UPPSALA DISSERTATIONS IN MATHEMATICS

150

Mass and Causality in Geometric Relativity

Benjamin Meco

Department of Mathematics
Uppsala University
UPPSALA 2026

Dissertation presented at Uppsala University to be publicly examined in Sonja Lyttkens, Ångström, Regementsvägen 10, Uppsala, Friday, 5 June 2026 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English. Faculty examiner: Professor Lan-Hsuan Huang (University of Connecticut).

Abstract

Meco, B. 2026. Mass and Causality in Geometric Relativity. *Uppsala Dissertations in Mathematics* 150. 60 pp. Uppsala: Department of Mathematics, Uppsala University. ISBN 978-91-506-3175-3.

In this thesis we study a variety of problems in Geometric Relativity concerned with the notion of mass and its positivity, and with developing a metric geometry approach to comparison of spacetimes.

Paper I is concerned with the positive mass theorem for asymptotically anti-de Sitter initial data sets. We suggest a new coupled system of partial differential equations, which yields a proof of this result whenever it is solvable. Similar to some previous approaches, our system involves the generalized Jang equation but is arguably simpler from an analytic and geometric perspective. We establish the existence of geometric solutions to the generalized Jang equation for a given warping factor with suitable asymptotics.

In Paper II we give a new proof of the positive mass theorem for three dimensional asymptotically Euclidean manifolds. Using the minimal Green function of the Laplace operator, we define a several parameter family of vector fields which can be employed to derive a family of geometric inequalities, using integration by parts. We show that these inequalities can be used to prove the positive mass theorem and several other geometric inequalities.

In Paper III we propose a new definition of ADM mass for asymptotically Euclidean manifolds which allows for metrics of low regularity. Our definition is inspired by the definition of mass for weakly regular asymptotically hyperbolic manifolds of Gicquaud and Sakovich. Under suitable asymptotic assumptions we show that our weak ADM mass is well defined, finite, coordinate invariant, and that it agrees with the classical notion of ADM mass when the metric is smooth.

In Paper IV we construct uniform Temple charts for spacetimes equipped with weak temporal functions. We prove that these charts are bi-Lipschitz and that causality is encoded by a weak temporal function and the associated null distance within a uniform Temple chart. Related to this, we prove a Lorentzian isometry theorem generalizing an earlier result of Sakovich and Sormani. These results will be used in our future work to define a notion of spacetime intrinsic flat convergence in the Lorentzian setting.

Keywords: Positive mass theorem, Initial data sets, Prescribed mean curvature equation, Jang equation, C-almost minimizing currents, Asymptotically hyperbolic manifolds, ADM-mass, Asymptotically Euclidean manifolds, Riemannian geometry in low regularity, Temple coordinate charts, Null Distance, Time functions, Encoding of Causality

Benjamin Meco, Department of Mathematics, Geometri and Physics, Box 480, Uppsala University, SE-751 06 Uppsala, Sweden.

ISSN 1401-2049
ISBN 978-91-506-3175-3
URN urn:nbn:se:uu:diva-583550 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-583550)

To my mother, to my father,
to Vedran, and to Elnaz.
You are the ones I think of first.

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Benjamin Meco. "On the existence and properties of solutions of the generalized Jang equation with respect to asymptotically anti-de Sitter initial data". In: *Annals of Global Analysis and Geometry* 68:9 (2025)

II Carla Cederbaum, Ariadna León Quirós and Benjamin Meco. "Applications of vector field methods to problems on Riemannian manifolds with non-negative scalar curvature". Manuscript (2026)

III Stig Lundgren and Benjamin Meco. "A generalization of the ADM mass for asymptotically Euclidean manifolds of weak regularity". In: *Geometriae Dedicata* 220:18 (2026)

IV Benjamin Meco, Anna Sakovich and Christina Sormani. "Existence of uniform Temple charts and applications to null distance". Submitted (2026)

Reprints were made with permission from the publishers.

Contents

1. Introduction

The central objects of study in Mathematical General Relativity are *spacetimes*. Roughly speaking, a spacetime is a model of all *events* in our universe, which are specified by a time and a place. Formally, a spacetime is a pair (N^{n+1}, h) where N^{n+1} is manifold of dimension $n+1 \geq 2$ and h is a so called *Lorentzian metric*. It is common to study spacetimes satisfying the *Einstein equations*:

$$\mathrm{Ric}^h - \frac{\mathrm{Scal}^h h}{2} + \Lambda h = \mathrm{T}. \tag{1.1}$$

Here, Ric^h and Scal^h are the Ricci and scalar curvatures respectively of the metric h, Λ is referred to as the *cosmological constant* and T is the so called *stress-energy tensor*. The tensor T describes how mass and energy are distributed at different times and places while the Einstein equations (1.1) describe how the curvature of the universe (the left hand side) is determined by its mass and energy content (the right hand side).

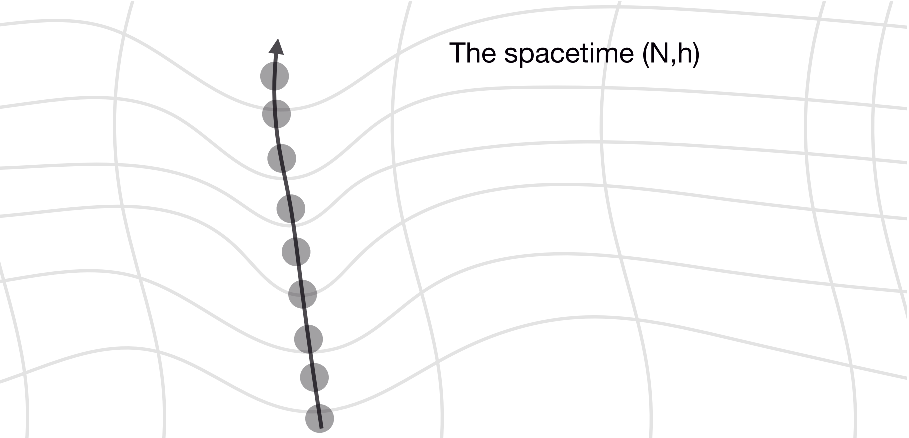

Figure 1.1. A spacetime curving due to a massive body. The trajectory of the body is at the same time determined by the curvature of the spacetime.

As in any physical theory, it is standard to ask *"given data about the present, what can be said about the future?"*. Such problems are in general referred to as *Cauchy problems* or *initial value problems*. There is more than one way to formulate this question rigorously in Mathematical General Relativity. One formulation involves a so called *initial data set*, which is a triple (M^n, g, k) that consists of a smooth manifold M^n, a Riemannian metric g and a symmetric $\binom{2}{0}$ tensor field k. The Riemannian manifold (M^n, g) is interpreted as the "initial position" of the universe while the tensor k is thought of as the "initial velocity". The Cauchy problem of Mathematical General Relativity amounts to finding a spacetime (N^{n+1}, h) satisfying the Einstein equations and an embedding $\iota : M^n \hookrightarrow N^{n+1}$ such that $\iota^* h = g$ and such that the second fundamental form of the manifold $\iota(M^n) \subset N^{n+1}$ is $\iota_* k$.

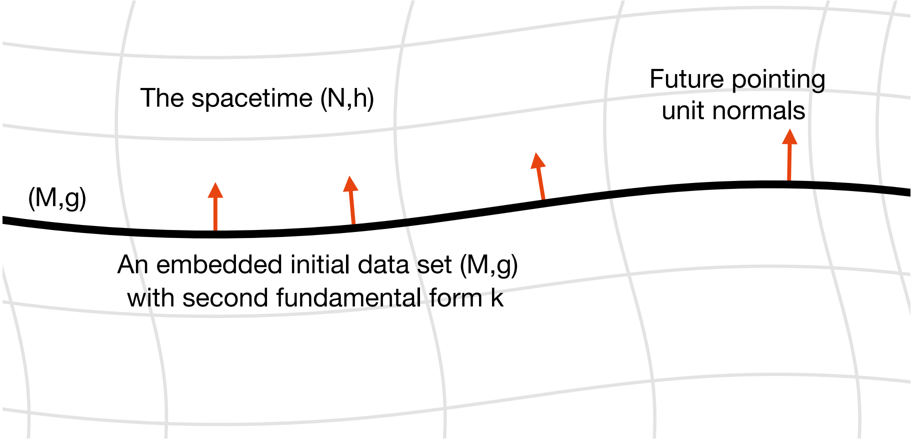

Figure 1.2. An initial data set (M^n, g, k) embedded in a spacetime (N^{n+1}, h).

For a Cauchy problem to be a reasonable physical model, it is important that the problem be *well-posed*. In this context, one needs to understand for what initial data sets (M^n, g, k) the following are true.

- There exists a solution to the Cauchy problem.
- Solutions to the Cauchy problem are unique.
- Solutions to the Cauchy problem are "continuous in the initial data".

To highlight the significance of the above, we note that if the there is no solution to the Cauchy problem, then one can make no predictions and if there are more than one solution then one does not know which prediction to trust. The third point above says that if we want more accurate predictions of the future, then it suffices to have a more accurate model and sufficiently precise measurements of the present.

Due to the Cauchy problem being a central subject in the field of Mathematical General Relativity, these questions have been extensively studied and the progress in this area is due to many people. That being said, a very important figure in the field was Yvonne Choquet-Bruhat. The first very general results showing that a spacetime can be constructed from an initial data set were proven by Choquet-Bruhat in 1952 [31] and subsequently generalized by Choquet-Bruhat and Geroch [24]. To give a fair and complete view of this topic is outside the scope of this introduction, we refer the reader to the text by Ringström [55] and the references therein for further reading.

Since Mathematical General Relativity is a theory of gravity, the most important physical quantities that one would like to study are mass and energy. This thesis consists of four papers which try to address the following questions in the field.

- How can the mass of a spacetime be defined and what properties should the mass have?
- Is the mass of a physically reasonable spacetime non-negative?
- How can spacetimes be compared, and is it possible to define a notion of convergence for spacetimes?

To give some context to the above questions, recall that the Einstein equations relate the mass and energy content of the universe to the curvature. So the first question might be understood as follows: "*knowing only the geometry of a spacetime, is it possible to calculate the energy content?*". This turns out to be a surprisingly difficult question to answer even the most commonly used notions of mass in Mathematical General Relativity are not non-negative for any obvious reason, giving rise to the second question above. Lastly, a tool for comparing spacetimes is needed in order to study questions such as "Do solutions to the Cauchy problem depend continuously on the initial data?" and "Does the mass depend continuously on the spacetime in some reasonable topology?".

We are not the first to be interested in these questions. The goal of this introduction to the thesis is to briefly review relevant theory, to give overview of previous work and to indicate how the work of this thesis fits into this larger body of knowledge.

2. Preliminaries

2.1 Spacetimes

A *Lorentzian manifold* is a pair (N^{n+1}, h) where N^{n+1} is a connected $n+1 \geq 2$ dimensional smooth manifold and h is a symmetric $\binom{2}{0}$ tensor field h of signature $(-,+,\dots,+)$, called a *Lorentzian metric*. The most common example of a Lorentzian manifold is *Minkowski space*.

Example 2.1. Minkowski space, $\mathbb{M}^{n+1}$, is the manifold $\mathbb{R}^n \times \mathbb{R}$, where the metric η is given by

$$\eta = -dt^2 + \sum_{i=1}^{n} dx_i^2,$$

where t and $x_1, \dots, x_n$ are the usual coordinates on $\mathbb{R}$ and $\mathbb{R}^n$ respectively.

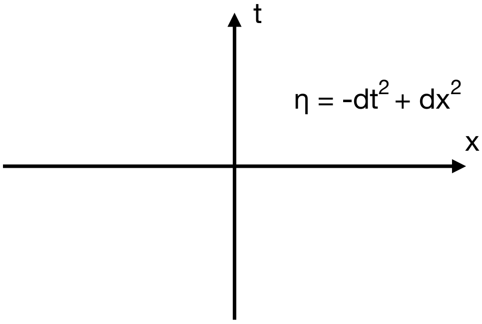

Figure 2.1. An illustration of 2-dimensional Minkowski space, $\mathbb{M}^{1+1}$.

A point $p \in N$ is a representation of a time and place that a particle could inhabit. A curve $\gamma : I \subset \mathbb{R} \to N^{n+1}$ represents the history of a particle moving in the universe. The metric h determines what curves $\dot{\gamma}$ represent physically reasonable particles. In general, we use the following terms.

Definition 2.2. Let $v \in TN^{n+1}$ be a tangent vector, let $\gamma : I \subset \mathbb{R} \to N^{n+1}$ be a differentiable curve and let $\Sigma \subset N^{n+1}$ be a hypersurface.

- We say that a vector v is *timelike* if $h(v,v) < 0$, it is *null* if $h(v,v) = 0$, it is *causal* if $h(v,v) \leq 0$ and lastly that it is *spacelike* if $h(v,v) > 0$.
- The curve γ is causal/null/spacelike or timelike if $\dot{\gamma}(t)$ is always such.
- The hypersurface Σ is called spacelike if every tangent vector $v \in T\Sigma$ is spacelike.

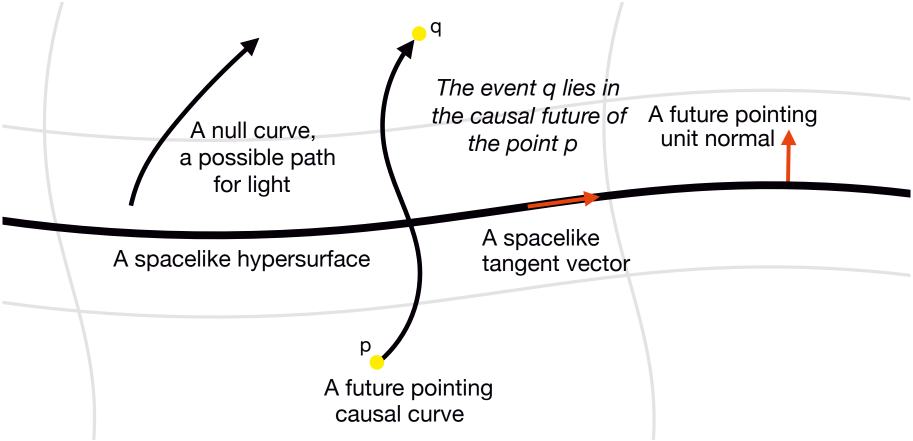

Figure 2.2. An illustration of the concepts in Definition 2.2.

A causal tangent vector $v \in TN^{n+1}$ represents a direction in which a particle slower than (or as fast as) light might move, while a null vector represents a direction that light might move in. By definition, a tangent vector $v \in TN^{n+1}$ is causal iff $-v$ is causal. Keeping Minkowski space in mind, one is tempted to say that one of the vectors v and $-v$ represents movement into the future and the other into the past. When such a distinction can be made continuously and globally, we say that a Lorentzian manifold is *time-orientable*.

Definition 2.3. A *spacetime* is a time-orientable Lorentzian manifold.

To summarize, a spacetime is a collection of points for which there is a notion of "particles moving as fast as or slower than light" and a notion of "particles moving into the future or past". A curve $\gamma : I \subset \mathbb{R} \to N^{n+1}$ such that $\dot{\gamma}(t)$ is causal and future pointing for all t represents the history of a physical particle moving slower than (or as fast as) light.

In Mathematical General Relativity, one often talks about events in spacetimes being able to influence each other. We say that $p \leq q$, meaning that p could influence q, if there exists a future pointing causal curve γ from p to q. In physical terms, this means that it is possible to send a signal from the event p and for someone to receive it at q. If $p \leq q$ or $q \leq p$, the points p and q are said to be *causally related*. The information about what events can and cannot influence each other is referred to as the *causal structure* of the spacetime.

2.2 Initial data sets

In classical mechanics, a system can have a certain energy or momentum or angular momentum and so on, at a certain *instant of time*. Similarly, physical quantities of spacetimes are often calculated at the level of so called *initial data sets*.

Definition 2.4. By an *initial data set* we mean a triple (M^n, g, k) where M^n is an n-dimensional manifold, g is a Riemannian metric and k is a symmetric $\binom{2}{0}$ tensor field.

Recall that in the context of the initial value problem of Mathematical General Relativity, the Riemannian manifold (M^n, g) is interpreted as an "initial position" of the universe while the tensor k is thought of as an "initial velocity". A natural way to construct an initial data set is to let (N^{n+1}, h) be a spacetime, let $M^n \hookrightarrow N^{n+1}$ be any spacelike hypersurface, let g be the induced metric and let k be the second fundamental form, as in the following example.

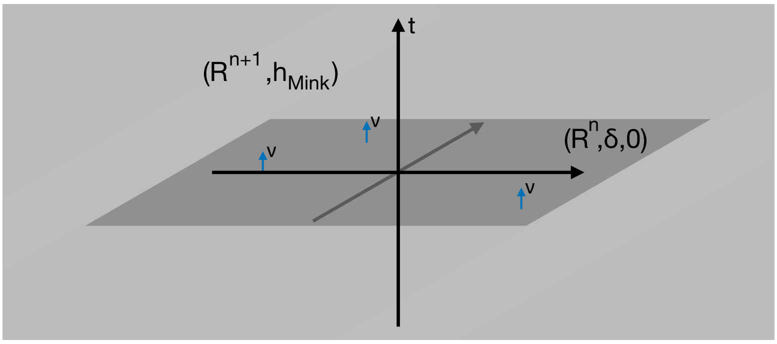

Figure 2.3. An illustration of the initial data set $(\mathbb{R}^n, \delta, 0)$ as a spacelike slice in Minkowski space.

Certain types of initial data sets model special systems in special spacetimes. *Asymptotically Euclidean initial data sets* model the initial conditions of an isolated system in a vacuum of a non-expanding universe. These may have very wild behaviour inside of a compact set $K \subset M$ but resemble $(\mathbb{R}^n, \delta, 0)$ more as one approaches infinity.

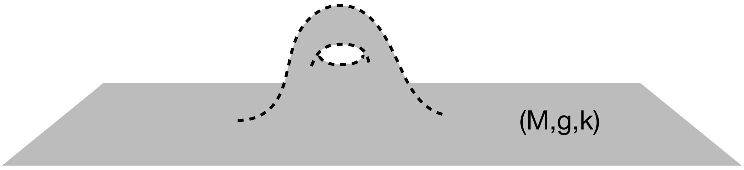

Figure 2.4. An illustration of an asymptotically Euclidean initial data set (M^n, g, k). Outside of a compact set, it looks very much like $(\mathbb{R}^n, \delta, 0)$.

Formally, we say that a Riemannian manifold (M^n, g) is asymptotically Euclidean if there is a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_R}$ defined outside of a compact set $K \subset M$ such that in the coordinates of Φ, the components g_{ij} tend to δ_{ij} at infinity. An initial data set (M^n, g, k) is called asymptotically Euclidean if (M^n, g) is an asymptotically Euclidean manifold and in the coordinates of the diffeomorphism Φ, the components k_{ij} tend to zero at infinity.

A special case of asymptotically Euclidean manifolds and initial data sets are the asymptotically Schwarzschildean ones. These are asymptotically Euclidean manifolds respectively initial data sets that strongly resemble the exterior of an isolated non-rotating black hole in a vacuum.

