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Existence and uniqueness of exact Borel subalgebras

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Abstract
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The main focus of this thesis is the study of exact Borel subalgebras of quasi-hereditary algebras. Existence results and constructions of such subalgebras are given in several special settings, in particular in the context of skew group and tensor algebra constructions, as well as for quasi-hereditary monomial algebras. Moreover, it is shown that regular exact Borel subalgebras are unique up to inner automorphism.

Keywords: quasi-hereditary algebras, exact Borel subalgebras

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1. Introduction

The main aim of this thesis is to contribute to the study of quasi-hereditary algebras and their exact Borel subalgebras. A particular focus of the thesis is on providing tools for calculating exact Borel subalgebras in concrete examples by showing their compatibility with certain constructions and investigating them in classes of examples. Another aim is to investigate their uniqueness properties of exact Borel subalgebras.

Quasi-hereditary algebras are certain finite-dimensional algebras, equipped with an additional partial order, and their study is therefore a special field within the study of finite-dimensional algebras in general. The key idea of representation theory is to replace the study of an algebra A by the study of its module category A- mod, also called the category of representations of A.

This category contains all finite-dimensional vectors spaces on which A acts, together with all linear maps between them that are compatible with this action. This perspective has proven very useful in the study of finite-dimensional algebras, and indeed of mathematical objects in general.

Nevertheless, not all information about an algebra is encoded in its category of representations. It may happen that two non-isomorphic algebras have equivalent categories of representations, which makes them basically indistinguishable from a representation-theoretic standpoint. For example, if we consider the complex numbers $\mathbb{C}$ on the one hand, and the matrix ring $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ on the other, we can associate to any representation M of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ a vector space $M_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M$, which is nothing but a $\mathbb{C}$-module, and to any $\mathrm{Mat}_{2\times 2}(\mathbb{C})$-linear map $f : M \to N$ a $\mathbb{C}$-linear map

$$f_1 : M_1 \to N_1, x \mapsto f(x).$$

Similarly, we can associate to any $\mathbb{C}$-representation V a representation $\mathbb{C}^2 \otimes V$ of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$, where $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ acts on the first factor, and to any $\mathbb{C}$-linear map φ a $\mathrm{Mat}_{2\times 2}(\mathbb{C})$-linear map $\mathbb{C}^2 \otimes \varphi$. Moreover, for any $\mathrm{Mat}_{2\times 2}(\mathbb{C})$-module M and any $\mathbb{C}$-module V, there are isomorphisms

$$\alpha_M : \mathbb{C}^2 \otimes M_1 \to M, e_1 \otimes m \mapsto m, e_2 \otimes m \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} m$$

of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$-modules, and

$$\beta_V : (\mathbb{C}^2 \otimes V)_1 \to V, e_1 \otimes v \mapsto v$$

of $\mathbb{C}$-modules such that for any homomorphism $f : M \to N$ of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$-modules and any homomorphism $\varphi : V \to W$ of $\mathbb{C}$-modules, we have

$$f = \alpha_N \circ (\mathbb{C}^2 \otimes f_1) \circ \alpha_M^{-1}$$
$$\varphi = \beta_W \circ (\mathbb{C}^2 \otimes \varphi)_1 \circ \beta_V^{-1}.$$

In particular, the maps

$$\mathrm{Hom}_{\mathrm{Mat}_{2\times 2}(\mathbb{C})}(M,N) \to \mathrm{Hom}_{\mathbb{C}}(M_1,N_1), f \mapsto f_1$$
$$\mathrm{Hom}_{\mathbb{C}}(V,W) \to \mathrm{Hom}_{\mathrm{Mat}_{2\times 2}(\mathbb{C})}(\mathbb{C}^2 \otimes V, \mathbb{C}^2 \otimes W), \varphi \mapsto \mathbb{C}^2 \otimes \varphi$$

are isomorphisms. Therefore, instead of studying modules over $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ and morphisms between them, we can study the associated vector spaces over $\mathbb{C}$ and morphisms between them, and the other way around. Algebras which have the same representations in this sense are called Morita equivalent algebras.

Fortunately, quasi-hereditary structures, in contrast to e.g. cellular structures [23, 34], can be transferred along Morita equivalences. This means that being quasi-hereditary is less a property of the algebra, and more a property of its category of representations. Hence it is possible to study quasi-hereditary algebras with the tools of a representation theory. A main object of interest from this perspective is a certain subcategory $\mathrm{F}(\Delta)$ of A-mod, which inherits some but not all of the structure of A-mod, and which characterizes the quasi-hereditary algebra up to Morita equivalence.

On the other hand, exact Borel subalgebras, which are the main topic of this thesis, are certain nice subalgebras of quasi-hereditary algebras which are by no means transferrable along Morita equivalences. This creates a dichotomy in the study of exact Borel subalgebras. On the one hand, one may study them up to Morita equivalence of the quasi-hereditary algebra, trading, in many cases, better theoretical results for the ability to explicitly compute and work with the actual algebra at hand. On the other hand, one may study them concretely for a given algebra, where one can compute explicitly, but e.g. existence may fail.

In this thesis, both approaches are applied at different points in time, depending on the problem at hand. Article 2 stands out as the most theoretical, where an abstract uniqueness result is investigated. In the other articles, the focus is rather on obtaining tools to investigate exact Borel subalgebras in concrete settings.

2. Representation-theoretic background

The study of quasi-hereditary algebras and their exact Borel subalgebras has numerous connections to other areas of representation theory. In many cases, theoretical background from such areas is of vital importance in the study of quasi-hereditary algebras. Vice versa, it is also sometimes the case that tools developed in the context of quasi-hereditary algebras can be transferred to other areas, or that the study of quasi-hereditary algebras is interesting from the standpoint of those areas for other reasons. Particularly strong ties of this kind exist between the study of quasi-hereditary algebras and the study of exact categories, to deformation theory and to the study of A-∞ algebras.

From now on, let us assume that the underlying field k is algebraically closed.

2.1 Exact categories

Recall that from a representation-theoretic point of view, the core object of inquiry associated to an algebra A is its category of representations, A-mod. From a category-theoretic perspective, A-mod is a particularly nice category, namely an essentially small abelian category. In studying the structure of this category, one is often interested in certain subcategories of A-mod which don't inherit this nice structure.

In representation theory, one most commonly considers full subcategories, which are easier to manage than general subcategories. One reason for this is that they are compatible with equivalences in the following sense:

If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence, then for any full subcategory $\mathscr{A}$ of $\mathscr{C}$, the collection of objects of the form $F(A)$, A object in $\mathscr{A}$, gives rise to a full subcategory $F(\mathscr{A})$ of $\mathscr{D}$, and F restricts to an equivalence between $\mathscr{A}$ and $F(\mathscr{A})$. From this one can deduce that instead of studying full subcategories of $\mathscr{C}$, one also may study full subcategories of any equivalent category, with essentially the same results. For non-full subcategories, this is not true in general.

For example, the trivial category, with one object and only the identity endomorphism is equivalent to the category with exactly two objects which each have only the identity endomorphism, and where there is additionally exactly one morphism in either direction between the two objects. Graphically, this may be represented as follows:

The former category has no subcategories except itself and the empty category, while the latter also has a subcategory consisting of two non-isomorphic objects, whose endomorphisms consist only of the identity, and who admit no morphisms in either direction (or alternatively, one morphism in one direction).

Since full subcategories are in one-to-one correspondence with sub-collections of the class of objects, it is generally not feasible to classify them in a reasonable way. One therefore usually is interested in full subcategories which are closed under one or more additional constructions, such as direct sums, direct summands, extensions, kernels, cokernels, subobjects, quotients and the like. Different combinations of these properties define, for example, Serre subcategories, and torsion and torsion-free classes.

In this thesis, the full subcategories of A-mod which are of the most interest are those which are closed under extensions. This means that for any short exact sequence

$$(0) \to X \to Y \to Z \to (0)$$

in A-mod, such that X and Z are in the subcategory, so is Y. While such subcategories are in general not abelian, they inherit from A-mod the structure of an **exact category**.

Exact categories are a notion due to Quillen, axiomatizing the behaviour of just such extension closed subcategories of abelian categories. Their data consists of an additive category $\mathscr{E}$ together with a collection of short exact sequences S in $\mathscr{E}$, called admissible exact sequences, fulfilling the following axioms:

1. S is closed under isomorphism and contains all split exact sequences.
2. Pushouts along admissible epimorphisms exist and are admissible epimorphisms, and pullbacks along admissible monomorphisms exist and are admissible monomorphisms.
3. Compositions of admissible epimorphisms are admissible epimorphisms and compositions of admissible monomorphisms are admissible monomorphisms.

The definition originally included a fourth axiom which was afterwards shown to be redundant [50, 30, 4]. In the case of extension closed subcategories of abelian categories, admissible exact sequences in the subcategory are exactly the short exact sequences in the subcategory which are also short exact sequences in the ambient abelian category. As is easy to check, this gives indeed rise to an exact category, and in fact, by the Gabriel-Quillen embedding theorem [41, 47], any exact category can be obtained in this way up to equivalence.

Given a collection $M = (M_i)_{i \in I}$ of objects in an abelian category, one may form their extension closure, $\mathrm{F}(M)$, which is the smallest extension-closed subcategory containing all M_i, $i \in I$, and which consists of all objects X which admit a filtration

$$(0) = X_0 \subseteq X_1 \subseteq X_2 \cdots \subseteq X_n = X$$

such that for all $1 \leq j \leq n$ there is an $i \in I$ with $X_j/X_{j-1} \cong M_i$.

Even if M is a finite collection of objects, it is in general not very well understood how properties of M relate to properties of $\mathrm{F}(M)$. For example, it is, to our best knowledge, an open question which property of M is needed for $\mathrm{F}(M)$ to be closed under direct summands.

