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RESEARCH ARTICLE

On the L^p-semigroups for Stokes equations with dynamic slip boundary conditions in the half-space

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Abstract

We consider the evolutionary Stokes system, coupled with the so-called dynamic slip boundary condition, in the simple geometry of a d-dimensional half-space. Using the standard technique of Fourier transform in tangential directions, we obtain an explicit formula for the resolvent. We then deduce estimates for both the weak (i.e. $W^{1,p}$) and strong (hence $W^{2,p}$) solutions, which are optimal in terms of the data belonging to an appropriate negative Sobolev or fractional Besov space. In the latter case L^p-integrability of the pressure gradient is included. We allow for solutions with non-zero divergence, thus preparing the way for extensions to general domains. As a by-product, we show that the system generates an analytic semigroup in $L^p(\Omega) \times L^p(\partial\Omega)$. Our approach remains elementary in the sense that only the classical Mikhlin multiplier theorem will be used. The methods of $\mathcal{H}^\infty$-calculus are implicitly present; but we stay away from the concept of R-boundedness and related heavy functional analytic machinery.

Keywords Stokes equations · Dynamic slip boundary condition · Unbounded domains · Optimal regularity · Analytic semigroup

Mathematics Subject Classification 76D07 · 47D03 · 35B65

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1 Introduction

This paper is motivated by the system

$$\begin{aligned} \partial_t \boldsymbol{u} - \operatorname{div} 2\nu \mathrm{D}\boldsymbol{u} + \nabla \pi &= \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega, \\ \beta\partial_t \boldsymbol{u} + \alpha\boldsymbol{u} + 2\nu[(\mathrm{D}\boldsymbol{u})\boldsymbol{n}]_\tau &= \beta\boldsymbol{h}, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } (0,T)\times\Gamma, \end{aligned} \tag{1}$$

where α, β, and ν are positive parameters, $\Omega = \{x_d > 0\}$ is the d-dimensional half-space, and $\Gamma = \{x_d = 0\}$ the corresponding boundary. Furthermore, $\boldsymbol{n}$ is the outer normal vector, and $\tau : \mathbb{R}^d \to \mathbb{R}^{d-1}$ denotes the tangential component, i.e.,

$$[(\mathrm{D}\boldsymbol{u})\boldsymbol{n}]_\tau = (\mathrm{D}\boldsymbol{u})\boldsymbol{n} - ((\mathrm{D}\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n}.$$

Note that (1) consists of two evolutionary equations, coupled via the trace operator. The problem can be efficiently treated in the Hilbert L^2-setting, since there is an underlying symmetric operator $A\colon V \to V^*$, where V is a suitable subspace of $W^{1,2}(\Omega)$, see e.g., [1], cf. also the weak formulation (6) below. However, the L^p-theory is expectedly more difficult if $p \neq 2$ and up to our knowledge, problem (1) has not been treated in the literature (although a number of closely related results are available, see also the discussion below). In the present paper, we intend to contribute to this, with two main results being: the existence of an analytic semigroup in $L^p(\Omega) \times L^p(\Gamma)$, and the optimal $W^{2,p}$- and $W^{1,p}$-regularity estimates for the corresponding stationary (resolvent) problem.

Our paper is organized as follows. In Sect. 2, we introduce the function spaces and state the two main results: Theorems 2.1 and 2.2, providing resolvent estimates of optimal regularity for both the weak and strong solutions. In Theorem 2.3, we show the existence of an analytic semigroup; we also identify the domain of its generator. Main technical results of the paper follow: in Sect. 3, we use (partial) Fourier transform, in the spirit of [9], to express the resolvent in terms of the boundary right-hand side. Key estimates in the L^p spaces are then obtained in Sect. 4. Our approach remains more elementary in that we only rely on the classical Mikhlin multiplier theorem. One can, however, note that Lemma 4.1 in fact asserts the existence of $\mathcal{H}^\infty$-calculus for the operator $(-\Delta)^{1/2}$. Finally, the proofs of the main theorems are completed in Sect. 5, essentially by combining the results of Sect. 4 with the well-known L^p-theory for the (homogeneous) Dirichlet problem, as given e.g., in [10].

Although our results are formulated for the case of the half-space $\Omega = \{x_d > 0\}$, an extension to more general domains is straightforward, using standard localization and perturbation arguments. Crucially, this requires extension of our estimates to the case of functions $\boldsymbol{u}$ with non-zero divergence. We treat this situation explicitly in the part (iii) of Theorem 2.2.

Let us conclude the introduction by discussing some related, both classical and more recent results. Concerning the Stokes system subject to the homogeneous Dirichlet boundary condition, the existence of an L^p-analytic semigroup was proven in [12] via the theory of pseudo-differential operators. For a more elementary approach based on the Fourier transform and classical multiplier theorems, see [9]. Recently, a different method using the maximal operator was proposed in [18]; however, the method seems to be limited to bounded domains.

There is also an extensive literature devoted to problems similar to (1), but crucially, without the time derivative in the boundary equation. For example, a fairly complete L^p-theory, including the existence of an analytic semigroup, has been established in [2], see also [11] and the references therein; yet the results again seem to be limited to bounded Ω only. In the case of a half-space, the problem was treated in [17], albeit for $\boldsymbol{h} = 0$ only. A more modern approach based on the $\mathcal{H}^\infty$-calculus was employed here. More recently, a closely related class of problems with dynamic boundary conditions was treated in [6], and maximal regularity results were obtained.

A rather comprehensive functional setup for treating resolvent problems related to dynamic boundary conditions is described in [8]; see also the references therein. This method does provide sharp regularity estimates in fractional Sobolev spaces, yet direct application to the Stokes problem — due to the incompressibility condition and the pressure term — does not seem viable.

Let us also mention the recent paper [7], where the existence of an analytic semigroup for (1) is proven in the case of smooth bounded domains, for the Hilbert setting of the maximal regularity space $L^2(\Omega) \times W^{1/2,2}(\Gamma)$.

Concerning possible applications, we remark that optimal regularity estimates of the stationary Stokes problem (cf. Theorem 2.2) can be used to obtain higher regularity of evolutionary nonlinear problems, via the standard bootstrap argument. See [16] for the corresponding analysis in bounded domains, where the smoothness of solutions was used to estimate the dimension of the global attractor, in the case of two-dimensional Navier–Stokes equations with dynamic boundary conditions.

2 Notation and main results

We set $\Omega = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R};\ x_d > 0\}$ and identify $\Gamma = \partial\Omega$ with $\mathbb{R}^{d-1}$. Furthermore, $\boldsymbol{n} = (0, -1)$ is the outer normal, and $\tau\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ denotes the tangential component. For $p \in (1, \infty)$ and integer $k \geqslant 1$, we denote by L^p and $W^{k,p}$ the usual Lebesgue and Sobolev spaces. The negative Sobolev space $W^{-1,p}$ is defined as the completion of L^p with respect to the norm

$$\|\boldsymbol{u}\|_{W^{-1,p}(\Omega)} = \sup\left\{ \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} \,\mathrm{d}x;\ \ \|\boldsymbol{\varphi}\|_{W^{1,p'}(\Omega)} \leqslant 1 \right\}.$$

It is well known (see e.g., [3, Section 3.14]) that $W^{-1,p}$ can be identified with the dual to $W^{1,p'}$.

