



Convergence of trees with a given degree sequence and of their associated laminations

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ABSTRACT

In this paper, we study uniform rooted plane trees with given degree sequence. We show that, under some natural hypotheses on the degree sequences, these trees converge towards the so-called Inhomogeneous Continuum Random Tree after renormalization. Our proof relies on the convergence of a modification of the well-known Łukasiewicz path. We also give a unified treatment of the limit, as the number of vertices tends to infinity, of the fragmentation process derived by cutting down the edges of a tree with a given degree sequence, including its geometric representation by a lamination-valued process. The latter is a collection of nested laminations, which are compact subsets of the unit disk made of non-crossing chords. In particular, we prove an equivalence between planar Gromov-weak convergence of discrete trees and the convergence of their associated lamination-valued processes.

1. Introduction

In his seminal papers [1–3], Aldous introduced the so-called Brownian Continuum Random Tree (Brownian CRT) as the limit — after renormalization of distances — of a uniform tree with n vertices, and more generally, of critical size-conditioned Galton–Watson trees with finite offspring variance. The Brownian CRT has appeared since then as the limit of various random tree-like structures such as multi-type Galton–Watson trees [4] or unordered binary trees [5]. Over the last decade, the study of scaling limits of large discrete random trees towards a random continuum tree has seen numerous developments and found extensive applications in the study of other random structures. These include, among others, random planar maps [6], random dissections of regular polygons [7,8], fragmentation and coalescent processes [9] or Erdős–Rényi random graphs in the critical window [10]. In this work, we investigate the scaling limit of trees with a given degree sequence, along with their associated laminations (sets of chords in the unit disk).

For a finite rooted plane tree \mathbf{t} , let $V(\mathbf{t})$ denote its number of vertices. If $V(\mathbf{t}) \geq 1$, then for $i \geq 0$, let $N_i(\mathbf{t})$ be the number of vertices in \mathbf{t} having i children (or out-degree i). The sequence $(N_i(\mathbf{t}), i \geq 0)$ is called the degree sequence of \mathbf{t} , and satisfies $V(\mathbf{t}) = \sum_{i \geq 0} N_i(\mathbf{t}) = 1 + \sum_{i \geq 0} iN_i(\mathbf{t})$. Moreover, for $n \in \mathbb{N}$, a sequence $\mathbf{s}_n = (N_i^n, i \geq 0)$ of non-negative integers is the degree sequence of some finite rooted plane tree if and only if $\sum_{i \geq 0} N_i^n = 1 + \sum_{i \geq 0} iN_i^n < \infty$. Then, a random tree with given degree sequence (TGDS) \mathbf{s}_n is a random variable whose law is uniform on the set $\mathbb{T}_{\mathbf{s}_n}$ of rooted plane trees with $V_n := \sum_{i \geq 0} N_i^n$ vertices amongst which N_i^n have i offspring for every $i \geq 0$, and $E_n := \sum_{i \geq 0} iN_i^n$ edges.

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1.1. Scaling limits of trees

Scaling limits for trees with given degree sequences were first studied by Broutin & Marckert [11]. Let s_n be a degree sequence and \mathbf{t}_n be a random tree sampled uniformly at random in \mathbb{T}_{s_n} . We see \mathbf{t}_n as a rooted metric measure space $(\mathbf{t}_n, r_n^{\text{gr}}, \rho_n, \mu_n)$, i.e. \mathbf{t}_n is identified with its set of V_n vertices, r_n^{gr} is the graph-distance on \mathbf{t}_n (that is, all edges have length 1), ρ_n is its root, and μ_n is the uniform measure on the set of vertices of \mathbf{t}_n . Consider the global variance term $\sigma_n^2 = \sum_{i \geq 1} i(i-1)N_i^n$ for the degree sequence s_n (with $\sigma_n \geq 0$), and the maximum degree $\Delta_n = \max\{i \geq 0 : N_i^n > 0\}$ of any tree with degree sequence s_n . Under technical assumptions on s_n , in particular $\sigma_n^2 \sim \sigma^2 V_n$ as $n \rightarrow \infty$ for some $\sigma \in (0, \infty)$ and $\lim_{n \rightarrow \infty} \sigma_n^{-1} \Delta_n = 0$, Broutin & Marckert [11] showed the convergence in distribution,

$$\left(\mathbf{t}_n, \frac{\sigma_n}{V_n} r_n^{\text{gr}}, \rho_n, \mu_n \right) \xrightarrow{d} (\mathcal{T}_{\text{Br}}, r_{\text{Br}}, \rho_{\text{Br}}, \mu_{\text{Br}}), \text{ as } n \rightarrow \infty,$$

for the so-called Gromov–Hausdorff–Prohorov topology, where $(\mathcal{T}_{\text{Br}}, r_{\text{Br}}, \rho_{\text{Br}}, \mu_{\text{Br}})$ is the Brownian CRT. In particular, μ_{Br} is a probability measure supported on the leaves of \mathcal{T}_{Br} .

Marzouk [12] proved a weaker convergence (in the sense of subtrees spanned by finitely many random vertices) by requiring only the condition $\lim_{n \rightarrow \infty} \sigma_n^{-1} \Delta_n = 0$. To be precise, fix $q \geq 1$ and let u_1, \dots, u_q be q i.i.d. uniform random vertices of \mathbf{t}_n . The reduced tree $\mathbf{t}_n^{(q)}$ is obtained by keeping only the root of \mathbf{t}_n , these q vertices, the branching points (if any), and then connecting by a single edge two of these vertices if one is the ancestor of the other in \mathbf{t}_n and there is no other vertex of $\mathbf{t}_n^{(q)}$ inbetween. We define the length of an edge e in $\mathbf{t}_n^{(q)}$ as the number of edges in \mathbf{t}_n between the endpoints of e . In particular, the combinatorial structure of $\mathbf{t}_n^{(q)}$ is that of a rooted plane tree with at most q leaves, so there are only finitely many possibilities, and thus there are a bounded number of edge lengths to record. Following Aldous [3], if $\mathbf{t}_n^{(q)}$ has k vertices (and thus $k-1$ edges), one can formally regard $\mathbf{t}_n^{(q)}$ as a rooted plane tree with edge lengths, that is, a vector $(\hat{\mathbf{t}}, \ell_1, \dots, \ell_{k-1}) \in \mathbb{T}_k \times \mathbb{R}_+^{k-1}$, where \mathbb{T}_k is the set of rooted plane trees with k vertices, $\hat{\mathbf{t}}$ is the tree $\mathbf{t}_n^{(q)}$ without edge lengths and the ℓ_i 's are the edge lengths. The space of trees with edge lengths is thus endowed with the natural product topology (i.e., \mathbb{T}_k is equipped with the discrete topology and \mathbb{R}_+^{k-1} with the usual one). For x_1, \dots, x_q i.i.d. random points of the Brownian CRT \mathcal{T}_{Br} sampled from its mass measure μ_{Br} , one can construct similarly a discrete tree with edge lengths $\mathcal{T}_{\text{Br}}^{(q)}$; see [3]. If $\lim_{n \rightarrow \infty} \sigma_n^{-1} \Delta_n = 0$, Marzouk [12] proved that, for every $q \geq 1$, one has the convergence in distribution,

$$\frac{\sigma_n}{V_n} \mathbf{t}_n^{(q)} \xrightarrow{d} \mathcal{T}_{\text{Br}}^{(q)}, \text{ as } n \rightarrow \infty.$$

In this work, we go one step further and, under the existence of at most countably many large degree vertices (see (A.2) and (A.4)), we prove weak convergence of \mathbf{t}_n towards the associated Inhomogeneous continuum random tree (Inhomogeneous CRT, which may be different from the Brownian CRT). The Inhomogeneous CRT, introduced in [13,14], arises as the scaling limit of another model of random trees called \mathbf{p} -trees (or birthday trees). The simplest description of the Inhomogeneous CRT is via a line-breaking construction based on a Poisson point process in the plane which can be found in [14,15]. The spanning subtree description is set out in [13], and its description via an exploration process is given in [16]. An Inhomogeneous CRT $(\mathcal{T}_\theta, r_\theta, \rho_\theta, \mu_\theta)$ is uniquely defined by a parameter set $\theta := (\theta_0, \theta_1, \dots)$ such that

$$\theta_1 \geq \theta_2 \geq \dots \geq 0, \quad \theta_0 \geq 0, \quad \sum_{i \geq 0} \theta_i^2 = 1 \quad \text{and either } \theta_0 > 0 \quad \text{or} \quad \sum_{i \geq 1} \theta_i = \infty;$$

see Fig. 1, left for a simulation of an Inhomogeneous CRT. In the special case $\theta = (1, 0, 0, \dots)$, \mathcal{T}_θ is precisely the Brownian CRT. For $q \geq 1$, a discrete tree with edge lengths, $\mathcal{T}_\theta^{(q)}$, can be constructed from q i.i.d. random points of \mathcal{T}_θ sampled from the mass measure μ_θ . Its distribution is described in [13–15].

Theorem 1. For $n \in \mathbb{N}$, let $s_n = (N_i^n, i \geq 0)$ be a degree sequence. Let $(d^n(i), 1 \leq i \leq V_n)$ denote the associated child sequence, obtained by writing N_0^n zeros, N_1^n ones, etc., and ordering the resulting sequence decreasingly. Assume that there exists a sequence $(b_n, n \geq 1)$ with $b_n \rightarrow \infty$ such that, as $n \rightarrow \infty$:

- (A.1) **Size.** $V_n \rightarrow \infty$;
- (A.2) **Hubs.** For all $i \geq 1$, the sequence $(d^n(i)/b_n, n \geq 1)$ converges to a limit $\theta_i \geq 0$;
- (A.3) **Degree variance.** $\frac{1}{b_n^2} \sum_{i \geq 0} (i-1)^2 N_i^n \rightarrow 1$;
- (A.4) $\theta_0 = \sqrt{1 - \sum_{i \geq 1} \theta_i^2} > 0$ and $\sum_{i \geq 1} \theta_i < \infty$.

Then, for all $q \geq 1$, we have that

$$\frac{b_n}{V_n} \mathbf{t}_n^{(q)} \xrightarrow{d} \mathcal{T}_\theta^{(q)}, \text{ as } n \rightarrow \infty,$$

where \mathcal{T}_θ is an Inhomogeneous CRT with parameter set $\theta = (\theta_0, \theta_1, \dots)$ and the convergence is in distribution within the space of trees with edge lengths, equipped with the product topology.

By Fatou’s lemma and (A.2)–(A.3), we have that $\sum_{i \geq 1} \theta_i^2 \leq 1$ and thus, θ_0 in (A.4) is well-defined. Observe also that the maximum degree Δ_n is equal to $d^n(1)$ and thus, (A.2) implies that $\Delta_n/b_n \rightarrow \theta_1$ as $n \rightarrow \infty$. In particular, if $\theta_1 = 0$ in (A.2), the hypotheses made in Theorem 1 correspond to the setting studied by Marzouk [12]. Indeed, by (A.3), the global variance $\sigma_n^2 = \sum_{i \geq 1} i(i-1)N_i^n$ of the

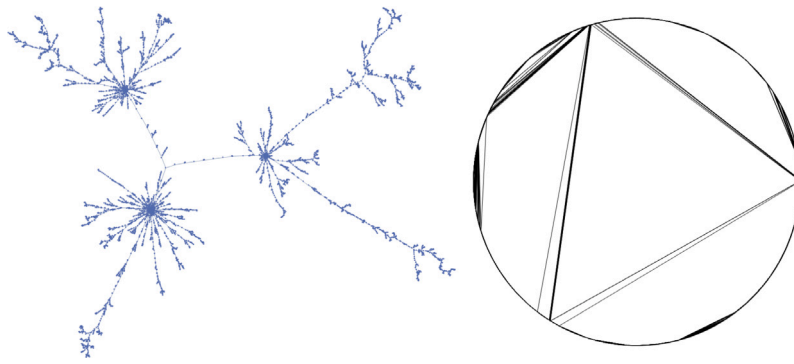


Fig. 1. Simulation of an Inhomogeneous CRT \mathcal{T}_θ and its associated lamination $\mathbb{L}(\mathcal{T}_\theta)$, for $\theta = (1/\sqrt{7}, 2/\sqrt{7}, 1/\sqrt{7}, 1/\sqrt{7}, 0, \dots)$.

degree sequence s_n satisfies that $\sigma_n^2/b_n^2 \rightarrow 1$, as $n \rightarrow \infty$. On the other hand, Theorem 1 together with [17, Theorem 5] implies the convergence in distribution

$$\left(\mathbf{t}_n, \frac{b_n}{V_n} r_n^{\text{gr}}, \rho_n, \mu_n \right) \xrightarrow{d} (\mathcal{T}_\theta, r_\theta, \rho_\theta, \mu_\theta), \quad \text{as } n \rightarrow \infty, \tag{1}$$

for the so-called Gromov-weak topology (often cited as Gromov–Prohorov topology); see e.g. Section 4.4 for background.

In the regime of possibly countably many large degree vertices (A.2) and (A.4), Theorem 1 characterizes the possible scaling limits of this model of random trees. It is then natural to wonder whether (1) can be reinforced to hold for the stronger Gromov–Hausdorff–Prohorov topology under the assumptions (A.1)–(A.4). To achieve this, one would need to prove the tightness of the sequence of discrete trees (see e.g. [3, Equation 25]), which requires precise estimates of the height of \mathbf{t}_n . However, as pointed out in [12, Section 1.2], there are cases where the maximal height of the tree can be much larger than V_n/b_n and thus no general tightness result as in [11] holds.

Recently, Blanc-Renaudie [18, Theorem 5 (a)–(b)] independently proved a general result (including the case $\sum_{i \geq 1} \theta_i = \infty$). Indeed, [18, Theorem 6 (b)] shows that, under [18, Assumption 2 (b)] (this assumption is similar to our conditions (A.1)–(A.3)), \mathbf{t}_n converges, after normalization by V_n/σ_n , towards the Inhomogeneous CRT, for the Gromov-weak topology; see also the remarks after [18, Theorem 6 (a)–(b)]. Nevertheless, our methods are completely different. In contrast to Blanc-Renaudie’s recursive construction for TGDSs based on a modified Aldous–Broder algorithm, this work adopts a classical approach centered around the Łukasiewicz path and its associated height process. The height process is a significant object of study in its own right and its analysis is central to the present work.

On the other hand, Blanc-Renaudie [18, Theorem 7] implies, under additional tightness conditions ([18, Assumption 7]), the convergence of \mathbf{t}_n for the Gromov–Hausdorff–Prohorov topology. To be precise, Blanc-Renaudie provides under those conditions an upper bound for the height of \mathbf{t}_n , and uses similar estimates to control the Gromov-Hausdorff distance between $\mathbf{t}_n^{(q_n)}$ and \mathbf{t}_n , for a well-chosen sequence $(q_n, n \geq 1)$. The height estimate in [18] is the ingredient we are missing to prove the tightness of the sequence of discrete trees (i.e., [3, equation (25)]), and thus, under some additional technical conditions, the convergence for the Gromov–Hausdorff–Prohorov topology.

A natural question would be to deduce an analogous result to Theorem 1 in the case $\sum_{i \geq 1} \theta_i = \infty$ by studying the asymptotic behavior the height process of the discrete trees. This broader setting, however, introduces additional technical complications. For instance, defining the height process of the Inhomogeneous CRT in this context remains an open problem.

Finally, we expect that similar results also hold for forests with given degree sequence. This has only been investigated under the assumption of no large degree vertices by Lei [19] and Marzouk [12]. In particular, they view the forest as a single tree by attaching all the roots to an extra root vertex. In this framework, the limit is a different continuum tree that is encoded by a certain Brownian first-passage bridge.

1.2. Fragmentations and laminations

Aldous, Evans and Pitman [9,20,21] initiated the study of fragmentation processes derived by deleting the edges of (random) trees one by one, uniformly at random. As time passes, the deletion of edges creates more and more connected components whose ordered sequence of sizes is called the fragmentation process of the tree. Aldous, Evans and Pitman studied the case of a uniform random tree with n labeled vertices and showed that the associated fragmentation process, suitably rescaled, converges to the fragmentation process of the Brownian CRT as $n \rightarrow \infty$; see also [22,23]. This latter is connected to the standard additive coalescent via a deterministic time-change and it is constructed by cutting down the skeleton of the Brownian CRT in a Poisson manner. Aldous and Pitman [15] (see also [20]) established a similar result in the broader context of \mathbf{p} -trees. They showed that this fragmentation process converges after rescaling to the fragmentation process of the Inhomogeneous CRT. Recently, the first authors [24] studied the case of critical Galton–Watson trees conditioned on having n vertices, whose offspring distribution μ belongs to the domain of

attraction of a stable law of index $\alpha \in (1, 2]$ (α -stable Galton–Watson trees). In this case, the limit is the fragmentation process of the so-called α -stable Lévy tree constructed by cutting down its skeleton in a Poisson manner.

It turns out that the fragmentation process of a tree can be coded by a non-decreasing process of subsets of the unit disk called laminations. A lamination is a closed subset of the closed unit disk $\bar{\mathbb{D}}$ made of the union of the unit circle \mathbb{S}^1 and a set of chords that do not intersect in the open unit disk \mathbb{D} . A face of a lamination L is a connected component of the complement of L in $\bar{\mathbb{D}}$. Laminations appear for instance in topology and hyperbolic geometry, see [25] and references therein. We denote by $\mathbb{L}(\bar{\mathbb{D}})$ the set of laminations of $\bar{\mathbb{D}}$ and equip it with the usual Hausdorff topology on the compact subsets of $\bar{\mathbb{D}}$.

The idea of coding (random) trees by (random) laminations of $\bar{\mathbb{D}}$ goes back to Aldous [7,8] in his study of a uniform triangulation of a large polygon. Since then, laminations have appeared in different contexts, as limits of discrete structures [26–28] or in the theory of random maps [29]. Roughly speaking, each chord of the lamination corresponds to an edge of the tree. Then, by adding chords one by one in the order in which the corresponding edges are removed, we code the fragmentation of the tree by a random process taking its values in $\mathbb{L}(\bar{\mathbb{D}})$ (see Section 4 for a rigorous definition of this process). Furthermore, at any given time in the process, there is a one-to-one correspondence between faces of the lamination and connected components of the fragmented tree. In the case of α -stable Galton–Watson trees, the third author proves in [28] the convergence of this lamination-valued process, towards a limiting process that can be constructed directly from the corresponding α -stable Lévy tree and encodes its fragmentation process.

A natural way of extending the previous investigations is to study the asymptotic behavior of the fragmentation process and the lamination-valued process derived by cutting-down a rooted plane tree, and in particular a tree with given degree sequence. This is the second goal of this paper. To present the main result of this section, we need some notation and background that some readers may not be familiar with. We refer to Section 4 for proper definitions. For τ a finite rooted plane tree, let $(\mathbb{L}_t(\tau), t \geq 0)$ be the lamination-valued process associated to the fragmentation of τ (that is, for all $t \geq 0$, $\mathbb{L}_t(\tau)$ is obtained by removing the first $\lfloor t \rfloor \wedge (\zeta(\tau) - 1)$ edges from τ , where $\zeta(\tau)$ denotes the number of vertices of τ , and drawing the corresponding chords in the disk); see Definition 1. For $I \subseteq \mathbb{R}_+$ a closed interval, let $\mathbf{D}(I, \mathbb{M})$ be the space of càdlàg functions (that is, right-continuous with left limits) from I to a metric space \mathbb{M} . We equip $\mathbf{D}(I, \mathbb{M})$ with the J_1 Skorohod topology; see e.g. [30, Section 5 in Chapter 3] or [31, Chapter 3] for details on this space. We denote by $\mathcal{T} = (\mathcal{T}, r, \rho, \mu)$ a plane continuum tree, that is, (\mathcal{T}, r) is a metric space, ρ is a distinguished element of \mathcal{T} called the root, μ is a probability measure on the set of leaves of \mathcal{T} , and \mathcal{T} is endowed with a compatible total order; see Definition 5. As in the case of finite trees, it is possible to define from \mathcal{T} a lamination-valued process $(\mathbb{L}_t(\mathcal{T}), t \geq 0)$ obtained by cutting \mathcal{T} in a Poissonian way and associating to each cutpoint a chord in the disk; see Section 4.3.1.

For $n \geq 1$, let τ_n be a (possibly random) rooted plane tree. We view it as a rooted metric measure space $(\tau_n, r_n^{\text{gr}}, \theta_n, \mu_n)$, i.e., τ_n is identified as its set of vertices, r_n^{gr} is the graph distance on τ_n , θ_n is the root of τ_n and μ_n is the uniform measure on the set of vertices of τ_n . In this work, whenever we consider a random rooted plane tree τ , we always assume that the number of its vertices, $\zeta(\tau)$, is deterministic. Similarly, for a sequence of such trees $(\tau_n, n \geq 1)$, we also assume that $\zeta(\tau_n)$ is deterministic for all $n \geq 1$.

The following result, which in particular can be applied to trees with given degree sequence, states the equivalence of (a planar version of) the Gromov-weak convergence of a sequence of plane trees (see Definition 9) and the convergence of its associated lamination-valued processes. Planar Gromov-weak convergence is weaker than the convergence used in Theorem 1 (i.e., of trees with edge lengths) because, in the former, branching points may merge at the limit.

Theorem 2. *Let $(\tau_n, n \geq 1)$ be a sequence of random rooted plane trees, \mathcal{T} be a random plane continuum tree and $(a_n, n \geq 1)$ be a sequence of non-negative real numbers satisfying $a_n \rightarrow \infty$ and $\zeta(\tau_n)/a_n \rightarrow \infty$, as $n \rightarrow \infty$. Then, the following assertions are equivalent:*

- (C.1) $(\mathbb{L}_{t/a_n}(\tau_n), t \geq 0) \xrightarrow{d} (\mathbb{L}_t(\mathcal{T}), t \geq 0)$, as $n \rightarrow \infty$, in $\mathbf{D}(\mathbb{R}_+, \mathbb{L}(\bar{\mathbb{D}}))$.
- (C.2) $\left(\tau_n, \frac{a_n}{\zeta(\tau_n)} r_n^{\text{gr}}, \theta_n, \mu_n \right) \rightarrow (\mathcal{T}, r, \rho, \mu)$, as $n \rightarrow \infty$, in the planar Gromov-weak sense.

To prove Theorem 2, we develop a general approach based on the notion of reduced laminations that may be of independent interest. These reduced laminations are constructed considering reduced trees obtained by sampling only a finite number of vertices in the tree. In particular, the more vertices are sampled, the closer one is to the lamination-valued process associated with the entire tree.

Theorem 2 and Theorem 5 (in Section 3.2) together imply the following result regarding the lamination-valued process associated with the fragmentation of a tree with a given degree sequence \mathbf{t}_n . Specifically, Theorem 5 establishes the convergence of $(\mathbf{t}_n, (b_n/V_n)r_n^{\text{gr}}, \rho_n, \mu_n)$ towards the Inhomogeneous CRT \mathcal{T}_θ in the planar Gromov-weak sense (Definition 9). In contrast to Theorem 1, Theorem 5 takes the order into account.

Corollary 1. *Suppose that \mathbf{s}_n satisfies (A.1)–(A.4). Let \mathcal{T}_θ be an Inhomogeneous CRT with parameter set $\theta = (\theta_0, \theta_1, \dots)$. Then, jointly with the convergence of Theorem 1, we have that*

$$(\mathbb{L}_{t/b_n}(\mathbf{t}_n), t \geq 0) \xrightarrow{d} (\mathbb{L}_t(\mathcal{T}_\theta), t \geq 0), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \mathbb{L}(\bar{\mathbb{D}})).$$

In Fig. 1, one can see a simulation of $\mathbb{L}_\infty(\mathcal{T}_\theta) := \overline{\lim_{t \rightarrow \infty} \mathbb{L}_t(\mathcal{T}_\theta)}$, for a given parameter set θ . Theorem 2 (or Corollary 1) does not directly imply the convergence, after proper rescaling, of the fragmentation process associated with a tree \mathbf{t}_n having a given degree sequence. However, this convergence can be established independently, without relying on the scaling limit of \mathbf{t}_n . The precise statement (Theorem 8) is deferred to Section 5. We highlight here that the limiting process, say $(\mathbf{F}(t), t \geq 0)$, of this rescaled fragmentation process corresponds exactly to Bertoin’s construction [32,33] using partitions of the unit interval induced by

specific bridges with exchangeable increments (see Section 5). Furthermore, $(\mathbf{F}(t), t \geq 0)$ coincides with the fragmentation process $(\mathbf{F}_{\mathcal{T}_\theta}(t), t \geq 0)$ of the Inhomogeneous CRT \mathcal{T}_θ , with parameter set $\theta = (\theta_0, \theta_1, \dots)$, as shown in [15].

On the other hand, one can readily observe that the rescaled size of a component in the fragmentation process of \mathfrak{t}_n corresponds to the mass of the associated face in the lamination process, defined as $(2\pi)^{-1}$ times the fraction of its perimeter lying on the unit circle. Then, as a consequence of Theorem 8, we can relate the sequence of masses of the faces of $(\mathbb{L}_t(\mathcal{T}_\theta), t \geq 0)$ with the fragmentation process $(\mathbf{F}(t), t \geq 0)$. For any lamination $L \in \mathbb{L}(\mathbb{D})$, let $\mathbb{M}\text{ass}[L]$ denote the sequence of the masses of its faces, sorted in non-increasing order. Consider also the infinite ordered set

$$\Delta := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0 \text{ and } \sum_{i=1}^{\infty} x_i < \infty \right\}.$$

We equip Δ with the ℓ^1 -norm, $\|\mathbf{x}\|_1 = \sum_{i=1}^{\infty} |x_i|$ for $\mathbf{x} \in \Delta$.

Corollary 2. Suppose that s_n satisfies (A.1)–(A.3) and

(A.5) either $\theta_0 := \sqrt{1 - \sum_{i \geq 1} \theta_i^2} > 0$ or $\sum_{i \geq 1} \theta_i = \infty$.

Then,

$$(\mathbb{M}\text{ass}[\mathbb{L}_{t_n}(\mathfrak{t}_n)], t \geq 0) \xrightarrow{d} (\mathbf{F}(t), t \geq 0), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \Delta).$$

Furthermore, the limiting process satisfies that, almost surely, $(\mathbf{F}_{\mathcal{T}_\theta}(t), t \geq 0) = (\mathbb{M}\text{ass}[\mathbb{L}_t(\mathcal{T}_\theta)], t \geq 0)$.

Note that there is a priori no connection between the convergence of the fragmentation process in Corollary 2 and the convergence of the lamination-valued process in Corollary 1. Roughly speaking, Corollary 2 controls how small edges disappear at the limit, along the convergence of the lamination-valued processes in Corollary 1.

