FLOOR CROSSING FOR LEGENDRIAN CLASP MOVE

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Abstract. We investigate the effect of the clasp move of Legendrian submanifolds of dimension 2 and higher on the moduli spaces of pseudo-holomorphic curves that contribute to the differential of the Chekanov-Eliashberg algebra. As an application, we compute the Chekanov-Eliashberg algebra differential of twist spheres Λ_k of dimension greater than 1. Moreover, we construct an infinite family of Legendrians $\mathcal{T}(\Lambda, k)$ from any horizontally displaceable Legendrian Λ in $P \times \mathbb{R}$, such that the Chekanov-Eliashberg algebra of $\mathcal{T}(\Lambda, k)$ admits an augmentation, while $\mathcal{T}(\Lambda, k)$ has no exact embedded Lagrangian filling of vanishing Maslov class in the symplectization of $P \times \mathbb{R}$ for sufficiently large k.

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1. Introduction

Let $(P, d\eta)$ be an exact symplectic manifold and $P \times \mathbb{R}$ be its contactisation. We call the canonical projection $\Pi_P : P \times \mathbb{R} \to P$ the Lagrangian projection and we consider embedded Legendrian submanifolds Λ of $P \times \mathbb{R}$ so that their Lagrangian projection $L = \Pi_P(\Lambda)$ is an exact closed immersed Lagrangian with transverse self-intersections. The intersection points of L correspond to the Reeb chords of Λ of the Reeb vector field of the contact form $\alpha = dz - \eta$. We denote the set of the Reeb chords $\mathcal{R}(\Lambda)$. The Chekanov-Eliashberg algebra $(\mathcal{A}(\Lambda), \partial)$ is

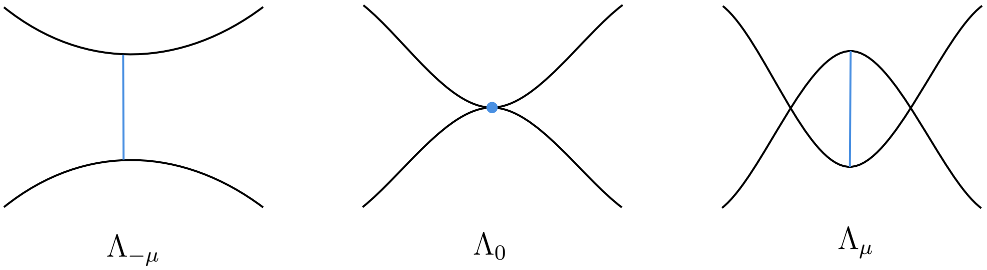


Figure 1. Representation of the (un)clasp move at a Reeb chord in the front projection of a Legendrian knot if read from left to right (vice versa).

a tensor algebra over $\mathbb{Z}_2$ of the vector space spanned by the elements of $\mathcal{R}(\Lambda)$ with grading determined by the Conley-Zehnder index of the chords and differential counting the pseudo-holomorphic polygons in P with boundary on L. Note that the stable tame isomorphism class of $\mathcal{A}(\Lambda)$ is a Legendrian isotopy invariant, see [Che02],[EES05a],[EES05b],[EES05c].

In this paper, we will investigate the change of the differential ∂ under the (un)clasp move, a specific regular homotopy through Legendrian immersions. Let us give a brief description of this move (for more details see Section 2). The action $\mathfrak{a}(c) = \int c^*\alpha$ of the Reeb chord c corresponds to the length of the projection of c onto the z-coordinate. Let us assume that Λ has a Reeb chord s_0 of action 0, meaning Λ is an immersed Legendrian with a transverse self-intersection point s_0. We construct a family of immersed Legendrians Λ_μ so that the family coincides with Λ everywhere except for a small neighbourhood of the self-intersection and so Λ_μ is a Legendrian embedding for all μ except for $\mu = 0$. In the family Λ_μ, we realise the self-intersection s_0 of Λ_0 by contracting a Reeb chord s_μ of Λ_μ, where the sheets with the end-points of s_μ locally pass through each other along the clasp move. For $\mu > 0$, we say that Λ_μ was obtained by a clasp move of $\Lambda_{-\mu}$ centred at s_μ. Conversely, $\Lambda_{-\mu}$ is obtained by performing an unclasp move on Λ_μ centered at $s_{-\mu}$. Figure 1 illustrates the move from left to right; the reverse direction corresponds to the unclasp move.

In general, the (un)clasp move might produce a loose Legendrian out of a non-loose one and vice versa. As an immediate corollary of this fact, the h-principle for loose Legendrians (see [Mur19]) yields, in Section 3.1, bounds on the higher-dimensional Legendrian unknotting number for Legendrian spheres in terms of the self-linking of its Lagrangian projection. And so – in contrast to its smooth equivalent for knots – the higher-dimensional Legendrian unknotting number of Legendrian spheres can attain only a finite number of values for a fixed Thurston-Bennequin invariant, see Proposition 3.2.

An augmentation of $\mathcal{A}(\Lambda)$ is a degree 0 map of differential graded algebras

$$\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0),$$

where $\mathbb{Z}_2$ is concentrated in degree 0. We denote with $Aug(\Lambda)$ the set of augmentations of $\mathcal{A}(\Lambda)$. The clasp move of connected Legendrian knots was studied in [PR19]. There, one considers the immersed Lagrangian cobordism Σ in the symplectization of $P \times \mathbb{R}$ induced by the clasp move from $\Lambda_{-\mu}$ to Λ_μ. Counting certain Ekholm's flow-trees [Ekh07], one defines the so-called immersed cobordism map, and one obtains an algebraic subset I_Σ of $Aug(\Lambda_\mu)$. Note that I_Σ might not, in general, coincide with $Aug(\Lambda_\mu)$. In this paper, we describe how the contributions to the differential of Chekanov-Eliashberg algebra of $\Lambda_{-\mu}$ (of dimension at least two) change under the clasp move. This means that we understand the full change of $Aug(\Lambda_{-\mu})$ to $Aug(\Lambda_\mu)$. It is unknown to the author how the inclusion $I_\Sigma \subset Aug(\Lambda_\mu)$ and the

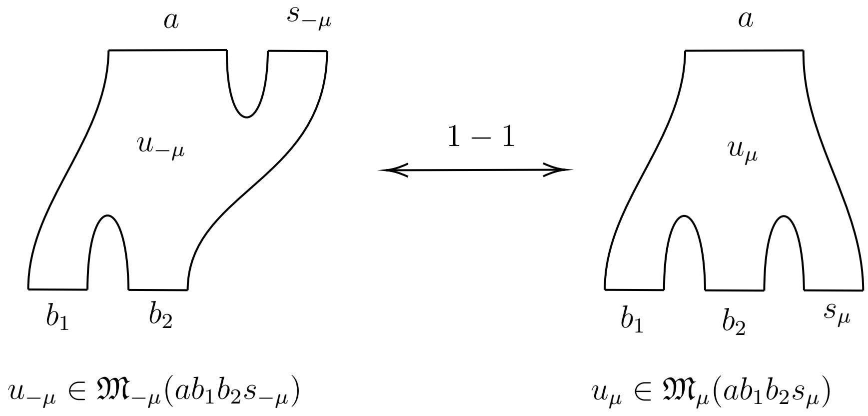


Figure 2. Illustration of the floor-crossing changing the clasping puncture from positive to negative.

variety $Aug(\Lambda_{-\mu})$ relate to each other. Our methods rely on a certain genericity condition for J-holomorphic curves (see Section A.5) that fail in the case of Legendrian knots, and so the full effect of the clasp move on the augmentation variety remains open for $dim\, \Lambda_\mu = 1$.

Let us summarise the effect that the clasp move has on the contributions to the differential of Chekanov-Eliashberg algebra before and after the clasp move; new contributions are coming from the curves with possibly multiple positive punctures asymptotic to the clasping chord, and the old curves with a negative puncture asymptotic to the clasping chord stop contributing.

More precisely, fix μ. Let $\mathbf{w}$ be a word in Reeb chords of Λ_μ containing exactly one letter a, possibly several letters s_μ, and possibly several letters b_i distinct from a and s_μ. We abuse the notation and denote the Reeb chord of Λ_μ and the corresponding intersection point of L_μ with the same symbol. Let $\mathfrak{M}_\mu(\mathbf{w})$ denote the moduli space of pseudo-holomorphic polygons in P with boundary on L_μ and with boundary punctures are asymptotic to the letters of $\mathbf{w}$ in the order prescribed by $\mathbf{w}$. Where all elements of $\mathfrak{M}_\mu(\mathbf{w})$ have

- exactly one positive puncture asymptotic to a,
- negative punctures asymptotic to b_i,
- negative (positive) puncture asymptotic to s_μ if μ is positive (non-positive).

Moreover, the order of letters in $\mathbf{w}$ determines counterclockwise the order of boundary punctures in the domain of the polygon. Consult Section 4.1 for more details and Figure 2 for illustration. We can proceed with the statement of the main theorem of this paper.

Theorem 1.1 (Floor-crossing)**.** *Let $n > 1$ be the dimension of the Legendrian Λ, $\mu > 0$, and $\boldsymbol{w}$ be a word of the Reeb chords of $\Lambda_{\pm\mu}$ so that $\dim \mathfrak{M}_{\mu}(\boldsymbol{w}) = 0$, then the moduli spaces $\mathfrak{M}_{-\mu}(\boldsymbol{w})$ and $\mathfrak{M}_\mu(\boldsymbol{w})$ of rigid curves are compact and diffeomorphic.*

Observe that (un)linking is an obvious example of the (un)clasp move for disconnected Legendrians. Here, Theorem 1.1 follows from [Avd25, Lemma 3.19], where the pseudo-holomorphic curves are counted in the symplectization. The main point is that even though changing the sign of the asymptotic does not fit into any standard analytic framework in the symplectization, the Fredholm index of the problem in the Lagrangian projection does not change as μ varies. One can then relate the curves in the symplectization and in the Lagrangian projection via the lifting procedure of [DR16b]. The proof of [Avd25] relies heavily on the fact that for unlinking, the Lagrangian projection does not change, which is obviously untrue under the clasp move of connected Legendrians.

When dealing with pseudo-holomorphic discs of multiple positive boundary punctures, one can a priori expect the boundary of the moduli spaces to degenerate so that contributions interact with the string topology of the Legendrian, see [CL07], [Ng10], [Duk24], [Avd25]. We rule out these contributions in Section 4.1.

The Lagrangian projections L_μ do not change too much, and so we prove the floor-crossing considering the count of pseudoholomorphic curves in P with boundary on L_0, the Lagrangian projection of the *immersed* Legendrian Λ_0 representing the instance of the (un)clasp move when the action of the Reeb chord that we contract vanishes. The proof follows to a major extent the analytical set-up of [EES05a], however, we have to modify it a bit since the original set-up uses the assumptions of the Legendrian being embedded in several key constructions. After that, the proof boils down to the fact that the Cauchy-Riemann equation on disks with boundary punctures mapping the boundary on the Lagrangian projection of Legendrians forming the (un)clasp move produces a continuous path in the space of Fredholm operators of certain regularity, which substitutes the argument of the invariance of the Lagrangian projection of [Avd25, Lemma 3.19].

As an application of Theorem 1.1, we generalise the computation of Chekanov-Eliashberg algebra of twist knots from [Mey99], see Figure 3, to arbitrary dimension.

Proposition 1.2. *Let $n \geq 1$ and k be a natural number. The Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda_k)$ of $(n+1)$-dimensional Legendrian k-twist sphere Λ_k in $\mathbb{R}^{2n+3}$ has precisely one augmentation ε for $k > 2$. Moreover, the linearised Legendrian contact homology has Poincaré-Chekanov polynomial*

$$P_{\Lambda_k}(t) = \sum_j t^j \dim LCH_j^{\varepsilon}(\Lambda_k) = t^{-k} + t^{n+1} + t^{n+k}.$$

For references on Legendrian twist knots, see [Mey99], [EFM01], [ENV13].

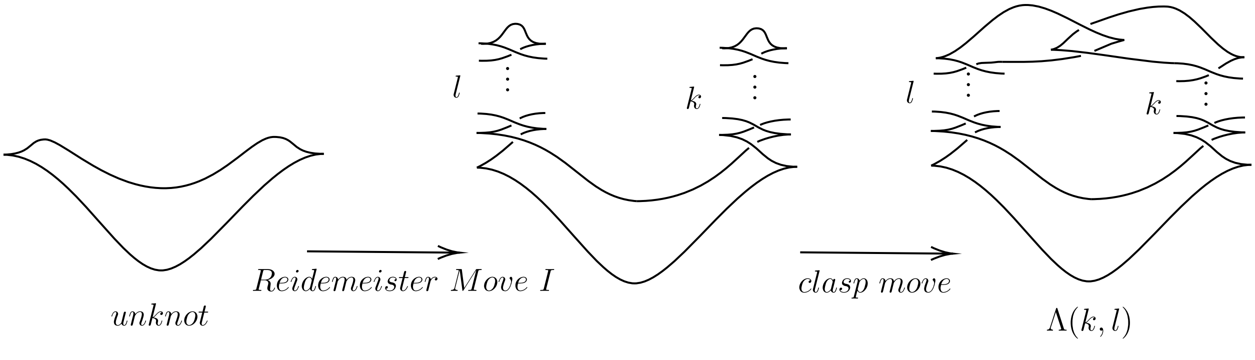


FIGURE 3. Meyer's examples of Legendrian twist knots $\Lambda(k,l)$ in the front projection.

We say that a Legendrian Λ in $P \times \mathbb{R}$ is fillable if $\Lambda = \partial L_\Lambda$ of a cylindrical exact embedded Lagrangian L_Λ with Maslov class 0 in the symplectization $\mathbb{R} \times P \times \mathbb{R}$. The filling L_Λ induces an augmentation ε_{L_Λ} of $\mathcal{A}(\Lambda)$. We say that the augmentation ε_{L_Λ} is a geometric augmentation. If there exists a Hamiltonian isotopy ϕ_H^t in P so that $\phi_H^1(\Pi_P(\Lambda))$ and $\Pi_P(\Lambda)$ are disjoint, we say that Λ is horizontally displaceable. Using the Seidel-Ekholm-DimitroglouRizell isomorphism, see [DR16b], and the duality long-exact sequence of linearised Legendrian homology of horizontally displaceable Legendrian, see [EES09], provide examples of non-geometric augmentations of Legendrian knots. To the best of the author's knowledge, the only examples of non-geometric augmentations of Legendrians of arbitrary dimension were produced in [Gol23], using the results on the spinning construction, and so importing the results from the low-dimensional case to arbitrary dimension. Nevertheless, this approach restricts the possible topological type of the Legendrian to be a product of spheres. Using the ambient Legendrian surgery, see [DR16a], we are now able to produce plenty of non-geometric augmentations; what is more, we produce classes of Legendrians that are not loose but not fillable in arbitrary topological type of arbitrary dimension $n > 1$.

Corollary 1.3. *Let n be an odd number, k a natural number, P an exact symplectic manifold of dimension $2n$ with finite geometry at infinity, and $\Lambda \subset P \times \mathbb{R}$ be a closed embedded horizontally displaceable Legendrian so that its Chekanov-Eliashberg algebra admits an augmentation. Then, there is a class of Legendrians $\mathcal{T}(\Lambda, k)$ so that the following conditions are satisfied.*

- *All $\mathcal{T}(\Lambda, k)$ are diffeomorphic to Λ.*
- *Both Λ and $\mathcal{T}(\Lambda, k)$ have the same rotation class and the same Thurston-Bennequin invariant.*
- *For $k > 2$, all $\mathcal{T}(\Lambda, k)$ are mutually Legendrian non-isotopic and none of them is Legendrian isotopic to Λ.*
- *None of $\mathcal{T}(\Lambda, k)$ is loose.*
- *For $k > 2$, none of $\mathcal{T}(\Lambda, k)$ admits a single augmentation coming from embedded exact Lagrangian filling in $\mathbb{R} \times P \times \mathbb{R}$ of Maslov class 0.*

Acknowledgements. The author thanks his supervisors Frédéric Bourgeois and Luís Diogo for their patience with his first steps involving pseudo-holomorphic curves and for their helpful advice during this project. The author is grateful to Russell Avdek and Milica Đukić for conversations about the pseudo-holomorphic curves with multiple positive ends and words of encouragement. Finally, the author thanks Georgios Dimitroglou-Rizell and Tobias Ekholm for the conversations about their work.

Structure of the paper. In Section 2 we construct the local model of the higher-dimensional clasp-move and prove lemmas controlling basic properties of the clasping chord and provide a contact topological proof of a standard result from differential topology.

In Section 3, we prove the main applications of this paper. Namely, if we obtain a loose Legendrian performing a clasp move, we obtain bounds on Legendrian unknotting number in Section 3.1. For non-loose Legendrians, we provide a generalisation of Legendrian twist spheres from a particular presentation of a Legendrian unknot to arbitrary dimension in Section 3.2 and compute their Chekanov-Eliashberg algebra.

In Section 4 we prove the main result of this paper, that is Theorem 1.1. We start with uniform compactness in Section 4.1, and then sketch the proof of implicit function theorem in Section 4.2.

The details of the Banach manifold set-up and the geometric set-up necessary are included in the Appendix A. Appendix B includes a user's guide to Ekholm's flow trees necessary for computations in Section 3.2.

2. Clasp move

To construct the twist knot, we had to change one overcrossing to an undercrossing. This change is called the clasp move and was first studied in the Legendrian context by Pan and Rutherford in [PR19], see Figure 1. The trace of the clasp move is an example of an exact immersed Lagrangian cobordism with one transverse self-intersection. Here we generalise the clasp move to an arbitrary dimension.

2.0.1. *The local model of the clasp move.* Let Λ be a two-sheeted (non-compact disconnected) Legendrian submanifold of $J^1(\mathbb{R}^n)$ with one sheet being the zero section and the other sheet parametrized as the one-jet graph of the quadratic form

$$h(\mathbf{x}) = -(x_1^2 + \ldots + x_r^2) + (x_{r+1}^2 + \ldots + x_n^2).$$

There is a unique Reeb chord s of action 0 above the origin corresponding to the critical point of index r. Consult Figure 4 for illustration.

We introduce a plateau function $\rho : \mathbb{R}^n \to \mathbb{R}$ to locally shift the zero section up, or down, to resolve the intersection point when $\mu = 0$. So, we define the front projection of the local model Λ_μ via the parametrising functions of the upper and lower sheet, respectively,

$$h_u(\mathbf{x}) = h(\mathbf{x}),$$
$$h_l(\mathbf{x}) = \mu\rho(\mathbf{x}),$$

where ρ is a plateau function that interpolates in between 0 and 1, $\rho(x) = 1$ for $||x|| < \varepsilon$, and ρ is supported on the disc $||x|| < 3\varepsilon$ centered at the projection of the self-intersection point.

The local model contains a Reeb chord for every point of the base over which the tangent planes are parallel, that is, when the following two vectors in $\mathbb{R}^{n+1}$ are co-linear

$$(\nabla h, -1) = (\mu\nabla\rho, -1).$$

This happens over the origin, where we obtain a Reeb chord s_μ of action μ. As we might observe in Figure 4 for L_-, for big values of $|\mu|$, one might introduce new Reeb chords. Equivalently, if the differential of ρ was not chosen carefully enough.

Lemma 2.1. *There is a choice of $\rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ and $\mu_0 > 0$ so that no new Reeb chords of Λ are introduced during the clasp move $L_\mu = L + \mu d\rho_\varepsilon$ for all $|\mu| < \mu_0$.*

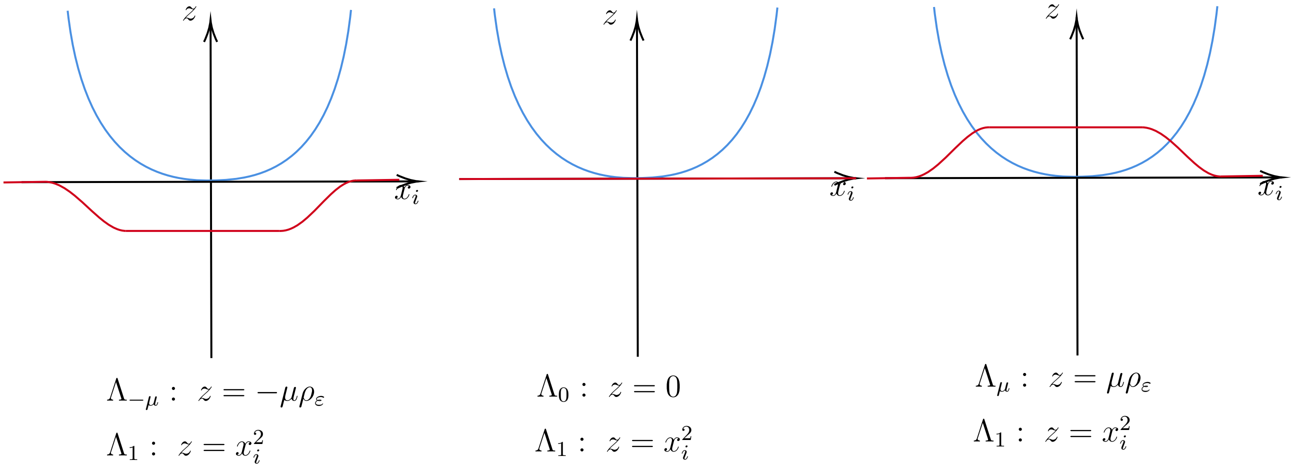
Proof. The new chord would be created when the normal lines to the sheets over the same point were co-linear. That is, $\nabla h = \mu\nabla\rho$ for some $\lambda \neq 0$. We can construct ρ so that the interior of the support of $\nabla\rho$ is A_ε, the annular region $\varepsilon < ||x|| < 3\varepsilon$ in $\mathbb{R}^n$. So that the maximal $||\nabla\rho||$ is attained along the locus $||x|| = 2\varepsilon$. Moreover, as we move along the rays from the origin, the direction of $\nabla\rho$ is constant, but its norm depends on the radial coordinate.

Now, for simplicity, we assume that $||\nabla h|| \neq 0$ except at the origin. Consider the graphs

$$\Gamma_f = \left\{ \frac{(\nabla f(x), -1)}{||(\nabla f(x), -1)||} : x \in A_\varepsilon \right\} \subset A_\varepsilon \times S^{n+1}$$

for $f = h, \mu\rho$. Note that any Reeb chord over A_ε will be the intersection of these two graphs. As the direction is the same on the radial coordinate, we see that we can retract the graphs onto the graphs Γ' over the $(n-1)$-sphere $||x|| = 2\varepsilon$, without the danger of introducing new intersection points, so now $\Gamma'_h, \Gamma'_{\mu\rho} \subset S^{n-1} \times S^{n+1}$. Note that the graph $\Gamma'_{\mu\rho}$ is just

Front projection



Lagrangian projection

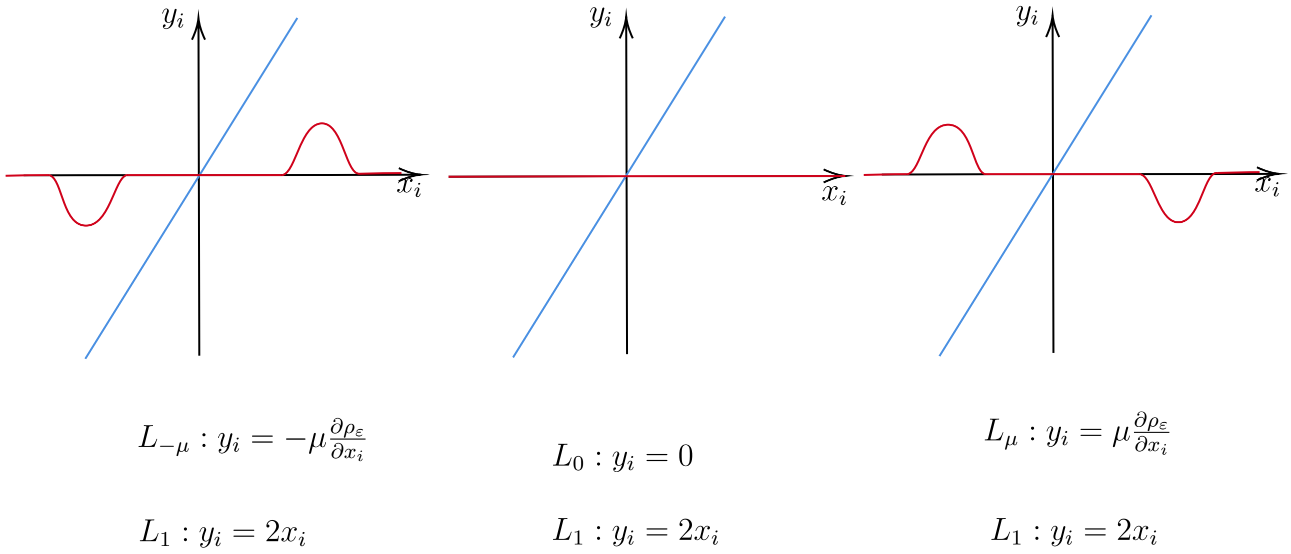


FIGURE 4. Illustration of the local model for $r+1 \leq i \leq n$. The objects with index one are blue and the object with index zero are red.