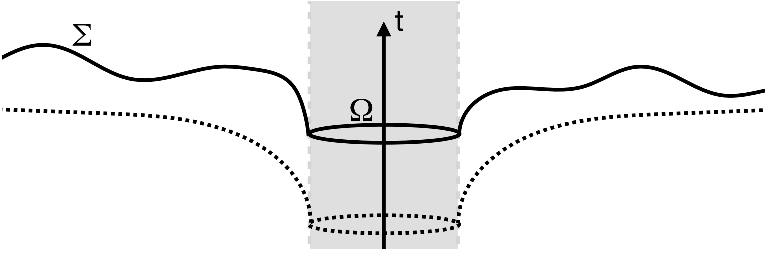

Figure 2.5. An illustration of a spacelike hypersurface Σ in the Schwarzschild spacetime. The gray region represents the interior of a black hole. The dotted surface represents Schwarzschild *space*, an initial data set.

Apart from the asymptotically Euclidean ones, we highlight the so called *asymptotically hyperbolic* manifolds and asymptotically *anti-de Sitter* initial data sets. These are initial data sets modelled on the initial data set $(\mathbb{H}^n, b, 0)$ consisting of hyperbolic space $(\mathbb{H}^n, b)$ equipped with the tensor $k = 0$. The initial data set $(\mathbb{H}^n, b, 0)$ is a spacelike hypersurface in anti-de Sitter space.

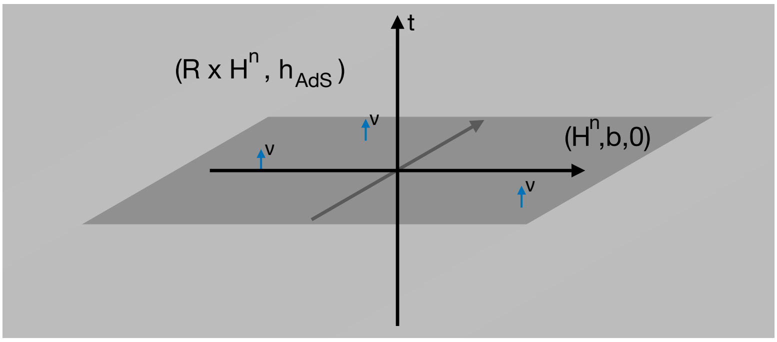

Figure 2.6. The initial data set $(\mathbb{H}^n, b, 0)$ as a space like slice in anti-de Sitter space.

These can be similarly defined in terms of coordinates at infinity. An asymptotically anti-de Sitter initial data set models an isolated system sitting in a vacuum of an *expanding* universe.

2.3 Mass in Mathematical General Relativity

We will now focus on the most important physical quantities in the Mathematical General Relativity, namely the mass and the energy. Perhaps the most commonly encountered notion of mass in the field is the eponymous *ADM mass of an asymptotically Euclidean manifold*, given by Arnowitt, Deser, and Misner [4, 5, 6, 7].

The definition of the ADM mass can be derived using a Hamiltonian formalism, see for example Chruściel [25]. Exactly how the mass is defined for an initial data set depends on what type of initial data set is being dealt with. Michel [52] gives a general argument for how an "ADM style" mass of an initial data set (M^n, g, k) can be defined by comparing it with a model initial data set (M_0^n, g_0, k_0) seen as a slice of some model spacetime (N_0, h_0). It is possible to view the model slice (M_0^n, g_0, k_0) as a sort of "lowest energy state" and the ADM mass then becomes a measure of how far a given initial data set (M^n, g, k) differs from this lowest energy state.

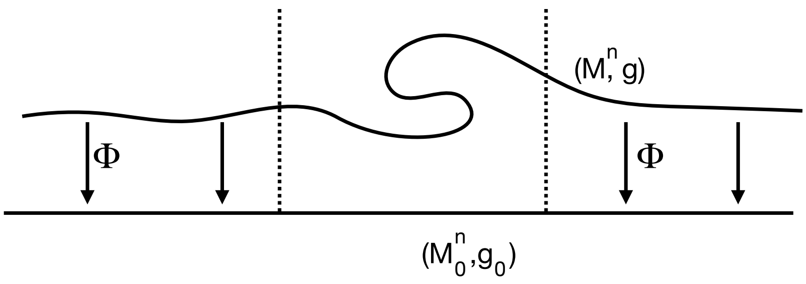

Figure 2.7. A representation of a "coordinate chart at infinity", comparing the initial data set (M^n, g, k) to the reference (M_0^n, g_0, k_0).

In classical mechanics, there is a notion of a local energy density of the gravitational field which can be integrated in order to recover the mass of any given part of a system. In general relativity, the situation is not as simple and there is no notion of a local energy density of an initial data set (M^n, g, k) which can be integrated to yield the total mass of a system. In fact the standard definitions of mass in Mathematical General Relativity are made to detect the total mass of isolated systems such as black holes, galaxies or stars that are thought to be alone in an universe which is otherwise empty.

In order to describe these definitions we will focus on the following two quantities coming from the stress-energy tensor T in the Einstein equations, namely the *energy density* μ and the *current density* J, given by

$$\begin{aligned} \mu &= \frac{1}{2}\left(\mathrm{Scal}^g - n\Lambda + \mathrm{tr}_g(k)^2 - |k|_g^2\right), \\ J &= \mathrm{div}_g\, k - d\,\mathrm{tr}_g\, k. \end{aligned} \tag{2.1}$$

In the case of asymptotically Euclidean manifolds (M^n, g), the cosmological constant is $\Lambda = 0$ and we think of the tensor k as simply being 0, so that the current density vanishes $J = 0$ and the energy density satisfies $2\mu = \mathrm{Scal}^g$. As mentioned, this is not a local energy density that may be integrated as in classical mechanics. Nevertheless, we can interpret this in the following way. If we think of an asymptotically Euclidean manifold $(M^{\cdot} g)$ as a perturbation of the slice $(\mathbb{R}^n, \delta, 0)$ in Minkowski space, which is "the lowest energy state", it is natural to relate the energy of this slice to the deviation of the energy density μ from that of Euclidean space. Since $2\mu = \mathrm{Scal}^g$, and since g is asymptotic to δ , it seems that a good approximation to the energy density μ can be obtained by considering the linearization of the scalar curvature Scal^g as a function of the metric g, about the metric δ.

We will now formalize this intuitive approach in the special case of Riemannian manifolds (M^n, g), following the argument presented by Michel [52].

Let (M_0^n, g_0) be a *reference manifold*, for example Euclidean space $(\mathbb{R}^n, \delta)$ or hyperbolic space $(\mathbb{H}^n, b)$. Let (M^n, g) be a Riemannian manifold and suppose for the sake of simplicity that $M^n = M_0^n$. In this context, we say that g is asymptotic to g_0 if $g - g_0$ tends to zero at infinity. This can be used to define asymptotically Euclidean and asymptotically hyperbolic manifolds. We have the following equality

$$\mathrm{Scal}^g = \mathrm{Scal}^{g_0} + D\,\mathrm{Scal}\,|_{g_0}(e) + Q(g, g_0), \tag{2.2}$$

where $e := g - g_0$ and $Q(g, g_0)$ represents terms that are quadratic in $g - g_0$ and its derivatives and $D\,\mathrm{Scal}\,|_{g_0}$ is the linearization of the scalar curvature at the metric g_0. Equation (2.2) can be thought of as a first order Taylor expansion of the scalar curvature with a second order error term. Integrating (2.2) against a function $V : M_0^n \to \mathbb{R}$ to be determined, over a domain $\Omega \subset M_0^n$ and with respect to the volume form of the metric g_0, one finds

$$\int_\Omega V\left(\mathrm{Scal}^g - \mathrm{Scal}^{g_0} - Q(g, g_0)\right) d\mu_{g_0} = \int_\Omega V\, D\,\mathrm{Scal}\,|_{g_0}(g - g_0)\, d\mu_{g_0}. \tag{2.3}$$

Note that the right hand side here we are integrating the linearization of the scalar curvature, as promised. Applying the divergence theorem in the right hand side of (2.3) one obtains a boundary term and the resulting expression looks as follows.

$$\begin{aligned}
&\int_{\Omega} V\left(\operatorname{Scal}^{g}-\operatorname{Scal}^{g_0}-Q(g,g_0)\right)d\mu_{g_0}\\
&=\int_{\partial\Omega}\left(V(\operatorname{div}_{g_0}(g)-d\operatorname{tr}_{g_0}(g))-e(\nabla^{g_0}V,\cdot)+(\operatorname{tr}_{g_0}e)dV\right)(\nu^{\Omega})d\mu_{g_0} \qquad (2.4)\\
&+\int_{\Omega}\langle e,(D\operatorname{Scal}|_{g_0})^*(V)\rangle_{g_0}d\mu_{g_0}.
\end{aligned}$$

Here, $(D\operatorname{Scal}|_{g_0})^*$ is the formal adjoint of $D\operatorname{Scal}|_{g_0}$. If $V(\operatorname{Scal}^g-\operatorname{Scal}^{g_0})\in L^1(M_0^n)$ and if $g-g_0$ falls of quickly enough so that $VQ(g,g_0)\in L^1(M_0^n)$, then the integral in the left hand side above converges to an integral over all of M_0^n if one chooses larger and larger domains $\Omega\subset M_0^n$ that exhaust M_0^n . If additionally $V\in\mathscr{N}:=\ker((D\operatorname{Scal}|_{g_0})^*)$ then the second integral on the right hand side above vanishes identically, and all that is left of the integral of the linearization of the scalar curvature is a so called "flux integral at infinity".

All in all, under the above described conditions, one can define the *mass functional* as a linear map $\mathscr{M}_{g,g_0}:\mathscr{N}\to\mathbb{R}$ given by

$$\begin{aligned}
&2(n-1)\omega_{n-1}\mathscr{M}_{g,g_0}(V)\\
&:=\lim_{k\to\infty}\int_{\partial\Omega_k}\left(V(\operatorname{div}_{g_0}(e)-d\operatorname{tr}_{g_0}(e))-e(\nabla^{g_0}V,\cdot)+\operatorname{tr}_{g_0}(e)dV\right)(\nu^{\Omega_k})d\mu_{g_0}.
\end{aligned}$$

where $\{\Omega_k\}_{k=1}^{\infty}$ is any increasing sequence of domains that exhausts M_0^n. In particular, we see from (2.4) that the choice of domains $\{\Omega_k\}_{k=1}^{\infty}$ that exhaust M_0^n does not influence $\mathscr{M}_{g,g_0}$. The constant $2(n-1)\omega_{n-1}$ is inserted for normalization.

To compute $\mathscr{M}_{g,g_0}$ one needs to understand the kernel $\mathscr{N}=\ker((D\operatorname{Scal}|_{g_0})^*)$. The formal adjoint of $D\operatorname{Scal}|_{g_0}$ can be computed (see [52, Section 4]) and is given by

$$(D\operatorname{Scal}|_{g_0})^*(V):=\operatorname{Hess}^{g_0}V+(\Delta^{g_0}V)g_0-V\operatorname{Ric}^{g_0}.$$

When (M^n,g) is asymptotically Euclidean, the background manifold (M_0^n,g_0) is nothing but Euclidean space $(\mathbb{R}^n,\delta)$. Since Euclidean space is flat, its Ricci curvature vanishes: $\operatorname{Ric}^{\delta}=0$ and so it follows that $\mathscr{N}$ is given by those functions $V:\mathbb{R}^n\to\mathbb{R}$ satisfying

$$\operatorname{Hess}^{\delta}V+(\Delta^{\delta}V)\delta=0.$$

It can be shown that the set of functions satisfying the above are precisely the affine functions $V=c+b_ix^i$. We will refer to the quantity $\mathscr{M}_{g,\delta}(1)$ as the

ADM mass[1] of (M^n, g), it is given by

$$m_{\mathrm{ADM}}(g) := \frac{1}{2(n-1)\omega_{n-1}} \lim_{k\to\infty} \int_{\partial\Omega_k} \big(\mathrm{div}_\delta(e) - d\,\mathrm{tr}_\delta(e)\big)(\nu^{\Omega_k})\, d\mu_\delta. \quad (2.5)$$

The quantities $\mathscr{M}_{g,\delta}(x^i)$, $i = 1, \dots, n$, are usually referred to as the components of the *ADM center of mass*, if the ADM mass of (M^n, g) is non-zero, then the components of the center of mass of (M^n, g) are given by

$$\begin{aligned} C^i_{\mathrm{ADM}}(g) &= \frac{\mathscr{M}_{g,\delta}(x^i)}{2(n-1)\omega_{n-1} m_{\mathrm{ADM}}(g)} \\ &= \lim_{k\to\infty} \frac{1}{2(n-1)\omega_{n-1} m_{\mathrm{ADM}}(g)} \int_{\partial\Omega_k} \big(x^i(\mathrm{div}_\delta(e) - d\,\mathrm{tr}_\delta(e)) \\ &\qquad\qquad - e(\partial_i, \cdot) + \mathrm{tr}_\delta(e) dx^i\big)(\nu^{\Omega_k})\, d\mu_\delta. \end{aligned}$$

In the above derivation there is no mention of the chart at infinity Φ, but in general the functional $\mathscr{M}_{g,g_0}$ might a priori depend on Φ.

When the manifold (M^n, g) is asymptotically hyperbolic, the background manifold (M^n_0, g_0) is hyperbolic space $(\mathbb{H}^n, b)$. We use the model of hyperbolic space where the underlying manifold is $\mathbb{R}^n$ and the hyperbolic metric b is given in geodesic coordinates by $b = dr^2 + \sinh(r)^2\sigma$, where σ is the standard round metric on the unit sphere S^{n-1}. In this case, the space $\mathscr{N}$ is

$$\mathscr{N} = \mathrm{span}\{V_0 = \cosh(r), V_1 = x^1 \sinh(r), \dots V_n = x^n \sinh(r)\},$$

and the mass functional is in this case given by

$$\begin{aligned} H_g(V) &:= \mathscr{M}_{g,b}(V) \\ &= \lim_{k\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{\partial\Omega_k} \big(V(\mathrm{div}_b(e) - d\,\mathrm{tr}_b(e)) \qquad (2.6) \\ &\qquad\qquad - e(V, \cdot) + \mathrm{tr}_b(e) dV\big)(\nu^{\Omega_k})\, d\mu_b. \end{aligned}$$

We refer the reader to Chruściel and Herzlich [27] for the definition and proofs of important properties of the mass functional in this setting. We also refer the reader to Paper I and Sakovich [58] and the references therein for a more thorough discussion about the history of this notion of mass.

The purpose of the above discussion is not only to give an overview of where the expression for the ADM mass in Mathematical General Relativity comes from, but also to highlight the following essential ingredients for showing that the ADM mass is well defined.

---

[1] It is in fact more appropriate to call this invariant at infinity the ADM *energy*, since it represents the first component of the energy-momentum vector, whose length is the ADM mass, see [42].

- It is necessary that the scalar curvature Scal^g is defined, for it is the object being linearized.
- It is necessary that $V(\mathrm{Scal}^g - \mathrm{Scal}^{g_0}) \in L^1(M_0^n)$ and that $g - g_0 \to 0$ at infinity sufficiently quickly, so that we may take a limit when choosing larger and larger domains Ω to integrate over.
- It is important to show that the mass does not depend on the choice of a chart $\Phi : M^n \setminus K \to M_0^n \setminus K_0$ used to compare the metric g to g_0.

Other notions of mass

The ADM formalism is not the only way to define a notion of mass for initial data sets and Riemannian manifolds. There is a definition of mass involving the Ricci tensor of (M^n, g) which has been shown to agree with the ADM mass, see Huang [37], Miao and Tam [51] and Herzlich [35]. For an asymptotically Euclidean Riemannian manifold (M^n, g) this so called *Ricci version of the ADM mass* is defined as

$$m_{\mathrm{R}}(g) := \frac{-1}{(n-1)(n-2)\omega_{n-1}} \lim_{k\to\infty} \int_{\Omega_k} G^g(rDr, \nu^{\Omega_k})\, d\mu_\delta, \tag{2.7}$$

where $\{\Omega_k\}$ is once again an increasing sequence of pre-compact open sets that exhaust $\mathbb{R}^n$. We refer the reader to Huang [37], Miao and Tam [51] and Herzlich [35] and the references therein for an overview of the history and properties of this notion of mass.

We would also like to mention that there are ways of defining mass for Riemannian manifold (M^n, g) that are not directly related to the scalar curvature. For example, Huisken [38] defines a notion of *isoperimetric mass* by comparing the volume of large geodesic spheres in (M^n, g) to the volume of spheres in $(\mathbb{R}^n, \delta)$ of the same radius.

Mass for metrics of lower regularity

The above presented derivation of the ADM mass can be made rigorous. Some of the first results that establish the well-definedness and coordinate invariance of the mass are by Chruściel [25] and Bartnik [8]. For different classes of asymptotically Euclidean manifolds, they showed that the limit in (2.5) defining the ADM mass exists, is independent of the choice of domains $\{\Omega_k\}_{k=1}^{\infty}$ and independent of the choice of a chart Φ at infinity.

In recent years, metrics of low regularity have become an increasingly important area of study in Mathematical General Relativity. In the context of spacetimes, understanding such metrics would allow one to study metrics with geometric singularities such as for example impulsive gravitational waves, see

[44]. Consequently it is important to define a notion of mass for such low regularity metrics.

The first steps in this direction were made by Bartnik [8], extending the notion ADM mass to asymptotically Euclidean manifolds of local regularity $W^{2,p}_{\mathrm{loc}}$, for $p > n$. More recently, Lee and LeFloch [43] have defined a notion of mass for metrics of local regularity $W^{1,n}_{\mathrm{loc}} \cap C^0$ which extends the classical definition in (2.5).

It is natural to ask: *how regular does the metric g have to be in order for the ADM mass to be defined?* An obvious difficulty here is that if the metric g has only one derivative, then the scalar curvature Scal^g is no longer defined as a function though it may be defined as a distribution, see for example by Lee and LeFloch [43] and Hafemann [34].

Inspecting the definition of the mass functional in (2.5) and (2.6), one can see that the integrand is defined when $g \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and $g^{-1} \in L^\infty_{\mathrm{loc}}$. In the case of asymptotically hyperbolic manifolds, Gicquaud and Sakovich [33] have shown that in this regularity it is still possible to define the mass functional, which under some additional assumptions turns out to be coordinate invariant.

Related to the above, in Paper III we address the following questions for asymptotically Euclidean manifolds.

- Can the ADM mass be defined when the only local assumption on the metric is that $g \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and $g^{-1} \in L^\infty_{\mathrm{loc}}$?
- Can the scalar curvature be given a reasonable distributional definition as in Lee and LeFloch [43] and Gicquaud and Sakovich [33], which can be used to show well-definedness of the ADM mass in this setting?
- Is this definition of the ADM mass independent of the choice of coordinates at infinity Φ?