2.2 Exceptional sequences, bocses and quasi-hereditary algebras

One setting where $\mathrm{F}(M)$ is better behaved is when $(M_i)_{1\leq i\leq n}$ is a finite **exceptional sequence**. This means that:

1. For $n \geq 1$ and $i \geq j$, $\mathrm{Ext}^n(M_i, M_j) = (0)$.
2. For $i > j$, $\mathrm{Hom}(M_i, M_j) = (0)$.
3. For all i, $\mathrm{End}(M_i, M_i) \cong \mathrm{k}$.

In this case, $\mathrm{F}(M)$ is an exact category with the Jordan-Hölder property in the sense of [19]. In fact, it is equivalent, as an exact category, to the category of representations of a directed bocs $\mathfrak{B}$ with projective kernel [16, 33].

A bocs is a pair $\mathfrak{B} = (B, V)$ of an algebra B and a comonoid object V in the category of B-B-bimodules, and its category of representations consists of B-modules as objects, and homomorphism spaces given by

$$\mathrm{Hom}_{\mathscr{B}}(X,Y) := \mathrm{Hom}_B(V \otimes_B X, Y)$$

with composition

$$(g, f) \mapsto g \circ (V \otimes f) \circ (\mu_V \otimes X).$$

It is said to have a **projective kernel** if the counit is an epimorphism with its kernel $\overline{V}$ being a projective bimodule. A **directed** bocs with projective kernel is additionally equipped with a partial order $\leq$ on the isomorphism classes of simple B-modules such that both $\mathrm{Ext}^1_B(L_i, L_j) = (0)$ and $\mathrm{Hom}_{\mathscr{B}}(L_i, L_j) = (0)$ for $i < j$.

Under the equivalence mentioned above, the modules M_i are mapped to the simple B-modules.

For any bocs $\mathfrak{B}$, its category of representations becomes a full, extension closed subcategory of the module category of its right algebra

$$R^{\mathfrak{B}} := \mathrm{End}_{\mathfrak{B}}(B, B)^{\mathrm{op}} \cong \mathrm{Hom}_B(V, B)^{\mathrm{op}}$$

via the induction functor

$$\begin{aligned} \mathfrak{B}\text{-}\mathrm{mod} &\to R^{\mathfrak{B}}\text{-}\mathrm{mod}, \\ M &\mapsto \mathrm{Hom}_B(V, M), \\ f &\mapsto (g \mapsto f \circ (V \otimes g) \circ \mu_V). \end{aligned}$$

The right algebra of a directed bocs with projective kernel is a so-called **quasi-hereditary algebra**. This means that there is a partial order on the isomorphism classes of simple $R^{\mathfrak{B}}$-modules, and for each simple $R^{\mathfrak{B}}$-module $L_i^{R^{\mathfrak{B}}}$, there is an associated **standard module** $\Delta(L_i^{R^{\mathfrak{B}}}) = \Delta_i^{R^{\mathfrak{B}}}$, in this case given by $R^{\mathfrak{B}} \otimes_B L_i \cong \mathrm{Hom}_B(V, L_i)$, such that $(\Delta_i^{R^{\mathfrak{B}}})_i$ is an exceptional sequence with respect to the partial order, and such that, additionally, $R^{\mathfrak{B}} \in \mathrm{F}(\Delta^{R^{\mathfrak{B}}})$. While this is one possible definition of quasi-heredity, in fact there are many alternative characterizations, from the original definition in terms of heredity ideals [10] to recollements (see e.g. [2]), or via the closely related notion of highest weight categories [10]; and quasi-hereditary algebras exhibit many desirable representation theoretic properties, most notably, being of finite global dimension [16]. Much of this thesis is concerned with the representation theory of quasi-hereditary algebras.

Note that a consequence of the above, which, however, may be shown more elementarily (see [16]) is that for any (finite) exceptional sequence $(M_i)_i$, there is a quasi-hereditary algebra $R^{\mathfrak{B}}$ such that $\mathrm{F}(M) \cong \mathrm{F}(\Delta^{R^{\mathfrak{B}}})$. In fact, this quasi-hereditary algebra is unique up to Morita equivalence preserving the standard modules [16], so that, in the case one starts with a collection of standard modules (Δ_i) for some quasi-hereditary algebra A, one obtains a Morita equivalent quasi-hereditary algebra $R^{\mathfrak{B}}$, which is the right algebra of a directed bocs $\mathfrak{B} = (B, V)$ with projective kernel.

Since ε is a B-B-bimodule homomorphism, there is an algebra homomorphism

$$\begin{aligned} &B \to R^{\mathfrak{B}} = \mathrm{Hom}_B(V, B), \\ &b \mapsto \varepsilon b, \end{aligned}$$

and since ε is surjective, this is an embedding. In fact, B becomes a **regular exact Borel subalgebra** of $R^{\mathfrak{B}}$.

2.3 Exact Borel subalgebras

The notion of an exact Borel subalgebra of a quasi-hereditary algebra, which is due to König [32], was defined in analogy to Borel subalgebras of semisimple Lie algebras. An **exact Borel subalgebra** of a quasi-hereditary algebra $(A, \leq)$ is a subalgebra $B \subseteq A$ such that

1. there is an isomorphism

$$\mathrm{Sim}(B) \to \mathrm{Sim}(A), L_i^B \mapsto L_i^A$$

between the isomorphism classes of simple B-modules and the isomorphism classes of simple A-modules, such that B is **directed**, i.e. quasi-hereditary with simple standard modules, with respect to the partial order on $\mathrm{Sim}(B)$ defined via

$$L_i^B \leq L_j^B :\Leftrightarrow L_i^A \leq L_j^A,$$

2. $A \otimes_B L_i^B \cong \Delta_i^A$,
3. $A \otimes_B -$ is exact.

Recall that, for a Lie algebra, a Borel subalgebra is defined as a maximal solvable sublagebra, and that, for a semisimple complex Lie algebra $\mathfrak{g}$, Borel subalgebras are the subalgebras of the form $\mathfrak{n}^+ \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra, and $\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ for an associated root system R with positive roots R^+.

A fundamental example is the case where $\mathfrak{g} = \mathfrak{gl}_n$ is the Lie algebra of $n \times n$-matrices, where, in the classical setup, one chooses diagonal matrices as the Cartan subalgebra, and upper triangular matrices as the Borel subalgebra. In fact, for any semisimple complex Lie algebra, one can find a basis such that the Cartan subalgebra acts via diagonal, and the Borel subalgebra via upper triangular matrices with the adjoint action on $\mathfrak{g}$, using the root space decomposition above.

In this sense, Borel subalgebras of Lie algebras are also directed in a way. In order to explain the criterion for standard modules, let us recall that for a semisimple complex Lie algebra, the associated Bernstein-Gelfand-Gelfand category $\mathscr{O}$ is the full subcategory of $\mathfrak{g}$-mod of finitely-generated $\mathfrak{g}$-modules with weight space decompositions over $\mathfrak{h}$, such that each weight space is finite-dimensional, and such that $\mathfrak{n}^+$ acts nilpotently on every element of the module.

An important collection of modules in category $\mathscr{O}$ are the **Verma-modules**. For any element $\lambda \in \mathfrak{h}^*$, denote by $\mathbb{C}_\lambda$ the associated one-dimensional $\mathfrak{b}$-module where $\mathfrak{n}^+$ acts via zero. Then the associated Verma module is defined as $V_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. Each Verma module V_λ has a simple top L_λ, and this gives rise to a bijection between simple objects in category $\mathscr{O}$ and weights $\lambda \in \mathfrak{h}^*$. The weights naturally are a partially ordered set, where $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a non-negative integer linear combination of positive roots.

This induces a partial order on the simple modules, and with respect to this partial order, the Verma modules are an infinite exceptional sequence, that is, they fulfill the same conditions as a finite exceptional sequence, but are indexed by an infinite poset. In fact, category $\mathscr{O}$ decomposes into blocks—full subcategories that do not admit any morphisms between each other—so that each of these blocks is equivalent to the module category of a finite-dimensional quasi-hereditary algebra, where the partial order is the restriction of the partial order on weights, and the standard modules of these algebras, under the above mentioned equivalence, correspond to the Verma modules. Note that Verma modules are induced by one-dimensional modules over the enveloping algebra of the Borel subalgebra. Mirroring this property, simple modules over an exact Borel subalgebras induce to standard modules over the quasi-hereditary algebra in the definition by König.

On the other hand, both exactness, which is the last criterion in the definition of an exact Borel subalgebra, and regularity, which is an additional property important in the study of exact Borel subalgebras, are homological criteria which are not reflected in the Lie algebra setting. Exactness, which is the last criterion in the definition of an exact Borel subalgebra above, means that short exact sequences induce to short exact sequences, so that the induction functor is compatible with the exact structure on B- mod and $\mathrm{F}(\Delta)$. On the other hand, **regularity** means that the induced maps $\mathrm{Ext}^n_B(L,L) \to \mathrm{Ext}^n_A(\Delta,\Delta)$ on Ext-groups are isomorphisms for all $n \geq 1$. This means that the exact structure is essentially the same. For example, if B is a regular exact Borel subalgebra B of a quasi-hereditary algebra $(A, \leq)$ then any short exact sequence

$$(0) \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow (0)$$

in $\mathrm{F}(\Delta)$ is isomorphic to a short exact sequence

$$(0) \longrightarrow A \otimes_B N' \xrightarrow{A \otimes_B i} A \otimes_B N \xrightarrow{A \otimes_B p} A \otimes_B N'' \longrightarrow (0)$$

where

$$(0) \longrightarrow N' \xrightarrow{i} N \xrightarrow{p} N'' \longrightarrow (0)$$

is a short exact sequence in $\mathrm{mod}\, B$. This can be seen as follows (see also [3]):

If B is regular, then, using long exact sequences in homology, we obtain that the induced maps

$$\mathrm{Ext}^1_B(N',N'') \to \mathrm{Ext}^1_A(A \otimes_B N', A \otimes_B N'')$$

are epimorphisms for all B-modules N' and N''. In particular, using induction on the number of standard composition factors, one can show that any $M \in \mathrm{F}(\Delta)$ is isomorphic to a module of the form $A \otimes_B N$ for $N \in B$- mod. Hence, for any $M', M'' \in \mathrm{F}(\Delta)$, there are $N', N'' \in B$- mod with $A \otimes_B N' \cong M'$ and $A \otimes_B N'' \cong M''$. Moreover, since

$$\mathrm{Ext}^1_B(N',N'') \to \mathrm{Ext}^1_A(A \otimes_B N', A \otimes_B N'')$$

is an epimorphism, there is, for any extension

$$(0) \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow (0)$$

in $\mathrm{F}(\Delta)$ an extension

$$(0) \longrightarrow N' \xrightarrow{i} N \xrightarrow{p} N'' \longrightarrow (0)$$

in $\mathrm{mod}\, B$ such that

$$(0) \longrightarrow A \otimes_B N' \xrightarrow{A \otimes_B i} A \otimes_B N \xrightarrow{A \otimes_B p} A \otimes_B N'' \longrightarrow (0)$$

is isomorphic to

$$(0) \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow (0).$$

Since every module in B-mod is filtered by simple modules, and since the simple modules have no homomorphisms between each other except scalar multiples of identities, extensions carry a lot of information about the module category B-mod. In fact, when viewed with the right structure, such extensions are enough to reconstruct the module category.