We also need Besov spaces $B_p^{s,p}$ with a general real s; we allow for non-integer s in the spirit of [4]. The subscripts σ or n stand for the solenoidality, and zero normal component, respectively. Finally, we will extensively use the partial Fourier transform $\mathcal{F}_{x'\to\xi}$, which is defined by

$$\mathcal{F}_{x'\to\xi}\, \boldsymbol{u}(x', x_d) = \widehat{\boldsymbol{u}}(\xi, x_d) = \int_{\mathbb{R}^{d-1}} \boldsymbol{u}(x', x_d)\, e^{-ix'\cdot\xi} \,\mathrm{d}x', \quad \xi \in \mathbb{R}^{d-1}.$$

Since it will not play any role in our paper, we set $\beta = \nu = 1$ in (1). We will be concerned with the related resolvent problem, i.e.,

$$(\lambda - \Delta)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{4}$$

$$(\lambda + \alpha)\boldsymbol{u} + 2[(\mathrm{D}\boldsymbol{u})\boldsymbol{n}]_\tau = \boldsymbol{h} \qquad \text{on } \Gamma, \tag{5}$$

where λ is a complex number. Note that $\operatorname{div} 2\mathrm{D}\boldsymbol{u} = \Delta\boldsymbol{u}$. Multiplying (2) by a divergence-free, smooth, and compactly supported function $\boldsymbol{\varphi}$ with $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ and integrating by parts, one obtains the integral formulation

$$\begin{aligned}\lambda \int_\Omega \boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + (\lambda+\alpha)\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}S + \int_\Omega 2\mathrm{D}\boldsymbol{u} : \mathrm{D}\boldsymbol{\varphi}\,\mathrm{d}x \\ = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Gamma \boldsymbol{h}\cdot\boldsymbol{\varphi}\,\mathrm{d}S.\end{aligned} \tag{6}$$

This problem is easy to handle in the L^2-setting, using Korn's inequality, see [2] or [1]. However, our objective is to obtain the L^p-theory, as stated in the following two main theorems.

Theorem 2.1 *Let $\alpha \geqslant 0$, $p \in (1, +\infty)$, $\boldsymbol{f} \in L^p_{\sigma,n}(\Omega)$ and $\boldsymbol{h} \in L^p_n(\Gamma)$. Then for any complex λ with $\mathfrak{Re}\,\lambda > 0$ there exists a unique $\boldsymbol{u} \in W^{1,p}_{\sigma,n}(\Omega)$ weak solution to (2)–(5).*

If, moreover, $|\lambda| > \omega$ for some fixed $\omega > 0$, then

$$\|\boldsymbol{u}\|_{L^p(\Omega)} + \|\boldsymbol{u}\|_{L^p(\Gamma)} \leqslant \frac{C_0}{|\lambda|}\big(\|\boldsymbol{f}\|_{L^p(\Omega)} + \|\boldsymbol{h}\|_{L^p(\Gamma)}\big), \tag{7}$$

where the constant C_0 depends on d, p and ω, but is independent of λ and α.

Theorem 2.2 *Let $\alpha \geqslant 0$, $p \in (1, +\infty)$ and $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,\lambda > 0$.*

(i) *Let $\boldsymbol{f} \in W^{-1,p}(\Omega)$ and $\boldsymbol{h} \in B^{-1/p,p}_{p;n}(\Gamma)$. Then one has the estimate*

$$\|\boldsymbol{u}\|_{W^{1,p}(\Omega)} \leqslant C_1\big(\|\boldsymbol{f}\|_{W^{-1,p}(\Omega)} + \|\boldsymbol{h}\|_{B^{-1/p,p}_p(\Gamma)}\big). \tag{8}$$

(ii) *Let $\boldsymbol{f} \in L^p_{\sigma,n}(\Omega)$ and $\boldsymbol{h} \in B^{1-1/p,p}_{p;n}(\Gamma)$. Then the solution to resolvent problem (2)–(5) is strong, and one has the estimate*

$$\|\boldsymbol{u}\|_{W^{2,p}(\Omega)} + \|\nabla\pi\|_{L^p(\Omega)} \leqslant C_2\big(\|\boldsymbol{f}\|_{L^p(\Omega)} + \|\boldsymbol{h}\|_{B^{1-1/p,p}_p(\Gamma)}\big). \tag{9}$$

(iii) *If (3) is generalized to $\operatorname{div}\boldsymbol{u} = g$, the estimates remain valid, with the norm of g in $L^p(\Omega)$ or $W^{1,p}(\Omega)$ added to the right-hand side of (8) or (9), respectively.*

The constants C_1, C_2 depend on d, p and $|\lambda|$.

Postponing proofs of these theorems to Sect. 5, we can now establish the existence of an analytic semigroup.

Corollary 2.3 *System (1) generates an analytic semigroup in $L^p_{\sigma,n}(\Omega) \times L^p_n(\Gamma)$, where the domain of the generator is contained in $B_p^{1+1/p-\varepsilon,p}(\Omega)$, $\varepsilon > 0$ arbitrary.*

Proof Set $X = L^p_{\sigma,n}(\Omega) \times L^p_n(\Gamma)$, and define $A\boldsymbol{u} = (\boldsymbol{f}, \boldsymbol{h})$ by the (weak) formulation of the resolvent, i.e., (6). We set

$$\mathcal{D}(A) = \{\boldsymbol{u} \in W^{1,p}_{\sigma,n}(\Omega);\ A\boldsymbol{u} \in X\},$$

with a slight abuse of notation, we identify the Sobolev function $\boldsymbol{u}$ with the couple $(\boldsymbol{u}, \operatorname{tr} \boldsymbol{u})$. It is easy to verify that A is densely defined, and the closedness follows from the resolvent estimate (7); see also (37) below. By a standard result (see e.g., [5, Corollary 3.7.14]), the existence of semigroup follows.

Let us identify the domain of A. In view of uniqueness (cf. the argument after (37) below), we can treat the regularity with respect to $\boldsymbol{f} \in L^p_{\sigma,n}(\Omega)$ and $\boldsymbol{h} \in L^p_n(\Gamma)$ separately. If $\boldsymbol{h} = 0$, if follows from (9) that $\boldsymbol{u} \in W^{2,p}(\Omega)$. However, in contrast to the Dirichlet problem, this cannot be the domain since (5) would then imply $\boldsymbol{h} \in B_p^{1-1/p,p}(\Gamma)$.