Let us finish with a remark on the assumptions that we make on the degree sequences. The hypotheses size (A.1), hubs (A.2), degree variance (A.3) and unbounded variation (A.5) are exactly those made in [34] to study the profile of a TGDS. They are necessary to apply the characterization and convergence results for exchangeable increments processes of [35] that are crucial to understand the shape of a TGDS via its Łukasiewicz path.

1.3. Organization of the paper

In Section 2, we first recall the definition of rooted plane trees and their encoding by paths. In Section 3, we prove Theorem 1 by studying the behavior of a modified version of the Łukasiewicz paths associated with trees with given degree sequence. Section 4 is devoted to the study of lamination-valued processes of rooted plane trees and plane continuum trees; we prove in particular Theorem 2. Finally, in Section 5, we prove Theorem 8, showing the convergence of the associated fragmentation processes, as well as Corollary 2.

Notation. For $I = [a, b]$ for some $a < b$, or $I = [a, +\infty)$ for some $a \in \mathbb{R}_+$, for $f \in \mathbf{D}(I, \mathbb{R})$, we denote by $f(t)$ the value of f at $t \in I$, by $f(t-)$ its left-hand limit at time t (with the convention $f(a-) = f(a)$) and by $\Delta f(t) = f(t) - f(t-)$ the size of the jump (if any) at t .

We write \xrightarrow{d} , $\xrightarrow{\mathbb{P}}$ and $\xrightarrow{a.s.}$ to denote convergence in distribution, probability and almost surely, respectively.

2. Plane trees and their encoding paths

We provide here some background on finite rooted trees and recall how they can be coded by different integer-valued paths.

Following Neveu’s formalism [36], let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers and consider the set of labels $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ (with the convention that $\mathbb{N}^0 = \{\emptyset\}$). An element $u \in \mathbb{U}$ is a sequence $u = (u_1, \dots, u_n)$ of positive integers. If $v = (v_1, \dots, v_m) \in \mathbb{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ be the concatenation of u and v . By a slight abuse of notation, if $z \in \mathbb{N}$, we let $uz = (u_1, \dots, u_n, z)$. A rooted plane tree is a non-empty, finite subset $\tau \subset \mathbb{U}$, whose elements are called vertices, such that: (i) $\emptyset \in \tau$; (ii) if $v \in \tau$ and $v = uz$ for some $z \in \mathbb{N}$, then $u \in \tau$; (iii) if $u \in \tau$, then there exists an integer $k_u \geq 0$ such that $ui \in \tau$ if and only if $1 \leq i \leq k_u$. We view each vertex of τ as an individual of a population whose genealogical tree is τ . The vertex \emptyset is called the root of the tree. For every $u = (u_1, \dots, u_n) \in \tau$, the vertex $pr(u) = (u_1, \dots, u_{n-1})$ is its parent, k_u represents the number of children of u (if $k_u = 0$, then u is called a leaf, otherwise, u is called an internal vertex), and $|u| = n$ represents the length (or generation, or height) of u . We let $\chi_u \in \{1, \dots, k_{pr(u)}\}$ be the only index such that $u = pr(u)\chi_u$, which is the relative position of u amongst its siblings. For two vertices $u, v \in \tau$, denote by $u \wedge v$ the first (highest) common ancestor of u and v . The total progeny (or size) of τ will be denoted by $\zeta(\tau) = \text{Card}(\tau)$ (i.e., the number of vertices of τ). In the following, by tree, we will always mean a finite rooted plane tree and we denote the set of all trees by \mathbb{T} . By a slight abuse, we consider a tree τ as a metric space, by drawing an edge of length 1 between each non-root vertex u and its parent $pr(u)$.

For $u, v \in \tau$, we denote by $\llbracket u, v \rrbracket$ the unique geodesic path between u and v in τ , and $\llbracket u, v \rrbracket = \llbracket u, v \rrbracket \setminus \{v\}$. In particular, we write $\llbracket \emptyset, u \rrbracket$ for the ancestral line of u . For $u \in \tau$, let us denote by $L(u)$ and $R(u)$ respectively the number of vertices whose parent is a strict ancestor of u and which lie strictly to the left, respectively to the right, of the ancestral line $\llbracket \emptyset, u \rrbracket$. To be precise, for some $n \in \mathbb{N}$,

let $u = (u_1, \dots, u_n) \in \tau$ where $u_i \in \mathbb{N}$, for $1 \leq i \leq n$. For $1 \leq i \leq n$, let $v_{(i)} = u$ and $v_{(n-j)} = pr(v_{(n+1-j)}) = (u_1, \dots, u_{n-j})$, for $1 \leq j \leq n$ (with the convention $v_{(0)} = \emptyset$, the root of τ). So, $\llbracket \emptyset, u \rrbracket = \{v_{(i)} : 0 \leq i < n\}$,

$$L(u) = \sum_{i=0}^{n-1} (u_{i+1} - 1) \quad \text{and} \quad R(u) = \sum_{i=0}^{n-1} (k_{v_{(i)}} - u_{i+1}). \tag{2}$$

Then, we set $LR(u) = L(u) + R(u)$, the total number of individuals branching-off the ancestral line of u .

We will use three different orderings of the vertices of a tree $\tau \in \mathbb{T}$:

- (i) **Lexicographical ordering.** Given $v, w \in \tau$, we write $v <_{\text{lex}} w$ if there exists $u \in \tau$ such that $v = u(v_1, \dots, v_n)$, $w = u(w_1, \dots, w_m)$ and $v_1 < w_1$.
- (ii) **Reverse-lexicographical ordering.** Given $v, w \in \tau$, we write $v <_{\text{rev}} w$ if $w <_{\text{lex}} v$.
- (iii) **Prim ordering.** Let $\text{edge}(\tau)$ be the set of edges of τ and consider a sequence of distinct and positive weights $\mathbf{w} = (w_e : e \in \text{edge}(\tau))$ (i.e., each edge e of τ is marked with a different and positive weight w_e). Given two distinct vertices $u, v \in \tau$, we write $\{u, v\}$ for the edge connecting u and v in τ - if it exists. Let us describe the Prim order $<_{\text{prim}}$ of the vertices in τ , that is, $\emptyset = u(0) <_{\text{prim}} u(1) <_{\text{prim}} \dots <_{\text{prim}} u(\zeta(\tau) - 1)$. First set $u(0) = \emptyset$ and $D_1 = \{u(0)\}$. Suppose that for some $1 \leq i \leq \zeta(\tau) - 1$, the vertices $u(0), \dots, u(i - 1)$ have been defined. We will use the notation D_i for the set $\{u(0), \dots, u(i - 1)\}$, for $0 \leq i \leq \zeta(\tau) - 1$. Consider the minimum of the set of weights $\{w_{\{u,v\}} : u \in D_i, v \notin D_i\}$ of the edges between a vertex of D_i and another outside of D_i . Since all the weights are distinct, this minimum is reached at a unique edge $\{\bar{u}, \bar{v}\}$ where $\bar{u} \in D_i$ and $\bar{v} \notin D_i$. Then set $u(i) = \bar{v}$. This iterative procedure completely determines the Prim order $<_{\text{prim}}$.

Lukasiewicz path, reverse-Lukasiewicz path and Prim path. Fix $\tau \in \mathbb{T}$, and for $*$ in $\{\text{lex}, \text{rev}, \text{prim}\}$, associate to the ordering $\emptyset = u(0) <_* u(1) <_* \dots <_* u(\zeta(\tau) - 1)$ of its vertices a path $W_\tau^* = (W_\tau^*(i), 0 \leq i \leq \zeta(\tau))$, by letting $W_\tau^*(0) = 0$ and for $0 \leq i \leq \zeta(\tau) - 1$, $W_\tau^*(i + 1) = W_\tau^*(i) + k_{u(i)} - 1$. Observe that $W_\tau^*(i + 1) - W_\tau^*(i) = k_{u(i)} - 1 \geq -1$ for every $0 \leq i \leq \zeta(\tau) - 1$, with equality if and only if $u(i)$ is a leaf of τ . Note also that $W_\tau^*(i) \geq 0$ for every $0 \leq i \leq \zeta(\tau) - 1$, and $W_\tau^*(\zeta(\tau)) = -1$. We shall think of such a path as the step function on $[0, \zeta(\tau)]$ given by $s \mapsto W_\tau^*(\lfloor s \rfloor)$. The path W_τ^{lex} is usually called the Łukasiewicz path of τ and we will refer to W_τ^{rev} and W_τ^{prim} as the reverse-Łukasiewicz path and the Prim path of τ , respectively.

In particular, if $\emptyset = u(0) <_{\text{lex}} u(1) <_{\text{lex}} \dots <_{\text{lex}} u(\zeta(\tau) - 1)$ is the sequence of vertices of τ in lexicographical order, then

$$R(u(i)) = W_\tau^{\text{lex}}(i), \quad \text{for } 0 \leq i \leq \zeta(\tau) - 1. \tag{3}$$

Similarly, if $\emptyset = u(0) <_{\text{rev}} u(1) <_{\text{rev}} \dots <_{\text{rev}} u(\zeta(\tau) - 1)$ are listed in reverse-lexicographical order, then

$$L(u(i)) = W_\tau^{\text{rev}}(i), \quad \text{for } 0 \leq i \leq \zeta(\tau) - 1. \tag{4}$$

Height process. Let $\emptyset = u(0) <_{\text{lex}} u(1) <_{\text{lex}} \dots <_{\text{lex}} u(\zeta(\tau) - 1)$ be the sequence of vertices of $\tau \in \mathbb{T}$ in lexicographical order. The height process $H_\tau = (H_\tau(i) : 0 \leq i \leq \zeta(\tau))$ of τ is defined by letting $H_\tau(i) = |u(i)|$, for every $i \in \{0, \dots, \zeta(\tau) - 1\}$, and $H_\tau(\zeta(\tau)) = 0$. We sometimes think of H_τ as a continuous function on $[0, \zeta(\tau)]$, obtained by linear interpolation.

Contour function. The contour function $C_\tau = (C_\tau(s), s \in [0, 2\zeta(\tau)])$ of $\tau \in \mathbb{T}$ is defined as follows. We see τ embedded in the oriented half-plane, with each edge having length one. We then imagine a particle exploring continuously all edges of τ from left to right at unit speed, going backwards when it reaches a leaf. For all $s \in [0, 2\zeta(\tau) - 2]$, let $C_\tau(s)$ be the distance from the particle to the root \emptyset at time s . By convention, we set $C_\tau(s) = 0$ for $s \in [2\zeta(\tau) - 2, 2\zeta(\tau)]$. In particular, C_τ is continuous and $C_\tau(0) = C_\tau(2\zeta(\tau)) = 0$.

3. Convergence of the trees with given degree sequences

In this section, we prove [Theorem 1](#) which states the convergence of trees with a given degree sequence towards the Inhomogeneous CRT. In [Section 3.1](#), we first recall the definition of the exploration process that encodes the Inhomogeneous CRT. In [Section 3.2](#), we introduce the discrete version of the above exploration process, the so-called modified Łukasiewicz path, which encodes a TGDS. We then prove that this modified Łukasiewicz path, suitably rescaled, converges to the exploration process of the Inhomogeneous CRT, which implies [Theorem 1](#). Finally, in [Section 3.3](#), we consider a specific case in which we can prove the convergence of the trees for the Gromov–Hausdorff–Prohorov topology.

3.1. The exploration process of the inhomogeneous CRT

Let us start with some definitions. A bridge with exchangeable increments (abridged EI bridge) is a continuous-time stochastic process $X^{\text{bg}} = (X^{\text{bg}}(t), t \in [0, 1])$ with paths in $\mathbf{D}([0, 1], \mathbb{R})$, of the form

$$X^{\text{bg}}(t) = \theta_0 B^{\text{bg}}(t) + \sum_{i=1}^{\infty} \theta_i (\mathbf{1}_{\{U_i \leq t\}} - t), \quad t \in [0, 1],$$

where $B^{\text{bg}} = (B^{\text{bg}}(t), t \in [0, 1])$ is a Brownian bridge on $[0, 1]$, $(U_i, i \geq 1)$ are i.i.d. random variables with uniform law on $[0, 1]$ independent of B^{bg} , and $\theta_0 \in \mathbb{R}_+$, $\theta_1 \geq \theta_2 \geq \dots \geq 0$ are constants such that $\sum_{i \geq 0} \theta_i^2 < \infty$. We say that X^{bg} is an EI bridge with parameters $(\theta_i, i \geq 0)$; see e.g. [\[37, Theorem 16.21\]](#).

The so-called Vervaat transform (or Vervaat excursion) was introduced by Takács [38] and used by Vervaat [39] (see also [33, Section 3] or [40, Section 4.2]) to change a bridge-type process with paths in $\mathbf{D}([0, 1], \mathbb{R})$ into an excursion-type process (i.e., a non-negative process that is equal to 0 at times 0 and 1). More precisely, let $X = (X(t), t \in [0, 1])$ be a stochastic process with paths in $\mathbf{D}([0, 1], \mathbb{R})$ such that $X(0) = 0$. Then, the Vervaat transform of X is the stochastic process $\text{Ver}_X := (\text{Ver}_X(t), t \in [0, 1])$ with paths in $\mathbf{D}([0, 1], \mathbb{R})$, defined by

$$\text{Ver}_X(t) = \begin{cases} X(t + \rho_X) - \inf_{0 \leq s \leq 1} X(s) & \text{if } t + \rho_X \leq 1, \\ X(t + \rho_X - 1) + X(1) - \inf_{0 \leq s \leq 1} X(s) & \text{if } t + \rho_X \geq 1 \end{cases} \tag{5}$$

where $\rho_X = \inf \{u \in [0, 1] : \min(X(u-), X(u)) = \inf_{0 \leq s \leq 1} X(s)\}$.

Through this manuscript, unless otherwise specified, we always consider an EI bridge X^{bg} with parameters $(\theta_i, i \geq 0)$ such that $\sum_{i \geq 0} \theta_i^2 = 1$ and (A.5) is satisfied (i.e. either $\theta_0 > 0$ or $\sum_{i \geq 1} \theta_i = \infty$). This is a necessary and sufficient condition for X^{bg} to have paths of infinite variation. More importantly, by [41] or [33, Proof of Lemma 6], it is well-known that under this condition X^{bg} almost surely achieves its infimum at a unique time $\rho_{X^{\text{bg}}}$ and continuously. Then, we let $X^{\text{exc}} = (X^{\text{exc}}(t), t \in [0, 1])$ be the excursion-type process associated to X^{bg} via its Vervaat transform. Note that $X^{\text{exc}}(0) = X^{\text{exc}}(1) = 0$ and $X^{\text{exc}}(t) > 0$ for all $t \in (0, 1)$.

The excursion process X^{exc} is not necessarily continuous. However, following [16, Section 2], one can also associate to X^{bg} a continuous excursion process H^{exc} . For $i \geq 1$ such that $\theta_i > 0$, write $t_i = \{U_i - \rho_{X^{\text{bg}}}\}$ (the fractional part of $U_i - \rho_{X^{\text{bg}}}$) for the location of the jump with size θ_i in X^{exc} . For each $i \geq 1$ such that $\theta_i > 0$, write $T_i = \inf\{t \in (t_i, 1] : X^{\text{exc}}(t) = X^{\text{exc}}(t_i-)\}$, which exists since the process X^{exc} has no negative jumps and goes back to 0 at time 1. In particular, all t_i 's and T_i 's are distinct almost surely. For $i \geq 1$ such that $\theta_i > 0$, let $R_i = (R_i(u), u \in [0, 1])$ be the process defined by

$$R_i(u) = \begin{cases} \inf_{t_i \leq s \leq u} X^{\text{exc}}(s) - X^{\text{exc}}(t_i-) & \text{if } u \in [t_i, T_i], \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

If $\theta_i = 0$ then let R_i be the null process on $[0, 1]$. Define the process $H^{\text{exc}} = (H^{\text{exc}}(u), u \in [0, 1])$ by

$$H^{\text{exc}}(u) = X^{\text{exc}}(u) - \sum_{i \geq 1} R_i(u), \quad \text{for } u \in [0, 1]. \tag{7}$$

As explained in [16, Section 2], H^{exc} is a well-defined continuous excursion-type process. Furthermore, Aldous, Miermont and Pitman [16, Theorem 1] also showed that, when $\theta_0 > 0$ and $\sum_{i \geq 1} \theta_i < \infty$, H^{exc} is the height process (up to a scaling factor) of an Inhomogeneous CRT with parameter set $\theta = (\theta_0, \theta_1, \dots)$.

3.2. The modified Łukasiewicz path

For $n \in \mathbb{N}$, let s_n be a degree sequence satisfying (A.1)–(A.3) and (A.5). We then sample a tree \mathbf{t}_n uniformly at random from \mathbb{T}_{s_n} and let $W_n^{\text{lex}} = (W_n^{\text{lex}}(V_n u), u \in [0, 1])$ denote its time-rescaled Łukasiewicz path. Recall that V_n denotes the number of vertices in \mathbf{t}_n . Let $(b_n, n \geq 1)$ be a sequence satisfying (A.2), and X be an EI bridge with parameters $(\theta_i, i \geq 0)$ as in (A.2).

Theorem 3. *Suppose that s_n satisfies (A.1)–(A.3) and (A.5). Then,*

$$(b_n^{-1} W_n^{\text{lex}}(V_n u), u \in [0, 1]) \xrightarrow{d} (X^{\text{exc}}(u), u \in [0, 1]), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}([0, 1], \mathbb{R}).$$

Proof. Let π_n be a uniform permutation of $\{1, \dots, V_n\}$. Recall that $(d^n(i), 1 \leq i \leq V_n)$ denotes the associated child sequence of s_n . Define the process $Y_n = (Y_n(u), u \in [0, 1])$ by letting

$$Y_n(u) = \sum_{i=1}^{\lfloor V_n u \rfloor} (d^n(\pi_n(i)) - 1), \quad \text{for } u \in [0, 1].$$

For $1 \leq i \leq V_n$, define $\Delta Y_n(i/V_n) = Y_n(i/V_n) - Y_n((i-1)/V_n) = d^n(\pi_n(i)) - 1$. Note that the increments $(\Delta Y_n(1/V_n), \Delta Y_n(2/V_n), \dots, \Delta Y_n(1))$ are exchangeable (i.e., their distribution is invariant under permutation). Note also that $\Delta Y_n(i/V_n) = d^n(\pi_n(i)) - 1 \geq -1$, for every $1 \leq i \leq V_n$, and furthermore, $Y_n(1) = -1$. Define

$$\rho_n = \min \left\{ i \in \{1, \dots, V_n\} : Y_n(i/V_n) = \min_{1 \leq j \leq V_n} Y_n(j/V_n) \right\}.$$

and let $Y_n^{\text{exc}} = (Y_n^{\text{exc}}(u), u \in [0, 1])$ be the Vervaat transform of Y_n ; see (5). Note that $Y_n^{\text{exc}}(u) \geq 0$, for $u \in [0, 1)$, and $Y_n^{\text{exc}}(1) = -1$.

Under our assumptions (A.1)–(A.3) and (A.5), it follows from [35, Theorem 2.2] that

$$(b_n^{-1} Y_n(u), u \in [0, 1]) \xrightarrow{d} (X^{\text{bg}}(u), u \in [0, 1]), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}([0, 1], \mathbb{R}).$$

Since the process X^{bg} achieves its infimum in a unique time and continuously, [33, Lemma 3] implies that $b_n^{-1} Y_n^{\text{exc}}$ converges to X^{exc} , the Vervaat transform of X^{bg} . Finally, our claim follows by noticing (see e.g. [11, Proof of Lemma 7]) that Y_n^{exc} has the same distribution as the Łukasiewicz path W_n^{lex} . \square

We now describe a modification of W_n^{lex} from which we construct the discrete analog of H^{exc} . Here we make the convention $\inf \emptyset = 1$ and $\sup \emptyset = 0$. For $\delta > 0$, define

$$t_0^{n,\delta} := 0 \quad \text{and} \quad t_i^{n,\delta} = \inf\{u \in (t_{i-1}^{n,\delta}, 1] : |\Delta W_n^{\text{lex}}(V_n u)| > \delta b_n\}, \quad \text{for } i \geq 1,$$

and

$$T_i^{n,\delta} = \inf\{u \in (t_i^{n,\delta}, 1] : W_n^{\text{lex}}(V_n u) - W_n^{\text{lex}}(V_n t_i^{n,\delta}-) = -1\}, \quad \text{for } i \geq 1.$$

Set $I_n^\delta = \sup\{i \geq 0 : 0 < t_i^{n,\delta} < 1\}$. In particular, $I_n^\delta \leq V_n$. Note that $t_i^{n,\delta} = T_i^{n,\delta} = 1$ for $i > I_n^\delta$. If $I_n^\delta > 0$, then for $i \geq 1$, we let $R_i^{n,\delta} = (R_i^{n,\delta}(u), u \in [0, 1])$ be the process given by

$$R_i^{n,\delta}(u) = \begin{cases} \inf_{t_i^{n,\delta} \leq s \leq u} W_n^{\text{lex}}(V_n s) - W_n^{\text{lex}}(V_n t_i^{n,\delta}-) & \text{if } u \in [t_i^{n,\delta}, T_i^{n,\delta}], \\ 0 & \text{otherwise.} \end{cases}$$

For $i > I_n^\delta$ (even when $I_n^\delta = 0$), we set $R_i^{n,\delta}(u) = 0$, for all $u \in [0, 1]$. Then, for $\delta > 0$, we define the modified Łukasiewicz path $G_n^\delta = (G_n^\delta(V_n u), u \in [0, 1])$ of t_n by letting

$$G_n^\delta(V_n u) = W_n^{\text{lex}}(V_n u) - \sum_{i \geq 1} R_i^{n,\delta}(u). \tag{8}$$

Note that the sum on the right hand side is always finite for every $u \in [0, 1]$, since the number of nonzero summands is I_n^δ .

Let $(H^{\text{exc}}(u), u \in [0, 1])$ be the continuous excursion process associated to an EI bridge $(X^{\text{exc}}(u), u \in [0, 1])$ with parameters $(\theta_i, i \geq 0)$ defined in (7). Recall also the process $R_i = (R_i(u), u \in [0, 1])$ defined in (6). For $\delta > 0$, we define the process $G^\delta = (G^\delta(u), u \in [0, 1])$ by

$$G^\delta(u) = H^{\text{exc}} + \sum_{i \geq 1} R_i(u) \mathbf{1}_{\{\theta_i \leq \delta\}} = X^{\text{exc}}(u) - \sum_{i \geq 1} R_i(u) \mathbf{1}_{\{\theta_i > \delta\}}. \tag{9}$$

Theorem 4. *Suppose that s_n satisfies (A.1)–(A.4). Then, for $\delta \notin \{\theta_i : i \geq 1\}$,*

$$(b_n^{-1} G_n^\delta(V_n u), b_n^{-1} W_n^{\text{lex}}(V_n u) : u \in [0, 1]) \xrightarrow{d} (G^\delta(u), X^{\text{exc}}(u) : u \in [0, 1]), \quad \text{as } n \rightarrow \infty,$$

in $\mathbf{D}([0, 1], \mathbb{R}) \times \mathbf{D}([0, 1], \mathbb{R})$. Moreover, as $\delta \downarrow 0$ (still with the assumption that $\delta \notin \{\theta_i : i \geq 1\}$), $(G^\delta(u), u \in [0, 1])$ converges uniformly to H^{exc} .

Observe that (A.4) is included in (A.5), so that in particular Theorem 3 holds under the assumptions of Theorem 4.

Proof of Theorem 4. Recall that, if $\theta_i > 0$, then t_i denotes the location of the jump with size θ_i of X^{exc} , i.e. $t_i = \{U_i - \rho_{X^{\text{bg}}}\}$ (the fractional part of $U_i - \rho_{X^{\text{bg}}}$). If $\theta_i = 0$, then we let $t_i = 1$. For $\delta > 0$, note that $I^\delta := \sup\{i \geq 1 : \theta_i > \delta\} < \infty$. By convenience, if $\theta_1 \leq \delta$ (and thus, $\theta_i \leq \delta$, for all $i \geq 1$), then we set $I^\delta = 0$. On the other hand, for $\delta > 0$, define

$$t_0^\delta := 0 \quad \text{and} \quad t_i^\delta = \inf\{u \in (t_{i-1}^\delta, 1] : |\Delta X^{\text{exc}}(u)| > \delta\}, \quad \text{for } i \geq 1.$$

Here we also make the convention $\inf \emptyset = 1$. If $I^\delta > 0$, then the sequence of times $(t_i^\delta, 1 \leq i \leq I^\delta)$ is the sequence $(t_i, 1 \leq i \leq I^\delta)$ in increasing order. For each $i \geq 1$, write $T_i^\delta = \inf\{t \in (t_i^\delta, 1] : X^{\text{exc}}(t) = X^{\text{exc}}(t_i^\delta-)\}$ and let $R_i^\delta = (R_i^\delta(u), u \in [0, 1])$ be the process given by

$$R_i^\delta(u) = \begin{cases} \inf_{t_i^\delta \leq s \leq u} X^{\text{exc}}(s) - X^{\text{exc}}(t_i^\delta-) & \text{if } u \in [t_i^\delta, T_i^\delta], \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if $t_i^\delta = 1$, then $R_i^\delta(u) = 0$ for all $u \in [0, 1]$ and, if $I^\delta = 0$, then $R_i^\delta(u) = 0$, for all $u \in [0, 1]$ and $i \geq 1$. Then, it should be clear that

$$G^\delta(u) = X^{\text{exc}}(u) - \sum_{i \geq 1} R_i^\delta(u), \quad \text{for } u \in [0, 1]. \tag{10}$$

Note that, under our assumptions (A.1)–(A.4), the result in Theorem 3 holds, i.e. the rescaled Łukasiewicz path $b_n^{-1} W_n^{\text{lex}}$ converges towards X^{exc} . By the Skorokhod representation theorem, there exists a probability space on which this convergence holds almost surely. Let us work on this space from now on. We claim that, for $\delta \notin \{\theta_i : i \geq 1\}$,

- (i) for all $i \geq 1$, $\lim_{n \rightarrow \infty} t_i^{n,\delta} = t_i^\delta$ a.s.,
- (ii) if $I^\delta > 0$, then for all $1 \leq i \leq I^\delta$, $\lim_{n \rightarrow \infty} T_i^{n,\delta} = T_i^\delta$ a.s., and
- (iii) for all $i \geq 1$, $(b_n^{-1} R_i^{n,\delta}(u), u \in [0, 1]) \rightarrow (R_i^\delta(u), u \in [0, 1])$ a.s., as $n \rightarrow \infty$, in $\mathbf{D}([0, 1], \mathbb{R})$.

Note that (i) follows from Theorem 3 and [42, Proposition VI.2.7]. Indeed, [42, Proposition VI.2.7] consider paths in $\mathbf{D}(\mathbb{R}_+, \mathbb{R})$. However, a close inspection of its proof shows that the result holds for paths in $\mathbf{D}([0, 1], \mathbb{R})$ with minor modifications.

