume by contradiction that there exists a sequence $v_i \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ such that

$$(-\Delta)_p^s v_i = \lambda_i |v_i|^{q-2} v_i \quad \text{in } \Omega,$$

$$\|v_i\|_{\mathrm{L}^q(\Omega)} = 1, \qquad \lambda_i \to \Lambda_{p,q},$$

and each v_i changes sign. Since $\{v_i\}$ is bounded in $V_0^{s,p}(\Omega|\mathbb{R}^n)$, up to a subsequence and a sign change, we obtain

$$v_i \rightharpoonup u \quad \text{weakly in } V_0^{s,p}(\Omega|\mathbb{R}^n),$$

$$v_i \to u \quad \text{strongly in } \mathrm{L}^r(\Omega),$$

for every $1 < r < p_s^*$. Let $v_i^- := \max\{-v_i, 0\}$ and denote

$$\Omega_i^- := \{x \in \Omega : \, v_i(x) < 0\}.$$

Testing the equation with v_i^- gives

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v_i(x) - v_i(y)|^{p-2}(v_i(x) - v_i(y))(v_i^-(x) - v_i^-(y))}{|x-y|^{n+ps}} \, dx\, dy = \lambda_i \|v_i^-\|_{\mathrm{L}^q(\Omega)}^q.$$

Hence, we obtain

$$[v_i^-]_{s,p}^p \le \lambda_i \|v_i^-\|_{\mathrm{L}^q(\Omega)}^q \le \lambda_i. \tag{4.3.5}$$

Using Hölder's inequality,

$$\|v_i^-\|_{\mathrm{L}^q(\Omega)} \le |\Omega_i^-|^{\frac{1}{q} - \frac{1}{p_s^*}} \|v_i^-\|_{\mathrm{L}^{p_s^*}(\Omega)}.$$

By the fractional Sobolev embedding,

$$\|v_i^-\|_{\mathrm{L}^{p_s^*}(\Omega)} \le C[v_i^-]_{s,p},$$

for a constant $C > 0$ independent of i. Therefore,

$$\|v_i^-\|_{\mathrm{L}^q(\Omega)}^q \le C^q |\Omega_i^-|^{1 - \frac{q}{p_s^*}} [v_i^-]_{s,p}^q. \tag{4.3.6}$$

Combining (4.3.5) and (4.3.6),

$$[v_i^-]_{s,p}^p \leq C^q \lambda_i |\Omega_i^-|^{1-\frac{q}{p_s^*}} [v_i^-]_{s,p}^q.$$

Since $v_i^- \not\equiv 0$, dividing by $[v_i^-]_{s,p}^q$ gives

$$[v_i^-]_{s,p}^{p-q} \leq C^q \lambda_i |\Omega_i^-|^{1-\frac{q}{p_s^*}}.$$

By $q > p$ and (4.3.5), we imply that $[v_i^-]_{s,p}^{p-q} \geq \lambda_i^{\frac{p-q}{p}}$. Hence,

$$|\Omega_i^-| \geq C_0, \tag{4.3.7}$$

for a constant $C_0 > 0$, since $\lambda_i \to \Lambda_{p,q} > 0$. Thus the negativity set has a uniform positive lower bound. On the other hand, $v_i \to u$ strongly in $\mathrm{L}^q(\Omega)$ and $u \geq 0$, which implies

$$|\{x \in \Omega : v_i(x) < 0\}| \to 0,$$

contradicting (4.3.7). Therefore, no such sign-changing sequence exists and $\Lambda_{p,q}$ is isolated.

5. Summary of the articles

5.1 Article I

In this article, we proved the Kato square root problem for the parabolic operators

$$\mathscr{H} = \partial_t u - w^{-1} \operatorname{div}_x(A\nabla_x u),$$

where $w(x)$ is time-independent Muckenhoupt class $A_2(\mathbb{R}^n, \mathrm{d}x)$, and A is measurable with complex matrices, satisfying the weighted ellipticity condition (2.2.1).