$S^{n-1} \times (0,\dots,0,-1)$ for $\mu = 0$. And as $|\mu|$ grows, the distance of the points of $\Gamma'_{\mu\rho}$ from $(0,\dots,0,-1)$ grows. Finally, Γ'_h does not hit the point $(0,\dots,0,-1)$. Therefore, there exists a $(n+1)$-disk of points in S^{n+1} that have strictly smaller distance than any point in Γ'_h, which provides us with our bound μ_0. We can embed in this disk as we please without intersecting the Γ'_h.

Observe, that this proof applies as well to the set-up when we have multiple sheets $h_1,\dots,h_k$ that might intersect each other. Then we would take the minimum distance over all the points of $\Gamma'_{h_1} \cup \dots \Gamma'_{h_k}$. □

Definition 2.2. We call $(\Lambda_\mu)_{|\mu|<\mu_0}$ the clasp-move of Λ centered at the chord s_μ, which we call s_μ the clasping chord.

Lemma 2.3. *The action and the Conley-Zehnder index of the clasping chord can be expressed as follows:*

$$\mathfrak{a}(s_\mu) = |\mu| \quad \textit{and} \quad CZ(s_\mu) = \begin{cases} r + \mathfrak{m}(u) - \mathfrak{m}(l), & \textit{if } \mu > 0, \\ (n-r) + \mathfrak{m}(l) - \mathfrak{m}(u), & \textit{if } \mu < 0. \end{cases}$$

Here, $\mathfrak{m}$ denotes the Maslov potential of the sheets in the local model. The upper sheet and lower sheet are denoted with u and l, respectively. In particular, for $\mu > 0$,

$$CZ(s_\mu) = n - CZ(s_{-\mu}), \tag{2.1}$$

$$|s_\mu| = (n-2) - |s_{-\mu}|, \tag{2.2}$$

where for a Reeb chord c the grading is defined as $|c| = CZ(c) - 1$.

Proof follows from [EES05b, Lemma 3.4].

Corollary 2.4. *The Thurston-Bennequin invariant of $\Lambda \subset \mathbb{R}^{2n+1}$ does not change under clasp move if n is even. Moreover,*

$$tb(\Lambda_\mu) = tb(\Lambda_{-\mu}) + 2(-1)^{\frac{(n-2)(n-1)}{2}}(-1)^{|s_{-\mu}|+1}, \ \textit{when } n \textit{ is odd.}$$

Proof. We know that

$$tb(\Lambda) = (-1)^{\frac{n}{2}+1}\frac{1}{2}\chi(\Lambda)$$

when n is even (see [EES05b, Proposition 3.2]). As clasp move does not change the topological type of Λ, it can not change the Thurston-Bennequin invariant.

Now, if n is odd, by [EES05b, Proposition 3.3], we have:

$$tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}} \sum_{c \in \mathcal{R}(\Lambda)} (-1)^{|c|}.$$

So,

$$\begin{aligned} tb(\Lambda_\mu) - tb(\Lambda_{-\mu}) &= (-1)^{\frac{(n-2)(n-1)}{2}}((-1)^{(n-2)-|s_{-\mu}|} - (-1)^{|s_{-\mu}|}), \\ &= (-1)^{\frac{(n-2)(n-1)}{2}}(-1)^{|s_{-\mu}|}((-1)^{(n-2)} - 1), \\ &= -(-1)^{\frac{(n-2)(n-1)}{2}}2(-1)^{|s_{-\mu}|}, \\ &= 2(-1)^{\frac{(n-2)(n-1)}{2}}(-1)^{|s_{-\mu}|+1}. \end{aligned}$$

□

Lemma 2.5. *The clasp move does not change the rotation class.*

Proof. The clasp move produces a regular homotopy through Legendrian immersions. The rotation class is an invariant up to regular homotopy through Legendrian immersions. For more details, see Section 3.3 in [EES05b]. □

The following lemma uses the higher-dimensional version of the Legendrian clasp move to reprove a smooth differential topological statement.

Lemma 2.6 (see [Wu58])**.** *Any two embeddings of M^n a smooth closed n-manifold into $\mathbb{R}^{2n+1}$ are smoothly isotopic if $n > 1$.*

The standard proof strategy involves first proving that the two embeddings are isotopic through immersions, then one modifies the path of immersions so that it passes through embeddings except for a finite number of instances. Finally, if necessary, one modifies this path through immersions using a local model, introducing an auxiliary instance when the path fails to pass through embeddings. The Whitney trick then finishes the proof. We use the clasp move to construct the local model and control the sign of the intersection.

Proof. Obviously, any two continuous maps $M^n \to \mathbb{R}^{2n+1}$ are continuously homotopic. By Whitney Immersion theorem, this continuous homotopy can be upgraded to a regular homotopy $H : M^n \times I \to \mathbb{R}^{2n+1}$ through immersions. Unfortunately, we are one codimension short to use the Whitney embedding theorem. Consider the trace of H in R^{2n+2}. By a standard general position argument, this can be made into an immersion $M^n \times I \to \mathbb{R}^{2n+2}$ with $\mathcal{P}$ finite set of transverse self-intersection points. Recall the Whitney invariant, as algebraic intersection number of H

$$I_W(H) = \sum_{p\in\mathcal{P}} I_H(p) \in \begin{cases} \mathbb{Z}, & \text{if } n+1 \text{ even},\\ \mathbb{Z}_2, & \text{if } n+1 \text{ odd}.\end{cases}$$

By Whitney, H is an isotopy if $I_W(H) = 0$. But H can be modified via the surgery of a Whitney disk carefully so that the surgered manifold is again a trace of family of smooth maps $M^n \to \mathbb{R}^{2n+1}$. Each surgery eliminates two intersection points (with different sign, if defined).

In case $n + 1$ is odd and the cardinality of $\mathcal{P}$ is even, we are done and we have an isotopy.

In case $n + 1$ is even, split $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$ into $\mathcal{P}_+$ the self-intersection points contributing to $I_W(H)$ with positive integer, similarly for $\mathcal{P}_-$. If the cardinalities of $\mathcal{P}_+$ and $\mathcal{P}_-$ are the same, then we are done.

The rest of cases will be tackled by introducing a new self-intersection point (of correct sign). This will be done using contact topology. First choose a neighbourhood of a point and isotope this neighbourhood so that it looks like a plane. Then, rotate the plane so that it is an open subset of a Legendrian plane in $\mathbb{R}^{2n+1}$. Then perform the higher-dimensional equivalent of the Reidemeister I move in $\mathbb{R}^{2n+1}$, consult Figure 5 for low-dimensional examples. Now, we will define two regular homotopies $\Sigma^{\pm}_t$, that vary through Legendrian embeddings except for $t = 0$. Fix the Maslov potential on the lowest sheet to be 0.

We construct the Σ^+ move as the concatenation of several moves (for low-dimensional example see Figure 6):

(1) introduce a small bump on the lowest sheet, this move is an isotopy and so the value of I_W does not change,
(2) move the top side of the bump upwards, until we clasp the sheet of Maslov potential 0 with the other sheet with Maslov potential 0, this is a regular homotopy with precisely one transverse self-intersection point that appears at $t = 0$.

Similarly, the Σ^- move is given by composition (see Figure 6)

(1) introduce a small negative bump on the sheet with Maslov potential 1, this move is an isotopy and so the value of I_W does not change,
(2) move the bottom side of the bump downwards, until we clasp the sheet of Maslov potential 1 with the sheet with Maslov potential 0 below. That is again a regular homotopy with precisely one transverse self-intersection point that appears at $t = 0$.

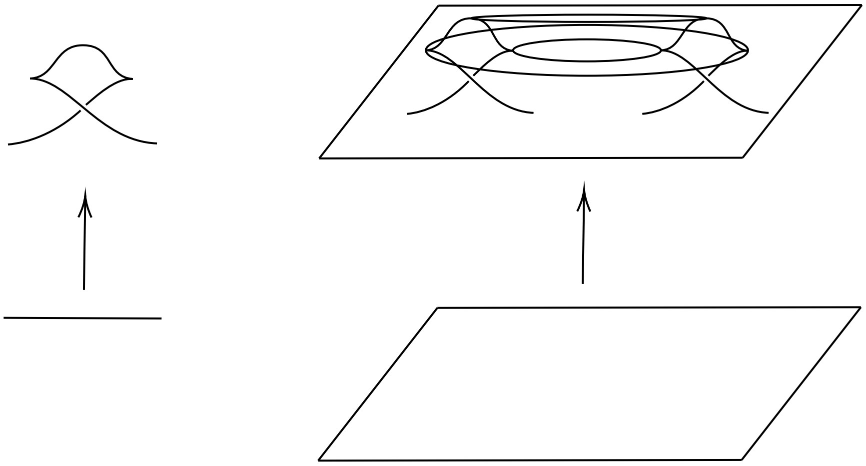
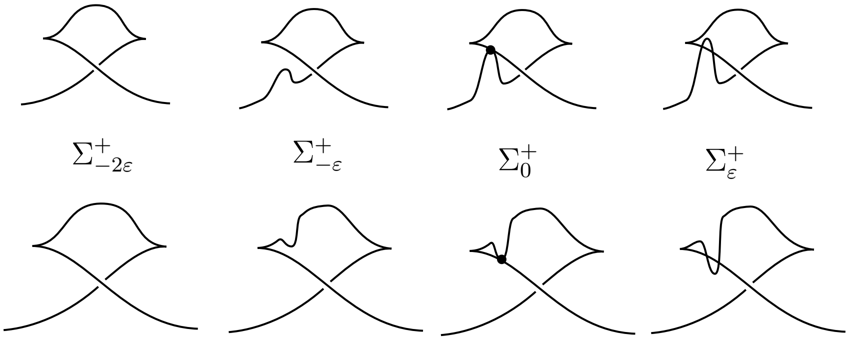


FIGURE 5. The front projection of the Reidemeister I move of a Legendrian knot on the left. On the right, the front projection of a higher-dimensional equivalent of the first Reidemeister move for a 2-dimensional Legendrian in $\mathbb{R}^5$.



$\Sigma^+_{-2\varepsilon}$ $\Sigma^+_{-\varepsilon}$ Σ^+_{0} Σ^+_{ε}

$\Sigma^-_{-2\varepsilon}$ $\Sigma^-_{-\varepsilon}$ Σ^-_{0} $\Sigma^-_{+\varepsilon}$

FIGURE 6. Slices of the clasp moves introducing the additional double point of correct sign

To finish the proof, we have to prove that the moves $\Sigma^\pm$ produce the correct sign of the contribution by the point $p_\pm$ they introduce. We can view the trace of $\Sigma^\pm$ as an exact immersed Lagrangian in $\mathbb{R}^{2n+2}$ with a single intersection point. It was observed in [EES05b, Proposition 3.2] that the Thurston-Bennequin invariant of the Legendrian lift of $\Sigma^\pm$ to $\mathbb{R}^{2n+3}$ is equal to $I_W(\Sigma^\pm)$ and that the contribution of the intersection point is equal to

$$I_H(p_\pm) = (-1)^{\frac{n(n-1)}{2}}(-1)^{|s_\pm|},$$

where $s_\pm$ is the Reeb chord coming from the intersection point $p_\pm$. Now, there is an obvious extension of [PR19, Proposition 8.3] to higher dimension stating that the grading

$$|s_\pm| = \mathfrak{m}(\text{upper sheet before clasping}) - \mathfrak{m}(\text{lower sheet before clasping})$$

is equal to the difference of the Maslov potential of the sheets that we clasped. To conclude, we point out that $|s_+| = 0 - 0 = 0$ and $|s_-| = 1 - 0 = 1$. Note that

$$\frac{(n-1)n}{2} \equiv 0 \pmod 2 \iff n \equiv 0 \text{ or } 1 \pmod 4.$$

So, for $n \equiv 1 \pmod 4$, the auxiliary point $p_\pm$ has the correct sign. In the other case, when $n \equiv 3 \pmod 4$, the $\pm$-notation above has to be flipped. ☐

3. Applications

3.1. **Legendrian Unknotting Number and Loose Legendrians.** As we have shown in Lemma 2.6, there is no smooth knotting of embeddings of S^n in $\mathbb{R}^{2n+1}$ for $n > 1$, and so there can not be any smooth unknotting number. However, we can restrict our attention to Legendrian embeddings as the clasp move provides us with a Legendrian model for the crossing change. We call the Legendrian lift of the Whitney immersion $S^n \to \mathbb{R}^{2n}$ the Legendrian unknot Λ_{un}.

Definition 3.1. Let Λ be a Legendrian n-sphere in $\mathbb{R}^{2n+1}$. Define the (Legendrian) unknotting number $U(\Lambda)$ of Λ as the minimal number of clasp moves or unclasp moves up to Legendrian isotopy one needs to perform on Λ to obtain the Legendrian unknot in $\mathbb{R}^{2n+1}$. If it is impossible to modify Λ by (un)clasp moves to the unknot, we define $U(\Lambda) = \infty$.

We will see that the unknotting number is bounded both from above and below by the self-intersection information of its Lagrangian projection for all the Legendrian spheres with finite unknotting number. Moreover, the clasp move can produce loose Legendrians and, using the h-principle results on the loose Legendrian embeddings, one can show that for a very negative Thurston-Bennequin invariant, the value of the unknotting number provides an obstruction to a Legendrian sphere being loose.

In [Mur19, Proposition A.4], we learn that there is only one formal Legendrian isotopy class of Λ, for each pair of values of tb and rot, where Λ a Legendrian submanifold in $\mathbb{R}^{2n+1}$ for n odd and Λ is a stably parallelizable manifold. In particular, all spheres are stably parallelizable, and so, we denote those formal Legendrian isotopy classes with $\ell(tb, rot)$.

Proposition 3.2. *For* $n \equiv 1 \pmod 4$ *and for any Legendrian sphere* Λ *so that* $rot(\Lambda) = rot(\Lambda_{un}) = 0$*, we have that*

$$\begin{cases} U(\Lambda) = \infty, & \text{if } tb(\Lambda) \text{ is even,} \\ m \le U(\Lambda) \le M, & \text{if } tb(\Lambda) = 2k-1, \end{cases}$$

where

$$m = \begin{cases} |k| & \text{if } \Lambda \text{ is not loose,} \\ |k+1| & \text{otherwise.} \end{cases} \quad M = \begin{cases} |k|+2 & \text{if } \Lambda \text{ is not loose,} \\ |k+1|+1 & \text{otherwise.} \end{cases}$$

In particular, if $U(\Lambda) > |k+1|+1$ *and* $tb(\Lambda) = 2k-1$*, then* Λ *can not be loose. Finally, no Legendrian of* $tb = -1$ *has* $U = 1$*.*

The proposition implies that for every odd Thurston-Bennequin invariant the unknotting number has only 3 possible values, except for $tb = -1$, where there are only two possible values. For better understanding of the bounds see Figure 7.

Proof. Take any Legendrian embedding of a sphere Λ choose an arbitrary point in the front projection which is not on a cusp-edge. Now, perform locally a higher-dimensional equivalent of the Reidemeister I move, and then clasp the upper sheet with the lower one as in the move Σ^- in the proof of Lemma 2.6. Call the composition of the previous two moves a loose clasp (and its inverse a loose unclasp). To see that the loose clasp move produces a loose chart, see [Sen23, Example 10.14].

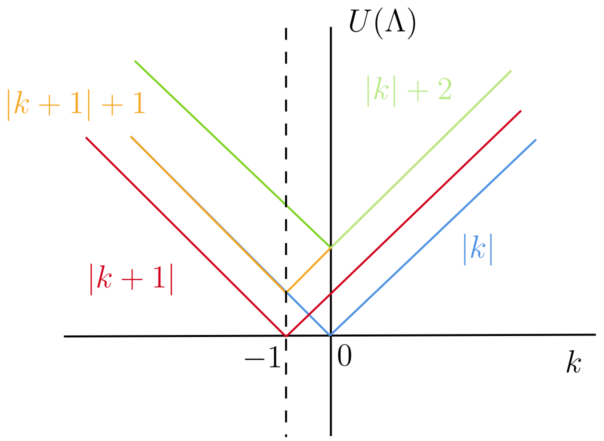


FIGURE 7. Plot of the possible bounds m and M from Proposition 3.2.

In Lemma 2.3 we learn that the loose clasp move around $s_{-\mu}$ of any grading preserves the parity of tb and so, Legendrian spheres of even tb can not be clasped to the Legendrian unknot, so for those, we have defined $U(\Lambda) = \infty$.

Assume that $tb(\Lambda) = 2k - 1$. Now, the loose clasp around Reeb chord of even grading decreases the tb by 2 since $|s_{-\mu}| = 0$ and $\frac{(n-2)(n-1)}{2} \equiv 0 \pmod 2$. Thus, the minimal lower bound on the necessary number of clasp moves must be $|k|$. Therefore, performing a loose clasp on the unknot Λ_{un}, we obtain a loose Legendrian sphere isotopic to $\ell(-3, 0)$. Now, any Legendrian sphere of rotation number 0 falls into one of these two categories:

- loose, in which case isotopic to $\ell(2k - 1, 0)$ and so, isotopic to $\ell(-3, 0)$ after $|k + 1|$ loose clasps,
- not loose, in which case, we perform $|k| + 1$ loose clasps to decrease the tb enough to be able to isotope to $\ell(-3, 0)$.

To get from $\ell(-3, 0)$ to unknot we have to unclasp, this explains the $+1$ contribution to the bound M. ☐

Remark 3.3.

- The assumption $n \equiv 1 \pmod 4$ assures that the loose clasp decreases tb invariant. For $n \equiv 3 \pmod 4$, one obtains an analogous statement and proof, where the loose clasp will increase tb invariant, and we will work with $\ell(1, 0)$ instead of $\ell(-3, 0)$.
- So far, n was assumed to be odd. This assumption is significant since there are, in general, two classes of loose Legendrians for every pair of Thurston-Bennequin invariant and the rotation class if n is even. As commented in [Mur19, Appendix], we do not know how to distinguish these two classes.

The unknotting number is very difficult to compute; nevertheless, one could define an unobstructed unknotting number, where one forbids the regular homotopy through Legendrian immersions to pass through embedded loose Legendrians.

Question 3.4. Is there a method coming from the J-holomorphic curves count on a connected, immersed, Lagrangian with cylindrical ends over embedded Legendrians providing us with a bound from below on the unobstructed unknotting number?

3.2. **Twist Spheres.** We will define a class of Legendrian spheres Λ_k^u in $\mathbb{R}^{2n+3}$ isotopic to the Legendrian unknot in $\mathbb{R}^{2n+3}$ for $n \geq 1$. Then, we will perform a clasp move along a specific chord of Λ_k^u to produce a Legendrian sphere Λ_k. This will generalise the construction of twist knots in $\mathbb{R}^3$ (see Figure 3).

First, we take the Legendrian strand ending with a Reeb chord so that the front projection is as in Figure 8 in $\mathbb{R}^2$ with coordinates x_1, z. That is, we start with opened-up Legendrian unknot of $tb = -1$ and $rot = 0$, then we perform a Reidemeister I move. After that, we modify the sheets of Maslov potential 1 to create two Reeb chords in between them, s^0 and s^1. Then, perform k very small Reidemeister I moves close to the lower end of s^0. Having this one-dimensional front, we rotate the front around the axis given by the line which contains the Reeb chord bounding the Legendrian strand from the right in $x_1, x_2, \dots, x_{n+1}, z$ (for details see [Gol14, Section 2]). Since the tangent vector to the Legendrian strand is flat at the endpoints, the tangent plane to the rotated front is perpendicular to the z-axis, keeping the centre Reeb chord and proving that the higher-dimensional spin creates an embedded Legendrian $(n+1)$-sphere. This would not be true if the slope at the endpoints of the strand were non-zero. We obtain an S^n Morse-Bott family of Reeb chords for each chord in the front projection except for the Reeb chord that coincides with the axis of rotation. We choose a Morse function on S^n with precisely two critical points and perturb the Morse-Bott families. Say that the lowest sheet has the Maslov potential equal to 0. Then, we find the shortest chord s^0, which is the chord starting on the sheet with Maslov potential $k+1$ on the top of the k-times iterated Reidemeister I move and ending on the sheet of Maslov potential 1. This iterated Reidemeister I move makes the Lagrangian projection look like a gluing of two reflected G clefs, see the lower right part of Figure 9. Note that the chord s^0 corresponds to the self-intersection in the centre of Figure 9. We call this particular presentation of Legendrian unknot the k-clef unknot Λ_k^u.

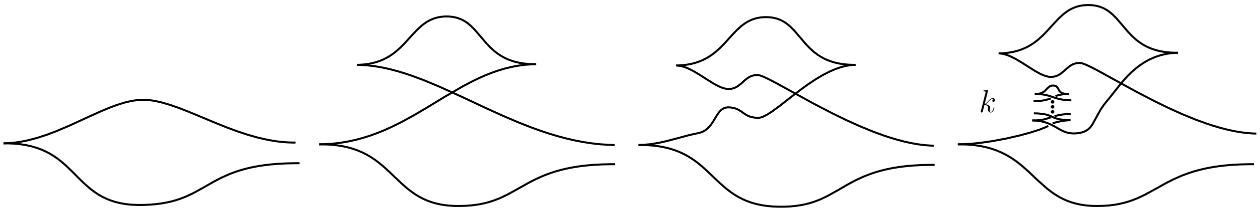


Figure 8. Clef k-sphere

Definition 3.5. The Legendrian $(n+1)$-sphere Λ_k in $\mathbb{R}^{2n+3}$ that is obtained by the clasp move of Λ_k^u centered at s^0 is called the k-twist sphere.

For an example of the front projection of $\Lambda_k \subset \mathbb{R}^5$ in $\mathbb{R}^3$ see Figure 10.

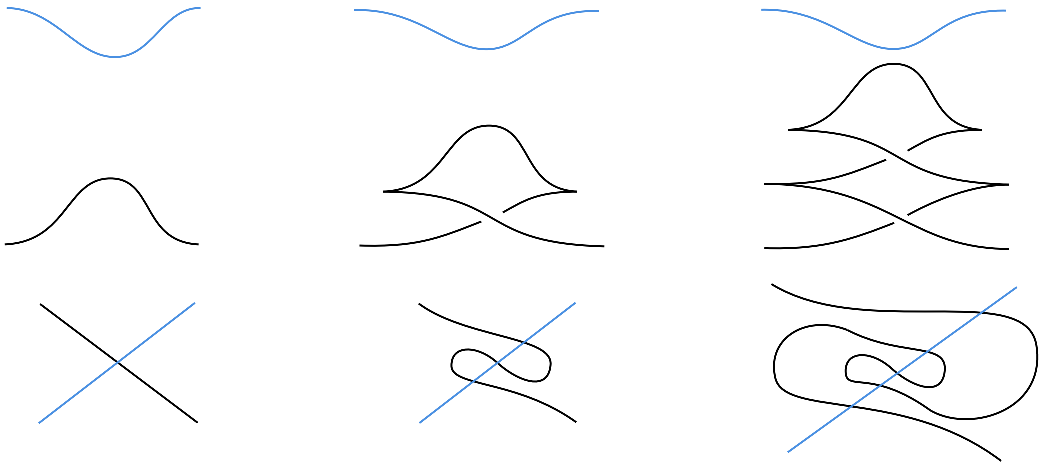


FIGURE 9. Reeb chord creation close to the Reidemeister I (RMI) moves. The upper part shows the RMI in the front projection, and the lower part shows RMI in the Lagrangian projection. We can observe the creation of the intersection points in the Lagrangian projection.

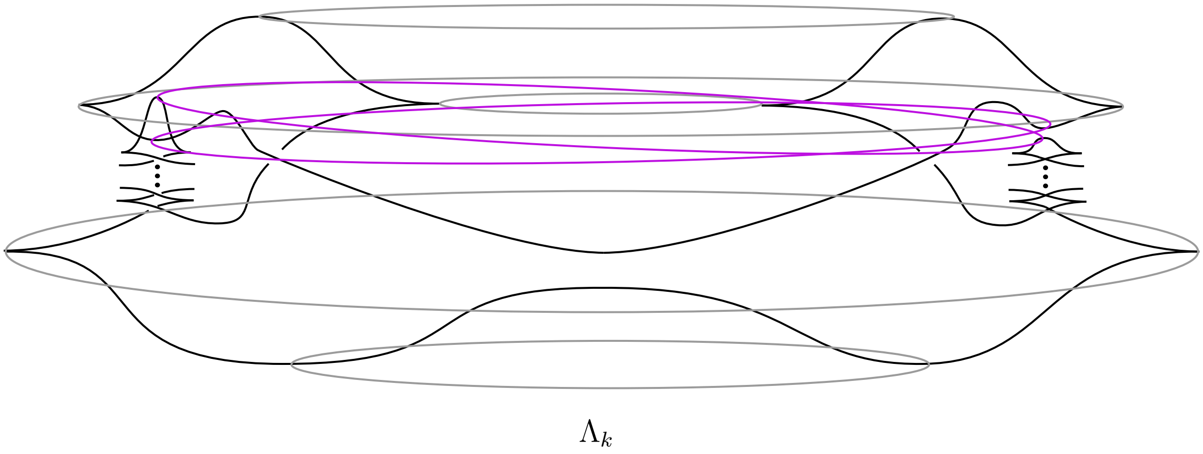


FIGURE 10. Front of twist spheres $\Lambda_k \subset \mathbb{R}^5$. Most of the sheets are rotated about the vertical axis. Two purple sheets exchange the vertical position along the rotation.