2.4 Bound quivers and Keller's reconstruction theorem

The idea that a representation may be understood via the simple representations it consists of, and how they are glued together, is a very old idea in representation theory. It can be seen already in the first definition of composition series due to Jordan in [28], and later became one of the main aims of homological algebra. A fundamental result in this area is Gabriel's structure theorem [21], which states that for any finite-dimensional algebra over an algebraically closed field, there is an associated bound quiver such that the module category is equivalent to the category of representations of the bound quiver. The earliest version of this seems to be due to Hochschild [26], where it is shown that a finite-dimensional algebra A over a perfect field can be written as the quotient of a tensor algebra of the form $T_L(J/J^2)$, where L is a maximal semisimple subalgebra and J is the Jacobson radical. While the visualization via quivers is not present yet, this result is essentially a non-basic version of Gabriel's structure theorem.

Recall that given a quiver $Q = (Q_0, Q_1, s, t)$ with finitely many vertices Q_0 and finitely many edges Q_1, as well as functions $s, t : Q_1 \to Q_0$ giving the start respectively terminal vertex of an arrow, its path algebra $\mathrm{k}Q$ is the k-vector space spanned by all paths in Q, including one trivial path at every vertex, together with a multiplication

$$\mathrm{k}Q \otimes \mathrm{k}Q \to \mathrm{k}Q$$

defined on basis elements as concatenation of paths when possible, and zero when not. For every $n \in \mathbb{N}$, the span I_n of all paths of length at least n defines an ideal in $\mathrm{k}Q$.

A bound quiver consists of the data of a quiver Q and an ideal $I \subseteq \mathrm{k}Q$, such that $I \subseteq I_2$, and such that there is some $n \geq 2$ with $I_n \subseteq I$, and its category of representations is simply the module category of the algebra $\mathrm{k}Q/I$.

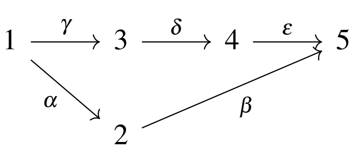
Describing an algebra up to Morita equivalence in this way can be extremely useful, since many important classes of modules, such as injective, projective and simple modules can be described explicitly in this setting. For example, simple modules are exactly those modules where all paths act by zero except one of the trivial paths associated to a vertex, which acts by one. Indecomposable projective modules are isomorphic to modules given by the span of all paths starting in a fixed vertex, with the module structure given by the restriction of the multiplication of A. These explicit descriptions make it much easier to access information about the module category. Additionally, rather a lot can even be read off directly by looking at the quiver and ideal itself. This makes bound quivers very useful in constructing examples and counterexamples, something that has been used many times throughout this thesis.

In the construction by Gabriel, the vertices of the quiver associated to the algebra A correspond to isomorphism classes of simple A-modules, while the arrows correspond to one-extensions between simple modules, that is, to non-split short exact sequences of the form

$$(0) \longrightarrow L_i \longrightarrow E \longrightarrow L_j \longrightarrow (0)$$

where L_i and L_j are both simple modules. More casually worded, one-extensions give information about how two simple modules may be glued together. However, the classical theory does not give a complete description for the relations in these terms. While it has been long known that there is a vector space isomorphisms between relations $e_j(I/II_1 + I_1I)e_i$ and $\mathrm{Ext}^2(L_i, L_j)$ [1, 22], this is not enough to fully reconstruct the relations. Instead, they are usually described as the kernel of an epimorphism from the path algebra of the quiver to a basic representative of the algebra.

Indeed, it is a well known fact that even the Yoneda algebra structure on $\mathrm{Ext}^*_A(L, L)$ is not enough to determine A up to isomorphism. To see this, consider for example the quiver Q



and the ideals $I = \langle \beta\alpha \rangle$ and $I' := \langle \beta\alpha - \varepsilon\delta\gamma \rangle$, and let $A := \mathrm{k}Q/I$, $A' := \mathrm{k}Q/I'$.

Then, if we denote by L_i respectively L'_i the simple A- respectively A'-module corresponding to the vertex i, we have

$$\begin{aligned}
\mathrm{Ext}^1_A(L_i, L_j) &\cong \mathrm{Ext}^1_{A'}(L'_i, L'_j) \cong \mathrm{k} \text{ for } (i, j) \in \{(1,2),(2,5),(1,3),(3,4),(4,5)\},\\
\mathrm{Ext}^1_A(L_i, L_j) &\cong \mathrm{Ext}^1_{A'}(L'_i, L'_j) \cong (0) \text{ else},\\
\mathrm{Ext}^2_A(L_1, L_5) &\cong \mathrm{Ext}^2_{A'}(L'_1, L'_5) \cong \mathrm{k},\\
\mathrm{Ext}^2_A(L_i, L_j) &\cong \mathrm{Ext}^2_{A'}(L'_i, L'_j) \cong (0) \text{ for } (i, j) \neq (1,5).
\end{aligned}$$

In particular, $\mathrm{Ext}^*_A(L,L) \cong \mathrm{Ext}^*_{A'}(L,L')$ as $\mathrm{k}\,Q_0$-bimodules, and, as is easy to check, even as algebras with the Yoneda algebra structure. The relevant information here is that in both cases, the multiplication in the Yoneda algebra restricts to an isomorphism

$$\mathrm{Ext}^1_A(L_2,L_5) \otimes \mathrm{Ext}^1_A(L_1,L_2) \to \mathrm{Ext}^2_A(L_1,L_5)$$

respectively

$$\mathrm{Ext}^1_{A'}(L'_2,L'_5) \otimes \mathrm{Ext}^1_{A'}(L'_1,L'_2) \to \mathrm{Ext}^2_{A'}(L'_1,L'_5).$$

However, the two algebras are not isomorphic, since any such isomorphism would induce an equivalence on the module categories. In particular, this equivalence would map indecomposable projective modules to indecomposable projective modules and simple modules to simple modules, and, since the quiver Q has no symmetries, it would in fact, up to isomorphism, map L_i to L'_i and the projective cover P_i of L_i to the projective cover P'_i of L'_i for every i. However, in the module category of A, there exists a short exact sequence

$$(0) \to P_2 \to P_1 \to X \to (0),$$

while in the module category of A', there is no injective map $P_2 \to P_1$.

In order to reconstruct the algebra A up to Morita equivalence, the Yoneda algebra structure is therefore not enough. However, it turns out that it is possible to reconstruct A up to Morita equivalence from $\mathrm{Ext}^*_A(L,L)$, if one equips the latter with additional structure, namely that of a so-called A-∞ algebra. This result, known as Keller's reconstruction theorem, was achieved much more recently [31].

2.5 A-infinity algebras

An A-∞ algebra is a graded vector space $A = (A_n)_{n\in\mathbb{Z}}$ together with an infinite collection of multiplications $(m_n : A^{\otimes n} \to A)_{n\in\mathbb{N}}$ of degree $2-n$, such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

Here, m_1 is often thought of as a differential, m_2 as a multiplication, and m_n for $n > 2$ are thought of as higher multiplications. In particular, dg-algebras, that is, differential graded algebras, are A-∞-algebras where all higher multiplications are zero.

A morphism of A-∞ algebras $f : A \to B$ is a collection $f = (f_n)$, of maps

$$f_n : A^{\otimes n} \to B$$

of degree $1-n$, such that

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{j_1+\cdots+j_k=l} (-1)^{\sum_{l=1}^{k}(1-j_l)\sum_{l'=1}^{l} j_{l'}} m_l(f_{j_1} \otimes \cdots \otimes f_{j_k}).$$

It turns out that f is invertible if and only if f_1 is an isomorphism of vector spaces. Moreover, f is called a quasi-isomorphism if and only if

$$f_1 : (A, m_1^A) \to (B, m_1^B)$$

is a quasi-isomorphism of cochain complexes.

A-∞ algebras were first defined in a topological context [45, 46], and can be viewed as deformations of dg-algebras. Every A-∞ algebra is quasi-isomorphic to a dg-algebra, and by Kadeishvili's theorem [29], for any dg-algebra, its homology obtains the structure of an A-∞ algebra which is quasi-isomorphic as an A-∞ algebra to the original dg-algebra.

In particular, for any algebra A, and any A-module M, $\mathrm{Ext}_A^*(M,M)$ obtains the structure of an A-∞ algebra which is quasi-isomorphic to the dg-algebra $\mathrm{End}_A^*(P^\cdot(M))$, where $P^\cdot(M)$ is a projective resolution of M.

As such, $\mathrm{Ext}_A^*(M,M)$ carries the whole information of $\mathrm{End}_A^*(P^\cdot(M))$ up to homotopy.

If one picks $M = L := A/\mathrm{rad}(A)$, then this is enough to reconstruct A-mod, and the reconstruction can be done via twisted modules.