2.4 The positive mass theorem

We recall that in the Einstein equations (1.1), the tensor T in the right hand side describes the distribution of matter in the universe at different times and places. Not every distribution of matter is physically reasonable. For example, we would like to rule out any distribution of matter that implies the existence of particles moving faster than light. A condition on the stress energy tensor T that achieves this is the so called *dominant energy condition*. In most physically motivated matter models this condition is satisfied and so it is standard to assume. On an initial data set (M^n, g, k) which is embedded in a spacetime, the dominant energy condition can be stated in terms of the metric g and the second fundamental form k. It amounts to the inequality $\mu \geq |J|_g$ where

$$\begin{aligned} \mu &= \frac{1}{2}\left(\mathrm{Scal}^g - n\Lambda + \mathrm{tr}_g(k)^2 - |k|_g^2\right), \\ J &= \mathrm{div}_g(k) - d\,\mathrm{tr}_g(k), \end{aligned}$$

are the energy density and the current density mentioned in the previous section. The positive mass conjecture can informally be stated as follows.

Conjecture 2.5. *If a complete initial data set* (M^n, g, k) *has well-defined mass and satisfies the dominant energy condition* $\mu \geq |J|_g$ *, then its mass is non-negative and is equal to zero if and only if* M^n *embeds in the model spacetime* (N_0, h_0) *as a spacelike slice with the induced metric* g *and the second fundamental form of the embedding equal to* k*.*

So, even though the ADM mass is a quantity that can be defined for *manifolds* (M^n, g), the appropriate hypothesis for the positive mass theorem should in general be phrased for *initial data sets* (M^n, g, k). This conjecture can be more easily understood in a special case. Given an asymptotically Euclidean manifold (M^n, g), the initial data set $(M^n, g, 0)$ will automatically be asymptotically Euclidean as well. In this case the dominant energy condition takes the form $\mathrm{Scal}^g \geq 0$, which is a condition on the manifold (M^n, g). Thus Conjecture 2.5 has the following specialization, called the Riemannian positive mass conjecture for asymptotically Euclidean manifolds.

Conjecture 2.6. *If a complete asymptotically Euclidean manifold* (M^n, g) *has well-defined mass and non-negative scalar curvature* $\mathrm{Scal}^g \geq 0$*, then* $m_{ADM}(g) \geq 0$ *with equality if and only if* $(M^n, g) \cong (\mathbb{R}^n, \delta)$.

We note that the Conjecture 2.6 says that if an asymptotically Euclidean manifold (M^n, g) satisfies $\mathrm{Scal}^g \geq 0$ and $m_{\mathrm{ADM}} = 0$, then $(M^n, g) \cong (\mathbb{R}^n, \delta)$ which is nothing but the slice $t = 0$ in Minkowski space, consistent with the conclusion of Conjecture 2.5.

A landmark result in this area is that of Schoen and Yau [61], where the authors proved Conjecture 2.6 in dimension $n = 3$ using variational techniques from the theory of minimal surfaces. Independently, Witten [67] proved Conjecture 2.6 in all dimensions $n \geq 3$ under the topological assumption that the manifold (M^n, g) is spin. Subsequently, Schoen and Yau [62] have proven the more general Conjecture 2.5 for 3-dimensional asymptotically Euclidean initial data sets (M^3, g, k) by reducing it to Conjecture 2.6, using the so called *Jang equation reduction argument*, which we discuss below.

Since then, a lot of attention has been directed to Conjecture 2.5 and related results. The Jang equation reduction argument of Schoen and Yau [62] has been generalized to dimensions $3 \leq n \leq 7$ by Eichmair [29] and the variational techniques employed by Schoen and Yau in [61] have been generalized by Eichmair, Huang, Lee and Schoen [30] to the case of initial data sets of dimensions $3 \leq n \leq 7$.

Recently some new methods for proving Conjecture 2.5 have been introduced, such as for example level set methods. In particular Bray, Khuri, Kazaras, and Stern [11] and Hirch, Kazaras, and Khuri [36] proved the positive mass theorem using level sets of linearly growing harmonic functions and their generalizations. Agostiniani, Mazzieri and Oronzio [1] give a proof of the positive mass theorem for asymptotically Euclidean Riemannian manifolds by producing a monotone quantity along the level sets of a Green function of the manifold (M^3, g).

Furthermore, the classes of initial data sets considered have been extended. For example, Lesourd, Unger and Yau [45] showed that the positive mass theorem holds for asymptotically Euclidean manifolds which have other additional ends. There are proofs of the positive mass theorem under weaker regularity assumptions on the metric g, see for example Lee and LeFloch [43] for a proof when (M^n, g) is spin and Hafemann [34] for a proof where the spin assumption is replaced by the assumption that the metric g is smooth outside of a compact set. Furthermore, the positive mass theorem for so called asymptotically hyperboloidal and asymptotically anti-de Sitter initial data sets has been studied as well. We refer the reader to Sakovich [58], Paper I and Lundberg [46] and the references therein for a more thorough discussion of the history of this subject.

Traditionally, positive mass theorems for non-spin manifolds (M^n, g) include the assumption that the dimension satisfies $3 \leq n \leq 7$. This restriction comes from the regularity theory of minimal surfaces and related objects such as MOTS. The result of Paper I is obtained under this assumption as well. Recently there have been a number of important break troughs which has resulted in relaxing the assumption that $3 \leq n \leq 7$. Most notably Chodosh, Mantoulidis and Shulze [22] and Chodosh, Mantoulidis and Wang [23] have been able to prove that minimal surfaces are generically smooth in ambient dimension ≤ 11. Bi, Hao, He, Shi and Zhu [9] have used these results to prove the positive mass theorem for asymptotically Euclidean manifolds in dimensions $3 \leq n \leq 19$ without assuming the manifold (M^n, g) is spin. Moreover, using a technique of Lesourd, Unger and Yau [45], Brendle and Wang [14] proved the positive mass theorem for asymptotically Euclidean manifolds in all dimensions $n \geq 3$, also without assuming that (M^n, g) is spin.

Below, we give an overview of the so-called Jang equation reduction argument which was first employed by Schoen and Yau [62] to reduce the positive mass theorem for 3-dimensional asymptotically Euclidean *initial data sets* (M^3, g, k) to the positive mass theorem for 3-dimensional asymptotically Euclidean *manifolds* (M^3, g). As mentioned, this argument has subsequently been extended by Eichmair [29] to asymptotically Euclidean initial data sets (M^n, g, k) of dimension $3 \leq n \leq 7$. This technique has also been used to prove the positive mass theorem for the so called asymptotically *hyperboloidal* initial data sets. The original argument of Sakovich [58] in dimension 3 is adapted to dimensions $4 \leq n \leq 7$ by Lundberg [46].

We also give a brief overview of the method used by Robinson [56] to prove the black-hole uniqueness theorem of Israel [40]. The method of Robinson has recently inspired several new results in the field, see for example the work of Cederbaum, Cogo, Leandro, and Paolo dos Santos [18], Cederbaum, Cogo, and Fehrenbach [17], Cederbaum and Miehe [20], and Cederbaum and León Quirós [19].

The Jang equation reduction argument

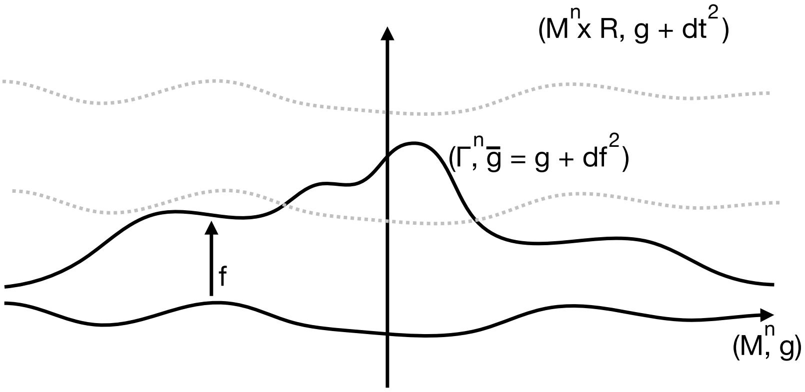

Figure 2.8. An illustration of a graphical deformation of (M^n, g) by the graphing function $f : M^n \to \mathbb{R}$ in the product space $(M^n \times \mathbb{R}, \widetilde{g} = g + dt^2)$.

Supposing that Conjecture 2.6 is known and that an asymptotically Euclidean *initial data set* (M^n, g, k) satisfying the dominant energy condition $\mu \geq |J|_g$ is given, one can ask if it is possible to deform the given initial data set into a Riemannian manifold $(\overline{M}^n, \overline{g})$ with the same mass as (M^n, g) and with $\mathrm{Scal}^{\overline{g}} \geq 0$? The deformation considered by Schoen and Yau [62] is a graphical one, given (M^n, g, k) one asks: how should the function $f : M^n \to \mathbb{R}$ be chosen so that the graph $\Gamma = \{(p, f(p)) : p \in M^n\} \subset M^n \times \mathbb{R}$, equipped with the induced metric $\overline{g} := g + df^2$ satisfies $\mathrm{Scal}^{\overline{g}} \geq 0$?

Schoen and Yau proved the so called *Schoen-Yau identity* which is an expression for the scalar curvature $\mathrm{Scal}^{\overline{g}}$ of the metric $\overline{g} = g + df^2$.

$$\begin{aligned}\mathrm{Scal}^{\overline{g}} &= 2(\mu - J(w)) + |A-k|^2_{\overline{g}} + 2|q|^2_{\overline{g}} - 2\,\mathrm{div}^{\overline{g}} q \\ &\quad + (H^\Gamma)^2 - (\mathrm{tr}^\Gamma k)^2 + 2k(\nu,\nu)(H^\Gamma - \mathrm{tr}^\Gamma k) + 2\nu(H^\Gamma - \mathrm{tr}^\Gamma k).\end{aligned} \tag{2.8}$$

Here, μ and J are the energy and current density, w is a vector defined in terms of ∇f and satisfies $|w|_g \leq 1$, A is the second fundamental form of the graph $\Gamma \hookrightarrow M^n \times \mathbb{R}$, q is a one form defined in terms of ∇f and the difference $A - k$, H^Γ is the mean curvature of the graph $\Gamma \subset M \times \mathbb{R}$ and lastly $\mathrm{tr}^\Gamma k$ is the trace of the tensor k along Γ. We note that if $H^\Gamma = \mathrm{tr}^\Gamma k$, then:

$$\mathrm{Scal}^{\overline{g}} = 2(\mu - J(w)) + |A-k|^2_{\overline{g}} + 2|q|^2_{\overline{g}} - 2\,\mathrm{div}^{\overline{g}} q. \tag{2.9}$$

The dominant energy condition implies that the first term above is non-negative, the second and third terms are squares and hence also non-negative. The remaining term is a divergence term and therefore referred to as "almost non-negative". All in all, if the graphing function f in Figure 2.4 is chosen so that $H^\Gamma = \mathrm{tr}^\Gamma k$, then the graph $(\Gamma, \overline{g})$ has "almost $\mathrm{Scal}^{\overline{g}} \geq 0$". The equality $H^\Gamma = \mathrm{tr}^\Gamma k$ can be written in terms of the graphing function $f : M^n \to \mathbb{R}$ and the tensor k from the initial data set (M^n, g, k) and amounts to the following *prescribed mean curvature equation* on (M^n, g).

$$\left(g^{ij} - \frac{\nabla^i f \nabla^j f}{1 + |\nabla f|^2}\right)\left(\frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij}\right) = 0. \tag{2.10}$$

The above equation is the eponymous Jang equation, which was introduced by Jang [41] to study which initial data sets might be embedded as spacelike slices in Minkowski space. Schoen and Yau [62] rediscovered and employed this equation in the following way.

Suppose that the initial data set (M^3, g, k) is asymptotically Euclidean and that $f : M^3 \to \mathbb{R}$ is a solution to (2.10). Inspecting the definition of the ADM mass (2.5), one sees that $m_{\mathrm{ADM}}(g)$ is defined by the behaviour of the metric g near infinity. If f tends to zero sufficiently quickly, then roughly $g + df^2$ asymptotes to g. This can be made precise and if f falls off fast enough then the graph $(\Gamma, \overline{g})$ is asymptotically Euclidean as well and moreover $(\Gamma, \overline{g})$ and (M^n, g) have the same ADM mass. Due to the "almost non-negativity" of $\mathrm{Scal}^{\overline{g}}$, the positive mass theorem for asymptotically Euclidean *manifolds* can be used to show that

$$m_{\mathrm{ADM}}(g) = m_{\mathrm{ADM}}(\overline{g}) \geq 0.$$

Furthermore, if $m_{\mathrm{ADM}}(g) = 0$, then $m_{\mathrm{ADM}}(\overline{g}) = 0$ as well and moreover it follows that $(\Gamma, \overline{g}) \cong (\mathbb{R}^3, \delta)$. A change of perspective now gives the desired embedding into the Minkowski spacetime as follows.

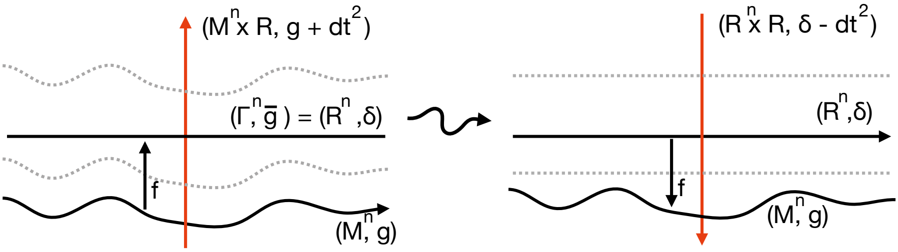

Figure 2.9. An illustration of a graphical embedding of (M^n, g) in Minkowski space.

In the left part of Figure 2.4, we view $(\Gamma, \overline{g}) = (\mathbb{R}^n, \delta)$ as a graph over (M^n, g) in the product space $(M^n \times \mathbb{R}, g + dt^2)$. Equivalently we can view (M^n, g) as a graph over $(\mathbb{R}^n, \delta)$ in the product space $(\mathbb{R}^n \times \mathbb{R}, \delta - dt^2)$, as in the right part of Figure 2.4. But this is exactly what it means to view M^n as a graph in Minkowski space. That the metric g is indeed the induced metric for this embedding is due to the equality

$$g = \overline{g} - df^2 = \delta - df^2 = (\delta - dt^2)|_{M^n}.$$

Lastly, the Schoen-Yau identity (2.9) implies that the second fundamental form of the embedding $(M^n, g) \hookrightarrow (\Gamma \times \mathbb{R}, \overline{g} - dt^2)$ above is nothing but the desired second fundamental form k. In this way, solving the Jang equation (2.10) on (M^n, g, k) not only shows that $m_{\mathrm{ADM}}(g) \geq 0$, but also provides the embedding of (M^n, g, k) as a spacelike slice in Minkowski space in the case when the mass vanishes. We have omitted certain important technical details in this discussion but the above is the main idea of the Jang equation reduction argument. For more details, see Schoen and Yau [62], Eichmair [29] and Sakovich [58].

A refinement of the positive mass theorem for asymptotically Euclidean initial data sets (M^3, g, k) satisfying the dominant energy condition is the so called *Penrose inequality*:

$$m_{\mathrm{ADM}}(g) \geq \sqrt{\frac{A(\Omega)}{16\pi}}, \tag{2.11}$$

where $\Omega \subset M$ is a hypersurface modelling the event horizon of a black hole, with equality if and only if (M^3, g, k) can be embedded as a spacelike slice in the Schwarzschild spacetime. For more details, see Mars [48] and the references therein.

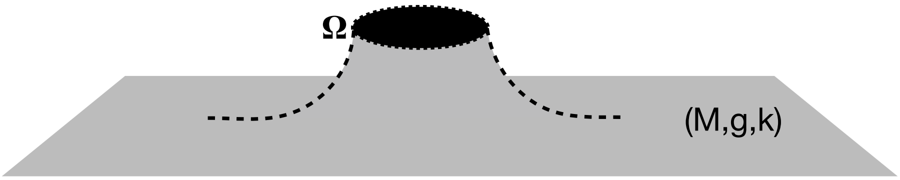

Figure 2.10. An illustration of an asymptotically Euclidean initial data set (M^n, g, k) with a boundary representing a black hole.

In the case of asymptotically Euclidean manifolds ($k = 0$), this inequality has been proven by Huisken and Ilmanen [39] and by Bray [10] but the general case is still open. The possibility of adapting the Jang equation reduction argument to prove this so called spacetime Penrose inequality has been discussed by Malec and O'Murchadha [47], with the conclusion that a substantial modification of the method is needed.

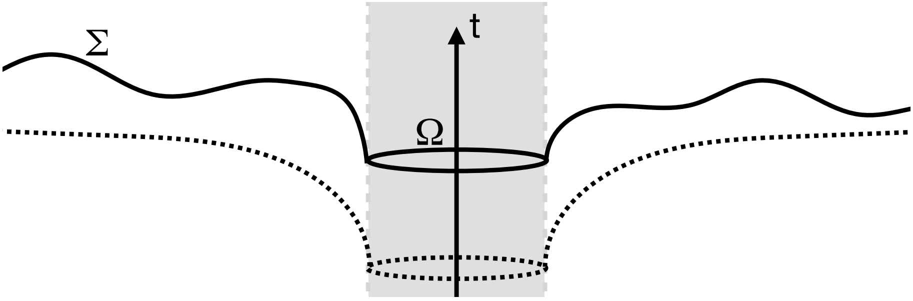

Figure 2.11. An illustration of a spacelike hypersurface Σ in the Schwarzschild spacetime. The gray region represents the interior of a black hole.

In an attempt to adapt the Jang equation reduction in this setting, Bray and Khuri [12] pointed out that the Schwarzschild spacetime is a *warped* product, while the classical Jang equation argument always gives rise to an embedding in a *non-warped* product space. Consequently their suggestion was to modify the Jang equation so that the reduction is carried out in a warped product setting instead.

The idea is as follows, one considers the Jang graph $(\Gamma, \bar{g} := g + u^2 dt^2)$ in the warped product space $M^n \times_u \mathbb{R}$, defined as the Riemannian manifold consisting of $M^n \times \mathbb{R}$ equipped with the metric $\tilde{g} := g + u^2 dt^2$ and where $u : M^n \to (0, \infty)$ is a function to be determined.

Just like in the classical Jang equation reduction argument, one requests that the graph $(\Gamma, \overline{g})$ has (almost) non-negative scalar curvature.

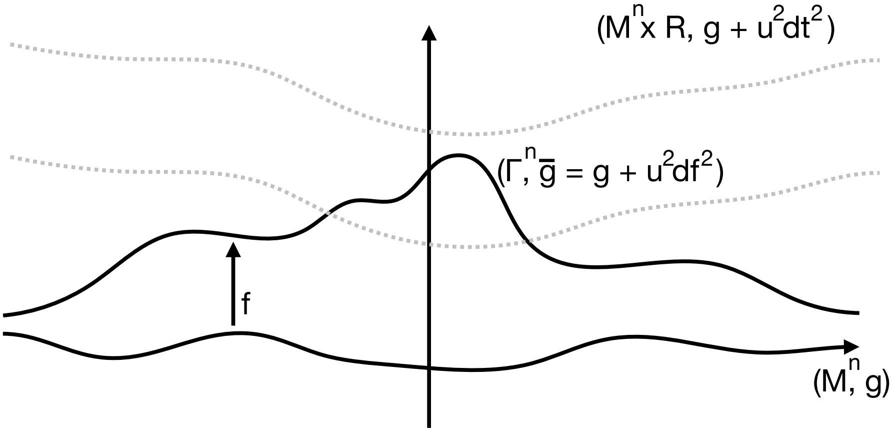

Figure 2.12. An illustration of a graphical deformation of (M^n, g) in the *warped* product space $M^n \times_u \mathbb{R}$.