Next, we prove (ii). It follows from Theorem 3 that $\liminf_{n \rightarrow \infty} T_i^{n,\delta} \geq T_i^\delta$. Suppose that $t' = \limsup_{n \rightarrow \infty} T_i^{n,\delta} > T_i^\delta$ and up to extraction suppose that t' is actually the limit of $(T_i^{n,\delta}, n \geq 1)$. Since (i) and [42, Proposition VI.2.1] imply that

$\lim_{n \rightarrow \infty} b_n^{-1} W_n^{\text{lex}}(V_n T_i^{n,\delta}) = X^{\text{exc}}(t_i^\delta -)$, we would find that $t' > T_i^\delta$ with $X^{\text{exc}}(t') = X^{\text{exc}}(t_i^\delta -)$ and $X^{\text{exc}}(s) \geq X^{\text{exc}}(t_i^\delta -)$, for $s \in [T_i^\delta, t']$. This shows that $X^{\text{exc}}(t_i^\delta -)$ is a local minimum of X^{exc} , attained at time T_i^δ , which is a.s. impossible by (A.4) and [16, Lemma 1]. Therefore, $\lim_{n \rightarrow \infty} T_i^{n,\delta} = T_i^\delta$ which proves (ii).

We now prove (iii). Suppose first that $I^\delta = 0$. Then $t_i^\delta = 1$ and $R_i^\delta(u) = 0$, for all $i \geq 1$ and $u \in [0, 1]$. Moreover, for $i \geq 1$, a.s.,

$$\sup_{0 \leq u \leq 1} |b_n^{-1} R_i^{n,\delta}(u)| \leq b_n^{-1} \max(|\Delta W_n^{\text{lex}}(V_n t_i^{n,\delta})|, 1) \rightarrow |\Delta X^{\text{exc}}(1)| = 0, \quad \text{as } n \rightarrow \infty,$$

by [42, Proposition VI.2.1]. Thus, (iii) holds whenever $I^\delta = 0$. Suppose now that $I^\delta > 0$. By Theorem 3, there exists a sequence of strictly increasing, continuous mappings $(\lambda_n, n \geq 1)$ of $[0, 1]$ onto itself such that, a.s.,

$$\lim_{n \rightarrow \infty} \sup_{u \in [0,1]} |\lambda_n(u) - u| = 0 \text{ and } \lim_{n \rightarrow \infty} \sup_{u \in [0,1]} |b_n^{-1} W_n^{\text{lex}}(V_n \lambda_n(u)) - X^{\text{exc}}(u)| = 0. \tag{11}$$

For $1 \leq i \leq I^\delta$, we shall prove that, a.s.,

$$\lim_{n \rightarrow \infty} \sup_{u \in [0,1]} |b_n^{-1} R_i^{n,\delta}(\lambda_n(u)) - R_i^\delta(u)| = 0. \tag{12}$$

Suppose by contradiction that (12) does not hold. Then there exists $\varepsilon > 0$ and a sequence $(z_n, n \geq 1) \in [0, 1]^{\mathbb{N}}$ such that

$$|b_n^{-1} R_i^{n,\delta}(\lambda_n(z_n)) - R_i^\delta(z_n)| > \varepsilon. \tag{13}$$

for infinitely many indices n . Extracting a subsequence if necessary, let us assume that this holds for all n and that either $z_n \uparrow z$ or $z_n \downarrow z$ for some $z \in [0, 1]$, as $n \rightarrow \infty$. If $z \notin [t_i^\delta, T_i^\delta]$, then, by (i) and (ii), for n large enough, $\lambda_n(z_n), z_n \notin [t_i^\delta, T_i^\delta] \cup [t_i^{n,\delta}, T_i^{n,\delta}]$, which leads to a contradiction of (13).

Now, assume that $z_n \uparrow t_i^\delta$, as $n \rightarrow \infty$ and $z_n \neq t_i^\delta$ for all $n \geq 1$. It follows also from (11) that $\lambda_n(z_n) \rightarrow t_i^\delta$ and $b_n^{-1} W_n^{\text{lex}}(V_n \lambda_n(z_n)) \rightarrow X^{\text{exc}}(t_i^\delta -)$, a.s., as $n \rightarrow \infty$. On the other hand, if $z_n \uparrow T_i^\delta$, as $n \rightarrow \infty$, then by (11) we have that $\lambda_n(z_n) \rightarrow T_i^\delta$ and $b_n^{-1} W_n^{\text{lex}}(V_n \lambda_n(z_n)) \rightarrow X^{\text{exc}}(T_i^\delta)$ a.s., as $n \rightarrow \infty$ (note that $X^{\text{exc}}(T_i^\delta) = X^{\text{exc}}(t_i^\delta -)$). Thus, in both cases, $b_n^{-1} R_i^{n,\delta}(\lambda_n(z_n)) \rightarrow 0$ and $R_i^\delta(z_n) \rightarrow 0$ a.s., as $n \rightarrow \infty$, which contradicts (13). If $z_n \uparrow t_i^\delta$, as $n \rightarrow \infty$ and $z_n = t_i^\delta$, for n large enough, then it follows from the definition of $t_i^{n,\delta}$, [42, Proposition VI.2.1 and Proposition VI.2.7] that $\lambda_n(z_n) = \lambda_n(t_i^\delta) = t_i^{n,\delta}$, for n large enough. In particular, for n large enough, $R_i^{n,\delta}(\lambda_n(z_n)) = \Delta W_n^{\text{lex}}(V_n t_i^{n,\delta})$, and thus, by [42, Proposition VI.2.7], $b_n^{-1} R_i^{n,\delta}(\lambda_n(z_n)) \rightarrow \Delta X^{\text{exc}}(t_i^\delta) = R_i^\delta(t_i^\delta)$ a.s., as $n \rightarrow \infty$, which contradicts (13).

Suppose finally that $z_n \uparrow z \in (t_i^\delta, T_i^\delta)$, then, by (i) and (ii), for n large enough, $\lambda_n(z_n) \in [t_i^{n,\delta}, T_i^{n,\delta}]$ and $z_n \in [t_i^\delta, T_i^\delta]$. Then,

$$|b_n^{-1} R_i^{n,\delta}(\lambda_n(z_n)) - R_i^\delta(z_n)| \leq \left| \inf_{t_i^{n,\delta} \leq u \leq \lambda_n(z_n)} b_n^{-1} W_n^{\text{lex}}(V_n u) - \inf_{t_i^\delta \leq u \leq z_n} X^{\text{exc}}(u) \right| + |b_n^{-1} W_n^{\text{lex}}(V_n t_i^{n,\delta}) - X^{\text{exc}}(t_i^\delta -)|. \tag{14}$$

On the other hand, note that

$$\begin{aligned} & \left| \inf_{t_i^{n,\delta} \leq u \leq \lambda_n(z_n)} b_n^{-1} W_n^{\text{lex}}(V_n u) - \inf_{t_i^\delta \leq u \leq z_n} X^{\text{exc}}(u) \right| \\ & \leq \left| \inf_{\lambda_n^{-1}(t_i^{n,\delta}) \leq u \leq z_n} b_n^{-1} W_n^{\text{lex}}(V_n \lambda_n(u)) - \inf_{\lambda_n^{-1}(t_i^\delta) \leq u \leq z_n} X^{\text{exc}}(u) \right| + \left| \inf_{\lambda_n^{-1}(t_i^{n,\delta}) \leq u \leq z_n} X^{\text{exc}}(u) - \inf_{t_i^\delta \leq u \leq z_n} X^{\text{exc}}(u) \right| \\ & \leq \sup_{\lambda_n^{-1}(t_i^{n,\delta}) \leq u \leq z_n} |b_n^{-1} W_n^{\text{lex}}(V_n \lambda_n(u)) - X^{\text{exc}}(u)| + \left| \inf_{\lambda_n^{-1}(t_i^{n,\delta}) \leq u \leq z_n} X^{\text{exc}}(u) - \inf_{t_i^\delta \leq u \leq z_n} X^{\text{exc}}(u) \right|, \end{aligned} \tag{15}$$

where λ_n^{-1} denotes the inverse of λ_n . It follows from (11), (14), (15), (i) and [42, Proposition VI.2.1] that $b_n^{-1} R_i^{n,\delta}(\lambda_n(z_n)) - R_i^\delta(z_n)$ converges to 0, a.s., as $n \rightarrow \infty$, and this yields to a contradiction of (13).

Similarly, if $z_n \downarrow z \in [0, 1]$, as $n \rightarrow \infty$, then one can check that (13) does not hold by considering the cases $z = t_i^\delta$, $z = T_i^\delta$, $z \in (t_i^\delta, T_i^\delta)$ and $z \notin [t_i^\delta, T_i^\delta]$, separately. We leave the details to the reader.

Therefore, we have showed (12), that is, (iii), for every $1 \leq i \leq I^\delta$. For $i > I^\delta$, the proof of (iii) follows as in the case $I^\delta = 0$.

Finally, we have all the ingredients to prove Theorem 4. Again, by Theorem 3, we know that there exists a sequence of strictly increasing, continuous mappings $(\lambda_n, n \geq 1)$ of $[0, 1]$ onto itself such that (11) holds. Note that we have actually shown during the proof of (iii) that (12) holds for every $i \geq 1$. Hence the triangle inequality shows that for every fixed $q \geq 1$ that, a.s.,

$$\lim_{n \rightarrow \infty} \sup_{u \in [0,1]} \left| b_n^{-1} W_n^{\text{lex}}(V_n \lambda_n(u)) - \sum_{i=1}^q b_n^{-1} R_i^{n,\delta}(\lambda_n(u)) - X^{\text{exc}}(u) + \sum_{i=1}^q R_i^\delta(u) \right| = 0. \tag{16}$$

Note that there exists q such that, for n large enough, $t_i^{n,\delta} = t_i^\delta = 1$, for $i > q$, which implies that $R_i^{n,\delta}(\lambda_n(u)) = R_i^\delta(u) = 0$, for all $u \in [0, 1]$ and $i > q$. Therefore, the first part of Theorem 4 follows from (10) and (16). The second part of Theorem 4 follows by Dini's theorem (see for e.g., [43, Theorem 7.13]). \square

Let $(H_n(V_n u), u \in [0, 1])$ be the time-rescaled height process associated to \mathbf{t}_n .

Theorem 5. Suppose that s_n satisfies assumptions (A.1)–(A.4). Fix $q \geq 1$, let U_1, \dots, U_q be i.i.d. uniform random variables in $[0, 1]$ independent of the rest and denote by $0 = U_{(0)} < U_{(1)} < \dots < U_{(q)}$ their order statistics. Then,

$$\frac{\theta_0^2}{2} \frac{b_n}{V_n} \left(H_n(V_n U_{(i)}), \inf_{U_{(i-1)} \leq u \leq U_{(i)}} H_n(V_n u) \right)_{1 \leq i \leq q} \xrightarrow{d} \left(H^{\text{exc}}(U_{(i)}), \inf_{U_{(i-1)} \leq u \leq U_{(i)}} H^{\text{exc}}(u) \right)_{1 \leq i \leq q}, \text{ as } n \rightarrow \infty,$$

holds jointly with the convergence in distribution of the Łukasiewicz path from Theorem 4.

Assuming Theorem 5, we have now all the ingredients to establish Theorem 1.

Proof of Theorem 1. It follows from Theorem 5 and [16, Theorem 1]. Indeed, [16, Theorem 1] shows that, under our assumptions, the exploration process of the Inhomogeneous CRT \mathcal{T}_θ with parameter set $\theta = (\theta_0, \theta_1, \dots)$ is distributed as $\frac{2}{\theta_0} H^{\text{exc}}$. Further details can be found in [16, Section 4]. \square

Note that, in Theorem 5, we assumed that the degree sequence satisfies (A.4). But clearly, (A.4) is included in (A.5).

As a preparation for the proof of Theorem 5, we deduce the following property for degree sequences that satisfy (A.1)–(A.3) and (A.5).

Lemma 1. Suppose that s_n satisfies assumptions (A.1)–(A.3) and (A.5). Then, $\lim_{n \rightarrow \infty} V_n/b_n = \infty$.

Proof. First, suppose that $\sum_{i \geq 1} \theta_i = \infty$ in (A.5). Then, our claim follows from (A.1)–(A.2) since $V_n \geq 1 + \sum_{i=1}^k d^n(i)$, for every fixed $1 \leq k \leq V_n$. Suppose now that $\theta_0 > 0$ in (A.5) and that our claim does not hold, i.e., there exists a constant $C > 0$ such that, along a subsequence, $V_n < C b_n$. Then, for any $\varepsilon > 0$,

$$\begin{aligned} b_n^{-2} \sum_{i \geq 0} (i-1)^2 N_i^n &\leq b_n^{-2} N_0^n + b_n^{-2} \sum_{i \geq 0} i^2 N_i^n \leq C b_n^{-1} + b_n^{-2} \sum_{i \leq \lfloor \varepsilon b_n \rfloor} i^2 N_i^n + b_n^{-2} \sum_{i > \lfloor \varepsilon b_n \rfloor} i^2 N_i^n \\ &\leq C b_n^{-1} + \varepsilon C + b_n^{-2} \sum_{i > \lfloor \varepsilon b_n \rfloor} i^2 N_i^n, \end{aligned} \tag{17}$$

where we have used that $V_n = \sum_{i \geq 0} N_i = 1 + \sum_{i \geq 0} i N_i$. Since we have assumed $V_n < C b_n$, we necessarily have $\sum_{i > \lfloor \varepsilon b_n \rfloor} N_i \leq \lceil C/\varepsilon \rceil$ (by considering the number of children of these vertices). Hence,

$$b_n^{-2} \sum_{i > \lfloor \varepsilon b_n \rfloor} i^2 N_i \leq b_n^{-2} \sum_{i \leq \lceil C/\varepsilon \rceil} (d^n(i))^2.$$

By (A.2), we get that

$$\limsup_{n \rightarrow \infty} b_n^{-2} \sum_{i > \lfloor \varepsilon b_n \rfloor} i^2 N_i \leq \sum_{i \leq \lceil C/\varepsilon \rceil} \theta_i^2 \leq \sum_{i \geq 1} \theta_i^2.$$

Taking $\varepsilon < \theta_0^2/C$ provides a contradiction between (17) and (A.3). \square

Proof of Theorem 5. We will follow an argument similar to that of the proof of [12, Theorem 2.4]. Let U be uniformly distributed on $[0, 1]$ independently of \mathbf{t}_n and let u_n be the $\lfloor V_n U \rfloor + 1$ -st vertex of \mathbf{t}_n in lexicographical order, so that it has the uniform distribution in \mathbf{t}_n . Recall that $H_n(\lfloor V_n U \rfloor) = |u_n|$ and that $R(u_n)$ denotes the number of individuals branching-off strictly to the right of the ancestral line $\llbracket \emptyset, u_n \rrbracket$ in \mathbf{t}_n ; see (2). Recall that, by (3), $R(u_n) = W_n^{\text{lex}}(V_n U)$. Fix $\delta > 0$ and henceforth consider n large enough such that $\delta b_n > 1$. Let $R^\delta(u_n)$ be the number of individuals branching-off strictly to the right of the ancestral line $\llbracket \emptyset, u_n \rrbracket$ in \mathbf{t}_n , except for vertices that are children of vertices with degree larger than $\delta b_n + 1$, i.e.,

$$R^\delta(u_n) = G_n^\delta(\lfloor V_n U \rfloor), \tag{18}$$

where G_n^δ is the modified Łukasiewicz defined in (8). Define also

$$\sigma_{n,\delta}^2 = \sum_{k=0}^{\lfloor \delta b_n + 1 \rfloor} k(k-1) N_k^n.$$

We claim that, for every $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\left| R^\delta(u_n) - \frac{\sigma_{n,\delta}^2}{2E_n} |u_n| \right| > \varepsilon \sigma_{n,\delta} \right) = 0. \tag{19}$$

We will prove (19) later; assume it for now. Note that (A.2)–(A.4) imply that

$$\lim_{n \rightarrow \infty} b_n^{-2} \sigma_{n,\delta}^2 = 1 - \sum_{i \geq 1} \theta_i^2 \mathbf{1}_{\{\theta_i > \delta\}} \quad \text{and} \quad \lim_{\delta \downarrow 0} \left(1 - \sum_{i \geq 1} \theta_i^2 \mathbf{1}_{\{\theta_i > \delta\}} \right) = \theta_0^2. \tag{20}$$

Recall the definition of G^δ in (9) and that by Theorem 4, G^δ converges uniformly to H^{exc} , as $\delta \downarrow 0$. Therefore, a combination of (18), (19), (20), the union bound and Theorem 4 shows that, jointly with the convergence of the Łukasiewicz path from Theorem 3,

$$\frac{\theta_0^2}{2} \frac{b_n}{V_n} (H_n(V_n U_{(i)}))_{1 \leq i \leq q} \xrightarrow{d} (H^{\text{exc}}(U_{(i)}))_{1 \leq i \leq q}, \quad \text{as } n \rightarrow \infty,$$

that is, the convergence of the first marginal in Theorem 5.

Now we proceed to prove (19). Recall the notation $\text{LR}(u_n) = \text{L}(u_n) + \text{R}(u_n)$ for the total number individuals branching-off the ancestral line of u_n introduced after (2) in Section 2. Fix $\varepsilon, \eta > 0$. By [12, Propositions 4.4 and 4.5] and our assumptions, we can and will consider $K > 0$ such that

$$\mathbb{P}(\text{LR}(u_n) \leq K b_n \text{ and } |u_n| \leq K b_n^{-1} V_n) \geq 1 - \eta.$$

for every n large enough. Then,

$$\mathbb{P}\left(\left|\text{R}^\delta(u_n) - \frac{\sigma_{n,\delta}^2}{2E_n} |u_n|\right| > \varepsilon \sigma_{n,\delta}\right) \leq \eta + \mathbb{P}\left(\left|\text{R}^\delta(u_n) - \frac{\sigma_{n,\delta}^2}{2E_n} |u_n|\right| > \varepsilon \sigma_{n,\delta}, \text{LR}(u_n) \leq K b_n \text{ and } |u_n| \leq K b_n^{-1} V_n\right).$$

Suppose that an urn contains initially $k N_k^n$ balls labeled k for every $k \geq 1$, so E_n balls in total. Let us pick balls repeatedly one after the other without replacement. For every $1 \leq i \leq E_n$, we denote the label of the i th ball by $\xi_n(i)$. Conditionally on $(\xi_n(i), 1 \leq i \leq E_n)$, let us sample independent random variables $(\chi_n(i), 1 \leq i \leq E_n)$ such that each $\chi_n(i)$ is uniformly distributed in $\{1, \dots, \xi_n(i)\}$. The spinal decomposition obtained in [12, Lemma 4.1] (see also [11, Section 3]) with $q = 1$ shows that the probability of the event that $|u_n| = h$ and that for all $0 \leq i < h$, the ancestor of u_n at generation i has k_i offspring and its j_i -th is the ancestor of u_n at generation $i + 1$ is bounded by

$$\frac{1 + \sum_{1 \leq i \leq h} (k_i - 1)}{V_n} \cdot \mathbb{P}\left(\bigcap_{i \leq h} \{(\xi_n(i), \chi_n(i)) = (k_i, j_i)\}\right).$$

Note that, in the previous event, $\text{R}^\delta(u_n) = \sum_{i \leq h} (k_i - j_i) \mathbf{1}_{\{k_i \leq \delta b_{n+1}\}}$. To see this, compare $\text{R}^\delta(u_n)$ with $\text{R}(u_n)$ defined in (2). Note also that, in the previous event, $\text{LR}(u_n) = \sum_{1 \leq i \leq h} (k_i - 1) \leq K b_n$. Thus, by decomposing according to the height of u_n and taking the worst case (i.e. the union bound), we then obtain that,

$$\mathbb{P}\left(\left|\text{R}^\delta(u_n) - \frac{\sigma_{n,\delta}^2}{2E_n} |u_n|\right| > \varepsilon \sigma_{n,\delta}\right) \leq \eta + \left(\frac{K}{b_n} + K^2\right) \sup_{h \leq K V_n / b_n} \mathbb{P}\left(\left|\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}} - \frac{\sigma_{n,\delta}^2}{2E_n} h\right| > \varepsilon \sigma_{n,\delta}\right). \tag{21}$$

Hence to prove (19), it is enough to check the following two estimates:

- (i) $\mathbb{E}\left[\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}}\right] = \frac{\sigma_{n,\delta}^2}{2E_n} h$, and
- (ii) $\text{Var}\left(\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}}\right) \leq \frac{\sigma_{n,\delta}^2}{E_n} \delta b_n h$, for $h \geq 1$.

Indeed, (21) and Chebychev's inequality imply that

$$\mathbb{P}\left(\left|\text{R}^\delta(u_n) - \frac{\sigma_{n,\delta}^2}{2E_n} |u_n|\right| > \varepsilon \sigma_{n,\delta}\right) \leq \eta + \left(\frac{K}{b_n} + K^2\right) \frac{K \delta \sigma_{n,\delta}^2 V_n b_n}{\varepsilon^2 \sigma_{n,\delta}^2 E_n b_n} \leq \eta + \left(\frac{K}{b_n} + K^2\right) \frac{K \delta V_n}{\varepsilon^2 E_n}.$$

Therefore, given that $\eta > 0$ is arbitrary and (A.1), (19) follows by letting $n \rightarrow \infty$ and then $\delta \downarrow 0$.

We start with the proof of (i). Note that, for every $1 \leq i \leq E_n$, $\mathbb{P}(\xi_n(i) = k) = k N_k^n / E_n$, for $k \geq 1$. Moreover, for every $k \geq 1$, $\mathbb{P}(\chi_n(i) = j | \xi_n(i) = k) = k^{-1}$, for $1 \leq j \leq k$. Hence,

$$\mathbb{E}\left[\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}}\right] = h \sum_{k=1}^{\lfloor \delta b_{n+1} \rfloor} \sum_{j=1}^k \frac{(k-j) k N_k^n}{k E_n} = \frac{\sigma_{n,\delta}^2}{2E_n} h.$$

which proves (i). Now, we show (ii). Note that

$$\begin{aligned} \text{Var}\left(\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}}\right) &= \text{Var}\left(\mathbb{E}\left[\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}} \mid (\xi_n(i), 1 \leq i \leq h)\right]\right) \\ &\quad + \mathbb{E}\left[\text{Var}\left(\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}} \mid (\xi_n(i), 1 \leq i \leq h)\right)\right] \\ &= \text{Var}\left(\sum_{i \leq h} \sum_{j=1}^{\xi_n(i)} \frac{(\xi_n(i) - j)}{\xi_n(i)} \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}}\right) \\ &\quad + \mathbb{E}\left[\text{Var}\left(\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_{n+1}\}} \mid (\xi_n(i), 1 \leq i \leq h)\right)\right]. \end{aligned} \tag{22}$$

On the one hand, the variables $\chi_n(1), \dots, \chi_n(E_n)$ are conditionally independent. Then,

$$\begin{aligned}
 & \mathbb{E} \left[\text{Var} \left(\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \mid (\xi_n(i), 1 \leq i \leq h) \right) \right] \\
 &= \mathbb{E} \left[\mathbb{E} \left[\left(\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} - \mathbb{E} \left[\sum_{i \leq h} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \mid (\xi_n(i), 1 \leq i \leq h) \right] \right)^2 \mid (\xi_n(i), 1 \leq i \leq h) \right] \right] \\
 &= \mathbb{E} \left[\mathbb{E} \left[\sum_{i \leq h} \left((\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} - \mathbb{E} \left[(\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \mid (\xi_n(i), 1 \leq i \leq h) \right] \right)^2 \mid (\xi_n(i), 1 \leq i \leq h) \right] \right] \\
 &= \mathbb{E} \left[\sum_{i \leq h} \text{Var} \left((\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \mid (\xi_n(i), 1 \leq i \leq h) \right) \right] \\
 &\leq \mathbb{E} \left[\sum_{i \leq h} \mathbb{E} \left[\left((\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \right)^2 \mid (\xi_n(i), 1 \leq i \leq h) \right] \right] \\
 &= \mathbb{E} \left[\sum_{i \leq h} \sum_{j=1}^{\xi_n(i)} \frac{(\xi_n(i) - j)^2}{\xi_n(i)} \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \right] \\
 &\leq \delta b_n h \mathbb{E} \left[\sum_{j=1}^{\xi_n(1)} \frac{(\xi_n(1) - j)^2}{\xi_n(1)} \mathbf{1}_{\{\xi_n(1) \leq \delta b_n + 1\}} \right] \\
 &\leq \frac{\sigma_{n,\delta}^2}{2E_n} \delta b_n h.
 \end{aligned} \tag{23}$$

On the other hand, recall that the variables $\xi_n(1), \dots, \xi_n(E_n)$ are obtained by successive picks without replacement in an urn, and therefore they are negatively correlated. Then,

$$\begin{aligned}
 \text{Var} \left(\sum_{i \leq h} \sum_{j=1}^{\xi_n(i)} \frac{(\xi_n(i) - j)}{\xi_n(i)} \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} \right) &\leq h \text{Var} \left(\sum_{j=1}^{\xi_n(1)} \frac{(\xi_n(1) - j)}{\xi_n(1)} \mathbf{1}_{\{\xi_n(1) \leq \delta b_n + 1\}} \right) \\
 &\leq h \mathbb{E} \left[\left(\sum_{j=1}^{\xi_n(1)} \frac{(\xi_n(1) - j)}{\xi_n(1)} \mathbf{1}_{\{\xi_n(1) \leq \delta b_n + 1\}} \right)^2 \right] \\
 &\leq h \mathbb{E} \left[\sum_{j=1}^{\xi_n(1)} \frac{(\xi_n(1) - j)^2}{\xi_n(1)} \mathbf{1}_{\{\xi_n(1) \leq \delta b_n + 1\}} \right] \\
 &\leq \frac{\sigma_{n,\delta}^2}{2E_n} \delta b_n h.
 \end{aligned} \tag{24}$$

Therefore, the combination of (22), (23) and (24) proves (ii).

The proof of the full statement in Theorem 5 follows as explained at the end of the proof of [12, Theorem 2.4], which is based on an extension of the previous argument and the spinal decomposition in [12, Lemma 4.1]. Fix $q \geq 2$ an integer and let U_1, \dots, U_q be q i.i.d. uniform random variables in $[0, 1]$ independently of \mathbf{t}_n . For $i = 1, \dots, q$, let $u_{i,n}$ be the $\lfloor V_n U_i \rfloor + 1$ -st vertex of \mathbf{t}_n in lexicographical order, so that $u_{1,n}, \dots, u_{q,n}$ are q i.i.d. uniform random vertices of \mathbf{t}_n (possibly with repetition). Let $\mathbf{T}_n^{(q)}$ be the reduced tree built from \mathbf{t}_n by keeping only the root of \mathbf{t}_n and these q i.i.d. vertices together with their ancestors. Note that $\mathbf{T}_n^{(q)}$ has a random number of branching points $b \in \{0, \dots, q - 1\}$ and a random number of leaves $\hat{q} \in \{1, \dots, q\}$.