Theorem 5.1.1. *The operator $\mathscr{H}$ can be defined as a maximal accretive operator in L^2_μ associated with an accretive sesquilinear form with domain E_μ. The domain of its unique maximal accretive square root is the same as the form domain, that is $\mathsf{D}(\sqrt{\mathscr{H}}) = \mathrm{E}_\mu$, and*

$$\|\sqrt{\mathscr{H}}u\|_{\mathrm{L}^2_\mu} \sim \|\nabla_x u\|_{\mathrm{L}^2_\mu} + \|D_t^{1/2} u\|_{\mathrm{L}^2_\mu} \qquad (u \in \mathrm{E}_\mu)$$

holds with an implicit constant that depends on dimensions, ellipticity parameters of A and the A_2-constant for w.

The solution of the Kato problem for unweighted elliptic operators led to the development of analytic methods that have since found wide application in harmonic analysis and partial differential equations. This, in turn, has motivated the search for Kato-type estimates for more general operators, including parabolic ones, where such results are expected to play an important role in boundary value problems for weighted second-order equations.

In the elliptic setting, A_2-weighted estimates are by now well established, and in the parabolic case a sequence of works has progressively relaxed the assumptions on the coefficients. Theorem 5.1.1 completes this line of research by allowing coefficients that are measurable in all variables together with spatial A_2-weighted degeneracy. At the same time, the present paper introduces a new proof strategy: rather than relying on a first-order Dirac operator framework, we adopt a second-order approach that is conceptually simpler and, in the unweighted case, leads to a substantial simplification of earlier arguments.

The key structural insight behind the proof is that, despite full measurable dependence of the coefficients, the argument can be organized so as to almost completely separate time and space. This separation appears at several stages of the proof: Littlewood–Paley theory combines weighted elliptic estimates in space with classical Fourier analysis in time; off-diagonal bounds involve only spatial derivatives and follow directly from the equation with the correct parabolic scaling; and the Tb-argument admits test functions with a product structure. As a result, delicate harmonic analysis is confined to the spatial variables, while time derivatives enter only through resolvent blocks controlled by elementary L^2_μ estimates based on maximal accretivity and hidden coercivity. This principle is further highlighted in the final section in the proof of the Carleson measure estimate.

5.2 Article II

In this article, we maintain our ambition of proving Kato square root problem for parabolic operators with rough coefficients, where now the following linear operator is considered

$$\mathscr{H} = \partial_t u - \operatorname{div}_x(A\nabla_x u).$$

Here, it is assumed that $A = S + D$ where S is symmetric with complex entries satisfying the ellipticity condition (2.2.2) and D is a real-valued anti-symmetric matrix with entries satisfying the BMO condition (2.2.3).

Theorem 5.2.1. *The operator $\mathscr{H}$ in $\mathrm{L}^2(\mathbb{R}^{n+1})$ can be defined as maximal accretive. Furthermore, there is a well-defined square root of $\mathscr{H}$ whose domain satisfies $\mathsf{D}(\sqrt{\mathscr{H}}) = \mathsf{E}(\mathbb{R}^{n+1})$, and*

$$\|\sqrt{\mathscr{H}}u\|_2 \sim \|\nabla_x u\|_2 + \|D_t^{1/2}u\|_2 \qquad (u \in \mathsf{E}(\mathbb{R}^{n+1})),$$

holds with implicit constants that only depend on the dimension, the ellipticity parameters of S, and the BMO *constant of D.*

To prove Theorem 1.1, we build on the second-order approach developed in Article I, which traces its origins to the solution of the elliptic Kato problem. This framework was designed for parabolic operators with time-dependent measurable coefficients and was shown to considerably simplify earlier arguments by organizing the main square function estimates so as to nearly decouple time and space. In particular, off-diagonal bounds are required only for operators involving spatial derivatives; these follow directly from the equation with the correct parabolic scaling and bypass the more intricate estimates for non-local time derivatives.

A central structural ingredient is the hidden coercivity of parabolic operators on the full time line, revealed through the Hilbert transform in the time variable. This allows time derivatives to be grouped into blocks that are controlled by elementary L^2 resolvent estimates stemming from maximal accretivity, while refined harmonic analysis is confined to the spatial variables.

Compared to earlier works, a key technical contribution of the present paper is the derivation of off-diagonal estimates for operators with time-dependent coefficients whose antisymmetric part belongs to BMO. These estimates are obtained by a careful separation of spatial and temporal components and combine ideas from previous approaches. Since the presence of the Hilbert transform prevents the use of compactly supported test functions, the argument avoids direct applications of Lax–Milgram and instead relies on this structural decomposition.