In fact, we claim that for any $\varepsilon > 0$

$$B^{1+1/p+\varepsilon,p}_{p;\sigma,n}(\Omega) \subset \mathcal{D}(A) \subset B^{1+1/p-\varepsilon,p}_{p;\sigma,n}(\Omega).$$

Indeed, interpolating (8), (9) with $\boldsymbol{f} = 0$, we get the second inclusion. On the other hand, if $\boldsymbol{u} \in B_p^{1+1/p+\varepsilon,p}(\Omega)$, then the left-hand side of (5) belongs to $B_p^{\varepsilon,p}(\Gamma)$, in contradiction to the fact that in general, $\boldsymbol{h} \in L^p(\Gamma)$ only.

Note that if $p = 2$, one has the equality $\mathcal{D}(A) = W^{3/2,2}_{\sigma,n}(\Omega)$, as follows from Lemma 4.5 below. □

3 Fundamental solution

Here, we deal with (2)–(5) for $\boldsymbol{f} = 0$ and $\boldsymbol{h} = \boldsymbol{\Phi} = (\boldsymbol{\phi}, 0)$, i.e., we consider the system

$$(\lambda - \Delta)\boldsymbol{u} + \nabla\pi = 0 \qquad \text{in } \Omega, \tag{10}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{11}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{12}$$

$$(\lambda + \alpha)\boldsymbol{u} + 2[(\mathrm{D}\boldsymbol{u})\boldsymbol{n}]_\tau = \boldsymbol{\Phi} \qquad \text{on } \Gamma. \tag{13}$$

We will (formally) solve the problem, using Fourier analysis.

3.1 Reducing the equations

By div', Δ' we denote the corresponding operators with respect to x'. Let us now simplify (12)–(13). First, $\boldsymbol{n} = (0, -1)$, and thus

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \Leftrightarrow \quad u_d = 0 \ \text{ on } \ \Gamma.$$

Hence,

$$2(\mathrm{D}\boldsymbol{u})\,\boldsymbol{n} = -\begin{pmatrix} \partial_d u_1 + \partial_1 u_d \\ \partial_d u_2 + \partial_2 u_d \\ \vdots \\ \partial_d u_{d-1} + \partial_{d-1} u_d \\ 2\partial_d u_d \end{pmatrix} = -\begin{pmatrix} \partial_d \boldsymbol{u}' \\ 2\partial_d u_d \end{pmatrix}$$

and

$$2\,[(\mathrm{D}\boldsymbol{u})\,\boldsymbol{n}]_\tau = 2\,(\mathrm{D}\boldsymbol{u})\,\boldsymbol{n} - 2\,((\mathrm{D}\boldsymbol{u})\,\boldsymbol{n} \cdot \boldsymbol{n})\,\boldsymbol{n} = -\begin{pmatrix} \partial_d \boldsymbol{u}' \\ 0 \end{pmatrix}.$$

Therefore, we can rewrite (13) as follows

$$(\lambda + \alpha)\,\boldsymbol{u}' - \partial_d\,\boldsymbol{u}' = \boldsymbol{\phi} \quad \text{on} \ \ \Gamma. \tag{14}$$

Now, we apply $\partial_d \operatorname{div}'$ to the first $d-1$ equations of (10) and $-\Delta'$ to the last one; adding these equations together we obtain

$$\begin{aligned}(\lambda - \Delta)\big[\partial_d(\partial_1 u_1 + \partial_2 u_2 + \cdots + \partial_{d-1} u_{d-1}) - \Delta' u_d\big] \\ + \big[\partial_d(\partial_1^2 \pi + \partial_2^2 \pi + \cdots + \partial_{d-1}^2 \pi) - \Delta' \partial_d \pi\big] = 0.\end{aligned}$$

The pressure term clearly vanishes and in the first bracket we use $\operatorname{div} \boldsymbol{u} = 0 \Leftrightarrow \operatorname{div}' \boldsymbol{u}' = -\partial_d\, u_d$ to get

$$\begin{aligned} -(\lambda - \Delta)\,\Delta u_d &= 0 \quad \text{in} \ \ \Omega, \\ u_d &= 0 \quad \text{on} \ \ \Gamma. \end{aligned}$$

Finally, we deal with the boundary condition (14). Applying $-\operatorname{div}'$ yields

$$-\,(\lambda + \alpha)\operatorname{div}' \boldsymbol{u}' + \partial_d \operatorname{div}' \boldsymbol{u}' = -\operatorname{div}' \boldsymbol{\phi},$$

and thanks to $\operatorname{div}' \boldsymbol{u}' = -\partial_d\, u_d$ we achieve

$$(\lambda + \alpha - \partial_d)\,\partial_d\, u_d = -\operatorname{div}' \boldsymbol{\phi} \quad \text{on} \ \ \Gamma.$$

Therefore, we reduced the whole problem to the question of solving the following system for u_d:

$$\begin{aligned} -(\lambda-\Delta)\,\Delta u_d &= 0 && \text{in } \Omega, \\ (\lambda+\alpha-\partial_d)\,\partial_d u_d &= -\operatorname{div}'\boldsymbol{\phi} && \text{on } \Gamma, \\ u_d &= 0 && \text{on } \Gamma. \end{aligned} \tag{15}$$

3.2 Fourier transform

We solve (15) by applying the Fourier transform $\mathcal{F}_{x'\to\xi}$. We need to deal with the system

$$\begin{aligned} (\lambda+|\xi|^2-\partial_d^2)(|\xi|^2-\partial_d^2)\,\widehat{u_d} &= 0 && \text{for } x_d>0, \\ (\lambda+\alpha-\partial_d)\,\partial_d\,\widehat{u_d} &= -\,i\xi\cdot\widehat{\boldsymbol{\phi}} && \text{for } x_d=0, \\ \widehat{u_d}(\xi,0) &= 0, \end{aligned} \tag{16}$$

for any $\xi\in\mathbb{R}^{d-1}$. Here, the characteristic roots are

$$\pm\sqrt{\lambda+|\xi|^2} \quad \text{and} \quad \pm|\xi|;$$

certainly, only the terms with the minus sign are relevant. It will be useful to introduce the function

$$m_0(\lambda,\xi,x_d) = \frac{e^{-x_d\sqrt{\lambda+|\xi|^2}} - e^{-x_d|\xi|}}{\lambda+(\lambda+\alpha)(\sqrt{\lambda+|\xi|^2}-|\xi|)}, \tag{17}$$

which is nothing else than the fundamental solution to (16), i.e., one has

$$\begin{aligned} (\lambda+|\xi|^2-\partial_d^2)(|\xi|^2-\partial_d^2)\,m_0 &= 0 && \text{for } x_d>0, \\ (\lambda+\alpha-\partial_d)\,\partial_d\,m_0 &= -1 && \text{for } x_d=0, \\ m_0(\lambda,\xi,0) &= 0. \end{aligned}$$

The solution for the last component is thus

$$\widehat{u_d}(\xi,x_d) = i\xi\cdot m_0(\lambda,\xi,x_d)\,\widehat{\boldsymbol{\phi}}(\xi).$$

Now, we need to express $\widehat{\pi}$ and $\widehat{\boldsymbol{u}'}$ using m_0. We start with the pressure. Let us apply div' to the first $d-1$ equations of (10); we get

$$\Delta'\pi = -\operatorname{div}'(\lambda-\Delta)\,\boldsymbol{u}' \quad \text{in } \Omega.$$