The idea of twisting known structures appears in many mathematical contexts, however, the version we refer to here originates in the works of Maurer [40] and Cartan [5] on Lie-groups. In honour of their work, the equations which describe twists in various settings are called Maurer-Cartan equations. The original Maurer-Cartan equation describes flatness of a connection on a principal G-bundle for a Lie group G. This relates to the Maurer-Cartan equation

$$dw + 1/2[w,w] = 0,$$

for elements $w \in L_1$ of a dg-Lie algebras L, where elements fulfilling this equation, called twists, give rise to a twisted differential

$$d^w(x) = d(x) + [w,x],$$

on L. This Maurer-Cartan equation, which is of great importance in deformation theory (see e.g. [18]), has been generalized to many other algebraic contexts, including dg-algebras and A-∞ algebras (see [17] for a more detailed account of this, and how it relates to the theory of operads).

Let A be a dg-algebra unital over a semisimple algebra L. Then a twisted module over A is a pair consisting of an L-module X and a twist

$$w^X \in A \otimes_{L \otimes L^{\mathrm{op}}} \mathrm{End}_k(X)$$

fulfilling the Maurer-Cartan equation

$$w_X^2 + d(w_X) = 0$$

as well as a triangularity condition, stating that there is a decomposition of X into simple L-modules $X = X_1 \oplus \cdots \oplus X_k$ such that the component

$$(w_X)_{i,j} := (1_A \otimes \mathrm{id}_{X_j}) \cdot w_X \cdot (1_A \otimes \mathrm{id}_{X_i})$$

vanishes for $i \geq j$.

The dg-category $\mathrm{twmod}_L(A)$ of twisted modules has twisted modules over A as objects, while Hom-complexes are given by

$$\mathrm{twmod}_L(A)((X, w_X), (Y, w_Y)) = A \otimes_{L \otimes L^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{k}}(X, Y)$$

as graded vector spaces with differential

$$d^{\mathrm{twmod}}(f) = (d_A \otimes \mathrm{id})(f) + w_Y \cdot f + (-1)^{|f|} f \cdot w_X$$

and composition is induced by multiplication in A and composition in k-mod.

This is related to reconstructing all modules from simple modules as follows: Let A be a finite-dimensional algebra with maximal semisimple subalgebra L, and let $P(L)$ be a projective resolution of L as an A-L-bimodule. Then the horseshoe lemma yields that every A-module M has a projective resolution given by

$$P(M) = (P(L) \otimes_{L\, L|} M, d_{P(L)} \otimes_L \mathrm{id}_M + w_M)$$

where

$$\begin{aligned} w_M \in \mathrm{End}_A^1(P(L) \otimes_{L\, L|} M) \\ \cong \mathrm{End}_A(P(L)) \otimes_{L \otimes L^{\mathrm{op}}} \mathrm{End}_{\mathrm{k}}(M) \end{aligned}$$

is a twist fulfilling the Maurer-Cartan equation and the triangularity condition. For more details on this, see [43, Section 4].

In this way, M corresponds to the twisted module $({}_{L|}M)$ over the dg-algebra $\mathrm{End}_A(P(L))$. Moreover, homomorphisms between two modules correspond to elements in

$$H^0(\mathrm{Hom}_A(P(M), P(N))) \cong \mathrm{End}(P(L)) \otimes_{L \otimes L^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{k}}(M, N).$$

This gives a description of A-mod as the homology in degree zero of the twisted module category $\mathrm{twmod}_L(\mathrm{End}_A(P(L)))$.

Similarly, one may define the category of twisted modules over an A-∞ algebra strictly unital over L, and it is possible to show that quasi-isomorphisms of A-∞ algebras induce quasi-isomorphism of twisted module categories. In particular, the quasi-isomorphism between $\mathrm{Ext}_A^*(L, L)$ and $\mathrm{End}(P(L))$ induces a quasi-isomorphism between twisted module categories, so that

$$A\text{-mod} \cong H^0(\mathrm{twmod}_L(\mathrm{Ext}_A^*(L, L))).$$

Since the module category contains enough information to recover a basic representative of the algebra A, a basic algebra A can be reconstructed, up to isomorphism, from $\mathrm{Ext}_A^*(L,L)$. More concretely, the reconstructed algebra is the quotient of the tensor algebra $T_L((\mathrm{Ext}_A^1(L,L))^*)$ by the ideal generated by elements of the form

$$\sum_{n=0}^{\infty}(-1)^{n(n-1)/2}m_n^*(x)$$

for $x \in \mathrm{Ext}_A^2(L,L)^*$. Here, $*$ denotes the k-linear duality. This reconstruction plays an important role in Article 2.

Taking up the example from before of two non-isomorphic basic algebras A and A' with isomorphic Yoneda algebras, one can see that m_3 is zero on $\mathrm{Ext}_A^*(L,L)$, while m_3 is non-zero on $\mathrm{Ext}_{A'}^*(L,L)$, so that the above relations indeed give the relations for A respectively A'.

2.6 Twisted modules and exact Borel subalgebras

The description of a module category A-mod via twisted modules over the A-∞ algebra $\mathrm{Ext}_A^*(L,L)$ can also be generalized to filtered module categories by replacing the simple modules $L_1,\dots,L_n$ with the collection of modules $M_1,\dots,M_m$, and the semisimple algebra L by $L_M := \mathrm{End}(M)/\mathrm{rad}(\mathrm{End}(M))$. Then, one obtains that $H^0(\mathrm{twmod}_{L_M}(\mathrm{Ext}_A^*(M,M))) \cong \mathrm{F}(M)$. This is of great importance in the construction of exact Borel subalgebras in [33]. There, the inclusion of A-∞ algebras

$$L_\Delta \oplus \mathrm{Ext}_A^{>0}(\Delta,\Delta) \to \mathrm{Ext}_A^*(\Delta,\Delta)$$

is used to obtain an essentially surjective, faithful functor

$$H^0(\mathrm{twmod}_{L_\Delta}(L_\Delta \oplus \mathrm{Ext}_A^{>0}(\Delta,\Delta)) \to \mathrm{F}(\Delta).$$

The former turns out to be a finite abelian category, equivalent to B-mod, where B is the finite-dimensional algebra obtained from the A-∞ algebra

$$L_\Delta \oplus \mathrm{Ext}_A^{>0}(\Delta,\Delta)$$

via the construction used in Keller's reconstruction theorem. The article then proceeds to show that the category $H^0(\mathrm{twmod}_{L_\Delta}(\mathrm{Ext}_A^*(\Delta,\Delta)))$ is equivalent to the module category of a directed bocs $\mathfrak{B}$ over B with projective kernel, so that $\mathrm{F}(\Delta) \cong \mathfrak{B}$. The right algebra of this bocs, in turn becomes a quasi-hereditary algebra Morita-equivalent to A, admitting a regular exact Borel subalgebra B.

This construction, which is due to König, Külshammer and Ovsienko [33], is the main existence result for exact Borel subalgebras. It proves existence of exact Borel subalgebras up to Morita equivalence, and also shows that these exact Borel subalgebras have the additional property of regularity. It had been known previously for a long time that exact Borel subalgebras do not always exist on the level of algebras [32], and that they are not necessarily unique up to isomorphism. However, restricting the attention to regular exact Borel subalgebras, it became possible to prove their uniqueness up to isomorphism, and even up to isomorphism compatible with the embedding [36, 12]. Exact Borel subalgebras of Lie algebras, however, are even unique up to inner automorphism, and in fact, it was conjectured by Külshammer and Miemietz in [36] that the same should be true for regular exact Borel subalgebras.

In this context, it is helpful to note that the construction procedure relates the embedding of a basic regular exact Borel subalgebra B into quasi-hereditary algebra A to the embedding

$$L_\Delta \to \mathrm{End}_A(\Delta)$$

of the maximal semisimple quotient $L_\Delta = \mathrm{End}_A(\Delta)/\operatorname{rad}(\mathrm{End}_A(\Delta))$ of $\mathrm{End}_A(\Delta)$ into $\mathrm{End}_A(\Delta)$. The latter is by the Wedderburn-Malcev theorem unique up to inner automorphism [49, 38]. This observation is the key to the proof of the above mentioned conjecture in Article 2.

2.7 Skew group constructions

Another main topic of the thesis are skew group constructions and their relationship to quasi-hereditary algebras. Such constructions are of great importance in geometric and algebro-geometric contexts. For example, skew group constructions on the coordinate ring of an algebraic variety correspond to orbit constructions on the variety, and the latter appear naturally e.g. in the context of covering spaces.

Given an algebra A with a group action by a finite group G, the skew group algebra $A*G$ is defined as the vector space $A \otimes \mathrm{k}G$ together with multiplication

$$(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh.$$

Note that this is very similar to the construction of semi-direct products of groups.

Modules over the skew group algebra correspond exactly to A-modules M with an additional G-action on M such that $a \cdot g(m) = g(a \cdot m)$.

The two module categories of A and $A*G$ are related by a pair of adjoint functors (I,R), the induction and restriction functor

$$I : A\text{-}\mathrm{mod} \to A*G\text{-}\mathrm{mod}, M \mapsto \mathrm{k}G \otimes M, f \mapsto \mathrm{k}G \otimes f$$

and

$$R : A*G\text{-}\mathrm{mod} \to A\text{-}\mathrm{mod}, M \mapsto_{A|} M, f \mapsto f$$

which are both exact. In characteristic zero, where the group algebra $\mathrm{k}G$ is semisimple, it is possible to relate homological properties of A to those of $A * G$, via these functors, which in this case both preserve and reflect projective modules. For example, it is easy to see that, in the characteristic zero case, the projective dimension of $A * G$ is the same as the projective dimension of A. In particular, A is hereditary if and only if $A * G$ is. This leads naturally to the question whether the same holds for quasi-heredity. In Article 1, we give a positive answer to that question, assuming a certain compatibility between the partial order and the group action. We additionally investigate exact Borel subalgebras of skew group algebras, a question which is taken up again in Article 2, where we show an existence result up Morita equivalence for regular exact Borel subalgebras which are compatible with a G-action.

Skew group algebras have been investigated in depth, notably in [44]. There, it is also shown, supposing additionally that the group is solvable, how to construct the quiver of $A * G$ based on the quiver of A and the group action, something that is very useful in many examples. Skew group algebras of quasi-hereditary algebras also show up as one of the steps of the wreath product construction used in the proof of Broué's abelian defect group conjecture [7, 8, 9].