Finally, the proof of the Kato estimate requires suitable Tb-type arguments and a Carleson measure estimate for operators of the form $\mathscr{U}_\lambda A$, where

$$\mathscr{U}_\lambda := \lambda (I + \lambda^2 \mathscr{H})^{-1} \operatorname{div}_x .$$

The main difficulty arises from the BMO nature of the antisymmetric part of the coefficients. This is overcome by a local modification of the coefficients and an application of the John–Nirenberg inequality, which allows one to control the relevant approximations and complete the argument.

5.3 Article III

In this work, we again consider the weighted parabolic operator

$$\mathscr{H} = \partial_t - w^{-1} \operatorname{div}_x(A\nabla_x),$$

with same assumptions for A, w as in Article I. The main result is as follows:

Theorem 5.3.1. *Given $f \in \mathrm{L}^2_w(\mathbb{R}^n)$ and $T > 0$, there exists a unique weak solution to*

$$\begin{aligned} &\text{(i) } \mathscr{H}u = \partial_t u - w^{-1} \operatorname{div}_x(A(x,t)\nabla_x u) = 0 \text{ in } \mathbb{R}^n \times (0,T), \\ &\text{(ii) } \lim_{t\to 0} u(x,t) = f(x), \end{aligned}$$

such that

$$u \in \mathrm{L}^\infty([0,T], \mathrm{L}^2_w(\mathbb{R}^n)) \cap \mathrm{L}^2((0,T], \mathrm{H}^1_w(\mathbb{R}^n)),$$

and

$$u(\cdot,t) \to f(\cdot) \text{ in } \mathrm{L}^2_w(\mathbb{R}^n) \text{ as } t \to 0^+.$$

The unique solution u can be represented as

$$u(x,t) = \int_{\mathbb{R}^n} K_t(x,y) f(y)\, w(y)\, \mathrm{d}y, \text{ for all } (x,t) \in \mathbb{R}^n \times (0,T),$$

where $K_t(x,y) = K(x,t,y,0)$ *is the fundamental solution of* $\mathscr{H}$*, satisfying*

$$\int_{\mathbb{R}^n} K_t(x,y)w(y)\,\mathrm{d}y = 1, \textit{ for all } (x,t) \in \mathbb{R}^n \times (0,T).$$

Furthermore, there exist c*,* $1 \le c < \infty$*, and* $\nu > 0$*, both depending only on the structural constants, such that*

$$K_t(x,y) \le \frac{c}{\sqrt{w_t(x)w_t(y)}} e^{-\frac{|x-y|^2}{ct}},$$

for all $t > 0, x, y \in \mathbb{R}^n$*, and*

$$|K_t(x+h,y) - K_t(x,y)| \le \frac{c}{\sqrt{w_t(x)w_t(y)}} \left(\frac{|h|}{t^{1/2}+|x-y|}\right)^{\nu} e^{-\frac{|x-y|^2}{ct}},$$
$$|K_t(x,y+h) - K_t(x,y)| \le \frac{c}{\sqrt{w_t(x)w_t(y)}} \left(\frac{|h|}{t^{1/2}+|x-y|}\right)^{\nu} e^{-\frac{|x-y|^2}{ct}},$$

for all $t > 0, x, y, h \in \mathbb{R}^n$*, satisfying* $2|h| \le t^{1/2} + |x-y|$.

This work extends the results of Cruz-Uribe and Rios [7] to operators with time-dependent, possibly non-symmetric coefficients. Since the semigroup approach used in the time-independent case is no longer available, we instead rely on Kato's abstract theory for evolution equations. Here, the operator $\mathscr{A}(t)$ is defined formally via

$$\langle \mathscr{A}(t)u, v\rangle := \int_{\mathbb{R}^n} A(x,t)\nabla_x u \cdot \overline{\nabla_x v}\,\mathrm{d}x,$$

which is considered on a suitable domain in $\mathrm{L}^2_w(\mathbb{R}^n)$, and ellipticity, along with sufficient regularity in t, allows the application of Kato's existence and uniqueness results for the corresponding abstract Cauchy problem; see [12].

To construct the fundamental solution, we first regularize the time-dependent coefficients and solve the approximating problems. Off-diagonal estimates are then established following Davies' approach, which allows us to derive upper Gaussian bounds by methods analogous to Cruz-Uribe and Rios [7]. Finally, the regularization is removed via a convergence argument, yielding the fundamental solution for the original operator.

