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Multiple orthogonal polynomials on the real line and the unit circle

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Abstract

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The thesis presents a series of papers in the field of orthogonal polynomials and approximation theory. The majority of research on orthogonal polynomials can be divided into two parallel groups, where the first one studies orthogonality with respect to measures supported on the real line (OPRL), while the other one studies measures supported on the unit circle (OPUC).

The main topic of the thesis is multiple orthogonal polynomials, on the real line (MOPRL) and the unit circle (MOPUC). The former theory is well-developed, while the latter had seen very little development prior to this work. The thesis contains several fundamental results on MOPUC, which I believe will be its main contribution.

Paper I derives recurrence relations for MOPUC that generalize the Szegő recurrence relations of OPUC, while standing in direct analogue with the nearest-neighbour recurrence relation of MOPRL. Partial difference equations for the recurrence coefficients followed from these relations, along with a Christoffel–Darboux formula.

Paper II-IV study Christoffel transforms in MOPRL, and the more general Uvarov transforms. The papers give formulas for the transformed polynomials and recurrence coefficients, and illustrate a connection to zero location and interlacing. Applications to the zero behaviour of some families of multiple orthogonal polynomials are given. The papers also contain applications to the computation of recurrence coefficients and orthogonal polynomials of OPRL.

Paper V-VII takes a new approach to MOPUC, working with some multiple orthogonal Laurent polynomials of integer and semi-integer degree. Analogues of the Angelesco and AT systems of MOPRL are given, together with important results on perfectness and zero location. The recurrence relations of Paper I also generalize to some of these Laurent polynomials. Szegő mapping relations are given, along with Geronimus relations for the recurrence coefficients, illustrating a connection to MOPRL that was not established for the polynomials studied in Paper I. The connection with two-point Hermite–Padé approximation is also outlined.

The papers are complemented by an introduction with background content, followed by a summary of each paper. The first two chapters provide a friendly introduction to orthogonal and multiple orthogonal polynomials, first through a problem from the thesis, and then through motivating applications to diophantine approximation and numerical integration. The applications relate to Padé approximation and Gauss quadrature, which play a central role in the theory of orthogonal polynomials, and are used frequently in the discussions throughout the introductory chapters.

Keywords: Mathematical Analysis, Approximation Theory, Orthogonal Polynomials, Multiple Orthogonal Polynomials, Padé Approximation, Gauss Quadrature, Partial Difference Equations

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I Rostyslav Kozhan and Marcus Vaknäs, Szegő recurrence for multiple orthogonal polynomials on the unit circle, *Proceedings of the American Mathematical Society* **152** (2024), no. 11, 2983–2997
- II Rostyslav Kozhan and Marcus Vaknäs, Christoffel transform and multiple orthogonal polynomials, *Journal of Computational and Applied Mathematics* **476** (2026), Paper No. 117121
- III Rostyslav Kozhan and Marcus Vaknäs, Determinantal formulas for rational perturbations of multiple orthogonality measures, arXiv:2407.13961 (accepted for publication in *Journal of Approximation Theory*)
- IV Rostyslav Kozhan and Marcus Vaknäs, Zeros of multiple orthogonal polynomials: location and interlacing, *Bulletin of the London Mathematical Society* **58** (2026), e70281
- V Rostyslav Kozhan and Marcus Vaknäs, Angelesco and AT systems on the unit circle, arXiv:2410.12094
- VI Rostyslav Kozhan and Marcus Vaknäs, Szegő mapping and Hermite–Padé Polynomials for multiple orthogonality on the unit circle, arXiv:2601.04783
- VII Rostyslav Kozhan and Marcus Vaknäs, Zeros of Laurent multiple orthogonal polynomials on the unit circle, arXiv:2603.21468

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So, What?

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1. Introduction

The thesis contains research on multiple orthogonal polynomials, on the real line (MOPRL), and on the unit circle (MOPUC). The papers discuss topics such as recurrence relations, location and interlacing of zeros, normality, Christoffel transforms, and the connection to Hermite–Padé approximation.

These opening chapters attempt to introduce the papers contained in the thesis. Chapters 3-5 give the necessary prerequisite content, while chapter 2 provides context through two applications. This chapter introduces some of the thesis papers through an abstract problem. The first two chapters should be somewhat accessible with no more than a basic one-variable calculus background. A more detailed description of each paper is given in Chapter 6.

1.1 Angelesco systems on the real line

For the beginning sections, our polynomials will be generated by a real-valued function $w(x) \geq 0$ on a bounded real interval $[a, b]$, and

$$0 < \int_a^b w(x)dx < \infty. \quad (1.1)$$

Two complex-valued functions f and g are orthogonal with respect to the weight distribution w if

$$\int_a^b \overline{f(x)}g(x)w(x)dx = 0. \quad (1.2)$$

A polynomial Q_n of degree n with real coefficients is an orthogonal polynomial with respect to w if it is orthogonal to all polynomials of lower degree, or equivalently,

$$\int_a^b Q_n(x)x^p w(x)dx = 0, \quad p = 0, 1, \dots, n-1. \quad (1.3)$$

This determines Q_n up to multiplication by a non-zero constant. Unless stated otherwise, we choose to work with the unique monic polynomial Q_n , i.e., the one with leading coefficient equal to 1.

The polynomials studied in the thesis are not usually orthogonal polynomials, but rather multiple orthogonal polynomials. They split the orthogonality relations (1.3) between several weight functions w_1, \dots, w_r . Here we have

$n = n_1 + \dots + n_r$ for a multi-index of non-negative integers $\mathbf{n} = (n_1, \dots, n_r)$, and the multiple orthogonal polynomials $Q_{\mathbf{n}}$ of degree n are defined by

$$\int_{a_j}^{b_j} Q_{\mathbf{n}}(x) x^p w_j(x) dx = 0, \quad p = 0, 1, \dots, n_j - 1, \quad j = 1, \dots, r. \quad (1.4)$$

Unlike the usual orthogonal polynomials, there may not be a unique monic $Q_{\mathbf{n}}$ for some \mathbf{n} . The uniqueness is an essential property of most studies of multiple orthogonal polynomials. Nearly every result from the thesis papers either requires uniqueness, or establishes conditions for uniqueness to hold.

One of the most interesting things about orthogonal polynomials is their zero behaviour. A basic property is that all zeros lie on the open interval (a, b) , each with multiplicity 1. Another property relevant to the thesis is the interlacing of zeros, i.e., between every pair of zeros of Q_n there is a zero of Q_{n-1} .

For multiple orthogonal polynomials, zeros are not always real, but there are known large classes of weights for which the zeros stay on the real line. An important example was given by Romanian mathematician Angelescu. Here the weights w_j are defined on intervals $[a_j, b_j]$ such that the open intervals (a_j, b_j) are pairwise disjoint. These systems of weight functions $\mathbf{w} = (w_1, \dots, w_r)$ are nowadays known as Angelesco systems (for some reason, the name is spelled differently).

Theorem 1.1.1 (Angelescu, 1918 [1]) *If \mathbf{w} is an Angelesco system, then there is a unique monic polynomial $Q_{\mathbf{n}}$ of degree n satisfying (1.4), and all zeros of $Q_{\mathbf{n}}$ are real and simple.*

The proof is so simple I may as well include it. Suppose $Q_{\mathbf{n}}$ has exactly k sign changes on (a_j, b_j) , at the points x_1, \dots, x_k . Then the polynomial $Q_{\mathbf{n}}(x) \prod_{i=1}^k (x - x_i)$ does not change sign on $[a_j, b_j]$, so that

$$\int_{a_j}^{b_j} Q_{\mathbf{n}}(x) \prod_{i=1}^k (x - x_i) w_j(x) dx \neq 0. \quad (1.5)$$

By the orthogonality relations (1.4) we must have $k \geq n_j$ zeros on (a_j, b_j) . Since the intervals are disjoint we find n real and simple zeros of $Q_{\mathbf{n}}$.

The above proof also shows that there is a unique monic $Q_{\mathbf{n}}$. Indeed, if there is another monic solution, say $P_{\mathbf{n}}$, then the difference $Q_{\mathbf{n}} - P_{\mathbf{n}}$ has degree $< n$ but still satisfies the same orthogonality relations. The above zero counting proof then finds n different zeros, so $Q_{\mathbf{n}} = P_{\mathbf{n}}$ is the only possibility. Existence follows similarly, after solving for the $n + 1$ coefficients of $Q_{\mathbf{n}}$ in the n linear equations (1.4). More generally, any complex polynomial of degree n satisfying (1.4) is a complex multiple of $Q_{\mathbf{n}}$, by the same proof applied to the real and imaginary part separately.

1.2 Angelesco systems on the unit circle

Theory centered around the polynomials Q_n of (1.3) is known as orthogonal polynomials on the real line (OPRL). The orthogonality relations can be modified to consider orthogonality on other subsets of the complex plane. In particular, OPRL has a well-developed dual theory on the unit circle (OPUC). Here the monic orthogonal polynomials Φ_n of degree n are defined by

$$\int_a^b \Phi_n(e^{i\theta}) e^{-ip\theta} w(\theta) d\theta = 0, \quad p = 0, 1, \dots, n-1, \quad (1.6)$$

for w with the same properties as before, and $[a, b] \subseteq [0, 2\pi]$.

Such polynomials were initially studied by Szegő, and are often referred to as Szegő polynomials. As with OPRL, these polynomials are unique. The zeros always lie inside the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (but are not necessarily simple), see [36] for several basic proofs. A large part of the thesis is devoted to exploring multiple orthogonality analogues of these polynomials.

Despite the popularity of MOPRL over the last few decades, MOPUC has not seen much development. When I started out as a PhD student, only the two papers [14, 34] had been written on MOPUC (although we also recently found related constructions in the papers [33, 39]). These papers studied polynomials Φ_n of degree n satisfying

$$\int_{a_j}^{b_j} \Phi_n(e^{i\theta}) e^{-ip\theta} w_j(\theta) d\theta = 0, \quad p = 0, 1, \dots, n_j - 1, \quad j = 1, \dots, r, \quad (1.7)$$

with respect to the index $\mathbf{n} = (n_1, \dots, n_r)$ and the system of weight functions $\mathbf{w} = (w_1, \dots, w_r)$.

Paper I also studies these polynomials, where we derive some interesting recurrence relations. After this we shifted our study of MOPUC to consider alternative constructions. While (1.7) appears reasonable, it should relate to something like an application, or satisfy desirable properties. In Paper V we suggest shifting the attention to polynomials Φ_n of degree n satisfying the alternative orthogonality relations

$$\int_{a_j}^{b_j} e^{-in/2} \Phi_n(e^{i\theta}) e^{-ip\theta} w_j(\theta) d\theta = 0, \quad p = -\frac{n_j}{2}, -\frac{n_j}{2} + 1, \dots, \frac{n_j}{2} - 1, \quad j = 1, \dots, r. \quad (1.8)$$

If $r = 1$, both (1.7) and (1.8) reduce to the usual orthogonal polynomials (1.6). For the new approach, we found the following result.

Theorem 1.2.1 *If the intervals $(a_j, b_j) \subseteq [0, 2\pi]$ are pairwise disjoint, then there is a unique monic polynomial Φ_n of degree n satisfying (1.8). All zeros of Φ_n lie in \mathbb{D} .*

The uniqueness was a main result of Paper V, which relies on a very different argument to the one given in the previous section on the real line. Paper VII then uses the uniqueness to locate the zeros, with parts of the proof using ideas from Paper IV, which discusses zero location in MOPRL. This result was found only some weeks before the thesis was finalized. Paper VII is likely to be updated with more results in the near future.

With the definition (1.7), numerical simulations show that zeros do not necessarily lie in \mathbb{D} for Angelesco systems, and no uniqueness of those polynomials has been established for any large class of weights.

The picture on the cover shows a plot of the zeros for an Angelesco system (w_1, w_2) with w_1 defined on the interval $[0, 2\pi/3]$ and w_2 defined on $[5\pi/6, 11\pi/6]$. It may seem like some of the zeros lie on the circle, but they only lie very close to it. This is not so surprising as there are results on this for OPUC, such as Widom's theorem [46], which states that the number of zeros of Φ_n in $\{z \in \mathbb{C} : |z| < 1 - \varepsilon\}$ is bounded in n for every $\varepsilon > 0$.

As MOPUC is very new, there are many promising directions for future research. Currently, there are no asymptotic results on MOPUC. Also, the polynomials have not been computed explicitly for any special choice of weights. Hopefully there will be some development on this in the future.

2. Some approximation theory

We discuss problems in number theory and numerical analysis, and then show their connection to orthogonal polynomials in section 2.3. The chapter is partially based on talks I have given to my fellow PhD students over the years, in an attempt to motivate interest in orthogonal and multiple orthogonal polynomials, not just for others but also for myself.

The common theme here is the connection to approximation theory, which is closely tied with orthogonal polynomials. We discuss connections to Padé approximation, Fourier approximation, and Lagrange interpolation. At first, they do not seem related, but they turn out to all be naturally described in terms of orthogonal polynomials.

For more on Padé approximations and number theory applications, see e.g. [35]. For the numerical analysis applications and the convergence of interpolation polynomials, see e.g. [30, 41].

2.1 Irrationality and transcendence

A real number x is rational if we eventually find an integer when we run through the infinite sequence of numbers $x, 2x, 3x, 4x, \dots$. If x is irrational we will never find an integer, but we may come very close, such as $7\pi \approx 21.991$, or $113\pi \approx 354.99997$. A natural question to ask is if we can find a number as close as we want to an integer, by scanning far enough into the sequence. In other words, find integers q_n and p_n , such that

$$(I1) \lim_{n \rightarrow \infty} (q_n x - p_n) = 0,$$

$$(I2) q_n x - p_n \neq 0.$$

The existence of such numbers would immediately prove that x is irrational. Although finding them tends to be very difficult, they do exist for every irrational x . In fact, a simple and clever application of the pigeonhole principle (Dirichlet approximation theorem) shows the existence of infinite sequences $q_n \rightarrow \infty$ and p_n with

$$|q_n x - p_n| < 1/q_n. \quad (2.1)$$

A much more advanced result (Roth's theorem) states that this can barely be improved if x is algebraic, in the sense that there are no constants $c > 0$ and $\varepsilon > 0$ such that $0 < |q_n x - p_n| < c/q_n^{1+\varepsilon}$.