Remove from $\mathbf{T}_n^{(q)}$ its \hat{q} leaves and b branching points (if any) to obtain a random number $\hat{q} + b$ of single branches. Denote by $B_n = \{B_n^1, \dots, B_n^{\hat{q}+b}\}$ this collection (i.e., each B_n^i , for $i = 1, \dots, \hat{q} + b$, is a set of vertices). It is worth noting that the sets (“branches”) B_n^i may be empty. In the case $q = 1$, with a single vertex $u_{1,n}$, the set B_n^1 is the ancestral line $\llbracket \emptyset, u_{1,n} \rrbracket$. For every $i = 1, \dots, \hat{q} + b$, we let $v_i(1), \dots, v_i(\#B_n^i)$ be the vertices in B_n^i (if $\#B_n^i \geq 1$) in lexicographical order. So, any vertex $v_i(m)$, for $m = 1, \dots, \#B_n^i$, has $k_{m,i}^n \geq 1$ offspring in the original tree \mathbf{t}_n , and only one of them, the $j_{m,i}^n$ -th say, is an ancestor of (or equal to) at least one the random vertices $u_{1,n}, \dots, u_{q,n}$. In particular, let $\hat{v}_i(\#B_n^i)$ be the $j_{m,i}^n$ -th child of $v_i(\#B_n^i)$.

As before, for a vertex u (say, the k th vertex of \mathbf{t}_n in lexicographical order) that belongs to one of the single branches in B_n , let $R^{\delta}(u)$ be the number of individuals branching-off strictly to the right of the ancestral line $\llbracket \emptyset, u \rrbracket$ in \mathbf{t}_n , except for vertices that are children of vertices with degree larger than $\delta b_n + 1$, i.e.,

$$R^{\delta}(u) = G_n^{\delta}(k), \tag{25}$$

where G_n^{δ} is the modified Łukasiewicz defined in (8). For $i = 1, \dots, \hat{q} + b$, recall that $pr(v_i(1))$ denotes the parent of the vertex $v_i(1)$ and let $\chi_{v_i(1)} \in \{1, \dots, k_{pr(v_i(1))}\}$ be the only index such that $v_i(1) = pr(v_i(1))\chi_{v_i(1)}$ (here, $k_{pr(v_i(1))}$ denotes the number of children of $pr(v_i(1))$). Now, let

$$\hat{R}_{i,n}^{\delta} = \begin{cases} R^{\delta}(\hat{v}_i(\#B_n^i)) - R^{\delta}(pr(v_i(1))) - (k_{pr(v_i(1))} - \chi_{v_i(1)}) \mathbf{1}_{\{|k_{pr(v_i(1))} - 1| > \delta b_n\}} & \text{if } \#B_n^i \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

for $i = 1, \dots, \hat{q} + b$. We claim that, for every $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{i=1, \dots, \hat{q}+b} \left| \hat{R}_{i,n}^\delta - \frac{\sigma_{n,\delta}^2}{2E_n} \#B_n^i \right| > \varepsilon \sigma_{n,\delta} \right) = 0. \tag{27}$$

Therefore, a combination of (25), (26), (27), (20), the union bound and Theorem 4 shows Theorem 5 for $q \geq 2$.

We prove (27) along the lines of the proof of (19). Fix $\varepsilon, \eta > 0$. By [12, Propositions 4.4 and 4.5] and our assumptions, we can and will consider $K > 0$ such that

$$\mathbb{P} \left(\sum_{i=1}^{\hat{q}+b} \sum_{m=1}^{\#B_n^i} (k_{m,i}^n - 1) \leq K b_n \quad \text{and} \quad \sum_{i=1}^{\hat{q}+b} \#B_n^i \leq \frac{K V_n}{b_n} \right) \geq 1 - \eta. \tag{28}$$

for n large enough. Since we have assumed that $b_n \rightarrow \infty$, as $n \rightarrow \infty$, we have that, under the above event in particular, with high probability none of the q random vertices is an ancestor of another (i.e., the q i.i.d. vertices are leaves in the reduced tree), so that $q = \hat{q}$. Recall the definition of the random variables $(\xi_n(i), 1 \leq i \leq E_n)$, and $(\chi_n(i), 1 \leq i \leq E_n)$. The spinal decomposition in [12, Lemma 4.1] with $q \geq 2$ shows that under the above event (i.e., (28)), for any h_0, h_1, \dots, h_{q+b} with $h_0 = 0$ and $h_1 + \dots + h_{q+b} \leq K b_n^{-1} V_n$, and any integers $k_{m,i} \geq j_{m,i} \geq 1$ for every $m \in \{1, \dots, q+b\}$ and $i \in \{1, \dots, h_m\}$, the probability that the tree reduced to the q random vertices and their ancestors has b branching points and that, for every m , we have $\#B_n^m = h_m$ and $(k_{m,i}^n, j_{m,i}^n)_{1 \leq i \leq \#B_n^m} = (k_{m,i}, j_{m,i})_{1 \leq i \leq h_m}$ is upper bounded by

$$C \left(\frac{b_n}{V_n} \right)^{q+b} \mathbb{P} \left(\bigcap_{m=1}^{q+b} \bigcap_{i=1}^{h_m} \{(\xi_n(i), \chi_n(i)) = (k_{m,i}, j_{m,i})\} \right),$$

for some constant $C > 0$ that depends on K (and n large enough). Thus, by the union bound,

$$\begin{aligned} & \mathbb{P} \left(\sup_{i=1, \dots, \hat{q}+b} \left| \hat{R}_{i,n}^\delta - \frac{\sigma_{n,\delta}^2}{2E_n} \#B_n^i \right| > \varepsilon \sigma_{n,\delta} \right) \\ & \leq \eta + C \sum_{b=0}^{q-1} \left(\frac{b_n}{V_n} \right)^{q+b} \sum_{h_1 + \dots + h_{q+b} \leq K b_n^{-1} V_n} \sum_{m=1}^{q+b} \mathbb{P} \left(\left| \sum_{1 \leq i \leq h_m} (\xi_n(i) - \chi_n(i)) \mathbf{1}_{\{\xi_n(i) \leq \delta b_n + 1\}} - \frac{\sigma_{n,\delta}^2}{2E_n} h_m \right| > \varepsilon \sigma_{n,\delta} \right). \end{aligned} \tag{29}$$

Hence, by (i) and (ii) and the Chebychev’s inequality, we have that, for some constant $C' > 0$

$$\mathbb{P} \left(\sup_{i=1, \dots, \hat{q}+b} \left| \hat{R}_{i,n}^\delta - \frac{\sigma_{n,\delta}^2}{2E_n} \#B_n^i \right| > \varepsilon \sigma_{n,\delta} \right) \leq \eta + C' \sum_{b=0}^{q-1} \frac{K^{q+b+1} \delta V_n}{\varepsilon^2 E_n}. \tag{30}$$

Therefore, given that $\eta > 0$ is arbitrary, using (A.1) and Lemma 1, (27) follows by letting $n \rightarrow \infty$ and then $\delta \downarrow 0$. \square

3.3. Gromov–Hausdorff–Prohorov convergence of a specific model of TGDS

In this section, we consider a specific case in which the convergence of trees with given degree sequence holds for the Gromov–Hausdorff–Prohorov topology.

Proposition 1. *Suppose that s_n satisfies (A.1)–(A.4). If moreover,*

$$\lim_{n \rightarrow \infty} \frac{b_n^2}{V_n} = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{N_1^n}{V_n} < 1,$$

then

$$(\mathbf{t}_n, b_n^{-1} r_n^{\text{gr}}, \rho_n, \mu_n) \xrightarrow{d} (\mathcal{T}_\theta, r_\theta, \rho_\theta, \mu_\theta), \quad \text{as } n \rightarrow \infty,$$

for the Gromov–Hausdorff–Prohorov topology, where \mathcal{T}_θ is an Inhomogeneous CRT with parameter set $\theta = (\theta_0, \theta_1, \dots)$.

As mentioned in the introduction, Blanc-Renaudie [18, Theorem 7 (a)] established necessary conditions for Gromov–Hausdorff–Prohorov convergence. Our assumptions (A.1)–(A.4) and the condition $b_n^2/V_n \rightarrow 1$, as $n \rightarrow \infty$, align with [18, Assumption 2] (denoted as $\mathcal{D}_n \Rightarrow \theta$ in [18, Theorem 7]). However, [18] imposed additional technical requirements, specifically [18, Assumption 7]. In particular, [18, Assumption 7 (ii)] imposes a condition on vertices with an out-degree greater than one (corresponding to $\sum_{i \geq 2} N_i^n$ in our notation), which is not a requirement for Proposition 1. Furthermore, [18, Assumption 7 (i)] presents another technical condition that does not appear straightforward to verify under the assumptions of Proposition 1. Therefore, we have included this specific case because its proof, based on the convergence of height processes (see (32) below), differs from the approach in [18] and may be of independent interest. The proof of Proposition 1 builds upon arguments found in the proofs of [11, Theorems 1 and 3].

Proof of Proposition 1. We prove that the family of processes $((b_n^{-1} H_n(V_n u), u \in [0, 1]) : n \geq 1)$ is tight. Since $H_n(0) = 0$, it is enough to check that for any $\varepsilon, \varepsilon' > 0$, there exists $0 < \eta < 1$ such that

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\omega_\eta(H_n) \geq \varepsilon b_n) \leq \varepsilon', \tag{31}$$

where $\omega_\eta(g) = \sup_{|u-u'| \leq \eta} |g(u) - g(u')|$, for a continuous function $g : [0, 1] \rightarrow \mathbb{R}$; see [31, Theorem 2.7.3].

Once tightness is in place, Theorem 5 implies that any subsequential weak limit of $((b_n^{-1} H_n(V_n u), u \in [0, 1]) : n \geq 1)$ is distributed as $\frac{2}{\theta_0^2} H^{\text{exc}}$, i.e.,

$$\left(\frac{\theta_0^2}{2b_n} H_n(V_n u), u \in [0, 1] \right) \xrightarrow{d} (H^{\text{exc}}(u), u \in [0, 1]), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}([0, 1], \mathbb{R}). \tag{32}$$

Then, the claim in Proposition 1 follows, from e.g. [44, Lemma 3.19] and [45, Proposition 3.3].

For two vertices $u, v \in \mathfrak{t}_n$, recall that $u \wedge v$ denotes the first (highest) common ancestor of u and v . Recall also that $|u|$ denotes the height of $u \in \mathfrak{t}_n$. Then,

$$\|u\| - \|v\| \leq \|u\| - \|u \wedge v\| + \|v\| - \|u \wedge v\|. \tag{33}$$

Let $u(0) <_{\text{lex}} u(1) <_{\text{lex}} \dots <_{\text{lex}} u(V_n - 1)$ be the sequence of vertices of \mathfrak{t}_n in lexicographical order, where in particular $u(0)$ is the root of \mathfrak{t}_n . By (33), we deduce that

$$\sup_{|i-j| \leq \eta V_n} |H_n(i) - H_n(j)| \leq 2 + 2 \sup_{\substack{|i-j| \leq \eta V_n \\ u(j) \in \llbracket u(0), u(i) \rrbracket}} |H_n(i) - H_n(j)|, \tag{34}$$

where the supremum is over $i, j \in \{0, \dots, V_n - 1\}$ such that $u(j) \in \llbracket u(0), u(i) \rrbracket$. Recall that $\llbracket u, v \rrbracket$ denotes the unique geodesic path between u and v in \mathfrak{t}_n . For $i, j \in \{0, \dots, V_n - 1\}$ such that $u(j) \in \llbracket u(0), u(i) \rrbracket$, we see that every $v \in \llbracket u(j), u(i) \rrbracket$ which has degree more than one contributes at least one to the number of vertices branching-off the path $\llbracket u(j), u(i) \rrbracket$. So, for $\delta > 0$ and $i, j \in \{0, \dots, V_n - 1\}$ such that $u(j) \in \llbracket u(0), u(i) \rrbracket$, we have that

$$H_n(i) - H_n(j) - J_n(i, j) \leq 1 + \sum_{v \in \llbracket u(j), u(i) \rrbracket} (k_v - 1) \mathbf{1}_{\{|k_v - 1| \leq \delta b_n\}} + \sum_{v \in \llbracket u(j), u(i) \rrbracket} \mathbf{1}_{\{|k_v - 1| > \delta b_n\}}, \tag{35}$$

where

$$J_n(i, j) = \sum_{v \in \llbracket u(j), u(i) \rrbracket} \mathbf{1}_{\{k_v = 1\}}. \tag{36}$$

For $\delta > 0$, let G_n^δ be the modified-Lukasiewicz path defined in (8). We define also $\hat{G}_n^\delta = (\hat{G}_n^\delta(V_n u), u \in [0, 1])$ as the version of the modified-Lukasiewicz path defined as in (8) but with the time-rescaled reverse-Lukasiewicz path $W_n^{\text{rev}} = (W_n^{\text{rev}}(V_n u), u \in [0, 1])$ of \mathfrak{t}_n instead of W_n^{lex} . In particular, the claim in Theorem 4 remains valid if we replace W_n^{lex} and G_n^δ by W_n^{rev} and \hat{G}_n^δ , respectively. It follows from (2), (3) and (4) that, $i, j \in \{0, \dots, V_n - 1\}$ such that $u(j) \in \llbracket u(0), u(i) \rrbracket$,

$$\sum_{v \in \llbracket u(j), u(i) \rrbracket} (k_v - 1) \mathbf{1}_{\{|k_v - 1| \leq \delta b_n\}} \leq G_n^\delta(i + 1) - G_n^\delta(j + 1) + \hat{G}_n^\delta(i + 1) - \hat{G}_n^\delta(j + 1) + 2(k_{u(j)} - 1) \mathbf{1}_{\{|k_{u(j)} - 1| \leq \delta b_n\}} \tag{37}$$

and

$$\sum_{v \in \llbracket u(j), u(i) \rrbracket} \mathbf{1}_{\{|k_v - 1| > \delta b_n\}} \leq \sum_{r=0}^i \mathbf{1}_{\{|\Delta W_n^{\text{lex}}(r+1)| > \delta b_n\}} \leq I_n^\delta = \max\{i \in \{1, \dots, V_n\} : |\Delta W_n^{\text{lex}}(i)| > \delta b_n\}. \tag{38}$$

Then, (34), (35), (37) and (38) allow us to deduce that, for $0 < \eta < 1$,

$$\begin{aligned} & \sup_{|i-j| \leq \eta V_n} |H_n(i) - H_n(j)| \\ & \leq 4 + 2 \sup_{\substack{|i-j| \leq \eta V_n \\ u(j) \in \llbracket u(0), u(i) \rrbracket}} |G_n^\delta(i) - G_n^\delta(j)| + 2 \sup_{\substack{|i-j| \leq \eta V_n \\ u(j) \in \llbracket u(0), u(i) \rrbracket}} |\hat{G}_n^\delta(i) - \hat{G}_n^\delta(j)| + 2I_n^\delta + 4\delta b_n + \sup_{|i-j| \leq \eta V_n} J_n(i, j) \\ & \leq 4 + 2b_n \omega_\eta(b_n^{-1} G_n^\delta) + 2b_n \omega_\eta(b_n^{-1} \hat{G}_n^\delta) + 2I_n^\delta + 4\delta b_n + \sup_{|i-j| \leq \eta V_n} J_n(i, j). \end{aligned} \tag{39}$$

For $0 < \eta < 1$ and $k \in \mathbb{N}$, a sequence $\Delta_k = \{0 = t_0 < t_1 < \dots < t_k = 1\}$ of subdivisions of $[0, 1]$ is called η -sparse if it satisfies $\min_{1 \leq i \leq k} (t_i - t_{i-1}) \geq \eta$. The so-called modified modulus of continuity in $\mathbb{D}([0, 1], \mathbb{R})$ is given by

$$\bar{\omega}_\eta(g) := \inf_{\Delta_k} \max_{1 \leq i \leq k} \sup_{r, r' \in [t_{i-1}, t_i]} |g(r) - g(r')|, \quad \text{for } g \in \mathbb{D}([0, 1], \mathbb{R}),$$

where the infimum extends over all η -sparse sets Δ_k . Note that (see e.g. [31, equation (12.9)])

$$\omega_\eta(b_n^{-1} G_n^\delta) \leq 2\bar{\omega}_\eta(b_n^{-1} G_n^\delta) + \delta \quad \text{and} \quad \omega_\eta(b_n^{-1} \hat{G}_n^\delta) \leq 2\bar{\omega}_\eta(b_n^{-1} \hat{G}_n^\delta) + \delta.$$

On the other hand, it is not difficult to see that, for n large enough, by (A.2)–(A.4) that $I_n^\delta \leq I^\delta = \max\{i \geq 1 : \theta_i > \delta\}$. Hence, by (39),

$$\sup_{|i-j| \leq \eta V_n} b_n^{-1} |H_n(i) - H_n(j)| \leq 4b_n^{-1} + 4\bar{\omega}_\eta(b_n^{-1} G_n^\delta) + 4\bar{\omega}_\eta(b_n^{-1} \hat{G}_n^\delta) + 2I^\delta b_n^{-1} + 6\delta + b_n^{-1} \sup_{|i-j| \leq \eta V_n} J_n(i, j). \tag{40}$$

Given $0 < \eta < 1$ arbitrary, one can choose δ small enough so that the first five terms in (40) are arbitrary small (in probability) when n is large. Indeed, Theorem 3 allows us to bound the terms $\bar{\omega}_\eta(b_n^{-1} G_n^\delta)$ and $\bar{\omega}_\eta(b_n^{-1} \hat{G}_n^\delta)$. It only remains to control the sixth term in

(40), that is, the term concerning the number of vertices of degree one. This is done as in the last part of the [11, Proof of Lemma 8] to which we refer for a detailed argument. Informally, the idea is first sample a tree t_n with degree sequence s_n and then remove from it all the vertices of degree one, then the tightness of the associated height processes follows by (40); one then needs to plug back these vertices of degree one. \square

Interestingly, Proposition 1 shows that the limit of certain TGDSs can be the Brownian CRT. Specifically, it suffices to consider degree sequences where the number of vertices with degree 1 is asymptotically negligible compared to the total number of vertices in the TGDS.

Corollary 3. *Suppose that s_n satisfies (A.1)–(A.4). If moreover,*

$$\lim_{n \rightarrow \infty} \frac{b_n^2}{V_n} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{N_1^n}{V_n} = 0, \tag{41}$$

then

$$(t_n, b_n^{-1} r_n^{gr}, \rho_n, \mu_n) \xrightarrow{d} (\mathcal{T}_{Br}, r_{Br}, \rho_{Br}, \mu_{Br}), \quad \text{as } n \rightarrow \infty,$$

for the Gromov–Hausdorff–Prohorov topology, where \mathcal{T}_{Br} is the Brownian CRT.

Proof. By Proposition 1, it is enough to identify that the limiting Inhomogeneous CRT is indeed the Brownian CRT. This is equivalent to showing that $\sum_{i \geq 1} \theta_i^2 = 0$ (i.e. $\theta_0 = 1$). To see this, note that

$$\begin{aligned} \sum_{i \geq 0} (i-1)^2 N_i^n - V_n &= \sum_{i \geq 0} (i-1)^2 N_i^n - \sum_{i \geq 0} N_i^n = -N_1^n + \sum_{i \geq 3} i(i-2) N_i^n \\ &= -N_1^n + \sum_{i \geq 1} d_n(i)(d_n(i)-2) \mathbf{1}_{\{d_n(i) \geq 3\}} \geq -N_1^n + \sum_{i \geq 1} d_n(i)(d_n(i)-2) \mathbf{1}_{\{d_n(i) > \delta b_n\}}, \end{aligned}$$

for all $\delta > 0$ and n such that $\delta b_n > 2$. Then, (41) and our assumptions (A.1)–(A.4) imply that

$$\sum_{i \geq 1} \theta_i^2 \mathbf{1}_{\{\theta_i > \delta\}} = \liminf_{n \rightarrow \infty} \sum_{i \geq 1} \frac{d_n(i)(d_n(i)-2)}{V_n} \mathbf{1}_{\{d_n(i) > \delta b_n\}} \leq \liminf_{n \rightarrow \infty} \frac{N_1^n}{V_n} = 0,$$

for all $\delta > 0$. This implies that $\sum_{i \geq 1} \theta_i^2 = 0$, which concludes our proof. \square

4. Lamination-valued processes

This section is devoted to the proof of Theorem 2, which establishes the convergence of lamination-valued processes associated with plane trees. We will start by defining the set of laminations in Section 4.1. Then, we will rigorously define laminations for discrete and continuum trees in Sections 4.2 and 4.3, respectively, before presenting the proof of Theorem 2 in Section 4.4.

Throughout this section, whenever we consider a random rooted plane tree τ , we assume that the number of its vertices, $\zeta(\tau)$, is deterministic. Similarly, for a sequence of such trees $(\tau_n, n \geq 1)$, we also assume that $\zeta(\tau_n)$ is deterministic for all $n \geq 1$.

4.1. The set of laminations

Recall that a lamination is a closed subset of the closed unit disk \mathbb{D} which can be written as the union of the unit circle \mathbb{S}^1 and a collection of chords which do not intersect in the open unit disk \mathbb{D} . By definition, a lamination is always compact. In this section, we denote by d_H the Hausdorff distance on the set $\mathbb{K}(\mathbb{D})$ of compact subsets of \mathbb{D} . Since $\mathbb{L}(\mathbb{D}) \subset \mathbb{K}(\mathbb{D})$, $\mathbb{L}(\mathbb{D})$ is also naturally equipped with d_H . We denote by d_{Sk}^L the J_1 Skorohod distance on $\mathbf{D}(\mathbb{R}_+, \mathbb{L}(\mathbb{D}))$; see e.g. [30, Section 5 in Chapter 3] or [31, Chapter 3] for a precise definition.

4.2. The discrete setting

4.2.1. Laminations associated to plane trees

We start by considering rooted plane trees. In this setting, as for fragmentation processes, there are two natural ways to define a lamination-valued process: either the one obtained from removing edges one by one at integer times, or the one that we get when putting i.i.d. variables on edges and removing those whose variable is smaller than a given value.

Definition 1 (Discrete Lamination-Valued Process). Let τ be a rooted plane tree with contour function C_τ and let $(e_1, \dots, e_{\zeta(\tau)-1})$ be a random uniform ordering of its edges. For $k = 1, \dots, \zeta(\tau) - 1$, let g_k and d_k be the first and last times at which the contour function C_τ visits the endpoint of the edge e_k further from the root \emptyset . Associate to e_k the chord $c_k := [e^{-2\pi i g_k / 2\zeta(\tau)}, e^{-2\pi i d_k / 2\zeta(\tau)}] \subset \mathbb{D}$. We define the lamination-valued process $(\mathbb{L}_t(\tau), t \geq 0)$ associated to τ by letting

$$\mathbb{L}_t(\tau) := \mathbb{S}^1 \cup \bigcup_{k=1}^{\lfloor t \wedge (\zeta(\tau)-1) \rfloor} c_k, \quad \text{for } t \in [0, \infty].$$

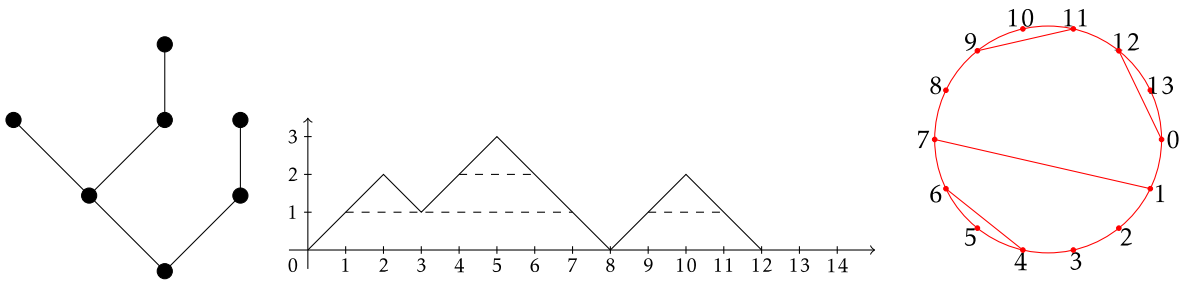


Fig. 2. A tree τ , its contour function C_τ , and the associated lamination $\mathbb{L}(\tau)$.

In particular, the process $(\mathbb{L}_t(\tau), t \geq 0)$ interpolates between \mathbb{S}^1 and $\mathbb{L}(\tau) := \mathbb{L}_\infty(\tau)$; see Fig. 2. We also consider a dynamic continuous-time version of the lamination-valued process $(\mathbb{L}_t(\tau), t \geq 0)$.

Definition 2 (Dynamic Discrete Lamination-Valued Process). Let τ be a rooted plane tree with contour function C_τ and denote by $(e_1, \dots, e_{\zeta(\tau)-1})$ its edges (their ordering is irrelevant). Given the tree τ , its edges are equipped with i.i.d. exponential random variables of parameter 1, say $(\gamma_1, \dots, \gamma_{\zeta(\tau)-1})$. For $k = 1, \dots, \zeta(\tau) - 1$, let g_k and d_k be the first and last times at which the contour function C_τ visits the endpoint of the edge e_k further from the root \emptyset . For $t \geq 0$, we associate to e_k the chord $c_k(t) := [e^{-2\pi i g_k/2\zeta(\tau)}, e^{-2\pi i d_k/2\zeta(\tau)}] \subset \mathbb{D}$ whenever $\gamma_k \leq t$, and otherwise we set $c_k(t) = \mathbb{S}^1$. We define the dynamic lamination-valued process $(\mathbb{L}_t^d(\tau), t \geq 0)$ by letting

$$\mathbb{L}_t^d(\tau) := \mathbb{S}^1 \cup \bigcup_{k=1}^{\zeta(\tau)-1} c_k(t), \quad \text{for } t \in [0, \infty].$$

The process $(\mathbb{L}_t^d(\tau), t \geq 0)$ also interpolates between \mathbb{S}^1 and $\mathbb{L}(\tau) := \mathbb{L}_\infty^d(\tau)$. Furthermore, it is possible to couple $(\mathbb{L}_t(\tau), t \geq 0)$ and $(\mathbb{L}_t^d(\tau), t \geq 0)$ in a natural way so that, under mild assumptions, they are asymptotically close. Indeed, with the same notation as in Definition 2, since the i.i.d. exponential variables $(\gamma_k, 1 \leq k \leq \zeta(\tau) - 1)$ are a.s. distinct, their ordering induces a uniform ordering on the edges of τ . In what follows, we always implicitly consider that we work under this coupling.

