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Structural identifiability in fractional-order networks☆

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ABSTRACT

Identifying parameters of fractional-order discrete-time dynamical networks from input–output data is crucial for modeling complex systems in neuroscience and biology. Despite existing methods, fundamental conditions ensuring structural identifiability based solely on network structure remain unexplored.

This paper establishes identifiability theory for fractional-order networks by proving that a network is structurally identifiable if and only if all its subsystems are identifiable. We introduce a graph-theoretical hierarchical algorithm that systematically partitions networks into identifiable subsystems via input–output reachability, enabling decomposition-based parameter learning with provable guarantees. Extensive simulation experiments validate that this approach preserves identifiability while achieving significant computational speedup, demonstrating scalability for large-scale systems.

1. Introduction

Fractional-order network provide mathematical frameworks that extend classical integer-order differential and difference equations to non-integer orders, enabling the modeling of phenomena with memory, hereditary properties, and multi-scale dynamics (Monje et al., 2010; Podlubny, 1999).

In discrete-time domain, these networks have become increasingly relevant for modeling dynamics exhibiting memory and complex interdependencies across various domains: from capturing long-term memory effects in neural dynamics (Reed et al., 2022) and biological processes (Ionescu et al., 2017), to modeling complex behaviors in engineering systems (Ortigueira & Machado, 2022) and economic phenomena (Tarasov, 2019). Therefore, the ability to accurately characterize such networks through discrete-time fractional-order models is fundamental to understanding and predicting their behavior, enabling advancements in monitoring, control, and optimization tasks.

To reliably utilize these fractional-order models, it is imperative to develop system identification methods capable of extracting accurate parameters from observed input–output data. While several approaches have been proposed for the identification of fractional-order network parameters (Chatterjee & Pequito, 2022), there is still a notable absence of fundamental conditions that guarantee the identifiability of these parameters. Particularly lacking are criteria that ensure the computational feasibility of parameter estimation and verify that such estimation is fundamentally possible, given the intrinsic complexity of fractional-order dynamics.

To reliably utilize these fractional-order models, it is imperative to develop system identification methods capable of extracting accurate parameters from observed input–output data. While several approaches have been proposed for the identification of fractional-order network parameters (Chatterjee & Pequito, 2022), there is still a notable absence of fundamental conditions that guarantee the identifiability of these parameters. Particularly lacking are criteria that ensure the computational feasibility of parameter estimation and verify that such estimation is fundamentally possible, given the intrinsic complexity of fractional-order dynamics.

In this paper, we extend classical structural identifiability theory to fractional-order networks and establish the relationship between network-level and subnetwork-level identifiability properties. Specifically, we significantly advance the field by unveiling the intrinsic role of input–output decompositions—previously used primarily in hierarchical decentralized control and estimation—as fundamental building blocks for structural identifiability to establish necessary conditions for identifiability based purely on network structure. This decomposition naturally breaks the overall identification problem into manageable subproblems, suggesting that we can first identify parameters within clearly defined subsystems and, subsequently, determine the inter-subsystem connections. Consequently, our approach not only provides essential theoretical insights into the identifiability conditions but also introduces a systematic, efficient, and

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dynamic method for parameter learning with guaranteed identifiability. The proposed methodology significantly enhances computational tractability and paves the way for practical applications involving large-scale fractional-order discrete-time networks.

Our main contributions are as follows:

1. **Identifiability Theory for Fractional-Order Networks**
    - First formal adaptation of structural and output identifiability concepts to discrete-time fractional-order networks
    - Novel sufficient conditions for testing identifiability through sensitivity matrices accounting for fractional dynamics and memory effects
2. **Network–Subnetwork Identifiability Relationship**
    - Proved bidirectional relationship: a fractional-order network is identifiable if and only if all its subsystems are identifiable
    - Provides rigorous foundation for decomposition-based parameter learning with provable identifiability preservation
3. **Graph-Theoretical Decomposition Framework**
    - Extended input–output decomposition from (Agbi & Krogh, 2014) to fractional-order networks with memory dynamics
    - Introduced hierarchical decomposition algorithm that systematically partitions networks into identifiable subsystems based on input–output reachability
4. **Computational Efficiency and Validation**
    - Demonstrated significant computational reduction through subsystem-based identification
    - Validated approach with simulations and reconstruction accuracy

2. Problem statement

Consider the following linear, discrete-time fractional-order network model described by the following model $M(\theta)$:

$$\Delta^{\alpha}x[k+1] = A(\theta_A)x[k] + B(\theta_B)u[k] + G(\theta_G)w[k] \tag{1}$$

$$y[k] = Cx[k] + e[k], \tag{2}$$

parameterized by $\theta = [\alpha, \theta_A, \theta_B, \theta_G] \in \mathbb{R}^q$, which includes the fractional-order exponents $\alpha \in \mathbb{R}^n_+$ and the parameter vectors θ_A, θ_B, and θ_G representing the unknown entries of the state matrix $A(\theta_A) \in \mathbb{R}^{n\times n}$, input matrix $B(\theta_B) \in \mathbb{R}^{n\times m}$, and disturbance matrix $G(\theta_G) \in \mathbb{R}^{n\times v}$, respectively. The remaining variables are $x[k] \in \mathbb{R}^n$ which is the state vector, $u[k] \in \mathbb{R}^m$ is the actuated input, $w[k] \in \mathbb{R}^v$ is the disturbance input, $y[k] \in \mathbb{R}^o$ is the measured output, $e[k] \in \mathbb{R}^o$ is the measurement noise, and $C \in \mathbb{R}^{o\times n}$ is the output matrix.

The fractional difference operator $\Delta^{\alpha}x[k+1] = [\Delta^{\alpha_1}x_1[k+1], \ldots, \Delta^{\alpha_n}x_n[k+1]]^{\top}$ is characterized by the different entries of the state $x[k]$ and defined element-wise as

$$\Delta^{\alpha_i}x_i[k+1] = x_i[k+1] + \sum_{j=0}^{k} \psi(\alpha_i, j+1)x_i[k-j], \tag{3}$$

where

$$\psi(\alpha_i, j) = \frac{\prod_{l=0}^{j-1}(l-\alpha_i)}{j!}, \tag{4}$$

and for $0 < \alpha_i < 1$ and integers $j \geq 0$, it leads to the well known discretization of the Grünwald–Letnikov derivative (Baleanu et al., 2011).