Lemma 3.6. *The k-clef sphere has Reeb chords of the following grading*

$$\begin{gathered}
|s^0_{(k+1)1}[i]| = i-(k+1),\ |s^1_{11}[i]| = i,\ |t^0_{m1}[i]| = |r^0_{m1}[i]| = i-m, \\
|c^1_{02}[i]| = i+2,\ |c^0_{01}| = 0,\ |g^1_{01}[i]| = i+1,\ |h^1_{01}[i]| = i+1, \\
|f^0_{01}[i]| = i,\ |f^1_{01}[i]| = i+1,\ |f^0_{12}[i]| = i,\ |f^1_{12}[i]| = i+1, \\
|d^0_{01}[i]| = i,\ |d^1_{01}[i]| = i+1,\ |e^0_{01}[i]| = i,\ |e^1_{01}[i]| = i+1, \\
|d^0_{12}[i]| = i,\ |d^1_{12}[i]| = i+1,\ |e^0_{12}[i]| = i,\ |e^1_{12}[i]| = i+1,
\end{gathered}$$

for $i = 0, n$ *and* $0 \leq m \leq k$.

Moreover, the Chekanov-Eliashberg algebra of the k-clef sphere has the differential given on generators as follows:

The minimal part

$$\begin{aligned}
\partial c^0_{01} &= \partial s^0_{(k+1)1}[0] = 0\\
\partial r^0_{m1}[0] &= \partial t^0_{m1}[0] = \begin{cases} r^0_{(m+1)1}[0] + t^0_{(m+1)1}[0], & \text{if } m < k,\\ s^0_{(k+1)1}[0], & \text{if } m = k.\end{cases}\\
\partial f^0_{m(m+1)}[0] &= 0 \text{ and } \partial f^1_{m(m+1)}[0] = 1 + f^0_{m(m+1)}[0] \text{ for } m = 0,1\\
\partial s^1_{11}[0] &= \partial e^0_{12}[0] = \partial e^0_{01}[0] = \partial d^0_{01}[0] = \partial d^0_{12}[0] = r^0_{11}[0] + t^0_{11}[0]\\
\partial g^1_{01}[0] &= d^0_{12}[0] + d^0_{01}[0]\\
\partial h^1_{01}[0] &= f^0_{01}[0] + f^0_{12}[0]c^0_{01}[0]\\
\partial d^1_{01}[0] &= c^0_{01} + e^0_{01}[0] + s^1_{11}[0]f^0_{01}[0] + (t^0_{11}[0] + r^0_{11}[0])f^1_{01}[0],\\
\partial d^1_{12}[0] &= 1 + e^0_{12}[0] + f^0_{12}[0]s^1_{11}[0] + f^1_{12}[0](t^0_{11}[0] + r^0_{11}[0]).\\
\partial c^1_{02}[0] &= g^1_{01}[0] + h^1_{01}[0] + e^1_{01}[0] + e^1_{12}[0] + f^1_{12}[0]e^0_{01}[0] + e^0_{12}[0]f^1_{01}[0] + f^0_{12}[0]d^1_{01}[0] + d^1_{12}[0]f^0_{01}[0]\\
&\quad + f^1_{12}[0]t^0_{11}[0]f^1_{01}[0] + f^1_{12}[0]r^0_{11}[0]f^1_{01}[0].
\end{aligned}$$

The maximal part:

$$\begin{aligned}
\partial s^0_{(k+1)1}[n] &= \partial f^0_{01}[n] = \partial f^0_{12}[n] = 0\\
\partial r^0_{m1}[n] &= \partial t^0_{m1}[n] = \begin{cases} r^0_{(m+1)1}[n] + t^0_{(m+1)1}[n], & \text{if } m < k,\\ s^0_{(k+1)1}[n], & \text{if } m = k.\end{cases}\\
\partial f^1_{m(m+1)}[n] &= f^0_{m(m+1)}[n] \text{ for } m = 1,2\\
\partial e^0_{m(m+1)}[n] &= \partial d^0_{m(m+1)}[n] = r^0_{11}[n] + t^0_{11}[n] \text{ for } m = 1,2\\
\partial e^1_{m(m+1)}[n] &= e^0_{m(m+1)}[n] + d^0_{m(m+1)}[n]\\
\partial g^1_{01}[n] &= d^0_{01}[n] + d^0_{12}[n]\\
\partial h^1_{01}[n] &= f^0_{01}[n] + f^0_{12}[n]c^0_{01}
\end{aligned}$$

$$\begin{aligned}
\partial d^1_{01}[n] &= e^0_{01}[n] + s^1_{11}[n]f^0_{01}[0] + s^1_{11}[0]f^0_{01}[n] + (t^0_{11}[0] + r^0_{11}[0])f^1_{01}[n] + (t^0_{11}[n] + r^0_{11}[n])f^1_{01}[0]\\
\partial d^1_{12}[n] &= e^0_{12}[n] + f^0_{12}[0]s^1_{11}[n] + f^0_{12}[n]s^1_{11}[0] + f^1_{12}[n](t^0_{11}[0] + r^0_{11}[0]) + f^1_{12}[0](t^0_{11}[n] + r^0_{11}[n])\\
\partial c^1_{01}[n] &= g^1_{01}[n] + h^1_{01}[n] + e^1_{01}[n] + e^1_{12}[n]\\
&\quad + f^1_{12}[n]e^0_{01}[0] + f^1_{12}[0]e^0_{01}[n] + e^0_{12}[n]f^1_{01}[0] + e^0_{12}[0]f^1_{01}[n]\\
&\quad + f^0_{12}[0]d^1_{01}[n] + f^0_{12}[n]d^1_{01}[0] + d^1_{12}[n]f^0_{01}[0] + d^1_{12}[0]f^0_{01}[n]\\
&\quad + f^1_{12}[n](t^0_{11}[0] + r^0_{11}[0])f^1_{01}[0] + f^1_{12}[0](t^0_{11}[0] + r^0_{11}[0])f^1_{01}[n] + f^1_{12}[0](t^0_{11}[n] + r^0_{11}[n])f^1_{01}[0]
\end{aligned}$$

If $k = 0$, then $t^0_{11}[i] + r^0_{11}[i] = s^0_{11}[i]$ in the formulas above.

Proof. First, we identify all the Reeb chords of the clef sphere before the rotation, see Figure 11. The labels in the notation x^a_{cb} correspond to: a is the Morse index of the difference function of the local height functions, c is the Maslov potential of the starting sheet of the Reeb chord, b is the Maslov potential of the sheet where the Reeb chord ends. Now,

- the chord c^1_{02} is the unknot Reeb chord that we start with (see the left-most part of Figure 8),

- the chords g_{01}^1 and h_{01}^1 are the Reeb chords created by the big RMI move (see the part of Figure 8 that is second from the left),
- the chords x_{cb}^a for $x \in \{d, e, f, s\}$ come from the arrangement of the slopes of the front (see the part of Figure 8 that is third from the left),
- the chords x_{cb}^a for $x \in \{t, r\}$ come from the small iterative RMI moves (see the part of Figure 8 that is the last from the left).

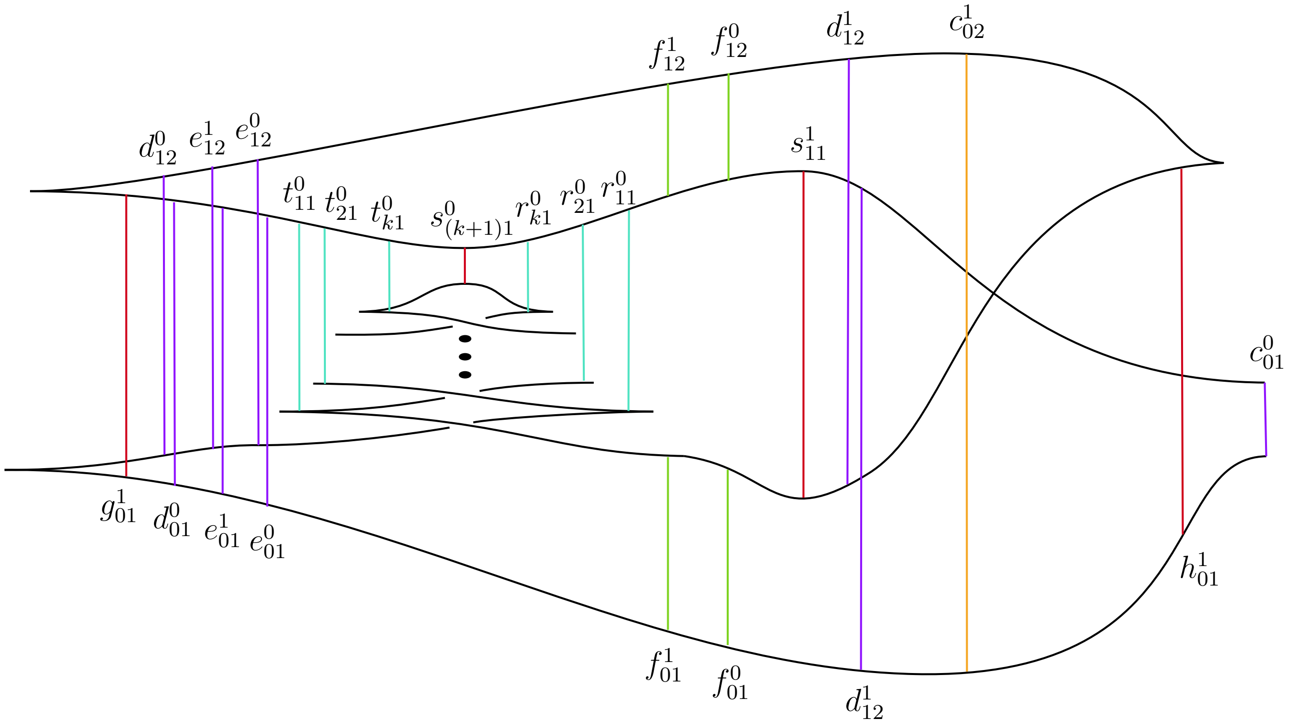


Figure 11. Reeb chords of the clef sphere. Note that this figure is strictly diagrammatic. The top-most and the bottom-most sheets have much bigger slopes in reality.

We perturb out the Morse-Bott loci, the Reeb chords x corresponding to the critical point of index i will be denoted with $x[i]$. We determine the grading of the Reeb chords of Λ_k^u, via the formula

$$|x_{cb}^a[i]| = i + a + b - c - 1,$$

where x is a letter representing a labelling of a Reeb chord, $i = 0, n$ is the index of the critical point of the Morse function used for the perturbation of the Morse-Bott loci and $0 \leq m \leq k$.

We organise the Morse-Bott perturbation so that the minima are close to each other and are separated from the maxima by a hyperplane in $\mathbb{R}^{n+1} \times \mathbb{R}$. This immediately implies that no Reeb chords with $i = n$ can contribute to the differential of a Reeb chord with $i = 0$.

To compute the contributions to the differential of the Chekanov-Eliashberg algebra, we will use Ekholm's flow trees as the rigid flow trees are in bijective correspondence with rigid pseudo-holomorphic curves; for a brief overview, see Appendix B, and for details, see [Ekh07].

As the Legendrian sphere was defined via spinning construction of the front around a sphere S^n, we define an "angular slice" of this Legendrian to be an intersection with the ray given fixed values of all the angles giving us the higher-dimensional spherical coordinates used for spinning. This gives us a half-plane $\mathbb{R}_{\geq 0} \times \mathbb{R}_z$ in $\mathbb{R}^{n+1} \times \mathbb{R}_z$, there the front of the sphere coincides with the one-dimensional front.

Now, we will explain that the flow-tree contributions of the Reeb chords corresponding to the minima of the Morse-Bott perturbation can be seen in the one-dimensional front corresponding to one angular slice. This is because there is a bijection of rigid flow trees.

Note that the Morse-Bott loci of the Reeb chord form a system of concentric spheres S^n in the base $\mathbb{R}^{n+1}$. Before the Morse-Bott perturbation, the Morse-Bott loci have index either 0 or 1 with their trajectories of the local difference functions parallel to the radial direction emanating from the origin. We can easily arrange the perturbation of the Morse-Bott locus to be such that for different local parametrizing functions, the radial trajectories after perturbation intersect the Morse-Bott loci of all other difference functions transversally.

The bijection is given by projection onto one angular slice, as any trajectories leaving the perturbed Morse-Bott locus intersect the Morse-Bott loci of other sheets transversally at a point in the stable manifold of the minimum, inducing a unique and so rigid flow-line to the minimum. The bijection sends the flow-trees of the angular slice with a P_1-vertex (which is a Y_0-vertex with a ghost edge leading to a P_0-vertex, see Figure 16) to flow trees, where this vertex is substituted with a Y_0-vertex and the trajectory contained in the former Morse-Bott locus leading to the P_0-vertex is uniquely determined by the intersection of the radial trajectory with this locus. This means that the ghosts in the angular slice trees are substituted by trajectories in the Morse-Bott loci when lifted to the higher-dimensional front, see Figure 12. The bijection also substitutes the P_0-vertex in the angular slice for a Y_0-vertex and with a P_0-vertex at the Morse-Bott minimum and edge ending at an E-vertex.

The differential of the Chekanov-Eliashberg algebra of Λ_k^u is computed as follows:

$$\partial s^0_{(k+1)1}[0] = 0$$

This is because there is no other chord of lower grading.

$$\partial r^0_{m1}[0] = \partial t^0_{m1}[0] = \begin{cases} r^0_{(m+1)1}[0] + t^0_{(m+1)1}[0], & \text{if } m < k, \\ s^0_{(k+1)1}[0], & \text{if } m = k. \end{cases}$$

We prove the first identity, the second one follows from a symmetric argument. The index of the critical point representing the Reeb chord $t^0_{m1}[0]$ is 0 and so the only possibility is to split the Reeb chord along a sheet that it intersects with combination of P_0 vertex going to Y_0 vertex along a trajectory of length 0. In between the sheets with Maslov potential m and $m+1$, there is a unique trajectory going to the edge. In between the sheet 1 and $m+1$, there is a unique trajectory going to the minimum in between those two sheets, which is the chord $r^0_{(m+1)1}[0]$ or $s^0_{(k+1)1}[0]$ depending on the Maslov potential.

The contribution $t^0_{(m+1)1}[0]$ is given by a Y_1-vertex at the singular edge, the lower part in between sheets of Maslov potential $m+2$ and $m+1$, the lower in between the sheets of Maslov potential $m+1$ and 1.

$$\partial s^1_{11}[0] = t^0_{11}[0] + r^0_{11}[0]$$

The Morse index of the chord is 1 and both emanating trajectories are rigid. The trajectory going towards the axis of rotation loses energy at the intersection locus of sheets with Maslov potential 1. This intersection is not an edge and so the tree is not admissible. When following the other trajectory, Y_1 splitting along the singular edges in between the sheets with Maslov potential 1 and 2 of the small Reidemeister move yields the first contribution. Moreover,

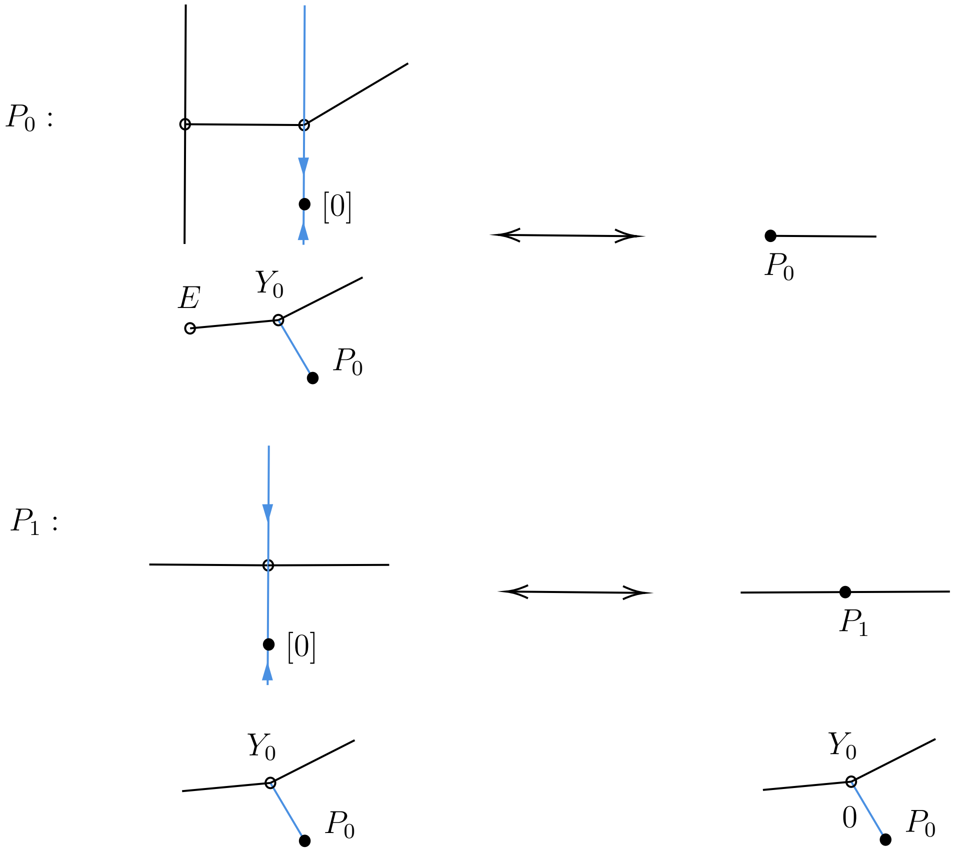


FIGURE 12. The bijection of flow trees of the whole Legendrian (left) and the angular slice (right). The black vertical edge corresponds to the projection of the singular locus of the front projection of the front to the base. The vertical blue lines correspond to the trajectories created by the perturbation of the Morse-Bott locus. The circles correspond to the image of vertices in the base, and the full circles correspond to the P-type vertices.

one has one obvious contribution via going to the minimum in between the sheets of Maslov potential 1, which is the chord $t^0_{11}[0]$.

$$\partial f^0_{m(m+1)}[0] = 0 \text{ and } \partial f^1_{m(m+1)}[0] = 1 + f^0_{m(m+1)}[0] \text{ for } m = 0, 1$$

This is obvious as $f^0[0]$ are minima with no intermediate sheets to split at and f^1 are of Morse index 1. The contributions are given by two trees each of them with two vertices. One tree of two P_0 vertices and the other tree of one P_0 vertex and one E vertex.

$$\partial c^0_{01} = 0$$

The chord is represented by a minimum and there are no intermediate sheets to interact with.

$$\partial e^0_{12}[0] = \partial e^0_{01}[0] = \partial d^0_{01}[0] = \partial d^0_{12}[0] = r^0_{11}[0] + t^0_{11}[0]$$

The contribution comes from splitting the minimal chord along the sheet with Maslov potential 1 with a Y_0 vertex and a ghost trajectory, and then flowing to the minimum in between sheets of Maslov potential 1 along one edge and to the singular edge locus of the front along the other edge. The $t^0_{11}[0]$ contribution comes from a tree with Y_1-vertex along the edge of the sheet with Maslov potential 1 and 2 as in $\partial s^1_{11}[0]$.

$$\partial e^1_{m(m+1)}[0] = e^0_{m(m+1)}[0] + d^0_{m(m+1)}[0] \text{ for } m = 0, 1.$$

The Morse index of $e^1_{m(m+1)}$ is too big to have a rigid ghost splitting. Thus the only two contributions can come from the only two rigid trajectories.

$$\partial g^1_{01}[0] = d^0_{12}[0] + d^0_{01}[0]$$

The Morse index of the chord is 1 and so taking the radial trajectory towards the axis of rotation, we arrive at the minimum $d^0_{01}[0]$ along a rigid trajectory, which yields the $d^0_{01}[0]$ contribution. Now, taking the opposite trajectory, we arrive at the edge, where we perform a Y_0-vertex splitting along the sheet of Maslov potential 1. The lower part of the tree follows to the edge in between the sheet of Maslov potential 0 and the intermediary sheet of Maslov potential 1. The upper part of the tree after splitting flows to the minimum representing $d^0_{12}[0]$.

$$\partial h^1_{01}[0] = f^0_{01}[0] + f^0_{12}[0]c^0_{01}[0]$$

This is analogous to the previous case, with one difference. That is, instead of the edge vertex, we will have a P_0-vertex since we flow with a unique trajectory to the minimum representing c^0_{01}, yielding the quadratic term.

$$\begin{aligned}\partial d^1_{01}[0] &= c^0_{01} + e^0_{01}[0] + s^1_{11}[0]f^0_{01}[0] + (t^0_{11}[0] + r^0_{11}[0])f^1_{01}[0],\\ \partial d^1_{12}[0] &= 1 + e^0_{12}[0] + f^0_{12}[0]s^1_{11}[0] + f^1_{12}[0](t^0_{11}[0] + r^0_{11}[0]).\end{aligned}$$

We will comment on the first differential. First two contributions come from obvious trees ending with a P_0-vertex. The first two quadratic terms come from a tree that is a path with P_0,P_1, and P_0 in this order. The last term comes from a tree with P_1-vertex and then a Y_1-vertex as in the differential of $s^1_{11}[0]$. The second differential follows by analogous argument.

$$\begin{aligned}\partial c^1_{02}[0] = {} & g^1_{01}[0] + h^1_{01}[0] + e^1_{01}[0] + e^1_{12}[0]\\ & + f^1_{12}[0]e^0_{01}[0] + e^0_{12}[0]f^1_{01}[0] + f^0_{12}[0]d^1_{01}[0] + d^1_{12}[0]f^0_{01}[0]\\ & + f^1_{12}[0]t^0_{11}[0]f^1_{01}[0] + f^1_{12}[0]r^0_{11}[0]f^1_{01}[0].\end{aligned}$$

First eight contributions are coming from trees (isomorphic to a path with three vertices). First four have P_0-vertex at $c^1_{02}[0]$, P_1-vertex at the respective chords, followed by the E-vertex. The quadratic terms are given by trees with P_0-vertex at $c^1_{02}[0]$, P_1-vertex at the respective chord, ending with P_0-vertex at the respective chord. The first cubic contributions is given by a tree isomorphic to a path with four vertices. Starting with P_0-vertex at $c^1_{02}[0]$, then two P_1-vertices at the f-chords, and ending with a P_0-vertex at $t^0_{11}[0]$. The last cubic contribution differs by a Y_1-vertex as in the differential of $\partial s^1_{11}[0]$.

The differential of Reeb chords corresponding to the maximum (denoted with $[n]$) is more involved and uses the additional rotation dimensions.

- Constant terms: If there is a constant contribution to the differential of $c[0]$, then there is no contribution to the differential of $c[n]$ because of the grading shift. The only negatively graded Reeb chords are the $r[0], t[0]$-chords and the $s^0[0]$-chord. And for those, it is impossible to obtain the constant term even after the degree shift by n

because we enumerated all possible single positive puncture trees in the computation of their differential in the [0]-case.

- Linear terms: The term c^0_{01} can not contribute to any linear term of the differential of the $[n]$-chords. Either the grading is too big, or it is impossible to find a rigid intersection of flow-lines. The rest of the linear terms are going to be preserved with the n-grading shift since there is a bijection of flow-trees realising the linear contributions to the [0]-chords and the linear contributions of the $[n]$-chords, see Figure 13.