Proposition 2. Let $(\tau_n, n \geq 1)$ be a sequence of random rooted plane trees and $(a_n, n \geq 1)$ be a sequence of positive real numbers such that $a_n \rightarrow \infty$ and $\zeta(\tau_n)/a_n \rightarrow \infty$, as $n \rightarrow \infty$. Then, under the coupling defined above, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(d_{\text{Sk}}^{\mathbb{L}} \left(\left(\mathbb{L}_{\frac{t a_n}{\zeta(\tau_n)}}^d(\tau_n), t \geq 0 \right), \left(\mathbb{L}_{t a_n}(\tau_n), t \geq 0 \right) \right) > \varepsilon \right) = 0.$$

Proof. For $1 \leq k \leq \zeta(\tau_n) - 1$, let $\kappa_{n,k}$ be the time at which the k th chord is added in $(\mathbb{L}_{t a_n / \zeta(\tau_n)}^d(\tau_n), t \geq 0)$. Then,

$$\mathbb{L}_{t a_n}(\tau_n) = \mathbb{L}_{\frac{\kappa_{n, \lfloor s a_n \rfloor}}{\zeta(\tau_n)}}^d(\tau_n), \quad \text{for all } t \geq 0.$$

Thus, our claim follows from [31, Theorem 3.9], [46, Theorem 3.1] and [42, Theorem 1.14 in Chapter VI] provided that, for each $t \geq 0$,

$$\sup_{s \in [0, t]} \left| \kappa_{n, \lfloor s a_n \rfloor} - s \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty. \tag{42}$$

Let $\gamma_1, \dots, \gamma_{\zeta(\tau_n)-1}$ be the i.i.d. exponential random variables of parameter 1 of Definition 2, and define the process

$$N_n(s) = \frac{1}{a_n} \sum_{i=1}^{\zeta(\tau_n)-1} \mathbf{1}_{\{E_i \leq s a_n / \zeta(\tau_n)\}}, \quad s \geq 0.$$

An application of the Chebyshev inequality shows that, for $s \geq 0$, $N_n(s) \rightarrow s$, in probability, as $n \rightarrow \infty$. On the other hand, observe that $\kappa_{n, \lfloor s a_n \rfloor} = \inf \{u \geq 0 : a_n N_n(u) \geq \lfloor s a_n \rfloor\}$. Then, by inversion, we have that $\kappa_{n, \lfloor s a_n \rfloor} \rightarrow s$, in probability, as $n \rightarrow \infty$. Moreover, since $\kappa_{n, \lfloor s a_n \rfloor}$ is non-decreasing as a function of s , we obtain (42); see e.g. [47, Lemma 2.2 in Chapter 5]. \square

4.2.2. Reduced tree and reduced lamination built from discrete trees

Fix $q \geq 1$ (an integer) and let u_1, \dots, u_q be q i.i.d. uniform random vertices of a rooted plane tree τ . The reduced tree $\tau^{(q)}$ of τ is obtained by keeping only the root \emptyset of τ , these q vertices and the branching points (if any), i.e. the vertices $w \in \tau$ such that $\llbracket \emptyset, u_i \rrbracket \cap \llbracket \emptyset, u_j \rrbracket = \llbracket \emptyset, w \rrbracket$ for some $1 \leq i < j \leq q$. Then one puts an edge between two vertices of $\tau^{(q)}$ if one is the ancestor of the other in τ , and there is no other vertex of $\tau^{(q)}$ inbetween. The length of an edge e in $\tau^{(q)}$ is defined as the number of edges between the vertices of τ corresponding to the endpoints of e . The tree $\tau^{(q)}$ is rooted at \emptyset and has a plane structure induced by that of τ ; see Fig. 3. Note that its number of vertices is a priori random.



Fig. 3. A tree τ with 4 marked vertices, and the reduced tree $\tau^{(4)}$ with its edge lengths.

The notion of reduced tree naturally translates in the lamination setting into the notion of reduced lamination. Suppose that $\tau^{(q)}$ has exactly q leaves. Let $u_{0,q} := \emptyset$ and $u_{1,q}, \dots, u_{q,q}$ be the q leaves of $\tau^{(q)}$ listed in lexicographical order. Let a_1, \dots, a_q be q i.i.d. uniform random points on the unit circle \mathbb{S}^1 . We define $a_{0,q} = 1$, and let $a_{1,q}, \dots, a_{q,q}$ be these q points sorted in clockwise order. Set $A_q = \{a_{0,q}, a_{1,q}, \dots, a_{q,q}\}$ and $E_q = \{u_{0,q}, u_{1,q}, \dots, u_{q,q}\}$. Observe that removing any edge of $\tau^{(q)}$ splits E_q into two subsets, which are made of consecutive elements of E_q in lexicographical order (up to cyclic shift), corresponding to two subsets of A_q of consecutive points. We now associate to a reduced tree $\tau^{(q)}$ a lamination $\mathbb{L}(\tau^{(q)})$ as follows:

Definition 3 (Discrete reduced laminations). By convention, if $\tau^{(q)}$ does not have exactly q leaves, we set $\mathbb{L}(\tau^{(q)}) := \mathbb{S}^1$. Otherwise, for each $0 \leq i \leq j \leq q$, denote by $A_{i,j}$ the event that there exists an edge in $\tau^{(q)}$ splitting the set E_q into $\{u_{i+1,q}, \dots, u_{j,q}\}$ and $E_q \setminus \{u_{i+1,q}, \dots, u_{j,q}\}$. Define $y_{i,q} := e^{-2i\pi(a_{i,q} + a_{i+1,q})/2}$ and let

$$\mathbb{L}(\tau^{(q)}) := \mathbb{S}^1 \cup \{[y_{i,q}, y_{j,q}] \mid A_{i,j} \text{ holds}\}.$$

It is clear that $\mathbb{L}(\tau^{(q)})$ is a lamination satisfying the following property: for any $0 \leq i < j \leq q$, there exists a chord in $\mathbb{L}(\tau^{(q)})$ between the open arcs $(a_{i,q}, a_{i+1,q})$ and $(a_{j,q}, a_{j+1,q})$ (with the convention that $a_{q+1,q} = a_{0,q}$), if and only if $A_{i,j}$ holds.

We then associate to $\tau^{(q)}$ a random lamination-valued process. Recall that $\ell(e)$ is defined as the number of edges between the vertices of τ corresponding to the endpoints of e . Equip the edges of τ with i.i.d. exponential random variables of parameter 1, and for each edge e of $\tau^{(q)}$, denote by γ_e the minimum of the $\ell(e)$ exponential random variables associated to the edges of τ between the endpoints of e . In particular, $(\gamma_e, e \in \tau^{(q)})$ is a sequence of independent exponential random variables of respective parameters $(\ell(e), e \in \tau^{(q)})$.

Definition 4 (Discrete reduced lamination-valued process). Consider the lamination $\mathbb{L}(\tau^{(q)})$. We define the reduced lamination-valued process $(\mathbb{L}_t^{(q)}(\tau), t \geq 0)$ from $\mathbb{L}(\tau^{(q)})$. Specifically, $\mathbb{L}_t^{(q)}(\tau)$ is the unit circle \mathbb{S}^1 if $\tau^{(q)}$ does not have exactly q leaves. However, if $\tau^{(q)}$ has precisely q leaves, then $\mathbb{L}_t^{(q)}(\tau)$ is the union of \mathbb{S}^1 and the set of chords of $\mathbb{L}(\tau^{(q)})$ corresponding to an edge e if and only if $\gamma_e \leq t$.

In particular, if $\tau^{(q)}$ has precisely q leaves, then this process interpolates between \mathbb{S}^1 ($t = 0$) and the lamination $\mathbb{L}(\tau^{(q)})$ ($t \rightarrow \infty$).

4.3. The continuum setting

In this section, we define lamination-valued processes associated to so-called plane continuum random trees. Let us first recall the notion of \mathbb{R} -tree. A metric space (\mathcal{T}, r) is an \mathbb{R} -tree, if for every $x, y \in \mathcal{T}$: (i) there exists a unique isometry $f_{x,y} : [0, r(x, y)] \rightarrow \mathcal{T}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(r(x, y)) = y$; (ii) for any continuous injective function $g : [0, 1] \rightarrow \mathcal{T}$ such that $g(0) = x$ and $g(1) = y$, we have $g([0, 1]) = f_{x,y}([0, r(x, y)])$. The range of the mapping $f_{x,y}$ is the geodesic between $x, y \in \mathcal{T}$ and is denoted by $\llbracket x, y \rrbracket$. A point $x \in \mathcal{T}$ is called a leaf if $\mathcal{T} \setminus \{x\}$ is connected, and a branching point if $\mathcal{T} \setminus \{x\}$ has at least three disjoint connected components. We denote by $\text{Lf}(\mathcal{T})$ the set of leaves of \mathcal{T} , by $\text{Br}(\mathcal{T})$ its set of branching points, and by $\text{Skel}(\mathcal{T}) := \mathcal{T} \setminus \text{Lf}(\mathcal{T})$ its skeleton. The distance r in \mathcal{T} induces a length measure λ_r on $\text{Skel}(\mathcal{T})$ given by $\lambda_r(\llbracket x, y \rrbracket) = r(x, y)$ for all $x, y \in \mathcal{T}$. A rooted \mathbb{R} -tree (\mathcal{T}, r, ρ) is a \mathbb{R} -tree (\mathcal{T}, r) with a distinguished point $\rho \in \mathcal{T}$ called the root of \mathcal{T} . For $x, y \in \mathcal{T}$, we define $x \wedge y$ as the most recent common ancestor of x and y , that is, the unique $z \in \mathcal{T}$ such that $\llbracket \rho, z \rrbracket = \llbracket \rho, x \rrbracket \cap \llbracket \rho, y \rrbracket$. Finally, for each $x \in \mathcal{T}$, let

$$\mathcal{T}(x) := \{y \in \mathcal{T} : x \in \llbracket \rho, y \rrbracket\}$$

be the subtree of \mathcal{T} above x rooted at x .

Definition 5. A (rooted) continuum tree is a quadruple $(\mathcal{T}, r, \rho, \mu)$, where (\mathcal{T}, r, ρ) is a rooted \mathbb{R} -tree and μ is a non-atomic Borel probability measure on \mathcal{T} such that $\mu(\text{Lf}(\mathcal{T})) = 1$ and for every non-leaf vertex $x \in \mathcal{T}$, $\mu(\mathcal{T}(x)) > 0$. We call μ the mass measure of \mathcal{T} .

In [3], Aldous makes slightly different definitions of these quantities which, in particular, restricts his discussion to binary trees, but the theory can be easily extended. Note that the definition of a continuum tree implies that the \mathbb{R} -tree \mathcal{T} satisfies certain extra properties; for example, $\text{Lf}(\mathcal{T})$ must be uncountable, have no isolated point and λ_r must be σ -finite. In what follows, \mathcal{T} will always denote a continuum tree $(\mathcal{T}, r, \rho, \mu)$. The following result is well known when the space is locally compact, see e.g. [48, Lemma 3.1].

Lemma 2. *The set of branching points $\text{Br}(\mathcal{T})$ of a continuum tree \mathcal{T} is at most countable.*

Proof. By definition, for any branching point y of \mathcal{T} , all connected components of $\mathcal{T} \setminus \{y\}$ which do not contain the root have non-zero μ -mass. For all $k \geq 1$, let B_k be the set of branching points of \mathcal{T} such that at least two connected components of $\mathcal{T} \setminus \{y\}$ not containing the root have μ -mass $> 1/k$. Then, the number of points in B_k has to be less than k . Our claim follows by taking the union over all $k \geq 1$. \square

Lemma 2 will become relevant later when we define a lamination-valued process on a continuum tree \mathcal{T} by adding chords according to a Poisson point process on $\mathcal{T} \times \mathbb{R}_+$ with intensity measure $\lambda_r \times dt$, where dt is the Lebesgue measure on \mathbb{R}_+ (see Section 4.3.1). Since λ_r is the length measure on \mathcal{T} , Lemma 2 ensures that no chord is associated with a branching point of \mathcal{T} . This is convenient because whenever a chord appears, removing the corresponding points in \mathcal{T} splits it into exactly two connected components, and the connection to the fragmentation process arises from this splitting.

The continuum trees that we consider here are equipped with what we call a compatible total (or linear) order; see [49].

Definition 6. We say that a relation \leq on a probability space \mathcal{X} is a total order if:

- For all $x \in \mathcal{X}$, $x \leq x$;
- For all $x, y \in \mathcal{X}$, if $x \leq y$ and $y \leq x$, then $x = y$;
- For all $x, y, z \in \mathcal{X}$, if $x \leq y$ and $y \leq z$, then $x \leq z$;
- The set $\{(x, y) \in \mathcal{X} \times \mathcal{X} : x \leq y\}$ is measurable.

A total order on a continuum tree \mathcal{T} is compatible with (\mathcal{T}, ρ, μ) if the following hold:

- For all $x_1, x_2 \in \mathcal{T}$, if $x_1 \in \llbracket \rho, x_2 \rrbracket$, then $x_1 \leq x_2$.
- For $x_1, x_2, x_3 \in \mathcal{T}$ such that $x_1 \leq x_2 \leq x_3$, if y is the branching point of x_1 on the subtree spanned by ρ, x_2 and x_3 (i.e., $\llbracket \rho, y \rrbracket = \llbracket \rho, x_1 \rrbracket \cap (\llbracket \rho, x_2 \rrbracket \cup \llbracket \rho, x_3 \rrbracket)$), then $y \in \llbracket \rho, x_2 \rrbracket$.
- For any distinct $x_1, x_2 \in \mathcal{T}$ such that $x_1 < x_2$, we have $\mu(\{x \in \mathcal{T} : x_1 < x < x_2\}) > 0$.

In what follows, unless specified, we always consider continuum trees endowed with a compatible linear order \leq . Observe that if μ satisfies the last property in Definition 6, then its topological support $\text{Supp}(\mu)$ is equal to \mathcal{T} . In particular, by adapting e.g. [49, Proposition 2.6], a compatible linear order corresponds to a certain choice of orderings of the I_x 's, $x \in \text{Br}(\mathcal{T}) \cup \{\rho\}$, where I_x stands for the set of connected components of $\mathcal{T} \setminus \{x\}$ that do not contain the root ρ . By a small abuse of notation, we call *plane continuum tree* a continuum tree endowed with a compatible linear order, that is, (\mathcal{T}, \leq) , and call \leq the lexicographical order on \mathcal{T} .

4.3.1. Laminations associated to plane continuum trees

We show here how to associate to a plane continuum tree (\mathcal{T}, \leq) a lamination-valued process $(\mathbb{L}_t(\mathcal{T}), t \geq 0)$. In particular, this construction is valid even when the tree \mathcal{T} is not compact. For $x \in \mathcal{T}$, set

$$G(x) := \mu(\{y \in \mathcal{T} : y \leq x\}), \quad D(x) := \mu(\{y \in \mathcal{T} : y \leq x\} \cup \mathcal{T}(x)), \tag{43}$$

and define the chord $c_x := [e^{-2i\pi G(x)}, e^{-2i\pi D(x)}]$. Note that the set $\{y \in \mathcal{T} : y \leq x\}$ is a Borel set (see for example, the representation in [49, Proof of Proposition 2.10] which is valid even for non-compact plane continuum trees). Note also that if $x, x' \in \mathcal{T}$ such that $x < x'$, then the last property in Definition 6 implies that $G(x) < G(x')$.

Proposition 3. *For any plane continuum tree (\mathcal{T}, \leq) , the set*

$$\mathbb{L}(\mathcal{T}) = \mathbb{S}^1 \cup \overline{\bigcup_{x \in \mathcal{T}} c_x}.$$

is a lamination.

Proof. The set $\mathbb{L}(\mathcal{T})$ is clearly closed. The only thing that we need to prove is that the chords do not cross. Take $x, y \in \mathcal{T}$ such that $G(x) < G(y) < D(x)$. We want to prove that $D(y) \leq D(x)$. Since \leq is a total order and $G(x) < G(y)$, we have that $x \leq y$. First, we claim that x is an ancestor of y . Indeed, assume that $x \wedge y \neq x$ and take $z \in \mathcal{T}(x)$. By definition, $x \leq z$. If $x \leq y \leq z$ then by definition $x := x \wedge z \in \llbracket \rho, y \rrbracket$, which contradicts our assumption. Hence, for all $z \in \mathcal{T}(x)$, $z < y$. But then $D(x) \leq G(y)$ which contradicts our initial assumption. Therefore, x is an ancestor of y . Now take a point $w \in \mathcal{T}$ such that $x < w \leq y$. By Definition 6, the branching point of x on the subtree spanned by ρ, w and y is on $\llbracket \rho, w \rrbracket$. In particular, $w \in \mathcal{T}(x)$. This shows that $\mu(\{x' \in \mathcal{T} : x < x' \leq y\}) \leq \mu(\mathcal{T}(x)) - \mu(\mathcal{T}(y))$ (since $y \leq w$ for all $w \in \mathcal{T}(y)$). This can be rephrased in $D(y) \leq D(x)$. Thus, c_x and c_y do not cross inside the disk. \square

Let also Π be a Poisson point process on $\mathcal{T} \times \mathbb{R}_+$, with intensity measure $\lambda_r \times dt$, where dt is the Lebesgue measure on \mathbb{R}_+ . For any $t \geq 0$, set $\Pi_t := \{x \in \mathcal{T} : \exists s \leq t, (x, s) \in \Pi\}$. We define a lamination-valued process $(\mathbb{L}_t(\mathcal{T}), t \geq 0)$ by letting, for all $t \geq 0$:

$$\mathbb{L}_t(\mathcal{T}) := \mathbb{S}^1 \cup \overline{\bigcup_{x \in \Pi_t} c_x}. \tag{44}$$

4.3.2. Compact continuum trees and excursion-type functions

We consider here a particular case of plane continuum trees, the compact ones. A common way to construct compact \mathbb{R} -trees is from continuous excursion-type functions, i.e. continuous functions $f : [0, 1] \rightarrow \mathbb{R}_+$ such that $f(0) = f(1) = 0$ and $f(x) \geq 0$ for all $0 \leq x \leq 1$. Let f be such a function, consider the pseudo-distance on $[0, 1]$,

$$r_f(x, y) := f(x) + f(y) - 2 \inf_{z \in [x \wedge y, x \vee y]} f(z), \quad \text{for } x, y \in [0, 1],$$

and define an equivalence relation on $[0, 1]$ by setting $x \sim_f y$ if and only if $r_f(x, y) = 0$. The image of the projection $p_f : [0, 1] \rightarrow [0, 1] \setminus \sim_f$ endowed with the pushforward of r_f (again denoted r_f), i.e. $\mathcal{T}_f = (\mathcal{T}_f, r_f, \rho_f) := (p_f([0, 1]), r_f, p_f(0))$, is a rooted plane \mathbb{R} -tree, with the linear order induced by the usual order on $[0, 1]$ (we say that, for $x, y \in \mathcal{T}_f$, $x \leq y$ if $\inf\{p_f^{-1}(x)\} \leq \inf\{p_f^{-1}(y)\}$); see [50, Lemma 3.1]. In particular, (\mathcal{T}_f, r_f) is a compact and connected metric space. Conversely, it has been noted in [51, Remark following Theorem 2.2] (see also [49, Corollary 1.2]) that for every compact \mathbb{R} -tree (\mathcal{T}, r) there exists a continuous excursion-type function $f : [0, 1] \rightarrow \mathbb{R}$ such that (\mathcal{T}, r) and (\mathcal{T}_f, r_f) are isometric. We can endow \mathcal{T}_f with the probability measure μ_f given by the pushforward of the Lebesgue measure on $[0, 1]$ under the projection p_f . Suppose furthermore that the set of one-sided local minima of f has Lebesgue measure 0 (recall that $x \in [0, 1]$ is a one-sided local minimum of f if there exists $\varepsilon > 0$ such that $f(x) = \inf\{f(y) : x \leq y \leq x + \varepsilon\}$ or $f(x) = \inf\{f(y) : x - \varepsilon \leq y \leq x\}$). Then, μ_f is a non-atomic measure and $\mu_f(\text{Lf}(\mathcal{T}_f)) = 1$; see [3, Proof of Theorem 13]. Moreover, $\mathcal{T}_f = (\mathcal{T}_f, r_f, \rho_f, \mu_f)$ is a (rooted) continuum tree.

In this setting, we can construct a lamination $\mathbb{L}(f)$ and a lamination-valued process $(\mathbb{L}_t(f), t \geq 0)$ associated to the continuous excursion-type function f (and thus to \mathcal{T}_f). Let us recall the definition of $(\mathbb{L}_t(f), t \geq 0)$ and refer to [28] for further details. First, define the epigraph of f as the set of points below its graph, that is,

$$\mathcal{EG}(f) := \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), 0 \leq y < f(x)\}.$$

To each $(x, y) \in \mathcal{EG}(f)$, associate the chord $c(x, y) := [e^{-2\pi i g(x, y)}, e^{-2\pi i d(x, y)}] \in \mathbb{D}$, where $g(x, y) := \sup\{z \leq x : f(z) < y\}$ and $d(x, y) := \inf\{z \geq x : f(z) < y\}$. Consider now a Poisson point process \mathcal{N}^f on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure

$$\frac{1}{d(x, y) - g(x, y)} \mathbf{1}_{\{(x, y) \in \mathcal{EG}(f)\}} dx dy ds.$$

Here, ds denotes the Lebesgue measure on \mathbb{R}_+ , while dx and dy both denote the Lebesgue measure on \mathbb{R} . For $t \geq 0$, consider also the Poisson point process $\mathcal{N}_t^f(\cdot) := \mathcal{N}^f(\cdot \times [0, t])$ on $\mathcal{EG}(f)$ and construct the lamination-valued process $(\mathbb{L}_t(f), t \geq 0)$ associated to f as follows. For all $t \geq 0$,

$$\mathbb{L}_t(f) = \mathbb{S}^1 \cup \overline{\bigcup_{(x, y) \in \mathcal{N}_t^f} c(x, y)}.$$

Clearly, this process is non-decreasing for the inclusion. Moreover, define

$$\mathbb{L}_\infty(f) := \mathbb{L}(f) = \bigcup_{t \geq 0} \mathbb{L}_t(f).$$

It is straightforward that chords of the lamination $\mathbb{L}_\infty(f)$ are in bijection with points of \mathcal{T}_f . In particular it is useful to define the process directly from \mathcal{T}_f . For all $x \in \mathcal{T}_f$, recall the definition of $G(x)$ in (43) and observe that $G(x) := \inf\{p_f^{-1}(x)\}$.

Proposition 4. *We have that $(\mathbb{L}_t(\mathcal{T}_f), t \geq 0) \stackrel{d}{=} (\mathbb{L}_t(f), t \geq 0)$.*

Proof. The idea consists in coupling $(\mathbb{L}_t(\mathcal{T}_f), t \geq 0)$ and $(\mathbb{L}_t(f), t \geq 0)$. For any $(x, y) \in \mathcal{N}^f$, let $w(x, y) \in \mathcal{T}_f$ be the equivalence class of $g(x, y)$ with respect to \sim_f . Then, the chord $c(x, y)$ is exactly the chord $c_{w(x, y)}$. Thus, we only need to check that the image of \mathcal{N}^f under the projection p_f is a Poisson point process on $\text{Skel}(\mathcal{T}_f)$ with the correct intensity. To this end, observe that, for any $x', y' \in \mathcal{T}_f$, we have that

$$\int_{E(x', y')} \frac{1}{d(x, y) - g(x, y)} \mathbf{1}_{\{(x, y) \in \mathcal{EG}(f)\}} dx dy = \lambda_r(\llbracket x', y' \rrbracket),$$

where $E(x', y') := \{(x, y) \in \mathbb{R}^2 : w(x, y) \in \llbracket x', y' \rrbracket\}$ (this representation can be found, for example, in [52, Example 4.34]). The result follows. \square

Remark 1. In fact, under the natural coupling defined in the proof of Proposition 4, the lamination-valued processes $(\mathbb{L}_t(\mathcal{T}_f), t \geq 0)$ and $(\mathbb{L}_t(f), t \geq 0)$ are almost surely equal.

In particular, $(\mathbb{L}_t(\mathcal{T}_f), t \geq 0)$ is non-decreasing and it interpolates between \mathbb{S}^1 ($t = 0$) and $\mathbb{L}(\mathcal{T}_f)$ ($t \rightarrow \infty$). Indeed, by [28, Proposition 2.2 (ii)], we have that $\mathbb{L}_t(\mathcal{T}_f) \rightarrow \mathbb{L}(\mathcal{T}_f) = \mathbb{L}_\infty(f)$, as $t \rightarrow \infty$, on $(\mathbb{L}(\mathbb{D}), d_H)$, whenever \mathcal{T} is compact.

4.3.3. Reduced tree and reduced lamination built from continuum trees

For $q \geq 1$ (an integer), let x_1, \dots, x_q be q i.i.d. random leaves of a plane continuum tree \mathcal{T} sampled from its mass measure μ . Observe that they are a.s. all distinct, and set $E_q := \{x_i : 1 \leq i \leq q\} \cup \{\rho\}$. The reduced tree $\mathcal{T}^{(q)}$ of \mathcal{T} is the rooted plane tree with

edge lengths whose vertices are the leaves x_1, \dots, x_q , the root ρ of \mathcal{T} and all branching points. The length of an edge is simply the length measure of the unique geodesic path in \mathcal{T} between the corresponding endpoints.

We can also define the notion of reduced lamination and reduced lamination-valued process in the continuum setting. For $x \in \mathcal{T}$, recall the definition of $G(x)$ in (43). Let $x_{0,q} = \rho$ be the root of $\mathcal{T}^{(q)}$ and $x_{1,q}, \dots, x_{q,q}$ be its q leaves listed in lexicographical order. Set $a_{j,q} := e^{-2i\pi G(x_{j,q})}$, for $0 \leq j \leq q$ (in particular, $a_{0,q} = 1$). The following result must be clear since μ is non-atomic.

Lemma 3. *If $x \in \text{Lf}(\mathcal{T})$ is distributed according to μ , then $G(x)$ is uniformly distributed on $[0, 1]$.*

As a consequence, for all $q \geq 1$, if the q leaves of \mathcal{T} are sampled in an i.i.d. way according to μ , then $a_{1,q}, \dots, a_{q,q}$ are the order statistics of q i.i.d. uniform variables on the unit circle.

Analogously to the discrete case, we now associate to $\mathcal{T}^{(q)}$ a lamination $\mathbb{L}(\mathcal{T}^{(q)})$ which satisfies the following property: for $0 \leq i < j \leq q$, there exists a chord in $\mathbb{L}(\mathcal{T}^{(q)})$ between open arcs $(a_{i,q}, a_{i+1,q})$ and $(a_{j,q}, a_{j+1,q})$ (with the convention that $a_{q+1,q} = a_{0,q}$) if and only if there exists an edge in $\mathcal{T}^{(q)}$ splitting E_q into $\{x_{i+1,q}, \dots, x_{j,q}\}$ and $E_q \setminus \{x_{i+1,q}, \dots, x_{j,q}\}$ (denote this last event by $A_{i,j}$).