A complex number x is algebraic if there are integers a_0, a_1, \dots, a_r , not all 0, such that

$$a_0 + a_1 x + \dots + a_r x^r = 0. \quad (2.2)$$

Note that the case $r = 1$ corresponds to rationality. If x is not algebraic it is called transcendental. The approach (I1)-(I2) extends to proving transcendence, by showing that (2.2) cannot hold for a_0, a_1, \dots, a_r if there are integers q_n and $p_{n,1}, \dots, p_{n,r}$ such that

$$(T1) \lim_{n \rightarrow \infty} (q_n x^j - p_{n,j}) = 0 \text{ for each } j = 1, \dots, r.$$

$$(T2) a_0 q_n + a_1 p_{n,1} + \dots + a_r p_{n,r} \neq 0.$$

In other words, one should look for good rational approximants $x^j \approx p_{n,j}/q_n$ with common denominator q_n .

To verify (I1)-(I2), one typically starts by approximating some function and then evaluate at a point. For Euler's number e , the classic Taylor approximation $e^z \approx \sum_{k=0}^n z^k/k!$ will be sufficient. It is the best approximation of e^z as $z \rightarrow 0$, in the sense that it is the only polynomial P_n of degree at most n with $|e^z - P_n(z)| \leq c|z|^{n+1}$ for some constant c , when $|z|$ is small enough. Evaluating at $z = 1$ produces rational approximants p_n/q_n with $q_n = n!$. Taylor's theorem gives the error bound

$$0 < n!e - p_n < 3/n. \quad (2.3)$$

This is a lot worse than (2.1), but still barely satisfies (I1)-(I2). The same approach does not prove irrationality of e^k for integers $k > 1$.

An improvement can be made via Padé approximation. Here one looks for polynomials $Q_{n,m} \neq 0$ and $P_{n,m}$ with $\deg Q_{n,m} \leq n$ and $\deg P_{n,m} \leq m$, such that

$$Q_{n,m}(z)f(z) - P_{n,m}(z) = \mathcal{O}(z^{n+m+1}), \quad z \rightarrow 0. \quad (2.4)$$

The rational function $P_{n,m}/Q_{n,m}$ is the unique best approximation of f as $z \rightarrow 0$, with the given degree constraints (although the order may decrease in (2.4) when you divide by $Q_{n,m}$). Note that the case $n = 0$ corresponds to Taylor approximation.

One typically considers the diagonal case $n = m$. Hermite computed these polynomials explicitly for $f(z) = e^z$ [25]. Evaluating at a positive integer k produces rational numbers $P_{n,n}(k)/Q_{n,n}(k)$ satisfying (I1)-(I2) for e^k , and significantly improving (2.3) for $k = 1$. At $n = 3$ we find $71e \approx 192.998$.

Theorem 2.1.1 (Hermite, 1873 [25]) *e is transcendental.*

Hermite obtained his result through a more general version of (2.4), approximating a system of functions (f_1, \dots, f_r) with a common denominator Q . Here one works with polynomials $Q_{\mathbf{n},\mathbf{m}}$ and $P_{\mathbf{n},\mathbf{m}}$ with respect to multi-indices $\mathbf{n} = (n_1, \dots, n_r)$ and $\mathbf{m} = (m_1, \dots, m_r)$ such that $\deg Q_{\mathbf{n},\mathbf{m}} \leq |\mathbf{n}| = n_1 + \dots + n_r$ and $\deg P_{\mathbf{n},\mathbf{m},j} \leq m_j$, $j = 1, \dots, r$. They should satisfy the approximation condition

$$Q_{\mathbf{n},\mathbf{m}}(z)f_j(z) - P_{\mathbf{n},\mathbf{m},j}(z) = \mathcal{O}(z^{n_j+m_j+1}), \quad z \rightarrow 0. \quad (2.5)$$

This is known as Hermite–Padé approximation of type II, at $z = 0$. Of course one can approximate functions at other points, which would be related through a substitution $z \mapsto z - z_0$, or $z \mapsto 1/z$ to approximate at large values of z .

For the functions $e^z, e^{2z}, \dots, e^{rz}$, Hermite computed the polynomials $Q_{n,m}$ and $P_{n,m}$ explicitly, for indices \mathbf{n} and \mathbf{m} with $n_j + m_j = |\mathbf{n}| + N$ for all $j = 1, \dots, r$ and some $N \geq 0$. At $z = 1$, this generates rational approximants verifying (T1)-(T2). The approach was extended by Lindemann, which proves transcendence of e^α if $\alpha \neq 0$ is algebraic. In particular, π is transcendental, since $e^{i\pi}$ is rational. Hermite–Padé approximation has also been used to verify (I1)-(I2) for certain constants, such as generating the Apéry numbers approximating $\zeta(3)$. See e.g. [44] for this and several other examples.

Polynomials such as $Q_{n,m}$, defined by simultaneous approximation conditions, are the objects of study in this thesis. For the Padé case $r = 1$, they are naturally described by orthogonal polynomials, for a large class of functions. For general r , they form multiple orthogonal polynomials. This connection is described in section 2.3, and further in the subsequent chapters. A connection between two-point Padé approximation and OPUC is described in Chapter 5.

2.2 Interpolation and quadrature

To compute functions one may want to approximate them with polynomials, preferably of small degree, although this would sacrifice some accuracy in the approximation. For example, the Taylor approximation

$$e^{-x^2} \approx 1 - x^2 + 0.5x^4 \quad (2.6)$$

has a maximum error of ≈ 0.13 on the interval $[-1, 1]$. Since the Taylor polynomial is designed to be as good as possible as $x \rightarrow 0$, it may not be a particularly good approximation away from 0.

A somewhat less familiar approximation would be

$$e^{-x^2} \approx 1 - 0.961x^2 + 0.335x^4. \quad (2.7)$$

This gives an error of ≈ 0.006 on $[-1, 1]$. The uniform bound is particularly important when integrating over the interval. By integrating the right hand side of (2.7) we get

$$\int_{-1}^1 e^{-x^2} dx \approx 1.493, \quad (2.8)$$

with an error no more than $\approx 0.006 \times 2 = 0.012$ (the actual error is ≈ 0.0003).

The polynomial in (2.7) was constructed via polynomial interpolation, with zeros of some orthogonal polynomials as interpolation nodes. In general, for a function $f(x)$, match it with a polynomial $L_n(f;x)$ of degree $\leq n - 1$ at n different points $x_{n,1}, \dots, x_{n,n}$, i.e., $f(x_{n,j}) = L_n(f;x_{n,j})$, $j = 1, \dots, n$. This always uniquely determines $L_n(f;x)$. It can be written as

$$L_n(f;x) = \sum_{j=1}^n f(x_{n,j}) \ell_{n,j}(x), \quad (2.9)$$

where $\ell_{n,k}$ is the polynomial of degree $\leq n-1$ equal to 1 at $x_{n,k}$ and 0 at $x_{n,j}$ for $j \neq k$. $\ell_{n,k}$ can be computed explicitly by

$$\ell_{n,k}(x) = \frac{\prod_{j \neq k} (x - x_{n,j})}{\prod_{j \neq k} (x_{n,k} - x_{n,j})}, \quad k = 1, \dots, n. \quad (2.10)$$

To approximate a function on $[-1, 1]$ using (2.9) it makes sense to distribute the points $x_{n,j}$ over the entire interval. Equally spaced nodes seem like a natural choice, i.e., $x_{n,j+1} - x_{n,j} = 2/n$. Of course, it is not clear that this controls how much $L_n(f)$ jumps between two nodes $x_{n,j}$ and $x_{n,j+1}$, and indeed we may observe arbitrarily large jumps of $L_n(f)$ as $n \rightarrow \infty$. A classical example is given by Runge's function $f(x) = 1/(1 + 25x^2)$.

In (2.7), I used the Chebyshev nodes

$$x_{n,j} = \cos \frac{(2j-1)\pi}{2n}, \quad j = 1, \dots, n. \quad (2.11)$$

This actually tends to give a much better approximation than equally spaced nodes. One positive result in this direction is the following.

Theorem 2.2.1 *If f is k times differentiable on $[-1, 1]$ and $f^{(k)}$ is continuous, then*

$$\max_{x \in [-1, 1]} |f(x) - L_n(f; x)| = \mathcal{O}\left(\frac{\log n}{n^k}\right), \quad n \rightarrow \infty. \quad (2.12)$$

More generally, for any polynomial P_n of degree at most $n-1$,

$$\max_{x \in [-1, 1]} |f(x) - L_n(f; x)| \leq \left(\mathcal{O}(1) + \frac{2}{\pi} \log n\right) \max_{x \in [-1, 1]} |f(x) - P_n(x)|, \quad (2.13)$$

suggesting that these approximating polynomials are close to optimal. It follows from a result of Bernstein [7], combined with Lebesgue's lemma. (2.12) follows from (2.13) by Jackson's inequality.

If we integrate the interpolation polynomial (2.9) we get a numerical quadrature rule

$$\int_{-1}^1 f(x) dx \approx \sum_{j=1}^n w_{n,j} f(x_{n,j}). \quad (2.14)$$

By (2.12) the Chebyshev nodes (2.11) are very good for smooth functions, typically better than e.g. the classical trapezoid rule. The following integration technique corresponds to Fejér quadrature, closely related to Clenshaw–Curtis quadrature. Quadrature using equally spaced nodes is typically referred to as Newton–Cotes rules.

The Chebyshev nodes are the zeros of the Chebyshev polynomials, which are orthogonal polynomials with respect to the weight $w(x) = 1/\sqrt{1-x^2}$. Zeros of orthogonal polynomials turn out to be quite good for interpolation and quadrature, for functions that are well approximated by polynomials. If we use zeros of Legendre polynomials ($w(x) = 1$) in (2.14) we get Gauss–Legendre quadrature. It gives equality for polynomials up to the highest degree possible, which is $2n-1$ for each n .

2.3 The connection to orthogonal polynomials

Orthogonal polynomials are certainly most familiar through the Fourier approximation

$$f(x) \approx s_n(f; x) = \sum_{j=0}^n c_j Q_j(x), \quad c_j = \frac{\int_a^b f(x) Q_j(x) w(x) dx}{\int_a^b Q_j(x)^2 w(x) dx}. \quad (2.15)$$

For real-valued functions f on $[a, b]$ with $\int_a^b f(x)^2 w(x) dx < \infty$, $g = s_n(f)$ is the unique polynomial of degree $\leq n$ minimizing the distance

$$d(f, g) = \sqrt{\int_a^b (f(x) - g(x))^2 w(x) dx}. \quad (2.16)$$

A great revelation to me as a young student was that (2.15) is algebraically similar to projecting a Euclidean vector f onto a subspace spanned by perpendicular vectors Q_0, Q_1, \dots, Q_n .

The Fourier series associated with Chebyshev polynomials satisfy similar convergence properties to the interpolation polynomials of the previous section. Another great revelation to me as a young student was that this allows polynomial approximation of quite complicated functions. The Fourier series converges for any differentiable function on $[-1, 1]$, and the convergence is uniform when the derivative is continuous. Moreover, a monumental result of Carleson [11] (Uppsala University!) proves the almost everywhere convergence for any continuous function.

In general, Fourier series have a close connection with polynomial interpolation using zeros of orthogonal polynomials. To see this, we first integrate $L_n(f; x)$ with respect to the distribution $w(x) dx$ to obtain the Gauss quadrature rule

$$\int_a^b f(x) w(x) dx \approx \sum_{j=1}^n \lambda_{n,j} f(x_{n,j}), \quad (2.17)$$

where $x_{n,j}$ are chosen to be the zeros of Q_n with respect to w . These nodes $x_{n,j}$ and weights $\lambda_{n,j}$ are characterized as the only choice giving equality in (2.17) for all polynomials of degree $\leq 2n - 1$. The weights are indeed always positive, which may not be clear from the current discussion.

The polynomials $s_n(f; x)$ have the integral representation

$$s_n(f; x) = \int_a^b f(t) K_n(x, t) w(t) dt, \quad (2.18)$$

where $K_n(x, t)$ is the Christoffel–Darboux kernel

$$K_n(x, t) = \sum_{j=0}^n \frac{Q_j(x) Q_j(t)}{\int_a^b Q_j(x)^2 w(x) dx} = \frac{Q_{n+1}(x) Q_n(t) - Q_n(x) Q_{n+1}(t)}{(x - t) \int_a^b Q_n(x)^2 w(x) dx}. \quad (2.19)$$

The second equality is known as the Christoffel–Darboux formula. Now by applying Gauss quadrature to the integral (2.18) we actually get exactly the interpolation polynomial $L_{n+1}(f; x)$.

There is also a close connection between the interpolation polynomials and Padé approximation. This is through the Padé approximation problem at ∞ , which seeks polynomials $Q_n \neq 0$ and P_n with $\deg Q_n \leq n$ and

$$Q_n(z)f(z) - P_n(z) = \mathcal{O}(z^{-n-1}), \quad z \rightarrow \infty. \quad (2.20)$$

Here, Q_n are orthogonal polynomials with respect to w on $[a, b]$ when f is the Stieltjes function

$$f(z) = \int_a^b \frac{w(x)dx}{z-x}, \quad z \in \bar{\mathbb{C}} \setminus [a, b]. \quad (2.21)$$

By applying Gauss quadrature to this function we actually obtain exactly the Padé approximants, i.e., we have the partial fraction decomposition

$$\frac{P_n(z)}{Q_n(z)} = \sum_{j=1}^n \frac{\lambda_{n,j}}{z - x_{n,j}}. \quad (2.22)$$

Theorem 2.3.1 *The Gauss quadrature rule implies the following two results.*

- $\lim_{n \rightarrow \infty} d(L_n(f), f) = 0$ for every continuous function f on $[a, b]$.
- The Padé approximants (2.22) converge to the Stieltjes function (2.21) uniformly on compact subsets of $\bar{\mathbb{C}} \setminus [a, b]$.