In terms of the graphing function f and the warping factor u, Bray and Khuri [12, 13] derive the *generalized Schoen-Yau identity* which in turn motivates the so called *generalized Jang equation*:

$$\left(g^{ij} - \frac{u^2 \nabla^i f \nabla^j f}{1+u^2|\nabla f|^2}\right)\left(\frac{u \mathrm{Hess}_{ij} f + u_i f_j + f_i u_j}{\sqrt{1+u^2|\nabla f|^2}} - k_{ij}\right) = 0. \tag{2.12}$$

If this equation is satsified, then $\mathrm{Scal}^{\overline{g}}$ is "almost non-negative". Still it is necessary to relate the mass of the initial data (M^n, g, k) to that of the graph $(\Gamma, \overline{g})$. Addressing this issue, Bray and Khuri [12, 13] proposed several coupled systems of partial differential equations involving f and u which when solved, would yield a proof of the spacetime Penrose inequality. In the spherically symmetric case, one of these coupled system reduces to a single ODE, establishing the spacetime Penrose inequality in this special case, see Bray and Khuri [12] further details.

Somewhat similar to the spacetime Penrose inequality in spirit is the positive mass conjecture for asymptotically anti-de Sitter initial data sets. The Riemannian case ($k = 0$) has been proven by Chruściel and Herzlich [27] under a spin assumption and by Anderson, Cai and Galloway [2] and Chruściel and Galloway [26] without the spin assumption, but instead imposing rather restrictive asymptotic conditions at infinity. Regarding the general case, unlike the positive mass theorem for asymptotically hyperboloidal initial data sets discussed above, this inequality cannot be established using the classical Jang equation reduction argument, since the anti-de Sitter spacetime is a warped product spacetime.

Similar in spirit to the systems of Bray and Khuri [12, 13], Cha and Khuri [21] propose a coupled system involving the generalized Jang equation that whenever solvable reduces the positive mass theorem for anti-de Sitter initial data sets to the Riemannian case. We refer the reader to Cha and Khuri [21] and to the introduction of Paper I for more details. The results of Paper I contribute to this area of research in the two following ways.

- First, we give a rigorous analysis of the existence and properties of solutions to the generalized Jang equation for asymptotically anti-de Sitter initial data sets.
- Second, we provide a new reduction argument involving a coupled system that is simpler than that of Cha and Khuri [21] and which if solved would reduce the positive mass theorem for asymptotically anti-de Sitter initial data sets to the Riemannian case.

The vector field method of Robinson

We give an overview of the so called vector field method of Robinson [56], that was originally used to give a simple proof of the *static vacuum black hole uniqueness theorem* of Israel [40].

We recall that a spacetime (N^{n+1}, h) is *vacuum* if it satisfies the Einstein equations when the stress energy tensor vanishes: $\mathrm{T} = 0$, which is the case if and only if the Ricci curvature of the spacetime is zero: $\mathrm{Ric}^h = 0$. A spacetime is *static* if it admits a *timelike Killing* vector field X that is *irrotational*. Roughly speaking, this means that X defines a flow whose integral curves are future pointing timelike curves and along which the metric h does not change and that the hypersurfaces of the resulting foliation are spacelike. Sánchez [60] has shown that if the flow defined by the vector field X is complete and if the manifold N^{n+1} is simply connected, then the spacetime splits as a warped product as follows:

$$N^{n+1} = M^n \times \mathbb{R}, \quad h = -u^2 dt^2 + g, \tag{2.13}$$

where $u : M^n \to \mathbb{R}$ is a function and (M^n, g) is a Riemannian manifold.

Vacuum spacetimes model "empty universes" and static spacetimes model "universes that do not evolve". Thus, a static vacuum spacetime represents an empty universe in which there is no evolution. Related to this, one of the central questions in Mathematical General Relativity is "*What does a static vacuum spacetime look like?*". Two examples of such spacetimes are Minkowski space and the Schwarzschild spacetime, the latter of which models the exterior region of a black hole, see Figure 2.4.

Israel [40] proved that any 4-dimensional static vacuum spacetime (N^{3+1}, h) which is asymptotic to the Schwarzshild spacetime, is in fact *isometric* to the Schwarzschild spacetime, see also see also Bunting and Masood-ul Alam [15]. Subsequently, Robinson [56] gave a short proof of the result of Israel [40] using a method which we outline below.

The condition $\mathrm{Ric}^h = 0$ is equivalent to the function u in (2.13) satisfying

$$\Delta^g u = 0, \quad \mathrm{Hess}^g\, u = u \mathrm{Ric}^g\,.$$

We assume that u is not identically a constant, that M^3 has a boundary given by $\partial M = \{u = 0\}$ and that there is a constant $W_0 > 0$ such that $|\nabla u|^2 \equiv W_0$ on ∂M. The last condition ensures that the boundary ∂M represents the surface of a black hole. Since (N^{3+1}, h) is asymptotically Schwarzschildean, it follows that $u \to 1$ at infinity. Moreover since u is harmonic, it cannot attain an interior local maximum or minimum and so $u : M^3 \to (0,1)$.

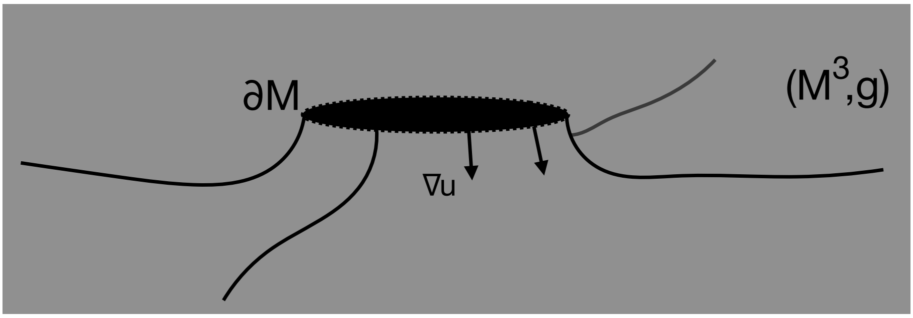

Figure 2.13. A Riemannian manifold (M^3, g) with a black hole boundary given by $\partial M = \{p \in M : u(p) = 0\}$.

For the function u as above, Robinson [56] introduces a vector field for the function u of the form

$$Z := F(u)\frac{\nabla(|\nabla u|^2)}{u} + G(u)|\nabla u|^2 \nabla u,$$

for $F, G : (0,1) \to \mathbb{R}$ are functions to be determined and finds a family of functions F and G, parametrized by $c, d \in \mathbb{R}$ such that $d \geq 0$ and $c \geq -d$, for which $\mathrm{div}_g(Z) \geq 0$. For this family, an application of the divergence theorem yields

$$0 \leq \int_M \mathrm{div}_g(Z)\, d\mu_g = \lim_{x \to 1} \int_{\{u=x\}} g(\nu^{\{u=x\}}, Z)\, d\mu_g - \int_{\partial M} g(\nu^{\partial\Omega}, Z)\, d\mu_g, \tag{2.14}$$

for any choice of the parameters as described above. In the view of $|\nabla u|^2 \equiv W_0$ on ∂M, Robinson [56] obtains

$$\int_{\partial M} g(\nu^{\partial\Omega}, Z)\, d\mu_g = 8\pi d W_0^{1/2} - (6d - 2c) W_0^{3/2} S_0,$$

where S_0 is the area of the surface ∂M. As for the remaining integrals on the right hand side of (2.14), Robinson [56] showed that

$$\lim_{x\to 1} \int_{\{u=x\}} g(\nu^{\{u=x\}}, Z)\, d\mu_g = \frac{\pi(c+d)}{2m},$$

where m is the ADM mass of the asymptotically Schwarzschildean spacetime (N^{3+1}, h). We note that the parameters c, d must satisfy $d \geq 0$ and $c \geq -d$. All in all, for this set of parameters one arrives at the inequality

$$\frac{\pi(c+d)}{2m} \leq 8\pi d W_0^{1/2} - (6d-2c) W_0^{3/2} S_0.$$

Specifically, in the case when $-c = d = 1$ one obtains

$$W_0 S_0 \leq \pi, \tag{2.15}$$

and in the case when $d = 0$, $c = 1$, one gets

$$\frac{\pi}{4m} \leq W_0^{3/2} S_0. \tag{2.16}$$

Recalling that u is harmonic, one can integrate $\operatorname{div}_g(\nabla u) = 0$ from $\{u = 0\}$ to $\{u = 1\}$, similar to the above computation and infinity. One then obtains the identity $W_0^{1/2} S_0 = 4\pi m$. This combined with (2.16) yields

$$\pi \leq S_0 W_0. \tag{2.17}$$

Together the inequalities (2.15) and (2.17) imply that equality in (2.14) holds when $c = -1$ and $d = 1$. Consequently, for this choice of parameters we have $\operatorname{div}_g(Z) = 0$, which can be used to show that the vector field Z is spherically symmetric and that the underlying manifold (M^3, g) is isometric to Schwarzschild space.

The purpose of Paper II is to use this method to give a conceptually simple proof that reproduces several known geometric inequalities in Riemannian geometry and to use these geometric inequalities to give a proof of the positive mass theorem. We emphasize the following steps in the above argument.

1. The key ansatz is to define a several parameter family of vector fields Z in terms of a geometrically defined function such as the function u as described above.
2. Imposing a lower bound on $\operatorname{div}_g(Z)$ and applying the divergence theorem yields a several parameter family of geometric inequalities.
3. Analysing the cases of equality in these inequalities one obtains rigidity statements for the underlying manifold (M^n, g).

2.5 Distance functions and comparison of spacetimes

The need for comparing spacetimes in Mathematical General Relativity is motivated by questions such as:

- Is our spacetime well-approximated by cosmological models?
- Can spacetimes with different topology (e.g. Minkowski space and a Schwarzschild spacetime with small mass) be compared?
- Can singular and smooth spacetimes be compared?
- Are geometric inequalities that hold for physically reasonable spacetimes such as the positive mass theorem and the Penrose inequality stable?

In particular, we recall that the positive mass theorem (see Conjecture 2.5) for asymptotically Euclidean initial data sets (M^n, g, k) states that if the dominant energy condition is satisfied, then $m_{\mathrm{ADM}}(g) \geq 0$ and $m_{\mathrm{ADM}}(g) = 0$ if and only if (M^n, g) is a spacelike slice of Minkowski space with second fundamental form k. Therefore it is natural to ask: "*If an initial data set (M^n, g, k) has small mass, does it mean that (M^n, g, k) is close to a slice in Minkowski space?*".

To formulate this rigorously one needs a notion of distance between initial data sets. In the case of *Riemannian manifolds* (M^n, g), there are many reasonable notions of distance between spaces, such as the Gromov-Hausdorff distance, the metric measure distance and the Intrinsic Flat distance to name a few, see Sakovich and Sormani [59, Section 2] for further details. The main idea underlying all of these notions of distance is that Riemannian manifolds are converted into metric spaces, possibly with some additional structure, which are then compared using one of the above mentioned notions of distance for metric spaces.

However, initial data sets come equipped with a tensor k as well. Since this tensor is really carrying information about the extrinsic geometry of (M^n, g) inside a spacetime, it seems natural to compare initial data sets not by comparing them directly, but by comparing the *spacetimes* that they give rise to. The difficulty here is that spacetimes are not natural metric spaces, so tools like the Gromov-Hausdorff distance cannot be directly applied. For example, the Lorentzian distance function in the definition below (see [53, Section 2.9]) is not a true metric distance.

Definition 2.7. Let $\gamma : [a,b] \to N$ be a future causal curve. The *Lorentzian length* of γ is defined as

$$L_h(\gamma) := \int_a^b |h(\dot{\gamma}, \dot{\gamma})|^{1/2} dt.$$

The *Lorentzian distance* $d_h : N \times N \to [0,\infty]$ is defined as

$$d_h(p,q) := \begin{cases} \sup L_h(\gamma) & \text{if } p \leq q \\ 0 & \text{otherwise} \end{cases},$$

where the supremum is taken over all future causal curves $\gamma\colon [0,1] \to N$ such that $\gamma(0) = p$ and $\gamma(q) = 1$.

In this situation, a natural approach would be to convert spacetimes into metric spaces in a suitable way and apply one of the notions of distance between metric spaces above. It turns out that a conversion of a spacetime into a metric spacetime can be achieved given a time function and using the null distance introduced by Sormani and Vega [64]. We briefly review this construction.

In Minkowski space there is an intuitive notion of "time", namely the coordinate t (see Example 2.1), but in general there may not be a unique notion of time. However it is natural to request that any reasonable notion of time should be increasing for physical particles, leading to the following definition.

Definition 2.8. Let $\tau : N \to \mathbb{R}$ be a function. We say
- τ is a *generalized time function* if it strictly increasing along future causal curves,
- τ is a *time function* if it is a continuous generalized time function.

In general, time functions are not smooth as the following example shows.

Example 2.9. A "canonical" time function is the *cosmological time function* defined by Andersson, Galloway and Howard [3]:

$$\tau_h(q) := \sup_{\substack{\gamma\colon[0,1]\to N \\ \gamma \text{ future causal} \\ \gamma(1)=q}} L_h(\gamma).$$

Roughly speaking, τ_h measures the time elapsed "since the big bang". The cosmological time function is called *regular* if it takes values in $(0,\infty)$.

In [3] it is shown that the cosmological time function τ_h is locally Lipschitz with gradient $|\nabla^h \tau_h|_h = 1$ almost everywhere. An earlier approach to converting spacetimes into metric spaces that was based on replacing the Lorentzian metric h by the Riemannian metric $h_{\mathrm{R}} := h + 2d\tau_h^2$ using a cosmological time function τ_h, see Sormani [63]. However this approach would only apply in the case when τ_h is smooth, so it has not been pursued further. Instead Sormani and Vega [64] went on to define the so called *null distance*, which we will now describe.

Not any two points $p, q \in N$ can be connected by a causal curve, only causally related ones. But one can nevertheless connect any two points using *piecewise causal curves* [64, Lemma 3.5], which are curves that "zig-zag" forward and backward in time.

Definition 2.10. Let $\gamma : [a,b] \to N$ be a curve into a spacetime (N^{n+1}, h). We say that γ is *piecewise causal* if there are points $a = s_0 < s_1 < \ldots < s_m = b$ such that each curve $\gamma|_{[s_{i-1}, s_i]}$, $i = 1, \ldots, m$ is causal.

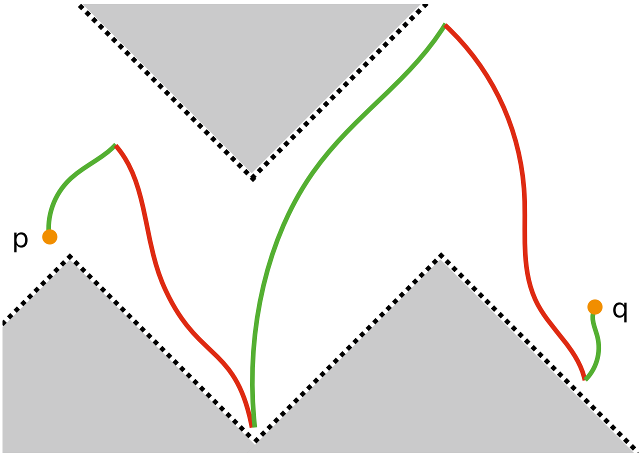

Figure 2.14. A piecewise causal curve connecting points in a spacetime.

In a connected Riemannian manifold (M^n, g), two points $p, q \in M$ can be connected by many different curves and the distance between them is given by the infimum of the length of such curves. The idea underlying the null distance $\hat{d}_\tau$ is similar once the length to a piecewise causal curve has been defined. The following definition of Sormani and Vega [64] has been inspired by the Manhattan distance.

Definition 2.11. [64, Definition 3.2] Let (N^{n+1}, h) be a spacetime, let $\tau : N \to \mathbb{R}$ be a generalized time function and let $\gamma : [a,b] \to N$ be a piecewise causal curve. The *null length* of γ with breaks at $s_0, s_1, \ldots, s_m$ is given by

$$\hat{L}_\tau(\gamma) := \sum_{i=1}^{m} |\tau(s_i) - \tau(s_{i-1})|,$$

The *null distance* between $p, q \in N$ is defined as

$$\hat{d}_\tau(p,q) := \inf_{\gamma:[0,1]\to N} \hat{L}_\tau(\gamma),$$

where the infimum is taken over all piecewise causal curves $\gamma : [0,1] \to \mathbb{R}$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Example 2.12. In Minkowski space $(\mathbb{R} \times \mathbb{R}^n, \eta)$, the null distance with respect to the time function $\tau = t$ is given by

$$\hat{d}_t((t_1, \vec{x}_1), (t_2, \vec{x}_2)) = \max(|t_1 - t_2|, |\vec{x}_1 - \vec{x}_2|).$$

The following properties of the null distance have been established by Sormani and Vega [64].

Theorem 2.13. *Let (N^{n+1}, h) be a spacetime, let $\tau : N \to \mathbb{R}$ be a generalized time function and let $p, q \in N$.*

- *$\hat{d}_\tau$ is symmetric, semidefinite and satisfies the triangle inequality.*
- *$\hat{d}_\tau$ induces the manifold topology if and only if $\hat{d}_\tau$ is definite and τ is continuous on N with respect to the manifold topology.*

Thus in order for $\hat{d}_\tau$ to be a true metric distance, one just has to show that it is *definite*:

$$\hat{d}_\tau(p,q) = 0 \implies p = q.$$

This is not true for all generalized time functions τ, thus it is common to make some additional assumptions on the time function.

Definition 2.14. Let $\tau : N \to \mathbb{R}$ be a generalized time function. We say that τ is a *temporal function* if $\nabla\tau$ exists on all of N and $h(\nabla\tau, \nabla\tau) < 0$ always. We say that τ is a *weak temporal function* if for all $p \in N$ there is a neighbourhood $U \subset N$ of p, a Riemannian metric g on U and a constant $C > 0$ such that for all $p, q \in U$ such that $p \leq q$:

$$C^{-1} d_g(p,q) \leq \tau(q) - \tau(p) \leq C d_g(p,q), \tag{2.18}$$

where $d_g : U \times U \to [0, \infty)$ is the Riemannian distance function of g on U.

We note the following corollary of the results of Sormani and Vega [64], and Burtcher and Heveling [16, Proposition 2.11].

Corollary 2.15. *Let (N^{n+1}, h) be a spacetime and let $\tau : N \to \mathbb{R}$ be a weak temporal function. Then τ is locally Lipschitz and $\hat{d}_\tau$ is a definite distance function that induces the manifold topology.*

Suppose that two spacetimes (N_1^{n+1}, h_1) and (N_2^{n+1}, h_2) are equipped with weak temporal functions τ_1 and τ_2. We could potentially define the distance between these spacetimes by computing the Gromov-Hausdorff distance between the converted metric spaces $(N_1, \hat{d}_{\tau_1})$ and $(N_2, \hat{d}_{\tau_2})$. However the following example of Sakovich and Sormani [59, Example 5.1] illustrates that this approach may sometimes fail.