In what follows, we consider the following assumptions:

Assumption 1. An actuated input variable is applied to exactly one state variable, such that the input matrix $B(\theta_B)$ has exactly one non-zero entry per column. ∘

Assumption 2. An output variable measures exactly one state variable, such that the output matrix C has exactly one non-zero entry per row. ∘

Furthermore, let the predicted output of $M(\theta)$ at time k be $\hat{y}[k; \theta, x[0], u, w] = Cx[k]$, where $x[0]$ represents the initial condition, $u = \{u[k] : \forall k \geq 0\}$, and $w = \{w[k] : \forall k \geq 0\}$.

Notably, the presence of fractional-order derivatives introduces long-term memory effects and coupling between fractional exponents α and system matrices, fundamentally altering the identifiability landscape compared to classical integer-order systems. Furthermore, fractional-order networks can be represented as infinite-dimensional linear systems, posing challenges and questions that do not arise in the finite-dimensional LTI counterpart, with no a priori reason to expect that identifiability methods developed for LTI systems would remain valid.

In this context, we focus on addressing the following two problems:

Problem 1. Given a fractional-order network model $M(\theta)$ as described in (1)–(2), determine necessary and/or sufficient conditions under which the parameters θ can be uniquely identified from input–output data.

In other words, on the one hand, Problem 1 addresses the fundamental question of structural identifiability for fractional-order networks. On the other hand, the second problem addresses the computational efficiency of identifiability analysis for large-scale networks. By decomposing a complex network into manageable subcomponents, we aim to significantly reduce computational complexity while preserving the ability to accurately identify system parameters. Specifically, we have the following.

Problem 2. Given a large-scale fractional-order network model $M(\theta)$, determine whether it can be decomposed into smaller subnetworks $M_i(\theta_i)$ such that the identifiability of the overall network can be inferred from the identifiability of its subnetworks.

Thus, the solutions to these problems provide us with a framework that not only provides theoretical insights into the identifiability of fractional-order systems but also offers practical methods for analyzing complex networks across various applications.

3. Structural identifiability in fractional-order networks

This section develops the theoretical foundation for assessing structural identifiability in fractional-order network models. It is organized into three main parts. Section 3.1 introduces the notion of identifiability for fractional-order systems, adapting classical definitions of structural and output identifiability to this setting and presenting sufficient conditions for testing them. Section 3.2 establishes the relationship between the identifiability of the full network and that of its subnetworks, formalizing the idea that network-level identifiability can be inferred from properly structured and identifiable subsystems. Finally, Section 3.3 presents a graph-theoretical framework for decomposing fractional-order networks into input–output reachable subnetworks, culminating in a hierarchical algorithm that guarantees identifiability is preserved throughout the decomposition.

3.1. Identifiability of fractional-order network models

Given a fractional-order network model $M(\theta)$ as described in (1)–(2), we explore and adapt the notions of model identifiability previously established for linear and nonlinear systems (see Ljung (1999), van Doren et al. (2009)) to the context of fractional-order networks.

Definition 1 (*Structural Identifiability*)**.** The fractional-order network model $M(\theta)$, shown in (1), is structurally identifiable at $\theta = \theta^*$ if for all parameters θ_1, θ_2 in a neighborhood of θ^*, then $M(\theta_1) = M(\theta_2) \Rightarrow \theta_1 = \theta_2$. ∘

Definition 2 (*Output Identifiability*)**.** The predicted output of the fractional-order network (1), $\hat{y}[k; \theta, x[0], u, w]$, is output identifiable at $\theta = \theta^*$ if for all parameters θ_1, θ_2 in a neighborhood of θ^* and for a given $x[0]$, u, and w, then $\hat{y}[k; \theta_1, x[0], u, w] = \hat{y}[k; \theta_2, x[0], u, w]$ leads to $\theta_1 = \theta_2$. ∘

Notice that it readily follows that an output identifiable model is also structurally identifiable.

In what follows we show that structural identifiability and output identifiability of the fractional-order network model $M(\theta)$ can be verified using Proposition 1 and Proposition 2, respectively. These propositions are sufficient conditions for model identifiability.

Proposition 1 (*Test for Structural Identifiability*)**.** *Let the state transition relationship for fractional-order systems be described by Reed et al. (2023):*

$$x[k] = F(\alpha, \theta_A, k)x[0] + \sum_{j=0}^{k-1} F(\alpha, \theta_A, k-1-j)B(\theta_B)u[j], \tag{5}$$

where $F(\alpha, \theta_A, k)$ represents the transition matrix defined as

$$F(\alpha, \theta_A, k) = \begin{cases} I_n, & k = 0, \\ \sum_{j=0}^{k-1} A_j(\alpha, \theta_A) \cdot F(\alpha, \theta_A, k-1-j), & k \geq 1, \end{cases} \tag{6}$$

with I_n the identity matrix, $A_0(\alpha, \theta_A) = A(\theta_A) - D(\alpha, 1)$ and $A_j(\alpha) = -D(\alpha, j+1)$ for $j \geq 1$, where

$$D(\alpha, j) = \begin{pmatrix} \psi(\alpha_1, j) & 0 & \dots & 0 \\ 0 & \psi(\alpha_2, j) & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \psi(\alpha_n, j) \end{pmatrix},$$

$\psi(\alpha_i, j) = \frac{\Gamma(j-\alpha_i)}{\Gamma(-\alpha_i)\Gamma(j+1)}$, with $\Gamma(\cdot)$ denoting the Gamma function (Arfken et al., 2013).

Additionally, suppose that $W(q, \theta) = [G_U(q, \theta), G_W(q, \theta)]$ represents $M(\theta)$ in (1) such that $G_U(q, \theta) = \sum_{i=1}^{k} C \cdot F(\alpha, \theta_A, i) \cdot B(\theta_B) \cdot q^{-i}$ and $G_W(q, \theta) = \sum_{i=1}^{k} C \cdot F(\alpha, \theta_A, i) \cdot G(\theta_G) \cdot q^{-i}$, where q^{-i} is a delay operator (i.e., $u[k-1] = q^{-1}u[k]$).

Given the matrix $I_{ss}(\theta) = \left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_\theta^\intercal \left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_\theta \in \mathbb{R}^{q\times q}$ expressing the model structure sensitivity to parameter changes, then $M(\theta)$ is structurally identifiable at $\theta = \theta^*$ if rank$(I_{ss}(\theta^*)) = q$. ∘

Proof. Consider the transfer function matrix $W(q, \theta) = [G_U(q, \theta), G_W(q, \theta)]$ which characterizes the input–output behavior of the fractional-order system. For the system to be structurally identifiable, different parameter values must lead to different input–output behaviors.

Let θ_1 and θ_2 be two parameter vectors in a neighborhood of θ^*. Structural identifiability requires that $W(q, \theta_1) = W(q, \theta_2)$ implies $\theta_1 = \theta_2$.