3. Summary of the articles

The thesis consists of four articles. The first article investigates skew group constructions on quasi-hereditary algebras and their exact Borel subalgebras. The second shows uniqueness up to inner automorphism of regular exact Borel subalgebras. The third is concerned with exact Borel subalgebras of quasi-hereditary monomial algebras, and the fourth deals with tensor algebras of quasi-hereditary algebras and exact Borel subalgebras.

Throughout the thesis, modules and algebras (but not dg- or A-∞-algebras) are always finite-dimensional, and the underlying field k is algebraically closed.

3.1 Article 1

In this article, skew group constructions on quasi-hereditary algebras over algebraically closed fields of characteristic zero are investigated in relation to their effect on exact Borel subalgebras. To this end, it is assumed that A is an algebra with a group action by a finite group G such that the group action is compatible with the partial order, in the sense that

$$L < L' \Leftrightarrow gL < hL' \text{ for all} g, h \in G,$$

where gM for an A-module M denotes the module with the same underlying vector space, and with A-action given by

$$A \otimes gM \to gM, a \otimes m \mapsto g(a)m.$$

If one assumes no compatibility condition, it is not clear which partial order on $\mathrm{Sim}(A * G)$ should be induced by $\leq$. For example, consider the algebra $A = \mathrm{k}Q$ for the quiver

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

with the partial order on $\mathrm{Sim}(A)$ induced by the natural order on the vertices of the quiver, that is, denoting the simple corresponding to the i-th vertex by L_i we have

$$L_1 < L_2 < L_3.$$

Then the group $G = \{1, g\} \cong \mathbb{Z}/2\mathbb{Z}$ acts on A via

$$g(e_1) = e_3$$
$$g(e_3) = e_1$$
$$g(e_2) = e_2$$
$$g(\alpha) = \beta$$
$$g(\beta) = \alpha.$$

Then, up to isomorphism, the simple modules of the skew group algebra are given by

- $L_a = L_1 \oplus L_3$ as an A-module, with G-action $g(e_1) = e_3$,
- $L_b = L_2$ as an A-module, with G-action $g(e_2) = e_2$,
- $L_c = L_2$ as an A-module, with G-action $g(e_2) = -e_2$.

It does not seem clear which partial order to induce on these modules based on $\leq$.

However, if one assumes the criterion

$$L_i < L_j \Leftrightarrow gL_i < hL_j \text{ for all} g, h \in G \text{ and } L_i, L_j \in \mathrm{Sim}(A)$$

it is possible to define a corresponding partial order on $\mathrm{Sim}(A * G)$.

For example, taking the same algebra as above, but with partial order $L_2 < L_1$ and $L_2 < L_3$, this naturally induces the partial order given by

$$L_b < L_a, L_c.$$

There is a similar criterion on a partial order on $\mathrm{Sim}(A * G)$, called G-stable in the article, such that one obtains a bijection between partial orders on $\mathrm{Sim}(A)$ fulfilling the above compatibility criterion, and G-stable partial orders on $\mathrm{Sim}(A * G)$. The first main theorem of the article then is the following:

Theorem 3.1.1. *Let A be a finite-dimensional algebra with an action by a finite group G. Suppose that $\leq$ is a partial order on* $\mathrm{Sim}(A)$ *such that*

$$L < L' \Leftrightarrow gL < hL' \text{ for all} g, h \in G.$$

and $\leq_G$ is the induced G-stable partial order on $\mathrm{Sim}(A * G)$. *Then $(A, \leq)$ is quasi-hereditary if and only if $(A * G, \leq_G)$ is.*

The second main goal of the article, was to study exact Borel subalgebras of quasi-hereditary algebras. In this context the main theorem is the following:

Theorem 3.1.2. *Let $(A, \leq)$ be a finite-dimensional quasi-hereditary algebra with an action by a finite group G such that*

$$L_i < L_j \Leftrightarrow gL_i < hL_j \text{ for all } g, h \in G \text{ and } L_i, L_j \in \mathrm{Sim}(A).$$

Let $\leq_G$ be the induced partial order on $\mathrm{Sim}(A*G)$.

If B is an exact Borel subalgebra of $(A,\leq)$ such that $g(B)=B$ for every $g\in G$, then $B*G$ is an exact Borel subalgebra of $(A*G,\leq_G)$. Moreover, B is regular if and only if $B*G$ is.

In writing this introduction, it seemed likely one may instead assume a slightly weaker compatibility condition for the partial order $\leq$, namely that

$$L_i < L_j \Leftrightarrow gL_i < gL_j \text{ for all } g\in G \text{ and } L_i, L_j \in \mathrm{Sim}(A).$$

In this case $\Delta(gL_i)\cong g\Delta(L_i)$ for every $L_i\in\mathrm{Sim}(A)$, and if $(A,\leq)$ is quasi-hereditary, then there should be an induced partial order $\leq_G$ on $\mathrm{Sim}(A*G)$ such that $(A*G,\leq_G)$ is quasi-hereditary. The following is a sketch of the main adaptations which seem necessary in this setting.

Suppose that A is a bound quiver algebra, and that the G-action on A is induced from a G-action on the quiver Q. Note that we can always make this assumption by [44], and since quasi-hereditary structures transfer along Morita equivalences. Consider the quiver Q' with vertices Q_0, and with the span of the arrow spaces from the vertex i to j corresponding to $\mathrm{Hom}_A(P(L_i),\mathrm{rad}(\Delta(L_j)))\oplus \mathrm{Ext}^1_A(\Delta(L_i),L_j)^*$, where $*$ denotes the k-linear duality. Define a pre-order $\leq'$ by $i\leq' j$ if and only if there is a path from i to j in Q'. Then $\leq'$ is coarser than $\leq$: Clearly, if $\mathrm{Hom}_A(P(L_i),\Delta(L_j))\neq(0)$, then L_i is composition factor of $\Delta(L_j)$, so that $L_i\leq L_j$. On the other hand, if $\mathrm{Ext}^1_A(\Delta(L_i),L_j)$, then there is a non-split extension

$$(0)\longrightarrow L_j\longrightarrow E\longrightarrow \Delta(L_i)\longrightarrow (0).$$

Since L_j is simple, the extension being non-split implies that

$$\mathrm{top}(E)\cong\mathrm{top}(\Delta(L_i))\cong L_i.$$

Since all composition factors of $\Delta(L_i)$ are less than or equal to L_i with respect to $\leq$, and since $\leq$ is adapted, this implies that L_j and L_i are comparable. However, if $L_j\leq L_i$, then E is a quotient of the projective $P(L_i)$ of L_i, with all composition factors being less than or equal to L_i, so that E would be a quotient of $\Delta(L_i)$. Since this is impossible for dimension reasons, we see that $L_j>L_i$. Transitivity of $\leq$ now implies that $\leq'$ is coarser than $\leq$. In particular, $\leq'$ is a partial order. Moreover, it gives rise to the same standard modules as $\leq$. The reason is the following: Denote for every $i\in Q_0$ by $\Delta'(L_i)$ the standard module with respect to $\leq'$. Since $\leq'$ is coarser than $\leq$, $\Delta'(L_i)$ is a quotient of $\Delta(L_i)$ for every $i\in Q_0$. On the other hand, by definition of $\leq'$, each composition factor of $\Delta(L_i)$ is less than or equal to L_i with respect to $\leq'$, so that $\Delta(L_i)$ is a quotient of $\Delta(L_i')$. By finite-dimensionality, they are therefore isomorphic.

Since $\leq'$ is a partial order, there are no cycles in Q' of length at least two. On the other hand, since $(A,\leq)$ is quasi-hereditary, there are no loops in Q', so that Q' is directed and hence $\mathrm{k}Q'$ is a finite-dimensional hereditary algebra. The group G acts on $\mathrm{k}Q'$, and the skew group algebra $\mathrm{k}Q' * G$ is also a finite-dimensional hereditary algebra, and hence Morita equivalent to the path algebra of a quiver Q'_G. Moreover, since $Q'_0 = Q_0$ with the same G-action, the vertices of Q'_G correspond to isomorphism classes of simple $A * G$-modules, so that the directed partial order on Q'_G induces a partial order on $\mathrm{Sim}(A * G)$.

In Article 1 (see also [39]), it is shown that a complete set of representatives of the simple modules of $A * G$ is given by modules of the form $\mathrm{k}G \otimes_{\mathrm{k}H_i} (L_i \otimes V)$, where H_i is the stabilizer of the vertex i, and V is a simple H_i-representation. Using the definition of Q', it should be possible to show that the arrow space of Q'_G between two vertices corresponding to simple modules $\mathrm{k}G \otimes_{H_i} (L_i \otimes V)$ and $\mathrm{k}G \otimes_{H_j} (L_j \otimes W)$ correspond, in turn, the vector space

$$\begin{aligned}\mathrm{Hom}_{A*G}((\mathrm{k}G \otimes_{H_i} (P(L_i) \otimes V)), \mathrm{k}G \otimes_{H_j} (\mathrm{rad}(\Delta(L_j) \otimes W)))\\ \oplus \mathrm{Ext}^1_{A*G}((\mathrm{k}G \otimes_{H_i} (\Delta(L_i) \otimes V)), \mathrm{k}G \otimes_{H_j} (L_j \otimes W))^*.\end{aligned}$$

This means that the modules $\mathrm{k}G \otimes_{H_i} (\Delta(L_i) \otimes V)$ have only composition factors which are less than or equal to their top with respect to the partial order $\leq_G$ induced by Q'_G, and that they have extensions

$$\mathrm{Ext}^1_{A*G}((\mathrm{k}G \otimes_{H_i} (\Delta(L_i) \otimes V)), \mathrm{k}G \otimes_{H_j} (L_j \otimes W))$$

with simple modules only if the simple modules are greater than their top. This implies that the modules $\mathrm{k}G \otimes_{H_i} (\Delta(L_i) \otimes V)$ are the standard modules of $A * G$ with respect to the partial order induced by Q'_G, up to isomorphism. The rest of the proof should be analogous to the proof of [42, Theorem 3.13].