Because of $\operatorname{div}'\boldsymbol{u}' = -\partial_d u_d$ we rewrite it in the form

$$\Delta'\pi = (\lambda-\Delta)\,\partial_d u_d \quad \text{in } \Omega,$$

and use the Fourier transform to achieve

$$\widehat{\pi} = -\frac{1}{|\xi|^2}(\lambda + |\xi|^2 - \partial_d^2)\,\partial_d\,\widehat{u_d}.$$

In other words,

$$\begin{aligned}\widehat{\pi}(\xi, x_d) &= -(\lambda + |\xi|^2 - \partial_d^2)\,\partial_d\, m_0(\lambda, \xi, x_d)\,\frac{i\xi}{|\xi|^2}\cdot\widehat{\boldsymbol{\phi}}(\xi)\\ &= -\frac{\lambda e^{-x_d|\xi|}}{\lambda + (\lambda+\alpha)(\sqrt{\lambda + |\xi|^2} - |\xi|)}\,\frac{i\xi}{|\xi|}\cdot\widehat{\boldsymbol{\phi}}(\xi).\end{aligned}$$

Up to now, everything was dimension-independent. At this point, we need to express the function $\boldsymbol{u}'$. If $d = 2$, then everything is simpler. From (11) we easily find that

$$\widehat{u_1}(\xi, x_d) = -\partial_2 m_0(\lambda, \xi, x_2)\,\widehat{\boldsymbol{\phi}}(\xi).$$

In the general case $d > 2$ we start in the same way, i.e., we take the Fourier transform of (11) and express our unknown vector field

$$i\xi\cdot\widehat{\boldsymbol{u}'} + \partial_d\,\widehat{u_d} = 0 \quad\Leftrightarrow\quad \widehat{\boldsymbol{u}'} = \frac{i\xi}{|\xi|^2}\,\partial_d\,\widehat{u_d},$$

which yields

$$\widehat{\boldsymbol{u}'}(\xi, x_d) = -\partial_d\, m_0(\lambda, \xi, x_d)\,\frac{\xi\otimes\xi}{|\xi|^2}\,\widehat{\boldsymbol{\phi}}(\xi).$$

This yields the solution to the Stokes system, but our condition (13) is not satisfied (in general). We need to add a correction term $\boldsymbol{v} = (\boldsymbol{v}', 0)$. For this purpose, consider the following problem:

$$\begin{aligned}(\lambda - \Delta)\,\boldsymbol{v}' &= 0 \quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v}' &= 0 \quad \text{in } \Omega,\\ (\lambda+\alpha)\,\boldsymbol{v}' - \partial_d\,\boldsymbol{v}' &= \boldsymbol{h}' \quad \text{on } \Gamma,\end{aligned}$$

where

$$\widehat{\boldsymbol{h}'} = \widehat{\boldsymbol{\phi}} - \frac{\xi\otimes\xi}{|\xi|^2}\,\widehat{\boldsymbol{\phi}}.$$

Now, by the Fourier transform we get

$$\begin{aligned}(\lambda + |\xi|^2 - \partial_d^2)\,\widehat{\boldsymbol{v}'} &= 0 \quad \text{in } \Omega,\\ (\lambda + \alpha - \partial_d)\,\widehat{\boldsymbol{v}'} &= \widehat{\boldsymbol{h}'} \quad \text{on } \Gamma.\end{aligned}$$

The characteristic roots are now $\pm\sqrt{\lambda+|\xi|^2}$, and thus

$$\widehat{\boldsymbol{v}'}(\xi, x_d) = \boldsymbol{b}(\xi)\, e^{-x_d\sqrt{\lambda+|\xi|^2}}.$$

From the boundary condition we find

$$\boldsymbol{b}(\xi) = \frac{1}{\lambda+\alpha+\sqrt{\lambda+|\xi|^2}}\,\widehat{\boldsymbol{h}'}(\xi).$$

Together,

$$\begin{aligned}\widehat{\boldsymbol{u}'}(\xi, x_d) = &-\partial_d\, m_0(\lambda, \xi, x_d)\,\frac{\xi\otimes\xi}{|\xi|^2}\,\widehat{\boldsymbol{\phi}}(\xi)\\ &+\frac{1}{\lambda+\alpha+\sqrt{\lambda+|\xi|^2}}\left(\mathrm{Id}_{d-1}-\frac{\xi\otimes\xi}{|\xi|^2}\right)e^{-x_d\sqrt{\lambda+|\xi|^2}}\,\widehat{\boldsymbol{\phi}}(\xi).\end{aligned}$$

To conclude this section, we summarize that for $d \geqslant 2$, the solution to (10)–(13) can be expressed as

$$\begin{aligned}\widehat{\boldsymbol{u}'}(\xi, x_d) = &-\partial_d\, m_0(\lambda, \xi, x_d)\,\frac{\xi\otimes\xi}{|\xi|^2}\,\widehat{\boldsymbol{\phi}}(\xi)\\ &+\frac{1}{\lambda+\alpha+\sqrt{\lambda+|\xi|^2}}\left(\mathrm{Id}_{d-1}-\frac{\xi\otimes\xi}{|\xi|^2}\right)e^{-x_d\sqrt{\lambda+|\xi|^2}}\,\widehat{\boldsymbol{\phi}}(\xi),\\ \widehat{u_d}(\xi, x_d) = &\ i\,\xi\cdot m_0(\lambda, \xi, x_d)\,\widehat{\boldsymbol{\phi}}(\xi),\\ \widehat{\pi}(\xi, x_d) = &-\frac{\lambda e^{-x_d|\xi|}}{\lambda+(\lambda+\alpha)(\sqrt{\lambda+|\xi|^2}-|\xi|)}\,\frac{i\xi}{|\xi|}\cdot\widehat{\boldsymbol{\phi}}(\xi),\end{aligned}\tag{18}$$

where m_0 was defined in (17). For $d = 2$, the second part of the first right-hand side vanishes.