Since the Gamma function is differentiable for its entire domain except at non-positive integer, the function $\psi(\alpha_i, j)$ is differentiable with respect to α_i. This ensures that $D(\alpha, j)$ is differentiable with respect to the fractional exponents, allowing us to perform Taylor expansions of $G(\alpha, k)$ and consequently $W(q, \theta)$.

Assuming $W(q, \theta_1) = W(q, \theta_2)$, we can perform a first-order Taylor series expansion around θ^* $W(q, \theta_1) \approx W(q, \theta^*) + \left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*} \cdot (\theta_1 - \theta^*)$ $W(q, \theta_2) \approx W(q, \theta^*) + \left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*} \cdot (\theta_2 - \theta^*)$.

Since $W(q, \theta_1) = W(q, \theta_2)$, we have: $\left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*} \cdot (\theta_1 - \theta_2) = 0$

For this equation to imply $\theta_1 = \theta_2$, the matrix $\left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*}$ must have full column rank, meaning that no non-zero parameter difference vector can be in its null space.

The sensitivity matrix $I_{ss}(\theta^*) = \left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*}^T \cdot \left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*}$ has the same rank as $\left[\frac{\partial W(q,\theta)}{\partial \theta}\right]_{\theta*}$. Therefore, if rank$(I_{ss}(\theta^*)) = q$ (the dimension of θ), then the system is structurally identifiable at $\theta = \theta^*$.

Proposition 2 (*Test for Output Identifiability*)**.** *Consider the model $M(\theta)$ in (1) where $\theta \in \mathbb{R}^q$, and let the matrix $I_{dd}(\theta) = \sum_k \left[\frac{\partial y[k]}{\partial \theta}\right]_\theta^T \left[\frac{\partial y[k]}{\partial \theta}\right]_\theta \in \mathbb{R}^{q\times q}$ express model output sensitivity to parameter changes given $x[0]$, u, and w. Then, $M(\theta)$ is output identifiable at $\theta = \theta^*$ if rank$(I_{dd}(\theta^*)) = q$.*

Proof. Output identifiability concerns whether the system parameters can be uniquely determined from the output sequence $y[k]$ given known inputs and initial conditions. Consider the predicted output sequence $\hat{y}[k; \theta, x[0], u, w]$ for $k = 1, 2, \dots, N$. For output identifiability, we require that if $\hat{y}[k; \theta_1, x[0], u, w] = \hat{y}[k; \theta_2, x[0], u, w]$ for all k, then $\theta_1 = \theta_2$.

Using a first-order Taylor expansion around θ^* $\hat{y}[k; \theta_1, x[0], u, w] \approx \hat{y}[k; \theta^*, x[0], u, w] + \left[\frac{\partial y[k]}{\partial \theta}\right]_{\theta*} \cdot (\theta_1 - \theta^*)$ $\hat{y}[k; \theta_2, x[0], u, w] \approx \hat{y}[k; \theta^*, x[0], u, w] + \left[\frac{\partial y[k]}{\partial \theta}\right]_{\theta*} \cdot (\theta_2 - \theta^*)$. Since $\hat{y}[k; \theta_1, x[0], u, w] = \hat{y}[k; \theta_2, x[0], u, w]$ for all k, we have $\left[\frac{\partial y[k]}{\partial \theta}\right]_{\theta*} \cdot (\theta_1 - \theta_2) = 0$ for all k.

For this to imply $\theta_1 = \theta_2$, the composite sensitivity matrix formed by stacking $\left[\frac{\partial y[k]}{\partial \theta}\right]_{\theta*}$ for all k must have full column rank. The matrix $I_{dd}(\theta^*) = \sum_k \left[\frac{\partial y[k]}{\partial \theta}\right]_{\theta*}^T \cdot \left[\frac{\partial y[k]}{\partial \theta}\right]_{\theta*}$ represents the cumulative output sensitivity to parameter changes across all time steps. Its rank equals the rank of the stacked sensitivity matrix, and if rank$(I_{dd}(\theta^*)) = q$, then the system is output identifiable at $\theta = \theta^*$.

3.2. Identifiability of fractional-order sub-network model

Consider a fractional-order network model $\bar{M}(\theta)$ that is written as the interconnection of N sub-networks of $M(\theta)$:

$$\Delta^{\bar{\alpha}}\bar{x}[k+1] = \bar{A}(\theta_A)\bar{x}[k] + \bar{B}(\theta_B)\bar{u}[k] + \bar{G}(\theta_G)\bar{w}[k], \tag{7}$$

$$\bar{y}[k] = \bar{C}\bar{x}[k] + \bar{e}[k], \tag{8}$$

such that the fractional-order exponents $\bar{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_N]^\intercal \in \mathbb{R}_+^{\bar{n}}$, where each α_i represents the fractional-order exponent associated with the ith subnetwork, $\bar{x}[k] = [x_1[k]^\intercal, \dots, x_N[k]^\intercal]^\intercal \in \mathbb{R}^{\bar{n}}$ where $x_i[k]$ is a subvector of $x[k]$, $\bar{u}[k] = [u_1[k]^\intercal, \dots, u_N[k]^\intercal]^\intercal \in \mathbb{R}^{\bar{m}}$ where $u_i[k]$ is a subvector of $u[k]$, $\bar{w}[k] = w[k]$, $\bar{y}[k] = [y_1[k]^\intercal, \dots, y_N[k]^\intercal]^\intercal \in \mathbb{R}^{\bar{o}}$ where $y_i[k]$ is a subvector of $y[k]$, $\bar{e}[k] = [e_1[k]^\intercal, \dots, e_N[k]^\intercal]^\intercal \in \mathbb{R}^{\bar{o}}$ where $e_i[k]$ is a subvector of $e[k]$.

For the subnetworks, we define matrices $\bar{A}_j(\bar{\alpha}, \theta_A)$ that are structured as

$$\bar{A}_j(\alpha, \theta_A) = \begin{bmatrix} [A_j]_{11}(\alpha_1, \theta_{A,1}) & \cdots & [A_j]_{1N}(\alpha_1, \theta_{A,1}) \\ \vdots & \ddots & \vdots \\ [A_j]_{N1}(\alpha_N, \theta_{A,N}) & \cdots & [A_j]_{NN}(\alpha_N, \theta_{A,N}) \end{bmatrix} \tag{9}$$

where each element $[A_j]_{mn}$ is a function of the corresponding submatrices of $\bar{A}(\theta_A)$ and $\bar{D}(\bar{\alpha}, j)$. Specifically, for $j = 0$, $[A_j]_{mn} = A_{mn}(\theta_{A,m}) - D_{mn}(\alpha_m, 1)$, and for $j \geq 1$, $[A_j]_{mn} = -D_{mn}(\alpha_m, j+1)$, following the structure described in Eq. (4). Here, $D_{mn}(\alpha_m, j)$ represents the component of $\bar{D}(\alpha, j)$ that corresponds to the relationship between the mth and nth subnetworks.