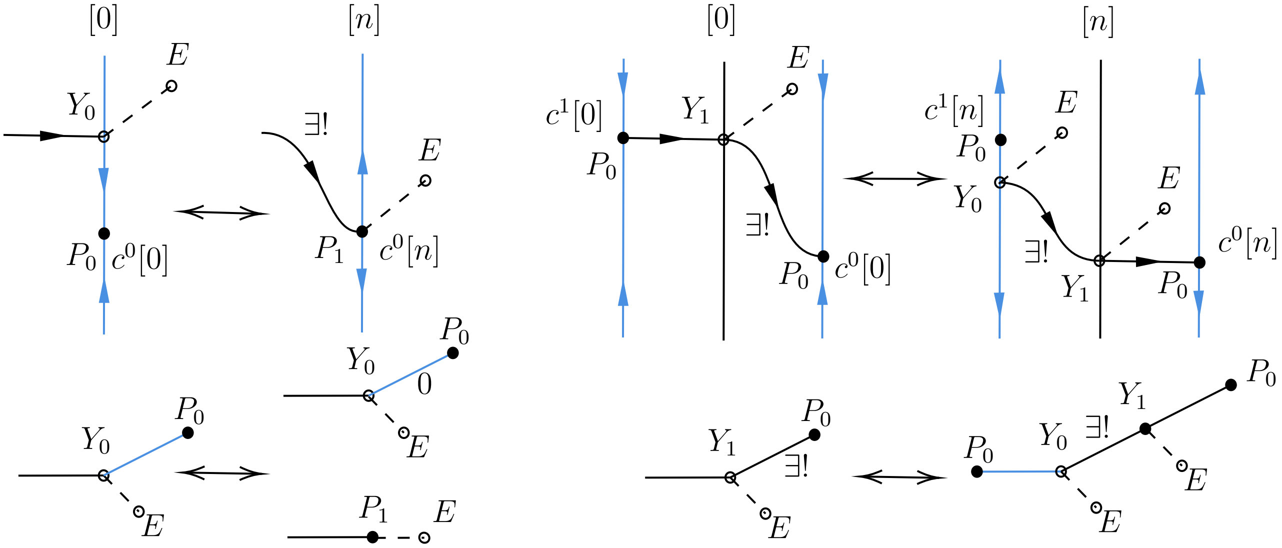


Figure 13. Bijection of the flow trees comparing the trees contributing to the differential of some $c[0]$ and $c[n]$. Here the notation c^i denotes the Reeb chords of the one-dimensional front given by a critical point of the local difference function of index i. The bijection for trees with positive puncture at c^0 is analogous with slightly more complicated trees.

- Quadratic terms: Only quadratic differential contribution to a differential of a [0]-chord containing the chord c^0_{01} is the contribution $f^0_{12}[0]c^0_{01}$ to $\partial h^1_{01}[0]$. Note that there is a point p that is the intersection of the rigid radial trajectory flowing to $f^0_{12}[n]$ and the singular edge of the sheets of Maslov potential 1 and 2. Now, $\partial h^1_{01}[n]$ contains a contribution $f^0_{12}[n]c^0_{01}$ that is given by first flowing along the unstable manifold of the critical point of index $n+1$ representing $h^1_{01}[n]$ to the point p. There we perform a rigid Y_0-vertex and flow to $f^0_{12}[n]$ along the upper branch, and to c^0_{01} along the lower branch that is the unique trajectory passing through p. All other quadratic terms contain only chords coming from Morse-Bott families.

  Any tree representing a quadratic differential must contain a Y_0-vertex, note that P_1-vertex contains a Y_0-vertex and a ghost. This Y_0 vertex must be performed at the $[n]$-part, otherwise the contribution would not have the correct grading due to the degree shift of the positive puncture. This intersection is going to be the intersection of the radial trajectory and an Morse-Bott unstable manifold. This means that for every contribution $c[0]d[0]$ of $\partial a[0]$ one obtains two contributions $c[n]d[0]$ and $c[0]d[n]$ of $\partial a[n]$.

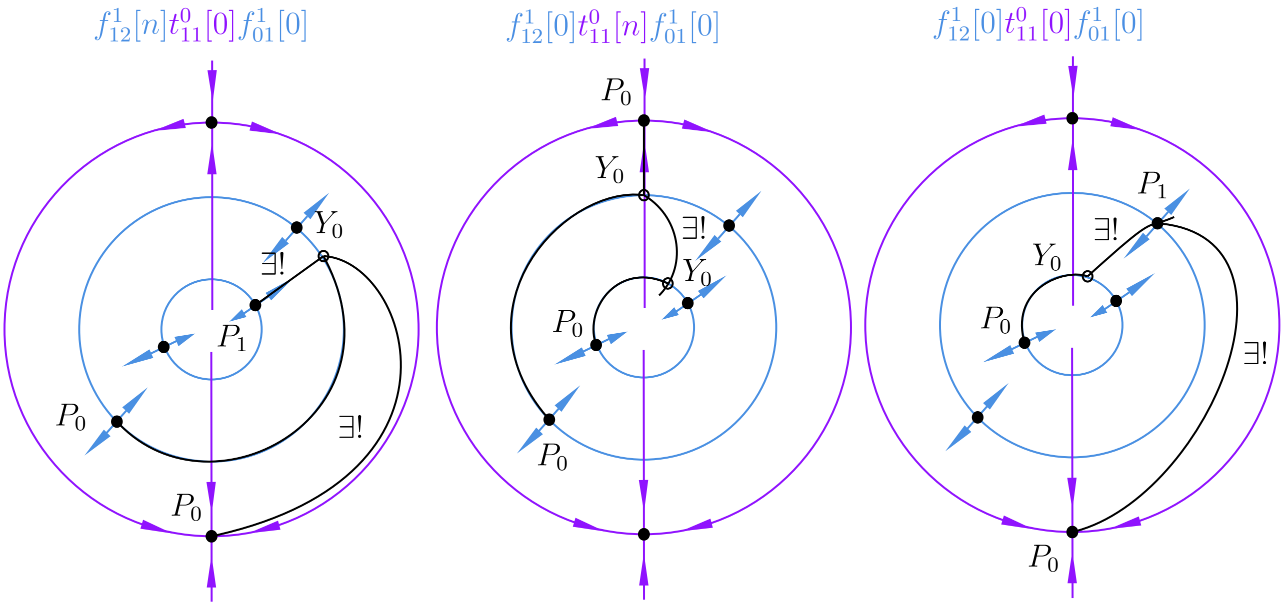


Figure 14. Here, we find some of the trees representing cubic contributions to $\partial c^1_{02}[n]$. This figure depicts the projection of the relevant rigid trajectories of 3 difference functions in between 4 sheets to the base. The circle loci denote the former Morse-Bott loci and the radial trajectories denote the trajectories in the angular slices. The locus drawn in black depicts the resulting flow-tree.

- Cubic terms: Here, a similar statement for trees with two Y_0-vertices in the $[n]$-part could be written. However, the statement would be quite involved because of the interaction of four sheets. Therefore, it is faster to explicitly find the trees. We do so for some of the contributions; the rest is analogous, see Figure 14.

□

3.3. **Floor-crossing and Chekanov-Eliashberg algebra.** The following lemma shows the utility of the floor-crossing in computation of Chekanov-Eliashberg algebra under the clasp move. Once one computes the differential of the original Legendrian before the clasp move one can obtain the differential of the Legendrian after the clasp move extremely easy if certain grading assumption is satisfied.

Lemma 3.7. *Let $\Lambda_{-\mu}$ be a $(n+1)$-dimensional Legendrian and the clasping chord has grading $|s_{-\mu}| = -(k+1)$. We denote $(\mathcal{A}(\Lambda_\mu), \partial_\mu)$ the Chekanov-Eliashberg algebra of Λ_μ. Denote by*

$$M = \max\{|c| : c \in \mathcal{R}(\Lambda_{-\mu})\}.$$

Let Λ_μ be the Legendrian obtained via the clasp move centered around $s_{-\mu}$ as in Definition 2.2 and let $(\mathcal{A}(\Lambda_\mu), \partial_\mu)$ be its Chekanov-Eliashberg algebra. If $M < n+k$, then the differential ∂_μ is obtained from $\partial_{-\mu}$ substituting $s_{-\mu}$ by 0 and defining $\partial s_\mu = 0$.

Proof. Lemma 2.3 implies that $|s_\mu| = n + k$ and Lemma 2.1 imply that the grading of all other Reeb chords of Λ_μ was left unchanged and at most M. Thus s_μ can not be in the image of ∂_μ. Moreover, $\mathfrak{a}(s_\mu) = |\mu|$ is the smallest of all Reeb chords of Λ_μ and so $\partial_\mu s_\mu = 0$.

Using the floor-crossing, see Theorem 1.1, and the correspondence of rigid flow-trees and rigid J-holomorphic curves from [Ekh07], we know that the differential of Λ_k can not obtain any new terms, since the grading of s_μ is too big. Therefore, the differential ∂_μ is given by the same expressions on generators as $\partial_{-\mu}$, except for the contributions containing $s_{-\mu}$, which will not contribute to the differential ∂_μ. ☐

Proof of Proposition 1.2. We can arrange so that the top sheet of Maslov potential $k+1$ and the sheet of Maslov potential 1 are arbitrarily close. Then perform the clasp move with $\Lambda_{-\mu} = \Lambda_k^u$ and $\Lambda_\mu = \Lambda_k$, at the chord $s_{-\mu} = s^0_{(k+1)1}[0]$. Lemma 3.7 implies that the only defining formula for the differential of Λ_k that is going to change is $\partial r^0_{k1}[0] = \partial t^0_{k1}[0] = 0$.

Now, considering the equation

$$\varepsilon \circ \partial = 0$$

one learns via a straight-forward computation that $\mathcal{A}(\Lambda_k)$ has a unique augmentation ε vanishing on all generators except for

$$\varepsilon(f^0_{01}[0]) = \varepsilon(f^0_{12}[0]) = \varepsilon(s^1_{11}[0]) = \varepsilon(c^0_{01}) = 1.$$

Therefore, Λ_k can not be loose. Since the differential is given explicitely, we can linearise it, and a straight-forward linear-algebra computation yields the resulting Poincaré-Chekanov polynomial. ☐

Proof of Corollary 1.3. Take a generic point of Λ that is not an endpoint of a Reeb chord. There are two options

- there is a Darboux ball centered at p so that p is a point on the singular locus of the front projection of Λ the Lagrangian projection $\Pi_P(\Lambda)$,
- if the previous option is not possible, take a small neighbourhood around p and perform a small Reidemeister I move to introduce a singular edge of the local front projection.

Therefore, we can perform the cusp-connected sum of Λ and a small Λ_k that we place in a Darboux ball close to the point p. Let us define

$$\mathcal{T}(\Lambda, k) = \Lambda \# \Lambda_k.$$

The fisrt property is obvious as we perform a connected sum of M with standard sphere S^n. The results of [DR16a, Section 4.5.1] imply that the rotation (Maslov) class does not change since we perform a cusp connected sum of two spheres and the result is a sphere and so simply connected. And since $n+1$ is even, we know (see [EES05b, Proposition 3.2]) that

$$tb(\Lambda) = (-1)^{\frac{n+1}{2}+1}\frac{1}{2}\chi(\Lambda).$$

The results in [DR16a] imply the following. Let $\varepsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ and $\varepsilon'_k : \mathcal{A}(\Lambda_k) \to \mathbb{Z}_2$ be augmentations. We know that the augmentations ε and ε'_k induce an augmentation

$$\varepsilon_k : \mathcal{A}(\mathcal{T}(\Lambda, k)) \to \mathbb{Z}_2,$$

and so $\mathcal{T}(\Lambda, k)$ can not be loose. On the level of linearised Legendrian contact homology, surgery yields a short exact sequence of chain complexes

$$0 \to (\mathbb{Z}_2\langle s\rangle, 0) \to (C_\bullet(\mathcal{T}(\Lambda, k)), \partial^{\varepsilon_k}) \to (C_\bullet(\Lambda), \partial^{\varepsilon}) \oplus (C_\bullet(\Lambda_k), \partial^{\varepsilon'_k}) \to 0.$$

As $|s| = n$, we obtain that, for $\bullet \neq n+1, n$,

$$LCH^{\varepsilon_k}_\bullet(\mathcal{T}(\Lambda, k)) = LCH^{\varepsilon}_\bullet(\Lambda) \oplus LCH^{\varepsilon'_k}_\bullet(\Lambda_k).$$

Thus, $dim(LCH^{\varepsilon_k}_{n+k}(\mathcal{T}(\Lambda,k))) = dim(LCH^{\varepsilon}_{n+k}(\Lambda)) + 1$. As all Poincaré-Chekanov polynomials of $\mathcal{T}(\Lambda,k)$ share this property for all the augmentations of the Chekanov-Eliashberg algebra of $\mathcal{T}(\Lambda,k)$, we obtain the third statement.

Note that as Λ is horizontally displaceable, some small normal neighbourhood of Λ is horizontally displaceable as well, and so $\mathcal{T}(\Lambda,k)$ is horizontally displaceable.

Finally, for the sake of contradiction assume that $\varepsilon_k = \varepsilon_L$ is a geometric augmentation induced by an embedded exact Lagrangian filling L of $\mathcal{T}_k = \mathcal{T}(\Lambda,k)$. The Seidel-Ekholm-Dimitroglou Rizell isomorphism relates the linearised Legendrian contact cohomology and relative homology of the filling (see [DR16b]);

$$LCH^{\bullet}_{\varepsilon_L}(\mathcal{T}_k;\mathbb{Z}_2) \cong H_{(n+1)-\bullet}(L;\mathbb{Z}_2),$$

here $n+1$ is the dimension of the Legendrian and note that the right-hand side is supported in non-negative degrees $0 \leq \bullet \leq n+2$, which forces $LCH^{\bullet}_{\varepsilon_L}(\mathcal{T}_k;\mathbb{Z}_2)$ to vanish for $n+1 < \bullet$ or $\bullet < -3$. The duality long exact sequence of Legendrian contact homology for horizontally displaceable Legendrians was established in [EES09] and reads;

$$\cdots \to H_{\bullet+1}(\mathcal{T}_k;\mathbb{Z}_2) \to LCH^{n-\bullet}_{\varepsilon}(\mathcal{T}_k;\mathbb{Z}_2) \to LCH^{\varepsilon}_{\bullet}(\mathcal{T}_k;\mathbb{Z}_2) \to H_{\bullet}(\mathcal{T}_k;\mathbb{Z}_2) \to \cdots .$$

Thus, for $\bullet < -1$ or $n < \bullet$, observe that we obtain a contradiction as

$$0 \cong H_{\bullet+1}(L;\mathbb{Z}_2) \cong LCH^{n-\bullet}_{\varepsilon_L}(\mathcal{T}_k;\mathbb{Z}_2) \cong LCH^{\varepsilon_L}_{\bullet}(\mathcal{T}_k;\mathbb{Z}_2),$$

and this can not be satisfied for $k > 2$ and $\bullet = -k, n+k$. □

4. Floor-crossing

4.1. **Uniform Compactness.** Let Λ be a closed embedded Legendrian submanifold of $P \times \mathbb{R}$ so that its set of Reeb chords is finite. Let $\mu > 0$ be a real number. Let $s_{-\mu}$ be a fixed Reeb chord of Λ so that it is a clasping chord for a clasp move $(\Lambda_\mu)_{|\mu|<\mu_0}$ providing us with a 1-parametric family of Legendrians Λ_μ. By Lemma 2.1, there is a natural bijection of the sets of the Reeb chords $\mathcal{R}(\Lambda_\mu)$ and $\mathcal{R}(\Lambda_{-\mu})$. We abuse the notation and denote the image and preimage with the same letter, except for the chord s_μ. Write $\mathbf{w}_\mu$ for a word in Reeb chords of Λ_μ for μ fixed so that

$$\mathbf{w} = a\mathbf{b}_1 s_\mu \mathbf{b}_2 \dots \mathbf{b}_{p-1} s_\mu \mathbf{b}_p,$$

where

- a is called the exceptional positive letter,
- $\mathbf{b}_i$ are possibly empty words in Reeb chords distinct from a and s_μ for all $0 < i \leq p$.

Let $\dot{D}_m$ denote a disk with m-boundary punctures, where $m = \ell(\mathbf{w})$, where ℓ denotes the number of letters in $\mathbf{w}$. We denote by $\mathfrak{M}_\mu(\mathbf{w})$ the space of pseudo-holomorphic curves

$$u_\mu : (\dot{D}_m, \partial\dot{D}_m) \to (P, L_\mu)$$

so that each $u_\mu \in \mathfrak{M}_\mu(\mathbf{w})$ has an exceptional positive puncture asymptotic to a and negative punctures asymptotic to all of the other letters of $\mathbf{w}$. More precisely, these curves solve the equation

$$du_\mu + J_\mu \circ du_\mu \circ j = 0$$

for an almost complex structure J_μ on P defined in Appendix A.1.3 and j a conformal structure on $\dot{D}_m$. The punctures are ordered along the boundary counterclock-wise in the same way as in the word $\mathbf{w}$ with the letter a being always the first letter. See Figure 2 for illustration. Similarly, denote by $\mathfrak{M}_{-\mu}(\mathbf{w})$ the moduli space of pseudo-holomorphic curves that have (possibly multiple) positive punctures; precisely one positively asymptotic to the exceptional letter a and the rest of the positive punctures asymptotic to the chord $s_{-\mu}$ as many times as $s_{-\mu}$ appears in the word $\mathbf{w}$ in the same order and position as in the word with chords labeled with positive μ. Note, that there is no configuration where s_μ is both a negative and positive puncture. In that case, our proof of compactness breaks down; see Remark 4.7.

The expected dimension of the moduli space follows from [EES05a, Section 6] and reads;

$$\dim \mathfrak{M}(a_1,\dots,a_j;b_1,\dots,b_k)=(1-j)n+\sum_{r=1}^{j} CZ(a_r)-\sum_{r=1}^{k} CZ(b_r)+\max\{0,j+k-3\}, \tag{4.1}$$

where

- n is the dimension of the Legendrian,
- j is the number of positive punctures,
- k is the number of negative punctures.

Now, Lemma 2.3 yields that $\dim \mathfrak{M}_{-\mu}(\mathbf{w}) = \dim \mathfrak{M}_\mu(\mathbf{w})$ for all $\mathbf{w}$.

For each fixed $\mu > 0$ the moduli space $\mathfrak{M}_\mu(\mathbf{w})$ was proven to be a compact metric space (see [EES05a]) when the disks have precisely one positive puncture using the standard Gromov compactness approach. We will extend this point-wise result to a compactness of the family $\mathfrak{M}_\mu(\mathbf{w})$. For interval I, let

$$\mathfrak{M}_I(\mathbf{w}) = \bigcup_{\mu\in I} \overline{\mathfrak{M}}_\mu(\mathbf{w}).$$

Fix $\mu_0 > 0$ and define $\overline{\mathfrak{M}}_{(0,\mu_0]}$ as the sequential compactification, that is all the possible limits of sequences of J_k-holomorphic curves $u_k \in \mathfrak{M}_{\mu_k}(\mathbf{w})$ where $\mu_k \to 0$, where we denoted $J_k = J_{\mu_k}$. In the limit, the action of s_μ goes to zero, thus, we might obtain broken ghost bubbles in the limit. These are components of the broken configuration of symplectic area equal to 0.

For disks $u \in \mathfrak{M}_{-\mu}$ with multiple positive punctures, one can import most of the standard arguments proving Gromov compactness results from the single positive punctured case, however, there is a class of so-called *pinched* configurations that one has to consider as a possible limit.

Definition 4.1. Let $\dot{D}_m$ be a disc with m boundary punctures. Let γ be a curve $\gamma(s) : [0,1] \to \dot{D}_m$ so that $\gamma(s) \in \partial\dot{D}_m$ for $s = 0,1$ and $\gamma(s) \in \mathrm{int}\dot{D}_m$ for $s \in (0,1)$ and so there is a positive puncture in each component of $\dot{D}_m \setminus \gamma([0,1])$. And let $\varphi_t : \dot{D}_m \to \dot{D}_m$ be a family of continuous maps that contract the curve γ and so that φ_t is a diffeomorphism for $t \in [0,1)$. We say that a (J_∞, j_∞)-holomorphic curve u is a pinched configuration if there is a (J_0, j_0)-holomorphic curve u' and a sequence $k \to \infty$ so that $(J_k, \varphi^*_{t_k} j)$-holomoprhic curves $u_k = u' \circ \varphi_{t_k}$ converge to u in Gromov topology as $t_k \to 1$.

For example, see Figure 15.

As above, we define $\overline{\mathfrak{M}}_{[-\mu_0,0)}$. Here, one has to prove that

- For $\mu > 0$ fixed, the pinched configurations do not appear in the broken limits $\overline{\mathfrak{M}}_{-\mu}$.
- The neither pinched configurations nor ghosts appear in the limit as $\mu_k \to 0$.

We can consider a configuration of punctures on the boundary of the disk so that multiple consecutive punctures are asymptotic to s^*,the double point corresponding to the Reeb chord s_μ, then a bigger portion of the curve might enter the ball $B^h_r(s^*)$ in the clasping region,

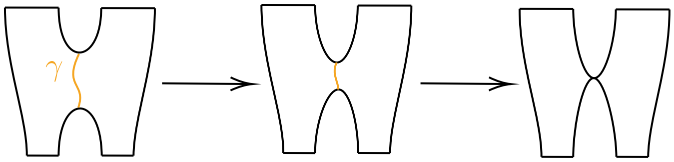


FIGURE 15. Example of a pinched configuration

where h is a metric on P (with finite geometry at infinity) compatible with ω and the almost complex structure J. We can prevent this by shrinking the ball so that it does not intersect the complement of the image of the asymptotic neighbourhood of the puncture. The key step in the proof of the following lemma is to realise, that the radius of the ball is bounded from below. That provides us with the information of how many times the curve leaves the neighbourhood.

Lemma 4.2. *Let $r > 0$ be a radius small enough so that $B_r(s^*)$ contains only local model of the clasp move, that is, only two transversely intersecting Lagrangians L_0 and L_1. For this r and any word $\boldsymbol{w}$ and every $u \in \overline{\mathfrak{M}}_\mu(\boldsymbol{w})$ it holds that $u^{-1}(\partial B_r) \cap \partial \dot{D}$ has at least $2k$-points for k the number of punctures asymptotic to s^*.*

Proof. Step 1: Fix $\mu \neq 0$ and let $u_k \to u_\infty$ be given as an uniform limit of elements in $\mathfrak{M}_\mu$. Then for all $k < \infty$ there is a point $p_k \in [p_j, p_{j+1}]$ such that $p_k \notin u^{-1}(B_r)$.

For the sake of contradiction assume that there is $k < \infty$ so that for all points $p \in [p_j, p_{j+1}]$ the points are in the pre-image $u_k^{-1}(B_r)$. Clearly, $u([p_j, p_{j+1}]) = s^*$, otherwise, there is a point p distinct from s^* on the component of the boundary joining p_j and p_{j+1}, call image of this curve γ. This means that γ starts at s^* and follows the Lagrangian L_0 (or L_1) in $B_{r_k}(s^*)$ and before it ends at s^* it follows the Lagrangian L_1 (or L_0) in $B_{r_k}(s^*)$ with no corner in between. Since γ is continuous, then $u(p)$ would have to lie on both L_0 and L_1, which is impossible since $(L_0 \cup L_1) \setminus \{s^*\}$ is disconnected. And so, γ has to leave and enter the neighbourhood $B_r(s^*)$. But this implies $u_\infty([p_j, p_{j+1}]) = s^*$ since u_∞ is a uniform limit. Therefore, the component of u_∞ containing $[p_j, p_{j+1}]$, denoted u_c, must be constant, thus $\text{Area}(u_c) = 0$.

Now, if all s^*-punctures are positive, then the constant component can not have a negative puncture. If it had a negative puncture asymptotic to an intersection point b different from s^*, then u would have to escape the neigbourgood of s^* and so could not be constant. If the negative puncture were s^*, then u_c would have to have at least two positive punctures and so $\text{Area}(u_c) \geq \mu > 0$.

If all s^*-punctures are negative, then the positive puncture must be distinct from s^* because of action reasons. But then the curve must leave a neighbourhood of s^* and so can not be constant.

Step 2: Let $\mu_k \to 0$ for $\mu_k \neq 0$, and $u_k \to u_\infty$ be a uniform limit for $u_k \in \mathfrak{M}_{\mu_k}$. Then for all $k \leq \infty$ there is a point $p_k \in [p_j, p_{j+1}]$ such that $p_k \notin u^{-1}(B_r)$.

Let $\mu_k \to 0$. By uniform convergence, if $u_\infty([p_j, p_{j+1}]) \in B_r(s^*)$, then for k big enough $u_k([p_j, p_{j+1}]) \in B_r(s^*)$. Consequently, we already obtain a ghost breaking for $\mu_k \neq 0$, which was precluded in *Step 1*. □

Lemma 4.3. *There is a* $\hbar > 0$ *so that for every word* $\boldsymbol{w}$ *all* $u \in \overline{\mathfrak{M}}_I(\boldsymbol{w})$, *for* $I = [0, \mu_0)$ *or* $I = (-\mu_0, 0]$, *satisfy*

$$Area(u) \geq k\hbar,$$

where k *is the number of letters of* $\boldsymbol{w}$ *coinciding with* s^*.