Definition 7 (Continuum Reduced Lamination). For each $0 \leq i \leq q$, let $y_{i,q} := e^{-2i\pi(a_{i,q} + a_{i+1,q})/2}$. We let

$$\mathbb{L}(\mathcal{T}^{(q)}) := \mathbb{S}^1 \cup \{[y_{i,q}, y_{i+1,q}] \mid A_{i,j} \text{ holds}\}.$$

It is clear that $\mathbb{L}(\mathcal{T}^{(q)})$ is a lamination satisfying the property that we want. Let us now state and prove a result which will be useful in what follows.

Lemma 4. *For all $a, a' \in \mathbb{S}^1$, let $d(a, a')$ denote the length of the shortest arc from a to a' in \mathbb{S}^1 . For all $\varepsilon > 0$,*

$$\lim_{q \rightarrow \infty} \mathbb{P} \left(\sup_{1 \leq j \leq q} d(a_{j,q}, e^{-2\pi i j/q}) < \varepsilon \right) = 1.$$

Proof. Fix $\varepsilon > 0$, choose an integer $K \geq 1$ such that $2/K < \varepsilon$ and take $q \geq K^8$. Let $A_q := \{a_{j,q}, 0 \leq j \leq q\}$. Split the unit circle into the K arcs of the form, $(e^{-2\pi i(k-1)/K}, e^{-2\pi i k/K})$ for $1 \leq k \leq K$. For $1 \leq j \leq q$, almost surely no point of the form $a_{j,q}$ is one of the endpoints of these arcs. For any $q \geq 2$ and $1 \leq k \leq K$, denote by $A_q(k)$ the set $A_q \cap (e^{-2\pi i(k-1)/K}, e^{-2\pi i k/K})$, and $M_q(k)$ the number of points in $A_q(k)$. In particular, $M_q(k)$ is distributed as a binomial random variable with parameters $(q, 1/K)$. Hoeffding's inequality implies that, for q large enough:

$$\mathbb{P}(|M_q(k) - q/K| \geq q^{3/4}) \leq 2e^{-2\sqrt{q}}.$$

Hence, the probability that $|M_q(k) - q/K| \leq q^{3/4}$, for all $1 \leq k \leq K$, is at least $1 - 2Ke^{-2\sqrt{q}}$ and our claim follows by choosing K large enough (depending on ε). Indeed, suppose that $|M_q(k) - q/K| \leq q^{3/4}$, for all $1 \leq k \leq K$. Then, for any $1 \leq k \leq K$ and any integer $(k-1)q/K < j \leq kq/K$, we necessarily have that $a_{j,q}$ is in $A_q(k-1), A_q(k)$ or $A_q(k+1)$ (with the convention that $A_q(0) = A_q(K)$ and $A_q(K+1) = A_q(1)$) and since $e^{-2\pi i j/q} \in A_q(k)$, the result in Lemma 4 holds. \square

We now associate to $\mathcal{T}^{(q)}$ a lamination-valued process. Recall that Π denotes a Poisson point process on $\mathcal{T} \times \mathbb{R}_+$, with intensity measure $\lambda_r \times dt$, where dt is the Lebesgue measure on \mathbb{R}_+ . For each edge $e \in \mathcal{T}^{(q)}$, let γ_e be the first time at which a point of Π falls on the geodesic of \mathcal{T} that corresponds to e . In particular, γ_e is an exponential random variable of parameter $\ell(e)$ the length of e , and $(\gamma_e, e \in \mathcal{T}^{(q)})$ is a collection of independent random variables.

Definition 8 (Continuum reduced lamination-valued process). We define the process $(\mathbb{L}_t^{(q)}(\mathcal{T}), t \geq 0)$ from $\mathbb{L}(\mathcal{T}^{(q)})$ by letting $\mathbb{L}_t^{(q)}(\mathcal{T})$ be the union of the unit circle \mathbb{S}^1 and the set of chords of $\mathbb{L}(\mathcal{T}^{(q)})$ corresponding to an edge e if and only if $\gamma_e \leq t$.

It turns out that these reduced processes actually approximate the usual lamination-valued process $(\mathbb{L}_t(\mathcal{T}), t \geq 0)$.

Proposition 5. *Let \mathcal{T} be a random rooted plane tree. Then,*

$$(\mathbb{L}_t^{(q)}(\mathcal{T}), t \geq 0) \xrightarrow{d} (\mathbb{L}_t(\mathcal{T}), t \geq 0), \quad \text{as } q \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \mathbb{L}(\mathbb{D})).$$

Proof. We only need to prove that, for all $\varepsilon > 0$ and for every $M > 0$,

$$\lim_{q \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [0, M]} d_H \left(\mathbb{L}_t^{(q)}(\mathcal{T}), \mathbb{L}_t(\mathcal{T}) \right) < \varepsilon \right) = 1.$$

Since the support of λ_r is $\text{Skel}(\mathcal{T})$, we only consider chords c_x that are coded by a point $x \in \text{Skel}(\mathcal{T})$. Moreover, by Lemma 2, a.s. no such x is a branching point. It follows from Lemmas 3 and 4 that for fixed $\varepsilon' > 0$, we can and will choose $Q > 0$ large enough such that, with probability at least $1 - \varepsilon'$, for any $q \geq Q$,

$$\sup_{1 \leq j \leq q} d(a_{j,q}, e^{-2\pi i j/q}) < \varepsilon'. \tag{45}$$

Fix $M > 0$ and consider a point $x \in \Pi_M$. There are two cases: either x falls in the geodesic of \mathcal{T} that corresponds to an edge of $\mathcal{T}^{(q)}$, or not. If x does not fall in such a geodesic, then removing x does not split the set of leaves of $\mathcal{T}^{(q)}$ and thus necessarily $d_H(c_x, \mathbb{S}^1) < 2\pi(2\epsilon' + 1/Q)$ by (45). Now, suppose that x falls in such a geodesic. By definition, since $x \in \Pi_M$, there exists a chord $c' \in \mathbb{L}_M^{(q)}(\mathcal{T})$ corresponding to a point in the same edge of $\mathcal{T}^{(q)}$ as x . Thus, there exist two arcs A_1, A_2 between clockwise consecutive $a_{j,q}$'s such that c_x and c' connect A_1 and A_2 . Hence, by (45) and e.g. [53, Lemma 5.2 (i)], $d_H(c', c_x) < 2\pi(2\epsilon' + 1/Q)$. Finally, our claim follows by choosing ϵ' so that $2\pi(2\epsilon' + 1/Q) < \epsilon$. \square

4.4. Convergence of the process of laminations

In this section, we prove Theorem 2, stating the equivalence between the planar version of the Gromov-weak convergence of trees and the convergence of their associated lamination-valued processes. We first need to introduce some notation and establish some additional geometric properties of reduced trees.

A rooted metric measure space is a quadruple $\mathcal{X} = (\mathcal{X}, r, \rho, \mu)$, where (\mathcal{X}, r) is a metric space such that $(\text{Supp}(\mu), r)$ is complete and separable, the so-called sampling measure μ is a finite measure on (\mathcal{X}, r) and $\rho \in \mathcal{X}$ is a distinguished point which is referred to as the root; the support $\text{Supp}(\mu)$ of μ is defined as the smallest closed set $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $\mu(\mathcal{X}_0) = \mu(\mathcal{X})$. Two rooted metric measure spaces $(\mathcal{X}, r, \rho, \mu)$ and $(\mathcal{X}', r', \rho', \mu')$ are said to be equivalent if there exists an isometry $\phi : \text{Supp}(\mu) \cup \{\rho\} \rightarrow \text{Supp}(\mu') \cup \{\rho'\}$ such that $\phi(\rho) = \rho'$ and $\phi_*\mu = \mu'$, where $\phi_*\mu$ is the pushforward of μ under ϕ . We denote by \mathbb{K}_0 the space of rooted metric measure spaces. We consider that \mathbb{K}_0 is equipped with the Gromov-weak topology; see Gromov's book [54] or [17,55]. In particular, the Gromov-weak topology is metrized by the so-called pointed Gromov-Prohorov metric d_{pGP} . Moreover, $(\mathbb{K}_0, d_{\text{pGP}})$ is a complete and separable metric space; see [55, Proposition 2.6]. Let us give a simple characterization for convergence in the Gromov-weak topology, see e.g. [17,55]. For each $n \in \mathbb{N} \cup \{\infty\}$, consider a rooted metric measure space $\mathcal{X}_n = (\mathcal{X}_n, r_n, \rho_n, \mu_n)$, set $\xi_n(0) = \rho_n$ and $(\xi_n(i), i \geq 1)$ i.i.d. random variables sampled according to μ_n . The convergence $\mathcal{X}_n \rightarrow \mathcal{X}_\infty$, as $n \rightarrow \infty$, for the Gromov-weak topology is equivalent to the convergence in distribution of the matrices

$$(r_n(\xi_n(i), \xi_n(j)) : 0 \leq i, j \leq q) \xrightarrow{d} (r_\infty(\xi_\infty(i), \xi_\infty(j)) : 0 \leq i, j \leq q), \quad \text{as } n \rightarrow \infty, \tag{46}$$

for every fixed integer $q \geq 1$. By Gromov's reconstruction theorem [54, Subsection 3 $\frac{1}{2}$.7], the distribution of $(r_n(\xi_n(i), \xi_n(j)) : i, j \geq 0)$ characterizes (the equivalence class of) \mathcal{X}_n .

We consider the following notion of convergence that is in a sense a planar version of (46).

Definition 9. For $n \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (\mathcal{X}_n, r_n, \rho_n, \mu_n)$ be a rooted metric measure space such that \mathcal{X}_n is a totally ordered set (or linearly ordered set). Set, for all n , $\xi_n(0) = \rho_n$ and $(\xi_n(i), i \geq 1)$ i.i.d. random variables sampled according to μ_n . Then, we say that $\mathcal{X}_n \rightarrow \mathcal{X}_\infty$, as $n \rightarrow \infty$, in the planar Gromov-weak sense if, for all fixed integer $q \geq 1$,

$$(r_n(\xi_n^l(i), \xi_n^l(j)) : 0 \leq i, j \leq q) \xrightarrow{d} (r_\infty(\xi_\infty^l(i), \xi_\infty^l(j)) : 0 \leq i, j \leq q), \quad \text{as } n \rightarrow \infty, \tag{47}$$

where $(\xi_n^l(i))_{0 \leq i \leq q}, (\xi_\infty^l(i))_{0 \leq i \leq q}$ are the respective order statistics of $(\xi_n(i))_{0 \leq i \leq q}, (\xi_\infty(i))_{0 \leq i \leq q}$.

The measurability of the total order ensures that the order statistics of q points are well-defined and measurable. Observe that, in particular, convergence in the planar Gromov-weak sense implies convergence for the Gromov-weak topology. A plane continuum tree $\mathcal{T} = (\mathcal{T}, r, \rho, \mu)$ is a particular case of rooted metric measure space. For $q \geq 1$, let x_1, \dots, x_q be q i.i.d. leaves sampled according to μ . Let $x_{0,q} = \rho$ and $x_{1,q}, \dots, x_{q,q}$ be the q leaves x_1, \dots, x_q of \mathcal{T} listed in lexicographical order. Set $E_q = \{x_{0,1}, \dots, x_{q,q}\}$. For $\omega \subset \{0, \dots, q\}$, let $\Xi_q(\omega)$ be the set of points in \mathcal{T} (if any) whose removal separates the set E_q into $E_q^{\omega,1} = \{x_{k,q} : k \in \omega\}$ and $E_q^{\omega,2} = \{x_{k,q} : k \notin \omega\}$, such that each of these subsets, $E_q^{\omega,1}$ and $E_q^{\omega,2}$, is contained within a single connected component of \mathcal{T} .

The following two lemmas show that the sets of points whose removal splits the set of leaves of the reduced trees into two given subsets are either empty, or a geodesic corresponding to an edge of the reduced tree.

Lemma 5. For $q \geq 1$ and $\omega \subset \{0, \dots, q\}$, we have that $\Xi_q(\omega)$ is either empty or a geodesic of \mathcal{T} that corresponds precisely to an edge e_ω in $\mathcal{T}^{(q)}$ of length $\ell(e_\omega) = \lambda_r(\Xi_q(\omega)) > 0$. Reciprocally, for every edge e in $\mathcal{T}^{(q)}$ of length $\ell(e)$ there exists $\omega_e \subset \{0, \dots, q\}$ such that $\Xi_q(\omega_e)$ is a geodesic of \mathcal{T} that corresponds precisely to e such that $\lambda_r(\Xi_q(\omega_e)) = \ell(e)$.

Furthermore, for any points $x_1, x_2 \in E_q^{\omega,1}$ and $y_1, y_2 \in E_q^{\omega,2}$, define

$$f(x_1, x_2; y_1, y_2) := \frac{r(x_1, y_1) + r(x_1, y_2) + r(x_2, y_1) + r(x_2, y_2)}{4} - \frac{r(x_1, x_2) + r(y_1, y_2)}{2}.$$

Then $\Xi_q(\omega)$ is not empty if and only if $g(\omega) := \min\{f(x_1, x_2; y_1, y_2) : x_1, x_2 \in E_q^{\omega,1}, y_1, y_2 \in E_q^{\omega,2}\} > 0$, in which case $g(\omega) = \ell(e_\omega)$.

Proof. For an edge e in $\mathcal{T}^{(q)}$, we denote by e_g the geodesic in \mathcal{T} that corresponds to e . Let us first prove the first part. Consider $\omega \subset \{0, \dots, q\}$ such that $\Xi_q(\omega)$ is not empty and recall that a point $x \in \Xi_q(\omega)$ splits E_q into $E_q^{\omega,1}$ and $E_q^{\omega,2}$. Observe that x cannot be a branching point of $\mathcal{T}^{(q)}$. Then, $x \in e_g$ for some edge e of $\mathcal{T}^{(q)}$. Take $y \in E_q^{\omega,1}$ and $y' \in E_q^{\omega,2}$. By definition, $x \in \llbracket y, y' \rrbracket$ (otherwise, y and y' are in the same connected component of $\mathcal{T} \setminus \{x\}$). Since the geodesic $\llbracket y, y' \rrbracket$ is injective and connects two vertices of $\mathcal{T}^{(q)}$ in \mathcal{T} , we have $e_g \subset \llbracket y, y' \rrbracket$. In particular, for any $x' \in e_g$ and for any $y \in E_q^{\omega,1}$ and $y' \in E_q^{\omega,2}$, we have that $x' \in \llbracket y, y' \rrbracket$. Thus, $e_g \subset \Xi_q(\omega)$.

Now, assume that there exists another edge $e' \neq e$ such that $\hat{y} \in e'_g$ and $\hat{y} \in \Xi_q(\omega)$. Choose $b \in \llbracket x, \hat{y} \rrbracket$ a branching point of $\mathcal{T}^{(q)}$, and let $z \in E_q$ such that it is not in the same connected component of $\mathcal{T} \setminus \{b\}$ as x nor as \hat{y} (if $b = \rho$, we take instead $z = b$). Assume

without loss of generality that $z \in E_q^{\omega,1}$. Then, for any $z' \in E_q^{\omega,2}$, x and \hat{y} both belong to $\llbracket z, z' \rrbracket$, which contradicts the fact that $\llbracket z, z' \rrbracket$ is a geodesic. Hence, $\Xi_q(\omega) = e_g$.

Conversely, by definition, it is not difficult to see that every edge of $\mathcal{T}^{(q)}$ corresponds to one $\Xi_q(\omega)$, for some $\omega \subset \{0, \dots, q\}$.

We now prove the second part. First, assume that $\Xi_q(\omega)$ is a geodesic that corresponds to an edge e_ω of $\mathcal{T}^{(q)}$ and let x, y be its endpoints, so that $\ell(e_\omega) = r(x, y)$. For $z \in \Xi_q(\omega)$, $\mathcal{T} \setminus \{z\}$ has two connected components, one of them containing x and the other containing y . Without loss of generality, suppose that x is in the connected component of all points of $E_q^{\omega,1}$, and y is in the one of all points of $E_q^{\omega,2}$. Then, for any $x' \in E_q^{\omega,1}$ and $y' \in E_q^{\omega,2}$, we have $r(x', y') = r(x', x) + r(x, y) + r(y, y')$. Thus, for all $x_1, x_2 \in E_q^{\omega,1}$, $y_1, y_2 \in E_q^{\omega,2}$, we get that

$$f(x_1, x_2; y_1, y_2) = \frac{2r(x_1, x) + 2r(x_2, x) + 4r(x, y) + 2r(y_1, y) + 2r(y_2, y)}{4} - \frac{r(x_1, x_2) + r(y_1, y_2)}{2}$$

$$= r(x, y) + \frac{r(x_1, x) + r(x_2, x) - r(x_1, x_2)}{2} + \frac{r(y_1, y) + r(y_2, y) - r(y_1, y_2)}{2}.$$

By the triangle inequality, $f(x_1, x_2; y_1, y_2) \geq r(x, y)$. Furthermore, if ω and $\{0, \dots, q\} \setminus \omega$ both have at least two elements, then x and y are branching points of $\mathcal{T}^{(q)}$, and we can find $x_1, x_2 \in E_q^{\omega,1}$, $y_1, y_2 \in E_q^{\omega,2}$ such that $f(x_1, x_2; y_1, y_2) = r(x, y)$. Otherwise, if ω is a singleton, say $\{i\}$, then $x = x_{i,q}$ and we can find two elements $y_1, y_2 \in E_q^{\omega,2}$ such that $f(x, x; y_1, y_2) = r(x, y)$. The case where $\{0, \dots, q\} \setminus \omega$ is a singleton is handled the same way. Hence, we have proved that if $\Xi(\omega)$ is not empty then $g(\omega) = \ell(e_\omega) > 0$.

Now we assume that $g(\omega) > 0$. If ω is a singleton, say $E_q^{\omega,1} := \{x\}$, then for all $y_1, y_2 \in E_q^{\omega,2}$ such that $y_1 \neq y_2$ we have

$$f(x, x; y_1, y_2) = \frac{r(x, y_1) + r(x, y_2) - r(y_1, y_2)}{2} \geq g(\omega).$$

Hence, by letting b be the branching point of y_1, y_2 and x in $\mathcal{T}^{(q)}$ (that is, $b = \llbracket x, y_1 \rrbracket \cap \llbracket x, y_2 \rrbracket \cap \llbracket y_1, y_2 \rrbracket$), we have $b \neq x$ and $r(x, b) \geq g(\omega)$. Remark that if $x = \rho$ then $x \neq b$ since a point of $\Xi_q(\omega)$ should belong to both $\llbracket \rho, y_1 \rrbracket$ and $\llbracket \rho, y_2 \rrbracket$. Then if x is a leaf of $\mathcal{T}^{(q)}$, for any edge $e \in \mathcal{T}^{(q)}$ and $z \in e_g$ such that $r(x, z) \in (0, g(\omega))$, we see that $z \in \Xi_q(\omega)$. If $x = \rho$, any $z \in \llbracket \rho, y \rrbracket$ for some $y \in E_q^{\omega,2}$ such that $r(x, z) \in (0, g(\omega))$, we have that $z \in \Xi_q(\omega)$.

Finally, assume that ω and $\{0, \dots, q\} \setminus \omega$ are not singletons. Fix $x_1, x_2 \in E_q^{\omega,1}$ such that $x_1 \neq x_2$. First, we prove that for all $y \in E_q^{\omega,2}$, the branching points of x_1, x_2 and y in $\mathcal{T}^{(q)}$ are the same. Indeed, let $y_1 \neq y_2$ be two elements of $E_q^{\omega,2}$ and z_1, z_2 their respective branching points with x_1 and x_2 . Then, if $z_1 \neq z_2$, using the fact that $z_1, z_2 \in \llbracket x_1, x_2 \rrbracket$, we get

$$f(x_1, x_2; y_1, y_2) = -r(z_1, z_2) < 0.$$

Let $a(x_1, x_2)$ be therefore the branching point of x_1, x_2 and y in $\mathcal{T}^{(q)}$, for all $y \in E_q^{\omega,2}$. Symmetrically, for any $y_1, y_2 \in E_q^{\omega,2}$, let $b(y_1, y_2)$ be the branching point of y_1, y_2 and x in $\mathcal{T}^{(q)}$, for all $x \in E_q^{\omega,1}$. If there exists $x_1 \neq x_2 \in E_q^{\omega,1}$, $y_1 \neq y_2 \in E_q^{\omega,2}$ such that $a(x_1, x_2) = b(y_1, y_2)$, then $f(x_1, x_2; y_1, y_2) = 0$ which contradicts our assumption. Choose $x_1, x_2 \in E_q^{\omega,1}$, $y_1, y_2 \in E_q^{\omega,2}$ such that $r(a(x_1, x_2), b(y_1, y_2)) > 0$ is minimum, and take $z \in \llbracket a(x_1, x_2), b(y_1, y_2) \rrbracket \setminus \{a(x_1, x_2), b(y_1, y_2)\}$. If $z \notin \Xi_q(\omega)$ then without loss of generality there exists $x \in E_q^{\omega,1}$ in the component of $\mathcal{T} \setminus \{z\}$ containing y_1 and y_2 . But in this case $a(x_1, x) \in \llbracket a(x_1, x_2), b(y_1, y_2) \rrbracket \setminus \{a(x_1, x_2), b(y_1, y_2)\}$ which contradicts the minimality assumption. Thus, $z \in \Xi_q(\omega)$ which concludes the proof. \square

The previous lemma admits a discrete counterpart. For $n \geq 1$, recall that a rooted plane tree τ_n can also be viewed as a rooted metric measure space $(\tau_n, r_n^{\text{gr}}, \emptyset_n, \mu_n)$. For $q \geq 1$, let u_1^n, \dots, u_q^n be q i.i.d. random vertices sampled according to μ_n . Let $u_{0,q}^n = \emptyset_n$ and $u_{1,q}^n, \dots, u_{q,q}^n$ be the q vertices u_1^n, \dots, u_q^n of τ_n listed in lexicographical order. Set $E_{q,n} = \{u_{0,q}^n, \dots, u_{q,q}^n\}$. For $\omega \subset \{0, \dots, q\}$, let $\Xi_{q,n}(\omega)$ be the set of edges (if any) of τ_n splitting the set $E_{q,n}$ into $E_{q,n}^{\omega,1} = \{u_{k,q}^n : k \in \omega\}$ and $E_{q,n}^{\omega,2} = \{u_{k,q}^n : k \notin \omega\}$.

Lemma 6. For $q \geq 1$ and $\omega \subset \{0, \dots, q\}$, we have that $\Xi_{q,n}(\omega)$ is either empty or a collection of neighboring edges in τ_n that corresponds precisely to an edge $e_{\omega,n}$ in $\tau_n^{(q)}$ of length $\ell(e_{\omega,n})$ given by the number of edges in the set $\Xi_{q,n}(\omega)$. Reciprocally, for every edge e_n in $\tau_n^{(q)}$ of length $\ell(e_n)$ there exists $\omega_n \subset \{0, \dots, q\}$ such that $\Xi_{q,n}(\omega_n)$ is a collection of neighboring edges in τ_n that corresponds precisely to e_n such that number of edges in the set $\Xi_{q,n}(\omega_n)$ is equal to $\ell(e_n)$.

Furthermore, for any points $x_1, x_2 \in E_{q,n}^{\omega,1}$ and $y_1, y_2 \in E_{q,n}^{\omega,2}$, define

$$f_n(x_1, x_2; y_1, y_2) := \frac{r_n^{\text{gr}}(x_1, y_1) + r_n^{\text{gr}}(x_1, y_2) + r_n^{\text{gr}}(x_2, y_1) + r_n^{\text{gr}}(x_2, y_2)}{4} - \frac{r_n^{\text{gr}}(x_1, x_2) + r_n^{\text{gr}}(y_1, y_2)}{2}.$$

Then $\Xi_{q,n}(\omega)$ is not empty if and only if $g_n(\omega) := \min\{f_n(x_1, x_2; y_1, y_2) : x_1, x_2 \in E_{q,n}^{\omega,1}, y_1, y_2 \in E_{q,n}^{\omega,2}\} > 0$, in which case $g_n(\omega) = \ell(e_{\omega,n})$.

Proof. It follows as in the proof of Lemma 5. \square

If $\Xi_{q,n}(\omega)$ is not empty, let $e_{\omega,n}$ be the corresponding edge of $\tau_n^{(q)}$ that splits $E_{q,n}$ into $E_{q,n}^{\omega,1}$ and $E_{q,n}^{\omega,2}$. Denote by $\ell(e_{\omega,n})$ the number of edges in $\Xi_{q,n}(\omega)$, that is, the length of the edge $e_{\omega,n}$, with the convention $\ell(e_{\omega,n}) = 0$ whenever $\Xi_{q,n}(\omega)$ is empty. If $\Xi_q(\omega)$ is not empty, let e_ω be the corresponding edge of $\mathcal{T}^{(q)}$ that splits E_q into $E_q^{\omega,1}$ and $E_q^{\omega,2}$. Denote by $\ell(e_\omega)$ the length of the geodesic $\Xi_q(\omega)$; with the convention $\ell(e_\omega) = 0$ whenever $\Xi_q(\omega)$ is empty. The following lemma, whose proof makes use of Lemmas 5 and 6, states the equivalence of the convergence of a sequence of discrete trees and the convergence of the lengths of edges of the reduced trees.

Lemma 7. Let $(\tau_n, n \geq 1)$ be a sequence of random rooted plane trees, \mathcal{T} be a random plane continuum tree and $(a_n, n \geq 1)$ a sequence of non-negative real numbers satisfying $a_n \rightarrow \infty$ and $\zeta(\tau_n)/a_n \rightarrow \infty$, as $n \rightarrow \infty$. Then, the following assertions are equivalent:

- (i) $\left(\tau_n, \frac{a_n}{\zeta(\tau_n)} t_n^{\text{gr}}, \theta_n, \mu_n\right) \rightarrow (\mathcal{T}, r, \rho, \mu)$, as $n \rightarrow \infty$, in the planar Gromov-weak sense.
- (ii) For every integer $q \geq 1$ fixed, $\left(\frac{a_n}{\zeta(\tau_n)} \ell(e_{\omega,n}) : \omega \subset \{0, \dots, q\}\right) \xrightarrow{d} (\ell(e_\omega) : \omega \subset \{0, \dots, q\})$, as $n \rightarrow \infty$.