Moreover, consider

$$\bar{B}(\theta_B) = \mathrm{diag}(B_{11}(\theta_{B,1}), \ldots, B_{NN}(\theta_{B,N})) \in \mathbb{R}^{\bar{n}\times\bar{m}}, \tag{10}$$

where $B_{ii}(\theta_{B,i})$ is the submatrix of $B(\theta_B)$ associated to $x_i[k]$ and $u_i[k]$, $\bar{G}(\theta_G) = [G_1^{\intercal}(\theta_{G,1}), \ldots, G_N^{\intercal}(\theta_{G,N})]^{\intercal} \in \mathbb{R}^{\bar{n}\times\bar{v}}$, where $G_i(\theta_{G,i})$ is the submatrix of $G(\theta_G)$ associated to $x_i[k]$ and $w[k]$, and $\bar{C} = \mathrm{diag}(C_{11}, \ldots, C_{NN}) \in \mathbb{R}^{\bar{o}\times\bar{n}}$, where C_{ii} is the submatrix of C associated to $x_i[k]$ and $y_i[k]$. We note that $\bar{n} \geq n$, $\bar{m} \geq m$, $\bar{v} = v$, and $\bar{o} \geq o$.

On the one hand, if the model $\bar{M}(\theta)$ is such that $x_i[k]$ and $x_j[k]$ do not share state variables for all $i, j \in \{1, 2, \ldots, N\}$, $\bar{x}[k] = x[k]$ such that $\bar{n} = n$, $\bar{u}[k] = u[k]$ such that $\bar{m} = m$, and $\bar{y}[k] = y[k]$ such that $\bar{o} = o$. Then, $\bar{M}(\theta)$ behaves identically to $M(\theta)$ given identical inputs, and $\bar{M}(\theta)$ exhibits the same properties of identifiability as $M(\theta)$.

On the other hand, assume that the model $\bar{M}(\theta)$ is such that $x_i[k]$ and $x_j[k]$ share state variables for all $i, j \in \{1, 2, \ldots, N\}$, $x[k]$ is a subvector of $\bar{x}[k]$ such that $\bar{n} > n$, $u[k]$ is a subvector of $\bar{u}[k]$ such that $\bar{m} > m$, and $y[k]$ is a subvector of $\bar{y}[k]$ such that $\bar{o} > o$. Under this assumption, we can prove that $M(\theta)$ and $\bar{M}(\theta)$ behave identically given identical inputs if and only if $\bar{M}(\theta)$ includes $M(\theta)$. We revisit the notion of model inclusion below (Siljak, 2011).

Definition 3 (*Inclusion*). $\bar{M}(\theta)$ includes $M(\theta)$ (denoted as $\bar{M}(\theta) \supset M(\theta)$) when there exist full rank matrices $\{V; R_U; R_W; S\}$ such that $\bar{x}[0] = Vx[0]$, $u[k] = R_U\bar{u}[k]$, and $w[k] = R_W\bar{w}[k]$ for $k \geq 0$, implies that $\bar{x}[k; \bar{x}[0], \bar{u}, \bar{w}] = Vx[k; x[0], u, w]$ and $\bar{y}[k] = Sy[k]$ for $k \geq 0$.

Therefore, assuming $\bar{M}(\theta) \supset M(\theta)$, we can also say that $\bar{M}(\theta)$ exhibits the same properties of identifiability as $M(\theta)$, and vice versa. We formalize this claim as follows.

Theorem 1. *Let $M(\theta)$ be a fractional-order network model as written in (1) where $\theta = [\alpha, \theta_A, \theta_B, \theta_G]$. Then, $M(\theta)$ is structurally (output) identifiable if and only if every subsystem of $M(\theta)$ is also structurally (output) identifiable.*

Proof. Let all the subsystems of $M(\theta)$ be contained in a single model $\bar{M}(\theta)$ as shown in van Doren et al. (2009) such that $x[k]$ is a subvector of $\bar{x}[k]$ where $\bar{n} \geq n$, $u[k]$ is a subvector of $\bar{u}[k]$ where $\bar{m} \geq m$, and $y[k]$ is a subvector of $\bar{y}[k]$ where $\bar{o} \geq o$. This implies that there exists a set of full rank matrices $\{V; R_U; R_W; S\}$ such that $\bar{x}[k] = Vx[k]$, $u[k] = R_U\bar{u}[k]$, $w[k] = R_W\bar{w}[k]$, and $\bar{y}[k] = Sy[k]$. Therefore, by Definition 3, $\bar{M}(\theta) \supset M(\theta)$.

Let the sensitivity of model outputs $y[k]$ and $\bar{y}[k]$ to parameter changes be $\partial Y = \left[\frac{\partial y[1]}{\partial\theta}^{\intercal} \ldots \frac{\partial y[k]}{\partial\theta}^{\intercal}\right]^{\intercal} \in \mathbb{R}^{ko\times q}$ and $\partial\bar{Y} = \left[\frac{\partial\bar{y}[1]}{\partial\theta}^{\intercal} \ldots \frac{\partial\bar{y}[k]}{\partial\theta}^{\intercal}\right]^{\intercal} \in \mathbb{R}^{k\bar{o}\times q}$ for $k \geq 1$.

Necessary condition: We first prove that if $M(\theta)$ is output identifiable, then $\bar{M}(\theta)$ is also output identifiable.

Assume $M(\theta)$ is output identifiable. Then, using Proposition 2, $\mathrm{rank}(I_{dd}) = \mathrm{rank}(\partial Y^{\intercal}\partial Y) = \mathrm{rank}(\partial Y) = q$. Since $\bar{M}(\theta) \supset M(\theta)$, there exists a matrix $S \in \mathbb{R}^{\bar{o}\times o}$ such that $\bar{y}[k] = Sy[k]$. This implies $\partial\bar{Y} = S_D\partial Y$, where $S_D = \mathrm{diag}(S, \ldots, S)$ with $\mathrm{rank}(S_D) = k\bar{o}$.

Since $\mathrm{rank}(\partial Y) = q$ and $k\bar{o} \geq q$, we have $\mathrm{rank}(\partial\bar{Y}) = \mathrm{rank}(S_D\partial Y) = q$. Therefore, $\bar{M}(\theta)$ is output identifiable.

Sufficient condition: We now prove that if $\bar{M}(\theta)$ is output identifiable, then $M(\theta)$ is also output identifiable.