Proof. Let $r > 0$ be the radius of $B_r(s^*)$ as in Lemma 4.2. We can assume that there is a constant $C > 0$ that depends on the explicit plateau function realizing the clasp move so that

$$(L_0 \cup L_1) \cap B_{\frac{r}{C}}(s^*)$$

are real analytic. This means that there is a sphere S on L_0 so that $S = L_0 \cap \partial B_{\frac{r}{2C}}(s^*)$ so that its neighbourhood is analytic. Now, denote with θ the angle in between Lagrangian planes L_0 and L_1, then for all R satisfying

$$R < \frac{r}{2C} \cos\theta$$

a ball of radius R centered at a point q of S intersects only the sheet L_0 and not the other sheet L_1.

Again thanks to Lemma 4.2, we know that we can choose asymptotic charts on the domains of the pseudo-holomorphic disks, so that the preimage $u^{-1}(B_r(s^*))$ is not going to contain a non-compact section of these charts. Now, the fact that the boundary condition is real analytic, we can double our J-holomorphic curve. Assume that our J-holomorphic curve $u : \mathbb{R} \times [0, 1] \to \mathbb{C}^n$ maps the upper line of the domain onto the Lagrangian L_0,

$$u((\mathbb{R} \times \{1\}) \cap u^{-1}(B_R(q))) = L_0 \cap B_R(q),$$

the other boundary condition analogously. Because the Lagrangian is real analytic and the u is a holomorphic curve we use the Schwartz reflection principle and extend the curve to its double

$$u^d(\tau + it) = \begin{cases} u(\tau + it) & \text{for } 0 \leq t \leq 1, \\ -\bar{u}(\tau + i(2 - t)) & \text{for } 1 < t \leq 2. \end{cases}$$

To see that the double is holomorphic (thus smooth by elliptic regularity) we refer the reader to [EES05a, Lemma 6.2]. Note that we can fix the almost complex structure and background metric in $B_r(s^*)$ to be the standard Hermitian pair.

$$\text{Area}(u^d) = \int_{\mathbb{R}\times[0,1]} (u^d)^*\omega = \frac{1}{2} \int_{\mathbb{R}\times[0,1]} |\partial_\tau u^d| + |\partial_t u^d| \; d\tau \wedge dt$$

Now, a straightforward computation yields that

$$(|\partial_\tau u^d| + |\partial_t u^d|)(\tau + i(2 - t)) = (|\partial_\tau u| + |\partial_t u|)(\tau + it)$$

and the symmetry of the domain now implies $\text{Area}(u^d) = 2\text{Area}(u)$.

Now the monotonicity lemma used for double u^d whose image contains q and on the interior of the strip intersects the ball $B_R(q)$ yields the estimate $\text{Area}(u^d) \geq C'' R^2$. Finally, this yields

$$\text{Area}(u) \geq \frac{C''}{2} R^2 = \hbar.$$

The proof is finished once one observes that for every puncture asymptotic to s^* we have one such q as above. □

Lemma 4.4. *There is $\mu_0 > 0$ so that for all $|\mu| < \mu_0$, only finitely many words $\boldsymbol{w}$ can be realised in the clasp move by a pseudo-holomorphic curve:*

- *either having all the punctures asymptotic to s^* negatively,*
- *or having all the punctures asymptotic to s^* positively.*

Proof. Let M be the biggest action of a Reeb chord of Λ_μ, then any curve with k positive punctures asymptotic to s^* has area

$$M + k|\mu| \geq \text{Area}(u) \geq k\hbar, \tag{4.2}$$

where the second inequality follows from Lemma 4.3. Choose $|\mu| < \hbar$, then inequality (4.2) is satisfied only for finitely many $0 \leq k$. Proof for k negative punctures follows similarly via the inequality $M \geq k\hbar$. ☐

Corollary 4.5. *The pinched pseudo-holomorphic curves are not a part of the compactification of curves $\overline{\mathfrak{M}}_{-\mu}(\boldsymbol{w})$ with multiple positive punctures for fixed $\mu > 0$ small enough.*

Proof. Assume that we have a limit of pseudo-holomorphic curves that converges to a pinched configuration, then we can choose $\mu = \mathfrak{a}(s_\mu)$ strictly smaller than $\hbar$ that we obtained in Lemma 4.3. As pinched component has k positive punctures at s_μ area is at most $k\mu$ by Stokes'. Thus, we obtain a contradiction

$$k\hbar \leq \text{Area}(u_{pinch}) = k\mathfrak{a}(s_\mu) < k\mu.$$

☐

Corollary 4.6. *Ghosts do not appear in the limit as $\mu_k \to 0$.*

Proof. This follows by *Step 2* of Lemma 4.2. ☐

Remark 4.7. Note that the assumption that the J-holomorphic curve in our moduli spaces can have only one sign of the asymptotic marker at all punctures asymptotic to the clasping chord is crucial in the proofs above. Otherwise, one can not, in general, prevent the appearance of ghosts and pinched configurations leading to beautiful connections with string topology, see [CL07], [Ng10], [Duk24], [Avd25].

4.2. **Analytic sketch.** At this point, we know that $\mathfrak{M}_{[-\mu_0,0)}(\mathbf{w})$ and $\mathfrak{M}_{(0,\mu_0]}(\mathbf{w})$ can be compactified. However, it is not obvious that these are 1 dimensional manifolds with boundary and that the compactifications match over $\mu = 0$ and give rise to a moduli space, which is a 1 dimensional manifold.

Proposition 4.8. *For $\mu_0 > 0$ denote*

$$\mathfrak{M}_{[-\mu_0,\mu_0]}(\boldsymbol{w}) = \overline{\mathfrak{M}}_{[-\mu_0,0)} \cup \overline{\mathfrak{M}}_{(0,\mu_0]} \quad \text{and} \quad \mathfrak{p} : \mathfrak{M}_{[-\mu_0,\mu_0]}(\boldsymbol{w}) \to [-\mu_0, \mu_0],$$

where $\mathfrak{p}(u) = \mu$ if $u \in \overline{\mathfrak{M}}_\mu(\boldsymbol{w})$. There is a $\mu_0 > 0$ so that if $\dim \mathfrak{M}_\mu(\boldsymbol{w}) = 0$ for all $|\mu| \leq \mu_0$, then the projection map $\mathfrak{p}$ is a local diffeomorphism onto $[-\mu_0, \mu_0]$.

Now, it is clear that Theorem 1.1 directly follows from the Proposition 4.8.

The strategy of the proof is as follows. Assume that we have a disc in the moduli space over $\mu = 0$, then we have to prove that the local structure of the moduli space close to the disc is the one of a 1 dimensional manifold.

The analytical framework for Legendrian contact homology of *embedded* Legendrians was established in [EES05a]. However, over 0, our Λ_0 is *immersed.* Moreover, one learns that the Banach manifolds used in [EES05a] allow only variations of the candidate maps from punctured discs for pseudo-holomorphic curves that are *tangent* to the Lagrangian projection along the boundaries of the disc. To model the clasp move, one has to define a vector field (called V called clasping vector field, see Appendix A.1.1) that is pointing in the *normal* direction along the boundary. And so, to set-up the analytical framework, one has to

1. extend the set-up of [EES05a] to include immersed Legendrians,
2. thicken the relative Banach manifolds to account for the normal variation given by the clasping vector field V.

These two tasks are done in the Appendix A so that the proofs of [EES05a] apply verbatim to this new set-up and contain no analytical innovation. Here, we sketch the main strategy of proof, deferring details to later sections of the Appendix A.

Proof. Fix $\mu = 0$ and consider u_0 to be a J-holomorphic map from a punctured disc that is asymptotic to the double point s corresponding to the clasping chord and with boundary on $L_0 = \Pi_P(\Lambda_0)$. That is, u is a zero of a Banach section

$$\mathcal{F}_0 : \mathcal{W}_0 \to \mathcal{E},$$

where $\mathcal{W}_0$ is the Banach manifold of so called candidate maps, and $\mathcal{F}_0$ is locally modelled by the Cauchy-Riemann operator $\bar{\partial}_{J,j}$. The proof of the surjectivity of $D\mathcal{F}_0$ follows verbatim from those in [EES05a] as all the strategies achieving this result concentrate close to the exceptional positive puncture that is not asymptotic to s. The proof of this fact is a simpler version of the one in Appendix A.3 and so we omit it.

We model the clasp move $(\Lambda_\mu)_{\mu\in\mathbb{R}}$ in the Lagrangian projection (see Definition A.4) using a so-called clasping vector field V (see Appendix A.1.1). On the Banach manifold side, this "thickens" our source Banach manifold $\mathcal{W}$ to

$$\mathcal{W}_\mu \to \mathcal{N} \xrightarrow{\mathfrak{p}} \mathbb{R},$$

an affine Banach fibre bundle of candidate maps for the J_μ-holomorphic curves (see Definition A.11), where $\mathcal{W}_\mu = \mathfrak{p}^{-1}(\mu)$ describes the set of maps with boundary on $L_\mu = \Pi_P(\Lambda_\mu)$. Let

$$\mathcal{F} : \mathcal{N} \to \mathcal{E}$$

be the section of affine Banach bundles so that

$$\mathcal{F}_\mu : \mathcal{W}_\mu \to \mathcal{E}$$

is the section corresponding to the operator $\bar{\partial}_{J_\mu,j}(\Psi_\mu(\cdot))$. We want to prove that $\mathcal{F}_\mu$ is a family of sections so that $D\mathcal{F}_\mu$ is a family of Fredholm operators that is $\mathcal{C}^1$ in the μ variable, that is

$$|\Psi_{-\mu} \circ \mathcal{F}_\mu \circ \Phi_\mu - \mathcal{F}_0| \leq C_0(\mu) \text{ and } |D_{u_\mu}\mathcal{F}_\mu - D_{u_0}\mathcal{F}_0| \leq C_1(\mu), \tag{4.3}$$

where C_i are continuous functions that vanish at $\mu = 0$. The proof of the inequalities 4.3 follow by expressing variations of all the first-order variations of $\Psi_{-\mu} \circ \mathcal{F}_\mu \circ \Phi_\mu$ up to order 1 in the coordinate μ. This is shown in Lemma A.15. Therefore, we obtain that $D_{u_\mu}\mathcal{F}_\mu$ must be surjective for μ small enough, as surjectivity is an open condition.

We care about moduli spaces $\mathfrak{M}_I(\mathbf{w})$ that are expected to be one-dimensional manifolds with boundary. Note that there is a unique smooth structure compatible with all C^1-structures on a one-dimensional topological manifold. That means that it is enough to obtain a C^1-structure.

For this, we need to use the infinite-dimensional version of the inverse function theorem, which requires the quadratic estimate and a uniform bound on right inverses.

Lemma 4.9 (Floer's Picard lemma)**.** *Let X, Y be Banach spaces and let $F : X \to Y$ be a continuous map. We write*

$$F(x) = F(0) + L(x) + N(x),$$

where $L(x) = (D_0F)(x)$ is the linearization of F and we suppose that there is a continuous $G : Y \to X$ and a constant C such that

(1) $L \circ G = Id_Y$,
(2) $||GN(x) - GN(y)|| \leq C(||x|| + ||y||)||x - y||$ for all $x, y \in B(0, r)$,
(3) $||GF(0)|| \leq \frac{\varepsilon}{2}$, where $\varepsilon = \min(r, \frac{1}{5C})$.

Then there exists a ***unique*** *$\alpha \in Im(G) \cap B(0, \varepsilon)$ such that $F(\alpha) = 0$. Moreover, we have that $||\alpha|| \leq 2||GF(0)||$.*

In our case, F is the operator given by the section $\mathcal{F}$ expressed in local coordinates of our Banach manifolds. Note that the first assumption of Picard's Floer Lemma is equivalent to the non-stationary transversality (that we comment on in Appendix A.3). The third assumption is obviously satisfied since we start with u_0 so that $\bar{\partial}_{J,j}u = 0$.

Lemma 4.10. *There is a constant $C_N > 0$ so that the non-linear term N of $\mathcal{F}$ in a neighbourhood of u_μ satisfies*

$$||N(v, \mu_v) - N(w, \mu_w)||_1 \leq C_N(||v||_2 + |\mu_v| + ||w||_2 + |\mu_w|)(||v - w||_2 + |\mu_v - \mu_w|)$$

for all variations v and w.

Proof. Follows almost verbatim the proof of [EES05a, Lemma 8.16], which uses the properties of the geometric set-up, expressing the delbar operator in Jacobi fields, reformulating the problem into a multivariable calculus one. ☐

To obtain the constant C in the Floer's Picard Lemma, we require also the uniform estimate (in μ) on the right inverses G . In the case of gluing, these are more challenging to obtain as the asymptotic weights interfere with the estimation. In our case, the clasp move happens in the compact part of the curve, and so we can dismiss the influence of the weights completely. What is more, one can find an explicit formula for approximate right inverses as the path of linearisations is C^1 in μ by Lemma A.15.

Lemma 4.11. *Let $T_\mu : X \to Y$ be a C^1-family of surjective Fredholm operators and let $K : X \to Y$ be the derivative of T_μ at $\mu = 0$. Then there exists a family of bounded operators G_μ so that*

(1) $T_\mu \circ G_\mu = 1$,
(2) there is an $\mu_0 > 0$ and a constant C so that for all $|\mu| < \mu_0$

$$||G_\mu|| < C. \tag{4.4}$$

Proof. The derivative yields that for every $\epsilon > 0$ there is a $\mu_0 > 0$ so that for all $|\mu| < \mu_0$

$$||(T_\mu - T_0) - \mu K|| < \epsilon.$$

Let $\tilde{G}_0$ be the right inverse to T_0, then

$$||T_\mu \tilde{G}_0 - (1 + \mu K \tilde{G}_0)|| < \epsilon.$$

Now, choose μ_0 small enough so that $||\mu K \tilde{G}_0|| < 1$, thus

$$Z = \sum_{n=0}^{\infty} (-1)^n \mu^n (K \tilde{G}_0)^n$$

satisfies $(1 + \mu K\tilde{G}_0)Z = 1$. The Banach space X splits as $X = X' \oplus \ker T_\mu$ for every μ and denote $\pi_\mu : X \to X'$ the canonical projection. Denote $G'_\mu = \pi_\mu \circ \tilde{G}_0 Z$. Let $\tilde{G}_\mu$ be any family of right inverses of T_μ, denote $G_\mu = \pi_\mu \circ \tilde{G}_\mu$.

Let G_μ and G'_μ be two maps, then

$$||T_\mu G'_\mu - T_\mu G_\mu|| < \epsilon \quad \text{implies} \quad ||\pi_\mu G'_\mu - \pi_\mu G_\mu|| < C'\epsilon$$

for some constant $C' > 0$. To see this, it is enough to realise that $T_\mu \circ \pi_\mu \circ G_\mu = T_\mu G_\mu$ and $T_\mu \circ \pi_\mu \circ G'_\mu = T_\mu G'_\mu$ and that, on the compact interval $[-\mu_0, \mu_0]$, the continuous function $||T_\mu||$ attains a maximum.

Finally, by elementary inequalities and the sum of geometric series, we obtain the uniform bound

$$||G_\mu|| \leq ||G_\mu - G'_\mu|| + ||G'_\mu|| < C'\epsilon + \frac{||\tilde{G}_0||}{1 - \mu||K||||\tilde{G}_0||} \leq C'\epsilon + \frac{||\tilde{G}_0||}{1 - \mu_0||K||||\tilde{G}_0||} = C$$

□

Lemma 4.12 (Implicit function theorem)**.** *Let us assume that the Lemma 4.9 holds. Then for* $\varepsilon < \frac{2}{5C}$*, the zero set* $F^{-1}(0) \cap B(0, \varepsilon)$ *is a smooth submanifold of dimension* $\dim(\ker(D_0 F))$ *diffeomorphic to the* ε*-ball in* $\ker(D_0 F)$*.*

This finishes the proof for words $\mathbf{w}$ containing letters s_μ. The claim for words $\mathbf{w}$ not containing the letter s_μ follows from the enhanced transversality Lemma A.17. □

Appendix A. Banach Manifold set-up for immersed Legendrians

The proof of the floor-crossing follows closely the foundational results in [EES05a] and [EES05c]. There is very little analytical innovation; the proof strategies and computations are standard and most likely non-surprising for the experts in the field. In this section, we will explain how to modify the foundational setup of [EES05a] so that it can deal with *immersed* Legendrians. The original work of [EES05a] constructs the domain-dependent metrics using the fact that one works with Legendrian *embedding*. More specifically, this assumption has an impact on the construction of Banach manifold charts and, consequently, all the analytical results that follow. Nevertheless, once one redefines the domain-dependent metrics appropriately, the proofs found in [EES05a] and [EES05c] follow verbatim. We will thus not reprove the key results as Fredholm property, transversality, and gluing, but we will show how to pose our problem so that the established analytical machinery can be used.

The task is then to prove that our setup is sufficiently regular in the variable μ, the clasp move parameter. For that, we will have to write down the expansion of all the trivializations of Banach fibre bundles in the μ variable with respect to metrics that depend as well on the μ variable, which will require some technical details.

Before we dive into the lengthy construction, let us outline the structure of the argument;

- Construct a suitable metric $\hat{g}$ that is used to import the local model of the clasp move to P using the Weinstein neighbourhood theorem and variation along the clasping vector field V,
- Define the family of metrics $\hat{g}_\mu$ and the corresponding almost complex structures J_μ so that some geometric properties are satisfied, and later necessary for our analytical setup,

- Define the appropriate metrics on the domain and target of a map $u_0 : D_m \to P$, eliminating the dependence on the Legendrian. Here, we will comment on why such a change is only cosmetic.
- Using the map $u_0 : D_m \to P$ as the centre of our future Banach manifold charts, we define a *domain-dependent* extension of the vector field V. We prove that this extension satisfies the boundary condition (this is why the domain-dependence is necessary).
- We will define the thickening $\mathcal{N}$ and expand the trivialisations in the μ-variable providing us with some regularity results.

Let us remark that our analytical problem is an order of magnitude simple than gluing. The reason is that the clasp move enters the analysis only in the compact part of the domain eliminating the difficulties with the asymptotic weights.

A.1. **Geometric preliminaries.**

A.1.1. *Normal bundle identification and the metric* $\hat{g}$. Here, we recall identification from Section 5.2. of [EES05a]. Let Λ be a closed manifold, and $\pi : T^*\Lambda \to \Lambda$ be its cotangent bundle. Let g be a metric on Λ. Cosider the bundle $T\pi : T(T^*\Lambda) \to T^*\Lambda$. Then for each $q \in \Lambda$ and $p \in T^*_q\Lambda$ we have a splitting $T_{(q,0)}(T\Lambda) = T\Lambda \oplus \ker T\pi$ into horizontal and vertical vectors respectively. Writing Λ for the zero section of $T(T^*\Lambda)$, the splitting reads

$$T_\Lambda(T^*\Lambda) = H_q \oplus V_q = T\Lambda \oplus T^*\Lambda.$$

Define J_g an almost complex structure by $J_g H = V$. Here the horizontal vectors are of form $h \in T_q\Lambda$ and the action of J_g on h is given as

$$J_g h = g(h, \cdot) \in T^*_q\Lambda. \tag{A.1}$$

Definition A.1. Let g be a Riemannian metric on Λ. We say that a Riemannian metric $\hat{g}$ on $T^*\Lambda$ is Kähler along the zero section if

- $\hat{g}|_\Lambda = g$,
- Λ is totally geodesic submanifold of $T^*\Lambda$ when embedded as zero section and the geodesics in Λ are exactly those in metric g,
- if γ is a geodesic in Λ and X a vector field in $\gamma^*(T^*\Lambda)$, then X is Jacobi if and only if J_gX is Jacobi.

Note that this metric always exists and can be explicitly constructed from g and its curvature tensor R, see Section 5.2 of [EES05a] for details.

A.1.2. *Metric* $\hat{g}$ *in flat chart and the clasping vector field* V. We can choose the metric g on Λ so that g is flat in the neighbourhood of the endpoints of the clasping chord. This assumption simplifies the metric $\hat{g}$ considerably. Now for the details; Therefore, the Christoffel symbols of g vanish. The covariant derivative of $\hat{g}$ on $T^*\Lambda$ then looks like

$$\nabla(a_i dq_i) = (\partial_j(a_i) - \Gamma^i_{jk} a_i) dq_j \otimes dq_k = \partial_j(a_i) dq_j \otimes dq_k.$$

The splitting into the vertical and horizontal subspace under this connection then looks like

$$V_{(q,p)} = \operatorname{span}\{\partial_{j^*}\} \cong T^*_q\Lambda, \qquad H_{(q,p)} = \operatorname{span}\{\partial_j\} \cong T_q\Lambda$$

The Sasakian almost complex structure J_g on $T^*\Lambda$ characterised by $J(V) = H$ can be expressed in the coordinates as

$$J\partial_{j^*} = -g^{kj}\partial_k, \qquad J\partial_j = g_{jk}\partial_{k^*}.$$

Now the metric $\hat{g}$ is just the product metric since the curvature term vanishes

$$\hat{g}(v)(X,Y) = g(p)(X^H, Y^H) + g^*(p)(X^V, Y^V),$$

where $g^*(p)(\mu,\nu) = g(p)(\mu^\#, \nu^\#)$ for $\mu, \nu \in T_p^*\Lambda$, where $\# : T^*\Lambda \to T\Lambda$ is defined by $\alpha(\cdot) = g(\alpha^\#, \cdot)$ for any $\alpha \in T^*\Lambda$. Dually, define the map $\flat : T\Lambda \to T^*\Lambda$ by $w^\flat(\cdot) = g(w, \cdot)$ for any $w \in T\Lambda$ and recall that for all $\alpha, \beta \in T^*\Lambda$ and $v, w \in T\Lambda$ it holds that

$$g^*(v^\flat, w^\flat) = g(v,w), \qquad g^*(\alpha, \beta) = g(\alpha^\#, \beta^\#),$$

where g^* denotes the dual metric to g. Furthermore, in the splitting $H \oplus V = T\Lambda \oplus T^*\Lambda$ the Sasakian almost complex acts like the following block antidiagonal operator

$$J = \begin{bmatrix} 0 & -\# \\ \flat & 0 \end{bmatrix}.$$

Now, we compute that the Sasakian almost complex structure is compatible with $\hat{g}$,

$$\begin{aligned} \hat{g}(v)(JX, JY) &= g(p)((JX)^H, (JY)^H) + g^*(p)((JX)^V, (JY)^V), \\ &= g(p)(JX^V, JY^V) + g^*(p)(JX^H, JY^H), \\ &= g(p)(-(X^V)^\#, -(Y^V)^\#) + g^*(p)((X^H)^\flat, (Y^H)^\flat), \\ &= g^*(p)(X^V, Y^V) + g(p)(X^H, Y^H), \\ &= \hat{g}(v)(X,Y). \end{aligned}$$

Let $\hat{\nabla}$ be the Levi-Civita connection of the metric $\hat{g}$ on $T^*\Lambda$.

Lemma A.2. *The Sasakian almost complex structure J_g is convariantly constant with respect to $\hat{g}$, that is*

$$\hat{\nabla} J = 0.$$

Proof. We derive the identity out of the local expression for J. Note that the product metric $\hat{g}$ is also flat since g is and so all the Christoffel symbols must vanish. Therefore,

$$\hat{\nabla}_{\partial_\alpha} \partial_\beta = 0,$$

for all $\alpha, \beta \in \{1, 2, \ldots, n, 1^*, 2^*, \ldots, n^*\}$. Moreover, let $G = (g_{ij})$ and $G^{-1} = (g^{ij})$ be the matrix of the Riemannian metric and its inverse, then

$$\begin{aligned} 1 &= G^{-1}G, \\ 0 &= \hat{\nabla}_{\partial_\alpha}(G^{-1})G + G^{-1}\hat{\nabla}_{\partial_\alpha}(G) = \hat{\nabla}_{\partial_\alpha}(G^{-1})G + 0, \end{aligned}$$

so $\hat{\nabla}_{\partial_\alpha} g^{ij} = 0$. Now, finally,

$$\begin{aligned} \hat{\nabla}_{\partial_\alpha} J\partial_j &= \hat{\nabla}_{\partial_\alpha}(g_{jk}\partial_{k^*}) = \hat{\nabla}_{\partial_\alpha}(g_{jk})\partial_{k^*} + g_{jk}\hat{\nabla}_{\partial_\alpha}(\partial_{k^*}) = 0, \\ \hat{\nabla}_{\partial_\alpha} J\partial_{j^*} &= \hat{\nabla}_{\partial_\alpha}(-g^{kj}\partial_k) = -\hat{\nabla}_{\partial_\alpha}(g^{kj})\partial_k - g^{kj}\hat{\nabla}_{\partial_\alpha}\partial_k = 0. \end{aligned}$$

□

Lemma A.3. *The bundle $T^*\Lambda$ is Kähler on T^*U, where U is the subset of Λ where the metric g is flat.*

Proof. We have shown that the Sasakian almost complex structure J is compatible with $\hat{g}$ and that it is covariantly constant. It is left to prove that $\omega(X,Y)=\hat{g}(JX,Y)$ is a closed form, which is obvious since ω is the standard exact symplectic form on $T^*\Lambda$. $\square$

To make the notation less cumbersome we adopt a slight abuse of notation and denote $\hat{\nabla}$ by ∇.