Proof of Lemma 7. For all $0 \leq i, j \leq q$, let $\Omega_n(i, j)$ be the set of subsets of $\omega \subset \{0, \dots, q\}$ such that $u_{i,q}^n \in E_{q,n}^{\omega,1}$ and $u_{j,q}^n \in E_{q,n}^{\omega,2}$. Similarly, let $\Omega(i, j)$ be the set of subsets of $\omega \subset \{0, \dots, q\}$ such that $x_{i,q} \in E_q^{\omega,1}$ and $x_{j,q} \in E_q^{\omega,2}$. Our claim then follows from (47), Lemma 5, Lemma 6 and the identities

$$r_n^{\text{gr}}(u_{i,q}^n, u_{j,q}^n) = \sum_{\omega \in \Omega_n(i,j)} \ell(e_{\omega,n}) \quad \text{and} \quad r(x_{i,q}, x_{j,q}) = \sum_{\omega \in \Omega(i,j)} \ell(e_\omega). \quad \square$$

The rest of the section is devoted to the proof of Theorem 2. Let us start by proving that (C.2) implies (C.1). As a preparation step, we need the following proposition.

Proposition 6. In the setting of Theorem 2, suppose that (C.2) is satisfied. Then, we have that:

- (i) there exists a coupling between the lamination-valued processes such that

$$\limsup_{n \rightarrow \infty} d_{\text{Sk}}^{\mathbb{L}} \left(\left(\mathbb{L}_{\frac{t a_n}{\zeta(\tau_n)}}^{(q)}(\tau_n), t \geq 0 \right), \left(\mathbb{L}_t^{(q)}(\mathcal{T}), t \geq 0 \right) \right) \xrightarrow{q \rightarrow \infty} \mathbb{P} 0;$$

- (ii) for any $\varepsilon > 0$,

$$\lim_{q \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left(d_{\text{Sk}}^{\mathbb{L}} \left(\left(\mathbb{L}_{\frac{t a_n}{\zeta(\tau_n)}}^{(q)}(\tau_n), t \geq 0 \right), \left(\mathbb{L}_{t a_n}(\tau_n), t \geq 0 \right) \right) < \varepsilon \right) = 1.$$

Roughly speaking, (i) states that, as $q \rightarrow \infty$, the time-rescaled discrete reduced lamination-valued process obtained from sampling q i.i.d. uniform vertices of τ_n is asymptotically close to its continuum counterpart $(\mathbb{L}_t^{(q)}(\mathcal{T}), t \geq 0)$. On the other hand, by (ii), the complete process $(\mathbb{L}_{t a_n}(\tau_n), t \geq 0)$ associated to τ_n can be approximated by the time-rescaled discrete reduced lamination-valued process, whenever q is large enough.

Proof of Theorem 2 ((C.2) \Rightarrow (C.1)). It follows from Propositions 5 and 6. \square

We now prove Proposition 6.

Proof of Proposition 6 (i). We can and will assume, by Skorohod’s representation theorem, that (C.2) holds almost surely. By (C.2), we have that for all $\varepsilon > 0$, $\mathbb{P}(|u_n| > \varepsilon \zeta(\tau_n)) \rightarrow 0$ as $n \rightarrow \infty$, where u_n denotes a uniform vertex of τ_n . This implies that, for all $q \in \mathbb{N}$ fixed, $\tau_n^{(q)}$ has q leaves with high probability as $n \rightarrow \infty$.

In the setting of Lemma 7, for any $\omega \subseteq \{0, \dots, q\}$, we define $\gamma_{\omega,n}$ an exponential variable of parameter $\ell(e_{\omega,n})$ associated to the edge $e_{\omega,n}$ of $\tau_n^{(q)}$ whenever $\Xi_{q,n}(\omega)$ is not empty, otherwise we let $\gamma_{\omega,n} = \infty$ almost surely. Those exponential random variables correspond to the ones used in the definition of $(\mathbb{L}_t^{(q)}(\tau_n), t \geq 0)$. Similarly, denote by γ_ω the exponential variable of parameter $\ell(e_\omega)$ associated to the edge e_ω of $\mathcal{T}^{(q)}$ whenever $\Xi_q(\omega)$ is not empty, otherwise $\gamma_\omega = \infty$ almost surely. The latter exponential random variables correspond to the ones used in the definition of $(\mathbb{L}_t^{(q)}(\mathcal{T}), t \geq 0)$. By Lemma 7, it follows, jointly with (C.2), that the following convergence holds in $[0, \infty]^{2^q}$, where Ω_q denotes the set of subsets of $\{0, \dots, q\}$:

$$\left(\frac{\zeta(\tau_n)}{a_n} \gamma_{\omega,n} : \omega \subset \{0, \dots, q\} \right) \xrightarrow{d} (\gamma_\omega : \omega \subset \{0, \dots, q\}), \quad \text{as } n \rightarrow \infty. \tag{48}$$

This implies that the “jump times” of the time-rescaled discrete reduced lamination-valued process converge to those of the continuum reduced lamination-valued process. Here, “jump times” refers to the times a new chord is added in the corresponding reduced lamination-valued processes. In particular, if $\gamma_{\omega,n} = \infty$ (resp. $\gamma_\omega = \infty$), then no chord associated to ω is added in the reduced lamination-valued processes (no jump), i.e., there is no edge in $\tau_n^{(q)}$ (resp. $\mathcal{T}^{(q)}$) associated to ω . In fact, the “actual jump times” that will count towards the limit are those $\gamma_{\omega,n}$ ’s and γ_ω ’s for which $\ell(e_\omega) > 0$.

We can now assume, by Skorohod’s representation theorem, that (C.2) and (48) hold almost surely. Denote by Θ_∞ the class of strictly increasing, continuous mappings $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\theta(0) = 0$ and $\theta(t) \uparrow \infty$, as $t \uparrow \infty$. Then to prove our claim, it is enough to show that there exists a sequence of functions $(\theta_n, n \geq 1) \in \Theta_\infty$ such that, for all $M \geq 0$,

$$\limsup_{n \rightarrow \infty} \sup_{t \geq 0} |\theta_n(t) - t| = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \sup_{t \in [0, M]} d_H \left(\mathbb{L}_{\frac{\theta_n(t) a_n}{\zeta(\tau_n)}}^{(q)}(\tau_n), \mathbb{L}_t^{(q)}(\mathcal{T}) \right) \xrightarrow{q \rightarrow \infty} \mathbb{P} 0;$$

see [30, Proposition 3.5.3] (or [42, Theorem 1.14 in Chapter VI]).

Let $(s_j : j = 1, \dots, J) := \{\gamma_\omega \mid \ell(e_\omega) > 0\}$ be the sequence of “actual jump times” of the continuum reduced lamination-valued process arranged in increasing order. Similarly, let $(s_j^n : j = 1, \dots, J) := \{\gamma_{\omega,n} \mid \ell(e_{\omega,n}) > 0\}$ be the corresponding sequence of “actual

jump times” of the time-rescaled reduced lamination-valued process arranged in increasing order. Set $s_0 = 0$ and define θ_n by letting $\theta_n(s_0) = 0$, $\theta_n(s_j) = s_j^n$, for $j = 1, \dots, J$, such that θ_n is linear on $[s_j, s_{j+1}]$ for $j = 0, 1, \dots, J - 1$ and $[s_J, \infty)$ (with slope 1 after s_J). Clearly, (48) implies that θ_n converges uniformly to the identity mapping on $[0, \infty)$, as $n \rightarrow \infty$. Thus, it only remains to check that the chords that we add at “actual jump times” are asymptotically close to each other. Consider $\omega \subset \{0, \dots, q\}$ such that $\ell(e_\omega) > 0$. Let $e_{\omega,n}$ be the corresponding edge of $\tau_n^{(q)}$ (which exists for n large enough). Denoting by $c_{\omega,n}$ and c_ω the chords coding respectively $e_{\omega,n}$ and e_ω , we have that

$$\sup_{t \in [0, M]} d_H \left(\mathbb{L}_{\frac{\theta_n(t)\omega_n}{\zeta(\tau_n)}}^{(q)}(\tau_n), \mathbb{L}_t^{(q)}(\mathcal{T}) \right) \leq \sup_{\omega \subset \{0, \dots, q\}} d_H(c_{\omega,n}, c_\omega). \tag{49}$$

Furthermore, Lemma 4 implies that $\limsup_{n \rightarrow \infty} \sup_{\omega \subset \{0, \dots, q\}} d_H(c_{\omega,n}, c_\omega) \rightarrow 0$, in probability, as $q \rightarrow \infty$. This concludes our proof. \square

In order to prove Proposition 6 (ii), we make use of the following lemma:

Lemma 8. *Let $(\tau_n, n \geq 1)$ be a sequence of random rooted plane trees such that $\zeta(\tau_n) \rightarrow \infty$, as $n \rightarrow \infty$. Suppose that, for any $\eta > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(|u_n| > \eta \zeta(\tau_n)) = 0, \tag{50}$$

where u_n is a uniformly distributed vertex in τ_n . For a vertex $u \in \tau_n$, let $\tau_n(u)$ be the subtree of τ_n rooted at u (i.e., $\tau_n(u)$ consists of u and all its descendants) and let $\zeta(\tau_n(u))$ be its size (i.e., number of vertices). Then, for all $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\zeta(\tau_n(u_n)) > \varepsilon \zeta(\tau_n)) = 0.$$

Proof. Fix $\varepsilon > 0$, and set $\Theta_n(\varepsilon) := \{v \in \tau_n : \zeta(\tau_n(v)) > \varepsilon \zeta(\tau_n)\}$. Call $v \in \Theta_n(\varepsilon)$ maximal if no child of v belongs to $\Theta_n(\varepsilon)$. Then, it is clear that all subtrees rooted at a maximal vertex of $\Theta_n(\varepsilon)$ are disjoint. Hence the number of maximal vertices is $\leq \varepsilon^{-1}$. Now observe that, as $n \rightarrow \infty$,

$$\sup_{v \in \Theta_n(\varepsilon)} |v| = o(\zeta(\tau_n)). \tag{51}$$

Indeed, clearly $\Theta_n(\varepsilon)$ is nonempty and, for any $v \in \Theta_n(\varepsilon)$, $\mathbb{P}(u_n \in \tau_n(v)) \geq \varepsilon > 0$. Furthermore, if $u_n \in \tau_n(v)$, then $|u_n| \geq |v|$. Thus, $\mathbb{P}(|u_n| \geq \sup_{v \in \Theta_n(\varepsilon)} |v|) \geq \varepsilon$, which implies (51) by the assumption (50). Finally, letting $M_n(\varepsilon)$ be the number of maximal vertices in $\Theta_n(\varepsilon)$, we have

$$\#\Theta_n(\varepsilon) \leq M_n(\varepsilon) \sup_{v \in \Theta_n(\varepsilon)} |v| = o(\zeta(\tau_n)).$$

In particular, $\mathbb{P}(u_n \in \Theta_n(\varepsilon)) \rightarrow 0$, as $n \rightarrow \infty$, which implies our claim. \square

Proof of Proposition 6 (ii). By Proposition 2, it is enough to prove that, for any $\varepsilon > 0$:

$$\lim_{q \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left(d_{\text{Sk}}^{\mathbb{L}} \left(\left(\mathbb{L}_{\frac{t\omega_n}{\zeta(\tau_n)}}^{(q)}(\tau_n), t \geq 0 \right), \left(\mathbb{L}_{\frac{t\omega}{\zeta(\tau_n)}}^d(\tau_n), t \geq 0 \right) \right) < \varepsilon \right) = 1.$$

Fix $q \geq 1$ and recall that, by (C.2), we can assume that $\tau_n^{(q)}$ has exactly q leaves. Consider $\omega \subset \{0, \dots, q\}$ such that $\Xi_{q,n}(\omega)$ is not empty. Then, the time at which the chord coding the edge $e_{\omega,n}$ in $\tau_n^{(q)}$ appears in the lamination-valued process $(\mathbb{L}_t^{(q)}(\tau_n), t \geq 0)$ is distributed as an exponential variable of parameter $\ell(e_{\omega,n})$ (which is the minimum of the $\ell(e_{\omega,n})$ exponential random variables of parameter 1 associated to the set of edges $\Xi_{q,n}(\omega)$; see also Definition 4). Hence, we only need to prove that for all $\varepsilon > 0$ there exists Q large enough so that with probability $> 1 - \varepsilon$, for all $q \geq Q$, all n large enough,

- (a) for any $\omega \subset \{0, \dots, q\}$ such that $\Xi_{q,n}(\omega)$ is not empty, and for any edge $e_n \in \Xi_{q,n}(\omega)$, $d_H(c_n, \tilde{c}_{\omega,n}) < \varepsilon$, where c_n is the chord of $\mathbb{L}(\tau_n)$ coding e_n and $\tilde{c}_{\omega,n}$ is the chord of $\mathbb{L}^{(q)}(\tau_n)$ coding $e_{\omega,n}$;
- (b) for any edge $e_n \in \tau_n$ whose removal does not split the set $E_{q,n}$, $d_H(c_n, \mathbb{S}^1) < \varepsilon$;

Fix $\varepsilon > 0$. Then, by Lemma 4, we can take Q large enough such that, for all $q \geq Q$,

$$\mathbb{P} \left(\sup_{1 \leq j \leq q} d(a_{j,q}, e^{-2\pi i j/q}) < \varepsilon \right) > 1 - \varepsilon, \tag{52}$$

where we recall that $(a_{j,q})_{1 \leq j \leq q}$ are q i.i.d. points on \mathbb{S}^1 sorted in clockwise order (starting from 1). In particular, observe that (a) follows directly from (52) along with Definition 3. So, it only remains to prove (b). For a vertex $u \in \tau_n$, recall that $\tau_n(u)$ is the subtree of τ_n rooted at u , i.e., $\tau_n(u)$ consists of u and all its descendants. Observe that the edges considered in (b) are of two kinds: either they are in a subtree of the form $\tau_n(u_{k,q}^n)$ for some $1 \leq k \leq q$, or they are in subtrees branching out of the set of edges that are in the geodesic paths of $\tau_n^{(q)}$.

To deal with the edges of the first kind, observe that with probability $1 - o(1)$, as $n \rightarrow \infty$, all subtrees $(\tau_n(u_{k,q}^n), 1 \leq k \leq q)$ have size $o(\zeta(\tau_n))$. This follows from (C.2), since $|u_n| = o(\zeta(\tau_n))$, in probability, where u_n denotes a uniform vertex of τ_n . Then, by Lemma 8, with probability $1 - o(1)$, all chords corresponding to edges of the first kind have length $o(1)$, as $n \rightarrow \infty$.

To deal with edges of the second kind, we use the definition of the lamination-valued process from the contour function C_{τ_n} of τ_n ; see Definition 1 or Definition 2. First, we recall a way of sampling a uniform vertex of τ_n . Consider a uniform random variable U on $[0, 1]$. Then, let e_U be the edge of τ_n visited at time $2\zeta(\tau_n)U$ by C_{τ_n} , and let v_U be the endpoint of e_U further from the root (if $2\zeta(\tau_n)U \geq 2\zeta(\tau_n) - 2$, set $v_U = \emptyset_n$). The vertex v_U is clearly uniform among the vertices of τ_n . We use this procedure to sample the q uniform vertices u_1^n, \dots, u_q^n of τ_n from q i.i.d. uniform random variables U_1, \dots, U_q . We get that chords that code edges of the second kind necessarily have their two endpoints between two consecutive points on \mathbb{S}^1 (in clockwise order) of the set $\{e^{-2\pi i U_k} : 0 \leq k \leq q\}$ (with the convention that $U_0 = 0$). Finally, we conclude by Lemma 4. \square

We can now prove the other implication in Theorem 2, that is, the convergence of the lamination-valued process implies the planar Gromov-weak convergence of the rooted plane trees.

Proof of Theorem 2 ((C.1) \Rightarrow (C.2)). By Skorohod’s representation theorem, suppose that (C.1) holds almost surely. In particular, (C.1) and Proposition 2 imply that

$$\left(\mathbb{L}_{\frac{\zeta(\tau_n)}{a_n}}^d(\tau_n), t \geq 0 \right) \xrightarrow{a.s.} (\mathbb{L}_1(\mathcal{T}), t \geq 0), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \mathbb{L}(\mathbb{D})). \tag{53}$$

Fix $q \geq 1$ and recall that we set $x_{0,q} = \rho$ and $x_{1,q}, \dots, x_{q,q}$ the q i.i.d. leaves of \mathcal{T} distributed according μ and listed in lexicographical order. Recall that for $x \in \mathcal{T}$, we denote by $G(x)$ the μ -mass of the set of leaves of \mathcal{T} that lie on the left of x ; see (43). For $n \geq 1$, we couple the reduced trees $\tau_n^{(q)}$ as follows. For $1 \leq k \leq q$, denote by $u_{k,q}^n$ the unique vertex of τ_n such that the edge between $u_{k,q}^n$ and its parent is visited at time $2\zeta(\tau_n) \cdot G(x_{k,q})$ by the contour function of τ_n . If $2\zeta(\tau_n)G(x_{k,q}) \geq 2\zeta(\tau_n) - 2$, set $u_{k,q}^n = \emptyset_n$. It follows from Lemma 3 that the vertices $u_{1,q}^n, \dots, u_{q,q}^n$ are i.i.d. uniform vertices of τ_n in lexicographical order. Recall also that we write $u_{0,q}^n = \emptyset_n$.

Let us prove that the sequence of properly rescaled rooted plane trees converges in the planar Gromov-weak sense towards \mathcal{T} . To be precise, we check that (46) is satisfied in this setting - or equivalently Lemma 7 (ii).

By Lemma 3, sampling $x_{1,q}, \dots, x_{q,q}$ is equivalent to sample the order statistics of q i.i.d. uniform points on \mathbb{S}^1 , say $a_{1,q}, \dots, a_{q,q}$, by letting $a_{k,q} = e^{-2\pi i G(x_{k,q})}$, for $1 \leq k \leq q$. We also set $a_{0,q} = e^{-2\pi i G(x_{0,q})} = 1$. Recall from Lemma 5, that for any $\omega \subset \{0, \dots, q\}$ such that $\Xi_q(\omega)$ is not empty, the set $\Xi_q(\omega)$ is a geodesic of \mathcal{T} that corresponds to an edge e_ω of $\mathcal{T}^{(q)}$ that splits E_q into $E_q^{\omega,1}$ and $E_q^{\omega,2}$. By the definition of the continuous lamination-valued process in (44), any point of the Poisson point process Π on \mathcal{T} falling on $\Xi_q(\omega)$ is coded by a chord splitting $A_q := \{a_{0,q}, \dots, a_{q,q}\}$ into $\{a_{k,q} : k \in \omega\}$ and $\{a_{k,q} : k \notin \omega\}$. Denote by c_ω the first such chord in the lamination-valued process $(\mathbb{L}_1(\mathcal{T}), t \geq 0)$ and by T_ω the time at which it appears. For all $n \geq 1$, let also $T_{\omega,n}$ be the time at which the first such chord, say $c_{\omega,n}$, appears in the lamination-valued process $(\mathbb{L}_1^d(\tau_n), t \geq 0)$. If there is no such edge e_ω (i.e., $\Xi_q(\omega)$ is empty) or no chord $c_{\omega,n}$, set $T_\omega = \infty$ and $T_{\omega,n} = \infty$, respectively. Therefore, Theorem 2, (C.1) \Rightarrow (C.2) follows by showing that the following convergence holds in $[0, \infty]^{2^q}$:

$$\left(\frac{\zeta(\tau_n)}{a_n} T_{\omega,n} : \omega \subset \{0, \dots, q\} \right) \xrightarrow{d} (T_\omega : \omega \subset \{0, \dots, q\}), \quad \text{as } n \rightarrow \infty. \tag{54}$$

Indeed, in the setting of Lemma 7, for $\omega \subset \{0, \dots, q\}$ such that $\Xi_q(\omega)$ and $\Xi_{q,n}(\omega)$ are not empty, observe that $T_{\omega,n}$ and T_ω are distributed as exponential random variables of respective parameters $\ell(e_{\omega,n})$ and $\ell(e_\omega)$. Then, it is a simple exercise to check that (54) implies the statement of Lemma 7 (ii) and therefore our result.

Let us then prove (54). Observe that (54) is clear for the ω ’s such that $T_\omega = \infty$. Indeed, if $T_{\omega,n}$ was bounded by some $K > 0$ along a subsequence then by (53) the sequence of associated chords would converge (up to taking again a subsequence) towards a chord which would appear in the continuous lamination-valued process before time K . Furthermore, this sequence of chords cannot degenerate into a point, since this point would be a leaf (or the root) of $\mathcal{T}^{(q)}$ and $T_\omega < \infty$ for any singleton ω . Therefore, we only have to focus on the case $T_\omega < \infty$.

By (53), necessarily $\liminf_{n \rightarrow \infty} (\zeta(\tau_n)/a_n) T_{\omega,n} \geq T_\omega$ almost surely. On the other hand, let us prove that for every $\omega \subset \{0, \dots, q\}$ such that $\Xi_q(\omega)$ is not empty, the chord c_ω is necessarily well approximated by a sequence of chords in the discrete lamination-valued processes. To this end, let (a_1, a_2) and (b_1, b_2) be the two arcs between consecutive points of A_q connected by c_ω (with $a_1, a_2, b_1, b_2 \in A_q$ in this clockwise order), and denote by p_ω the middle of the chord c_ω . Suppose that there exists a subsequence $(n_m, m \geq 1)$ of non-negative integers such that, for all $m \geq 1$, there exists a chord c_m in $\mathbb{L}_{\frac{\zeta(\tau_{n_m})}{a_{n_m}(T_\omega+1/m)}}^d(\tau_{n_m})$ satisfying $d(p_\omega, c_m) \leq m^{-1}$ and that does not connect the arcs (a_1, a_2) and (b_1, b_2) ; here $d(p_\omega, c_m)$ denotes the distance from the point p_ω to the set c_m . Up to taking a subsequence, we can assume that c_m has an endpoint in the arc (a_2, b_1) . Hence, since by (53), $\mathbb{L}_{\frac{\zeta(\tau_{n_m})}{a_{n_m}(T_\omega+1/m)}}^d(\tau_{n_m}) \rightarrow \mathbb{L}_{T_\omega}(\mathcal{T})$, as $m \rightarrow \infty$, almost surely, there would exist a chord in $\mathbb{L}_{T_\omega}(\mathcal{T})$ containing p_ω and with an endpoint in (a_2, b_1) , and thus crossing c_ω . However, the above necessarily does not happen and, along all sub-sequences $(n_m, m \geq 1)$, for m large enough, a chord c_m of $\mathbb{L}_{\frac{\zeta(\tau_{n_m})}{a_{n_m}(T_\omega+1/m)}}^d(\tau_{n_m})$ such that $d(p_\omega, c_m) < m^{-1}$ connects the arcs (a_1, a_2) and (b_1, b_2) . Therefore, we get that $\limsup_{n \rightarrow \infty} (\zeta(\tau_n)/a_n) T_{\omega,n} \leq T_\omega$ almost surely. This concludes the proof of (54). \square

5. Fragmentation and process of masses

The aim of this final section is to provide sufficient conditions on a sequence of rooted plane trees $(\tau_n, n \geq 1)$ to ensure that their appropriately rescaled fragmentation processes converge to a fragmentation process constructed from an excursion-type function, and consequently, to prove Corollary 2.

5.1. Convergence of the fragmentation process of trees

For convenience, we consider a slightly different version of the fragmentation process described at the beginning of Section 1.2, where edges are removed at i.i.d. uniform random times, rather than at integer times.

Let τ be a rooted plane tree and equip the edges $\text{edge}(\tau)$ of τ with i.i.d. uniform random variables (or weights) $\mathbf{w} = (w_e : e \in \text{edge}(\tau))$ on $[0, 1]$ independent of τ . In particular, for a vertex $v \in \tau$ with $k_v \geq 1$ children, we write $(w_{v,i}, 1 \leq i \leq k_v)$ for the weights of the edges connecting v with its children. For $s \in [0, 1]$, we then keep the edges of τ with weight smaller than s and discard the others. This gives rise to a forest $\mathbf{f}_\tau(s)$ with set of edges given by $\text{edge}(\mathbf{f}_\tau(s)) = \{e \in \text{edge}(\tau) : w_e \leq s\}$. Furthermore, each vertex $v \in \mathbf{f}_\tau(s)$ has $k_s(v) = \sum_{i=1}^{k_v} \mathbf{1}_{\{w_{v,i} \leq s\}}$ children if $k_v \geq 1$; otherwise, $k_s(v) = 0$ whenever $k_v = 0$. The forest $\mathbf{f}_\tau(s)$ associated to τ and \mathbf{w} is called the *fragmentation forest* at time s . Let $\mathbf{F}_\tau(u)$ be the sequence of sizes (number of vertices) of the connected components of the forest $\mathbf{f}_\tau(1-u)$, ranked in decreasing order. We view the sequence of sizes of the components of $\mathbf{f}_\tau(1-u)$ as an infinite sequence, by completing it with an infinite number of zero terms. In particular, $\mathbf{F}_\tau(0) = (\zeta(\tau), 0, 0, \dots)$ and $\mathbf{F}_\tau(1) = (1, 1, \dots, 1, 0, 0, \dots)$ (with $\zeta(\tau)$ 1's). We refer to $\mathbf{F}_\tau = (\mathbf{F}_\tau(u), u \in [0, 1])$ as the fragmentation process associated with τ .

Following [33, Section 3], we next explain how to construct fragmentation processes from excursion-type functions. A function $g = (g(s), s \in [0, 1]) \in \mathbf{D}([0, 1], \mathbb{R})$ is an excursion-type function if $g(0) = g(1) = g(1-) = 0$, it is non-negative and it makes only positive jumps (i.e. $g(s-) \leq g(s)$ for all $s \in (0, 1]$). For such a function g and every $t \geq 0$, define $g^{(t)} = (g^{(t)}(s), s \in [0, 1])$ and $I_g^{(t)} = (I_g^{(t)}(s), s \in [0, 1])$ as

$$g^{(t)}(s) = g(s) - ts \text{ and } I_g^{(t)}(s) = \inf_{u \in [0, s]} g^{(t)}(u), \quad \text{for } s \in [0, 1].$$

For $t \geq 0$, we write $\mathbf{F}_g(t) = (F_{g,1}(t), F_{g,2}(t), \dots)$ for the ranked sequence (in decreasing order) of the lengths of the interval components of the complement of the support of the Stieltjes measure $d(-I_g^{(t)})$; note that $s \mapsto -I_g^{(t)}(s) = \sup_{u \in [0, s]} -g^{(t)}(u)$ is an increasing process. The process $\mathbf{F}_g = (\mathbf{F}_g(t), t \geq 0)$ is the fragmentation process associated to the excursion-type function g . Let $\text{Supp}(d(-I_g^{(t)}))$ denote the support of $d(-I_g^{(t)})$ and note that $(0, 1) \setminus \text{Supp}(d(-I_g^{(t)}))$ is the union of all open intervals on which the function $-I_g^{(t)}$ is constant. We call constancy interval of $-I_g^{(t)}$ any interval component of $(0, 1) \setminus \text{Supp}(d(-I_g^{(t)}))$.