Assume $\bar{M}(\theta)$ is output identifiable. Then, using Proposition 2, $\mathrm{rank}(\bar{I}_{dd}) = \mathrm{rank}(\partial\bar{Y}^{\intercal}\partial\bar{Y}) = \mathrm{rank}(\partial\bar{Y}) = q$. Since $\bar{M}(\theta) \supset M(\theta)$, there exists a matrix $\bar{S} \in \mathbb{R}^{o\times\bar{o}}$ such that $y[k] = \bar{S}\bar{y}[k]$. This implies $\partial Y = \bar{S}_D\partial\bar{Y}$, where $\bar{S}_D = \mathrm{diag}(\bar{S}, \ldots, \bar{S})$ with $\mathrm{rank}(\bar{S}_D) = ko$.

Since $\mathrm{rank}(\partial\bar{Y}) = q$ and $ko \geq q$, we have $\mathrm{rank}(\partial Y) = \mathrm{rank}(\bar{S}_D\partial\bar{Y}) = q$. Therefore, $M(\theta)$ is output identifiable.

3.3. *Subnetwork partitioning of $M(\theta)$*

We briefly review the notation and properties for system digraphs below. A directed graph or digraph is an ordered pair, $G = \{V, E\}$, where V is a set of vertices or nodes and $E \subseteq V \times V$ is a set of edges between nodes. $G_s = \{V_s, E_s\}$ is a subgraph of G if $V_s \subseteq V$ and $E_s \subseteq V_s \times V_s \subseteq E$. If $V_s = V$, then G_s spans G. Removing the subgraph G_s from G is denoted as $G \setminus G_s = \{V_h, E_h\}$ where $V_h = V \setminus V_s$ and $E_h \subseteq V_h \times V_h \subseteq E$.

Suppose V is partitioned into a set of subsets V_i for $i \in \{1, 2, \ldots, p\}$ and let $E_i \subseteq V_i \times V_i$. Then, the set of subgraphs $G_i = \{V_i, E_i\}$ for all i is a graph partition of G.

A vertex $v_1 \in V$ is adjacent to vertex $v_2 \in V$ on G if there exists an edge $(v_1, v_2) \in E$ or an edge $(v_2, v_1) \in E$. The open neighborhood of the vertex set $V_s \subset V$ on the G is $N(V_s) = \{v \text{ adjacent to } v_i : v \in V \setminus V_s, v_i \in V_s\}$. An elementary path between vertices v_1 and v_k is a sequence of edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$ for which all the vertices are distinct. The path distance is defined as the number of edges in an elementary path. G is a weakly connected digraph if there exists a path from v_i to v_j or there exists a path from v_j to v_i for every pair of vertices $v_i, v_j \in V$ for $i \neq j$. G is a strongly connected digraph if there exists a path from v_i to v_j and there exists a path from v_j to v_i for all $v_i, v_j \in V$ for $i \neq j$. We simply refer to weakly connected digraphs as “connected digraphs”.

Prior works use digraphs to represent dynamic systems in order to analyze their structural properties (Dion et al., 2003; Liu et al., 2011; Pequito et al., 2013). We use digraphs to represent fractional-order network models $M(\theta)$, and we define fractional-order network model digraphs, $G(M(\theta))$ as follows.

Definition 4 (*Fractional-Order Network Model Digraph*). Given $M(\theta)$ in (1), the fractional-order network model digraph is $G(M(\theta)) = \{X \cup U \cup W \cup Y, E\}$ such that

- $x_i \in X$ represents $[x[k]]_i$ for $i = 1, \ldots, n$;
- $u_i \in U$ represents $[u[k]]_i$ for $i = 1, \ldots, m$;
- $w_i \in W$ represents $[w[k]]_i$ for $i = 1, \ldots, v$;
- $y_i \in Y$ represents $[y[k]]_i$ for $i = 1, \ldots, o$;
- $E = E_{x,x} \cup E_{u,x} \cup E_{w,x} \cup E_{x,y}$ such that $E_{x,x} = \{(x_i, x_j) : [A(\theta_A)]_{ij} \neq 0\} \cup \{(x_i, x_i) : \alpha_i \neq 0\}$, $E_{u,x} = \{(u_j, x_i) : [B(\theta_B) \neq 0]_{ij}\}$, $E_{w,x} = \{(w_j, x_i) : [G(\theta_G)]_{ij} \neq 0\}$, and $E_{x,y} = \{(x_j, y_i) : [C]_{ij} \neq 0\}$.

Although controllability and observability alone are neither necessary nor sufficient conditions for identifiability, Jacquez demonstrates that identifiability is linked to weaker notions of input reachability and output reachability of the model (Jacquez & Greif, 1985), which are defined below.

Definition 5 (*Input Reachability*). $M(\theta)$ is input reachable from an input $u_i[k]$ (or $w_i[k]$) if there exists a path from $u_i \in U$ ($w_i \in W$) to every state vertex $x_i \in X$ for all i on $G(M(\theta)) = \{X \cup U \cup W \cup Y, E\}$.

Definition 6 (*Output Reachability*). $M(\theta)$ is output reachable to the output $y_i[k]$ if there exists a path from every state vertex $x_i \in X$ for all i to $y_i \in Y$ on $G(M(\theta)) = \{X \cup U \cup W \cup Y, E\}$.

Jacquez observes that a system cannot be structurally (or output) identifiable if the system is not input-reachable or output-reachable (Jacquez & Greif, 1985). Therefore, we define a higher level undirected graph $G'(M(\theta))$ to easily evaluate the input and output reachability of the overall fractional-order network model $M(\theta)$. We refer to this particular graph as a network map since it also maps the connections in the network.

Definition 7 (*Network Map*)**.** Let $G'(M(\theta)) = \{V', E'\}$ be the network map of $M(\theta)$ if and only if $V' = \{i : \forall x_i \in X\}$ and $E' = \{(i, j) : \exists(x_i \to x_j) \vee \exists(x_j \to x_i), \forall i, j \in V' \wedge i \neq j\}$ where $v_i \to v_j$ denotes a path from node v_i to v_j on $G(M(\theta))$.

Given $M(\theta)$ is input and output reachable, we can partition $M(\theta)$ into input and output reachable subsystems $M_i(\theta_i)$ by creating an input–output reachable partition of the network map $G'(M(\theta))$.