4 Multipliers and L^p-bounds

The aim of this section is to establish L^p-estimates of the solution obtained above. To begin with, we note that m_0 only depends on $z = |\xi|$. Setting also for simplicity $y = x_d$, with a slight abuse of notation we can write

$$m_0 = m_0(\lambda, z, y) = \frac{\omega(\lambda, z)+z}{\alpha+\lambda+\omega(\lambda, z)+z}\cdot\frac{e^{-y\omega(\lambda,z)}-e^{-yz}}{\lambda},\tag{19}$$

where $\omega(\lambda, z) = \sqrt{\lambda + z^2}$. From relations (18) it follows that

$$\begin{aligned}
\widehat{\boldsymbol{u}'}(\xi, y) &= m_2(\lambda, |\xi|, y)\, \frac{\xi \otimes \xi}{|\xi|^2}\, \widehat{\boldsymbol{\phi}}(\xi) + m_3(\lambda, |\xi|, y) \left(\mathrm{Id}_{d-1} - \frac{\xi \otimes \xi}{|\xi|^2} \right) \widehat{\boldsymbol{\phi}}(\xi), \\
\widehat{u_d}(\xi, y) &= m_1(\lambda, |\xi|, y)\, \frac{i\xi}{|\xi|} \cdot \widehat{\boldsymbol{\phi}}(\xi), \\
\widehat{\pi}(\xi, y) &= m_4(\lambda, |\xi|, y)\, \frac{i\xi}{|\xi|} \cdot \widehat{\boldsymbol{\phi}}(\xi),
\end{aligned} \tag{20}$$

where we have set

$$\begin{aligned}
m_1(\lambda, z, y) &= z m_0(\lambda, z, y), \\
m_2(\lambda, z, y) &= -\partial_y m_0(\lambda, z, y), \\
m_3(\lambda, z, y) &= \frac{1}{\alpha + \lambda + \omega(\lambda, z)}\, e^{-y\omega(\lambda, z)}, \\
m_4(\lambda, z, y) &= -\frac{\omega(\lambda, z) + z}{\alpha + \lambda + \omega(\lambda, z) + z}\, e^{-yz}.
\end{aligned} \tag{21}$$

As the multipliers $\xi/|\xi|$ and $\xi \otimes \xi/|\xi|^2$ are easily seen to satisfy the Mikhlin condition (23), it is enough to treat just the terms m_i, $i = 1, \dots, 4$. Denote

$$\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\};\ |\arg z| < \theta\}.$$

Let $\mathcal{H}^\infty(\Sigma_\theta)$ be the set of all bounded and analytic functions in Σ_θ. We will use repeatedly the following estimate, which follows easily from the $\mathcal{H}^\infty$-calculus for the operator $(-\Delta)^{1/2}$ on L^p, see e.g., [15]. For the reader's convenience, we provide a short proof based on the classical Mikhlin's theorem.

Lemma 4.1 *Let* $\widehat{\boldsymbol{u}}(\xi, y) = m(|\xi|, y)\widehat{\boldsymbol{\phi}}(\xi)$*, where* $m(\cdot, y) \in \mathcal{H}^\infty(\Sigma_\theta)$ *for each fixed* $y \geqslant 0$*. Set*

$$k(y) = \sup_{z \in \Sigma_\theta} |m(z, y)|, \quad y \geqslant 0.$$

Then one has the estimate

$$\|\boldsymbol{u}\|_{L^p(\Gamma)} + \|\boldsymbol{u}\|_{L^p(\Omega)} \leqslant C_{p,d} \left(k(0) + \|k\|_{L^p(0,\infty)} \right) \|\boldsymbol{\phi}\|_{L^p(\mathbb{R}^{d-1})}.$$

Furthermore, if under these assumptions $\widehat{\boldsymbol{u}}(\xi, y) = m(|\xi|, y)\widehat{\boldsymbol{v}}(\xi, y)$ *for any fixed* $y > 0$*, then*

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leqslant C_{p,d} \|k\|_{L^\infty(0,\infty)} \|\boldsymbol{v}\|_{L^p(\Omega)}. \tag{22}$$

Proof Let $y \geqslant 0$ be fixed. We want to verify the Mikhlin condition

$$|\xi|^k |\nabla^k m(|\xi|, y)| \leqslant C_k(y), \quad \xi \neq 0, \quad k \geqslant 0, \tag{23}$$

see e.g., [13, Theorem 5.2.7]. It follows once we prove

$$|s^k m^{(k)}(s, y)| \leqslant C(y), \quad s \in (0, \infty).$$

Let $s \in (0, \infty)$ be fixed, and let Γ_s be the largest circle centered in s, and contained in Σ_θ. By Cauchy's theorem

$$s^k m^{(k)}(s, y) = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{s^k m(z, y)}{(z - s)^{k+1}} \, \mathrm{d}z.$$

Now it is easy to see that the right-hand side has a bound only depending on θ and the supremum of $m(\cdot, y)$.

As the Mikhlin condition is satisfied, we see that

$$\|\boldsymbol{u}(\cdot, y)\|_{L^p(\mathbb{R}^{d-1})} \leqslant C_{p,d}\, k(y)\, \|\boldsymbol{\phi}\|_{L^p(\mathbb{R}^{d-1})}$$

holds. The conclusion is immediate by Fubini's theorem. □

The following two lemmas are the main technical estimates of the paper.

Lemma 4.2 *Let $\theta \in (0, \pi/4)$ be fixed. Then there exist constants $C, c > 0$ and $\delta > 0$ such that*

$$\mathfrak{Re}\, \omega(\lambda, z) \geqslant c(z + \sqrt{|\lambda|}), \tag{24}$$

$$\left| \frac{z(e^{-y\omega(\lambda,z)} - e^{-yz})}{\omega(\lambda, z) - z} \right| \leqslant \frac{Ce^{-\delta y|z|}}{1 + y\sqrt{|\lambda|}}, \tag{25}$$

$$\frac{1}{|\alpha + \lambda + z + \omega(\lambda, z)|} \leqslant \frac{C}{\alpha + |\lambda|}, \tag{26}$$

$$\frac{1}{|\alpha + \lambda + z + \omega(\lambda, z)|} \leqslant \frac{C}{\alpha + |z|^2}, \tag{27}$$

for all $z \in \Sigma_\theta$, $y \geqslant 0$ and $\lambda \in \Sigma_{\pi/2}$.

Proof For (24) see [17, Lemma 5.2 (b), (c)], and (25) follows from Lemma 5.3 (b) therein. To show (26), we proceed as follows.

Since $\lambda \in \Sigma_{\pi/2}$ and $z \in \Sigma_\theta$ for some $\theta < \pi/4$, we have $\lambda + z^2 \in \Sigma_{\pi/2}$. Hence $\omega(\lambda, z) \in \Sigma_{\pi/4}$ and $z + \omega(\lambda, z) \in \Sigma_{\pi/4}$. As $\frac{1}{\alpha+\lambda} \in \Sigma_{\pi/2}$, we get $\frac{z+\omega(\lambda,z)}{\alpha+\lambda} \in \Sigma_{3\pi/4}$, which means that this value is strictly away from -1, in particular,

$$\left| 1 + \frac{z + \omega(\lambda, z)}{\alpha + \lambda} \right| \geqslant C$$

holds for some $C > 0$. Finally,

$$\begin{aligned} |\alpha + \lambda + z + \omega(\lambda, z)| &\geqslant C|\alpha + \lambda| = C\sqrt{(\alpha + \mathfrak{Re}\,\lambda)^2 + \mathfrak{Im}^2\,\lambda} \\ &\geqslant C\sqrt{\alpha^2 + |\lambda|^2} \geqslant C(\alpha + |\lambda|), \end{aligned}$$

where we used that α and $\mathfrak{Re}\,\lambda$ are positive.