The rate of convergence for the Padé approximants is geometric outside ellipses with foci at a and b , see e.g. [35]. The mean-squared convergence of the interpolation polynomials is a result of Erdős–Turán [17].

Multiple orthogonal polynomials connect to Hermite–Padé approximation at ∞ . Here, if Q_n satisfies the multiple orthogonality relations (1.4) then

$$Q_n(z)f_j(z) - P_{n,j}(z) = \mathcal{O}(z^{-n_j-1}), \quad z \rightarrow \infty, \quad j = 1, \dots, r, \quad (2.23)$$

where f_j is the Stieltjes function with respect to the weight w_j , and $P_{n,j}$ are some polynomials determined by Q_n . For convergence results, see e.g. [35].

There is also a connection with quadrature, through the approximation of integrals with respect to several weights

$$\int f(x)d\mu_j(x) \approx \sum_{k=1}^N \lambda_{j,k}f(x_k), \quad j = 1, \dots, r. \quad (2.24)$$

Assuming Q_n has n simple zeros x_1, \dots, x_n , we have equality in the integral in (2.24) for polynomials of degree up to $n + n_j - 1$, $j = 1, \dots, r$.

This is more efficient than applying Gauss quadrature to each integral, in the sense that we get equality in each integral for polynomials up to a certain degree with a smaller number of evaluations. It was discussed by Angelescu in his paper [1] with the convergence analyzed for Angelescu systems in [31]. It was also studied by Borges [8], motivated by a problem in computer graphics, and seemingly independent of the literature on Hermite–Padé approximation.

3. Orthogonality on the real line

This chapter discusses OPRL, in its most general form. Emphasis will be put on algebraic relations, uniqueness, and zero behaviour, along with the connection to Padé approximation and Gauss quadrature. Classical examples and asymptotic analysis is largely left out, as this is not as relevant to the thesis papers. For a more thorough discussion of the content of section 3.1, see e.g. [35, 43]. Section 3.2 relates closely with [38], while section 3.3 is more in the direction of [21].

3.1 Padé approximation and orthogonality

(2.21) describes a somewhat large class of functions for which Padé approximation is expressed through orthogonal polynomials, as we have many choices for the weight w . The positivity of the weight has a big impact on the limiting behaviour, but the connection between the Padé approximation problem (2.20) and the orthogonality relations (1.3) holds with any complex-valued weight function. We can extend this connection further to any formal power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^{-k-1}. \quad (3.1)$$

Define a linear functional L on the set of polynomials (with complex coefficients) by

$$c_k = L[x^k], \quad k \in \mathbb{N}. \quad (3.2)$$

Then any solution Q_n to the Padé approximation problem (2.20) satisfies the orthogonality relations

$$L[Q_n(x)x^p] = 0, \quad p = 0, \dots, n-1. \quad (3.3)$$

This corresponds to the orthogonality relations (1.3) when L is given by

$$L[P(x)] = \int_a^b P(x)w(x)dx. \quad (3.4)$$

For non-negative weights, Q_n is uniquely determined up to multiplication by a constant, and we always have $\deg Q_n = n$. In general there are some degenerate cases where we may have $\deg Q_n < n$. Solving (3.3) for the coefficients of a polynomial of degree $< n$, we get a system of equations with coefficient matrix equal to the Hankel matrix

$$H_n[L] = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{pmatrix}. \quad (3.5)$$

It follows that $\deg Q_n = n$ and uniqueness up to a constant holds exactly when this matrix is invertible. If $\det(H_n) \neq 0$ then $\det(H_{n+1}) = 0$ if and only if $P_{n+1}/Q_{n+1} = P_n/Q_n$.

L is called a quasi-definite functional if $\det(H_n) \neq 0$ for every $n > 0$. This is equivalent to $L[Q_n(x)^2] \neq 0$ for every $Q_n \neq 0$ satisfying (3.3). When L is quasi-definite, we always write Q_n for the unique monic orthogonal polynomial of degree n . Then Q_n has the Heine representation

$$Q_n(x) = \frac{1}{\det(H_n)} \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}. \quad (3.6)$$

L is positive-definite when $L[Q_n(x)^2] > 0$ for every n . This is equivalent to $\det(H_n) > 0$ for every $n > 0$, hence the Hankel matrices are positive-definite. Integration with respect to positive weights gives positive-definite functionals, but more generally we have the following representation.

Theorem 3.1.1 (Hamburger moment problem) *If the Hankel matrices (3.5) are positive-definite for every $n > 0$ then*

$$c_k = \int x^k d\mu(x), \quad k \in \mathbb{N}, \quad (3.7)$$

for some positive Borel measure μ on \mathbb{R} .

For the measure μ in (3.7), the support

$$\text{supp}(\mu) = \{x \in \mathbb{R} : \mu[(x - \varepsilon, x + \varepsilon)] > 0 \text{ for every } \varepsilon > 0\} \quad (3.8)$$

contains infinitely many points. μ is uniquely determined by L when $\text{supp}(\mu)$ is bounded, but this may not be true in the unbounded case.

For finite support measures $\mu = \sum_{j=1}^N w_j \delta_{x_j}$ (with $w_j > 0$, $j = 1, \dots, N$) the functional satisfies $\det H_n > 0$ for $n \leq N$ and $\det H_n = 0$ otherwise. Note that Q_N is the polynomial that vanishes on $\text{supp}(\mu) = \{x_1, \dots, x_N\}$. For any μ with infinite support, the Gauss quadrature rule can be formulated as the existence of a unique measure μ_N supported on N points such that

$$\int P(x) d\mu(x) = \int P(x) d\mu_N(x), \quad \deg P \leq 2N - 1. \quad (3.9)$$

As before, $\mu_N = \sum_{j=1}^N \lambda_{N,j} \delta_{x_{N,j}}$, where $x_{N,j}$ are the zeros of Q_N , and $\lambda_{N,j}$ are the residues of P_N/Q_N . By Weierstrass approximation theorem, μ_N converges weakly to μ in the bounded support case. Note that (3.9) implies that μ and μ_N have the same orthogonal polynomials of degrees $k \leq N$.

3.2 Recurrence relations and Jacobi matrices

For any quasi-definite functional L , the monic orthogonal polynomials Q_n satisfy the three-term recurrence relation

$$xQ_n(x) = Q_{n+1}(x) + b_n Q_n(x) + a_n Q_{n-1}(x). \quad (3.10)$$

For $n = 0$ the relation holds with the convention $a_0 = 0$ and/or $Q_{-1} = 0$. For $n > 0$ the coefficients a_n are given by

$$a_n = \frac{L[Q_n(x)^2]}{L[Q_{n-1}(x)^2]} = \frac{\det(H_{n+1}[L]) \det(H_{n-1}[L])}{\det(H_n[L])^2}. \quad (3.11)$$

From (3.11) we see that we always have $a_n \neq 0$ when $n > 0$, and when L is positive-definite we get $a_n > 0$. Conversely, any sequences $a_n \neq 0$ and b_n are the recurrence coefficients in (3.10) of some quasi-definite functional L . The representation theorem was reformulated by Favard [18] in terms of the recurrence coefficients to the following result.

Theorem 3.2.1 (Favard's Theorem) *If $a_n > 0$ for $n > 0$ and $b_n \in \mathbb{R}$ for all n , then $(Q_n)_{n=0}^\infty$ defined by (3.10) are the monic orthogonal polynomials with respect to some positive Borel measure μ on \mathbb{R} .*

The recurrence relation (3.10) implies that zeros are real and interlacing in the positive-definite case. Conversely, given any polynomials Q_n of degree n and Q_{n+1} of degree $n+1$ with real and interlacing zeros there is a measure μ such that Q_n and Q_{n+1} are orthogonal polynomials with respect to μ .

From the monic orthogonal polynomials Q_n with respect to μ , we can define the normalized polynomials $q_n = Q_n / \|Q_n\|_{L^2(\mu)}$. The three-term recurrence relation then translates to

$$xq_n(x) = \sqrt{a_{n+1}} q_{n+1}(x) + b_n q_n(x) + \sqrt{a_n} q_{n-1}(x). \quad (3.12)$$

Associated to this recurrence relation is the Jacobi matrix

$$J = \begin{pmatrix} b_0 & \sqrt{a_1} & & & \\ \sqrt{a_1} & b_1 & \sqrt{a_2} & & \\ & \sqrt{a_2} & b_2 & \sqrt{a_3} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \quad (3.13)$$

The weights w_j are uniquely determined by the points x_j , so we have a uniquely determined quadrature rule when \widehat{L} is quasi-definite.

We can also consider the generalized problem of finding optimal $(n+N)$ -point quadrature rules with N fixed nodes x_1, \dots, x_N . This leads to studying the Christoffel transforms $\widehat{L} = \prod_{j=1}^N (x - x_j)L$ defined by

$$\widehat{L}[P(x)] = L[P(x) \prod_{j=1}^N (x - x_j)]. \quad (3.18)$$

A natural question that arises is how associated objects change under this transform, such as orthogonal polynomials, recurrence coefficients, and Hankel determinants.

For the degree 1 transforms (3.17), some column operations on the Heine formula (3.6) reveal that

$$Q_n(x_0) = (-1)^{n+1} \det(H_n[(x - x_0)L]) / \det(H_n[L]). \quad (3.19)$$

In particular, if L is quasi-definite then $(x - x_0)L$ is quasi-definite if and only if $Q_n(x_0) \neq 0$ for every n . The polynomials are given by

$$\widehat{Q}_n(x) = \frac{Q_{n+1}(x)Q_n(x_0) - Q_n(x)Q_{n+1}(x_0)}{(x - x_0)Q_n(x_0)}. \quad (3.20)$$

Note that this corresponds to a constant multiple of the kernel polynomials $K_n(x, x_0)$ from (2.19). The recurrence coefficients \widehat{a}_n and \widehat{b}_n of \widehat{L} can be computed from the recurrence coefficients a_n and b_n of L , through the difference equations

$$\widehat{b}_n - r_{n+1} = b_{n+1} - r_n, \quad (3.21)$$

$$\widehat{a}_n - r_n \widehat{b}_n = a_{n+1} - r_n b_n, \quad (3.22)$$

$$r_{n-1} \widehat{a}_n = r_n a_n, \quad (3.23)$$

where $r_0 = x_0 - b_0$.

A determinantal formula given by Christoffel [12] generalizes (3.20) to the case $N > 1$ in (3.18). It was further generalized by Uvarov [42] to rational transformations

$$\widetilde{L}[P(x)] = \int P(x) \frac{\prod_{j=1}^N (x - x_j)}{\prod_{j=1}^M (x - y_j)} d\mu(x), \quad (3.24)$$

where $x_1, \dots, x_N, y_1, \dots, y_M$ are all distinct points.

Theorem 3.3.1 *If \tilde{L} is quasi-definite and $n \geq M$, then*

$$\tilde{Q}_n(x) = \frac{1}{\prod_{j=1}^N (x - x_j)} \det \begin{pmatrix} Q_{n+N}(x) & Q_{n+N-1}(x) & \cdots & Q_{n-M}(x) \\ Q_{n+N}(x_1) & Q_{n+N-1}(x_1) & \cdots & Q_{n-M}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n+N}(x_N) & Q_{n+N-1}(x_N) & \cdots & Q_{n-M}(x_N) \\ R_{n+N}(y_1) & R_{n+N-1}(y_1) & \cdots & R_{n-M}(y_1) \\ \vdots & \vdots & \ddots & \vdots \\ R_{n+N}(y_M) & R_{n+N-1}(y_M) & \cdots & R_{n-M}(y_M) \end{pmatrix} \quad (3.25)$$

is a degree n orthogonal polynomial with respect to \tilde{L} , where R_k is given by

$$R_k(z) = \int \frac{Q_k(x)}{z-x} d\mu(x), \quad k \in \mathbb{N}. \quad (3.26)$$

The result was originally stated for the case when x_1, \dots, x_N and y_1, \dots, y_M all lie outside $\text{supp}(\mu)$, so that \tilde{L} is positive-definite. We can also allow some factors to have higher multiplicity. In that case we would have to add rows corresponding to derivatives at these points. Furthermore, Uvarov derived a separate version of (3.25) for the case $n < M$.

The relation (3.19) provides a somewhat nonstandard approach to locating zeros, which is relevant to the thesis. If L is positive definite we see that if $[a, b] \supseteq \text{supp}(\mu)$ and $x_0 \in \mathbb{R} \setminus (a, b)$, then $(x - x_0)d\mu(x)$ or $(x_0 - x)d\mu(x)$ is a positive measure, so x_0 cannot be a zero of Q_n .

Also, if P divides Q_n and $\deg P \geq 1$ then $\tilde{L} = P(x)L$ is not quasi-definite. If $P(x) = (x - z_0)(x - \bar{z}_0)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$, then \hat{L} is positive-definite if L is positive-definite. It follows that all zeros of Q_n lie in the interior of the convex hull of $\text{supp}(\mu)$. Interlacing can also be analyzed, through the transforms $(x - x_0)^2 L$, $x_0 \in \mathbb{R}$. In this case the leading coefficient of (3.25) turns into the Wronskian $W(Q_{n+1}, Q_n)$ at x_0 .

4. Multiple orthogonality on the real line

The Hermite–Padé approximation, normality, Angelesco and AT systems, and nearest-neighbour recurrence relations of this chapter are the central topics of the thesis. For a more thorough treatment, the book [35] of Nikishin and Sorokin is a great source, although it predates several results mentioned within the chapter. Several further references are provided within these sections.