Definition 8 (*Input–Output Reachable Partition*)**.** Let $G'(M(\theta)) = \{V', E'\}$ be a network map of $M(\theta)$ and let $A = \{M_i^A(\theta) : \forall x_i \in X\}$ be the set of subnetworks. Then, the set of graphs $\{G_k'(M(\theta)) : \forall k\}$ is an input–output reachable partition of the network map $G'(M(\theta))$ if every graph $G_k'(M(\theta)) = \{V_k', E_k'\}$ satisfies the following:

- $G_k'(M(\theta))$ is a connected subgraph of $G'(M(\theta))$;
- $\exists i \in V_k$ such that $M_i^A(\theta) \in A$ is input reachable;
- $\exists j \in V_k$ such that $M_j^A(\theta) \in A$ is output reachable;
- The set $\{V_k' : \forall k\}$ is a partition of the set V';
- $E_k' \subseteq V_k' \times V_k' \subseteq E'$.

To identify the input–output reachable partitions of the network map, we introduce Algorithm 1, which implements a graph-theoretical hierarchical decomposition method. This algorithm identifies subsystems by iteratively decomposing the network based on input–output reachability.

Algorithm 1 Graph-Theoretical Hierarchical Decomposition of $G(M(\theta))$

```
Require: $G(M(\theta)) = \{X \cup U \cup W \cup Y, E\}$
Compute reachability relation for all vertices in $G(M(\theta))$
Construct Input–Output reachability matrix $H = (h_{ij})$ where $h_{ij} = 1$ if there exists a path from actuated input $u_i$, or disturbance input $w_j$, to output $y_k$
$k := 0$
$G_{rem} := G(M(\theta))$
while $G_{rem} \neq \emptyset$ do
    $Y_{nd} := \{y_i \in Y : y_i$ is non-dominated in $G_{rem}\}$
    for all $y_i \in Y_{nd}$ do
        $k := k + 1$
        $U_k := \{u_j \in U : u_j$ reaches $y_i$ in $G_{rem}\}$
        $X_k := \{x_j \in X : x_j$ is on a path from some $u \in U_k$ to $y_i$ in $G_{rem}\}$
        $Y_k := \{y_i\} \cup \{y_j \in Y : y_j$ is reached from some $x \in X_k$ in $G_{rem}\}$
        $E_k := \{(v_1, v_2) \in E : v_1, v_2 \in U_k \cup X_k \cup Y_k\}$
        $G_k := \{U_k \cup X_k \cup Y_k, E_k\}$
        Mark all vertices in $G_k$ as processed
    end for
    $G_{rem} := G_{rem} \setminus \{$all processed vertices and edges$\}$
end while
return Hierarchically ordered subsystems $G_1, G_2, \ldots, G_k$
```

Algorithm 1 identifies subsystems by first determining non-dominated outputs — those that are not influenced by other outputs but serve as terminal nodes for input signals. For each non-dominated output, the algorithm traces backward through the graph to identify all inputs that can reach it, and then determines all state variables that lie on paths connecting these inputs to the output.

This approach ensures that each identified subsystem is both input and output reachable. The algorithm proceeds iteratively, removing processed components from the graph until the entire network has been decomposed into hierarchically ordered subsystems.

The hierarchical nature of this decomposition is particularly valuable for fractional-order networks, as it captures the natural propagation of signals through the network while maintaining the identifiability properties established in Theorem 1.

The resulting subsystems can be individually identified, which is computationally more efficient than attempting to identify the entire network model at once. We note that these subnetworks are non-unique and may share state variables—a feature that provides flexibility when analyzing complex systems.

Remark 1. Multiple valid decompositions can exist for the same fractional-order network model, potentially with different hierarchical structures, allowing practitioners to select decompositions that align with the natural structure of the system under study. ○

Remark 2. The number of subnetworks resulting from a decomposition directly impacts the computational advantage of our approach. Networks that can be partitioned into multiple subnetworks represent favorable scenarios, as each subnetwork has reduced dimensionality compared to the full network, thereby significantly reducing the overall estimation problem complexity. Conversely, if the network structure only admits a single subnetwork (i.e., cannot be further decomposed), this represents the worst-case scenario where decomposition provides no computational advantage over identifying the entire network directly. ○

Despite the non-uniqueness, the algorithm properly accounts for interactions between subsystems, and as established by Theorem 1, the identifiability of the original network is preserved in its subnetworks, enabling efficient parameter estimation while maintaining mathematical guarantees. Subsequently, we introduce the following definition.

Definition 9 (*Subnetwork Model*)**.** Given a fractional-order network model $M(\theta)$, assume the following:

- $A = \{M_i^A(\theta) : \forall x_i \in X\}$ is the set of all subnetworks, where $G(M_i^A(\theta)) = \{X_i^A \cup U_i^A \cup W_i^A \cup Y_i^A, E_i^A\}$;
- $G'(M(\theta))$ is the network map of $M(\theta)$;
- $\{G_k'(M(\theta)) : \forall k\}$ is an input–output reachable partition of $G'(M(\theta))$ where $G_k'(M(\theta)) = \{V_k', E_k'\}$.

Then, $M_k(\theta_k)$ is a subnetwork model of $M(\theta)$ if $G(M_k(\theta_k)) = \{X^k \cup U^k \cup W^k \cup Y^k, E^k\}$ such that $X^k = \bigcup_{i \in V_k'} X_i^A$, $U^k = \bigcup_{i \in V_k'} U_i^A$, $W^k = \bigcup_{i \in V_k'} W_i^A \setminus X^k$, $Y^k = \bigcup_{i \in V_k'} Y_i^A$, and $E^k = \{(v_i, v_j) \in E : v_i, v_j \in X^k \cup U^k \cup Y^k\} \cup \{(w, x) \in E : x \in X^k, w \in W^k\}$.

4. Simulations results

This section presents a series of simulation experiments designed to validate the theoretical findings established in the previous sections. Specifically, the simulations aim to demonstrate that the proposed input–output decomposition strategy preserves the structural identifiability properties of fractional-order network models while significantly reducing computational complexity.

The section is structured into four parts. Section 6.1 describes the experimental setup of a representative fractional-order network model used as a testbed. Section 6.2 details the graph-theoretical decomposition of the network into input–output reachable subsystems using the hierarchical algorithm introduced in Section 5. Section 6.3 focuses on the parameter estimation process, both for the original system and for each decomposed subsystem individually. Finally, Section 6.4 provides quantitative and qualitative validation results, confirming that the reconstructed model accurately replicates the dynamics of the original system.

Together, these simulation results provide strong empirical support for the theoretical claims of the paper, illustrating the effectiveness and scalability of the proposed identifiability framework for large-scale fractional-order networks.