Example 2.16. Let $a > b > 0$ be real numbers and let (N_1, h_1) and (N_2, h_2) be the spacetimes defined by $N_1 = [0,a] \times [0,b]$, $N_2 = [0,b] \times [0,a]$ and $h_1 = h_2 = -dt^2 + dx^2$.

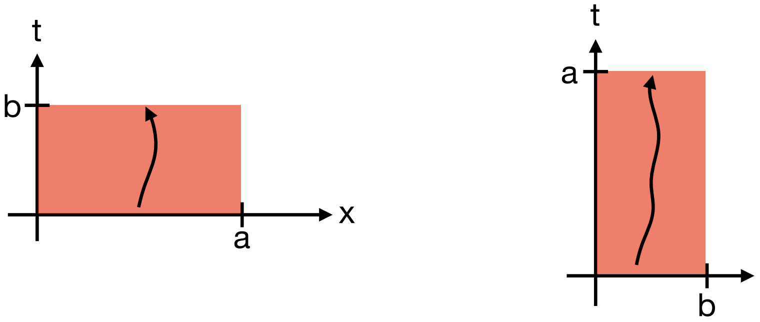

Figure 2.15. (N_1, h_1) and (N_2, h_2) together with future pointing causal curves.

The cosmological time functions (see Example 2.9) of the spacetimes (N_1, h_1) and (N_2, h_2) are given by $\tau_{h_1}(t,x) = \tau_{h_2}(t,x) = t$, and the null distances are:

$$\hat{d}_{\tau_{h_i}}((t_1,x_1),(t_2,x_2)) = \max(|t_1 - t_2|, |x_1 - x_2|), \quad i = 1,2.$$

If we now define the "spacetime Gromov-Hausdorff distance" between (N_1, h_1) and (N_2, h_2) by

$$d_{S-GH}((N_1,h_1),(N_2,h_2)) := d_{GH}((N_1, \hat{d}_{\tau_{h_1}}), (N_2, \hat{d}_{\tau_{h_2}})),$$

where d_{GH} denotes the Gromov-Hausdorff distance between metric spaces, it turns out that $d_{S-GH}((N_1,h_1),(N_2,h_2)) = 0$. Indeed, the map $\phi : (N_1, \hat{d}_{\tau_{h_1}}) \to (N_2, \hat{d}_{\tau_{h_2}})$ given by $\phi(t,x) = (x,t)$ is an isometry of metric spaces, which implies that $d_{GH}((N_1, \hat{d}_{\tau_1}), (N_2, \hat{d}_{\tau_2})) = 0$. However, there is no Lorentzian isometry between the spacetimes (N_1, h_1) and (N_2, h_2), as their causal curves are very different, see Figure 2.16.

The issue in this example appears to be related to the fact that when replacing (N_i, h_i) by $(N_i, \hat{d}_{\tau_{h_i}})$, we neglect their causal structures. Consequently, one can argue that an appropriate notion of distance on a spacetime should be capable of encoding the causal structure. Luckily this can be achieved by combining together the null distance $\hat{d}_\tau$ and its associated time function τ.

Recalling Example 2.12, we see that $p \leq q$ if and only if $\hat{d}_t(q,p) = t(q) - t(p)$ holds in Minkowski spacetime. Moreover Sormani and Vega showed [64, Lemma 3.11] that for a generalized time function $\tau : N \to \mathbb{R}$ and any $p, q \in N$ it holds

$$p \leq q \implies \hat{d}_\tau(p,q) = \tau(q) - \tau(p),$$

and that the converse holds when (N^{n+1}, h) is a warped product space [64, Lemma 3.24]. The inverse implication is not always true, see [57, Examples

2.1 and 2.2] for counterexamples. In general, given a spacetime (N^{n+1}, h) equipped with a generalized time function τ, we say that *the null distance encodes causality* if

$$\forall p, q \in N: \quad p \leq q \iff \hat{d}_\tau(p,q) = \tau(q) - \tau(p). \tag{2.19}$$

Sakovich and Sormani [57, Theorem 4.1] proved that if a spacetime, (N^{n+1}, h), has a *proper* locally anti-Lipschitz time function, τ, then (2.19) holds. This result was subsequently extended by Burtscher and García-Heveling [16] and by Galloway [32] using other global hypotheses on (N^{n+1}, h) and τ. Moreover Sakovich and Sormani showed that causality is always *locally* encoded when τ is a generalized anti-Lipschitz time function [57, Theorem 1.1]. Namely, for all $p \in N$ there is a neighbourhood U of p such that for all $q \in U$:

$$\begin{aligned} p \leq q &\iff \hat{d}_\tau(p,q) = \tau(q) - \tau(p) \quad \text{and} \\ q \leq p &\iff \hat{d}_\tau(q,p) = \tau(p) - \tau(q). \end{aligned}$$

A key ingredient in the proofs of Sakovich and Sormani [57] is a special type of coordinate charts introduced by Temple [66], which are useful for the local study of the causal structure. These coordinate charts are constructed by choosing a future timelike geodesic η satisfying $\eta(0) = p$, and parametrizing a tubular neighbourhood of η using outgoing null geodesics.

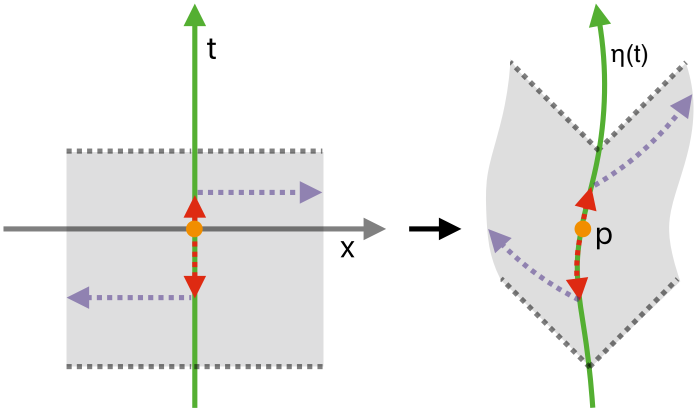

Figure 2.16. Graphical illustration of a Temple chart about a point in a spacetime.

Any chart Φ_p about a point $p \in N$ as above is referred to as a Temple chart about the point p. A key property of such a chart is that on its image $\mathrm{Im}(\Phi_p)$ one can define the *optical function* ω_p as

$$\omega_p(\Phi_p(t, x^1, \ldots, x^n)) := t.$$

The optical function ω_p serves as an indicator function of the causal future of the point p in the image $\mathrm{Im}(\Phi_p)$. Sakovich and Sormani showed [57, Lemma

3.6] that for all $q \in \mathrm{Im}(\Phi_p)$: $\omega_p(q) \geq 0 \iff p \leq q$.

The purpose of Paper IV is to show that Temple coordinate charts can be constructed locally uniformly. We then apply these results to and that on a spacetime equipped with a weak temporal function

- the causality is locally encoded by the time function and the associated null distance,
- our uniform Temple charts form a bi-Lipschitz atlas.

We also prove a Lorentzian isometry theorem, stating that a time function and null distance preserving bijection between spacetimes is in fact a Lorentzian isometry.

3. Summary of papers

3.1 Summary of Paper I

Paper I is also part of the licentiate thesis [49]. In Paper I we study asymptotically-anti de Sitter initial data sets (M^n, g, k). In Theorem 6.2 of Paper I, we suggest a new coupled system which if solved would give a proof of the positive mass theorem for asymptotically anti-de Sitter initial data sets.

Theorem 3.1. *If there are globally defined functions* $f, u : M^n \to \mathbb{R}$ *such that*

$$\begin{aligned}\left(g^{ij} - \frac{u^2 \nabla^i f \nabla^j f}{1 + u^2 |\nabla f|^2}\right)\left(\frac{u \operatorname{Hess}_{ij} f + u_i f_j + f_i u_j}{\sqrt{1 + u^2 |\nabla f|^2}} - k_{ij}\right) &= 0, \\ \Delta^{\overline{g}} u &= nu,\end{aligned} \tag{3.1}$$

where $\overline{g} = g + df^2$, f, u *satisfy* $f = O(e^{-(\tau+1)r})$, $u = \cosh(r) + O(e^{-(\tau-1)r})$. *Then the positive mass theorem for the initial data set* (M^n, g, k) *holds.*

For reasons which are outlined in Remark 6.1 in Paper I, one does not expect that a *globally* defined solution to the above system exists. In fact, in all previous applications of the Jang equation reduction argument the solution might only be defined outside of a compact set, see for example [62], [29] and [58].

Nevertheless, the above result serves as a proof of concept. The system in (3.1) features the same generalized Jang equation as in [12, 13] and [21] but a different equation for the warping factor. Since the reduction that we propose proceeds in just one step, the reduction appears to be simpler from an analytical perspective and the proposed coupled system is arguably more natural from a geometric point of view.

To further motivate the above result, in Theorem 5.1 of Paper I we show that when a suitable warping factor $u : M^n \to \mathbb{R}$ is given, there always exists a *geometric* solution to the generalized Jang equation.

Theorem 3.2. *Suppose that* (M^n, g, k) *is an asymptotically anti-de Sitter initial data of dimension* $3 \leq n \leq 7$ *and that a warping factor* $u = \cosh(r) + O(1)$ *is given. Then there is a geometric solution to the generalized Jang equation.*

Roughly speaking, a geometric solution to the generalized Jang equation consists the boundary $\Sigma := \partial E$ of an open subset of the warped product space

$E \subset M^n \times_u \mathbb{R}$. We show that Σ consists of graphical components and cylindrical components, all of which have prescribed mean curvature $H_\Sigma = \mathrm{tr}_\Sigma k$. The graphing function of the graphical components of Σ satisfies the generalized Jang equation (2.12).

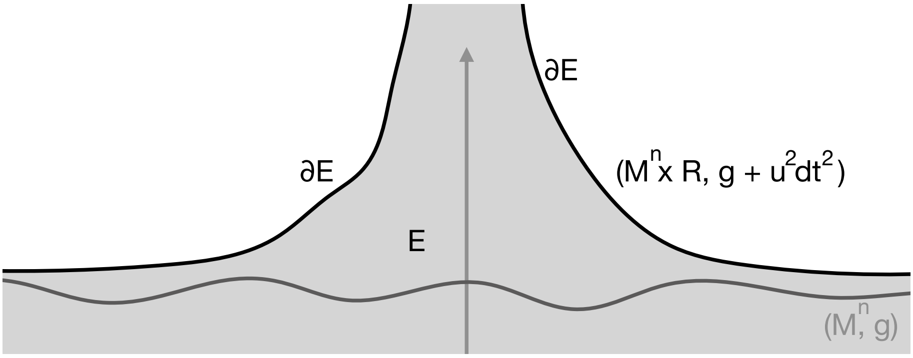

Figure 3.1. A geometric solution to the generalized Jang equation.

In the construction of this geometric solution, the graphing function is shown to satisfy all conditions necessary for Theorem 6.2, except being globally defined. Thus what is left in order to prove the positive mass theorem for asymptotically anti-de Sitter initial data sets in dimensions $3 \leq n \leq 7$ is to analyse the second equation in (3.1), and to show that the coupled system can in fact be solved. Furthermore one would have to show that the argument in the proof of Theorem 6.2 of Paper I can be adjusted to account for the lack of a *global* solution to (3.1).

In order to prove the existence of a geometric solution, we proceed as in Schoen and Yau [62] and Sakovich [58]. In Proposition 4.1 of Paper I we show that the following regularized boundary value problem has a solution on precompact domains Ω for $\varepsilon > 0$.

$$\begin{aligned} \left(g^{ij} - \frac{u^2 \nabla^i f \nabla^j f}{1+u^2|\nabla f|^2}\right)\left(\frac{u \operatorname{Hess}_{ij} f + u_i f_j + f_i u_j}{\sqrt{1+u^2|\nabla f|^2}} - k_{ij}\right) &= \varepsilon f, \quad \text{on } \Omega \\ f &= \phi, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

By choosing a sequence of domains $\{\Omega_k\}_{k=1}^\infty$ that exhausts M^n, and a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset (0,\infty)$ converging to zero, we show that the graphs of the corresponding solutions f_k to (3.2) subconverge to the above described geometric solution in a suitable sense, see Theorem 5.1 of Paper I.

There are a few of issues that need to be resolved in order to prove that such a limit exists. In fact, we have to use the convergence and regularity theory of C-almost minimizing currents developed by Duzar and Steffen [28] and Eichmair [29] in order to construct our geometric solution, which is also a C-almost minimizing current. The only reason for the dimensional restriction $3 \leq n \leq 7$ in Paper I is due to the regularity theory of these currents.

In order to show that the geometric solution has a graphical component defined outside of a compact set $K \subset M^n$ and to show that the solutions f_k do not "escape to infinity", it is necessary to provide barriers to the generalized Jang equation. These are sub and super solutions f_- and f_+ whose existence ensures that the solutions f_k to (3.2) satisfy $f_- \leq f_k \leq f_+$.

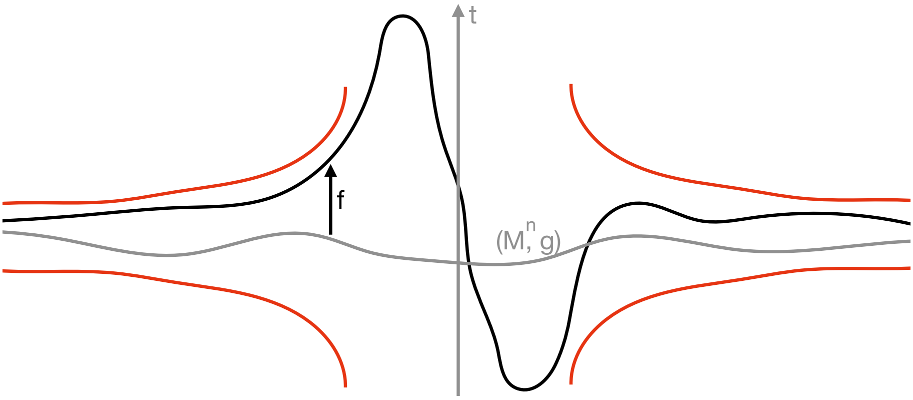

Figure 3.2. Barriers (in red) for the generalized Jang equation which force a solution to a regularized boundary value problem to lie between them.

One of the main results of Paper I is Proposition 3.1, which says that such barriers can be found under very general conditions.

Theorem 3.3. *Suppose that* (M^n, g, k) *is an asymptotically anti-de Sitter initial data set of order* $\tau > n/2$ *and dimension* $n \geq 3$ *and that the warping factor satisfies* $u = \cosh(r) + O_1(1)$. *Then there are barriers for the generalized Jang equation defined outside a compact set.*

We note that the barriers we construct are very general and work for all asymptotics unlike the previous barriers of [58] and [46], which require that one perturbs to the so called *Wang's asymptotics*. Due to generality of our result, we believe that that this barrier construction will have future applications such as finding a solution to solve the full coupled system (3.1).

3.2 Summary of Paper II

The main application of the results of Paper II is to give a new proof of the positive mass theorem for asymptotically Euclidean manifolds in dimension $n = 3$, using the vector field method which can be traced back to the work of Robinson [56].

In Paper II we deduce a several parameter family of geometric inequalities in terms of a minimal Greens function of the Laplace operator Δ^g of a Riemannian manifold (M^3, g). This result can be formulated as follows.

Theorem 3.4. *Let* (M^3, g) *be a complete, asymptotically Euclidean Riemannian manifold with non-negative scalar curvature* $\mathrm{Scal}^g \geq 0$ *and vanishing second homology group* $H_2(M^3, \mathbb{Z}) = 0$*. Let* $\mathcal{G} : M^3 \setminus \{o\} \to \mathbb{R}$ *be the minimal positive Green function for the Laplace operator* Δ^g *with pole at* $o \in M^3$*. Suppose that* $s < t$ *are regular values of* $u := 4\pi\mathcal{G}$*, then*

$$4\pi \int_s^t F(x)\,dx + \left[F(x) \int_{\Sigma_x} |\nabla u| H\, d\sigma + \left[G(x) + (\lambda_1 - 3)\frac{F(x)}{x} \right] \int_{\Sigma_x} |\nabla u|^2\, d\sigma \right]_s^t$$
$$\geq \int_s^t \left[\frac{(\lambda_2 - \lambda_1)F(x)}{x^2} - \frac{(3 - \lambda_1)F'(x)}{x} \right] \left(\int_{\Sigma_x} |\nabla u|^2\, d\sigma \right) dx,$$

where H *is the mean curvature of the level set* $\Sigma_x := u^{-1}(\{x\})$*, the constants* $\lambda_1, \lambda_2, \alpha \in \mathbb{R}$ *satisfy*

$$\alpha(\alpha - 1) + \lambda_1 \alpha = \lambda_1 - \lambda_2,$$

and the functions $F, G : (0, \infty) \to \mathbb{R}$ *are given by*

$$F(x) := x^\alpha, \quad G(x) := \beta\lambda_2 \begin{cases} \frac{x^{\alpha-1}}{\alpha-1} & \text{if } \alpha \neq 1 \\ 0, & \text{if } \alpha = 1 \end{cases}.$$

The condition that the second homology group of M^3 vanishes ensures that the level sets of the function u are connected. Such topological conditions are common whenever one uses level set methods like the above one, see for example [1, 54, 68, 50]. By defining a vector field Z in terms of the function u and by bounding its divergence from below, an application of the divergence theorem yields Theorem 3.4. The vector field Z that we use is irregular at the critical points of the function u and so the divergence theorem cannot be applied immediately. To address this we also prove a "singular" version of the divergence theorem adapted to our setting, see Theorem 3.4 of Paper II.

By specializing the constants λ_1, λ_2 and α, we are able to prove Theorem 4.2 of Paper II. This gives a new proof of the result of Munteanu and Wang [54, Corollary 3.2], under certain mild additional hypotheses. We prove the following.

Theorem 3.5. *With* (M^3,g) *and* $u : M^3 \setminus \{o\} \to \mathbb{R}$ *as in Theorem 3.4, the function u satisfies*

$$\int_{\Sigma_x} |\nabla u|^2 \, d\sigma \leq 4\pi x^2,$$

for every regular value $x \in (0,\infty)$ *of u.*

Using this inequality and Theorem 3.4 we are able to construct a one parameter family of monotone quantities defined in terms of the function u.

Theorem 3.6. *With* (M^3,g) *and* $u : M^3 \setminus \{o\} \to \mathbb{R}$ *as in Theorem 3.4, the quantity* $\mathscr{F}_\alpha$ *defined by*

$$\mathscr{F}_\alpha(x) := 4\pi(\alpha+1)x^{\alpha+1} + x^\alpha \int_{\Sigma_x} |\nabla u| H \, d\sigma - (\alpha+3)x^{\alpha-1} \int_{\Sigma_x} |\nabla u|^2 \, d\sigma,$$

is non-decreasing for any choice of $\alpha \in \mathbb{R} \setminus (-2,0)$.