Finally, as $z - z^2 + \lambda + \omega(\lambda, z) \in \Sigma_{\pi/2}$ and $\alpha + z^2 \in \Sigma_{2\theta}$, we have

$$\left|1 + \frac{z - z^2 + \lambda + \omega(\lambda, z)}{\alpha + z^2}\right| \geqslant C,$$

and the (27) follows as before. □

Lemma 4.3 *Let $\theta \in (0, \pi/4)$ be fixed. Then there exist constants $C > 0$ and $\delta > 0$ such that for all $z \in \Sigma_\theta$, $y \geqslant 0$ and $\lambda \in \Sigma_{\pi/2}$*

$$|m_i(\lambda, z, y)| \leqslant \frac{Ce^{-\delta y|z|}}{(1 + y\sqrt{|\lambda|})(\alpha + |\lambda|)}, \tag{28}$$

$$|zm_i(\lambda, z, y)| + |\partial_y m_i(\lambda, z, y)| \leqslant \frac{Ce^{-\delta y|z|}}{(1 + y\sqrt{|\lambda|})\sqrt{\alpha + |\lambda|}}, \tag{29}$$

for $i = 1, 2, 3$ and

$$|m_4(\lambda, z, y)| \leqslant Ce^{-\delta y|z|}.$$

Proof We will first prove the bounds for m_i and zm_i, for all i. The bound for $\partial_y m_i$ will be a simple consequence of these.

Ad m_1. Using (25) and (26) we find

$$\begin{aligned}|m_1(\lambda, z, y)| = |zm_0(\lambda, z, y)| &\leqslant \frac{C}{|\lambda|} \cdot \frac{e^{-\delta|z|y}}{1 + y\sqrt{|\lambda|}} \left|\frac{(\omega(\lambda, z) - z)(\omega(\lambda, z) + z)}{\alpha + \lambda + z + \omega(\lambda, z)}\right| \\ &\leqslant C\, \frac{e^{-\delta|z|y}}{1 + y\sqrt{|\lambda|}} \cdot \frac{1}{|\alpha + \lambda + z + \omega(\lambda, z)|} \\ &\leqslant \frac{C}{(1 + y\sqrt{|\lambda|})(\alpha + |\lambda|)}\, e^{-\delta|z|y}.\end{aligned}$$

Similarly, using (26) and (27) together, we get

$$\begin{aligned}|zm_1(\lambda, z, y)| &\leqslant C\, \frac{e^{-\delta|z|y}}{1 + y\sqrt{|\lambda|}} \cdot \frac{|z|}{|\alpha + \lambda + z + \omega(\lambda, z)|} \\ &\leqslant C\, \frac{e^{-\delta|z|y}}{1 + y\sqrt{|\lambda|}} \cdot \frac{1}{\sqrt{\alpha + |\lambda|}} \cdot \frac{|z|}{\sqrt{\alpha + |z|^2}},\end{aligned}$$

from which the estimate for zm_1 follows.

Ad m_3. Due to (24) and the inequality $(1 + cy\sqrt{|\lambda|})\, e^{-cy\sqrt{|\lambda|}} \leqslant 1 + \frac{1}{e} = C'$, we see that

$$|e^{-y\omega(\lambda,z)}| \leqslant e^{-cy(|z| + \sqrt{|\lambda|})} = e^{-cy\sqrt{|\lambda|}} \cdot e^{-cy|z|} \leqslant \frac{C'}{1 + cy\sqrt{|\lambda|}}\, e^{-\delta y|z|},$$

where we assumed δ to be small enough. Thanks to an obvious analogue of (26), we obtain

$$\begin{aligned}|m_3(\lambda, z, y)| &= \left|\frac{1}{\alpha+\lambda+\omega(\lambda, z)}\, e^{-y\omega(\lambda,z)}\right| \\ &\leqslant \frac{C'}{|\alpha+\lambda+\omega(\lambda, z)|}\cdot\frac{e^{-\delta y|z|}}{1+cy\sqrt{|\lambda|}} \leqslant \frac{C}{(1+cy\sqrt{|\lambda|})(\alpha+|\lambda|)}\, e^{-\delta y|z|}.\end{aligned}$$

Next, using a modification of both (26) and (27), we find

$$\begin{aligned}|zm_3(\lambda, z, y)| &\leqslant C'\frac{|z|}{|\alpha+\lambda+\omega(\lambda, z)|}\cdot\frac{e^{-\delta y|z|}}{1+cy\sqrt{|\lambda|}} \\ &\leqslant \frac{C}{(1+cy\sqrt{|\lambda|})\sqrt{\alpha+|\lambda|}}\, e^{-\delta y|z|}.\end{aligned}$$

Ad m_2. We have

$$\begin{aligned}m_2(\lambda, z, y) &= \partial_y m_0(\lambda, z, y) \\ &= \frac{1}{\lambda}\cdot\frac{z+\omega(\lambda, z)}{\alpha+\lambda+z+\omega(\lambda, z)}\cdot\frac{\partial}{\partial y}(e^{-y\omega(\lambda,z)}-e^{-yz}) \\ &= \frac{1}{\lambda}\cdot\frac{z+\omega(\lambda, z)}{\alpha+\lambda+z+\omega(\lambda, z)}\cdot\big(-\omega(\lambda, z)e^{-y\omega(\lambda,z)}+ze^{-yz}\pm ze^{-y\omega(\lambda,z)}\big) \\ &= \frac{z+\omega(\lambda, z)}{\alpha+\lambda+z+\omega(\lambda, z)}\cdot\frac{z(e^{-yz}-e^{-y\omega(\lambda,z)})+(z-\omega(\lambda, z))\,e^{-y\omega(\lambda,z)}}{\lambda} \\ &= -m_1(\lambda, z, y)-\frac{1}{\alpha+\lambda+z+\omega(\lambda, z)}\, e^{-y\omega(\lambda,z)}.\end{aligned}$$

We have already shown that the first term is bounded, and the second one can be estimated in the same way as $m_3(\lambda, z, y)$. Similarly,

$$\begin{aligned}zm_2(\lambda, z, y) &= z\partial_y m_0(\lambda, z, y) \\ &= -zm_1(\lambda, z, y)-\frac{z}{\alpha+\lambda+z+\omega(\lambda, z)}\, e^{-y\omega(\lambda,z)}\end{aligned}$$

is bounded as both terms are bounded due to the previous parts.

In the above, we proved (28) and the first part of (29); it remains to estimate $\partial_y m_i$. For $i = 1$, we have

$$\partial_y m_1(\lambda, z, y) = zm_2(\lambda, z, y),$$

hence, the bound of this follows from the already proven estimate. Next, for $i = 3$, we get

$$\partial_y m_3(\lambda, z, y) = -\frac{\omega(\lambda, z)}{\alpha+\lambda+\omega(\lambda, z)}\, e^{-y\omega(\lambda,z)}.$$

There obviously holds $|\omega(\lambda, z)| \leqslant C(\sqrt{|\lambda|} + |z|)$. Hence, the boundedness of $\partial_y m_3$ follows from already established inequalities for $i = 3$. Finally, for $i = 2$, we can compute that

$$\partial_y m_2(\lambda, z, y) = -\partial_y m_1(\lambda, z, y) + \frac{\omega(\lambda, z)}{\alpha + \lambda + z + \omega(\lambda, z)}\, e^{-y\omega(\lambda,z)}.$$

Invoking (27) for $i = 1, 3$, the estimate for m_2 is complete.