4.1 Hermite–Padé approximation and orthogonality

We now turn to simultaneous Padé approximation at ∞ , of power series of the type

$$f_j(z) = \sum_{k=0}^{\infty} c_{j,k} z^{-k-1}, \quad j = 1, \dots, r. \quad (4.1)$$

We define linear functionals L_1, \dots, L_r by

$$c_{j,k} = L_j[x^k], \quad k \in \mathbb{N}, \quad j = 1, \dots, r. \quad (4.2)$$

With respect to the multi-index of non-negative integers $\mathbf{n} = (n_1, \dots, n_r)$, the type II Hermite–Padé approximation problem seeks polynomials $Q_{\mathbf{n}} \neq 0$ and $P_{\mathbf{n},1}, \dots, P_{\mathbf{n},r}$, with $\deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + \dots + n_r$ that satisfy the simultaneous approximation condition (2.23). This is equivalent to solving the multiple orthogonality relations

$$L_j[Q_{\mathbf{n}}(x)x^p] = 0, \quad p = 0, \dots, n_j - 1, \quad j = 1, \dots, r. \quad (4.3)$$

$Q_{\mathbf{n}}(x)$ is nowadays typically referred to as a type II multiple orthogonal polynomial with respect to the system of functionals $\mathbf{L} = (L_1, \dots, L_r)$. The uniqueness of $Q_{\mathbf{n}}$ up to a constant multiple of a degree $|\mathbf{n}|$ polynomial is equivalent to $\det(H_{\mathbf{n}}[\mathbf{L}]) \neq 0$, where $H_{\mathbf{n}}[\mathbf{L}]$ is the generalized Hankel matrix

$$H_{\mathbf{n}}[\mathbf{L}] = \begin{pmatrix} c_{1,0} & c_{1,1} & \cdots & c_{1,|\mathbf{n}|-1} \\ c_{1,1} & c_{1,2} & \cdots & c_{1,|\mathbf{n}|} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,n_1-1} & c_{1,n_1} & \cdots & c_{1,|\mathbf{n}|+n_1-2} \\ \hline & \vdots & & \\ c_{r,0} & c_{r,1} & \cdots & c_{r,|\mathbf{n}|-1} \\ c_{r,1} & c_{r,2} & \cdots & c_{r,|\mathbf{n}|} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r,n_r-1} & c_{r,n_r} & \cdots & c_{r,|\mathbf{n}|+n_r-2} \end{pmatrix}. \quad (4.4)$$

Unlike the previous chapter, we may have $\det(H_{\mathbf{n}}[\mathbf{L}]) < 0$ for some \mathbf{n} when each L_j is positive-definite, or even $\det(H_{\mathbf{n}}[\mathbf{L}]) = 0$.

As the terminology suggests, there are also type I multiple orthogonal polynomials. They are vectors of polynomials $\mathbf{A}_{\mathbf{n}} = (A_{\mathbf{n},1}, \dots, A_{\mathbf{n},r}) \neq \mathbf{0}$, aside from the exception $\mathbf{A}_{\mathbf{0}} = \mathbf{0}$, with the degree restriction $\deg A_{\mathbf{n},j} \leq n_j - 1$, $j = 1, \dots, r$, and satisfying the orthogonality relations

$$\sum_{j=1}^r L_j[A_{\mathbf{n},j}(x)x^p] = 0, \quad p = 0, \dots, |\mathbf{n}| - 2. \quad (4.5)$$

The type I polynomials are characterized as the solutions to the type I Hermite–Padé approximation problem

$$\sum_{j=1}^r A_{\mathbf{n},j}(z)f_j(z) - B_{\mathbf{n}}(z) = \mathcal{O}(z^{-|\mathbf{n}|}), \quad z \rightarrow \infty. \quad (4.6)$$

Type I polynomials were given a systematic treatment by Mahler [32], who used the type I Hermite–Padé problem to study irrationality and transcendence.

The index \mathbf{n} is said to be normal with respect to \mathbf{L} when $\det(H_{\mathbf{n}}[\mathbf{L}]) \neq 0$. The system \mathbf{L} is said to be perfect when every multi-index is normal (also introduced by Mahler). Normality of \mathbf{n} is equivalent to the vector $\mathbf{A}_{\mathbf{n}}$ being unique up to multiplication by a non-zero constant and

$$\sum_{j=1}^r L_j[A_{\mathbf{n},j}(x)x^{|\mathbf{n}|-1}] \neq 0. \quad (4.7)$$

Indeed, solving for the coefficients of (4.5)+(4.7) results in a linear system with coefficient matrix $H_{\mathbf{n}}[\mathbf{L}]^T$. The duality between type I and type II polynomials will be on display further throughout the thesis. When \mathbf{n} is normal we always assume $Q_{\mathbf{n}}$ is normalized to be monic, and $\mathbf{A}_{\mathbf{n}}$ is normalized by

$$\sum_{j=1}^r L_j[A_{\mathbf{n},j}(x)x^{|\mathbf{n}|-1}] = 1. \quad (4.8)$$

Theorem 4.1.1 *The statements below are equivalent to perfectness.*

- (4.7) holds for every type I polynomial (except $\mathbf{A}_{\mathbf{0}}$).
- $\deg A_{\mathbf{n},j} = n_j - 1$ for every non-zero type I polynomial ($j = 1, \dots, r$).
- $\deg Q_{\mathbf{n}} = |\mathbf{n}|$ for every type II polynomial.
- $L_j[Q_{\mathbf{n}}(x)x^{n_j}] \neq 0$ for every type II polynomial ($j = 1, \dots, r$).
- $\det H_{\mathbf{n}}[\mathbf{L}] \neq 0$ for every \mathbf{n} .

4.2 Angelesco and AT systems

There are two main examples of large classes of perfect systems. The first one is Angelesco systems, which was discussed in the first chapter for orthogonality with respect to weight functions. More generally, they are systems of

infinitely supported positive measures $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$, such that the convex hulls Δ_j of $\text{supp}(\mu_j)$ have pairwise disjoint interiors.

The second one is AT systems. Here we work with non-negative weight functions w_1, \dots, w_r on an interval Δ , such that $d\mu_j = w_j d\mu$ for some measure μ with $\text{supp}(\mu) \subseteq \Delta$. $\boldsymbol{\mu}$ is an AT system on Δ if every non-trivial linear combination $\sum_{j=1}^r Q_j w_j$ has at most $|\mathbf{n}| - 1$ zeros in Δ whenever $\deg Q_j \leq n_j - 1$.

An equivalent condition is that

$$W_{\mathbf{n}}(x_1, \dots, x_{|\mathbf{n}|}) = \det \begin{pmatrix} w_1(x_1) & w_1(x_2) & \cdots & w_1(x_{|\mathbf{n}|}) \\ w_1(x_1)x_1 & w_1(x_2)x_2 & \cdots & w_1(x_{|\mathbf{n}|})x_{|\mathbf{n}|} \\ \vdots & \vdots & \ddots & \vdots \\ w_1(x_1)x_1^{n_1-1} & w_1(x_2)x_2^{n_1-1} & \cdots & w_1(x_{|\mathbf{n}|})x_{|\mathbf{n}|}^{n_1-1} \\ \hline & & & \vdots \\ w_r(x_1) & w_r(x_2) & \cdots & w_r(x_{|\mathbf{n}|}) \\ w_r(x_1)x_1 & w_r(x_2)x_2 & \cdots & w_r(x_{|\mathbf{n}|})x_{|\mathbf{n}|} \\ \vdots & \vdots & \ddots & \vdots \\ w_r(x_1)x_1^{n_r-1} & w_r(x_2)x_2^{n_r-1} & \cdots & w_r(x_{|\mathbf{n}|})x_{|\mathbf{n}|}^{n_r-1} \end{pmatrix} \neq 0 \quad (4.9)$$

for every set of distinct points $x_1, \dots, x_{|\mathbf{n}|}$ in Δ and every \mathbf{n} . The functions $W_{\mathbf{n}}$ naturally appear through the formula (4.10) below.

Theorem 4.2.1 *Angelesco and AT systems are perfect.*

The argument from section 1.1 shows that $Q_{\mathbf{n}}$ has at least n_j zeros of odd multiplicity in $\text{int}(\Delta_j)$, $j = 1, \dots, r$. The type I polynomial $A_{\mathbf{n},j}$ also has $n_j - 1$ zeros in Δ_j for Angelesco systems, and both the type I polynomials and the type II polynomials satisfy interlacing properties (see e.g. [3]). Similar zero counting proofs find $|\mathbf{n}|$ zeros of $Q_{\mathbf{n}}$ in $\text{int}(\Delta)$ for AT systems, and zero interlacing also holds [24]. Zeros may be complex for the type I polynomials, however.

An alternative proof of perfectness that is important to the thesis is through a direct computation of the determinant of the moment matrix $H_{\mathbf{n}}(\mathbf{L})$. It can be shown that

$$\det(H_{\mathbf{n}}[\boldsymbol{\mu}]) = \frac{1}{|\mathbf{n}|!} \int_{\Gamma^{|\mathbf{n}|}} W_{\mathbf{n}}(x_1, \dots, x_{|\mathbf{n}|}) \prod_{1 \leq j < k \leq |\mathbf{n}|} (x_k - x_j) \prod_{j=1}^{|\mathbf{n}|} d\mu(x_j), \quad (4.10)$$

where $W_{\mathbf{n}}$ is given by (4.9). For AT systems, this integrand has constant sign, which proves perfectness. These computations can be found in [28]. Similar arguments can also be made for Angelesco systems.

By the constant sign condition we can interpret a normalization of the integrand (4.10) as a probability distribution. In some cases, this corresponds to the eigenvalue distribution of certain random matrices, and the position of non-intersecting random paths [29].

An example of a large class of AT systems is provided by Nikishin systems. If $\text{supp}(\sigma_1)$ and $\text{supp}(\sigma_2)$ have disjoint convex hulls, then the system (μ_1, μ_2) with $\mu_1 = \sigma_1$ and $\mu_2 = \langle \sigma_1, \sigma_2 \rangle$, where

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t} d\sigma_1(x), \quad (4.11)$$

forms a Nikishin system. The perfectness of these systems is a result due to Driver and Stahl [16], improving results of Nikishin [35].

More generally, consider measures $\sigma_1, \dots, \sigma_r$ such that $\text{supp}(\sigma_{j+1})$ and $\text{supp}(\sigma_j)$ have disjoint convex hulls, $j = 1, \dots, r-1$. We inductively define $\langle \sigma_1, \dots, \sigma_s \rangle$ as $\langle \sigma_1, \langle \sigma_2, \dots, \sigma_s \rangle \rangle$ for $s \leq r$. The system (μ_1, \dots, μ_r) defined by

$$\sigma_1 = \mu_1, \quad \mu_2 = \langle \sigma_1, \sigma_2 \rangle, \quad \dots \quad \mu_r = \langle \sigma_1, \dots, \sigma_r \rangle, \quad (4.12)$$

is then a Nikishin system. Perfectness for general Nikishin systems was established by Fidalgo Prieto and Lopez Lagomasino [19].

4.3 The nearest-neighbour recurrence relations

The three-term recurrence relation generalizes to multiple orthogonal polynomials through the following result, which comes from Walter Van Assche's paper [45] (see also [26]).

Theorem 4.3.1 *Let L be a perfect system. Then we have the nearest neighbour recurrence relations*

$$xQ_n(x) = Q_{n+e_k}(x) + b_{n,k}Q_n(x) + \sum_{j=1}^r a_{n,j}Q_{n-e_j}(x), \quad (4.13)$$

$$xA_n(x) = A_{n-e_k}(x) + b_{n-e_k,k}A_n(x) + \sum_{j=1}^r a_{n,j}A_{n+e_j}(x). \quad (4.14)$$

In the case $n_\ell = 0$ we put $a_{n,\ell} = 0$ and $Q_{n-e_\ell} = 0$. Then (4.13) still holds for all k , but not (4.14) in the case $n_k = 0$. Some immediate consequences are the useful relations

$$Q_{n+e_k} - Q_{n+e_\ell} = (b_{n,\ell} - b_{n,k})Q_n, \quad (4.15)$$

$$A_{n-e_k} - A_{n-e_\ell} = (b_{n-e_\ell,\ell} - b_{n-e_k,k})A_n. \quad (4.16)$$

For an analogue of the Jacobi operator, see e.g. [13], which also relates the eigenvalues to the computation of multiple quadrature rules. It uses the type II polynomials along a path containing the diagonal indices (the step-line), which generates a banded Hessenberg matrix. There is also the more recent construction of Jacobi operators on trees [3], which directly uses the nearest-neighbour coefficients.

Theorem 3.2.1 does not quite generalize to MOPRL. Rather, the structure of the recurrence coefficients is characterized by the partial difference equations

$$b_{\mathbf{n}+\mathbf{e}_k,\ell} - b_{\mathbf{n}+\mathbf{e}_\ell,k} = b_{\mathbf{n},\ell} - b_{\mathbf{n},k}, \quad (4.17)$$

$$b_{\mathbf{n},k}b_{\mathbf{n}+\mathbf{e}_k,\ell} - b_{\mathbf{n},\ell}b_{\mathbf{n}+\mathbf{e}_\ell,k} = \sum_{j=1}^r a_{\mathbf{n}+\mathbf{e}_k,j} - \sum_{j=1}^r a_{\mathbf{n}+\mathbf{e}_\ell,j}, \quad (4.18)$$

$$a_{\mathbf{n}+\mathbf{e}_k,\ell}(b_{\mathbf{n}-\mathbf{e}_\ell,\ell} - b_{\mathbf{n}-\mathbf{e}_\ell,k}) = a_{\mathbf{n},\ell}(b_{\mathbf{n},\ell} - b_{\mathbf{n},k}). \quad (4.19)$$

One consequence of these relations is the Christoffel–Darboux formula of Daems–Kuijlaars [15]. From Van Assche’s paper [45], it takes the form

$$(x-y) \sum_{k=0}^{N-1} Q_{\mathbf{n}_k}(x) \mathbf{A}_{\mathbf{n}_{k+1}}(y) = Q_{\mathbf{n}_N}(x) \mathbf{A}_{\mathbf{n}_N}(y) - \sum_{j=1}^r a_{\mathbf{n}_N,j} Q_{\mathbf{n}_N-\mathbf{e}_j}(x) \mathbf{A}_{\mathbf{n}_N+\mathbf{e}_j}(y), \quad (4.20)$$

for any increasing path of indices $(\mathbf{n}_k)_{k=0}^N$ starting at $\mathbf{0}$.