When (M^3,g) is asymptotically Schwarzschildean, we show that

$$\lim_{x\to 0} x^{-\alpha-2} \mathscr{F}_\alpha(x) = 4\pi\alpha m,$$

where m is the ADM mass of the manifold (M^3,g). This allows us to show that for any $\alpha \leq -2$, the monotonicity of the quantity $\mathscr{F}_\alpha$ implies the positive mass theorem for asymptotically Schwarzschildean manifolds (M^3,g) satisfying the conditions in Theorem 3.4. Moreover we show that the monotonicity of the quantity $\mathscr{F}_{-2}$ implies that any asymptotically Schwarzschildean manifold (M^3,g) satisfying the conditions of Theorem 3.4 and with zero mass is isometric to $(\mathbb{R}^3,\delta)$.

We show that Theorem 3.6 can be used to deduce the monotonicity of some of the monotone quantities appearing in [1], [50] and [68]. More generally, we argue that the above method is an effective and conceptually clear way to generate monotone quantities on Riemannian manifolds.

3.3 Summary of Paper III

In Paper III we define a notion of mass for asymptocially Euclidean Riemannian manifolds with metrics g of local regularity $W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$. In the classical case, the ADM mass of an asymptotically Euclidean initial data set is given by

$$m_{\mathrm{ADM}}(g) := \frac{1}{2(n-1)\omega_{n-1}} \lim_{k\to\infty} \int_{\partial\Omega_k} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{ii}}{\partial x^j} \right) \nu_j^{\Omega_k} \, d\mu_\delta .$$

When the metric g is only known to satisfy $g \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_0$, the above definition becomes problematic. One issue is that the integrand could be ill-defined on $\partial\Omega_k$, since the smooth boundaries of the domains Ω_k have n-dimensional Hausdorff measure zero. Similarly to Gicquaud and Sakovich [33], we resolve this using cutoff functions. For a suitable sequence of compactly supported locally Lipschitz functions $\{\chi_\alpha\}_{\alpha=1}^\infty$, we define the *weak ADM mass* with respect to the chart Φ and the sequence $\{\chi_\alpha\}_{\alpha=1}^\infty$ as the following limit:

$$\begin{aligned} &m_{\mathrm{W}}(g, \Phi, \{\chi_\alpha\}_{\alpha=1}^\infty) \\ &:= \frac{1}{2(n-1)\omega_{n-1}} \lim_{\alpha\to\infty} \int_{\mathbb{R}^n \setminus B_R} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{ii}}{\partial x^j} \right) \left(-\frac{\partial \chi_\alpha}{\partial x^j} \right) d\mu_\delta , \end{aligned} \tag{3.3}$$

Roughly speaking, integrating against the gradient of a cut-off function can be interpreted as a mollified version of integrating against a unit normal on the boundary of a domain.

As mentioned in the end of Section 2.3 of this introduction, a crucial hypothesis for showing the well definedness of the ADM mass is that the scalar curvature $\mathrm{Scal}^g \in L^1(M^n, g)$ is well defined. When $g \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$, this might not be the case. Roughly speaking, we remedy this by treating the scalar curvature as a distribution, similar to that of Lee and LeFloch [43, Definition 2.1], see Definition 3.2 of Paper III.

Gicquaud and Sakovich [33, Remark 4.7] discuss different possible definitions of the scalar curvature in a low regularity setting. Using either of these definitions of *integrable distributional scalar curvature* and assuming an appropriate fall of condition on $g - \delta$, we show that the weak ADM mass in (3.3) is well defined, finite and independent of the choice of cutoff functions $\{\chi_\alpha\}_{\alpha=1}^\infty$, see Theorem 3.9 of Paper III.

Theorem 3.7. *If (M^n, g) is $W^{1,2}$-asymptotically Euclidean manifold of order $\tau \geq (n-2)/2$ and has integrable distributional scalar curvature, then the weak ADM mass $m_W(g, \Phi)$ is well defined and independent of the choice of cutoff functions.*

Furthermore, in Theorem 3.11 of Paper III we prove that under conditions in which the classical ADM mass in (2.5) is well-defined, so is the weak ADM mass in (3.3) and in this case they agree.

Theorem 3.8. *If* (M^n, g) *is a* C^1*-asymptotically Euclidean manifold of order* $\tau > (n-2)/2$ *such that* $\mathrm{Scal}^g \in L^1(M^n, g)$*, then the weak and the classical ADM mass agree:*

$$m_W(g, \Phi) = m_{ADM}(g, \Phi).$$

Having shown that our weak ADM mass is a well defined generalization of the classical ADM mass, we show that it is coordinate invariant. For this we use a result of Bartnik [8, Corollary 3.2], which says that if g is asymptotically Euclidean with respect to the two coordinate charts Φ and $\widetilde{\Phi}$, then the transition function $\widetilde{\Phi}^{-1} \circ \Phi$ can be decomposed into an isometry and an "almost identity", of the Euclidean space $(\mathbb{R}^n, \delta)$. Similar to Bartnik [8], at this stage we need to assume that g is $W^{1,p}$-asymptotically Euclidean for some $p > n$.

In Proposition 3.16 and Proposition 3.18 of Paper III we show that the weak ADM mass in (3.3) is invariant under isometries and under almost identities of Euclidean space. Combining these results we obtain the following.

Theorem 3.9. *If* (M^n, g) *is* $W^{1,p}$*-asymptotically Euclidean of order* $\tau \geq \frac{n-2}{2}$ *with respect to the two charts* Φ *and* $\widetilde{\Phi}$*, then the weak ADM mass is well defined with respect to* Φ *if and only if it is well defined with respect to* $\widetilde{\Phi}$*, in which case the two masses agree:*

$$m_W(g, \Phi) = m_W(g, \widetilde{\Phi}).$$

Finally, we turn to the Ricci version of the ADM mass studied by Miao and Tam [51] and Herzlich [35], for $g \in C^3_{\mathrm{loc}}$. We show that it can also be given a weak formulation, more specifically, we give a definition of a *Ricci version of the weak ADM mass* for metrics $g \in W^{2,2}_{\mathrm{loc}}$. For a suitable sequence of compactly supported functions $\{\chi_\alpha\}_{\alpha=1}^{\infty} \subset W^{1,\infty}_{\mathrm{loc}}$ and a chart at infinity Φ we define

$$m_{\mathrm{RW}}\big(g, \Phi, \{\chi_\alpha\}_{\alpha=1}^{\infty}\big) := \frac{-1}{(n-1)(n-2)\omega_{n-1}} \lim_{\alpha \to \infty} \int_{\mathbb{R}^n \setminus \overline{B}_R} G^g(rDr, -D\chi_\alpha)\, d\mu_\delta.$$

We prove that the weak Ricci version of the ADM mass is well defined and independent of choice of cutoff functions $\{\chi_\alpha\}_{\alpha=1}^{\infty}$, see Theorem 4.5 of Paper III, and that it agrees with the weak ADM mass, see Theorem 4.6 of Paper III.

As a corollary of our results, we conclude that when $g \in C^3_{\mathrm{loc}}$, then the ADM mass, the Ricci version of the ADM mass and their respective weak versions all agree and are independent of the choice of chart at infinity.

3.4 Summary of Paper IV

Paper IV is also part of the licentiate thesis [49]. The first major result of Paper IV is the following theorem, establishing the existence of uniform Temple charts and generalizing [57, Theorem 3.4], see also Temple [66].

Theorem 3.10. *Let (N^{n+1}, h) be a spacetime and let $p \in N$. Then there is a neighbourhood U_p of p such that for each $q \in U_p$, there exists a Temple chart $\Phi_{q,\eta_q} : W_q \to N$ that covers U_p:*

$$U_p \subset \Phi_{q,\eta_q}(W_q). \tag{3.4}$$

Here each η_q satisfies $\eta_q(0) = q$ and is a member of a smooth collection of timelike geodesics foliating a neighborhood of p and perpendicular to a fixed spacelike hypersurface, Σ_p, that contains p.

Any neighbourhood U_p with the properties described in Theorem 3.10 is called a *uniform Temple neighborhood* and any of its Temple charts satisfying (3.4) is called a *uniform Temple chart*. Uniform Temple neighbourhoods resemble uniformly normal neighbourhoods of Riemannian Geometry. Just like uniformly normal neighbourhoods are useful in the local analysis of Riemannian manifolds, uniform Temple charts are useful in the local analysis of spacetimes because they capture the local causal structure. As a consequence of the above result, we have the following result (cf. [57, Theorem 1.1], [16, Theorem 1.4], [32, Theorem 3]).

Theorem 3.11. *Let p be a point in a spacetime (N^{n+1}, h) equipped with a weak temporal function $\tau : N \to \mathbb{R}$. Then there is a neighbourhood $U \subset N$ of the point p such that*

$$\forall q, q' \in U : \quad q \le q' \iff \hat{d}_\tau(q, q') = \tau(q') - \tau(q).$$

This result says that if the time function τ is weak temporal then causality is always *locally* encoded, even though it might not be encoded *globally*, see [57, Examples 2.1 and 2.2] for counterexamples. We apply Theorem 3.11 to prove Theorem 4.8 of Paper IV, namely that a spacetime (N^{n+1}, h) equipped with a weak temporal function satisfying $|\nabla^h \tau|_h = 1$ can be recovered from the metric space $(N, \hat{d}_\tau)$ and the time function τ.

Theorem 3.12. *Suppose that a pair of spacetimes, (N_i^{n+1}, h_i), $i = 1, 2$, with $n \ge 2$, have weak temporal functions, $\tau_i : N_i^{n+1} \to \mathbb{R}$, such that $|\nabla^{g_i} \tau_i|_{g_i} = 1$ almost everywhere. If a bijection, $F : N_1 \to N_2$, preserves time and preserves distances:*

$$\tau_1(p_1) = \tau_2(F(p_1)), \quad \hat{d}_{\tau_1}(p_1, p_2) = \hat{d}_{\tau_2}(F(p_1), F(p_2)), \quad \text{for all } p_1, p_2 \in N,$$

then F is a diffeomorphism and a Lorentzian isometry.

Previously, Sakovich and Sormani [57, Theorem 1.3] proved Theorem 3.12 under the hypothesis that $\hat{d}_\tau$ encodes causality globally as in (2.19). Here, we are able to remove this additional hypothesis that $\hat{d}_\tau$ encode causality globally by using our uniform Temple charts.

Roughly speaking, Theorem 3.12 tells us that any time and null distance preserving bijection is a Lorentzian isometry. This in particular can be used to prove definiteness of intrinsic spacetime distances, cf. [59, Section 5]. Note in that in order to define the intrinsic flat distance between spacetimes, these need to be equipped with an integral current structure, see Sormani and Wenger [65] for details. In this regard we have the following theorem.

Theorem 3.13. *The uniform Temple charts constructed in the proof of Theorem 3.10 above form a bi-Lipschitz atlas on* N^{n+1}.

4. Svensk sammanfattning

Denna avhandling består av fyra artiklar inom differentialgeometri, allmän relativitet, och geometrisk analys där frågor rörande mass-begrepp och deras positivitet, samt utvecklingen av metrisk teori för jämförelse av rumtider behandlas.

Artikel I

I Artikel I studerar vi asymptotiskt anti-de Sitter initialdata (M^n, g, k). I Sats 6.2 i Artikel I föreslår vi ett nytt kopplat system som, om lösbart, skulle ge ett bevis av satsen om positiv massa för asymptotiskt anti-de Sitter initialdata.

Av skäl som redogörs för i Anmärkning 6.1 i Artikel I finns det emellertid inte anledning att förvänta sig att en globalt definierad lösning till systemet i (3.1) alltid existerar. I alla tidigare tillämpningar av reduktionsargumentet baserat på Jangs ekvation har lösningarna i allmänhet endast varit definierade utanför en kompakt mängd, se till exempel [62], [58] och [46]. Trots detta tjänar vårat resultat som en viktig indikation av metodens tillämpningsområden. Systemet i (3.1) innehåller samma generaliserade Jangekvation som i [12, 13] och [21], men en annan ekvation för krökningsfaktorn. Eftersom den reduktion vi föreslår genomförs i ett enda steg så verkar den enklare ur en analytisk synvinkel, och det föreslagna kopplade systemet kan dessutom sägas vara mer naturligt ur ett geometriskt perspektiv.

Som ytterligare motivation för resultatet ovan visar vi i Sats 5.1 i Artikel I att det i dimension $3 \leq n \leq 7$, givet en lämplig krökningsfaktor $u : M^n \to \mathbb{R}$, alltid existerar en *geometrisk lösning* till den generaliserade Jangekvationen. Grovt sett utgörs en geometrisk lösning av randen $\Sigma = \partial E$ till en öppen delmängd $E \subset M^n \times_u \mathbb{R}$. Vi visar att Σ består av endast grafiska och cylindriska komponenter, och att alla dessa har föreskriven medelkrökning $H_\Sigma = \mathrm{tr}_\Sigma k$, samt att graf-funktionen för de grafiska komponenterna av Σ uppfyller den generaliserade Jangekvationen (2.12).

I konstruktionen av denna geometriska lösning visar vi att graf-funktionen för de grafiska komponenterna av Σ uppfyller samtliga villkor som krävs i Sats 6.2, med undantag för att vara globalt definierad. För att erhålla ett bevis för satsen om positiv massa för asymptotiskt anti-de Sitter initialdata i dimensionerna $3 \leq n \leq 7$ återstår därför att analysera den andra ekvationen i (3.1) samt att visa att det kopplade systemet verkligen kan lösas. Därutöver krävs att argumentet i beviset av Sats 6.2 i Artikel I modifieras så att det även omfattar situationen där någon global lösning till (3.1) inte finns.

För att visa existensen av en geometrisk lösning följer vi i huvudsak samma strategi som Schoen och Yau [62] samt Sakovich [58]. I Proposition 4.1 i Artikel I visar vi att en följd av regulariserade randvärdesproblem (se (3.2)) kan lösas på kompakta domäner. Vi visar sedan att graferna till de motsvarande lösningarna till (3.2) konvergerar till en geometrisk lösning i lämplig mening, se Sats 5.1 i Artikel I.

Det finns ett antal tekniska svårigheter som måste hanteras för att visa att en sådan gräns verkligen existerar. I synnerhet behöver vi använda konvergens- och regularitetsteorin för C-nästan minimerande strömmar, utvecklad av Duzaar och Steffen [28] samt Eichmair [29], för att konstruera vår geometriska lösning, som också visar sig vara en C-nästan minimerande ström. Det enda skälet till dimensionsbegränsningen $3 \leq n \leq 7$ i Artikel I kommer från regularitetsteorin för dessa strömmar.

För att visa att den geometriska lösningen har en grafisk komponent definierad utanför en kompakt mängd och för att säkerställa att lösningarna inte "flyr mot oändligheten", måste man konstruera barriärer för den generaliserade Jangekvationen. Dessa utgörs av sub- och superlösningar som tvingar lösningarna till randvärdesproblemet (3.2) att förbli emellan dem.

Ett av huvudresultaten i Artikel I är Proposition 3.1, där vi visar att sådana barriärer kan konstrueras under mycket allmänna antaganden. Detta skiljer sig från de barriärer som tidigare konstruerats i [58] och [46] där man måste perturbera till så kallade *Wangs asymptotiker*. Denna allmänna karaktär gör att vår barriärkonstruktion skulle kunna få ytterligare tillämpningar i framtiden, exempelvis för att finna en lösning till det hela kopplade systemet (3.1).

Artikel II

Vi ger ett nytt bevis för satsen om positiv massa för asymptotiskt Euklidiska mångfalder i dimension $n = 3$. Vi gör detta med hjälp av en så kallad *vektorfältsmetod*, som kan spåras tillbaka till Robinsons arbete [56], med vilken vi härleder en flerparametrig familj av geometriska olikheter uttryckta i termer av den minimala Greenfunktionen $\mathscr{G}$ till Laplaceoperatorn Δ^g på en Riemannsk mångfald (M^3, g), se Sats 3.7 i Artikel II.

Genom att definiera ett vektorfält Z i termer av $\mathscr{G}$ erhålls Sats 3.7 i Artikel II som en konsekvens av divergenssatsen. Det vektorfält Z som vi använder är emellertid oregelbundet vid de kritiska punkterna för Greenfunktionen $\mathscr{G}$ och därför kan divergenssatsen inte tillämpas direkt. För att hantera detta bevisar vi även en singulär version av divergenssatsen anpassad till vår situation, se Sats 3.4 i Artikel II.

Genom att specialisera parametrarna i vår familj av geometriska olikheter kan vi därefter bevisa Sats 4.2 i Artikel II. Detta ger ett nytt bevis av ett resultat av Munteanu och Wang [55, Corollary 3.2], under vissa milda ytterligare antaganden. Med hjälp av denna olikhet och Sats 3.7 i Artikel II konstruerar vi en enparametrig familj $\{\mathscr{F}_\alpha\}_{\alpha \notin (-2,0)}$ av monotona storheter definierade i termer av $\mathscr{G}$. När (M^3, g) är asymptotiskt Schwarzschildsk visar vi att

$$\lim_{x \to 0} \mathscr{F}_\alpha(x) = 4\pi m \alpha,$$

där m är ADM massan för mångfalden (M^3, g). Detta tillsammans med att storheterna $\{\mathscr{F}_\alpha\}_{\alpha \notin (-2,0)}$ är monotona tillåter oss att bevisa satsen om positiv massa för asymptotiskt Euklidiska mångfalder, se Sats 1.5 i Artikel II.

Vi visar också att resultaten i Artikel II kan användas för att på nytt härleda några av de monotona storheter som förekommer i [1], [50] och [68]. Mer allmänt argumenterar vi för denna vektorfältsmetod metod är ett effektivt och begripligt sätt att generera monotona storheter på Riemannska mångfalder.

Artikel III

I Artikel III introducerar vi ett massbegrepp för asymptotiskt Euklidiska Riemannska mångfalder vars metrik g har lokal regularitet $W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$. I det klassiska fallet ges ADM-massan för asymptotiskt Euklidiska initialdata av formeln i (2.5). När metriken g ligger i en låg regularitetsklass blir denna definition problematisk. Ett av problemen är att integranden kan vara odefinierad på randen $\partial\Omega_k$.

På samma sätt som Gicquaud och Sakovich [33] hanterar vi detta genom att använda avskärningsfunktioner. För en lämplig följd av kompakt stödda, lokalt Lipschitz funktioner $\{\chi_\alpha\}_{\alpha=1}^\infty$ definierar vi den svaga ADM-massan med avseende på ett koordinatsystem vid oändligheten Φ och på följden $\{\chi_\alpha\}_{\alpha=1}^\infty$ som:

$$\begin{aligned} &m_{\mathrm{W}}(g, \Phi, \{\chi_\alpha\}_{\alpha=1}^\infty) \\ &:= \frac{1}{2(n-1)\omega_{n-1}} \lim_{\alpha \to \infty} \int_{\mathbb{R}^n \setminus B_R} \sum_{i,j=1}^n \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{ii}}{\partial x^j} \right) \left(-\frac{\partial \chi_\alpha}{\partial x^j} \right) d\mu_\delta, \end{aligned}$$

Grovt sett kan integration mot gradienten av en avskärningsfunktion tolkas som en mollifierad version av integration mot en enhetsnormal till randen av en domän. Som nämnt i slutet av Avsnitt 2.3 i denna sammanfattning är ett viktigt antagande för att visa att ADM-massan är väldefinierad att skalärkrökningen $\mathrm{Scal}^g \in L^1(M^n, g)$ är väldefinierad. När $g \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ behöver detta emellertid inte vara fallet. Grovt sett hanterar vi detta genom att betrakta skalärkrökningen som en distribution, på liknande sätt som Lee och LeFloch [43, Definition 2.1], se Definition 3.2 i Artikel III.