Concerning m_4, it is enough to invoke (21) and note that $|e^{-yz}| = e^{-(\cos\theta)y}$ whenever $z \in \Sigma_\theta$. □

We can now state and prove the following key result.

Lemma 4.4 *The solution $\boldsymbol{u}$ to (10)–(13) satisfies the estimates*

$$\|\boldsymbol{u}\|_{L^p(\Gamma)} + \|\boldsymbol{u}\|_{L^p(\Omega)} \leqslant \frac{K_1}{|\lambda|}\,\|\boldsymbol{\phi}\|_{L^p(\Gamma)}, \tag{30}$$

$$\|\nabla\boldsymbol{u}\|_{L^p(\Omega)} \leqslant \frac{K_2}{\sqrt{|\lambda|}}\,\|\boldsymbol{\phi}\|_{L^p(\Gamma)}. \tag{31}$$

Here the constants K_1 and K_2 depend on p and d, but are independent of λ.

Proof In view of (20) and Lemma 4.1, we have

$$K_1 \leqslant k(\lambda, 0) + \left(\int_0^\infty k^p(\lambda, y)\, dy\right)^{1/p},$$

where

$$k(\lambda, y) = \max_{i=1,2,3}\, \sup_{z\in\Sigma_\theta} |\lambda| m_i(\lambda, z, y).$$

By Lemma 4.3 we thus obtain

$$K_1 \leqslant \frac{c|\lambda|}{\alpha + |\lambda|}\left[1 + \left(\int_0^\infty \frac{\mathrm{d}y}{(1 + y\sqrt{|\lambda|})^p}\right)^{1/p}\right] \leqslant c\,\frac{|\lambda|^{1-1/2p}}{\alpha + |\lambda|}.$$

This is bounded independently of λ in a given range. The estimate of K_2 is obtained similarly. □

In the Hilbert case $p = 2$, one obtains the estimate of $\boldsymbol{u} \in W^{s,2}$ directly.

Lemma 4.5 *The solution $\boldsymbol{u}$ to (10)–(13) for $\lambda > 0$ satifies the estimates*

$$c(\lambda)\|\boldsymbol{\phi}\|_{W^{s-1/2,2}(\Gamma)} \leqslant \|\nabla\boldsymbol{u}\|_{W^{s,2}(\Omega)} \leqslant C(\lambda)\|\boldsymbol{\phi}\|_{W^{s-1/2,2}(\Gamma)} \tag{32}$$

for any $s \in [1/2, 3/2)$. In particular, $\boldsymbol{u} \in W^{s+1,2}(\Omega)$ if (and only if) $\boldsymbol{\phi} \in W^{s-1/2,2}(\Gamma)$.

Proof Let us comment first on the last part of the statement. If $\boldsymbol{\phi} \in W^{s-1/2,2}(\Gamma)$, then $\boldsymbol{u} \in W^{s+1,2}(\Omega)$ follows from (32) together with the energy inequality; see (6). Note that $\lambda > 0$ is needed since the Poincaré inequality in the half-space is not valid. On the other hand, if $\boldsymbol{u} \in W^{s+1,2}(\Omega)$, then $\boldsymbol{\phi} \in W^{s-1/2,2}(\Gamma)$ holds due to (13) and the trace theorem.

We will sketch the proof of (32) for $d = 2$, the general case being similar. The key point is that since $p = 2$, we can use the Fourier transform to characterize the Sobolev norm of ϕ as

$$\|\phi\|^2_{W^{s-1/2,2}(\Gamma)} = \int_{\mathbb{R}} (1 + |\xi|^2)^{s-1/2} \, |\widehat{\phi}(\xi)|^2 \, \mathrm{d}\xi,$$

which we want to relate to the velocity gradient norm

$$\|\nabla \boldsymbol{u}\|^2_{W^{s,2}(\Omega)} = \int_{\mathbb{R}^2} (1 + \xi^2 + \eta^2)^s \, |\mathcal{F}(\nabla \boldsymbol{u})(\xi, \eta)|^2 \, \mathrm{d}\xi \, \mathrm{d}\eta,$$

where $\mathcal{F}$ denotes the full Fourier transform in $\mathbb{R}^2$ of a function that is understood to be evenly extended with respect to the variable y. We will also frequently use the symbol $A \simeq B$ as a shorthand for $A \leqslant c_1 B$ and $B \leqslant c_2 A$, where $c_1, c_2 > 0$ are some irrelevant constants, and $A \lesssim B$ for a one-sided estimate.

Since $d = 2$, we have $\boldsymbol{u} = (u_1, u_2)$, and relations (19), (20) lead to formulas for the partial Fourier transform of $\widehat{\nabla} = (i\xi, \partial_y)$

$$\widehat{\nabla u_1}(\xi, y) = -\big(i\xi \partial_y m_0(z, y), \partial_y^2 m_0(z, y)\big)\widehat{\phi}(\xi),$$
$$\widehat{\nabla u_2}(\xi, y) = \big(i\xi z m_0(z, y), z\partial_y m_0(z, y)\big)\,\frac{i\xi}{|\xi|}\,\widehat{\phi}(\xi),$$

where $z = |\xi|$ and

$$m_0(z, y) = R(z)\big(e^{-\omega(z)y} - e^{-zy}\big).$$

Recall that $\omega(z) = \sqrt{\lambda + z^2}$; for simplicity, we omit the dependence on λ, which is now fixed. Note that likewise $R(z)$ is not relevant as $|R(z)| \simeq 1$.

It will be useful to set

$$Z_j(z, \eta) = \mathcal{F}_{y \to \eta}\left[\partial_y^j \left(e^{-\omega(z)y} - e^{-zy}\right)\right], \quad j = 0, 1, 2;$$

even extension with respect to y is used before applying the transform. The full Fourier transform of $\nabla \boldsymbol{u}$ now consists of

$$\mathcal{F}(\nabla u_1)(\xi, \eta) = -\big(i\xi Z_1(z, \eta), Z_2(z, \eta)\big)R(z)\widehat{\phi}(\xi),$$
$$\mathcal{F}(\nabla u_2)(\xi, \eta) = \big(i\xi z Z_0(z, \eta), z Z_1(z, \eta)\big)R(z)\widehat{\phi}(\xi).$$