4.1. Experimental system setup

We considered a 6 × 6 fractional-order network with the following parameters:

- State matrix A with sparse connections representing system dynamics:

$$A = \begin{bmatrix} -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.2 & -0.6 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.15 & 0.0 & -0.4 & 0.1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & -0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.7 & 0.12 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.15 & -0.6 \end{bmatrix}$$

- actuated input matrix B with $B_{11} = 1.0$ (input u_1 affecting state x_1) and $B_{32} = 1.0$ (input u_2 affecting state x_3);
- disturbance input matrix G with $G_{41} = G_{51} = G_{61} = 0.5$ (input w_1 affecting states x_4, x_5, and x_6);
- Fractional exponents $\alpha = [0.3, 0.5, 0.4, 0.6, 0.4, 0.5]$;
- Output matrix C with $C_{12} = 1.0$ (output y_1 measuring state x_2), $C_{24} = 1.0$ (output y_2 measuring state x_4), and $C_{35} = 1.0$ (output y_3 measuring state x_5).

In this example, we designed the connectivity to include unidirectional connections (e.g., $x_1 \rightarrow x_2$), bidirectional couplings (e.g., $x_3 \leftrightarrow x_4$ and $x_5 \leftrightarrow x_6$), and connections spanning multiple subsystems to examine Algorithm 1's ability to handle complex network structures and properly identify subsystem boundaries.

4.2. Graph decomposition

We applied our hierarchical decomposition algorithm (Algorithm 1) to partition the original network into subsystems based on input–output reachability. The decomposition process identified three meaningful subsystems with dimensions 2, 3, and 2 (corresponding to the number of states in each subsystem), which preserved the structural properties of the original network.

Fig. 1 illustrates the complete graph structure of our test system, while Fig. 2 shows its decomposition guided by the theoretical principles established in Section 5. The decomposition algorithm processes outputs in order (y_1, y_2, y_3) and performs backward reachability analysis to identify all states and inputs that influence each output, ensuring that each subsystem remains both input-reachable and output-reachable.

4.3. System identification

For both the entire network model $M(\theta)$ and the decomposed subnetworks $M_k(\theta_k)$, we employed a two-stage identification approach based on a gradient-based method with structural constraints described in Varalda and Pequito (2025); it is noteworthy that alternative methods, such as those in Gupta et al. (2018, 2019), could also be applied effectively for system identification, showcasing the flexibility of our approach.

To evaluate the robustness of our identification method, we incorporated realistic noise levels in our simulations: additive Gaussian noise with standard deviation $\sigma_w = 0.10$ for the disturbance inputs and $\sigma_e = 0.05$ for the output measurements.

Specifically, the identification process consisted of two sequential stages:

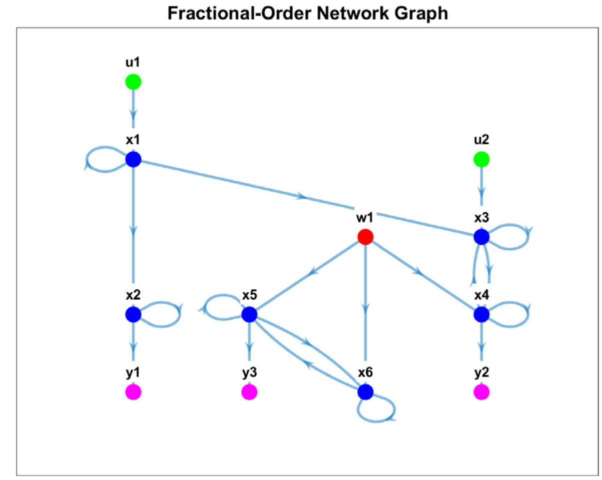

Fig. 1. Graph representation of the complete 6-state fractional-order network. The diagram uses circles for all nodes, with different colors indicating states (blue), actuated inputs (green), disturbance inputs (red), and outputs (magenta). The arrows show the interconnections between nodes. This network will be decomposed into three subsystems based on output reachability.

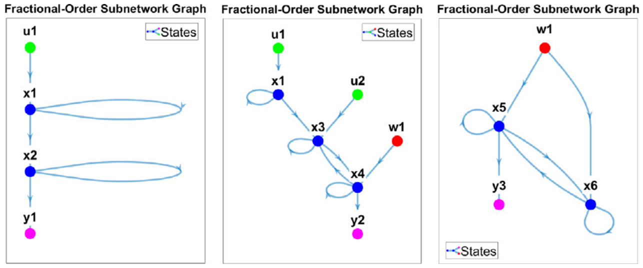

Fig. 2. Decomposition of the fractional-order network into three subsystems through backward reachability analysis. (Left) Subsystem 1 (dimension 2): output y_1 traces back through states x_2, x_1 to input u_1. (Center) Subsystem 2 (dimension 3): output y_2 reaches state x_4 via state x_3 and disturbance input w_1; x_3 connects to input u_2 and state x_1 from u_1. (Right) Subsystem 3 (dimension 2): output y_3 reaches state x_5, bidirectionally connected to x_6, both influenced by w_1.

- **Zero-input identification:** First, we simulated the system with zero input to identify the state matrix $A(\theta_A)$ and fractional-order exponents α using a gradient-based approach with structure masks that enforced the known network topology.
- **Input-driven identification:** Subsequently, we applied a step input to identify the input matrices $B(\theta_B)$ and $G(\theta_G)$ using a least-square approach with appropriate structure constraints derived from the network's graph representation.

While this approach provided the diagonal blocks $[A_j]_{kk}(\alpha_k, \theta_{A,k})$ for each subsystem, the off-diagonal blocks representing inter-subsystem connections remained to be identified. To complete the identification of $\bar{M}(\theta)$, we used an iterative approach:

(1) First, we combined the identified parameters from individual subsystems to form $\bar{\alpha}$, block-diagonal matrices $\bar{B}$ and $\bar{G}$, and the corresponding output matrix $\bar{C}$.
(2) For the state matrix $\bar{A}$, we iteratively identified the off-diagonal blocks by:
    - Starting with two subsystems and identifying their interconnection blocks using the same identification approach presented above,
    - progressively incorporating additional subsystems one at a time,