Now, we explain how we use this special flat model to import the local model of the clasp move (see Section 2) to the Lagrangian projection.

Let V be a vertical vector field defined on Λ supported on the neighbourhood where g is flat and $\Pi_P(\Lambda)$ is embedded. This means that it can be extended to a small neighbourhood of $\Pi_P(\Lambda)$ so that it looks constant on the cotangent fibres close to the zero section. We abuse the notation and denote this extension to a vector field on $\Pi_P(T^*\Lambda)$ with V. Let us note that in the neighbourhood, where the metric g is flat, the exponential map with respect to the metric $\hat{g}$ of the vector field V in time μ looks like a translation by μ in the direction of V, in local coordinates;

$$\phi_\mu(q,p)=(q,p+\mu V(q)).$$

Let $\rho:\Lambda\to\mathbb{R}$ be the plateau function defining the clasp move. Let us consider a form $d\rho\in T^*\Lambda$, then we get a gradient $\nabla^g\rho\in T\Lambda$ using the metric g on Λ as $d\rho=g(\nabla^g\rho,\cdot)$. We identify $\nabla^g\rho$ with its image under the embedding $T\Lambda$ into $T(T^*\Lambda)$ as horizontal vectors.

Definition A.4.

$$L_\mu=\hat{\iota}\Big(\exp^{\hat{g}}_{\Lambda}(\mu J\nabla^g\rho)\Big),$$

where we exponentiate $J_g\nabla^g\rho$ along the zero section of $T\Lambda$ with respect to the metric $\hat{g}$ on $T\Lambda$. We define Λ_μ as the lift of L_μ into $P\times\mathbb{R}$.

In local coordinates, L_μ looks like a graph of $\mu d\rho$ over the zero section giving the immersed exact Lagrangian L. Therefore, L_μ is still an exact immersed Lagrangian. One can observe that the following Lemma holds.

Lemma A.5. *The family of immersed exact Lagrangians L_μ lift to a clasp move as defined in Section 2.0.1.*

A.1.3. *Family of metrics $\hat{g}_\mu$.* Let g_μ be the metric on Λ given as the restriction of the metric $\hat{g}$ onto Λ_μ. This is possible since L_μ is embedded on the support of the variation and does not change around the immersed parts for μ small. This is the same as a metric induced by a diffeomorphism $\phi_{-\mu}:L_\mu\to L$ as the pull-back

$$g_\mu=\phi^*_{-\mu}g.$$

Now, construct $\hat{g}_\mu$ on $T^*\Lambda$ as in Section 5.2 of [EES05a] and we denote by ∇^μ their Levi-Civita connection. Furthermore, we construct the almost complex structures J_μ on P so that $\hat{g}_\mu$ is Kähler along L_μ. Then it follows from Lemma 5.6 in [EES05a] that

$$\nabla^\mu_{T_\mu}J_\mu X_\mu=J_\mu\nabla^\mu_{T_\mu}X_\mu, \tag{A.2}$$

for T_μ tangent to L_μ along L_μ. This is a foundational identity crucial for the construction of all the Banach manifold charts that in the end allows us to elegantly express the $\bar{\partial}_{J_\mu}$ using Jacobi fields.

Unfortunately, when dealing with the variation u_μ of J_0-holomorphic curves $u_0:(D_0,j)\to(P,J_0)$, one needs the variation vector field V to satisfy boundary conditions of being J_μ-holomorphic along the boundary, which are impossible to satisfy with a single vector field

V. Thus, we are forced to define a domain-dependent extension $V_\mu(\zeta, q)$ of the restriction of V to L_μ so that $V_\mu(\zeta, q)$ is holomorphic along L_μ. This will be made more precise in what follows.

A.1.4. *Cauchy-Riemann equation in the product.* Let (D_m, j, ω_0) and (P, J, ω) be symplectic manifolds with tame almost complex structures. We consider the symplectic product $(D_m \times P, \omega_0 \oplus \omega)$ with the almost complex structure $\tilde{J} = j \oplus J$. Observe that a (j, J)-holomorphic map $u_0 : D_m \to P$ induces a map

$$\begin{aligned} \tilde{u}_0 : D_m &\to D_m \times P, \\ \zeta &\mapsto (\zeta, u_0(\zeta)). \end{aligned}$$

In particular, the image of $\tilde{u}_0$ in $D_m \times P$ is an embedded submanifold and so any vector field in $T\tilde{u}_0(D_m)$ can be extended to a vector field in $TD_m \oplus TP$. In particular, the equation

$$d\tilde{u} \circ j = \tilde{J} \circ T\tilde{u}$$

translates to $du \circ j = J \circ du$.

Note that, by construction, we have the canonical splitting

$$\tilde{u}^*(TD_m \oplus TP) \cong TD_m \oplus u^*TP.$$

We have to make the set-up graphical to be able to define the extension of V from the boundary to the interior of D_m. This is impossible in non-graphical context as images might not be embedded.

Lemma A.6. *Let $u_\mu(\zeta) = \exp^{\hat{g}}_{u_0(\zeta)}(\mu V(u_0(\zeta)))$ for $\zeta \in \partial D_m$. Denote $\hat{\partial}_\tau = du_\mu \cdot \partial_\tau$ and $\hat{\partial}_t = du_\mu \cdot \partial_t$. There exists a μ-family of cut-off functions α_μ supported in the neighbourhood of the boundary component where $V(u_\mu(\zeta))$ is supported, and a μ-family of functions f_μ so that $f(\zeta) = 0$ for $\zeta \in \partial D_m$, $\hat{\partial}_\tau f_\mu = 0$, and $\hat{\partial}_t f_\mu = 1$. Moreover, the vector field defined as*

$$V_\mu(\zeta, u_\mu(\zeta)) = \Big(V(u_\mu(\zeta)) + f_\mu(\zeta) J_\mu \nabla^\mu_{\hat{\partial}_\tau} V(u_\mu(\zeta)) \Big) \alpha_\mu(\zeta)$$

satisfies the following two conditions for all $\zeta \in \partial D_m$;

$$V_\mu(\zeta, u_\mu(\zeta)) = V(u_\mu(\zeta)), \tag{A.3}$$

$$\nabla^\mu V_\mu(\zeta, q) + J_\mu \circ \nabla^\mu V_\mu(\zeta, q) \circ j = 0. \tag{A.4}$$

Proof. Let $\zeta = \tau + jt$ be conformal structure near the boundary of ∂D_m so that τ is the direction tangent to the boundary ∂D_m. Then it is enough to prove that

$$\nabla^\mu_{du_\mu \cdot \partial_\tau} V_\mu(\zeta, u_\mu(\zeta)) + J_\mu \circ \nabla^\mu_{du_\mu \cdot \partial_t} V_\mu(\zeta, u_\mu(\zeta)) = 0 \text{ for all } \zeta \in \partial D_m.$$

The existence of the families α_μ and f_μ is obvious, in conformal coordinates $f_\mu(\zeta) = t$. The proof follows from direct computation and assumptions on the families;

$$\begin{aligned} \overline{\nabla}^\mu_{J_\mu, j} V_\mu(\zeta, u_\mu(\zeta))(\partial_\tau) &= \Big(\nabla^\mu_{\hat{\partial}_\tau} + J_\mu \circ \nabla^\mu_{\hat{\partial}_t} \Big) \cdot V_\mu(\zeta, u_\mu(\zeta)) \\ &= \alpha(\zeta) \Big(\nabla^\mu_{\hat{\partial}_\tau} V + \hat{\partial}_\tau f_\mu(\zeta) J_\mu \nabla^\mu_{\hat{\partial}_\tau} V + f_\mu(\zeta) \nabla^\mu_{\hat{\partial}_\tau} \nabla^\mu_{\hat{\partial}_\tau} V + J_\mu \nabla^\mu_{\hat{\partial}_t} V \\ &\quad - \hat{\partial}_t f_\mu \cdot \nabla^\mu_{\hat{\partial}_\tau} V + f_\mu J_\mu \nabla^\mu_{\hat{\partial}_t} \nabla^\mu_{\hat{\partial}_\tau} J_\mu V \Big) + \hat{\partial}_\tau \alpha_\mu \Big(V + f_\mu(\zeta) J_\mu \nabla^\mu_{\hat{\partial}_\tau} V \Big) \\ &\quad + \hat{\partial}_t \alpha_\mu \Big(J_\mu V - f_\mu(\zeta) \nabla^\mu_{\hat{\partial}_\tau} V \Big) \end{aligned}$$

□

A.1.5. *Family of metrics on the pull-back bundle.* Here we construct a neighbourhood of the boundary ∂D_m and a domain-dependent family of metrics on the pull-back bundle $u^*(TP)$ so that the family of metrics is constant near the boundary and the partial derivatives of the family are uniformly bounded.

We diverge from the set-up of [EES05a, Section 5.3]. In particular, we are not constructing a family of metrics in P (that uses the embedding property of Λ, the Legendrian lift to the contactisation $P \times \mathbb{R}$). Instead of the geometric approach, we formulate the problem more virtually and construct a domain-dependent family of metrics on the pull-back bundle $\tilde{u}_0^*(TD_m \times TP)$. The sought family of metrics will be the product metric of the family of metrics on the domain from [EES05a] and the metrics constructed below.

Recall, that we denote by h a background metric on P. Denote by N_r the image of $D_r^*\Lambda$ in P under the composition $\Pi \circ \iota$. We choose r small enough so that the metric $\hat{g}$ on $T^*\Lambda$ is injective. Now, let $N_r(u) = u_0^{-1}(N_r)$, this forms a neighbourhood of the boundary ∂D_m. The boundary ∂D_m decomposes into components $\partial_j D_m$ spanning from the puncture p_j to the puncture p_{j+1}. For each $\partial_j D_m$ consider $Q_j = (\Pi_P \circ \iota)^{-1}(u(\partial_j D_m))$ a subset of the zero section of $D_r^*\Lambda$, denote by g^j the push-forward of the metric $\hat{g}$ under the map $\Pi_P \circ \iota$ restricted to $D_r^* Q_j$ and write $N_r^j(u)$ for the components of the set $u^{-1}(\Pi_P \circ \iota(D_r^* Q_j))$ that have non-empty intersection with Q_j. If the sets $N_r^j(u)$ are not pair-wise disjoint outside of the asymptotic neighbourhoods $\bigcup_{j=0}^m E_j[1]$, then choose r smaller to contain all the intersections in the respective asymptotic neighbourhoods. Let $\mathcal{G}$ be the space of Riemannian metrics on P. There is a smooth function $g^\sigma : D_m \to \mathcal{G}$ so that

$$g^\sigma(\zeta) = \begin{cases} h & \text{if } \zeta \notin \bigcup_{j=0}^m N_r^j(u), \\ g^j & \text{if } \zeta \in \bigcup_{j=0}^m N_{\frac{r}{2}}^j(u) \setminus \bigcup_{j=0}^m E_j[1], \\ g^j_{\tau+it} & \text{if } \zeta = \tau + it \in \bigcup_{j=0}^m E_j[1], \end{cases}$$

where $g^j_{\tau+it}$ is asymptotic family of metrics constructed as follows; assume that the lines $\mathbb{R}$ and $i\mathbb{R}$ map to the components Q_{j_0} and Q_{j_1}

- $g^j_{\tau+it} = g^{j_0}$ for t in a neighbourhood of 0,
- $g^j_{\tau+it} = g^{j_1}$ for t in a neighbourhood of 1,
- the family of metrics interpolates in between $g^j_{\tau+it} = g^{j_0}$ and $g^j_{\tau+it} = g^{j_1}$ in t so that the derivatives with respect to the coordinates τ, t are uniformly bounded,
- and, as we approach the boundary of the asymptotic neighbourhood with the compact part of the domain the neighbourhoods $N^{j_0}_{\frac{r}{2}}(u)$ and $N^{j_1}_{\frac{r}{2}}(u)$ become disjoint and we interpolate in between them to make g^σ smooth.

Repeat the construction above for every μ to obtain the family of the metrics $g^\sigma_\mu(\zeta)$.

A.1.6. *Modification of the standard set-up is cosmetic.* First let us recall some details from [EES05c]. We wish to construct a structure of Banach manifold of the space $\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa)$ consisting of pairs of maps

$$(u, f) : (D, \partial D_m) \to (P, \mathbb{R})$$

satisfying following two conditions;

$$(u,f)(\zeta) \in L, \text{ for all } \zeta \in \partial D_m; \tag{A.5}$$

$$\int_{\partial D_m} \langle \bar{\partial}_J(u), v\rangle ds = 0, \text{ for all } v \in C_0^0(\partial D_m, T^{*0,1}D_m \otimes u^*(TP)). \tag{A.6}$$

We denote by F the extension of $f : \partial D_m \to \mathbb{R}$ into $F : D_m \to \mathbb{R}$. The construction of the C^1-chart around some (u,f) goes via exponentiation of the elements the tangent space at (u,f),

$$\Psi : T_{(u,f)}\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa) \to \mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa),$$
$$\Psi(v)(\zeta) = \exp^{\rho(F(\zeta))}_{u(\zeta)}(v(\zeta)),$$

where $T_{(u,f)}\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa)$ consists of all $v \in \mathcal{H}_{2,\epsilon}(D, u^*(TP))$ so that

$$v(\zeta) \in T_{u(\zeta)}L \subset T_{u(\zeta)}P \text{ for all } \zeta \in \partial D, \tag{A.7}$$

$$\int_{\partial D_m} \langle \bar{\nabla}_J v, w\rangle ds = 0 \text{ for all } w \in C_0^0(\partial D_m, u^*(TP)), \tag{A.8}$$

denoting $\bar{\nabla}_J v = \nabla v + J \circ \nabla v \circ j$, where ∇ is the connection on $u^*(TP)$ given by the point-wise dependent metric $g^{\rho(F(\zeta))}$. This is well-defined only in some η-ball around (u,f) where η depends continuously on the minimal injectivity radius of the metrics g^σ for $0 \leq \sigma \leq 1$. Note that this construction depends on the extension F.

Remark A.7. The choice of a family of metrics in section A.1.5 induces an analytical set-up which is almost identical to the one of [EES05c] and [EES05a], so we use notation established there with one important difference. Instead of choosing a smooth function $\rho : P \times \mathbb{R} \to [0,1]$ which yields a $(P \times \mathbb{R})$-dependent family of metrics g^ρ on P, we defined a family of domain-dependent metrics g^σ on the bundle $u^*(TP)$. We eliminated this dependence carefully so that the analytic package of [EES05c] can still be used. All the proofs of [EES05c] translate *verbatim* into this slightly more general setup. This is possible since, in the crucial steps, the dependence of the family of metrics on the lift in $P \times \mathbb{R}$ is purely of convenience. Nevertheless, for the sake of completeness, we cover some of the crucial steps below.

Gluing: In [EES05c] asymptotic neighbourhood of the punctures on the disc has canonical neighbourhoods $E_j[0]$ conformally equivalent to $[0,+\infty)_\tau \times [0,1]_t$ and a complex structure so that $\zeta = \tau + it$ for $\zeta \in E_j[0]$. To be able to glue J-holomorphic curves together, one changes the asymptotic coordinates on $E_j[0]$ near negative asymptotics to $(-\infty, 0] \times [0,1]$ using the conformal equivalence

$$\tau + it \mapsto -\tau + i(1-t).$$

Then to construct the approximate solution one uses the domain-dependent family of metric $g^{\rho(F(\sigma))}$ to define the approximate solution on the interpolating region. Compare with Section 4.3. in [EES05c]. Note that here, the dependence on the lift F gives us a globally defined prescription of how the metrics change under the conformal equivalence of the asymptotic neighbourhoods $E_j[0]$. Therefore, in our more general set-up, we need to keep track of which boundary component gets mapped to the lower or upper sheet, which is the reason for the introduction of the sets Q_j above.

Fredholm problem: In Section 4.1. of [EES05c], there one writes out the linearization of the $\bar{\partial}_{J,j}$ at (u,f) in the form

$$D\bar{\partial}_{J,j}[v] = \bar{\nabla}_{J,j}v - \frac{1}{2}J \circ (\nabla_v J) \circ \partial_{J,j}u = \bar{\nabla}_{J,j}v + Kv,$$

where v is the tangent space of the Banach manifold of candidate maps $\mathcal{W}_{\epsilon,\mathbf{b}}$. Then one realises that for $D\bar{\partial}_{J,j}$ to be Fredholm, it is enough to prove that K is a compact operator. That is true, provided we have h a smooth matrix-valued function with all derivatives bounded satisfying $Kv = h(v) \cdot du \cdot v$. The derivatives of h depend on the metric g^σ, which we constructed carefully to retain this control.

Transversality: In Section 4.2. of [EES05c], one can achieve transversality by perturbing J near special regular oriented curves close to one distinguished positive puncture. Here, the linearization reads

$$D\bar{\partial}_{J,j}[v,\lambda] = \bar{\partial}_{J,j}v + Kv + K_S,$$

where the term K_S describing the variation of J does not depend on g^σ.

A.2. **Banach manifold charts.**

A.2.1. *Banach manifolds $\mathcal{V}$,$\mathcal{W}$, and $\mathcal{N}$.*

In this section, we first recall the construction of the Banach manifold $\mathcal{V}_{2,\epsilon}$ from [EES05c]. Then we will follow with a modified construction of the Banach manifold $\mathcal{W}_{2,\epsilon}$ (the original construction can be found in Section 3.1.2 of [EES05c]). We have to alter the former construction to mitigate the dependence on the lift to the contactisation $P \times \mathbb{R}$. Then, we define the Banach manifold $\mathcal{N}$, which is a "thickening" of $\mathcal{W}$ given via the variation of $\mathcal{W}$ along the vector field V defined in Section **??**. In the end, we describe the Banach bundles induced by the variation of the almost complex structures j and J and the invariance of the clasp move on thereof.

Denote by $\mathcal{H}_{2,\epsilon}(D_m, \mathbb{R}^N)$ the Sobolev space of functions that have k derivatives (possibly in the sense of regular distributions) in weighted L^2_ϵ-space with weight function $\exp(\epsilon|\tau|)$ in the asymptotic local models. Let us recall the construction of the Banach manifold $\mathcal{V}_{2,\epsilon}$ from [EES05c, Section 3.1].

Lemma A.8 (Lemma 3.1 of [EES05c])**.** *Let $M, N > 0$ be big enough and η a metric on $\mathbb{R}^N$ so that P is a totally geodesic submanifold. Fix $\boldsymbol{b} = b_1, \dots, b_m$ an ordered set of double points of $L \subset P$ and a reference function $f_{\boldsymbol{b}} : D_m \to \mathbb{R}^N$ that is constantly equal to b_j in a neighbourhood $E_j[M]$ of $p_j \in D_m$ for every j_κ on D_m. Then there is a C^1-Banach manifold $\mathcal{V}_{2,\epsilon}(\boldsymbol{b};\kappa)$ consisting of the continuous maps $u : D_m \to P \subset \mathbb{R}^N$ so that*

$$u - f_{\boldsymbol{b}} \in \mathcal{H}_{2,\epsilon}(D_m, \mathbb{R}^N).$$

In addition, the tangent space of $T_u\mathcal{V}_{2,\epsilon}(\boldsymbol{b};\kappa)$ at any $u \in \mathcal{V}_{2,\epsilon}(\boldsymbol{b};\kappa)$ is canonically isomorphic to the space $\mathcal{H}_{2,\epsilon}(D_m, u^*(TP))$ with C^1 local coordinate map around u being

$$\begin{aligned}\mathcal{H}_{2,\epsilon}(D_m, u^*(TP)) &\to \mathcal{V}_{2,\epsilon}(\boldsymbol{b};\kappa),\\ v &\mapsto (\zeta \mapsto \exp^\eta_{u(\zeta)}(v(\zeta))).\end{aligned}$$

Definition A.9. Define $\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa,J)$ as the subset of elements $u \in \mathcal{V}_{2,\epsilon}(\mathbf{b};\kappa)$ fulfilling the boundary condition of u and an L^2-vanishing condition on the trace of $\bar{\partial}u$:

$$u(\zeta) \in L, \text{ for all } \zeta \in \partial D_m; \tag{A.9}$$

$$\int_{\partial D_m} \langle \bar{\partial}_{J,j_\kappa} u, v\rangle ds = 0, \text{ for all } v \in C^0_0(\partial D_m, T^{*0,1}D_m \otimes u^*(TP)). \tag{A.10}$$

Lemma A.10. *For $M > 0$ big enough. The set $\mathcal{W}_{2,\epsilon}(\boldsymbol{b};\kappa,J)$ is a closed C^1-submanifold of $\mathcal{V}_{2,\epsilon}(\boldsymbol{b};\kappa)$ with a tangent space at u isomorphic to a closed subspace of $\mathcal{H}_{2,\epsilon}(D_m, u^*(TP))$*

consisting of v *satisfying the following conditions;*

$$\begin{aligned} &v(\zeta) \in T_{u(\zeta)}L, \text{ for all } \zeta \in \partial D_m,\\ &\int_{\partial D_m} \langle \bar{\nabla}_{J,j_\kappa} v, w\rangle ds = 0, \text{ for all } w \in C^0_0(\partial D_m, T^{*0,1}D_m \otimes u^*(TP)), \end{aligned}$$

with

$$\bar{\nabla}_{J,j_\kappa} v = \nabla v + J \circ \nabla v \circ j_\kappa,$$

where ∇ *is the Levi-Civita connection given by the projection from the pull-back bundle* $u^*(TD_m \times TP)$ *of the connection induced by the domain-dependent metric* g_{pr} *given by the maps* u *and* $\hat{\iota}$*. In addition, the local* C^1*-coordinate chart around* $u \in \mathcal{W}_{2,\epsilon}(\boldsymbol{b};\kappa,J)$ *is given by*

$$\Phi : T_u\mathcal{W}_{2,\epsilon}(\boldsymbol{b};\kappa,J) \to \mathcal{W}_{2,\epsilon}(\boldsymbol{b};\kappa,J) \tag{A.11}$$

$$\Phi[v](\zeta) = \exp^{g_{pr}}_{u(\zeta)}(v(\zeta)) \tag{A.12}$$

on an δ*-ball around* u*, where* $\delta > 0$ *depends continuously of the injectivity radius of the metrics* $g_{pr} = g(\kappa)(\zeta) \oplus g^\sigma(\zeta)$ *for any* $\zeta \in D_m$*.*

Proof. The proof follows from the proof of [EES05a, Proposition 5.9] translated to our notation and so we omit the details. ☐

Note that the variations v giving rise to the tangent space $T_u\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa,J)$ must be tangent to the Lagrangian projection L along the boundary. Nevertheless, the vector field V defining the clasp move is pointing in the normal direction. And so, we need to define a "thickening" of the Banach manifold $\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa,J)$ to account for the variation along V. To make our set-up fit into the foundational analytical set-up, we will first construct affine Banach fibre bundles accounting for variations of the geometric data and then we construct the thickening.