If all the coefficients $a_{\mathbf{n},j}$ are positive then the polynomials $Q_{\mathbf{n}+\mathbf{e}_k}$ and $Q_{\mathbf{n}}$ have real and interlacing zeros for every \mathbf{n} [24]. It also implies the interlacing of $Q_{\mathbf{n}+\mathbf{e}_k}$ and $Q_{\mathbf{n}+\mathbf{e}_\ell}$, in the sense that between consecutive zeros of one of the polynomials there is exactly one zero of the other polynomial. These interlacing properties may also hold without the positivity of every $a_{\mathbf{n},j}$, with several examples listed in [24]. However, the authors note that $\sum_{j=1}^r a_{\mathbf{n},j} > 0$ for each of these examples. This appears to be a necessary condition for interlacing, as I have been able to scribble down a somewhat convoluted proof of this.

From the relations (4.17)-(4.19) it is usually possible to compute all coefficients $a_{\mathbf{n},j}, b_{\mathbf{n},j}$ from the coefficients $a_{\mathbf{n}}(L_j), b_{\mathbf{n}}(L_j)$ of each individual functional [20]. The computation works assuming the condition

$$b_{\mathbf{n},k} \neq b_{\mathbf{n},\ell}, \quad k \neq \ell, \quad (4.21)$$

or equivalently, by (4.19), $a_{\mathbf{n},j} \neq 0$ and $b_{\mathbf{0},k} \neq b_{\mathbf{0},\ell}$. This condition always holds in perfect systems, and conversely it always generates a perfect system from the relations (4.17)-(4.19) [4]. $a_{\mathbf{n},j} > 0$ does not necessarily hold when every L_j is positive-definite, although it does in some cases, such as for Angelesco systems [3] and multiple Hermite polynomials [24].

5. Orthogonality on the unit circle

This section contains the part of OPUC theory that is relevant to the thesis. As asymptotics are not studied in the thesis, we leave out several interesting results such as Szegő's Theorem, Baxter's Theorem, and Rakhmanov's Theorem. For an exhaustive treatment of OPUC, see [36, 37]. The content of the paper [27] is particularly important here.

5.1 Two-point Padé approximation and orthogonality

For a measure μ supported on the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$, the orthogonal polynomial Φ_n is defined by $\deg \Phi_n = n$ and

$$\int \Phi_n(z) z^{-p} d\mu(z) = 0, \quad p = 0, \dots, n-1. \quad (5.1)$$

We will always assume that Φ_n is monic and μ has infinite support, in which case it is uniquely determined for every $n \in \mathbb{N}$. The zeros of Φ_n always lie inside the unit disc $\mathbb{D} = \{z : |z| < 1\}$, with several basic proofs given in [36].

The connection between Φ_n and rational approximation is through two-point Padé approximation at the points 0 and ∞ of the Carathéodory function

$$F(z) = \int \frac{w+z}{w-z} d\mu(w), \quad z \in \mathbb{C} \setminus \partial\mathbb{D}. \quad (5.2)$$

The approximation problem is given by

$$\Phi_n(z)F(z) - \Psi_n(z) = \mathcal{O}(z^n), \quad z \rightarrow 0, \quad (5.3)$$

$$\Phi_n(z)F(z) - \Psi_n(z) = \mathcal{O}(z^{-1}), \quad z \rightarrow \infty, \quad (5.4)$$

for some polynomial Ψ_n determined by Φ_n . Up to a constant, the unique solution Φ_n with $\deg \Phi_n \leq n$ is given by (5.1).

The paper [34] studied the type II multiple orthogonal polynomials $\Phi_{\mathbf{n}}$ satisfying $\deg \Phi_{\mathbf{n}} \leq |\mathbf{n}|$ and

$$\int \Phi_{\mathbf{n}}(z) z^{-p} d\mu_j(z) = 0, \quad p = 0, \dots, n_j - 1, \quad (5.5)$$

which they show give the solutions to the two-point Hermite–Padé approximation problem

$$\Phi_{\mathbf{n}}(z)F_j(z) - \Psi_{\mathbf{n},j}(z) = \mathcal{O}(z^{n_j}), \quad z \rightarrow 0, \quad (5.6)$$

$$\Phi_{\mathbf{n}}(z)F_j(z) - \Psi_{\mathbf{n},j}(z) = \mathcal{O}(z^{-1}), \quad z \rightarrow \infty, \quad (5.7)$$

where each F_j is a Carathéodory function with respect to a measure μ_j . Similarly, type I orthogonality relations

$$\sum_{j=1}^r \int \Lambda_{n,j}(z) z^{-p} d\mu_j(z) = 0, \quad p = 0, \dots, |\mathbf{n}| - 2, \quad (5.8)$$

were studied, for vectors $\Lambda_{\mathbf{n}} = (\Lambda_{n,1}, \dots, \Lambda_{n,r})$ with $\deg \Lambda_{n,j} \leq n_j - 1$.

Here $\mathbf{n} \in \mathbb{N}^r$ would be a normal index when the orthogonality relations force $\Phi_{\mathbf{n}}$ to have degree $|\mathbf{n}|$, and (5.8) cannot extend to $p = |\mathbf{n}| - 1$. One example of a perfect system was given in [14]. No large classes of perfect systems similar to the Angelesco and AT systems of MOPRL were identified, nor any general results about zero behaviour.

The Carathéodory function (5.2) has a power series expansion

$$F^{(0)}(z) = c_0 + 2 \sum_{k=1}^{\infty} c_k z^k, \quad z \rightarrow 0, \quad (5.9)$$

$$F^{(\infty)}(z) = -c_0 - 2 \sum_{k=1}^{\infty} c_{-k} z^{-k}, \quad z \rightarrow \infty, \quad (5.10)$$

where c_k are the moments

$$c_k = \int z^{-k} d\mu(z), \quad k \in \mathbb{Z}. \quad (5.11)$$

More generally, we can consider any pair $(F^{(0)}, F^{(\infty)})$ of formal power series on the form (5.9)-(5.10), and define a linear functional L on the space of Laurent polynomials by

$$c_k = L[z^{-k}], \quad k \in \mathbb{Z}. \quad (5.12)$$

The Padé approximation problem (5.3)-(5.4) with the pair $(F^{(0)}, F^{(\infty)})$ replacing (F, F) is then solved by Φ_n satisfying

$$L[\Phi_n(z) z^{-p}] = 0, \quad p = 0, \dots, n-1. \quad (5.13)$$

As with OPRL with respect to functionals, there may not exist a unique monic Φ_n with degree n . The uniqueness holds exactly when the Toeplitz matrix

$$T_n[L] = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{pmatrix} \quad (5.14)$$

is nonsingular, and we say that L is quasi-definite if this holds for every n . We also have the following analogue to Theorem 3.1.1.

Theorem 5.1.1 (Trigonometric moment problem) *If the matrices (5.14) are positive definite for a sequence $\{c_k\}_{k=-\infty}^{\infty}$ with $c_k = \bar{c}_{-k}$, then (5.11) holds for some measure μ supported on an infinite subset of $\partial\mathbb{D}$.*

In [40], Szegő provided a connection to OPRL through the mapping $\mu = \text{Sz}(\gamma)$, where γ is supported on $[-2, 2]$, μ is supported on $\partial\mathbb{D}$ and symmetric about the real line, and

$$\int P(2 \cos \theta) d\mu(e^{i\theta}) = \int P(x) d\gamma(x). \quad (5.15)$$

More generally, Paper VI studies functionals $L = \text{Sz}(M)$ defined by

$$L[z^k] = L[z^{-k}], \quad L[(z+z^{-1})^k] = M[x^k], \quad k \in \mathbb{N}. \quad (5.16)$$

If L is quasi-definite then so is M , but I am not aware of a converse result. Assuming L is quasi-definite, we then have the Szegő mapping relations

$$Q_n(z+z^{-1}) = \frac{1}{1+\Phi_{2n}(0)} (z^{-n}\Phi_{2n}(z) + z^n\Phi_{2n}(1/z)) \quad (5.17)$$

$$= z^{-n+1}\Phi_{2n-1}(z) + z^{n-1}\Phi_{2n-1}(1/z). \quad (5.18)$$

5.2 Szegő and Baxter recurrence relations

The monic orthogonal polynomials with respect to μ supported on the unit circle satisfy the Szegő recurrence relations

$$\Phi_n(z) = z\Phi_{n-1}(z) + \alpha_n\Phi_{n-1}^*(z), \quad (5.19)$$

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + \bar{\alpha}_n z\Phi_{n-1}(z), \quad (5.20)$$

for $n > 0$ where $\alpha_n = \Phi_n(0)$ and

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}. \quad (5.21)$$

Since the zeros of Φ_n are in \mathbb{D} , we have $|\alpha_n| < 1$, except for $\alpha_0 = 1$, and the zeros of Φ_n^* lie in $\mathbb{C} \setminus \bar{\mathbb{D}}$.

The polynomial Φ_n^* satisfies the related two-point Padé approximation problem

$$\Phi_n^*(z)F(z) - \Psi_n^*(z) = \mathcal{O}(z^{n+1}), \quad z \rightarrow 0, \quad (5.22)$$

$$\Phi_n^*(z)F(z) - \Psi_n^*(z) = \mathcal{O}(1), \quad z \rightarrow \infty, \quad (5.23)$$

which is easily seen from the property $\overline{F(1/\bar{z})} = -F(z)$ of the Carathéodory function. Φ_n^* is also characterized by $\deg \Phi_n^* \leq n$, $\Phi_n^*(0) = 1$, and

$$\int \Phi_n^*(z) z^{-p} d\mu(z) = 0, \quad p = 1, \dots, n. \quad (5.24)$$

We also have the inverse Szegő recurrence relations

$$\Phi_n(z) = \alpha_n\Phi_n^*(z) + \rho_n z\Phi_{n-1}(z), \quad (5.25)$$

$$\Phi_n^*(z) = \bar{\alpha}_n\Phi_n(z) + \rho_n\Phi_{n-1}^*(z), \quad (5.26)$$

where $\rho_n = 1 - |\alpha_n|^2$. The coefficients ρ_n behave similarly to a_n from MOPRL. For example, similarly to (3.11) we have

$$\rho_n = \frac{\|\Phi_n\|^2}{\|\Phi_{n-1}\|^2} = \frac{\det(T_{n+1}[\mu]) \det(T_{n-1}[\mu])}{\det(T_n[\mu])^2}. \quad (5.27)$$

The analogue of Favard's result for OPUC is the following result.

Theorem 5.2.1 (Verblunsky's Theorem) *If $|\alpha_n| < 1$ for every $n > 0$ then Φ_n defined by (5.19)-(5.20) are the monic orthogonal polynomials with respect to a measure μ supported on $\partial\mathbb{D}$.*

Instead of Φ_n and Φ_n^* , it may be preferable to work with the Laurent polynomials X_n defined by $X_{2n}(z) = z^{-n}\Phi_{2n}^*(z)$ and $X_{2n+1}(z) = z^{-n}\Phi_{2n+1}(z)$. By (5.1) and (5.24) this forms an orthogonal basis of $L^2(\mu)$. This basis gives an analogue of Jacobi matrices for OPUC, with multiplication by z being represented by the CMV matrices of [10].

If we work inside the more general framework of orthogonality with respect to quasi-definite functionals, then the generalized version of (5.22)-(5.23) is still equivalent to

$$L[\Phi_n^*(z)z^{-p}] = 0, \quad p = 1, \dots, n, \quad (5.28)$$

for a polynomial $\Phi_n^* \not\equiv 0$ of degree at most n . We can additionally impose $\Phi_n^*(0) = 1$, and then this uniquely determines Φ_n^* . This polynomial is not necessarily given by (5.21). The Szegő recurrence relations generalize to the form

$$\Phi_n(z) = z\Phi_{n-1}(z) + \alpha_n\Phi_{n-1}^*(z), \quad (5.29)$$

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + \beta_n z\Phi_{n-1}(z), \quad (5.30)$$

$$\Phi_n(z) = \alpha_n\Phi_n^*(z) + \rho_n z\Phi_{n-1}(z), \quad (5.31)$$

$$\Phi_n^*(z) = \beta_n\Phi_n(z) + \rho_n\Phi_{n-1}^*(z), \quad (5.32)$$

where $\rho_n = 1 - \alpha_n\beta_n \neq 0$, but we may have $\beta_n \neq \bar{\alpha}_n$. The coefficient β_n is the z^n -coefficient of $\Phi_n^*(z)$. These recurrence relations appeared initially in Baxter's work [6].

If $L = \text{Sz}(M)$ then we have $\alpha_n = \beta_n \in \mathbb{R}$. The recurrence coefficients $(a_n)_{n=1}^\infty$ and $(b_n)_{n=0}^\infty$ of M can be computed by

$$a_n = (1 + \alpha_{2n-2})(1 - \alpha_{2n-1}^2)(1 - \alpha_{2n}), \quad (5.33)$$

$$b_n = \alpha_{2n-1}(1 - \alpha_{2n}) - \alpha_{2n+1}(1 + \alpha_{2n}). \quad (5.34)$$

For measures, these relations appeared in the work of Geronimus [22].

5.3 Paraorthogonal polynomials and quadrature

We now shift to the polynomials $\Phi_n^{(\tau)}$, for $\tau \in \partial\mathbb{D}$, defined by $\deg \Phi_n^{(\tau)} = n$ and the orthogonality relations

$$\int \Phi_n^{(\tau)}(z) z^{-p} d\mu(z) = 0, \quad p = 1, \dots, n-1, \quad (5.35)$$

along with the τ -invariance

$$\overline{z^n \Phi_n^{(\tau)}(1/\bar{z})} = \bar{\tau} \Phi_n^{(\tau)}(z). \quad (5.36)$$

These polynomials will be referred to as paraorthogonal polynomials.