We remark that after the publication of our work, there were questions from other colleagues for providing more comprehensive details in Section 3. Hence, we submitted a new version to arXiv; see [3], where we have added more details in Section 3 of the paper. In particular, we have supplied more details on the proof of the uniqueness part of our main result, where the argument uses Steklov averages.

5.4 Article IV

This work contains three main results, namely isolation of the first eigenvalue problem for the fractional Laplacian, a generalization of the Hopf's lemma, and a boundary Harnack inequality. We were motivated by the first one which is as follows: Let

$$\Lambda_q := \inf_{\phi \in C_0^\infty(\Omega)} \Big\{ \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y : \|\phi\|_{\mathrm{L}^q(\Omega)} = 1 \Big\},$$

for a bounded open set $\Omega \subset \mathbb{R}^n$ and define $\mathscr{D}_0^{s,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$[\phi]_{\mathrm{H}^s(\mathbb{R}^n)} := \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2},$$

on $C_0^\infty(\Omega)$ defined for every $\phi \in C_0^\infty(\mathbb{R}^n)$.

Theorem 5.4.1. *Assume that $\Omega \subset \mathbb{R}^n$ has a Wiener regular boundary for s-fractional Laplacian. Then, there are no sequences $\lambda_i > \Lambda_q, v_i \in \mathscr{D}_0^{s,2}(\Omega)$ such that*

$$\begin{aligned} &\|v_i\|_{\mathrm{L}^q(\Omega)} = 1, \\ &(-\Delta)^s v_i = \lambda_i |v_i|^{q-2} v_i, \quad \text{in } \Omega, \\ &\lim_{i\to\infty} \lambda_i = \Lambda_q. \end{aligned} \tag{5.4.1}$$

The proof of Theorem 5.4.1 led us to the exotic behavior of the fractional Laplacian in comparison to the local Laplacian. We briefly mention the critical part of the proof where we argued by contradiction that there exists a sequence $v_i \in \mathscr{D}_0^{s,2}(\Omega)$ such that

$$\begin{aligned} &\|v_i\|_{\mathrm{L}^q(\Omega)} = 1, \\ &(-\Delta)^s v_i = \lambda_i |v_i|^{q-2} v_i, \quad \text{in } \Omega, \\ &\lim_{i\to\infty} \lambda_i = \Lambda_q. \end{aligned}$$

Then, by passing to a subsequence and a sign change, one can prove that $v_i \to u$ uniformly, where u is the first non-negative eigenvalue $u \in \mathscr{D}_0^{s,2}(\Omega)$. Now, we choose $x_i \in \Omega$ satisfying

$$\frac{1}{i}u(x_i) - v_i(x_i) = \max_{x\in\overline{\Omega}} \frac{1}{i}u(x) - v_i(x) = \max_{x\in\mathbb{R}^n} \frac{1}{i}u(x) - v_i(x) := m_i > 0,$$

where we used v_i changes sign. Then, up to subsequence, x_i converges to $x \in \partial\Omega$ and for every compact ball $K \subset \Omega$ we have

$$
\begin{aligned}
\frac{\lambda_i}{2}|v_i|^{q-2}v_i(x_i) &= \int_{\mathbb{R}^n} \frac{v_i(x_i)-v_i(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y = \int_{\mathbb{R}^n\setminus K} \frac{v_i(x_i)-v_i(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y + \int_K \frac{v_i(x_i)-v_i(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y \\
&\leq \frac{1}{i}\int_{\mathbb{R}^n\setminus K} \frac{u(x_i)-u(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y + \int_K \frac{v_i(x_i)-v_i(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y \\
&= \frac{1}{i}\int_{\mathbb{R}^n} \frac{u(x_i)-u(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y - \frac{1}{i}\int_K \frac{u(x_i)-u(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y + \int_K \frac{v_i(x_i)-v_i(y)}{|x_i-y|^{n+2s}}\,\mathrm{d}y \\
&:= \mathscr{I}_1(i) + \mathscr{I}_2(i) + \mathscr{I}_3(i).
\end{aligned}
$$