Let us denote $\mathcal{E}_{1,\epsilon}(\mathbf{b};\kappa,J)$ the Banach bundle over $\mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa,J)$, where the $\bar{\partial}_{J,j_\kappa}$-operator takes values. This can be identified with the bundle of complex anti-linear maps

$$T^*D_m \to \tilde{u}^*(TD_m \times TP)$$

with appropriate weighted Sobolev norm. In particular, the fibre over $u \in \mathcal{W}_{2,\epsilon}(\mathbf{b};\kappa,J)$ consists of elements

$$A \in \mathcal{H}_{1,\epsilon}(D_m, Hom^{0,1}(TD_m, \tilde{u}^*(TD_m \oplus TP)))$$

satisfying an L^2-vanishing boundary condition

$$\int_{\partial D_m} \langle A, W\rangle \text{ for all } W \in C^0_0(\partial D_m, T^{0,1}D_m \otimes \tilde{u}^*(TD_m \oplus TP)). \tag{A.13}$$

Note that the symbol $Hom^{0,1}$ refers to the (J,j_κ)-complex anti-linear maps. Denote Π_v be the parallel transport along the unique geodesic from $\tilde{u}(\zeta)$ to $\tilde{u}_\mu = \Phi[v](\zeta)$ in the metric g^{pr}. Let λ be a smooth path of sections of $\tilde{u}(TD_m \oplus TP)$. We set

$$J_\lambda = \Pi_\lambda^{-1} \circ J \circ \Pi_\lambda$$

To describe how the almost complex structure changes along the variation. And to trivialise, observe that we can write

$$J_\lambda = J(1+S_\lambda)^{-1}(1-S_\lambda) \text{ for } S_\lambda = (J+J_\lambda)^{-1}(J_\lambda - J).$$

It is a standard argument (see [EES05c, Section 3.2.1. and 3.2.2.]) that the following map sends (J, j_κ)-complex anti-linear maps to $(J^\lambda, j_{\kappa'})$-complex anti-linear maps

$$\begin{aligned}\beta : \mathcal{H}_{1,\epsilon}(D_m, Hom^{0,1}(TD_m, \tilde{u}^*(TD_m \oplus TP))) &\to \mathcal{H}_{1,\epsilon}(D_m, Hom^{0,1}(TD_m, \tilde{u}^*(TD_m \oplus TP))) \\ A &\mapsto (1 - S^\lambda)A(1 + \gamma)\end{aligned}$$

and $\gamma = (j_\kappa + j_{\kappa'})^{-1}(j_{\kappa'} - j_\kappa)$. To obtain the trivializations of the target $\mathcal{E}_{1,\epsilon}(\mathbf{b})$ we choose paths $\gamma(s)$ as above. And so the $\bar\partial$-operator gives rise to a section;

$$\mathcal{F} : \mathcal{W}_{2,\epsilon}(\mathbf{b}) \to \mathcal{E}_{1,\epsilon}(\mathbf{b}).$$

A.2.2. *Thickening $\mathcal{N}_{2,\epsilon}(\boldsymbol{b})$.*

Definition A.11. Define $\mathcal{N}_{2,\epsilon}(\mathbf{b}; \kappa, J) \subset \mathcal{W}_{2,\epsilon}(\mathbf{b}; \kappa, J) \times \mathbb{R}_\mu \subset \mathcal{V}_{2,\epsilon}(\mathbf{b}; \kappa)$ as follows; for every $u \in \mathcal{W}_{2,\epsilon}(\mathbf{b}; \kappa, J)$ and $\mu \in \mathbb{R}$ define the splitting

$$T_{(u,\mu)}\mathcal{N}_{2,\epsilon}(\mathbf{b}; \kappa, J) = T_u^{\mathfrak{h}}\mathcal{N}_{2,\epsilon}(\mathbf{b}; \kappa, J) \oplus T_\mu^{\mathfrak{v}}\mathcal{N}_{2,\epsilon}(\mathbf{b}; \kappa, J),$$

where $T_u^{\mathfrak{h}}\mathcal{N}_{2,\epsilon}(\mathbf{b}; \kappa, J) = T_u\mathcal{W}_{2,\epsilon}(\mathbf{b}; \kappa, J)$ and $T_\mu^{\mathfrak{v}}\mathcal{N}_{2,\epsilon}(\mathbf{b}; \kappa, J) = \mathbb{R}$, and a map

$$\Psi : T_u\mathcal{W}_{2,\epsilon}(\mathbf{b}; \kappa, J) \to \mathcal{V}_{2,\epsilon}(\mathbf{b}; \kappa), \tag{A.14}$$

$$\Psi[v, \mu](\zeta) = \exp^{g_{pr,\mu}(\zeta)}_{\exp_{u(\zeta)}(\mu V(\zeta))}(\Pi_{\mu V}\,(v(\zeta))). \tag{A.15}$$

To simplify the notation, we adopt the following abuse of notation $\Psi[v, \mu](\zeta) = u_\mu(\zeta)$.

Lemma A.12. *There are constants $\mu_0, M > 0$ so that $\mathcal{N}_{2,\epsilon}(\boldsymbol{b}; \kappa, J)$ is a C^1-Banach affine fibre bundle with horizontal tangent space canonically identified with a subspace of maps of $\mathcal{H}_{2,\epsilon}(D_m, u_\mu^*(TD_m \times TP))$ satisfying the following moving boundary conditions*

$$v_\mu(\zeta) \in T_{u_\mu(\zeta)}L_\mu, \textit{ for all } \zeta \in \partial D_m, \tag{A.16}$$

$$\int_{\partial D_m} \langle \overline{\nabla}_{J_\mu, j_\kappa} v_\mu, w_\mu \rangle = 0, \textit{ for all } w_\mu \in C^0_0(\partial D_m, T^{*0,1} \otimes u_\mu^*(TD_m \times TP)). \tag{A.17}$$

Moreover, $v_\mu = \Pi_{\mu V}\,(v)$ and Ψ is a C^1-map of Banach manifolds, which is C^2-with respect to the μ-coordinate.

Proof. First, let us check the boundary conditions. We have that

$$\bar\partial_{J_\mu, j_\kappa}\Psi[v, \mu](\zeta) = X^\mu_\tau(\zeta) + J_\mu X^\mu_t(\zeta),$$

where J_μ is the metric induced by $\hat{g}_\mu$ and $X^\mu_\tau(\zeta)$ is a Jacobi field of g^{pr} at time $s = 1$ along the geodesic $\gamma(s) = \exp^{g_{pr}(\zeta)}_{u_\mu(\zeta)}(s(\Pi_{\mu V} v(\zeta)))$ starting at $u(\zeta)$ given by the initial conditions

$$[X^\mu_\tau(\zeta)](0) = \partial_\tau u_\mu(\zeta), \text{ and } [\nabla_s X^\mu_\tau(\zeta)](0) = \nabla_{\partial_\tau}(\Pi_{\mu V}\, v).$$

For X_t substitute t for τ in the initial conditions. Now, as in [EES05a, Lemma 5.12], one finds that the second boundary condition holds if and only if $\overline{\nabla}_{J_\mu, j}\, v_\mu$ vanishes along the boundary. This is obvious in our case as we constructed our metrics $\hat{g}_\mu$ and subsequently the almost complex structures J_μ so that equation (A.2) is satisfied.

In the variation along the normal direction V, we have to show that $\bar\partial_{J_\mu, j} u_\mu(\zeta) = 0$ for $\zeta \in \partial D_m$, however, as above, this is equivalent to $\overline{\nabla}_{J_\mu, j} V_\mu(\zeta) = 0$ for $\zeta \in \partial D_m$. That is follows from the construction of V, see Lemma A.6.

Now, we will use the infinite dimensional implicit function theorem for maps with complemented image (see [Lan06, Corollary 5.5]). And so, we need to prove that there exists a neighbourhood of the origin in $T_{u,\mu}\mathcal{N}_{2,\epsilon}$ so that the map $D_{u,\mu}\Psi : T_{u,\mu}\mathcal{N}_{2,\epsilon} \to T_u\mathcal{V}_{2,\epsilon}$ is injective and has closed image. The rest of the proof is standard and follows analogously to [EES05a, Section 5].

□

Remark A.13. Note that we constructed the immersion $\iota : (D^*_r\Lambda, J_{Sas}) \to (P, J_P)$ so that ι is a (J_{Sas}, J_P)-holomorphic map, in particular, it depends on J_P. This forces us to fix the almost complex structure during the clasp move. That is we use only the auxiliary complex structures J_λ that come from the variation in the μ direction. In other words, the derivatives $\frac{\partial}{\partial\mu}$ and $\frac{\partial}{\partial\gamma}$ do not commute in general. This choice is a *key step in enhancing the transversality* considerations.

A.3. **Non-stationary transversality.** Here we discuss the fact that our problem is transverse and the transversality arguments from [EES05c] apply.

We have a smooth section $\bar\partial : \mathcal{N}_{2,\epsilon}(\mathbf{b}) \to \mathcal{E}_{1,\epsilon}(\mathbf{b})$.

Lemma A.14 (Non-stationary transversality)**.** *Let Λ be a chord generic unobstructed Legendrian immersion with a clasping chord s of length 0, let $(u, \mu, j_\kappa, J) \in \mathcal{N}_{2,\epsilon}(\boldsymbol{b})$ such that $\bar\partial_{J,j_\kappa} u_\mu = 0$. Denote by Λ_μ the clasped Legendrian.*

1. *Let $\mathcal{J}_{\Lambda,fam}$ be the space of admissible families almost complex structures to Λ_μ around J, then the linearization $D\bar\partial_{J,j_\kappa}$ at (u, μ, j_κ, J) is surjective.*
2. *Assume that u is not an exceptional disk and let $\mathcal{L}_{\Lambda,fam}$ be the space of admissible families of Legendrian immersions around Λ_μ, then the linearization $D\bar\partial_{J,j_\kappa}$ at (u, μ, j_κ, J) is surjective.*

Proof. Proof of both points almost verbatim follows from the proof of [EES05c, Lemma 4.5] and [EES05a, Theorem 7.12]. Nevertheless, the details of the non-stationary transversality in Theorem 7.12 were left out for notational reasons since no new thought is needed to finish the proof. And so, we will include a rough sketch of the proof in our case for completeness.

Let us concentrate on the second point, that is j_κ and J are fixed and we are varying the family Λ_μ. First, one constructs $Ham(\Lambda, \mu)$ a linear space of admissible perturbations $a : P \to \mathbb{R}$ so that

- a has support close to $\Pi_P(\Lambda_\mu)$,
- there is a small ball for each of the double points of $\Pi_P(\Lambda)$ so that a is real analytic there,
- the differential of a vanishes at the double points and also it sends them to zero.

Moreover, the balls and support are required to be uniformly bounded. So that, for a fixed μ, Λ_μ remains admissible after the perturbation of Λ by the Hamiltonian vector field induced by a. Then one fibers tivially those spaces over a small interval parametrised by μ as

$$pHam(\Lambda_\mu) = \bigcup_\mu Ham(\Lambda, \mu)$$

and defines a perturbed thickening $\mathcal{N}_{2,\epsilon,pHam(\Lambda_\mu)}$ by as a Banach bundle over $pHam(\Lambda_\mu)$ its tangents space given as

$$T_{(u,\mu,J,j_\kappa)}\mathcal{N}_{2,\epsilon,pHam(\Lambda_\mu)} = T_{(u,\mu,J,j_\kappa)}\mathcal{N}_{2,\epsilon} \oplus pHam(\Lambda_\mu)$$

One also has to fiber $\mathcal{E}_{1,\epsilon}$, the target of the $\bar\partial$-operator, over $pHam(\Lambda_\mu)$ analogously. And so, one obtains a section $\bar\partial_{ham} : \mathcal{N}_{2,\epsilon,pHam(\Lambda_\mu)} \to \mathcal{E}_{1,\epsilon,pHam(\Lambda_\mu)}$. We denote the projection onto $pHam(\Lambda_\mu)$ by pr. One seeks to prove that $(\bar\partial_{ham}, pr)$ is transverse to the zero-section, that is $D\bar{\partial}_{ham}$ is surjective. Since we are working with spaces with L^2-inner product, we can translate the former statement to the vanishing of the annihilator of the image of the linearization. The elements of the annihilator are represented by anti-holomorphic elements through an integration of parts argument.

It turns out, that it is enough to work in a very small neighbourhood of the unique marked positive double-point. Note that the family Λ_μ does not move any double-points and so we can take this ball uniformly small in the family. Because we have one special positive puncture c^* (that does not map to the clasping chord s since Λ is unobstructed) we can not have a branched cover appearing and so, it is enough to prove the transversality in the complement of exceptional disks. Then, one chooses balls around pre-images of c^* under u. There is only finite number of them because of the asymptotic estimates on the rate of convergence of u towards c^*. Now, in the clasping family, those preimages are not moved around since they lie outside of the asymptotic neighbourhood of the clasping chord. This allows us to complete the proof using the same local considerations using the Schwartz reflection principle as in [EES05a, Lemma 7.12].

□

A.4. **Stationary transversality.** Having the non-stationary transversality is not enough to reason that at $\mu = 0$ the linearization of the restriction of the $\bar{\partial}$-operator to the fiber over $\mu = 0$ of $\mathcal{N}_{2,\epsilon}(\mathbf{b})$ is surjective. One could rephrase the former in the smoothness properties of the moduli spaces as follows: the moduli space $\mathfrak{M}(\mathbf{b}, \mathbf{s})$ is a 1-dimensional manifold by the non-stationary transversality. Still, the projection onto $\mathbb{R}$ parametrizing the clasping parameter μ could have $\mu = 0$ as a critical value. Showing the stationary transversality at this point yields that $\mathfrak{M}(\mathbf{b}, \mathbf{s})$ is *diffeomorphic* to the collection of intervals close to $\mu = 0$.

A.4.1. *Variation of the geometric data.* By [EES05a, Lemma 5.4] we have

$$\hat{g}_{ij}(q,p) = g_{ij}(q) + \sum_{s,t=1}^{n} p_s p_t A^{i,j}_{s,t}(q),\ \hat{g}_{i^*j^*}(q,p) = g_{ij}(q),\ \hat{g}_{ij^*}(q,p) = \sum_{s,k=1}^{n} p_s g_{jk}(q)\Gamma^k_{is},$$

where

$$A^{i,j}_{s,t}(q) = \sum_{k,r=1}^{n} g_{kr}(q)\Gamma^k_{is}(q)\Gamma^r_{jt}(q) + R_{isjt}(q).$$

We know that in local coordinates $\Phi_\mu(p,q) = (q, p + \mu d\rho)$. We have defined $g_\mu(q,p) = \hat{g} \circ \Phi_\mu(q,p)$, therefore, by straightforward computation $\hat{g}_\mu(q,p) = g(q,p) + \mu E(q) + \mu^2 Q(q)$, where

$$E_{ij}(q) = \sum_{s,t=1}^{n} (p_t \partial_s \rho(q) + p_s \partial_t \rho(q)) A^{i,j}_{s,t}(q),\ E_{i^*j^*} = 0,\ E_{ij^*}(q) = \sum_{s,t=1}^{n} \partial_s \rho(q) g_{jk}(q) \Gamma^k_{is}(q).$$

$$Q_{ij}(q) = \sum_{s,t=1}^{n} \partial_t \rho(q)\, \partial_s \rho(q) \cdot A^{i,j}_{s,t}(q),\ Q_{i^*j^*}(q) = Q_{ij^*}(q) = 0.$$

Note that the operators E amd Q have their support contained in the support of ρ. Moreover, that $g_\mu^{-1}(q) = g^{-1}(q) - \mu g^{-1}(q) E(q) g^{-1}(q) + \mathcal{O}(\mu^2)$.

$$\begin{aligned}\Gamma^{\alpha}_{\beta\gamma}[\mu] &= \frac{1}{2}\sum_{\delta,\epsilon} g^{\alpha\delta}_{\mu}\Big(\partial_\epsilon g_{\mu,\delta\beta} + \partial_\beta g_{\mu,\delta\epsilon} - \partial_\delta g_{\mu,\beta\epsilon}\Big)\\ &= \Gamma^{\alpha}_{\beta\gamma} + \frac{1}{2}\mu\sum_{\delta,\epsilon}\Big(g^{\alpha\delta}\Big(\partial_\epsilon E^{g}_{\delta\beta} + \partial_\beta E^{g}_{\delta\epsilon} - \partial_\delta E^{g}_{\beta\epsilon}\Big) - E^{g^{-1}}_{\alpha\delta}\Big(\partial_\epsilon g_{\mu,\delta\beta} + \partial_\beta g_{\mu,\delta\epsilon} - \partial_\delta g_{\mu,\beta\epsilon}\Big)\Big) + \mathcal{O}(\mu^2)\\ &= \Gamma^{\alpha}_{\beta\gamma} + \mu E^{\Gamma}_{\alpha\beta\gamma} + \mathcal{O}(\mu^2)\end{aligned}$$

The Sasakian almost complex structure compatible with $\hat{g}$ is expressed in the local coordinates as

$$J(\partial_{j^*}) = -g^{kj}(\partial_k + p_k\Gamma^{r}_{ks}(q)\partial_{s^*}),\ J(\partial_j) = g_{jk}(q)\partial_{k^*} + g^{ls}(q)\Big(p_r\Gamma^{r}_{js}(q)\partial_l + p_r p_v\Gamma^{r}_{js}(q)\Gamma^{v}_{lt}(q)\partial_{t^*}\Big).$$

Thus, the Sasakian almost complex structure that arises from g_μ has the following expression in the same local coordinates;

$$\begin{aligned}J_\mu(\partial_{j^*}) &= -\sum_k g^{kj}_{\mu}(\partial_k + \sum_{r,s}(p+\mu d\rho)_k\Gamma^{r}_{ks}[\mu](q)\partial_{s^*})\\ &= J(\partial_{j^*}) - \mu\sum_k E^{g^{-1}}_{kj}(\partial_k + p_k\Gamma^{r}_{ks}(q)\partial_{s^*}) + g^{kj}(2\partial_k + (\partial_k\rho\cdot\Gamma^{r}_{ks}(q) + p_k E^{\Gamma}_{rks}(q))\partial_{s^*})) + \mathcal{O}(\mu^2)\\ &= J(\partial_{j^*}) + \mu E^{J}_{j} + \mathcal{O}(\mu^2)\end{aligned}$$

A.4.2. *Second variation along the clasping parameter.*

Lemma A.15. *The operator $\bar{\partial}_{J_\mu,j}(u_\mu)$ has the Taylor expansion at $\mu = 0$ equal to*

$$\bar{\partial}_{J_\mu,j}(u_\mu) = \bar{\partial}_{J,j}(u) + \mu(\overline{\nabla}_{J,j}V + \nabla_V J\cdot du\cdot j\eta - \frac{1}{2}J\mathcal{L}_V J\cdot\partial_{J,j}(u)) + \mathcal{O}(\mu^2). \tag{A.18}$$

In particular, the family of operators $(\bar{\partial}_{J_\mu,j}(u_\mu))_\mu$ is continuous in μ.

Moreover,

$$D_{u_\mu}\bar{\partial}_{J_\mu,j}[v,\lambda,\gamma] = D_u\bar{\partial}_{J,j}[v,\lambda,\gamma] + \mu K[v,\lambda,\gamma] + \mathcal{O}(\mu^2), \tag{A.19}$$

where $K[v,0,0] : \mathcal{H}_{2,\epsilon} \to \mathcal{H}_{1,\epsilon}$ is a compact operator. In particular, the family $(D_{u_\mu}\bar{\partial}_{J_\mu,j})_\mu$ is a C^1-family in μ of Fredholm operators with derivative at 0 with respect to μ equal to K.

Corollary A.16. *The clasp move induces a continuous path in Fredholm operators, and so does not change the index of the problem.*

Proof. The computation of the linearization of the $\bar{\partial}_{J,j}$-operator almost verbatim coincides with the computation in [EES05c, Section 3.5].

$$D_{u_\mu}\bar{\partial}[v,\lambda,\kappa]$$

is the linearization in variations with respect to the vector field v, variation of the clasping family Λ_μ in the space of admissible families of Legendrian immersions, and variation in the conformal structure κ. We will need to expand the total linearization as linearisation of the problem over the fibre over $\mu = 0$ plus a polynomial in μ with coefficients being operators independent of μ.

Variation in v: Under the trivializations above, we can write

$$\bar{\partial}_{J,j}(u) = (1 - S_{\lambda,\mu})^{-1}\Pi_{-\mu V}\circ\Pi_{-D\Phi_\mu(\lambda v)}(du_\mu + J_{\lambda,\mu}\circ du_\mu\circ j), \tag{A.20}$$

where $J_{\lambda,\mu} = \Pi_{\Pi_{\mu V}(\lambda v)} J_\mu \Pi_{\Pi_{\mu V}(\lambda v)}$. Recall,

$$J_\mu = J + \mu E^J + \mathcal{O}(\mu^2),$$

Now, $(1 - S_{\lambda,\mu})^{-1} = 1 + S_{\lambda,\mu} + \mathcal{O}(\lambda^2)$ and one obtains

$$\begin{aligned} S_{\lambda,\mu} &= -\lambda \frac{1}{2} J_\mu \nabla_{D\Phi_\mu v} J_\mu + \mathcal{O}(\lambda^2), \\ &= -\lambda \frac{1}{2} J_\mu (\nabla_v + \mu \nabla_{\nabla_V v}) J_\mu + \mathcal{O}(\lambda^2), \\ &= -\lambda \frac{1}{2} (J \nabla_v J + \mu (E^J \nabla_v J + J \nabla_v E^J + J \nabla_{\nabla_V v} J)) + \mathcal{O}(\lambda^2) + \lambda \mathcal{O}(\mu^2), \\ &= -\lambda \frac{1}{2} J \nabla_v J + \lambda\mu\, E_1[v] + \lambda \mathcal{O}(\mu^2) + \mathcal{O}(\lambda^2). \end{aligned}$$

We want to write the first-order approximation of $\Pi_{-\mu} \nabla_{du_\mu \cdot \eta} \Pi_\mu v$ in μ at 0 in Riemannian normal coordinates. Let $\gamma(\mu) = \exp_u(\mu V)$, denote $X(\mu) = \nabla_{du_\mu \cdot \eta} \Pi_\mu v$ and $Y(\mu) = \Pi_{-\mu} X(\mu)$. Then $X(0) = Y(0) = \nabla_{du \cdot \eta} V$ and

$$\begin{aligned} \frac{D}{d\mu} X(\mu) &= \nabla_{\dot\gamma(\mu)} \nabla_{du_\mu \cdot \eta} \Pi_\mu v, \\ &= \nabla_{\nabla_{\dot\gamma(\mu)} du_\mu \cdot \eta} \Pi_\mu v + \nabla_{du_\mu \cdot \eta} \nabla_{\dot\gamma(\mu)} \Pi_\mu v + R(\dot\gamma, du_\mu \cdot \eta) \Pi_\mu v, \\ \frac{D}{d\mu} X(0) &= \nabla_{\nabla_V du \cdot \eta}\, v + \nabla_{du \cdot \eta} \nabla_V\, v + R(V, du \cdot \eta)\, v. \end{aligned}$$

Let us denote

$$B_{V,w,v} = \nabla_{\nabla_V w}\, v + \nabla_w \nabla_V\, v + R(V, w)\, v.$$

Then $Y(\mu) = \nabla_{du \cdot \eta} V + \mu B_{V, du \cdot \eta, v} + \mathcal{O}(\mu^2)$. And so,

$$\begin{aligned} \Pi_{-\mu V} \circ \Pi_{-D\Phi_{\lambda \Pi_{\mu V} v}} (d\exp_{u_\mu}(\lambda \Pi_{\mu V} v) \cdot \eta) &= \Pi_{-\mu V} (du_\mu \cdot \eta + \lambda \nabla_{du_\mu \cdot \eta} \Pi_{\mu V} v + \mathcal{O}(\lambda^2)) \\ &= du \cdot \eta + \mu \nabla_{du \cdot \eta} V + \lambda \nabla_{du \cdot \eta} v + \lambda\mu B_{V, du \cdot \eta, v} \\ &\quad + \lambda \mathcal{O}(\mu^2) + \mathcal{O}(\lambda^2). \end{aligned}$$

Now, observe that for an endomorphism E and a vector field y it holds that

$$\Pi_{-\epsilon y} E - E \Pi_{-\epsilon y} = \epsilon \nabla_y E + \mathcal{O}(\epsilon^2),$$

and so we compute similarly to the identity above that

$$\begin{aligned} &\Pi_{-\mu V} \circ \Pi_{-D\Phi_{\lambda \Pi_{\mu V} v}} J_{\lambda,\mu} (d\exp_{u_\mu}(\lambda\, \Pi_\mu v) \cdot j\eta) = \\ &\quad = \Pi_{-\mu V} \Big(J_{\lambda,\mu} \Pi_{\Pi_{-\mu} \lambda v}\, d\exp_{u_\mu}(\lambda\, \Pi_\mu v) \cdot j\eta + \lambda \nabla_{\Pi_{-\mu} v} J \cdot d\exp_{u_\mu}(\lambda\, \Pi_\mu v) \cdot j\eta \Big) + \mathcal{O}(\lambda^2) \\ &\quad = J \Pi_{-\mu V} \Pi_{\lambda \Pi_{\mu V} v} d\exp_{u_\mu}(\lambda \Pi_{\mu V}\, v) \cdot j\eta - \mu \nabla_V J \cdot d\exp_{u_\mu}(\lambda \Pi_{\mu V}\, v) \cdot j\eta \\ &\quad + \lambda \nabla_{\lambda \Pi_{-\mu V} v} J \cdot \Pi_{-\mu V} d\exp_{u_\mu}(\lambda \Pi_{\mu V}\, v) \cdot j\eta \\ &\quad - \lambda\mu \nabla_V \Big(\nabla_{\lambda \Pi_{-\mu V} v} J \Big) \cdot d\exp_{u_\mu}(\lambda \Pi_{\mu V}\, v) \cdot j\eta + \lambda \mathcal{O}(\mu^2) + \mathcal{O}(\lambda^2) \\ &\quad = J(du \cdot j\eta) + \mu J \nabla_{du \cdot j\eta} V + \lambda J \nabla_{du \cdot j\eta} v + \lambda\mu J B_{V, du \cdot j\eta, v} - \mu \nabla_V J \cdot du \cdot j\eta - \lambda\mu \nabla_{du \cdot j\eta} V \\ &\quad + \lambda \nabla_v J \cdot du \cdot j\eta + \mu\lambda \nabla_v J \cdot \nabla_{du \cdot j\eta} V + \lambda\mu \nabla_{\nabla_V v} J \cdot du \cdot j\eta + \mathcal{O}(\mu^2) + \lambda \mathcal{O}(\mu^2) + \mathcal{O}(\lambda^2). \end{aligned}$$

Finally, compare the expansions above to define the operator $\bar{S}_1[v]$ and conclude that

$$\begin{align}
D_{u_\mu}\bar{\partial}_{J,j}[v,0,0] &= \bar{\nabla}v - \frac{1}{2}J(\nabla_v J)\partial_{J,j}u + \mu\bar{S}_1[v] + \mathcal{O}(\mu^2), \tag{A.21}\\
&= D_u[v,0,0] + \mu K_1[v] + \mathcal{O}(\mu^2). \tag{A.22}
\end{align}$$

Variation of Λ_μ in admissible families of Legendrian immersions: Let $\lambda \in \mathbb{R}$ describe the variation of Λ_0 in the space of admissible Legendrian immersions that

- outside of the clasping chart looks like variations through admissible Legendrian embeddings,
- keeps the clasping self-intersection point a self-intersection point,
- does not introduce any new self-intersection points.