3.2 Article 2

The main aim of this article is to sharpen the previously known uniqueness result for basic regular exact Borel subalgebras.

As mentioned in the introduction, one of the most important results in the area is the existence result up to Morita equivalence, due to Külshammer, König and Ovsienko [33], which gives a construction in terms of A-∞ Ext-algebras. On the basis of this, uniqueness up to isomorphism, which fails in general for exact Borel subalgebras, was shown for regular exact Borel subalgebras. In fact, in [36], Külshammer and Miemietz showed that if $(A, \leq)$ and $(A', \leq')$ are isomorphic quasi-hereditary algebras, with basic regular exact Borel subalgebras B and B' respectively, then there is an isomorphism $\varphi : A \to A'$ with $\varphi(B) = B'$. In their article, it was also conjectured that if $A = A'$, it should be possible to replace this isomorphism by an inner isomorphism. This conjecture is the main theorem of Article 2:

Theorem 3.2.1. *Let $(A, \leq)$ be a quasi-hereditary algebra with two basic regular exact Borel subalgebras B and B'. Then there is an invertible element $a \in A$ with $B' = aBa^{-1}$.*

The proof is based on the techniques used in the existence result in [33].

There is additionally a second part of Article 2, which uses similar techniques but ties thematically into Article 1, in that it is concerned with establishing an existence result in the spirit of [33] for G-invariant regular exact Borel subalgebras of quasi-hereditary algebras with a compatible G-action. It is shown that it is in general not the case that a quasi-hereditary algebra with a compatible G-action has an exact Borel subalgebra such that $g(B) = B$ for all $g \in G$, even if this quasi-hereditary algebra admits a regular exact Borel subalgebra. However, the following theorem holds:

Theorem 3.2.2. *Let $(A, \leq_A)$ be a quasi-hereditary algebra and G be a finite group such that $\mathrm{char}(k)$ does not divide $|G|$, acting on A via automorphisms such that $\leq_A$ is compatible with this action as in [42]. Then there is a Morita equivalent quasi-hereditary algebra $(R, \leq_R)$ with a G-action such that R has a regular exact Borel subalgebra B with $g(B) = B$ for all $g \in B$ and such that the Morita equivalence*

$$F : \mathrm{mod}\, A \to \mathrm{mod}\, R$$

is G-equivariant.

Note that it seems promising that the notion of compatibility here can be extended in the same way as suggested in the summary of Article 1, since the only necessary criterion for the construction in Article 2 is that

$$g\Delta(L_i) \cong \Delta(gL_i)$$

for every $g \in G$ and $L_i \in \mathrm{Sim}(A)$.

3.3 Article 3

While uniqueness and existence properties are much better for regular exact Borel subalgebras, such subalgebras are in general very hard to compute, as can be seen for example in the main example in Article 2 [43, Example 9.3], and indeed not much is known about how to effectively compute any exact Borel subalgebras, regular or not. Motivated by this difficulty, Article 3 investigates monomial quasi-hereditary algebras.

In the context of this thesis, a monomial algebra is an algebra of the form $A = \mathrm{k}Q/I$, where Q is a finite quiver and $I = \langle X \rangle$ is an admissible ideal which is generated by paths in Q. Monomial algebras allow for a notion of path, namely, a path in $\mathrm{k}Q/I$ is an element of the form $p+I$, where p is a path in Q with $p \notin I$, and paths form a basis of $\mathrm{k}Q/I$ such that the product of any two paths is either zero or another path. Moreover, monomial algebras are canonically graded by path length.

Quasi-hereditary monomial algebras were studied previously by Green and Schroll [24], where it was shown that given a monomial algebra $A = \mathrm{k}Q/I$ with a minimal generating set X of I consisting of paths, and a total order $\leq$ on $\mathrm{Sim}(A)$, $(A, \leq)$ is quasi-hereditary if and only if for any path in X, the maximal vertex with respect to the induced total order on the vertices of Q is not an inner vertex of the path. Note that the change from partial to total orders is not a serious restriction, since for any quasi-hereditary algebra $(A, \leq)$, and any refinement of $\leq$ to a total order $\leq'$, $(A, \leq')$ is also quasi-hereditary and has the same standard modules as $(A, \leq)$ [16].

The main theorem of Article 3 is the following:

Theorem 3.3.1. *Let $(A = \mathrm{k}Q/I, \leq)$ be a quasi-hereditary monomial algebra, where $\leq$ is a total order. Let B be the subalgebra of A spanned, as a vector space, by all elements $p+I$, where p is a path in Q such that the maximal vertex of p with respect to the induced total order on the vertices of Q is the terminal vertex of p. Then B is an exact Borel subalgebra of $(A, \leq)$, and it is the unique exact Borel subalgebra with a basis given by paths.*

More precisely, one even obtains a Reedy decomposition in the sense of Dalezios and Šťovíček [14] as follows. Let $(A, \leq)$ and B be as above. Additionally, let C be the subalgebra of A spanned, as a vector space, by all elements $p+I$, where p is a path in Q such that the maximal vertex of p with respect to the induced total order on the vertices of Q is the starting vertex of p. Then multiplication in A induces an isomorphism

$$C \otimes_L B \to A,$$

where $L = (\mathrm{k}Q_0 + I)/I$ is the maximal semisimple subalgebra of A spanned by the trivial paths. Reedy algebras, that is, algebras admitting a Reedy decomposition, were defined by Dalezios and Šťovíček as finite-dimensional algebras whose module categories are k-linear Reedy categories, that is, a k-linear generalization of Reedy categories, and their relationship to the theory of quasi-hereditary algebras is explained in [13]. Reedy categories are of importance in model theory, since the functor category from a Reedy category to a model category inherits a model structure [27, 25]. An analogous result hold for the category of k-linear functors between a k-linear Reedy category and a k-linear category with an abelian model structure by [14, Theorem 7.2].

Additionally to the decomposition result, regularity is studied for the exact Borel subalgebra B above, and an explicit, albeit technical condition for regularity is given in terms of paths in the quiver and the ideal. In case the underlying algebra A is not just monomial, but actually the path algebra of an acyclic quiver, we obtain the following specialization of the regularity result:

Theorem 3.3.2. *Suppose $A = \mathrm{k}Q$ is a finite-dimensional basic hereditary algebra, and $\leq$ is a total order on the vertices of Q. Then $(A, \leq)$ admits a regular exact Borel subalgebra if and only if the following two conditions hold:*

1. *For all paths $p : i \to k$ and $q : j \to k$ with $\max(p) = k > i$ and $j > k$ there is a path $r : j \to i$ such that $q = pr$.*
2. *For any path $q : j \to i$ with $j > i$, there is at most one path p starting in i such that $p : i \to k$ for some $j > k > i$ and $k' < i$ for every other vertex k' that p passes through.*

While this is not always easy to check for a given quiver, we give an application counting the number of quasi-hereditary structures on path algebras of quivers of Dynkin type D and E which admit a regular exact Borel subalgebra. This is similar to a result by Flores, Kimura and Ragnerud [20] which counts all quasi-hereditary structures on path algebras of quivers of Dynkin type D and E, and a result by Thuresson classifying quasi-hereditary structures on path algebras of quivers of Dynkin type A which admit a regular exact Borel subalgebra [48].

3.4 Article 4

The last article in the thesis is the most concrete. Its main aim is to help extend the set of examples of quasi-hereditary algebras with exact Borel subalgebras. Similarly to Article 1, it investigates compatibility of exact Borel subalgebras with a construction on quasi-hereditary algebras, in this case, with tensor algebra constructions.

The construction considered in this case is that of tensor algebras of generalized species. Species were first introduced by Gabriel in his classification of hereditary algebras over perfect fields [21], and later generalized in many different ways to different contexts (see for example [35, 11, 37]).

The notion used in Article 4 is the following: A generalized species consists of a quiver Q together with a collection of finite-dimensional algebras $(A_i)_{i\in Q_0}$, as well as a collection $(M_\alpha)_{\alpha\in Q_1}$ where M_α is a $A_{t(\alpha)}$-$A_{s(\alpha)}$ bimodule. Given a generalized species, one can consider the tensor algebra $T_{\bigoplus_{i\in Q_0} A_i}(\bigoplus_{\alpha\in Q_1} M_\alpha)$.

In the classical case studied by Gabriel, the algebras A_i are all division algebras over a perfect field. In case the field is algebraically closed, which is generally assumed throughout the thesis, the only finite-dimensional division algebra, up to isomorphism, is of course the field itself. In this case, the algebras are all just k, and the bimodules are vector spaces. Choosing a basis for each of these vector spaces, and considering the quiver Q_T obtained from Q by replacing any arrow α with $\dim_{\mathrm{k}} M_\alpha$-many arrows between the same vertices, labelled by the chosen basis elements, one obtains that the tensor algebra $T_{\bigoplus_{i\in Q_0} A_i}(\bigoplus_{\alpha\in Q_1} M_\alpha)$ in this case is nothing but the path algebra of Q_T.

Similarly, if one associates to every vertex of Q the same algebra A, and to any arrow in Q the A-A-bimodule A, then the tensor algebra of the species is isomorphic to the tensor product $\mathrm{k}Q\otimes A$.

Tensor products of quasi-hereditary algebras have been considered before in [9, 6], since they are of interest for wreath product constructions, which appear in the context of Broué's abelian defect group conjecture [7, 8].