The left hand side converges to $\frac{\Lambda_q}{2}u^{q-1}(\tilde{x}) = 0$. As for the right-hand side,

$$
\lim_{i\to\infty} \mathscr{I}_1(i) = \lim_{i\to\infty} \frac{\Lambda_q}{2i}u^{q-1}(x_i) = 0,
$$
$$
\lim_{i\to\infty} \mathscr{I}_2(i) = 0.
$$

Finally,

$$
\lim_{i\to\infty} \mathscr{I}_3(i) = \int_K \frac{u(\tilde{x})-u(y)}{|\tilde{x}-y|^{n+2s}}\,\mathrm{d}y = -\int_K \frac{u(y)}{|\tilde{x}-y|^{n+2s}}\,\mathrm{d}y < 0,
$$

leading to the contradiction that $\lim_{i\to\infty} \mathscr{I}_1(i) + \mathscr{I}_2(i) + \mathscr{I}_3(i) < 0$. This observation led us to prove a generalization of the Hopf's lemma and the boundary Harnack inequality for the fractional Laplacian. Define $u_{\mathrm{tor}} \in \mathscr{D}_0^{s,2}(\Omega)$ satisfying

$$
\begin{aligned}
(-\Delta)^s u_{\mathrm{tor}} &= 1, \quad \text{in } \Omega, \\
u_{\mathrm{tor}} &= 0, \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
$$

Lemma 5.4.2. *Let $u \in \mathrm{L}^1_{2s}(\mathbb{R}^n) \cap C(\overline{\Omega})$ be a non-negative function and $K \Subset \Omega$. Assume that $(-\Delta)^s u \geq f$ in Ω in the viscosity sense, where $f \in C(\Omega)$ satisfies*

$$
\begin{aligned}
\limsup_{\Omega\ni x\to x_0} f(x) &\geq -2\int_K \frac{u(y)}{|x_0-y|^{n+2s}}\,\mathrm{d}y, \quad \textit{if } x_0 \in \partial\Omega, u(x_0) = 0, \\
f(x_0) &> -2\int_{\mathbb{R}^n} \frac{u(y)}{|x_0-y|^{n+2s}}\,\mathrm{d}y, \quad \textit{if } x_0 \in \Omega,\, u(x_0) = 0.
\end{aligned}
$$

Then, $u > 0$ in Ω and

$$
u \geq C u_{\mathrm{tor}}, \quad \textit{in } \Omega,
$$

for a constant $C > 0$.

Theorem 5.4.3. *Let $u, v \in C(\overline{\Omega}) \cap \mathrm{L}^\infty(\mathbb{R}^n)$ satisfy*

$$
\begin{aligned}
u > 0, v > 0 \quad &\textit{in } \Omega, \\
0 \leq u = v \leq 1 \quad &\textit{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
$$

and

$$-2(\operatorname{diam}\Omega)^{-(n+2s)}\int_K u(y)\,\mathrm{d}y \le (-\Delta)^s u \le 1, \quad \text{in } \Omega,$$
$$-2(\operatorname{diam}\Omega)^{-(n+2s)}\int_K v(y)\,\mathrm{d}y \le (-\Delta)^s v \le 1, \quad \text{in } \Omega,$$

in the viscosity sense, where $K \Subset \Omega$. Assume that either $u(x_0) \ge D$, $v(x_0) \ge D$ for a fixed point $x_0 \in \Omega \setminus K$ or $\|u\|_{\mathrm{L}^p(\Omega\setminus K)} \ge D$, $\|v\|_{\mathrm{L}^p(\Omega\setminus K)} \ge D$ for $D > 0, 1 \le p < \infty$. Then,

$$C_1 \le \frac{u}{v} \le C_2, \quad \text{in } \Omega,$$

where C_1, C_2 are positive constants depending on Ω, K, n, s, D, x_0 or p.

The main achievement of this work, besides generalizing the previously known results in a way that is not applicable to the local Laplacian, is to connect three problems of different natures as an application of the same type of proof.