For details, see the construction in [EES05a, Section 5.4]. This yields a domain dependent family of difeomorphisms

$$\begin{aligned}
\Phi^\zeta_\lambda : D_m \times P &\to D_m \times P,\\
\zeta &\mapsto (\zeta, \Phi^\zeta_\lambda)
\end{aligned}$$

and realise that $\Phi_{\lambda,\mu} = \Phi^\zeta_\lambda \circ \Phi_\mu$ is a domain-dependent family of diffeomorphisms so that $\Phi_{\lambda,\mu}(\Lambda_\mu)$ is an admissible family of Legendrian immersions. Note that for μ and λ sufficiently small, we have that $\Phi^\zeta_\lambda \circ \Phi_\mu = \Phi_\mu \circ \Phi^\zeta_\lambda$ since the metric is flat near the clasping region and the diffeomorphism can be modelled as a flow of a domain-dependent vector field. Similarly to the above, we trivialise

$$\bar{\partial}_{J,j}(u) = (1 - S_{\lambda,\mu})^{-1}\Pi_{-\mu V} D\Phi^\zeta_{-\lambda}(d\Phi^\zeta_\lambda(u_\mu) + J \circ d\Phi^\zeta_\lambda(u_\mu) \circ j)$$

Write $\Phi^\zeta_\lambda = id + \lambda Y^\zeta_\lambda + \mathcal{O}(\lambda^2)$. Then as above,

$$\begin{aligned}
(1 - S_{\lambda,\mu})^{-1} &= 1 + S_{\lambda,\mu} + \mathcal{O}(\lambda^2),\\
S_{\lambda,\mu} &= -\lambda\frac{1}{2}J_\mu(\mathcal{L}_{Y^\zeta_\lambda}J_\mu) + \mathcal{O}(\lambda^2),\\
&= -\lambda\frac{1}{2}(J(\mathcal{L}_{Y^\zeta_\lambda}J) + \mu E_2[\lambda] + \mathcal{O}(\mu^2)) + \mathcal{O}(\lambda^2).
\end{aligned}$$

Therefore, we can compute that

$$\begin{aligned}
D_{u_\mu}\bar{\partial}_{J,j}[0,\lambda,0] &= -\frac{1}{2}J(\mathcal{L}_{Y^\zeta_\lambda}J)(du - Jduj) + (\frac{d}{d\zeta}Y^\zeta_\lambda)d\zeta + J(\frac{d}{d\zeta}Y^\zeta_\lambda)d\zeta \circ j + \mu K_2[\lambda] + \mathcal{O}(\mu^2),\\
&= D\bar{\partial}[0,\lambda,0] + \mu K_2[\lambda] + \mathcal{O}(\mu^2).
\end{aligned}$$

Variation of conformal structure κ: In our trivialisation this corresponds to

$$\bar{\partial}_{J_\mu,j}(u_\mu) = (1 - S_\mu)^{-1}\Pi_{-\mu V}(du_\mu + J \circ du_\mu \circ j(1+\epsilon\gamma)(1-\epsilon\gamma)^{-1})(1+\epsilon\gamma)^{-1}$$

Similarly to the computations above one obtains

$$\begin{aligned}
\bar{\partial}_{J,j}(u) &= (1-\mu\frac{1}{2}J\nabla_V J)(du\cdot\eta+\mu\nabla_{du\cdot\eta}V)\\
&\quad+(1-\mu\frac{1}{2}J\nabla_V J)(J(du\cdot j(1-\epsilon\gamma)\eta)+\mu(J\nabla_{du\cdot j(1-\epsilon\gamma)}V)+\nabla_V J\cdot du\cdot j(1-\epsilon\gamma))\\
&\quad+\epsilon\mathcal{O}(\mu^2)+\mathcal{O}(\epsilon^2),\\
D_{u_\mu}\bar{\partial}_{J,j}[0,0,\gamma] &= -\bar{\partial}_{J,j}(u)\circ\gamma-\mu(\nabla_{du\cdot\eta}V+J\nabla_{du\cdot j\gamma\eta}V+\nabla_V J\cdot du\cdot j\gamma\eta)+\mathcal{O}(\mu^2)\\
&= D_{u_\mu}\bar{\partial}_{J,j}[0,0,\gamma]+\mu K_3[\gamma]+\mathcal{O}(\mu^2)
\end{aligned}$$

Variation of Λ along the family Λ_μ: We will express the operator $\bar{\partial}_{J_\mu,j}u_\mu$ in the local trivialisations as the variation of $\bar{\partial}_{J,j}u$, that is

$$\bar{\partial}_{J_\mu,j}u_\mu=(1-S_\mu)^{-1}\Pi_{-\mu V}(du_\mu+J\circ du_\mu\circ j).$$

Analogously to the above, we obtain that

$$(1-S_\mu)^{-1}=1-\frac{1}{2}\mu J(\mathcal{L}_V J)+\mathcal{O}(\mu^2),$$

and that

$$\begin{aligned}
\bar{\partial}_{J_\mu,j}u_\mu &= \bar{\partial}_{J,j}(u)+\mu(\nabla_{du\cdot\eta}V+J\nabla_{du\cdot j\eta}V+\nabla_V J\cdot du\cdot j\eta-\frac{1}{2}J\mathcal{L}_V J\cdot\partial_{J,j}(u))+\mathcal{O}(\mu^2),\\
&= \bar{\partial}_{J,j}(u)+\mu(\overline{\nabla}_{J,j}V+\nabla_V J\cdot du\cdot j\eta-\frac{1}{2}J\mathcal{L}_V J\cdot\partial_{J,j}(u))+\mathcal{O}(\mu^2),
\end{aligned}$$

As we mentioned earlier, it is important to separate the variations of the Legendrian and the almost complex structure. And so, we should perform the same computation as the above for λ being a parameter in the almost complex structures. We will consider of only the variation with respect to J since the Taylor expansion of the other directional derivatives can be obtained by an argument analogous to the above.

Variation of the almost complex structure J: We vary the almost complex structure J through the ones for whom $\Pi_P(\Lambda)$ is admissible. Especially, close to the double points the almost complex structure does not change. Note that the construction of the immersion $\hat{\iota}$ and the associated metric g^{pr} depends on J and so we are forced to change it along the variation. Fortunately, we perform the clasp move sufficiently close to the double points where the almost complex structure J and so the metric g^{pr} does not change. We thus can assume that the variation of the almost complex structure J is given by a one-parametric family of diffeomorphisms

$$\Phi^\lambda: D_m\times P\to D_m\times P$$

that act as the identity on D_m and

- Φ^λ acts as the identity on small neighbourhoods of the double-points,
- in the complement of those neighbourhoods the Lagrangian projection $\Pi_P(\Lambda_\mu)$ is a submanifold and so has a small normal neighbourhood. There Φ^λ interpolates in between $S_{\lambda,\mu}S_0^{-1}$ close to the zero-section and the identity far enough along the normal fibres.

Then the variation of the almost complex structure J is given as

$$J_{\lambda,\mu}=D\Phi^\lambda\circ J_\mu\circ D\Phi^{-\lambda}.$$

In our trivialisation,

$$\bar{\partial}_{J_{\lambda,\mu},j}(u_\mu) = (1 - S_{\lambda,\mu})^{-1} D\Phi_{-\mu} D\Phi^{-\lambda}(d\Phi^\lambda(u_\mu) + J_{\lambda,\mu} d\Phi^\lambda(u_\mu) j).$$

Let us denote $\Phi^\lambda = id + \lambda Y_\lambda + \mathcal{O}(\lambda^2)$By computation analogous to the above we obtain

$$\begin{aligned} J_{\lambda,\mu} &= J_\mu(1 + 2S_{\lambda,\mu} + \mathcal{O}(\lambda^2)) \\ S_{\lambda,\mu} &= \lambda J_\mu(-\frac{1}{2}\mathcal{L}_{Y_\lambda} J_\mu) + \mathcal{O}(\lambda^2) \end{aligned}$$

Therefore,

$$\begin{aligned} D_{u_\mu}\bar{\partial}_{J_\mu,j}[0,\lambda,0] &= -\frac{1}{2}J(\mathcal{L}_{Y_\lambda} J)(\partial_{J,j} u) + S_\lambda(\partial_{J,j} u) + \mu K_4[\lambda] + \mathcal{O}(\mu^2), \\ &= D\bar{\partial}[0,\lambda,0] + \mu K_4[\lambda] + \mathcal{O}(\mu^2). \end{aligned}$$

□

A.5. **Enhanced transversality and marked points on the boundary.** The goal of this section is to prove that the clasp move does not influence the rigid curves without a puncture at the clasping chord.

Lemma A.17. *Let $n > 1$ be the dimension of the Legendrian submanifold and let $\boldsymbol{w}$ be a word in Reeb chords that does not contain s_μ and $\dim \mathfrak{M}_\mu(\boldsymbol{w}) = 0$. Then there is $\mu_0 > 0$ so that $\mathfrak{p} : \overline{\mathfrak{M}}_{[-\mu_0,\mu_0]} \to [-\mu_0, \mu_0]$ is a local diffeomorphism.*

The proof of this proposition is standard and follows the proof of [EES05b, Proposition 4.10]. It follows from the following two lemmas, whose proofs we include for completeness.

Lemma A.18 (Corollary 9.22 in [EES05b])**.** *No rigid J-holomorphic disk in P without a puncture at s_μ maps any boundary point to s_μ.*

This Lemma can not hold for knots, limiting our results about the clasp move to higher dimension.

Proof. Fix an arbitrary point p_μ on L_μ. Let ζ_0 be a marked point on the boundary of $\dot{D}_m$. Define the evaluation map

$$ev_{\zeta_0} : \mathcal{N}_{2,\epsilon}(\mathbf{c}) \to L_\mu.$$

The evaluation map is smooth and transverse to p_μ since for any direction, one can construct a variational vector field, thus spanning $T_{p_\mu} L_\mu$. Define

$$\mathcal{N}_{2,\epsilon}(\mathbf{c}; p_\mu, \zeta_0) = ev_{\zeta_0}^{-1}(p_\mu),$$

a closed submanifold of $\mathcal{N}_{2,\epsilon}(\mathbf{c})$ of codimension equal to $\dim L$. Observe that the tangent space of $\mathcal{N}_{2,\epsilon}(\mathbf{c}; p_\mu, \zeta_0)$ is spanned by the vector fields satisfying the standard boundary conditions and moreover that they vanish at ζ_0. One can define a new Banach manifold parametrised over the boundary of D_m with standard trivialisations

$$\mathcal{N}_{2,\epsilon}(\mathbf{c}; p_\mu) = \bigcup_{\zeta_0 \in \partial \dot{D}_m} \mathcal{N}_{2,\epsilon}(\mathbf{c}; p_\mu, \zeta_0).$$

Let $\mathcal{H}(p)$ be the space of suitable Hamiltonians so that the induced Hamiltonian diffeomorphisms fix p. One can see that the proof of transversality follows through as ζ_0 lies in the compact part of the curve. This yields a locally regular moduli space, denoted with $\mathfrak{M}_\mu(\mathbf{c}; p)$.

Therefore, we have a Baire set of admissible clasp moves L_μ so that the assumption of the Lemma is satisfied. Else, the transversality and the Fredholm property implies that

$$\dim \mathfrak{M}_\mu(\mathbf{c}; p) = ind\, D\Gamma,$$

where $\Gamma : \mathcal{N}_{2,\epsilon}(\mathbf{c}) \to \mathcal{H}_{1,\epsilon}$ is the section given by the perturbed $\bar{\partial}$ operator. Fixing μ the section looks in trivialisation as

$$\Gamma : \mathbb{R} \times \mathcal{W}_{2,\epsilon}(\mathbf{c}; p) \to \mathcal{H}_{1,\epsilon},$$

the first coordinate describes the variation of ζ_0 along the boundary and the evaluation cuts out a codimension $\dim L$ submanifold out of the second factor, so the dimension formula yields

$$\dim \mathfrak{M}_\mu(\mathbf{c}, p) = \dim \mathfrak{M}_\mu(\mathbf{c}) - \dim L + 1 = 0 - n + 1 < 0,$$

which can not generically occur. ☐

Lemma A.19 (proof of Lemma 4.4 in [EES05b])**.** *There is $d > 0$ so that no rigid disk without the clasping chord as a puncture maps any boundary point $B(s_\mu, d)$.*

Proof. For contradiction, let $v_j \to (v^1_\infty, \dots, v^N_\infty)$ be a limiting sequence of rigid curves so that one of the limit curves intersects all balls $B(s_\mu, d)$, which means that either the curve breaks at s or that it has a boundary point mapping to s.

The second case is ruled out by Lemma A.18 as

$$\sum_k ind\, v^k_\infty = ind\, v_j = 0,$$

and no rigid v^j can generically touch s, therefore, the broken sequence must contain a curve of negative formal dimension.

There are no ghosts in the limit, and so Stokes' inequality obstructs the possibility of s being a positive puncture with other negative punctures. We have shown in the proof of Lemma 4.2 that there can not be any disk with no negative puncture having s as the only positive puncture. This concludes the proof. ☐

Appendix B. Ekholm's flow trees

Here we will explain the Ekholm's flow tree technology (see [Ekh07]). This section does not contain any new results and should not be perceived as a complete survey of the topic. The aim is to explain the flow-tree technology sufficiently so that a reader not familiar with such techniques can read and understand the proof of Lemma 3.6.

Let Λ be a closed Legendrian embedding in $J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R}$ (with the standard contact form $dz - ydx$), where x are the coordinates on the base $\mathbb{R}^n$, y are the coordinates on the cotangent fibre, and z describes the contactization direction. Let $\Pi_F : J^1(\mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}$ be the so-called projection

$$(x, y, z) \mapsto (x, z).$$

We say that the set $\Pi_F(\Lambda)$ is the front of Λ. It is a stratified set, where each strata is a smooth manifold. The top-dimensional stratum is a smooth n-dimensional manifold (possibly with corners). We will call these sheets S. The lower-dimensional loci can be collected to a so-called singular set. For our methods to work, we need to assume that we work with a Legendrian submanifold has simple singularities of its front (for details see [Ekh07, Section 2.2]). In our case, the front has only cusp singularities that are simple.

Let U_i be open sets of $\mathbb{R}^n$ and $f_i : U_i \to \mathbb{R}$ be functions locally parametrising the sheet S_i over U_i. Say that U_i and U_j overlap, then we define the so-called local difference function

$$f_{ij} = f_i - f_j.$$

The critical points of the local difference function correspond to Reeb chords starting at the sheet S_j and ending at the sheet S_i.

Now, the flow trees are trees in the base $\mathbb{R}^n$, where the edges are negative gradient flow-lines of local difference functions so that the vertex is (generically) either

- critical point of the local difference function (we denote these with a full circle). See P_0, and P_1-vertices in Figure 16.
- a point, where three trajectories of three local difference functions meet, and which is not a point in the projection of the singular set. See vertex Y_0 in Figure 16 (denoted with a circle).
- a point, where three trajectories of three local difference functions meet, and which is a point in the projection of the singular set. See vertex Y_1 in Figure 16 (denoted with a circle).
- a point, that is in the projection of the singular set. See vertex E in Figure 16 (denoted with a circle).
- in general, we have to allow more vertices (S-vertex), but these do not appear in the computations of this paper, and so we leave them out (for details see [Ekh07, Remark 3.8])

Note that different edges can be flow-lines of different difference functions. We require that the indices of the local difference functions adjacent to the Y_0, Y_1, and P_1-vertices satisfy the relations as in Figure16. In the upper part, we draw the lift of the flow-tree into the front projection: the polygon coloured light blue, the sheets in grey, and the Reeb chords in dark blue. In the lower part of the figure, we draw the corresponding flow-tree in the base. Note that the P_1-vertex is a combination of a Y_0-vertex, where one of the outgoing trajectories is a constant trajectory (ghost) over a P_0-vertex.

The spaces of flow-trees have the same expected dimension formula as the spaces of pseudo-holomorphic curves, and the same Stokes bound on the quantity substituting area applies to the flow-trees. For the flow tree to be rigid, it has to lie in the appropriate intersections of the stable and unstable manifolds of the local difference functions. This means that a rigid tree can a priori contain a trajectory that is not rigid; however, the interaction with a different sheet (through P_1, Y_0, or Y_1 vertex) rigidifies the trajectory. We will draw the flow tree as the superposed polygons in the front projection that are the lift from the base. Note that along the flow, this polygon should have decreasing action along the fibres of $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. This action can be zero in one of the admissible vertices, else it is not an example of a flow-tree (this happens frequently in the presence of intersecting sheets).

Let $\mathfrak{T}(a; \mathbf{b})$ be a space of flow trees with a positive vertex at a and a negative punctures at Reeb chords $\mathbf{b} = b_1 \dots b_k$. The dimension of this space is $\dim \mathfrak{T}(a; \mathbf{b}) = |a| - \sum_{i=1}^{k} |b_i| - 1$. There is a bijection of rigid flow trees and the space of rigid pseudo-holomorphic curves in $T^*\mathbb{R}^n$ with positive puncture asymptotic to the double point corresponding to a, negative punctures asymptotic to the double points b_i's, and the boundary mapping to the Lagrangian projection $L = \Pi_P(\Lambda)$ (see [Ekh07, Theorem 1.1]):

$$\mathfrak{T}(a; \mathbf{b}) \xleftrightarrow{1-1} \mathfrak{M}(a; \mathbf{b}).$$

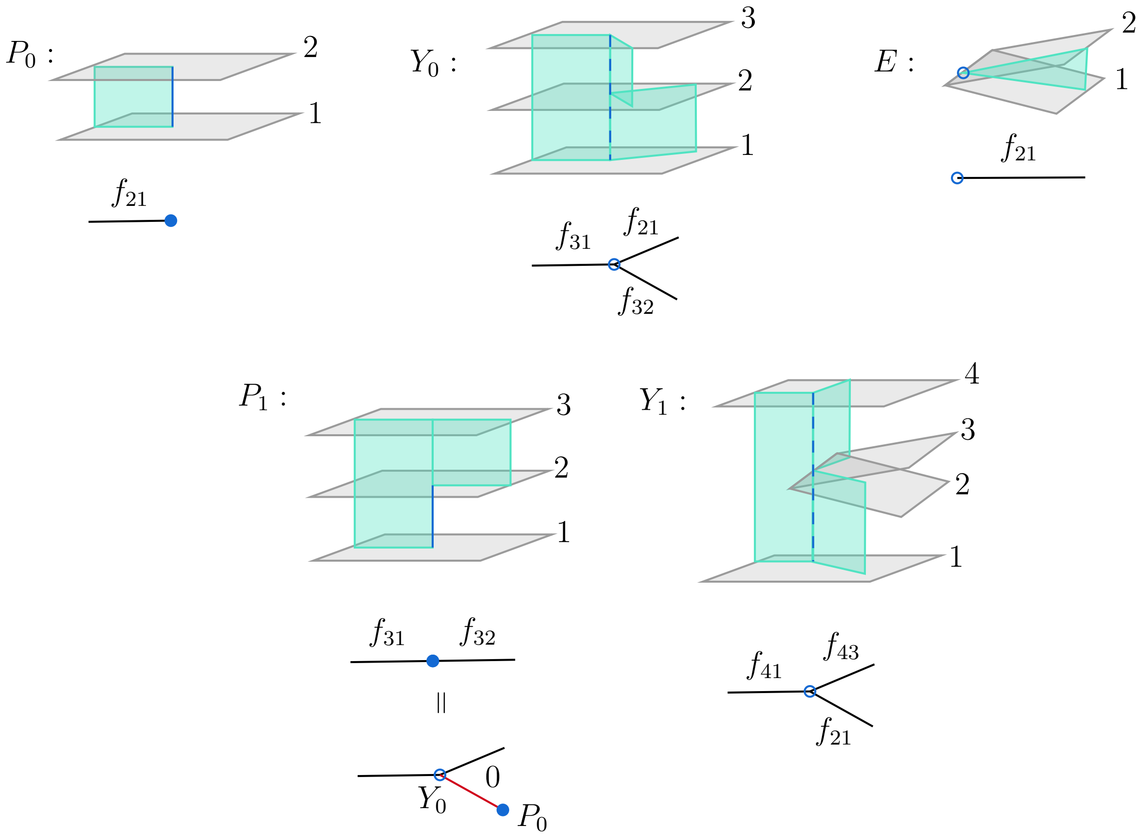


FIGURE 16. Vertices generically appearing in the flow-trees. Note that the pictures for P_1, Y_0 and Y_1 can be inverted along the vertical axis. The blue full dots denote the projection of Reeb chords that are denoted with blue line segments. The non-full dots represent some extraordinary point in the base.

We use this bijection to compute the differential of the Chekanov-Eliashberg algebra of the k-clef spheres in Lemma 3.6.

References

[Avd25] Russell Avdek, *A filtered generalization of the Chekanov–Eliashberg algebra*, Quantum Topology (2025).

[Che02] Yuri Chekanov, *Differential algebra of Legendrian links*, Inventiones Mathematicae **volume 150** (2002), 441–483.

[CL07] Kai Cieliebak and Janko Latschev, *The role of string topology in symplectic field theory*, 2007.

[DR16a] Georgios Dimitroglou Rizell, *Legendrian ambient surgery and Legendrian contact homology*, Journal of Symplectic Geometry **volume 14** (2016), no. 3, 811–901.

[DR16b] ———, *Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications*, Quantum Topology **volume 7** (2016), no. 1, 29–105.

[Duk24] Milica Dukic, *Extension of Chekanov-Eliashberg algebra using annuli*, 2024.

[EES05a] Tobias Ekholm, John Etnyre, and Michael Sullivan, *The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$*, Journal of Differential Geometry **volume 71** (2005), no. 2, 177 – 305.

[EES05b] ———, *Non-isotopic Legendrian submanifolds in $\mathbb{R}^{2n+1}$*, Journal of Differential Geometry **volume 71** (2005), no. 1, 85 – 128.

[EES05c] Tobias Ekholm, John Etnyre, and Michael G. Sullivan, *Legendrian Contact Homology in $P \times \mathbb{R}$*, 2005.

[EES09] Tobias Ekholm, John B. Etnyre, and Joshua M. Sabloff, *A duality exact sequence for Legendrian contact homology*, Duke Mathematical Journal **volume 150** (2009), no. 1, 1 – 75.

[EFM01] J. Epstein, D. Fuchs, and M. Meyer, *Chekanov–Eliashberg invariants and transverse approximations of Legendrian knots*, Pacific Journal of Mathematics **201** (2001), no. 1, 89–106.

[Ekh07] Tobias Ekholm, *Morse flow trees and Legendrian contact homology in 1–jet spaces*, Geometry & Topology **volume 11** (2007), no. 2, 1083–1224.

[ENV13] John B. Etnyre, Lenhard L. Ng, and Vera Vértesi, *Legendrian and transverse twist knots*, Journal of the European Mathematical Society **15** (2013), no. 3, 969–995.

[Gol14] Roman Golovko, *A note on the front spinning construction*, Bulletin of the London Mathematical Society **volume 46** (2014), no. 2, 258–268.

[Gol23] ———, *On non-geometric augmentations in high dimensions*, Geometriae Dedicata **217** (2023), no. 6.

[Lan06] Serge Lang, *Introduction to differentiable manifolds*, Springer Science & Business Media, 2006.

[Mey99] M. Meyer, *Computation of Chekanov–Eliashberg invariants for Legendrian knots*, Ph.D. thesis, University of California, Davis, Davis, CA, 1999.

[Mur19] Emmy Murphy, *Loose Legendrian embeddings in high-dimensional contact manifolds*, preprint,`https://arxiv.org/abs/1201.2245`, 2019.

[Ng10] Lenhard Ng, *Rational symplectic field theory for Legendrian knots*, Inventiones mathematicae **182** (2010), no. 3, 451–512.

[PR19] Yu Pan and Dan Rutherford, *Functorial LCH for immersed lagrangian cobordisms*, 2019.

[Sen23] Balarka Sen, *h-principle for loose Legendrian embeddings*, 2023, lecture notes.

[Wu58] Wen-Tsün Wu, *On the isotopy of C^r-manifolds of dimension n in euclidean $(2n+1)$-space*, Science Record (N.S.) **2** (1958), 271–275.