Theorem 6. *Let $(\tau_n, n \geq 1)$ be a sequence of random rooted plane trees. Suppose that there are a sequence $(a_n, n \geq 1)$ of positive real numbers and a random excursion-type function $X = (X(u), u \in [0, 1])$ satisfying*

(D.1) $a_n \rightarrow \infty$ and $\frac{\zeta(\tau_n)}{a_n} \rightarrow \infty$, as $n \rightarrow \infty$;

(D.2) for $n \geq 1$, let $(W_{\tau_n}^{\text{prim}}(\zeta(\tau_n)u), u \in [0, 1])$ be the (time-scaled) Prim path of τ_n with respect to \mathbf{w} . Then, $(a_n^{-1}W_{\tau_n}^{\text{prim}}(\zeta(\tau_n)u), u \in [0, 1]) \xrightarrow{d} (X(u), u \in [0, 1])$, as $n \rightarrow \infty$, in $\mathbf{D}([0, 1], \mathbb{R})$;

(D.3) for every fixed $t \geq 0$, $X^{(t)}(s) \wedge X^{(t)}(s-) > I_X^{(t)}(s)$, for $s \in (s', s'')$, whenever $(s', s'') \in [0, 1]$ is an interval of constancy of $-I_X^{(t)}$, almost surely.

Then, for every fixed $t > 0$,

$$\frac{1}{\zeta(\tau_n)} \mathbf{F}_{\tau_n} \left(t \frac{a_n}{\zeta(\tau_n)} \right) \xrightarrow{d} \mathbf{F}_X(t), \quad \text{as } n \rightarrow \infty, \text{ in } \Delta,$$

equipped with the topology of pointwise convergence. If moreover,

(D.4) for every fixed $t \geq 0$, $\mathbf{F}_X(t) \in \Delta_1$ almost surely, where $\Delta_1 \subset \Delta$ is the space of the elements of Δ with sum 1,

(D.5) $\mathbb{E}[r_{\tau_n}^{\text{gr}}(u_n, u'_n)] = O(\zeta(\tau_n)/a_n)$, where $r_{\tau_n}^{\text{gr}}(u_n, u'_n)$ denotes the graph distance between two independent uniformly random vertices u_n and u'_n of τ_n .

then

$$\left(\frac{1}{\zeta(\tau_n)} \mathbf{F}_{\tau_n} \left(t \frac{a_n}{\zeta(\tau_n)} \right), t \geq 0 \right) \xrightarrow{d} (\mathbf{F}_X(t), t \geq 0), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \Delta).$$

To prove Theorem 6, we follow the approach developed in [22], which was also recently used in [24]. Therefore, to avoid unnecessary repetitions, we only provide enough details to convince the reader that the arguments can be adapted from these works.

Let $\emptyset = u(0) <_{\text{prim}} u(1) <_{\text{prim}} \dots <_{\text{prim}} u(\zeta(\tau_n)-1)$ be the Prim order of the vertices of τ_n with respect to \mathbf{w} . Since $\mathbf{f}_{\tau_n}(s)$ and τ_n possess the same set of vertices, we can and will consider that the vertices of $\mathbf{f}_{\tau_n}(s)$ are ordered according to the Prim order of the vertices in τ_n . For $n \geq 1$ and $s \in [0, 1]$, we associate to the Prim order of the vertices of $\mathbf{f}_{\tau_n}(s)$ an exploration path $W_{\tau_n}^{(s)} = (W_{\tau_n}^{(s)}(i), 0 \leq i \leq \zeta(\tau_n))$ by letting $W_{\tau_n}^{(s)}(0) = 0$, and for $0 \leq i \leq \zeta(\tau_n) - 1$, $W_{\tau_n}^{(s)}(i+1) = W_{\tau_n}^{(s)}(i) + k_s(u(i)) - 1$, where $k_s(u(i))$ denotes the number of children of $u(i)$ in $\mathbf{f}_{\tau_n}(s)$. We shall think of such a path as the step function on $[0, \zeta(\tau_n)]$ given by $u \mapsto W_{\tau_n}^{(s)}(\lfloor u \rfloor)$. For fixed $t \geq 0$, consider the sequence $(s_n(t), n \geq 1)$ of positive times given by

$$s_n(t) = \max \left(1 - \frac{a_n}{\zeta(\tau_n)} t, 0 \right),$$

and define the process $X_n^{(t)} = (X_n^{(t)}(u), u \in [0, 1])$ by letting

$$X_n^{(t)}(u) = \frac{1}{a_n} W_{\tau_n}^{(s_n(t))}(\zeta(\tau_n)u), \quad \text{for } u \in [0, 1].$$

For simplicity, we use the notation $X_n = (X_n^{(t)}, t \geq 0)$. For any $u \in [0, 1]$, the mapping $t \mapsto X_n^{(t)}(u)$ is non-increasing in t , which implies that X_n has càdlàg paths. In particular, we can view the process $t \mapsto X_n^{(t)}$ as a random variable taking values in the space $\mathbf{D}(\mathbb{R}_+, \mathbf{D}([0, 1], \mathbb{R}))$. In other words, for fixed $t \geq 0$, $X_n^{(t)}$ is a random variable in $\mathbf{D}([0, 1], \mathbb{R})$.

Theorem 7. *In the setting of Theorem 6, we have that*

$$(X_n^{(t)}, t \geq 0) \xrightarrow{d} (X^{(t)}, t \geq 0), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \mathbf{D}([0, 1], \mathbb{R})).$$

Proof of Theorem 7. The proof of Theorem 7 follows from (D.1)–(D.2) by adapting the argument used in the proof of [24, Theorem 3] (see also [22, Section 5]). It involves two steps: convergence of the finite-dimensional distributions and tightness of the sequence of processes $(X_n, n \geq 1)$. Indeed, one only needs to be aware that, for fixed $t \geq 0$ and for $0 \leq i \leq \zeta(\tau_n) - 1$, the number of children $k_{s_n(t)}(u(i))$ of the vertex $u(i) \in \mathbf{f}_n(s_n(t))$ is distributed as a binomial random variable with parameters $(k_{u(i)}, s_n(t))$. Details are left to the interested reader. \square

Next we prove Theorem 6.

Proof of Theorem 6. For $t \geq 0$, define the process $I_n^{(t)} = (I_n^{(t)}(u), u \in [0, 1])$ by letting

$$I_n^{(t)}(u) = \inf_{s \in [0, u]} X_n^{(t)}(s), \quad \text{for } u \in [0, 1].$$

Similarly, we define the process $I^{(t)} = (I^{(t)}(u), u \in [0, 1])$ by letting $I^{(t)}(u) = \inf_{s \in [0, u]} X^{(t)}(s)$, for $u \in [0, 1]$.

For $t \geq 0$, we write $\mathbf{F}(-I_n^{(t)}) = (F_1(-I_n^{(t)}), F_2(-I_n^{(t)}), \dots)$ for the ranked sequence (in decreasing order) of the lengths of the intervals components of the complement of the support of the Stieltjes measure $d(-I_n^{(t)})$. By [24, Lemma 1], we know that

$$\frac{1}{\zeta(\tau_n)} \mathbf{F}_{\tau_n} \left(t \frac{a_n}{\zeta(\tau_n)} \right) = \mathbf{F}(-I_n^{(t)}), \quad \text{for } t \geq 0. \tag{55}$$

Observe that $X_n^{(t)}(0) = X^{(t)}(0) = 0$, for all $t \geq 0$. Our first claim follows from Theorem 7, (D.3) and [33, Lemma 4].

Next, we prove the second claim. By the Skorokhod representation theorem, we can and we will work in a probability space where the convergence in Theorem 7 together with (D.3) and (D.4) holds almost surely. By Theorem 7, there exists a dense subset D of \mathbb{R}_+ such that for any fixed $k \in \mathbb{N}$ and collection $0 \leq t_1 < t_2 < \dots < t_k < \infty$ with $t_1, \dots, t_k \in D$, we have that a.s.,

$$(I_n^{(t_1)}, \dots, I_n^{(t_k)}) \rightarrow (I^{(t_1)}, \dots, I^{(t_k)}), \quad \text{as } n \rightarrow \infty,$$

in $\mathbf{D}([0, 1], \mathbb{R})^k$ (i.e., the k -fold space of $\mathbf{D}([0, 1], \mathbb{R})$). Then [33, Lemma 4] implies that a.s.,

$$(\mathbf{F}(-I_n^{(t_1)}), \dots, \mathbf{F}(-I_n^{(t_k)})) \rightarrow (\mathbf{F}(-I^{(t_1)}), \dots, \mathbf{F}(-I^{(t_k)})), \quad \text{as } n \rightarrow \infty,$$

in Δ^k (i.e., the k -fold space of Δ equipped with the ℓ^1 -norm). Note that the conditions in [33, Lemma 4] are satisfied by our assumptions (in fact, one has to apply [33, Lemma 4] to $-I_n$ and $-J$). This shows the convergence of the finite-dimensional distributions of the sequence of processes $((\mathbf{F}(-I_n^{(t)}), t \geq 0))_{n \geq 1}$ to those of the process $(\mathbf{F}(-I^{(t)}), t \geq 0)$. On the other hand, (D.5) and [56, Corollary 5.5] imply that $((\mathbf{F}(-I_n^{(t)}), t \geq 0))_{n \geq 1}$ is tight in $\mathbf{D}(\mathbb{R}_+, \Delta)$, which concludes our proof (recall (55)). \square

The following result establishes the convergence, after proper rescaling, of the fragmentation process associated with a tree \mathbf{t}_n having a given degree sequence.

Theorem 8. *Suppose that s_n satisfies (A.1)–(A.3) and (A.5). Then,*

$$\left(\frac{1}{V_n} \mathbf{F}_{\mathbf{t}_n} \left(t \frac{b_n}{V_n} \right), t \geq 0 \right) \xrightarrow{d} (\mathbf{F}_{X^{\text{exc}}}(t), t \geq 0), \quad \text{as } n \rightarrow \infty, \text{ in } \mathbf{D}(\mathbb{R}_+, \Delta).$$

where X^{exc} is the Vervaat transform of an EI bridge with parameters $(\theta_i, i \geq 0)$.

Note that Theorem 8 holds under more general assumptions than Theorem 1. In essence, it requires the convergence of the Łukasiewicz path (recall Theorem 6), which we can prove under the weaker assumptions (see Theorem 3). The limiting process $(\mathbf{F}_{X^{\text{exc}}}(t), t \geq 0)$ corresponds precisely to the fragmentation process $(\mathbf{F}(t), t \geq 0)$ in the statement of Corollary 2, as we will see in its proof in the next section. We recall that $(\mathbf{F}_{X^{\text{exc}}}(t), t \geq 0)$ is the fragmentation process as constructed by Bertoin [32,33], and that it is also equivalent to the fragmentation process $(\mathbf{F}_{\mathcal{T}_\theta}(t), t \geq 0)$ of the Inhomogeneous CRT \mathcal{T}_θ , with parameters $\theta = (\theta_0, \theta_1, \dots)$, see [15].

Proof of Theorem 8. The assumptions (A.1)–(A.3) and (A.5) in Theorem 8 imply (D.1) and (D.2) in Theorem 6 with X given by the Vervaat transform of an EI bridge with parameters $(\theta_i, i \geq 0)$. Indeed, Lemma 1 and Theorem 3 imply (D.1) and (D.2), respectively. [33, Lemma 7] shows (D.3) and (D.4). On the other hand, it follows from [12, Proposition 4.5] that there exists two universal constants $c_1, c_2 > 0$ such that

$$\mathbb{P}(|u_n| \geq xV_n/b_n) \leq c_1 e^{-c_2 x} \tag{56}$$

uniformly for $x > 0$ and all n large enough, where u_n is a uniformly random vertex of \mathbf{t}_n . This implies (D.5). Therefore, Theorem 8 follows from Theorem 6. \square

5.2. Convergence of the process of masses

We conclude this section with the proof of Corollary 2. The proof makes use of Theorem 8.

Proof of Corollary 2. For all $n \geq 1$ and $1 \leq k \leq E_n$, let $\kappa_{n,k}$ be the time at which the k th edge of t_n is removed in the fragmentation process $(V_n^{-1}F_{t_n}(tb_n/V_n), t \geq 0)$. Then,

$$\text{Mass}[\mathbb{L}_{tb_n}(t_n)] = \frac{1}{V_n} F_{t_n} \left(\frac{b_n}{V_n} \kappa_{n, \lfloor tb_n \rfloor} \right), \text{ for all } t \geq 0.$$

Thus, the first claim of Corollary 2 follows from Theorem 8, [31, Theorem 3.9] and [46, Theorem 3.1] provided that, for each $t \geq 0$,

$$\sup_{s \in [0,t]} \left| \kappa_{n, \lfloor sb_n \rfloor} - s \right| \xrightarrow{\mathbb{P}} 0, \text{ as } n \rightarrow \infty. \tag{57}$$

This follows as in the proof of Proposition 2.

Finally, we prove the second claim of Corollary 2. Following Aldous–Pitman [15], we recall the construction of the fragmentation process $(F_{T_\theta}(t), t \geq 0)$ associated to the Inhomogeneous CRT $\mathcal{T}_\theta = (\mathcal{T}_\theta, r_\theta, \rho_\theta, \mu_\theta)$ by cutting-down its skeleton through a Poisson point process Π of cuts with intensity $\lambda_\theta \times dt$ on $\mathcal{T}_\theta \times \mathbb{R}_+$, where λ_θ denotes the length measure of \mathcal{T}_θ . For all $t \geq 0$, define an equivalence relation \sim_t on \mathcal{T}_θ by saying that $x \sim_t y$, for $x, y \in \mathcal{T}_\theta$, if and only if no atom of the Poisson process Π that has appeared before time t belongs to the geodesic $\llbracket x, y \rrbracket$. These cuts split \mathcal{T}_θ into a continuum forest, which is a countably infinite set of smaller connected components. Let $\mathcal{T}_{\theta,1}^{(t)}, \mathcal{T}_{\theta,2}^{(t)}, \dots$ be the distinct equivalence classes for \sim_t (connected components of \mathcal{T}_θ), ranked according to the decreasing order of their μ_θ -masses. So, $(F_{T_\theta}(t), t \geq 0)$ is the process given by $F_{T_\theta}(t) = (\mu_\theta(\mathcal{T}_{\theta,1}^{(t)}), \mu_\theta(\mathcal{T}_{\theta,2}^{(t)}), \dots)$, for $t \geq 0$; in particular, $F_{T_\theta}(0) = (1, 0, 0, \dots)$.

Consider now that the fragmentation process of \mathcal{T}_θ and its lamination valued-process $(\mathbb{L}_t(\mathcal{T}_\theta), t \geq 0)$ are constructed from the same Poisson point process Π . Observe that for any $t \geq 0$, the connected components associated to the fragmentation process of \mathcal{T}_θ at time t are in natural bijection with the faces of $\mathbb{L}_t(\mathcal{T}_\theta)$. Let $F^{(t)}$ be a face of $\mathbb{L}_t(\mathcal{T}_\theta)$ and $C_{F^{(t)}}$ be the connected component of $\mathcal{T}_\theta \setminus \Pi_t$ coding $F^{(t)}$. Moreover, let c_0 the unique chord in the boundary of $F^{(t)}$ separating $F^{(t)}$ from 1 and the other chords c_1, c_2, \dots bounding $F^{(t)}$, ranked according to the decreasing order of their lengths. Now fix $i \geq 0$. Suppose that c_i codes a point $x_i \in \Pi_t$. Then c_i splits the unit circle into two arcs of respective lengths $2\pi\ell_i := 2\pi(D(x_i) - G(x_i))$ and $2\pi(1 - \ell_i)$ that exactly corresponds to 2π times the μ_θ -masses of the two components of $\mathcal{T}_\theta \setminus \{x_i\}$; see (43). Suppose that c_i does not code a point of Π_t . Then, there exists a sequence of chords $(c_i^k, k \geq 1)$ in $\mathbb{L}_t(\mathcal{T}_\theta)$ coding respectively a sequence of points $(x_i^k, k \geq 1)$ of Π_t such that $c_i^k \rightarrow c_i$, as $k \rightarrow \infty$, for the Hausdorff distance. In particular, $G(x_i^k)$ and $D(x_i^k)$ converge to some values G_i, D_i such that c_i splits the unit circle into two arcs of lengths $2\pi\ell_i := 2\pi(D_i - G_i)$ and $2\pi(1 - \ell_i)$. On the other hand, $\mathcal{T}_\theta \setminus \bigcup_{k \geq 1} \{x_i^k\}$ also possesses a connected component of μ_θ -mass $(1 - \ell_i)$. Finally, observe that the mass of $F^{(t)}$ and the μ_θ -mass of $C_{F^{(t)}}$ both can be written as $\ell_0 - \sum_{i \geq 1} \ell_i$. This concludes our proof. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

- [1] D. Aldous, The continuum random tree. II. An overview, in: Stochastic Analysis (Durham, 1990), in: London Math. Soc. Lecture Note Ser., vol. 167, Cambridge Univ. Press, Cambridge, 1991, pp. 23–70, MR 1166406.
- [2] D. Aldous, The continuum random tree. I, Ann. Probab. 19 (1) (1991) 1–28, MR 1085326.
- [3] D. Aldous, The continuum random tree. III, Ann. Probab. 21 (1) (1993) 248–289, MR 1207226.
- [4] G. Miermont, Invariance principles for spatial multitype Galton-Watson trees, Ann. Inst. H. Poincaré Probab. Stat. 44 (6) (2008) 1128–1161, MR 2469338.
- [5] J.-F. Marckert, G. Miermont, The CRT is the scaling limit of unordered binary trees, Random Structures Algorithms 38 (4) (2011) 467–501, MR 2829313.
- [6] J.-F. Le Gall, Uniqueness and universality of the Brownian map, Ann. Probab. 41 (4) (2013) 2880–2960, MR 3112934.
- [7] D. Aldous, Recursive self-similarity for random trees, random triangulations and Brownian excursion, Ann. Probab. 22 (2) (1994) 527–545, MR 1288122.
- [8] D. Aldous, Triangulating the circle, at random, Amer. Math. Monthly 101 (3) (1994) 223–233, MR 1264002.
- [9] D. Aldous, J. Pitman, The standard additive coalescent, Ann. Probab. 26 (4) (1998) 1703–1726, MR 1675063.
- [10] L. Addario-Berry, N. Broutin, C. Goldschmidt, The continuum limit of critical random graphs, Probab. Theory Related Fields 152 (3–4) (2012) 367–406, MR 2892951.
- [11] N. Broutin, J.-F. Marckert, Asymptotics of trees with a prescribed degree sequence and applications, Random Structures Algorithms 44 (3) (2014) 290–316, MR 3188597.

- [12] C. Marzouk, On scaling limits of random trees and maps with a prescribed degree sequence, *Ann. H. Lebesgue* 5 (2022) 317–386, MR 4443293.
- [13] D. Aldous, J. Pitman, A family of random trees with random edge lengths, *Random Structures Algorithms* 15 (2) (1999) 176–195, MR 1704343.
- [14] M. Camarri, J. Pitman, Limit distributions and random trees derived from the birthday problem with unequal probabilities, *Electron. J. Probab.* 5 (2) (2000) 18, MR 1741774.
- [15] D. Aldous, J. Pitman, Inhomogeneous continuum random trees and the entrance boundary of the additive coalescent, *Probab. Theory Related Fields* 118 (4) (2000) 455–482, MR 1808372.
- [16] D. Aldous, G. Miermont, J. Pitman, The exploration process of inhomogeneous continuum random trees, and an extension of Jeulin’s local time identity, *Probab. Theory Related Fields* 129 (2) (2004) 182–218, MR 2063375.
- [17] A. Greven, P. Pfaffelhuber, A. Winter, Convergence in distribution of random metric measure spaces (A -coalescent measure trees), *Probab. Theory Related Fields* 145 (1–2) (2009) 285–322, MR 2520129.
- [18] A. Blanc-Renaudie, Limit of trees with fixed degree sequence, 2021, arXiv e-prints arXiv:2110.03378.
- [19] T. Lei, Scaling limit of random forests with prescribed degree sequences, *Bernoulli* 25 (4A) (2019) 2409–2438, MR 4003553.
- [20] S.N. Evans, J. Pitman, Construction of Markovian coalescents, *Ann. Inst. H. Poincaré Probab. Stat.* 34 (3) (1998) 339–383, MR 1625867.
- [21] J. Pitman, Coalescent random forests, *J. Combin. Theory Ser. A* 85 (2) (1999) 165–193, MR 1673928.
- [22] N. Broutin, J.-F. Marckert, A new encoding of coalescent processes: applications to the additive and multiplicative cases, *Probab. Theory Related Fields* 166 (1–2) (2016) 515–552, MR 3547745.
- [23] J.-F. Marckert, M. Wang, A new combinatorial representation of the additive coalescent, *Random Structures Algorithms* 54 (2) (2019) 340–370, MR 3912100.
- [24] G. Berzunza Ojeda, C. Holmgren, Invariance principle for fragmentation processes derived from conditioned stable Galton-Watson trees, *Bernoulli* 29 (4) (2023) 2745–2770, MR 4632119.
- [25] F. Bonahon, Geodesic laminations on surfaces, in: *Laminations and Foliations in Dynamics, Geometry and Topology* (Stony Brook, NY, 1998), in: *Contemp. Math.*, vol. 269, Amer. Math. Soc., Providence, RI, 2001, pp. 1–37, MR 1810534.
- [26] I. Kortchemski, Random stable laminations of the disk, *Ann. Probab.* 42 (2) (2014) 725–759, MR 3178472.
- [27] N. Curien, I. Kortchemski, Random non-crossing plane configurations: a conditioned Galton-Watson tree approach, *Random Structures Algorithms* 45 (2) (2014) 236–260, MR 3245291.
- [28] P. Thévenin, A geometric representation of fragmentation processes on stable trees, *Ann. Probab.* 49 (5) (2021) 2416–2476.
- [29] J.-F. Le Gall, F. Paulin, Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere, *Geom. Funct. Anal.* 18 (3) (2008) 893–918, MR 2438999.
- [30] S.N. Ethier, T.G. Kurtz, Markov processes, in: *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*, John Wiley & Sons, Inc., New York, 1986, p. x+534, MR 838085. Characterization and convergence.
- [31] P. Billingsley, Convergence of probability measures, second ed., in: *Wiley Series in Probability and Statistics: Probability and Statistics*, John Wiley & Sons, Inc., New York, 1999, p. x+277, MR 1700749. A Wiley-Interscience Publication.
- [32] J. Bertoin, A fragmentation process connected to Brownian motion, *Probab. Theory Related Fields* 117 (2) (2000) 289–301, MR 1771665.
- [33] J. Bertoin, Eternal additive coalescents and certain bridges with exchangeable increments, *Ann. Probab.* 29 (1) (2001) 344–360, MR 1825153.
- [34] O. Angtuncio, G. Uribe Bravo, On the profile of trees with a given degree sequence, 2020, arXiv e-prints arXiv:2008.12242.
- [35] O. Kallenberg, Canonical representations and convergence criteria for processes with interchangeable increments, *Z. Wahrscheinlichkeitstheorie Und Verw. Geb.* 27 (1973) 23–36, MR 394842.
- [36] J. Neveu, Arbres et processus de Galton-Watson, *Ann. Inst. H. Poincaré Probab. Stat.* 22 (2) (1986) 199–207, MR 850756.
- [37] O. Kallenberg, Foundations of modern probability, second ed., in: *Probability and its Applications* (New York), Springer-Verlag, New York, 2002, p. xx+638, MR 1876169.
- [38] L. Takács, On combinatorial methods in the theory of stochastic processes, in: *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability* (Berkeley, Calif., 1965/66), Univ. California Press, Berkeley, Calif., 1967, pp. 431–447, MR 0214129. Vol. II: Contributions to Probability Theory, Part 1.
- [39] W. Vervaat, A relation between Brownian bridge and Brownian excursion, *Ann. Probab.* 7 (1) (1979) 143–149, MR 515820.
- [40] T. Duquesne, A limit theorem for the contour process of conditioned Galton-Watson trees, *Ann. Probab.* 31 (2) (2003) 996–1027, MR 1964956.
- [41] F.B. Knight, The uniform law for exchangeable and Lévy process bridges, (236) 1996, pp. 171–188, MR 1417982. Hommage à P. A. Meyer et J. Neveu,
- [42] J. Jacod, A.N. Shiryaev, Limit theorems for stochastic processes, second ed., *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 288, Springer-Verlag, Berlin, 2003, p. xx+661, MR 1943877.
- [43] W. Rudin, Principles of mathematical analysis, third ed., *International Series in Pure and Applied Mathematics*, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, p. x+342, MR 385023.
- [44] J.-F. Marckert, A. Mokkadem, Limit of normalized quadrangulations: the Brownian map, *Ann. Probab.* 34 (6) (2006) 2144–2202, MR 2294979.
- [45] R. Abraham, J.-F. Delmas, P. Hoscheit, A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces, *Electron. J. Probab.* 18 (14) (2013) 21, MR 3035742.
- [46] W. Whitt, Some useful functions for functional limit theorems, *Math. Oper. Res.* 5 (1) (1980) 67–85, MR 561155.
- [47] A. Gut, Stopped random walks, second ed., *Springer Series in Operations Research and Financial Engineering*, Springer, 2009, p. xiv+263, MR 2489436. Limit theorems and applications.
- [48] T. Duquesne, M. Winkel, Growth of Lévy trees, *Probab. Theory Related Fields* 139 (3–4) (2007) 313–371, MR 2322700.
- [49] T. Duquesne, The coding of compact real trees by real valued functions, 2006, arXiv preprint math/0604106.
- [50] S.N. Evans, A. Winter, Subtree prune and regraft: a reversible real tree-valued Markov process, *Ann. Probab.* 34 (3) (2006) 918–961, MR 2243874.
- [51] J.-F. Le Gall, Random trees and applications, *Probab. Surv.* 2 (2005) 245–311, MR 2203728.
- [52] S.N. Evans, Probability and real trees, *Lecture Notes in Mathematics*, vol. 1920, Springer, 2008, p. xii+193, MR 2351587. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005.
- [53] V. Féray, B. Louf, P. Thévenin, Random generation and scaling limits of fixed genus factorizations into transpositions, *Probab. Theory Related Fields* 184 (3–4) (2022) 681–748, MR 4507933.
- [54] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, English, *Modern Birkhäuser Classics*, Birkhäuser Boston, Inc., Boston, MA, 2007, p. xx+585, MR 2307192. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [55] W. Löhr, G. Voisin, A. Winter, Convergence of bi-measure \mathbb{R} -trees and the pruning process, *Ann. Inst. Henri Poincaré Probab. Stat.* 51 (4) (2015) 1342–1368, MR 3414450.
- [56] G. Berzunza Ojeda, C. Holmgren, S. Janson, A tightness criterion for fragmentations, 2025, arXiv e-prints. arXiv:2507.05102.