Another example, which has been studied before in the context of quasi-hereditary algebras, are triangular matrix rings [51, 15], that is, rings of the form

$$\begin{pmatrix} A_2 & M \\ 0 & A_1 \end{pmatrix} = \left\{ \begin{pmatrix} a_2 & m \\ 0 & a_1 \end{pmatrix} | a_1 \in A_1, a_2 \in A_2, m \in M \right\}$$

where A_1 and A_2 are algebras, and M is an A_2-A_1-bimodule, with multiplication given by matrix multiplication. Such a triangular matrix ring is isomorphic to the tensor algebra of the species

$$A_1 \xrightarrow{M} A_2$$

on the quiver

$$1 \longrightarrow 2.$$

In Article 4, tensor algebras of species are investigated for the case that $(A_i, \leq_i)$ is a quasi-hereditary algebra for every vertex $i \in Q_0$. Similar to the results in [51, 6], one obtains the following result:

Theorem 3.4.1. *Let Q be a quiver and let $((A_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$ be a generalized species on Q, and let $\leq_Q$ be a partial order on Q_0. Suppose that, for every $i \in Q_0$, $\leq_i$ is a partial order on $\mathrm{Sim}(A_i)$ such that $(A_i, \leq_i)$ is quasi-hereditary. Moreover, assume that M_α is projective as a left and right module for every $\alpha \in Q_1$. Let $A := T_{\bigoplus_{i\in Q_0} A_i}(\bigoplus_{\alpha\in Q_1} M_\alpha)$ be the associated tensor algebra. Then there is an induced partial order $\leq$ on $\mathrm{Sim}(A)$ such that $(A, \leq)$ is quasi-hereditary.*

Moreover, exact Borel subalgebras are studied and the following is shown:

Theorem 3.4.2. *Suppose that we are in the setting of the previous theorem, and that, additionally, M_α is projective as a bimodule for every α. Assume that each of the quasi-hereditary algebras $(A_i, \leq_i)$ has an exact Borel subalgebra B_i. Then there is an associated exact Borel subalgebra B of $(A, \leq)$, which is also, up to isomoprhism, given by a tensor algebra of a generalized species.*

Regularity is investigated, and it is shown on a handful of examples that this is unfortunately not well-behaved with this respect to this construction. For example, even if all B_i are regular, and $\mathrm{k}Q$ has a regular exact Borel subalgebra B_Q with respect to $\leq_Q$, the tensor algebra A may not admit a regular exact Borel subalgebra.

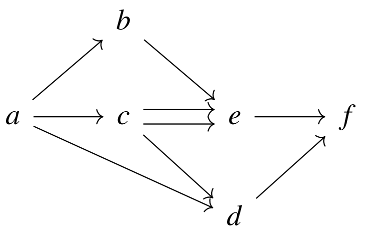
4. Sammanfattning på svenska

Forskningen som avhandligen består av handlar om kvasi-ärftliga algebror och deras exakta Borel delalgebror. En ärftlig algebra är en algebra A sådan att varje delmodul av en projektiv modul är i sin tur projektiv. Om kroppen k är algebraiskt sluten och algebran ändligdimensionell, så är en algebra ärftlig precis om den är Morita ekvivalent med en ändligdimensionell vägalgebra $\mathrm{k}Q$ av ett koger Q. De kan också beskrivas som de algebror vars projektiv dimension är högst ett.

I närmare detalj säger Gabriels struktursats att varje ändligdimensionell algebra över en algebraisk sluten kropp är Morita ekvivalent med en kvot $\mathrm{k}Q/I$ av en vägalgebra $\mathrm{k}Q$ av ett koger Q med ett ideal I som innehålls av spannet av alla vägar som har längd minst två, samt innehåller alla tillräckligt långa vägar. Både Q och $\mathrm{k}Q/I$ är dessutom unika up till isomorfi, och noder $x \in Q_0$ motsvarar isomorfiklasser av enkla A-moduler $L_x \in A\text{-}\mathrm{mod}$, medans mängden av pilar $\alpha \in Q_1$ mellan två noder $s(\alpha)$ och $t(\alpha)$ ger en bas till $\mathrm{Ext}^1_A(L_{s(\alpha}, L_{t(\alpha)})$, där varje pil motsvarar ett baselement [26].

Om Q är ett koger sådan att vägalgebran $\mathrm{k}Q$ är ändligdimensionell, så kan Q inte ha några riktade cykler, eftersom $\mathrm{k}Q$ annars skulle vara oändligdimensionell. Därför kan man definiera en partiell ordning på mängden Q_0 av noder i Q, genom att säga $x \leq y$ om och endast om det finns en väg i Q som börjar i x och slutar i y.

Example 4.0.1. *Om Q är kogret*



så är den partiella ordningen som beskrivits ovan given av

- $a \leq x \leq f$ *för alla* $x \in Q_0$*,*
- $b < e$*,* $c < d$ *och* $c < e$*.*

Notera att partiella ordningen är definierad så att $s(\alpha) < t(\alpha)$ för varje pil $\alpha \in Q_1$. Om A är en ändligdimensionell algebra som är Morita ekvivalent med $\mathrm{k}Q$, så ger Gabriels struktursats en korrespondens mellan partiella ordningar på Q_0 sådana att $s(\alpha) < t(\alpha)$ för varje pil $\alpha \in Q_1$, till partiella ordningar på mängden av isomorfiklasser $\mathrm{Sim}(A)$ av enkla A-moduler sådana att

$$\mathrm{Ext}^1(L_i, L_j) \neq (0) \Rightarrow i < j.$$

En algebra A tillsammans med en partiell ordning $\leq$ på $\mathrm{Sim}(A)$ sådan att

$$\mathrm{Ext}^1(L_i, L_j) \neq (0) \Rightarrow i < j$$

kallas för en riktad algebra $(A, \leq)$. Ordningen på $\mathrm{Sim}(A)$ inducerar en partiell ordning på Q_0, där Q är det associerade kogret som beskrivs i Gabriels struktursats, så att $s(\alpha) < t(\alpha)$ för varje pil $\alpha \in Q_1$. Alternativt kan riktade algebror beskrivas som par bestående av en algebra A och en partiell ordning $\leq$ på $\mathrm{Sim}(A)$, sådana att representanter av $\mathrm{Sim}(A)$ är en exceptionell följd i A-mod med hänsyn på $\leq$.

En exceptionell följd i A-mod är en sekvens $(M_i)_{i \in I}$ av A-moduler där I är en riktad mängd, så att:

1. $\mathrm{End}_A(M_i) \cong \mathrm{k}$ för alla $i \in I$,
2. $\mathrm{Hom}_A(M_i, M_j) = (0)$ för $i \not\leq j$,
3. $\mathrm{Ext}^1_A(M_i, M_j) = (0)$ för $i \not< j$.

I fallet att M_i är icke-isomorfa enkla moduler, så är första och andra kriterium alltid uppfyllda, och tredje kriterium säger då att $(A, \leq)$ är riktad.

Kvasi-ärftliga algebror är en generalisering av riktade algebror, där enkla moduler ersätts av så-kallade standardmoduler. Om A är en algebra och $\leq$ är en partiell ordning på $\mathrm{Sim}(A)$, så definierar vi, för varje enkel A-modul L standardmodulen $\Delta(L)$ associerad till L som den största kvoten av en valt projektiv täcke $P(L)$ av L så att varje kompositionsfaktor L' av $\Delta(L)$ uppfyller $L' \leq L$. Standardmodulen $\Delta(L)$ har toppen $\mathrm{top}(\Delta(L)) \cong L$, eftersom det är en kvot av $P(L)$. Per definitionem finns det därför inga nollskilda homomorfismer $\varphi : \Delta(L') \to \Delta(L)$ for $L' \not\leq L$. Standardmoduler bildar dock inte automatiskt exceptionella följder. Om $(\Delta(L))_{L \in \mathrm{Sim}(A)}$ däremot är en exceptionell följd, och om algebran A har dessutom en filtration via standardmoduler, det vill säga, om det finns en följd av delmoduler

$$(0) = A_0 \subset A_1 \subset \cdots \subset A_m = A$$

av A så att för varje $1 \leq i \leq m$ det finns någon $L \in \mathrm{Sim}(A)$ så att kvoten $A_i / A_{i-1} \cong \Delta(L)$, så kallas $(A, \leq)$ för en kvasi-ärftlig algebra.

Kvasi-ärftliga algebror dyker upp i många representations-teoretiska sammanhang, som till exempel i samband med Liealgebror, speciellt när man undersöker kategori $\mathcal{O}$, och de har bra homologiska egenskaper, till exempel ändlig projektiv dimension.

En exakt Borel delalgebra av en kvasi-ärftlig algebra $(A, \leq)$ är en delalgebra $B \subseteq A$ så att det finns en bijektion $\varphi : \mathrm{Sim}(B) \to \mathrm{Sim}(A)$, så att

1. $(B, \varphi^{-1}(\leq))$ är riktad,
2. $A \otimes_B L \cong \Delta(\varphi(L))$ för varje $L \in \mathrm{Sim}(B)$, och

3. $A \otimes_B -$ är exakt.

Sådana delalgebror har definierats av König [32], inspirerad av Borel delalgebror av Liealgebror och deras roll i kategori $\mathcal{O}$.

I motsats till Borel delalgebror av Liealgebror är existensen av exakta Borel delalgebror inte garanterat i det kvasi-ärftliga fallet. Dessutom är de inte alltid unika up till isomorfi [32]. Däremot bevisade König, Külshammer och Ovsienko existens up till Morita ekvivalens [33]. Det vill säga, de visade att för varje kvasi-ärftlig algebra $(A, \leq)$ finns det en kvasi-ärftlig algebra $(R, \leq_R)$ med en exakt Borel delalgebra $B \subseteq R$ och så att det finns en ekvivalens F : R- mod $\to A$- mod som uppfyller att

$$L < L' \Leftrightarrow F(L) < F(L').$$

Beviset är konstruktivt, och den exakta Borel delalgebran som erhålls är dessutom basisk och reguljär. Basisk betyder att $A \cong \bigoplus_{L \in \mathrm{Sim}(A)} P(L)$ är en summa av icke-isomorfa ouppdelbara projektiva moduler. Reguljär däremot är ett kriterium som kommer från terminologin om så-kallade bocses, vilket sammanhänger med kvasi-ärftilga algebror och exakta Borel delalgebror via konstruktionen i [33], och betyder här att avbildningarna

$$\mathrm{Ext}_B^n(L, L') \to \mathrm{Ext}_A^n(\Delta(\varphi(L)), \Delta(\varphi(L')))$$

som induceras av den exakta funktoren $A \otimes_B -$ är bijektiva för alla $L, L' \in \mathrm{Sim}(B)$ och $n \geq 1$.