5.5 Article V

In the final work, I generalize the results of Article IV to fractional p-Laplacian. Hence, I prove Hopf's lemma, boundary Harnack inequality, and isolation of the first (s,p)-eigenvalue for the s-fractional p-Laplacian equations. Before getting into the details, I mention that the last two results have not been studied before this work, even for simpler domains, in the context of fractional p-Laplacian, and I found a gap in the proof of the simplified version of Hopf's lemma [8, Lem. 4.1] in the context of fractional p-Laplacian. Hence, this work resolves the gap using a different type of proof.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p > 1$, and $0 < s < 1$. For $\delta > 0$, the δ-neighborhood of Ω, denoted by Ω_δ, is defined by $\{x \in \mathbb{R}^n : \operatorname{dist}(x, \overline{\Omega}) < \delta\}$. The torsion function $u_{\mathrm{tor}} \in \mathrm{L}^{p-1}_{ps}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ satisfies

$$u_{\mathrm{tor}} = 0, \quad \text{in } \mathbb{R}^n \setminus \Omega,$$
$$(-\Delta_p)^s u_{\mathrm{tor}} = 1, \quad \text{in } \Omega.$$

Lemma 5.5.1. *Let $u \in \mathrm{L}^{p-1}_{ps}(\mathbb{R}^n) \cap C(\overline{\Omega}_\delta)$ be a non-negative function for a $\delta > 0$ and $K \Subset \Omega$. Assume that $(-\Delta_p)^s u \ge f$ in Ω in the viscosity sense, where $f \in C(\Omega)$ satisfies*

$$f(x_0) > -2\int_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x_0 - y|^{n+ps}}\,\mathrm{d}y, \quad \text{if } x_0 \in \Omega, u(x_0) = 0,$$
$$\limsup_{\Omega \ni x \to x_0} f(x) \ge -2\int_K \frac{u^{p-1}(y)}{|x_0 - y|^{n+ps}}\,\mathrm{d}y, \quad \text{if } x_0 \in \partial\Omega, u(x_0) = 0.$$

Then, $u > 0$ *in* Ω *and*

$$u \geq C u_{\mathrm{tor}}, \quad \text{in } \Omega,$$

for a constant $C > 0$.

Theorem 5.5.2. *Let* $\delta > 0$, $u \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_u}(\Omega|\mathbb{R}^n), v \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_v}(\Omega|\mathbb{R}^n)$ *satisfy*

$$\begin{aligned} u > 0, v > 0 \quad &\text{in } \Omega, \\ 0 \leq \frac{1}{B} g_v \leq g_u \leq B g_v \leq M, \quad &\text{in } \mathbb{R}^n \setminus \Omega, \end{aligned}$$

for $B > 0, M \geq 0$, *and*

$$\begin{aligned} -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K u^{p-1}(y)\,\mathrm{d}y \leq (-\Delta_p)^s u \leq 1, \quad &\text{in } \Omega, \\ -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K v^{p-1}(y)\,\mathrm{d}y \leq (-\Delta_p)^s v \leq 1, \quad &\text{in } \Omega, \end{aligned}$$

in the locally weak sense, where $K \Subset \Omega$. *If either* $u(x_0) \geq D$, $v(x_0) \geq D$ *or* $\|u\|_{\mathrm{L}^q(\Omega\setminus K)} \geq D$, $\|v\|_{\mathrm{L}^q(\Omega\setminus K)} \geq D$ *for a fixed point* $x_0 \in \Omega \setminus K$ *and some constants* $D > 0, 1 \leq q < \infty$, *then*

$$C_1 \leq \frac{u}{v} \leq C_2, \quad \text{in } \Omega,$$

where C_1, C_2 *are positive constants depending on* $\Omega, \delta, K, n, s, p, D, B, M, x_0$ *or* q.

Theorem 5.5.3. *If* $1 < q \leq p$ *and* Ω *has regular boundary for the s-fractional p-Laplacian, then there exist no sequences* $\lambda_i > \Lambda_{p,q}, u_i \in V^{s,p}_0(\Omega|\mathbb{R}^n)$, *which satisfy*

$$\begin{aligned} \|u_i\|_{\mathrm{L}^q(\Omega)} &= 1, \\ \lim_{i\to\infty} \lambda_i &= \Lambda_{p,q}, \\ (-\Delta_p)^s u_i &= \lambda_i |u_i|^{q-2} u_i, \quad \text{weakly in } \Omega. \end{aligned}$$

The main novelty of the proof is the extension of the method in Article IV, which overcomes the difficulties arising from the nonlinearity of the fractional p-Laplacian. This is achieved by adapting the comparison principle for viscosity solutions, after which the argument proceeds similarly to [10, Lem. 3.1]. An appealing feature of this work is the simplicity of the proofs, despite working with the weakest solution concept, namely viscosity solutions.