Any $\Phi_n^{(\tau)}$ has the representation

$$\Phi_n^{(\tau)}(z) = a\Phi_n(z) + \bar{a}\tau\Phi_n^*(z), \quad a \in \mathbb{C} \setminus \{0\}, \quad (5.37)$$

or alternatively, for $n > 0$,

$$\Phi_n^{(\tau)}(z) = az\Phi_{n-1}(z) + \bar{a}\tau\Phi_{n-1}^*(z), \quad a \in \mathbb{C} \setminus \{0\}. \quad (5.38)$$

When $a = 1$ in (5.38) we get the unique monic τ -invariant paraorthogonal polynomial.

Theory of these polynomials is in many ways the analogue to OPRL. In particular, we have the following result.

Theorem 5.3.1 $\Phi_n^{(\tau)}$ has n simple zeros in $\partial\mathbb{D}$.

If $\tau_1 \neq \tau_2$ then the zeros of the monic versions of $\Phi_n^{(\tau_1)}$ and $\Phi_n^{(\tau_2)}$ have interlacing angles. If μ is supported on an arc $\Gamma \subseteq \partial\mathbb{D}$, then at least $n-1$ zeros lie in the interior of Γ (i.e., without its endpoints) for every paraorthogonal polynomial.

Write $z_{n,1}, \dots, z_{n,n}$ for the zeros of some $\Phi_n^{(\tau)}$. By interpolating with these zeros we get a quadrature rule

$$\int f(z) d\mu(z) \approx \sum_{j=1}^n \lambda_{n,j} f(z_{n,j}) \quad (5.39)$$

that is exact on $\text{span}\{z^{-n+1}, \dots, z^{-1}\}$. This is the best possible, in the sense that if we add z^n or z^{-n} to the span then exactness does not hold for any set of nodes.

We now shift perspective and define $S_n(\theta) = e^{-in\theta/2} \Phi^{(1)}(e^{i\theta})$, $\theta \in [0, 2\pi)$. $S_n(\theta)$ is then a trigonometric polynomial with

$$S_{2m}(\theta) \in \text{span}\{\cos p\theta, \sin p\theta\}_{p=0}^m, \quad (5.40)$$

$$S_{2m+1}(\theta) \in \text{span}\{\cos(p+1/2)\theta, \sin(p+1/2)\theta\}_{p=0}^m \quad (5.41)$$

In other words, for even n we get a trigonometric polynomial of integer degree, and for odd n we get a trigonometric polynomial of semi-integer degree. The orthogonality relations (5.35) imply that S_n is orthogonal to every trigonometric polynomial of lower degree with the same parity. By the invariance property (5.36) we get real coefficients in the bases (5.40)-(5.41).

The quadrature rule now shifts into one with equality for trigonometric polynomials of integer/semi-integer degree. In the case of semi-integer degree, this was extended to multiple quadrature in [33]. In the paper, the authors define a notion of trigonometric AT system, and give a zero counting proof analogue to that of MOPRL. There appears to be no connection between their construction and that of [34].

6. Summary of papers

This chapter discusses the main results of each paper from the thesis, following the context given by Chapter 3-5. Of course, a more detailed description can be found in the actual papers, along with more results left out of this chapter.

Paper I and Paper VI relate closely with the discussions in section 4.3 and section 5.1-5.2. Paper V uses an idea described in section 4.2 to prove perfectness in MOPUC, and the zero location results of Paper VII use the framework of section 5.3. Paper II-IV extend the results discussed in section 3.3.

I feel strongly that the results of Paper V-VII form the most interesting part of the thesis, particularly the parts related to the discussion in Chapter 1. I am however also quite fond of Paper I, which is short and elegant, and surprisingly similar to the results discussed in section 4.3.

6.1 Paper I

Paper I presents generalized Szegő recurrence relations for the type II and type I multiple orthogonal polynomials $\Phi_{\mathbf{n}}$ and $\Lambda_{\mathbf{n}} = (\Lambda_{\mathbf{n},1}, \dots, \Lambda_{\mathbf{n},r})$ of (5.5), (5.8). We complement these polynomials with $\Phi_{\mathbf{n}}^*$ and $\Lambda_{\mathbf{n}}^* = (\Lambda_{\mathbf{n},1}^*, \dots, \Lambda_{\mathbf{n},r}^*)$, defined by the degree restrictions

$$\deg \Phi_{\mathbf{n}}^* \leq |\mathbf{n}|, \quad (6.1)$$

$$\deg \Lambda_{\mathbf{n},j}^* \leq n_j - 1, \quad j = 1, \dots, r, \quad (6.2)$$

and the orthogonality relations

$$\int \Phi_{\mathbf{n}}^*(z) z^{-p} d\mu_j(z) = 0, \quad p = 1, \dots, n_j, \quad j = 1, \dots, r \quad (6.3)$$

$$\sum_{j=1}^r \int \Lambda_{\mathbf{n},j}^*(z) z^{-p} d\mu_j(z) = 0, \quad p = 1, \dots, |\mathbf{n}| - 1. \quad (6.4)$$

Note that there is no clear connection between $\Phi_{\mathbf{n}}^*(z)$ and $z^{|\mathbf{n}|} \overline{\Phi_{\mathbf{n}}(1/\bar{z})}$, and similarly for $\Lambda_{\mathbf{n}}^*$.

Normality, as it is defined in section 5.1, is equivalent to the existence and uniqueness of a $\Phi_{\mathbf{n}}^*$ with $\Phi_{\mathbf{n}}^*(0) = 1$, as well as to the existence and uniqueness of a $\Lambda_{\mathbf{n}}^*$ satisfying

$$\sum_{j=1}^r \int \Lambda_{\mathbf{n},j}^*(z) d\mu_j(z) = 1. \quad (6.5)$$

These normalizations on Φ_n^* and Λ_n^* will be used throughout the rest of the section.

Through these polynomials we were able to find the following nearest-neighbour type relations.

Theorem 6.1.1 *Assuming normality of the indices that appear in the respective relations below, we have the Szegő recurrence relations*

$$z\Lambda_n(z) = -\bar{\beta}_n\Lambda_n^*(z) + \sum_{j=1}^r \bar{\rho}_{n,j}\Lambda_{n+e_j}(z), \quad (6.6)$$

$$\Phi_n^*(z) = \Phi_{n-e_k}^*(z) + \beta_n z \Phi_{n-e_k}(z), \quad (6.7)$$

$$\Phi_n(z) = \alpha_n \Phi_n^*(z) + \sum_{j=1}^r \rho_{n,j} z \Phi_{n-e_j}(z), \quad (6.8)$$

$$\Lambda_n^*(z) = \Lambda_{n+e_k}^*(z) - \bar{\alpha}_n \Lambda_{n+e_k}(z). \quad (6.9)$$

When $r = 1$, (6.6)-(6.9) reduce to the usual Szegő recurrence relations. In general, we do not have $\beta_n = \bar{\alpha}_n$, as in the Baxter recurrence (5.29)-(5.32). Note that similar complications enter in MOPRL, where $a_{n,j} > 0$ does not necessarily hold when $r > 1$.

From these recurrence relations we were able to find the following partial difference equations for the recurrence coefficients (compare with (4.17)-(4.19)).

Theorem 6.1.2 *We have the compatibility relations*

$$\beta_{n+e_\ell+e_k}(\alpha_{n+e_k} - \alpha_{n+e_\ell}) = \alpha_n(\beta_{n+e_\ell} - \beta_{n+e_k}), \quad (6.10)$$

$$\alpha_{n+e_\ell}\beta_{n+e_\ell} - \alpha_{n+e_k}\beta_{n+e_k} = \sum_{j=1}^r \rho_{n+e_k,j} - \sum_{j=1}^r \rho_{n+e_\ell,j}, \quad (6.11)$$

$$\rho_{n+e_k,\ell}\alpha_n(\alpha_n - \alpha_{n+e_k-e_\ell}) = \rho_{n,\ell}\alpha_{n-e_\ell}(\alpha_{n+e_\ell} - \alpha_{n+e_k}). \quad (6.12)$$

The partial difference equations then imply the following Christoffel–Darboux formula (compare with (4.20)).

Theorem 6.1.3 *For any path of multi-indices $(\mathbf{n}_k)_{k=0}^N$ starting at $\mathbf{0}$, we have the CD-formula*

$$(1 - z\bar{\zeta}) \sum_{k=0}^{N-1} \Phi_{\mathbf{n}_k}(z) \overline{\Lambda_{\mathbf{n}_{k+1}}(\zeta)} = \Phi_{\mathbf{n}_N}^*(z) \overline{\Lambda_{\mathbf{n}_N}^*(\zeta)} - \sum_{j=1}^r \rho_{\mathbf{n}_N,j} z \Phi_{\mathbf{n}_N-e_j}(z) \overline{\Lambda_{\mathbf{n}_N+e_j}(\zeta)}. \quad (6.13)$$

Both Theorem 6.1.2 and Theorem 6.1.3 require normality of the indices appearing in the respective equations, along with their neighbours.

6.2 Paper II

Paper II studies MOPRL for measures with finite support. For a system $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$ containing a finite support measure μ_j , indices $\mathbf{n} = (n_1, \dots, n_r)$ cannot be normal if $n_j > \#\text{supp}(\mu_j)$. Hence we extend the notion of perfect system (only in this paper) to consider only the normality of all \mathbf{n} such that $n_j \leq \#\text{supp}(\mu_j)$, $j = 1, \dots, r$. This notion of perfectness is stronger than the normality established for the systems in [5]. More generally, the paper also works with functionals L_j with $\det(H_n[L_j]) \neq 0$ for $n \leq N_j$ and $\det(H_n[L_j]) = 0$ otherwise.

The main part of the paper presents new computation algorithms for the recurrence coefficients of Christoffel transforms, for OPRL as well as extensions to MOPRL. The following simple result captures the idea behind the algorithms.

Proposition 6.2.1 *For a perfect system $(\boldsymbol{\mu}, \lambda)$ with $\text{supp}(\lambda) = \{x_1, \dots, x_N\}$, we have*

$$Q_{n,N}(x) = \widehat{Q}_n(x) \prod_{j=1}^N (x - x_j), \quad (6.14)$$

where \widehat{Q}_n is the degree n orthogonal polynomial with respect to the Christoffel transform

$$\widehat{L}[P(x)] = \int P(x) \prod_{j=1}^N (x - x_j) d\boldsymbol{\mu}(x). \quad (6.15)$$

If $\boldsymbol{\mu}$ has infinite support then \widehat{L} is quasi-definite.

In the case $N = 1$, the compatibility relations (4.17)-(4.19) for the nearest-neighbour coefficients of $(\boldsymbol{\mu}, \delta_{x_1})$ reduce to (3.21)-(3.23). Hence the compatibility relations generalize Gautschi's algorithm to higher degree Christoffel transforms through Proposition 6.2.1 and the computation algorithm of [20]. Note that the result only depends on the support of λ , so choice of weights is entirely free (and positivity of the weights is not necessary).

More generally, we can consider $\widehat{\mathbf{L}} = (\widehat{L}_1, \dots, \widehat{L}_r)$ defined by

$$\widehat{L}_j[P(x)] = \int P(x) \prod_{k=1}^N (x - x_k) d\boldsymbol{\mu}_j(x), \quad j = 1, \dots, r. \quad (6.16)$$

The recurrence coefficients of $\widehat{\mathbf{L}}$ can similarly be computed from the recurrence coefficients of $\boldsymbol{\mu}$, by extending Proposition 6.2.1 to $(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r, \lambda)$. This is illustrated in the paper both for the nearest-neighbour coefficients and for the coefficients in the step-line recurrence. The paper also derives determinantal formulas for these systems, in the style of Theorem 3.3.1.

6.3 Paper III

In (6.16), each measure is modified by the same polynomial. Paper III considers the case when the different measures are modified by different polynomials. Although the computation algorithms of Paper II break down, the determinantal formulas can be modified. A motivating example was the following result.

Proposition 6.3.1 *Let (μ_1, μ_2) be a perfect system and define $(\widehat{\mu}_1, \widehat{\mu}_2)$ by*

$$d\widehat{\mu}_1(x) = (x - x_1)d\mu_1(x), \quad d\widehat{\mu}_2(x) = (x - x_2)d\mu_2(x), \quad x_1 \neq x_2. \quad (6.17)$$

Write $(A_{n,m}^{(1)}, A_{n,m}^{(2)})$ for the type I polynomials with respect to (μ_1, μ_2) and $Q_{n,m}$ for the type II polynomials with respect to (μ_1, μ_2) .

(i) For $(n, m) \neq (0, 0)$, define $(\widehat{A}_{n,m}^{(1)}, \widehat{A}_{n,m}^{(2)})$ by

$$\widehat{A}_{n,m}^{(j)}(x) = \frac{1}{x - x_j} \det \begin{pmatrix} A_{n,m}^{(j)}(x) & A_{n+1,m}^{(j)}(x) & A_{n,m+1}^{(j)}(x) \\ A_{n,m}^{(1)}(x_1) & A_{n+1,m}^{(1)}(x_1) & A_{n,m+1}^{(1)}(x_1) \\ A_{n,m}^{(2)}(x_2) & A_{n+1,m}^{(2)}(x_2) & A_{n,m+1}^{(2)}(x_2) \end{pmatrix}, \quad j = 1, 2. \quad (6.18)$$

If (n, m) is normal with respect to $(\widehat{\mu}_1, \widehat{\mu}_2)$, then $(\widehat{A}_{n,m}^{(1)}, \widehat{A}_{n,m}^{(2)})$ are the type I polynomials with respect to $(\widehat{\mu}_1, \widehat{\mu}_2)$. Conversely, if

$$\int \widehat{A}_{n,m}^{(1)}(x)x^{n+m-1}d\widehat{\mu}_1(x) + \int \widehat{A}_{n,m}^{(2)}(x)x^{n+m-1}d\widehat{\mu}_2(x) \neq 0, \quad (6.19)$$

then $(n, m) \neq (0, 0)$ is normal with respect to $(\widehat{\mu}_1, \widehat{\mu}_2)$.