Det är dock inte lätt at utföra konstruktionen i [33], eftersom den involverar en A-∞-struktur, det vill säga en struktur som har oändligt många operationer. En stor del av avhandligen handlar om att hitta och undersöka fall där det är lättare att hitta exakta Borel delalgebror, och där det, i bästa fallet, inte är nödvändigt att byta till en Morita-ekvivalent algebra.

Här, Artikel 1 och 4, och delar av Artikel 2, fokuserar på hur man kan hitta exakta Borel delalgebror för vissa kvasi-ärftliga algebror konstruerade från en eller fler andra kvasi-ärftliga algebror med exakta Borel delalgebror. Däremot fokuserar artikel 3 på en speciell klass av kvasi-ärftliga algebror.

Utöver existens, så är unikhet också en viktig fråga i detta kontext, och har undersökts för basiska reguljära exakta Borel delalgebror och för kvasi-ärftliga algebren $(R, \leq_R)$ som har en basisk reguljär exakt delalgebra och är Morita ekvivalent till $(A, \leq)$. Det har visats av Conde att $(R, \leq_R)$ är unik up till isomorfi och av Külshammer och Miemietz att paret $((R, \leq_R), B)$ är unik up till isomorfi, det vill säga att om $(R', \leq_{R'})$ är en till kvasi-ärftlig algebra med en exakt Borel delalgebra B', så att $(R', \leq_{R'})$ är Morita ekvivalent till $(A, \leq)$, då finns det en algebra isomorfism

$$f : R \to R'$$

så att $f(B) = B'$. I Artikel 2 etableras en relaterad resultat, som redan förmodades a Külshammer and Miemietz i [36], nämligen att om $(R, \leq_R)$ är en kvasi-ärftlig algebra med två basiska reguljära exakta Borel delalgebror B och B', då finns det en inverterbar element $a \in R$ med $B' = aBa^{-1}$.

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References

[1] K. Bongartz. Algebras and quadratic forms. *Journal of the London Mathematical Society*, s2-28(3):461–469, 1983.

[2] J. Brundan and C. Stroppel. Semi-infinite highest weight categories. *Memoirs of the American Mathematical Society*, 2018.

[3] T. Brzeziński, S. Koenig, and J. Külshammer. From quasi-hereditary algebras with exact Borel subalgebras to directed bocses. *Bulletin of the London Mathematical Society*, 52(2):367–378, 2020.

[4] T. Bühler. Exact categories. *Expositiones Mathematicae*, 28(1):1–69, 2010.

[5] E. Cartan. Sur la structure des groupes infinis de transformation. *Annales Scientifiques De L Ecole Normale Superieure*, 21:153–206, 1904.

[6] A. Chan. Some Homological Properties of Tensor and Wreath Products of Quasi-Hereditary Algebras. *Communications in Algebra*, 42:2368 – 2379, 2012.

[7] J. Chuang and R. Kessar. Symmetric Groups, Wreath Products, Morita Equivalences, and Broué's Abelian Defect Group Conjecture. *Bulletin of the London Mathematical Society*, 34, 2002.

[8] J. Chuang and R. Rouquier. Derived Equivalences for Symmetric Groups and sl_2-Categorification. *Annals of Mathematics*, 167:245–298, 2008.

[9] J. Chuang and K. M. Tan. Representations of Wreath Products of Algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 3(135), 2003.

[10] E. Cline, B. Parshall, and L. Scott. Finite dimensional algebras and highest weight categories. *Journal für die reine und angewandte Mathematik*, 391:85–99, 1988.

[11] F. U. Coelho and S. X. Liu. Generalized path algebras. In *Lecture Notes in Pure and Applied Mathematics*, 210, pages 53–66. Marcel Dekker, 2000.

[12] T. Conde. All quasihereditary algebras with a regular exact Borel subalgebra. *Advances in Mathematics*, 384:107751, 2021.

[13] T. Conde, G. Dalezios, and S. König. Triangular decompositions: Reedy algebras and quasi-hereditary algebras, 2024. Preprint, arXiv:2411.12912.

[14] G. Dalezios and J. Šťovíček. Linear Reedy categories, quasi-hereditary algebras and model structures. *Advances in Mathematics*, 481, 2025.

[15] R.F. Ochoa de la Cruz, M. Ortiz Morales, and V. Santiago Vargas. Triangular matrix categories over quasi-hereditary categories. *Glasgow Mathematical Journal*, pages 1–22, 2024.

[16] V. Dlab and C.-M. Ringel. The module theoretical approach to quasi-hereditary algebras. In *Representations of Algebras and Related Topics*, volume 168 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1992.

[17] V. Dotsenko, S. Shadrin, and B. Valette. *Maurer-Cartan methods in deformation theory: The twisting procedure*, volume 488 of *Lecture Note Series*. London Mathematical Society, 2023.

[18] M. Doubek, M. Markl, and P. Zima. Deformation theory (lecture notes). *Archivum mathematicum*, 43(5):333–371, 2007.

[19] H. Enomoto. The Jordan-Hölder property and Grothendieck monoids of exact categories. *Advances in Mathematics*, 396, 2022.

[20] M. Flores, Y. Kimura, and B. Rognerud. Combinatorics of quasi-hereditary structures. *Journal of Combinatorial Theory, Series A*, 187:105559, 2022.

[21] P. Gabriel. Représentations indécomposables. In *Séminaire Bourbaki. : vol. 1973/74, exposés 436-452*, number 16 in Séminaire Bourbaki, pages 143–169. Springer-Verlag, 1975. talk:444.

[22] V. E. Govorov. On the global dimension of an algebra. *Matematicheskie Zametki*, 14:399–406, 1973.

[23] J. J. Graham and G. I. Lehrer. Cellular algebras. *Inventiones Mathematicae*, 123:1–34, 1996.

[24] E. Green and S. Schroll. On quasi-hereditary algebras. *Bulletin des Sciences Mathématiques*, 157:102797, 2019.

[25] P. Hirschhorn. *Model categorier and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[26] G. Hochschild. On the structure of algebras with non-zero radical. *Bulletin of the American Mathematical Society*, 53:369–377, 1947.

[27] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[28] C. Jordan. *Traité des substitutions et des équations algébriques.* Gauthier-Villars, Paris, 1870.

[29] T. V. Kadeishvili. On the theory of homology of fibre spaces. *Uspekhi Mat. Nauk.*, 35(3):183–188, 1980.

[30] B. Keller. Chain complexes and stable categories. *Manuscripta Mathematica*, 67:379–417, 1990.

[31] B. Keller. Introduction to A-infinity algebras and modules. *Homology, Homotopy and Applications*, 3(1):1 – 35, 2001.

[32] S. Koenig. Exact Borel subalgebras of quasi-hereditary algebras, I. *Mathematische Zeitschrift*, 220:399–426, 1995.

[33] S. Koenig, J. Külshammer, and S. Ovsienko. Quasi-hereditary algebras, exact Borel subalgebras, A-∞-categories and boxes. *Advances in Mathematics*, 262:546–592, 2014.

[34] S. König and C. Xi. Cellular algebras: Inflations and Morita equivalences. *Journal of the London Mathematical Society*, 60(3):700–722, 1999.

[35] J. Külshammer. Pro-species of algebras I: Basic properties. *Algebras and Representation Theory*, 20, 2017.

[36] J. Külshammer and V. Miemietz. Uniqueness of exact borel subalgebras and bocses. *Memoirs of the AMS*, 2025.

[37] J. Lemay. Valued graphs and the representation theory of Lie algebras. *Axioms*, 1:111–148, 2011.

[38] A. I. Maltsev. On the Representation of an Algebra as a Direct Sum of the Radical and a Semi-simple Subalgebra. *Dokl. Akad. Nauk SSSR*, 36:42–45, 1942.

[39] R. Martínez-Villa. Skew Group Algebras and their Yoneda Algebras.

Mathematical Journal of Okayama University, 43:1–16, 2001.

[40] L. Maurer. Über allgemeinere Invarianten-Systeme. *Münch. Ber.*, 1888.

[41] D. Quillen. Higher algebraic K-theory i. In *Algebraic K-theory, I: Higher K-theories*, volume 341 of *Lecture Notes in Mathematics*. Battelle Memorial Institute, Seattle, Washington, Springer, Berlin, 1973.

[42] A. Rodriguez Rasmussen. Quasi-hereditary skew group algebras. *Nagoya Mathematical Journal*, 257:1–44, 2024.

[43] A. Rodriguez Rasmussen. Uniqueness up to inner automorphism of regular exact Borel subalgebras. *Advances in Mathematics*, 461:110049, 2025.

[44] I. Reiten and C. Riedtmann. Skew Group Algebras in the Representation Theory of Artin Algebras. *Journal of Algebra*, 92(1):224–282, 1985.

[45] J.D. Stasheff. Homotopy associativity of h-spaces, I. *Trans. Amer. Math. Soc.*, 108:275—-292, 1963.

[46] J.D. Stasheff. Homotopy associativity of h-spaces, II. *Trans. Amer. Math. Soc.*, 108:293—-312, 1963.

[47] R. D. Thomason. Higher algebraic K-theories of schemes and of derived categories. In *The Grothendieck Festschrift, vol. III*, volume 88 of *Progress in Mathematics*, pages 247—-435. Birkhäuser Boston, Boston, MA, 1990.

[48] M. Thuresson. Exact Borel subalgebras of path algebras of quivers of Dynkin type A. *Journal of Pure and Applied Algebra*, 228(5):107554, 2024.

[49] J.H.M. Wedderburn. On Hypercomplex Numbers. *Proc. London Math. Soc.*, 2:77–118, 1908.

[50] N. Yoneda. On Ext and exact sequences. *J. Fac. Sci. Univ. Tokyo*, 8:507–576, 1960. Section 1.

[51] B. Zhu. Triangular matrix algebras over quasi-hereditary algebras. *Tsukuba Journal of Mathematics*, 25(1):1–11, 2001.