Gicquaud och Sakovich [33, Remark 4.7] diskuterar olika möjliga definitioner av skalärkrökning i låg regularitet. Med utgångspunkt i en definition av integrerbar distributionell skalärkrökning som liknar dessa, och under antagandet att $g - \delta$ uppfyller ett lämpligt avtagningsvillkor, visar vi att den svaga ADM-massan i (3.3) är väldefinierad, ändlig och oberoende av valet av avskärningsfunktioner se Sats 3.9 i Artikel III.

Vidare bevisar vi Sats 3.11 Artikel III, som säger att under antaganden som garanterar att den klassiska ADM-massan är väldefinierad, så är den svaga ADM-massan i (3.3) också väldefinierad och att de i detta fall sammanfaller.

Efter att ha visat att vår svaga ADM-massa utgör en väldefinierad generalisering av den klassiska ADM-massan visar vi därefter att den är koordinatinvariant. För detta använder vi oss utav ett resultat av Bartnik [8, Corollary 3.2], som säger att om (M^n, g) är asymptotiskt Euklidisk med avseende på två koordinatsystem vid oändligheten Φ, Φ', då kan övergångsavbildningen $\Phi' \circ \Phi^{-1}$ uttryckas som en sammansättning av en isometri och en "nästan identitet" av Euklidiskt rum $(\mathbb{R}^n, \delta)$. I likhet med Bartnik [8] behöver vi i detta steg anta att g är $W^{1,p}$-asymptotiskt Euklidisk för något $p > n$.

I Proposition 3.16 och Proposition 3.18 i Artikel III visar vi att den svaga ADM-massan i (3.3) är invariant under isometrier och under nästan-identiteter av Euklidiskt rum. Genom att kombinera dessa resultat bevisar vi att den svaga ADM-massan är koordinatinvariant.

Slutligen vänder vi oss till Ricci versionen av ADM-massan som studerats av Miao och Tam [50] samt Herzlich [35] i fallet då $g \in C^3_{\mathrm{loc}}$. Vi visar att även denna kan ges en svag formulering. Vi introducerar en Ricci version av den svaga ADM-massan för metriker g som endast uppfyller $g \in W^{2,2}_{\mathrm{loc}}$. För en lämplig följd av kompakt stödda funktioner $\{\chi_\alpha\}_{\alpha=1}^\infty \subset W^{1,\infty}_{\mathrm{loc}}$ och ett koordinatsystem vid oändligheten Φ definierar vi

$$m_{\mathrm{RW}}\big(g, \Phi, \{\chi_\alpha\}_{\alpha=1}^\infty\big) := \frac{-1}{(n-1)(n-2)\omega_{n-1}} \lim_{\alpha\to\infty} \int_{\mathbb{R}^n \setminus \overline{B}_R} G^g(rDr, -D\chi_\alpha)\, d\mu_\delta.$$

Vi visar att den svaga Ricci-versionen av ADM-massan är väldefinierad och oberoende av valet av avskärningsfunktioner, se Sats 4.5 i Artikel III, samt att den sammanfaller med den svaga ADM-massan, se Sats 4.6 i Artikel III. Som en följd av våra resultat drar vi slutsatsen att när $g \in C^3_{\mathrm{loc}}$ så sammanfaller ADM-massan, Ricci-versionen av ADM-massan och deras respektive svaga versioner, och att de alla dessa är oberoende av valet av koordinater vid oändligheten.

Artikel IV

Det första huvudresultatet i Artikel IV är ett bevis för att Temple koordinater kan konstrueras lokalt uniformt på en rumtid (N^{n+1}, h). Detta generaliserar [57, Sats 3.4], se även Temple [66]. Vi visar nämligen att för varje punkt $p \in N$ så finns det ett område U_p kring p sådant att för alla $q \in U_p$ finns det ett Temple-koordinatsystem centrerat i q och som täcker U_p.

En sådan omgivning U_p kallas för en uniform Temple-omgivning, och varje av dess Temple-koordinatsystem som täcker U_p kallas ett uniformt Temple-koordinatsystem. Uniforma Temple-omgivningar kan ses som den Lorentziska motsvarigheten till uniformt normala omgivningar i Riemannsk geometri. På samma sätt som uniformt normala omgivningar är användbara vid lokal analys av Riemannska mångfalder, är uniforma Temple-kartor användbara vid lokal analys av rumtider eftersom de fångar den lokala kausalstrukturen. Som en följd av resultatet ovan bevisar vi att kausalitet är *lokalt* kodat av svagt temporala tidsfunktioner τ och det associerade null-avståndet $\hat{d}_\tau$ (trots att det inte nödvändigtvis är det *globalt*; se [57, Exempel 2.1 och 2.2] för motexempel), se Sats 4.6 i Artikel IV (jfr [57, Sats 1.1], [16, Sats 1.4], [32, Sats 3]).

Vi tillämpar Sats 4.6 i Artikel IV för att bevisa Sats 4.8 i Artikel IV, nämligen att en rumtid (N^{n+1}, h) försedd med en svagt temporal tidsfunktion τ som uppfyller $|\nabla^h \tau|^2_h = 1$ kan återhämtas från det metriska rummet $(N, \hat{d}_\tau)$ tillsammans med tidsfunktionen τ. Detta har tidigare bevisats av Sakovich och Sormani [57, Sats 1.3] under antagandet att kausalitet är globalt kodat. Med hjälp av våra uniforma Temple-koordinatsystem eliminerar vi detta ytterligare antagande om global kausal kodning.

Detta resultat kan i synnerhet användas för att visa definithet hos intrinsiska rumtidsavstånd, jfr [59, avsnitt 5]. Till exempel, för att det så kallade *intrinsic flat distance* ska kunna definieras mellan rumtider så måste rumtiderna vara försedda med en integral ström-struktur, se Sormani och Wenger [65] för detaljer. I detta avseende bevisar vi att de uniforma Temple-koordinatsystemen som vi konstruerar utgör en bi-Lipschitz atlas på N^{n+1}.

5. Acknowledgements

I could never have completed the work that resulted in this thesis if not for the support of many people around me.

First of all, I want to thank my advisor, *Anna Sakovich*. Since my first year at Uppsala University, your encouragement and challenging me have been invaluable and you have been a great mentor in more than one way. You have been generous in sharing your mathematical knowledge, you have been patient and sincere in answering and paying attention to all of my questions, you have supported me in both words and actions, you have provided me with opportunities and advice, and you have helped me grow with an immense amount of thoughtful feedback. I feel incredibly fortunate to have had an advisor as generous and as caring as you. Thank you for everything.

To my co-advisor, *Andreas Strömbergsson*, I am grateful that you showed me how fun and delicate analysis can be. It has become an integral part of how I think and while teaching I have hoped to do the same to others. Thank you.

To the opponent of my licentiate thesis, *Romain Gicquaud*, thank you for our discussions during your visit to Uppsala and for being so supportive and open with me despite our short time together.

I am grateful to the department of Mathematics at Uppsala University for providing me with my education and the opportunity to study and teach mathematics for a living. I want to thank the administrative staff for all the help they have given me throughout the years and my fellow PhD students for plenty of fika, lunches and fun conversations.

To *Seidon Alsaody*, I am grateful for your sharing all your thoughts about teaching and pedagogy. I am even more grateful for your being earnest and thoughtful every time we speak. Somehow our encounters have been few, but almost every one of them has left a lasting impression.

To *Jordi-Lluís Figueras* and *Martin Herschend*, I want to thank you for the Mathcamp you organized in 2017. That summer I experienced what became some of the most important memories that convinced me to pursue a PhD in Mathematics. Thank you for the conversations we have had in the years since.

To my co-author *Stig Lundgren*, I am very happy to have worked with you. Thank you for being a great colleague by navigating my impulsivity so well and for always taking the time to catch up on what has happened in between our meetings.

To my colleague *Ahmed Ellithy*, I wish you came to Uppsala earlier in my studies. I want to thank you for the honesty, sincerity and sensitivity that you always show me, and for poking so much fun with your constant humour.

During the Spring of 2025, I visited the research group “Geometric Analysis, Differential Geometry and Relativity Theory” at the University of Tübingen. I am grateful to *Carla Cederbaum*, for welcoming me to Tübingen, for sharing your experience and insight and letting me be part of such an interesting project. I want to thank *Ariadna León Quirós*, for so smoothly weathering all my confusion and for always answering a quip with a quip. I also want to thank *Alba, Anna, Andoni, Colin, Huisken, Namitha, Noah, Olivia, Rodrigo, Salvo, Sara, Simone*, and *Tejal* for making my time in Tübingen memorable.

To my office mates *Fabian*, *Jonathan* and *Rebekka*, thank you for keeping me company during all these years and for taking the time to talk about so many things not mathematics. My time with each one of you has been a pleasure.

To *Erik, Isak, Max* and *William*, thank you for being my dear friends. I have always felt appreciated and happy among your never ending sources of support and boyish challenges. More than once have you been the crucial reminders that I need to stop and take a breath. Thank you for bearing with me and for including me in your warmth.

To *Evana*, among my friends you are unique in how different we are yet how well we manage to understand each other. I am happy knowing that we share the joy of yet another milestone in our lives. Thank you for always being happy for me.

To *Elyar*, I cherish the tradition of you, Elnaz and I meeting weekly for dinner or a movie or just talking about what is happening in life. You bring a pleasant feeling of calm wherever you go and on more than one occasion have your words been those that untied the knot in my shoulders. Thank you.

To *Ganna* and to *Javidan*. When life brought me to Göteborg you welcomed me into your homes and to your tables, you cared for me, and you helped what had begun to grow to blossom. For your endless generosity during this critical year and which you continue to show, I am forever grateful.

To my parents, *Irfan* and *Vesna*. Malo je što bih postigao bez svega što ste mi vi dali. Bez samopouzdanja se ne bih usudio krenuti, a bez strpljenja bih davno odustao. Pokazali ste mi jedan neustrašiv, a blag način rada i borbe kojim se vodim u svemu što činim. I najvažnije od svega, dali ste mi obzira i razumijevanja, bez kojih bih se bunio u tim ključnim trenucima u kojima sam imao priliku rasti. Za ovo i za mnogo drugo ću biti uvijek zahvalan.

To the Joy of my life, *Elnaz*. In all that I do, you are my stalwart supporter. The next time I have an ocean to cross I know that I will do it equally calmly, knowing that your wind fills my sails. Not only of support, you are my greatest source of happiness. Looking back I see that because of you, all my bad days are less dim and all my good days are brighter. Looking forward I see everything is lit by that light of yours. Thank you for always standing by me.

Funding

My research was financed by the department of Mathematics at Uppsala University. For conference visits I received travel grants from the Swedish Mathematical Society, the Swedish Royal Academy of Sciences, the Liljewalch travel scholarships and the Lettersted travel scholarships. For my research visit to Tübingen i received financial support from Matariki Fellows.

References

[1] Virginia Agostiniani, Lorenzo Mazzieri, and Francesca Oronzio. A Green's function proof of the positive mass theorem. *Communications in Mathematical Physics*, 405(2):Paper No. 54, 23, 2024.

[2] Lars Andersson, Mingliang Cai, and Gregory J. Galloway. Rigidity and positivity of mass for asymptotically hyperbolic manifolds. *Annales Henri Poincaré. A Journal of Theoretical and Mathematical Physics*, 9(1):1–33, 2008.

[3] Lars Andersson, Gregory J. Galloway, and Ralph Howard. The cosmological time function. *Classical and Quantum Gravity*, 15(2):309–322, 1998.

[4] R. Arnowitt, S. Deser, and C. W. Misner. Dynamical structure and definition of energy in general relativity. *Physical Review. Series II*, 116:1322–1330, 1959.

[5] Richard Arnowitt, Stanley Deser, and Charles W. Misner. Energy and the criteria for radiation in general relativity. *Physical Review. Series II*, 118:1100–1104, 1960.

[6] Richard Arnowitt, Stanley Deser, and Charles W. Misner. Coordinate invariance and energy expressions in general relativity. *Physical Review. Series II*, 122:997–1006, 1961.

[7] Richard Arnowitt, Stanley Deser, and Charles W. Misner. The dynamics of general relativity. In *Gravitation: An introduction to current research*, pages 227–265. Wiley, New York-London, 1962.

[8] Robert Bartnik. The mass of an asymptotically flat manifold. *Communications on Pure and Applied Mathematics*, 39(5):661–693, 1986.

[9] Yuchen Bi, Tianze Hao, Shihang He, Yuguang Shi, and Jintian Zhu. A proof for the Riemannian positive mass theorem up to dimension 19, 2026. https://arxiv.org/abs/2603.02769.

[10] Hubert L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *Journal of Differential Geometry*, 59(2):177–267, 2001.

[11] Hubert L. Bray, Demetre P. Kazaras, Marcus A. Khuri, and Daniel L. Stern. Harmonic functions and the mass of 3-dimensional asymptotically flat Riemannian manifolds. *Journal of Geometric Analysis*, 32(6):Paper No. 184, 29, 2022.

[12] Hubert L. Bray and Marcus A. Khuri. A Jang equation approach to the Penrose inequality. *Discrete and Continuous Dynamical Systems. Series A*, 27(2):741–766, 2010.

[13] Hubert L. Bray and Marcus A. Khuri. P.D.E.'s which imply the Penrose conjecture. *Asian Journal of Mathematics*, 15(4):557–610, 2011.

[14] S. Brendle and Y. Wang. A dimension descent scheme for the positive mass theorem in high dimensions, 2026. https://arxiv.org/abs/2604.08473.

[15] Gary L. Bunting and A. K. M. Masood-ul Alam. Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time. *General Relativity and Gravitation*, 19(2):147–154, 1987.

[16] Annegret Burtscher and Leonardo García-Heveling. Global hyperbolicity through the eyes of the null distance. *Communications in Mathematical Physics*, 405(4):Paper No. 90, 35, 2024.

[17] Carla Cederbaum, Albachiara Cogo, and Axel Fehrenbach. Uniqueness of equipotential photon surfaces in 4-dimensional static vacuum asymptotically flat spacetimes for positive, negative, and zero mass—and a new partial proof of the Willmore inequality. *Journal of Mathematical Physics*, 66(5):Paper No. 052504, 24, 2025.

[18] Carla Cederbaum, Albachiara Cogo, Benedito Leandro, and João Paulo dos Santos. Uniqueness of static vacuum asymptotically flat black holes and equipotential photon surfaces in $n+1$ dimensions à la Robinson, 2024. https://arxiv.org/abs/2403.14422.

[19] Carla Cederbaum and Ariadna León Quirós. The Willmore-type inequality, Hamilton's Ricci pinching conjecture, and the enhanced Kasue theorem all in one. *In progress*, 2026.

[20] Carla Cederbaum and Anabel Miehe. A new proof of the Willmore inequality via a divergence inequality. *Transactions of the American Mathematical Society*, 378(9):6655–6676, 2025.

[21] Ye Sle Cha and Marcus Khuri. Transformations of asymptotically AdS hyperbolic initial data and associated geometric inequalities. *General Relativity and Gravitation*, 50(1):Paper No. 3, 48, 2018.

[22] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Generic regularity for minimizing hypersurfaces in dimensions 9 and 10, 2026. https://arxiv.org/abs/2302.02253.

[23] Otis Chodosh, Christos Mantoulidis, Felix Schulze, and Zhihan Wang. Generic regularity for minimizing hypersurfaces in dimension 11, 2025. https://arxiv.org/abs/2506.12852.

[24] Yvonne Choquet-Bruhat and Robert Geroch. Global aspects of the Cauchy problem in general relativity. *Communications in Mathematical Physics*, 14:329–335, 1969.

[25] Piotr Chruściel. Boundary conditions at spatial infinity from a Hamiltonian point of view. In *Topological properties and global structure of space-time (Erice, 1985)*, volume 138 of *NATO Adv. Sci. Inst. Ser. B: Phys.*, pages 49–59. Plenum, New York, 1986.

[26] Piotr T. Chruściel and Gregory J. Galloway. Positive mass theorems for asymptotically hyperbolic Riemannian manifolds with boundary. *Classical and Quantum Gravity*, 38(23):Paper No. 237001, 6, 2021.

[27] Piotr T. Chruściel and Marc Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific Journal of Mathematics*, 212(2):231–264, 2003.

[28] Frank Duzaar and Klaus Steffen. λ minimizing currents. *Manuscripta Mathematica*, 80(4):403–447, 1993.

[29] Michael Eichmair. The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight. *Communications in Mathematical Physics*, 319(3):575–593, 2013.

[30] Michael Eichmair, Lan-Hsuan Huang, Dan A. Lee, and Richard Schoen. The spacetime positive mass theorem in dimensions less than eight. *Journal of the European Mathematical Society (JEMS)*, 18(1):83–121, 2016.

[31] Y. Fourès-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Mathematica*, 88:141–225, 1952.

[32] Gregory J. Galloway. A note on null distance and causality encoding. *Classical and Quantum Gravity*, 41(1):Paper No. 017001, 5, 2024.

[33] Romain Gicquaud and Anna Sakovich. A definition of the mass aspect function for weakly regular asymptotically hyperbolic manifolds. *arXiv:2502.00125*, 2025. https://arxiv.org/abs/2502.00125.

[34] Eduardo Hafemann. A low-regularity riemannian positive mass theorem for non-spin manifolds with distributional curvature, 2026. https://arxiv.org/abs/2602.03451.

[35] Marc Herzlich. Computing asymptotic invariants with the Ricci tensor on asymptotically flat and asymptotically hyperbolic manifolds. *Annales Henri Poincaré. A Journal of Theoretical and Mathematical Physics*, 17(12):3605–3617, 2016.

[36] Sven Hirsch, Demetre Kazaras, and Marcus Khuri. Spacetime harmonic functions and the mass of 3-dimensional asymptotically flat initial data for the Einstein equations. *Journal of Differential Geometry*, 122(2):223–258, 2022.

[37] Lan-Hsuan Huang. On the center of mass in general relativity. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, volume 51, pt. 1, 2 of *AMS/IP Stud. Adv. Math.*, pages 575–591. Amer. Math. Soc., Providence, RI, 2012.

[38] Gerhard Huisken. An isoperimetric concept for the Mass in General Relativity. In *Calculus of variations*, volume 3, pages 1898–1899, 2006. Abstracts from the workshop held July 9–15, 2006, Organized by Giovanni Alberti, Robert McCann and Tristan Riviere, Oberwolfach Reports. Vol. 3, no. 3.