(ii) Define $R_{n,m}(z)$ by

$$R_{n,m}(z) = \int \frac{Q_{n,m}(x)}{z - x} d\mu_1(x). \quad (6.20)$$

For $(n, m) \in \mathbb{N}_{>0}^2$, the determinant

$$\det \begin{pmatrix} Q_{n+1,m+1}(x) & Q_{n+1,m}(x) & Q_{n,m+1}(x) & Q_{n+1,m-1}(x) & Q_{n-1,m+1}(x) \\ Q_{n+1,m+1}(x_1) & Q_{n+1,m}(x_1) & Q_{n,m+1}(x_1) & Q_{n+1,m-1}(x_1) & Q_{n-1,m+1}(x_1) \\ Q_{n+1,m+1}(x_2) & Q_{n+1,m}(x_2) & Q_{n,m+1}(x_2) & Q_{n+1,m-1}(x_2) & Q_{n-1,m+1}(x_2) \\ R_{n+1,m+1}(x_1) & R_{n+1,m}(x_1) & R_{n,m+1}(x_1) & R_{n+1,m-1}(x_1) & R_{n-1,m+1}(x_1) \\ R_{n+1,m+1}(x_2) & R_{n+1,m}(x_2) & R_{n,m+1}(x_2) & R_{n+1,m-1}(x_2) & R_{n-1,m+1}(x_2) \end{pmatrix} \quad (6.21)$$

defines a polynomial $(x - x_1)(x - x_2)\widehat{Q}_{n,m}(x)$. Then $\widehat{Q}_{n,m}$ has degree $n + m$ if and only if (n, m) is normal with respect to $(\widehat{\mu}_1, \widehat{\mu}_2)$, in which case $\widehat{Q}_{n,m}$ is the type II multiple orthogonal polynomial with respect to $(\widehat{\mu}_1, \widehat{\mu}_2)$ for the index (n, m) .

For the above example, the type I formula (6.18) is much less complicated than the type II formula (6.21). The type II formula uses (6.20), similarly to

Uvarov's result (3.25), even though we only modify the measures by polynomials. Motivated by formulas such as (6.21) we study rational transformations

$$\tilde{L}_j[P(x)] = \int P(x) \frac{\Phi_j(x)}{\Psi_j(x)} d\mu_j(x), \quad j = 1, \dots, r. \quad (6.22)$$

Uvarov's formula only generalizes to the case $\Phi_1 = \dots = \Phi_r$ for type II polynomials, and to the case $\Psi_1 = \dots = \Psi_r$ for type I polynomials. In the paper, we prove general determinantal formulas with these restrictions, along with perfectness conditions for the transformed measures/functionals. The formula (6.21) can then be obtained from the general formula by writing

$$d\hat{\mu}_1(x) = \frac{(x-x_1)(x-x_2)}{(x-x_2)} d\mu_1(x), \quad d\hat{\mu}_2(x) = \frac{(x-x_1)(x-x_2)}{(x-x_1)} d\mu_2(x). \quad (6.23)$$

The paper considers the more general rational perturbations of functionals, i.e., functionals \tilde{L}_j defined by

$$\tilde{L}_j[\Psi_j(x)P(x)] = L_j[\Phi_j(x)P(x)], \quad j = 1, \dots, r. \quad (6.24)$$

For these transformations we find a more general version of (3.26), which appears to be new even for OPRL. From this we can get formulas for more general transforms, such as

$$\tilde{L}_j[P(x)] = \int P(x) \frac{\Phi_j(x)}{\Psi_j(x)} d\mu_j(x) + \sum_{k=1}^{M_j} P(x_{j,k}), \quad j = 1, \dots, r, \quad (6.25)$$

where $x_{j,1}, \dots, x_{j,M_j}$ are the zeros of Ψ_j . This extension was never considered by Uvarov, but part of it is mentioned in Zhedanov's work [47].

Since we allow $\Phi_j = \Psi_j$, we obtain formulas for the transformed polynomials when we add only discrete point masses to μ_j . Strangely, Uvarov gave formulas for such transforms in his paper [42], but these were very different from his other formula (3.25).

The most general formulas presented in the paper allow general rational perturbations in the sense of (6.24), and a very general set of paths of indices used in the determinants (note that the polynomials in the determinant (6.21) do not even use indices that pass through $(n, m)!$). These formulas are quite a horrible sight, so we try to make it more presentable by stating several special cases in the paper, such as the ones stated in this section.

6.4 Paper IV

Proofs of normality often go through a zero counting approach, see e.g. [5, 26, 35]. Paper IV takes the opposite approach by proving zero location and

interlacing using normality, using Christoffel transforms for very special cases of the formulas from Paper III. We also illustrate that normality can be established by direct computation of the generalized Hankel determinants, such as through (4.10). We apply the approach to show interlacing of zeros of type I polynomials in Nikishin systems (only for the case $r = 2$), a result that we could not locate in the literature.

For type II polynomials the approach is described by the following basic result.

Proposition 6.4.1 *Let $\boldsymbol{\mu}$ be a perfect system, and write $Q_{\mathbf{n}}$ for the type II polynomials with respect to $\boldsymbol{\mu}$.*

- $z_0 \in \mathbb{R}$ is not a zero of multiplicity $\geq k$ of $Q_{\mathbf{n}}$ if \mathbf{n} is normal with respect to $\widehat{\boldsymbol{\mu}}$ given by

$$d\widehat{\boldsymbol{\mu}}_j(x) = (x - z_0)^k d\boldsymbol{\mu}_j(x), \quad j = 1, \dots, r. \quad (6.26)$$

- $z_0 \notin \mathbb{R}$ is not a zero of $Q_{\mathbf{n}}$ if \mathbf{n} is normal with respect to $\widehat{\boldsymbol{\mu}}$ given by

$$d\widehat{\boldsymbol{\mu}}_j(x) = |x - z_0|^2 d\boldsymbol{\mu}_j(x), \quad j = 1, \dots, r. \quad (6.27)$$

- Assume $Q_{\mathbf{n}+\mathbf{e}_j}$ and $Q_{\mathbf{n}}$ only have real zeros. Then the zeros interlace if \mathbf{n} is normal with respect to the systems $\widehat{\boldsymbol{\mu}}$ of the form

$$d\widehat{\boldsymbol{\mu}}_j(x) = (x - z_0)^2 d\boldsymbol{\mu}_j(x), \quad j = 1, \dots, r, \quad z_0 \in \mathbb{R}. \quad (6.28)$$

The analogue result for type I polynomials is the following.

Proposition 6.4.2 *Let $\boldsymbol{\mu}$ be a perfect system, and write $A_{\mathbf{n},j}$ for the type I polynomials with respect to $\boldsymbol{\mu}$.*

- $z_0 \in \mathbb{R}$ is not a zero of multiplicity $\geq k$ of $A_{\mathbf{n},1}$ if $\mathbf{n} - \mathbf{e}_1$ is normal with respect to $\widehat{\boldsymbol{\mu}}$ given by

$$d\widehat{\boldsymbol{\mu}}_1(x) = (x - z_0)^k d\boldsymbol{\mu}_1(x), \quad d\widehat{\boldsymbol{\mu}}_j(x) = d\boldsymbol{\mu}_j(x), \quad j = 2, \dots, r. \quad (6.29)$$

- $z_0 \notin \mathbb{R}$ is not a zero of $A_{\mathbf{n},1}$ if $\mathbf{n} - \mathbf{e}_1$ is normal with respect to $\widehat{\boldsymbol{\mu}}$ given by

$$d\widehat{\boldsymbol{\mu}}_1(x) = |x - z_0|^2 d\boldsymbol{\mu}_1(x), \quad d\widehat{\boldsymbol{\mu}}_j(x) = d\boldsymbol{\mu}_j(x) \quad j = 2, \dots, r. \quad (6.30)$$

- Assume $A_{\mathbf{n}-\mathbf{e}_{j,1}}$ and $A_{\mathbf{n},1}$ have real zeros. Then the zeros interlace if $\mathbf{n} - 2\mathbf{e}_1$ is normal with respect to the systems $\widehat{\boldsymbol{\mu}}$ of the form

$$d\widehat{\boldsymbol{\mu}}_1(x) = (x - z_0)^2 d\boldsymbol{\mu}_1(x), \quad d\widehat{\boldsymbol{\mu}}_j(x) = d\boldsymbol{\mu}_j(x), \quad j = 2, \dots, r, \quad z_0 \in \mathbb{R}. \quad (6.31)$$

By permutations on the measures, similar results follow for $A_{\mathbf{n}}^{(j)}$, $j > 1$.

These propositions easily reproduce well-known zero location and interlacing results for Angelesco and AT systems. With a bit more work, we find the following result for Nikishin systems.

Theorem 6.4.3 Let (μ_1, μ_2) be a Nikishin system generated by (σ_1, σ_2) , with Γ_2 the convex hull of $\text{supp}(\sigma_2)$.

- If $n_1 + 1 \leq n_2$ then all zeros of $A_{\mathbf{n}}^{(1)}$ lie in $\text{int}(\Gamma_2)$, and the zeros of $A_{\mathbf{n}-\mathbf{e}_j}^{(1)}$ and $A_{\mathbf{n}}^{(1)}$ interlace, $j = 1, 2$.
- If $n_1 + 1 \geq n_2$ then all zeros of $A_{\mathbf{n}}^{(2)}$ lie in $\text{int}(\Gamma_2)$ and the zeros of $A_{\mathbf{n}-\mathbf{e}_j}^{(2)}$ and $A_{\mathbf{n}}^{(2)}$ interlace, $j = 1, 2$.

The interlacing results prove that zeros of $A_{n,n}^{(1)}$ and $A_{n,n+2}^{(2)}$ are also real, since $\mathbf{n} - \mathbf{e}_j$ may fall out of the regions $\{(m_1, m_2) : m_1 + 1 \leq m_2\}$ or $\{(m_1, m_2) : m_1 + 1 \geq m_2\}$. Simulations show that zeros may be complex for other indices.

6.5 Paper V

In Paper V we study multiple orthogonality relations for measures supported on the unit circle. Rather than working with the polynomials $\Phi_{\mathbf{n}}$ of (5.5), we work with $\phi_{\mathbf{n}}$ defined by

$$\phi_{\mathbf{n}}(z) \in \text{span}\{z^{-|\mathbf{n}|/2}, z^{-|\mathbf{n}|/2+1}, \dots, z^{|\mathbf{n}|/2}\}, \quad (6.32)$$

and the orthogonality relations

$$\int \phi_{\mathbf{n}}(z) z^{-p} d\mu_j(z) = 0, \quad p = -n_j/2, -n_j/2 + 1, \dots, n_j/2 - 1, \quad j = 1, \dots, r, \quad (6.33)$$

for a choice of half-power functions

$$z^{p/2} = |z|^{p/2} \exp(ip \arg_{[t_0, t_0+2\pi)}(z)/2), \quad p \in \mathbb{Z}. \quad (6.34)$$

The paper also studies type I orthogonality relations in similar style, and obtains natural two-point Hermite–Padé approximation problems for both the type I and the type II polynomials.

Here we say that \mathbf{n} is normal (or ϕ -normal, to separate it from normality in the sense of [34]) if there is a unique solution to (6.32)-(6.33) of the form

$$\phi_{\mathbf{n}}(z) = z^{|\mathbf{n}|/2} + \kappa_{|\mathbf{n}|/2-1} z^{|\mathbf{n}|/2-1} + \dots + \kappa_{-|\mathbf{n}|/2} z^{-|\mathbf{n}|/2}. \quad (6.35)$$

When solving $\kappa_{-|\mathbf{n}|/2}, \dots, \kappa_{|\mathbf{n}|/2-1}$ in (6.33)-(6.35) we get a system of equations, say with coefficient matrix $T_{\mathbf{n}}$. When $r = 1$ we have $z^{n/2} \phi_{\mathbf{n}}(z) = \Phi_{\mathbf{n}}(z)$, so $T_{\mathbf{n}}$ reduces to (5.14), but for $r > 1$ this connection breaks.

By direct computation of $\det(T_{\mathbf{n}})$ we find formulas such as

$$\det(T_{\mathbf{n}}) = \frac{(-1)^{k_{\mathbf{n}}}}{n_1! \dots n_r!} \int_{\Gamma_1^{n_1}} \dots \int_{\Gamma_r^{n_r}} V_{|\mathbf{n}|}(\mathbf{z}_1, \dots, \mathbf{z}_r) \prod_{j=1}^r V_{n_j}(\mathbf{z}_j) \prod_{j=1}^r d^{n_j} \mu_j(\mathbf{z}_j), \quad (6.36)$$

for some integer k_n , where $d\mu_j(\mathbf{z}_j) = d\mu_j(z_{j,1}) \dots d\mu_j(z_{j,n_j})$, $\text{supp}(\mu_j) \subseteq \Gamma_j$, and V_N are Vandermonde-type determinants evaluating to

$$V_N(e^{i\theta_1}, \dots, e^{i\theta_N}) = (2i)^{N(N-1)/2} \prod_{1 \leq j < k \leq N} \sin \frac{\theta_k - \theta_j}{2},$$

$$t_0 \leq \theta_1 \leq \dots \leq \theta_N \leq t_0 + 2\pi. \quad (6.37)$$

From this we can show that the integrand does not change sign when the arcs $\Gamma_1, \dots, \Gamma_r$ are pairwise disjoint and $e^{it_0} \notin \Gamma_j$, $j = 1, \dots, r$ (the case when the arcs intersect at a point requires a more careful argument). Hence by the constant sign condition we identified an analogue of the perfectness of Angelesco systems for measures supported on the unit circle. A similar computation to (4.10) also identified a notion of AT system on the unit circle. No similar computations suggest any positive results on normality in the sense of [34].