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7. Summery in Swedish

Denna avhandling består av fem artiklar som undersöker samspelet mellan operatorteori och partiella differentialekvationer (PDE). En linjär PDE kan tolkas som en linjär operator på ett lämpligt funktionsrum, vilket möjliggör användning av operatorteoretiska metoder för att analysera existens, regularitet och spektrala egenskaper hos lösningar. Samtidigt kan PDE-tekniker ge avgörande information om spektrum och egenvärden för associerade operatorer.

Ett centralt tema i avhandlingen är Katos kvadratrotsproblem, som handlar om att identifiera definitionsmängden för kvadratroten av en operator och etablera normekvivalenser. Detta är särskilt viktigt vid studiet av randvärdesproblem, där kontroll av operatorns kvadratrot ger skarpa regularitetsresultat.

Artikel I

I den första artikeln behandlas Katos kvadratrotsproblem för paraboliska operatorer med degenerering styrd av en A_2-Muckenhoupt-vikt. Operatorn har mätbara, komplexa koefficienter och uppfyller ett viktat ellipticitetsvillkor.

Huvudresultatet visar att operatorn kan definieras som maximalt ackretiv i ett viktat L^2-rum och att definitionsmängden för dess kvadratrot sammanfaller med energirummet. Dessutom etableras en normekvivalens som kopplar kvadratroten till rumsliga gradienter och halvordningens tidsderivata.

Ett metodologiskt bidrag är en ansats som i stort sett separerar tids- och rumsvariabler, vilket leder till en förenkling jämfört med tidigare metoder baserade på första ordningens system.

Artikel II

Den andra artikeln behandlar Katos kvadratrotsproblem för paraboliska operatorer med icke-släta koefficienter där matrisen delas upp i en symmetrisk elliptisk del och en antisymmetrisk del med element i BMO (bounded mean oscillation). Även här visas att operatorn är maximalt ackretiv och att kvadratrotens definitionsmängd sammanfaller med energirummet. Beviset bygger vidare på metoden från Artikel I men utvecklar nya tekniker för att hantera den antisymmetriska BMO-delen, bland annat via John–Nirenberg-olikheten och Carlesonmåttuppskattningar.

Artikel III

Den tredje artikeln behandlar existens och egenskaper hos fundamentallösningar för viktade paraboliska operatorer med tidsberoende och eventuellt icke-symmetriska koefficienter. Med hjälp av Katos abstrakta teori för evolutionsproblem visas existens och entydighet av svaga lösningar till det paraboliska begynnelsevärdesproblemet. Vidare konstrueras en fundamentallösning och Gaussiska övre uppskattningar härleds, tillsammans med Hölder-kontinuitet i rumsvariablerna. Resultaten generaliserar tidigare arbeten till den tidsberoende och icke-symmetriska situationen.

Artikel IV

I den fjärde artikeln studeras den fraktionella Laplaceoperatorn. Huvudresultatet visar att det första egenvärdet är isolerat, det vill säga att det inte finns följder av större egenvärden som konvergerar mot det första. Beviset belyser det icke-lokala randbeteendet hos den fraktionella Laplaceoperatorn och leder vidare till en generalisering av Hopfs lemma samt en rand-Harnack-olikhet för positiva lösningar. Arbetet visar hur spektrala egenskaper och randbeteende är nära sammanlänkade i den icke-lokala teorin.

Artikel V

Den sista artikeln generaliserar resultaten från Artikel IV till den fraktionella p-Laplacianen, som är en icke-linjär och icke-lokal operator. Här bevisas Hopfs lemma, en rand-Harnack-olikhet samt isolering av det första (s,p)-egenvärdet. Dessa resultat var tidigare inte kända i denna allmänhet. Arbetet identifierar dessutom en lucka i tidigare litteratur och fyller denna genom en alternativ bevisstrategi baserad på jämförelseprinciper för viskositetslösningar.

Övergripande bidrag

De tre första artiklarna använder operatorteoretiska metoder för att analysera paraboliska PDE med icke-släta koefficienter, medan de två sista använder PDE-tekniker för att dra slutsatser om spektrala egenskaper hos icke-lokala och icke-linjära operatorer. Tillsammans belyser avhandlingen det djupa och fruktbara samspelet mellan operatorteori, harmonisk analys och partiella differentialekvationer.

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