[39] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[40] Werner Israel. Event horizons and gravitational collapse. *General Relativity and Gravitation*, 2:53–59, 1971.

[41] Pong Soo Jang. On the positivity of energy in general relativity. *Journal of Mathematical Physics*, 19(5):1152–1155, 1978.

[42] Dan A. Lee. *Geometric Relativity*, volume 201 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[43] Dan A. Lee and Philippe G. LeFloch. The positive mass theorem for manifolds with distributional curvature. *Communications in Mathematical Physics*, 339(1):99–120, 2015.

[44] Philippe G. LeFloch. Einstein spacetimes with weak regularity. In *Advances in Lorentzian geometry*, volume 49 of *AMS/IP Stud. Adv. Math.*, pages 81–96. Amer. Math. Soc., Providence, RI, 2011.

[45] Martin Lesourd, Ryan Unger, and Shing-Tung Yau. The positive mass theorem with arbitrary ends. *Journal of Differential Geometry*, 128(1):257–293, 2024.

[46] David Lundberg. On jang's equation and the Positive Mass Theorem for asymptotically hyperbolic initial data sets with dimensions above three and below eight, 2023. https://arxiv.org/abs/2309.11330.

[47] Edward Malec and Niall Ó Murchadha. The Jang equation, apparent horizons and the Penrose inequality. *Classical and Quantum Gravity*, 21(24):5777–5787,

2004.

[48] Marc Mars. Present status of the Penrose inequality. *Classical and Quantum Gravity*, 26(19):193001, 59, 2009.

[49] Benjamin Meco. On positivity of mass and metric structure of spacetimes, 2024. https://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-528134.

[50] Pengzi Miao. Mass, capacitary functions, and the mass-to-capacity ratio. *Peking Mathematical Journal*, 8(2):351–404, 2025.

[51] Pengzi Miao and Luen-Fai Tam. Evaluation of the ADM mass and center of mass via the Ricci tensor. *Proceedings of the American Mathematical Society*, 144(2):753–761, 2016.

[52] Benoît. Michel. Geometric invariance of mass-like asymptotic invariants. *Journal of Mathematical Physics*, 52(5):052504, 14, 2011.

[53] Ettore Minguzzi. Lorentzian causality theory. *Living reviews in relativity*, 22(1):1–202, 2019.

[54] Ovidiu Munteanu and Jiaping Wang. Comparison theorems for 3D manifolds with scalar curvature bound. *International Mathematics Research Notices. IMRN*, (3):2215–2242, 2023.

[55] Hans Ringström. *The Cauchy problem in general relativity*. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2009.

[56] D. C. Robinson. A simple proof of the generalization of Israel's theorem. *General Relativity and Gravitation*, 8(8):695–698, 1977.

[57] A. Sakovich and C. Sormani. The null distance encodes causality. *Journal of Mathematical Physics*, 64(1):Paper No. 012502, 18, 2023.

[58] Anna Sakovich. The Jang equation and the positive mass theorem in the asymptotically hyperbolic setting. *Communications in Mathematical Physics*, 386(2):903–973, 2021.

[59] Anna Sakovich and Christina Sormani. Introducing various notions of distances between space-times, 2025. https://arxiv.org/abs/2410.16800.

[60] Miguel Sánchez. On the geometry of static spacetimes. *Nonlinear Analysis: Theory, Methods & Applications*, 63(5):e455–e463, 2005. Invited Talks from the Fourth World Congress of Nonlinear Analysts (WCNA 2004).

[61] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, 1979.

[62] Richard Schoen and Shing Tung Yau. Proof of the positive mass theorem. II. *Communications in Mathematical Physics*, 79(2):231–260, 1981.

[63] Christina Sormani. Oberwolfach report: Spacetime intrinsic flat convergence, 2018. https://arxiv.org/abs/1805.08886.

[64] Christina Sormani and Carlos Vega. Null distance on a spacetime. *Classical and Quantum Gravity*, 33(8):085001, 29, 2016.

[65] Christina Sormani and Stefan Wenger. The intrinsic flat distance between Riemannian manifolds and other integral current spaces. *Journal of Differential Geometry*, 87(1):117–199, 2011.

[66] George Temple. New systems of normal co-ordinates for relativistic optics. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 168(932):122–148, 1938.

[67] Edward Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.

[68] Bo Zhu. Comparison theorem and integral of scalar curvature on three manifolds. *Journal of Geometric Analysis*, 32(7):Paper No. 197, 19, 2022.

UPPSALA DISSERTATIONS IN MATHEMATICS
Dissertations at the Department of Mathematics
Uppsala University

---

1. Torbjörn Lundh: *Kleinian groups and thin sets at the boundary*. 1995.
2. Jan Rudander: *On the first occurrence of a given pattern in a semi-Markov process*. 1996.
3. Alexander Shumakovitch: *Strangeness and invariants of finite degree*. 1996
4. Stefan Halvarsson: *Duality in convexity theory applied to growth problems in complex analysis*. 1996.
5. Stefan Svanberg: *Random walk in random environment and mixing*. 1997.
6. Jerk Matero: *Nonlinear elliptic problems with boundary blow-up*. 1997.
7. Jens Blanck: *Computability on topological spaces by effective domain representations*. 1997.
8. Jonas Avelin: *Differential calculus for multifunctions and nonsmooth functions*. 1997.
9. Hans Garmo: *Random railways and cycles in random regular graphs*. 1998.
10. Vladimir Tchernov: *Arnold-type invariants of curves and wave fronts on surfaces*. 1998.
11. Warwick Tucker: *The Lorenz attractor exists*. 1998.
12. Tobias Ekholm: *Immersions and their self intersections*. 1998.
13. Håkan Ljung: *Semi Markov chain Monte Carlo*. 1999.
14. Pontus Andersson: *Random tournaments and random circuits*. 1999.
15. Anders Andersson: *Lindström quantifiers and higher-order notions on finite structures*. 1999.
16. Marko Djordjević: *Stability theory in finite variable logic*. 2000.
17. Andreas Strömbergsson: *Studies in the analytic and spectral theory of automorphic forms*. 2001.
18. Olof-Petter Östlund: *Invariants of knot diagrams and diagrammatic knot invariants*. 2001.
19. Stefan Israelsson: *Asymptotics of random matrices and matrix valued processes*. 2001.
20. Yacin Ameur: *Interpolation of Hilbert spaces*. 2001.
21. Björn Ivarsson: *Regularity and boundary behavior of solutions to complex Monge-Ampère equations*. 2002.
22. Lars Larsson-Cohn: *Gaussian structures and orthogonal polynomials*. 2002.
23. Sara Maad: *Critical point theory with applications to semilinear problems without compactness*. 2002.
24. Staffan Rodhe: *Matematikens utveckling i Sverige fram till 1731*. 2002.
25. Thomas Ernst: *A new method for q-calculus*. 2002.
26. Leo Larsson: *Carlson type inequalities and their applications*. 2003.

27. Tsehaye K. Araaya: *The symmetric Meixner-Pollaczek polynomials*. 2003.
28. Jörgen Olsén: *Stochastic modeling and simulation of the TCP protocol*. 2003.
29. Gustaf Strandell: *Linear and non-linear deformations of stochastic processes*. 2003.
30. Jonas Eliasson: *Ultrasheaves*. 2003.
31. Magnus Jacobsson: *Khovanov homology and link cobordisms*. 2003.
32. Peter Sunehag: *Interpolation of subcouples, new results and applications*. 2003.
33. Raimundas Gaigalas: *A non-Gaussian limit process with long-range dependence*. 2004.
34. Robert Parviainen: *Connectivity Properties of Archimedean and Laves Lattices*. 2004.
35. Qi Guo: *Minkowski Measure of Asymmetry and Minkowski Distance for Convex Bodies*. 2004.
36. Kibret Negussie Sigstam: *Optimization and Estimation of Solutions of Riccati Equations*. 2004.
37. Maciej Mroczkowski: *Projective Links and Their Invariants*. 2004.
38. Erik Ekström: *Selected Problems in Financial Mathematics*. 2004.
39. Fredrik Strömberg: *Computational Aspects of Maass Waveforms*. 2005.
40. Ingrid Lönnstedt: *Empirical Bayes Methods for DNA Microarray Data*. 2005.
41. Tomas Edlund: *Pluripolar sets and pluripolar hulls*. 2005.
42. Göran Hamrin: *Effective Domains and Admissible Domain Representations*. 2005.
43. Ola Weistrand: *Global Shape Description of Digital Objects*. 2005.
44. Kidane Asrat Ghebreamlak: *Analysis of Algorithms for Combinatorial Auctions and Related Problems*. 2005.
45. Jonatan Eriksson: *On the pricing equations of some path-dependent options*. 2006.
46. Björn Selander: *Arithmetic of three-point covers*. 2007.
47. Anders Pelander: *A Study of Smooth Functions and Differential Equations on Fractals*. 2007.
48. Anders Frisk: *On Stratified Algebras and Lie Superalgebras*. 2007.
49. Johan Prytz: *Speaking of Geometry*. 2007.
50. Fredrik Dahlgren: *Effective Distribution Theory*. 2007.
51. Helen Avelin: *Computations of automorphic functions on Fuchsian groups*. 2007.
52. Alice Lesser: *Optimal and Hereditarily Optimal Realizations of Metric Spaces*. 2007.
53. Johanna Pejlare: *On Axioms and Images in the History of Mathematics*. 2007.
54. Erik Melin: *Digital Geometry and Khalimsky Spaces*. 2008.
55. Bodil Svennblad: *On Estimating Topology and Divergence Times in Phylogenetics*. 2008.
56. Martin Herschend: *On the Clebsch-Gordan problem for quiver representations*. 2008.
57. Pierre Bäcklund: *Studies on boundary values of eigenfunctions on spaces of constant negative curvature*. 2008.

58. Kristi Kuljus: *Rank Estimation in Elliptical Models*. 2008.
59. Johan Kåhrström: *Tensor products on Category O and Kostant's problem*. 2008.
60. Johan G. Granström*:* *Reference and Computation in Intuitionistic Type Theory*. 2008.
61. Henrik Wanntorp: *Optimal Stopping and Model Robustness in Mathematical Finance*. 2008.
62. Erik Darpö: *Problems in the classification theory of non-associative simple algebras*. 2009.
63. Niclas Petersson: *The Maximum Displacement for Linear Probing Hashing*. 2009.
64. Kajsa Bråting: *Studies in the Conceptual Development of Mathematical Analysis*. 2009.
65. Hania Uscka-Wehlou: *Digital lines, Sturmian words, and continued fractions*. 2009.
66. Tomas Johnson: *Computer-aided computation of Abelian integrals and robust normal forms*. 2009.
67. Cecilia Holmgren: *Split Trees, Cuttings and Explosions*. 2010.
68. Anders Södergren: *Asymptotic Problems on Homogeneous Spaces*. 2010.
69. Henrik Renlund: *Recursive Methods in Urn Models and First-Passage Percolation*. 2011.
70. Mattias Enstedt: *Selected Topics in Partial Differential Equations*. 2011.
71. Anton Hedin: *Contributions to Pointfree Topology and Apartness Spaces*. 2011.
72. Johan Björklund: *Knots and Surfaces in Real Algebraic and Contact Geometry*. 2011.
73. Saeid Amiri: *On the Application of the Bootstrap*. 2011.
74. Olov Wilander: *On Constructive Sets and Partial Structures*. 2011.
75. Fredrik Jonsson: *Self-Normalized Sums and Directional Conclusions*. 2011.
76. Oswald Fogelklou: *Computer-Assisted Proofs and Other Methods for Problems Regarding Nonlinear Differential Equations*. 2012.
77. Georgios Dimitroglou Rizell: *Surgeries on Legendrian Submanifolds*. 2012.
78. Boris Granovskiy: *Modeling Collective DecisionMaking in Animal Groups*. 2012.
79. Mahmoudreza Bazarganzadeh: *Free Boundary Problems of Obstacle Type, a Numerical and Theoretical Study*. 2012.
80. Qi Ma: *Reinforcement in Biology*. 2012.
81. Isac Hedén: *Ga-actions on Complex Affine Threefolds*. 2013.
82. Daniel Strömbom: *Attraction Based Models of Collective Motion*. 2013.
83. Bing Lu: *Calibration: Optimality and Financial Mathematics*. 2013.
84. Jimmy Kungsman: *Resonance of Dirac Operators*. 2014.
85. Måns Thulin: *On Confidence Intervals and Two-Sided Hypothesis Testing*. 2014.
86. Maik Görgens: *Gaussian Bridges – Modeling and Inference*. 2014.
87. Marcus Olofsson: *Optimal Switching Problems and Related Equations*. 2015.
88. Seidon Alsaody: *A Categorical Study of Composition Algebras via Group Actions and Triality*. 2015.

89. Håkan Persson: *Studies of the Boundary Behaviour of Functions Related to Partial Differential Equations and Several Complex Variables*. 2015.
90. Djalal Mirmohades: *N-complexes and Categorification*. 2015.
91. Shyam Ranganathan: *Non-linear dynamic modelling for panel data in the social sciences*. 2015.
92. Cecilia Karlsson: *Orienting Moduli Spaces of Flow Trees for Symplectic Field Theory*. 2016.
93. Olow Sande: *Boundary Estimates for Solutions to Parabolic Equations*. 2016.
94. Jonathan Nilsson: *Simple Modules over Lie Algebras*. 2016.
95. Marta Leniec: *Information and Default Risk in Financial Valuation*. 2016.
96. Arianna Bottinelli: *Modelling collective movement and transport network formation in living systems*. 2016.
97. Katja Gabrysch: *On Directed Random Graphs and Greedy Walks on Point Processes*. 2016.
98. Martin Vannestål: *Optimal timing decisions in financial markets*. 2017.
99. Natalia Zabzina: *Mathematical modelling approach to collective decision-making*. 2017.
100. Hannah Dyrssen: *Valuation and Optimal Strategies in Markets Experiencing Shocks*. 2017.
101. Juozas Vaicenavicius: *Optimal Sequential Decisions in Hidden-State Models*. 2017.
102. Love Forsberg: *Semigroups, multisemigroups and representations*. 2017.
103. Anna Belova: *Computational dynamics – real and complex.* 2017.
104. Ove Ahlman: *Limit Laws, Homogenizable Structures and Their Connections.* 2018.
105. Erik Thörnblad: *Degrees in Random Graphs and Tournament Limits.* 2018.
106. Yu Liu: *Modelling Evolution. From non-life, to life, to a variety of life.* 2018.
107. Samuel Charles Edwards: *Some applications of representation theory in homogeneous dynamics and automorphic functions*. 2018.
108. Jakob Zimmermann: *Classification of simple transitive 2-representations.* 2018.
109. Azza Alghamdi: *Approximation of pluricomplex Green functions – A probabilistic approach.* 2018.
110. Brendan Frisk Dubsky: *Structure and representations of certain classes of infinite-dimensional algebras*. 2018.
111. Björn R. H. Blomqvist: *Gaussian process models of social change*. 2018.
112. Konstantinos Tsougkas: *Combinatorial and analytical problems for fractals and their graph approximations*. 2019.
113. Yevgen Ryeznik: *Optimal adaptive designs and adaptive randomization techniques for clinical trials*. 2019.
114. Andrea Pasquali: *Constructions in higher-dimensional Auslander-Reiten theory*. 2019.
115. Daniah Tahir: *Multi-trait Branching Models with Applications to Species Evolution*. 2019.
116. Andreas Rocén: *7D supersymmetric Yang-Mills theory on toric and hypertoric manifolds*. 2019.

117. Filipe Mussini: *Selected Topics in Continuum Percolation. Phase Transitions, Cover Times and Random Fractals*. 2019.
118. Laertis Vaso: *Constructions of n-cluster tilting subcategories using representation-directed algebras.* 2020.
119. Anders Israelsson: *Regularity Properties of Oscillatory Integral Operators on Function Spaces.* 2020.
120. Johan Asplund: *Chekanov-Eliashberg dg-algebras and partially wrapped Floer cohomology*. 2021.
121. Elin Persson Westin: *Stratified algebras and classification of tilting modules*. 2022.
122. Marcus Westerberg: *Prostate cancer incidence, treatment and mortality. Empirical longitudinal register-based studies and methods for handling missing data*. 2022.
123. Dan Lilja: *Properties of invariant sets in certain two-dimensional dynamical systems. Renormalization and beyond*. 2022.
124. Oleksandra Gasanova: *On properties of monomial ideals and algebras.* 2022.
125. Colin Desmarais: *Decorating trees grown in urns*. 2022
126. Malte Litsgård: *On parabolic equations of Kolmogorov-Fokker-Planck type*. 2023
127. Carmina Fjellström: *Selected Topics in Mathematical Modelling: Machine Learning and Tugs-of-War*. 2023.
128. Robert Moscrop: *Quivers, superconformal field theories and higher symmetries.* 2023.
129. Markus Thuresson: *Ext-algebras of standard modules over quasi-hereditary algebras*. 2023.
130. Aksel Bergfeldt: *Regularity of multilinear oscillatory integral operators and nonlinear dispersive partial differential equations*. 2023.
131. Fabian Burghart: *Building and Destroying Urns, Graphs, and Trees.* 2023
132. Yuqiong Wang: *Optimal stopping, incomplete information, and stochastic games.* 2023.
133. Helena Jonsson: *Structure and representations of bimodule categories*. 2023.
134. Christoffer Olsson: *Symmetry in Randomness. Additive Functionals and Symmetries of Random Trees and Tree-Like Graphs*. 2023.
135. Linnéa Gyllingberg: *The Art of Modelling Oscillations and Feedback across Biological Scales.* 2024.
136. Joel Dahne: *Computer Assisted Studies in Fluid Mechanics and Spectral Geometry.* 2024.
137. Lukas Nakamura: *Skein-valued Gromov-Witten theory and Hofer geometry in contact manifolds*. 2024.
138. Axel Husin: *Generalized local systems and the Maslov class of exact Lagrangians. 2025.*

139. Rebekka Müller-Widmann: *Probabilistic Models of Genetic Variability in Sequence Evolution.* 2025.
140. Mingyi Hou: *Behind the training dynamics of neural networks. Analysis of Fokker-Planck equations and the path to metastability*. 2025.
141. Mateusz Stroiński: *Module categories in the absence of adjunctions.* 2025
142. Elias Riedel Gårding: *Aspects of Generalised Symmetries of Four-dimensional Field Theories. 2025.*
143. Darius Dramburg: *Higher representation infinite algebras from skew-group algebras. Higher preprojective gradings, Koszul gradings, and McKay quivers. 2025.*
144. Villhelm Agdur: *Community Structure in Graphs. Algorithms and Theoretical Guarantees. 2025.*
145. Aron Persson: *Adaptive Filtering on Manifolds. With Applications to Wireless Networks. 2025.*
146. Christoffer Söderberg: *Higher Auslander-Reiten Theory for Species with Relations. 2026.*
147. Martin Andersson: *Through a Gaussian, Darkly*. 2026.
148. Wei Xing: *Cluster tilting for higher Nakayama algebras*. 2026.
149. Anna Rodriguez Rasmussen: *Existence and uniqueness of exact Borel Subalgebras*. 2026.
150. Benjamin Meco: *Mass and Causality in Geometric Relativity* 2026.