Theorem 6.5.1 *Every $\mathbf{n} \in \mathbb{N}^r$ is ϕ -normal in Angelesco and AT systems on the unit circle.*

6.6 Paper VI

In paper VI we investigate a general two-point Hermite–Padé approximation problem of the form

$$\Phi_{\mathbf{n};\mathbf{m}}(z)F_j(z) + \Psi_{\mathbf{n};\mathbf{m},j}(z) = \mathcal{O}(z^{n_j}), \quad z \rightarrow 0, \quad (6.38)$$

$$\Phi_{\mathbf{n};\mathbf{m}}(z)F_j(z) + \Psi_{\mathbf{n};\mathbf{m},j}(z) = \mathcal{O}(z^{-m_j-1}), \quad z \rightarrow \infty, \quad (6.39)$$

for indices $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$, where F_j is the Carathéodory function (5.2) with respect to μ_j , $j = 1, \dots, r$.

Proposition 6.6.1 (6.38)-(6.39) has a solution $\Phi_{\mathbf{n};\mathbf{m}} \not\equiv 0$ with

$$\Phi_{\mathbf{n};\mathbf{m}}(z) \in \text{span}\{z^{-|\mathbf{m}|}, z^{-|\mathbf{m}|+1}, \dots, z^{|\mathbf{n}|}\}, \quad (6.40)$$

and the orthogonality relations

$$\int \Phi_{\mathbf{n};\mathbf{m}}(z)z^{-p}d\mu_j(z) = 0, \quad p = -m_j, \dots, n_j - 1, \quad j = 1, \dots, r. \quad (6.41)$$

We then say that the pair $(\mathbf{n}; \mathbf{m})$ is normal if there is a unique solution to (6.40)-(6.41) of the form

$$\Phi_{\mathbf{n};\mathbf{m}}(z) = z^{|\mathbf{n}|} + \dots + \alpha_{\mathbf{n};\mathbf{m}}z^{-|\mathbf{m}|} \quad (6.42)$$

We also study the related Hermite–Padé problem

$$\Phi_{\mathbf{n};\mathbf{m}}^*(z)F_j(z) + \Psi_{\mathbf{n};\mathbf{m},j}^*(z) = \mathcal{O}(z^{n_j+1}), \quad z \rightarrow 0, \quad (6.43)$$

$$\Phi_{\mathbf{n};\mathbf{m}}^*(z)F_j(z) + \Psi_{\mathbf{n};\mathbf{m},j}^*(z) = \mathcal{O}(z^{-m_j}), \quad z \rightarrow \infty, \quad (6.44)$$

which is solved by

$$\Phi_{\mathbf{n};\mathbf{m}}^*(z) = \overline{\Phi_{\mathbf{m};\mathbf{n}}(1/\bar{z})}. \quad (6.45)$$

This Laurent polynomial satisfies the shifted orthogonality relations

$$\int \Phi_{\mathbf{n};\mathbf{m}}^*(z) z^{-p} d\mu_j(z) = 0, \quad p = -m_j + 1, \dots, n_j, \quad j = 1, \dots, r. \quad (6.46)$$

Note that we have

$$\Phi_{\mathbf{n};\mathbf{0}}(z) = \Phi_{\mathbf{n}}(z), \quad \Phi_{\mathbf{n};\mathbf{0}}^*(z) = \Phi_{\mathbf{n}}^*(z), \quad (6.47)$$

and also

$$\Phi_{\mathbf{n};\mathbf{n}}(z) = \phi_{2\mathbf{n}}(z), \quad \Phi_{\mathbf{n};\mathbf{n}}^*(z) = \overline{\phi_{2\mathbf{n}}(1/\bar{z})}. \quad (6.48)$$

At $(\mathbf{n};\mathbf{0})$ the rational approximation is heavily weighted towards the point ∞ , while at $(\mathbf{n};\mathbf{m})$ with $\mathbf{n} \approx \mathbf{m}$ it approximates more evenly between the two points. For the latter case, we found the following relations for the Szegő mapping (5.15).

Theorem 6.6.2 *Let $\mu_j = \text{Sz}(\gamma_j)$, $j = 1, \dots, r$. If $(\mathbf{n};\mathbf{n})$ and $(\mathbf{n} + \mathbf{e}_j; \mathbf{n})$ are normal with respect to (μ_1, \dots, μ_r) for every $\mathbf{n} \in \mathbb{N}^r$, then $(\gamma_1, \dots, \gamma_r)$ is a perfect system and we have the Szegő mapping relations*

$$Q_{\mathbf{n}}(z + z^{-1}) = \frac{1}{1 + \alpha_{\mathbf{n};\mathbf{n}}} \left(\Phi_{\mathbf{n};\mathbf{n}}(z) + \Phi_{\mathbf{n};\mathbf{n}}(1/z) \right) \quad (6.49)$$

$$= \Phi_{\mathbf{n};\mathbf{n} - \mathbf{e}_j}(z) + \Phi_{\mathbf{n};\mathbf{n} - \mathbf{e}_j}(1/z). \quad (6.50)$$

Our Szegő recurrence relations of Paper I extend to any fixed \mathbf{m} , and we apply these relations to prove generalized Geronimus relations (5.33)-(5.34) for the recurrence coefficients. It should be noted however that normality in Angelesco and AT systems only follows from paper V for indices $(\mathbf{n};\mathbf{n})$, and similar arguments do not appear to show normality of indices $(\mathbf{n} + \mathbf{e}_j; \mathbf{n})$.

6.7 Paper VII

Paper VII settled two important questions that were left unanswered in Paper V-VI, with the following results.

Theorem 6.7.1 *For Angelesco and AT systems, the polynomial $z^{|\mathbf{n}|/2} \phi_{\mathbf{n}}(z)$ has $|\mathbf{n}|$ zeros in \mathbb{D} (with each zero counted as many times as its multiplicity).*

Theorem 6.7.2 *For Angelesco and AT systems, indices of the form $(\mathbf{n}; \mathbf{n} + \mathbf{e}_j)$ and $(\mathbf{n} + \mathbf{e}_j; \mathbf{n})$ are normal for every $\mathbf{n} \in \mathbb{N}^r$.*

The proofs go through some multiple paraorthogonal polynomials $X_{\mathbf{n}}^{(\tau)} \neq 0$, $\tau \in \partial\mathbb{D}$, that satisfy the τ -invariance

$$\overline{X_{\mathbf{n}}^{(\tau)}(1/\bar{z})} = \bar{\tau}_{\mathbf{n}} X_{\mathbf{n}}^{(\tau)}(z), \quad (6.51)$$

and the degree restriction

$$X_{\mathbf{n}}^{(\tau)} \in \text{span}\{z^{-(|\mathbf{n}|+1)/2}, \dots, z^{(|\mathbf{n}|+1)/2}\}, \quad (6.52)$$

together with the orthogonality relations

$$\int X_{\mathbf{n}}^{(\tau)}(z) z^{-p} d\mu_j(z) = 0, \quad p = -(n_j - 1)/2, \dots, (n_j - 1)/2. \quad (6.53)$$

ϕ -normality of \mathbf{n} is equivalent to existence and uniqueness of $X_{\mathbf{n}}^{(\tau)}$ satisfying (6.51)-(6.53) with $z^{(|\mathbf{n}|+1)/2}$ -coefficient equal to 1.

We can obtain paraorthogonal polynomials by

$$X_{\mathbf{n}}^{(\tau)}(z) = az^{1/2}\phi_{\mathbf{n}}(z) + \bar{a}\tau z^{-1/2}\overline{\phi_{\mathbf{n}}(1/\bar{z})}, \quad a \in \mathbb{C} \setminus \{0\}, \quad (6.54)$$

as well as

$$X_{2\mathbf{n}}^{(\tau)}(z) = az^{1/2}\Phi_{\mathbf{n}, \mathbf{n}+\mathbf{e}_j}(z) + \bar{a}\tau z^{-1/2}\overline{\Phi_{\mathbf{n}, \mathbf{n}+\mathbf{e}_j}(1/\bar{z})}, \quad a \in \mathbb{C} \setminus \{0\}. \quad (6.55)$$

If $X_{\mathbf{n}}^{(\tau)}$ has $|\mathbf{n}| + 1$ simple zeros on the unit circle then an application of the argument principle proves Theorem 6.7.1 and Theorem 6.7.2.

To locate zeros of $X_{\mathbf{n}}^{(\tau)}$ we use the same approach as Paper IV, which gives the following analogue results on the unit circle.

Theorem 6.7.3 *Assume \mathbf{n} is ϕ -normal for $\boldsymbol{\mu}$, and write $X_{\mathbf{n}}^{(\tau)}$ for any function satisfying (6.51)-(6.53) with respect to $\boldsymbol{\mu}$.*

- $z_0 = e^{i\theta_0} \in \partial\mathbb{D}$ is not zero of multiplicity ≥ 2 of $X_{\mathbf{n}}^{(\tau)}$ if \mathbf{n} is normal with respect to $\widehat{\boldsymbol{\mu}}$ given by

$$d\widehat{\boldsymbol{\mu}}_j(e^{i\theta}) = \sin^2 \frac{\theta - \theta_0}{2} d\mu_j(e^{i\theta}), \quad j = 1, \dots, r. \quad (6.56)$$

- $z_0 \notin \partial\mathbb{D}$ is not a zero of $X_{\mathbf{n}}^{(\tau)}$ if \mathbf{n} is normal with respect to $\widehat{\boldsymbol{\mu}}$ given by

$$d\widehat{\boldsymbol{\mu}}_j(z) = |z - z_0|^2 d\mu_j(z), \quad j = 1, \dots, r. \quad (6.57)$$

While working on this paper we found the papers [33, 39] that study multiple orthogonal trigonometric polynomials of semi-integer degree. It closely relates to our multiple paraorthogonal polynomials, similarly to the discussion in section 5.3. Their notion of trigonometric AT system is a weaker version of our definition from Paper V. Through their zero counting proof, it turns out they essentially proved ϕ -normality of AT systems on the unit circle in the case of all n_j even. In our paper, we complement this result with the following one, which also leads to an alternative proof of Theorem 6.5.1 for Angelesco systems.

Theorem 6.7.4 *If μ is an Angelesco system on the unit circle, then $X_n^{(\tau)}$ has at least n_j zeros of odd multiplicity in $\text{int}(\Gamma_j)$, for any arc $\Gamma_j \supseteq \text{supp}(\mu_j)$.*

I should also remark that for even n_j this follows from a result of [9].

Paper VII is based on ongoing work that will be updated in the future. The current version was finalized only a few days before handing in the first draft of the thesis. Hence the paper contains no analogue results for type I polynomials, no interlacing result, and no result on the asymptotic distribution of zeros. For Theorem 6.7.2, we only prove that zeros of $\Phi_{n;n+e_j}$ either all lie in \mathbb{D} or all lie in $\mathbb{C} \setminus \bar{\mathbb{D}}$. The zeros should lie in \mathbb{D} however, at least for Angelesco systems, and this will be addressed in a future version of the paper.

7. Sammanfattning på svenska

Denna avhandling avser forskning om ortogonala polynom och approximationsteori. Ortogonala polynom är följder av polynom som bland annat dyker upp i flera olika approximationstekniker. För avhandlingen är Padéapproximation särskilt relevant, som är motsvarigheten till Taylorapproximation för rationella funktioner. För en stor klass av funktioner har dessa rationella approximationer en naturlig beskrivning i termer av ortogonala polynom. Även Gausskvadratur är nära relaterat till innehållet i avhandlingen, som använder nollställena till ortogonala polynom som datapunkter för numerisk integration, och ger snabb konvergens vid integration av analytiska funktioner.

Forskning om ortogonala polynom kan delas upp i två separata grupper, där den ena studerar ortogonalitetsrelationer med avseende på mått på reella tallinjen (OPRL), medan den andra studerar mått på enhetscirkeln (OPUC). En viktig del av avhandlingen studerar lokalisering av nollställena. För mått på reella tallinjen är alla nollställena reella, och för mått på enhetscirkeln ligger alla nollställena i enhetsdisken.

Avhandlingen fokuserar på multipla ortogonala polynom, på reella tallinjen (MOPRL), och på enhetscirkeln (MOPUC). Idén bakom multipla ortogonala polynom (och även Padéapproximation) kommer från Hermites artikel [25], som bevisar att e^k är irrationellt för varje heltal k genom Padéapproximation, och att e är transcendent genom Hermite–Padéapproximation. Precis som Padéapproximation beskrivs av ortogonala polynom så beskrivs Hermite–Padéapproximation av multipla ortogonala polynom.

Multipla ortogonala polynom uppfyller blandade ortogonalitetsrelationer med avseende på flera mått. Till skillnad från vanliga ortogonala polynom så garanterar inte denna konstruktion att polynomen är unikt bestämda av sina ortogonalitetsrelationer. Denna fundamentala egenskap är väsentligen nödvändig för att kunna studera multipla ortogonala polynom i allmänhet. Angelescu [1, 2] presenterade en allmän klass mått på reella tallinjen där unikhetssegenskapen håller, nämligen när deras stöd ligger i parvis disjunkta intervall. I detta fall är alla nollställena reella, precis som för OPRL.

Mer än 100 år efter Angelescu presenterar denna avhandling direkt analoga resultat för mått på enhetscirkeln. Artikel V presenterar en lösning på unikhetsproblemet, medan Artikel VII visar att alla nollställena ligger i enhetsdisken, baserat på analoga resultat på reella tallinjen från Artikel IV. Artikel I, III och VI studerar algebraiska relationer för multipla ortogonala polynom, medan Artikel II studerar numeriska algoritmer. Artikel I är särskilt intressant, som visar rekursionsrelationer för MOPUC som starkt efterliknar både Szegőrekursionen för OPUC och Van Assches rekursionsrelationer för MOPRL från [45].

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