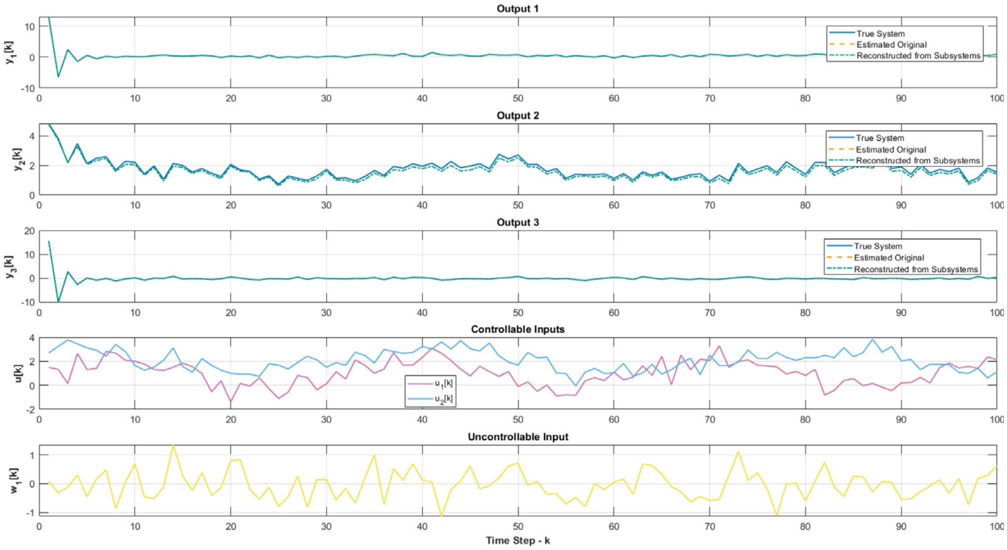

Fig. 3. Comparison of output responses from the true system (blue solid line), the directly identified system (red dashed line), and the reconstructed system from subnetwork estimates (green dash-dotted line). The bottom panel shows the two sinusoidal input signals and the white noise disturbance input used for validation, $(w_{\text{noise}}, e_{\text{noise}}) = (0.5, 0.25)$. The strong alignment among all responses demonstrates the accuracy and robustness of the proposed decomposition-based identification method under noisy conditions.

- for each new subsystem, identifying only the off-diagonal elements connecting it to the previously integrated subsystems.

This hierarchical identification process enabled us to efficiently determine all parameters of the complete network model while leveraging the computational advantages of the subsystem decomposition.

Remark on Computational Efficiency: This decomposition strategy guarantees reduced computational complexity. For a network with n states, direct least-squares estimation of the full system requires $\mathcal{O}(n^3)$ operations (Golub & Van Loan, 2013). In contrast, when decomposed into K subsystems with dimensions $n_1, \ldots, n_K$ where $\sum_{i=1}^{K} n_i = \bar{n} \geq n$ (accounting for shared states), the total computational cost becomes $\sum_{i=1}^{K} \mathcal{O}(n_i^3)$, which is significantly smaller since $\sum_{i=1}^{K} n_i^3 \ll n^3$ for modular networks. Although Fig. 2 shows that subnetworks may share state variables (e.g., state x_1 appears in both Subnetwork 1 and Subnetwork 2), we prevent redundant computations by fixing previously estimated parameters in subsequent subnetworks, ensuring the overall estimation time remains bounded by—and typically much smaller than—the cost of identifying the entire network directly. Moreover, reusing previously estimated parameters provides better initialization for the gradient descent-based identification method, accelerating convergence in subsequent subnetworks.

4.4. Validation results

To validate our approach, we compared the output response of three systems to a new input sequence:

(1) The true original system with known parameters,
(2) The directly identified original system,
(3) The reconstructed system from subnetwork estimates.

The validation was performed using a 100-step simulation with two sinusoidal test input signals defined as

$$u_{\text{test},1}(t) = 1 + \sin(0.2t) + w_{\text{noise}}(t),$$
$$u_{\text{test},2}(t) = 2 + \cos(0.15t) + w_{\text{noise}}(t),$$

where $w_{\text{noise}}(t) \sim \mathcal{N}(0, \sigma_w^2)$ represents Gaussian process noise. disturbance inputs were modeled as white noise, and measurement noise $e_{\text{noise}}(t)$ was also added with controlled variance (see Fig. 3).

Quantitative validation was performed by computing the mean squared error (MSE) and Pearson's correlation coefficients between the true system outputs and those of the identified systems under three noise levels:

$$(w_{\text{noise}}, e_{\text{noise}}) = (0.1, 0.05),\ (0.2, 0.1),\ (0.5, 0.25).$$

Across all noise configurations, both the directly identified and reconstructed systems exhibited nearly identical performance, with MSE values remaining below 10^{-2} and correlation coefficients consistently above 0.996. These results confirm that the reconstructed system faithfully reproduces the dynamics of the original network, even in the presence of significant process and measurement noise.

To verify the robustness and generalizability of our approach, we repeated the experiment on 50 randomly generated networks with larger dimensions: $n = 10$ states, $m = 3$ actuated inputs, $v = 2$ disturbance inputs, and $o = 3$ outputs. Each network was generated with random sparse connectivity patterns and subjected to medium noise levels $(w_{\text{noise}}, e_{\text{noise}}) = (0.2, 0.1)$.

Across all 50 trials, both the directly identified and reconstructed systems exhibited nearly identical performance, with MSE values remaining below 10^{-2} and correlation coefficients consistently above 0.996. Furthermore, the decomposition-based approach demonstrated significant computational efficiency gains: direct identification of the full network required 7.74 ± 1.85 seconds on average, while the decomposition–reconstruction strategy reduced this to 0.081 ± 0.016 seconds.

These results confirm that the reconstructed systems faithfully reproduce the dynamics of the original networks, demonstrating the consistency and scalability of our decomposition-based identification framework. The MATLAB implementation of our method is publicly available at https://github.com/alessandrovaralda9-web/fractional-order-networks.git.

5. Conclusion and future research

This paper established theoretical conditions for structural identifiability in fractional-order discrete-time networks through input–output decompositions based on network topology. This strategy partitions identification problems into simpler subproblems: first identifying subsystem parameters, then interconnections. Simulation experiments validated that this hierarchical decomposition preserves identifiability while achieving significant computational speedup, confirming accurate reconstruction from independently estimated subsystems.

Our framework assumes distinct parameter sets θ_A, θ_B, and θ_G for system matrices. While this facilitates decomposition with clear theoretical guarantees, extending to systems with parameter coupling remains important future work. Additionally, optimizing input–output placement to yield smaller, independently identifiable subnetworks can further enhance efficiency for practical applications across scientific and engineering disciplines.

Important directions for future research include: (i) extending our framework beyond Assumptions 1 and 2 to handle more general input/output configurations where inputs can influence multiple states or outputs measure combinations of states, (ii) developing optimization frameworks to systematically determine network decompositions that maximize the number of subnetworks while minimizing inter-subsystem coupling, and (iii) investigating the uniqueness of state-space representations for fractional-order networks, particularly whether memory effects impose stricter constraints on admissible similarity transformations compared to classical LTI systems.

Such extensions would broaden the applicability of our identifiability theory while further enhancing the computational scalability of our approach for large-scale fractional-order networks.

CRediT authorship contribution statement

Alessandro Varalda: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Sérgio Pequito:** Writing – review & editing, Writing – original draft, Validation, Supervision, Resources, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

I shared the code and data through a link in the manuscript itself.

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