The key step is to evaluate and estimate $Z_j(z, \eta)$. In view of the formula

$$\mathcal{F}_{y \to \eta}\left[\exp(-a|y|)\right] = \frac{a}{a^2 + \eta^2}, \quad \mathfrak{Re}\, a > 0,$$

we obtain

$$\begin{aligned}
Z_0(z,\eta) &= \frac{\omega(z)}{\omega^2(z)+\eta^2} - \frac{z}{z^2+\eta^2} = \frac{\lambda}{\omega(z)+z}\cdot\frac{\eta^2 - z\omega(z)}{(\omega^2(z)+\eta^2)(z^2+\eta^2)},\\
Z_1(z,\eta) &= \frac{z^2}{z^2+\eta^2} - \frac{\omega^2(z)}{\omega^2(z)+\eta^2} = \frac{-\lambda\eta^2}{(\omega^2(z)+\eta^2)(z^2+\eta^2)},\\
Z_2(z,\eta) &= \frac{\omega^3(z)}{\omega^2(z)+\eta^2} - \frac{z^3}{z^2+\eta^2}\\
&= \frac{\lambda(\eta^2(z^2+z\omega(z)+\omega^2(z)) + z^2\omega^2(z)))}{(z+\omega(z))(\omega^2(z)+\eta^2)(z^2+\eta^2)}.
\end{aligned}$$

Recalling now that $\lambda > 0$ is fixed, it is straightforward to deduce that $\omega(z) \simeq 1 + z$, $\omega^2(z) \simeq 1 + z^2$, and hence also $|\eta^2 - z\omega(z)| \lesssim \eta^2 + z^2 + z$, and $|\eta^2(z^2 + z\omega(z) + \omega^2(z)) + z^2\omega^2(z)| \simeq (\eta^2 + z^2)(1 + z^2)$. We thus can write

$$\begin{aligned}
|\mathcal{F}(\nabla u_1)(\xi,\eta)| &\simeq (|zZ_1(z,\eta)| + |Z_2(z,\eta)|)|\widehat{\phi}(\xi)| \simeq \frac{1+z}{1+z^2+\eta^2}\,|\widehat{\phi}(\xi)|,\\
|\mathcal{F}(\nabla u_2)(\xi,\eta)| &\simeq (|z^2Z_0(z,\eta)| + |zZ_1(z,\eta)|)|\widehat{\phi}(\xi)| \lesssim \frac{1+z}{1+z^2+\eta^2}\,|\widehat{\phi}(\xi)|.
\end{aligned}$$

This eventually leads to a key pointwise estimate

$$|\mathcal{F}(\nabla \boldsymbol{u})(\xi,\eta)| \simeq \frac{(1+|\xi|^2)^{1/2}}{1+|\xi|^2+\eta^2}\,|\widehat{\phi}(\xi)|.$$

It is now straightforward to deduce

$$\begin{aligned}
\|\nabla \boldsymbol{u}\|^2_{W^{s,2}(\Omega)} &= \int_{\mathbb{R}^2} (1+|\xi|^2+\eta^2)^s\,|\mathcal{F}(\nabla\boldsymbol{u})(\xi,\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\simeq \int_{\mathbb{R}} (1+|\xi|^2)\left(\int_{\mathbb{R}} (1+|\xi|^2+\eta^2)^{s-2}\,\mathrm{d}\eta\right)|\widehat{\phi}(\xi)|^2\,\mathrm{d}\xi\\
&\simeq \int_{\mathbb{R}} (1+|\xi|^2)^{s-1/2}\,|\widehat{\phi}(\xi)|^2\,\mathrm{d}\xi = \|\phi\|^2_{W^{s-1/2}(\Gamma)}.
\end{aligned}$$

Here we have also used that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\eta}{(b^2+\eta^2)^\sigma} \simeq b^{1-2\sigma}, \quad \sigma > 1/2. \qquad \square$$

5 Proof of the main results

Proof of Theorem 2.1 We start with formally recovering the resolvent estimate (7). Decompose the solution as $\boldsymbol{u} = \boldsymbol{u}^1 + \boldsymbol{u}^2$, where $\boldsymbol{u}^1$ solves the same problem, but with the zero Dirichlet boundary condition, and $\boldsymbol{u}^2$ has the zero right-hand side in Ω and takes care of the boundary condition. To be specific, the function $\boldsymbol{u}^1$ solves the system

$$\begin{aligned} (\lambda - \Delta)\boldsymbol{u}^1 + \nabla\pi^1 &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^1 &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^1 &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{33}$$

and $\boldsymbol{u}^2$ solves

$$\begin{aligned} (\lambda - \Delta)\boldsymbol{u}^2 + \nabla\pi^2 &= 0 \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^2 &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^2 \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma, \\ (\lambda + \alpha)\boldsymbol{u}^2 + 2\,[(\mathrm{D}\boldsymbol{u}^2)\boldsymbol{n}]_\tau &= \boldsymbol{\Phi} \quad \text{on } \Gamma, \end{aligned} \tag{34}$$

where

$$\boldsymbol{\Phi} = \boldsymbol{h} - 2\,[(\mathrm{D}\boldsymbol{u}^1)\boldsymbol{n}]_\tau. \tag{35}$$

Assume that $\lambda \in \mathbb{C}$ with $|\lambda| > \omega$, where $\omega > 0$ is fixed. Due to the standard theory (see e.g., [9, Theorem 1.2]) we know that

$$\|\lambda \boldsymbol{u}^1\|_{L^p(\Omega)} + \|\nabla^2 \boldsymbol{u}^1\|_{L^p(\Omega)} \leqslant c\|\boldsymbol{f}\|_{L^p(\Omega)}, \tag{36}$$

and, in particular,

$$\|\boldsymbol{u}^1\|_{L^p(\Omega)} \leqslant \frac{c}{|\lambda|}\,\|\boldsymbol{f}\|_{L^p(\Omega)}.$$

Because of the trace theorem we get

$$\|[(\mathrm{D}\boldsymbol{u}^1)\boldsymbol{n}]_\tau\|_{L^p(\Gamma)} \leqslant c\|\boldsymbol{f}\|_{L^p(\Omega)},$$

and thus, there also holds

$$\|\boldsymbol{\Phi}\|_{L^p(\Gamma)} = \|\boldsymbol{h} - 2\,[(\mathrm{D}\boldsymbol{u}^1)\boldsymbol{n}]_\tau\|_{L^p(\Gamma)} \leqslant c\,\|(\boldsymbol{f}, \boldsymbol{h})\|_{L^p(\Omega)\times L^p(\Gamma)}.$$

Note that c remains bounded as $|\lambda| \to \infty$. Hence, it suffices to show

$$\|(\boldsymbol{u}^2, \operatorname{tr} \boldsymbol{u}^2)\|_{L^p(\Omega)\times L^p(\Gamma)} \leqslant \frac{c}{|\lambda|}\,\|\boldsymbol{\Phi}\|_{L^p(\Gamma)}.$$

But this follows by Lemma 4.4 above, in particular, the estimate (30), because $\boldsymbol{\Phi} = (\boldsymbol{\phi}, 0)$.

So far, the argument was formal. Using a suitable approximation, one shows that the resolvent problem (2)–(5) has at least one solution. In view of (36) and the second part of Lemma 4.4, i.e., (31), we see that $\boldsymbol{u} \in W^{1,p}(\Omega)$. More precisely, we have

$$\|\boldsymbol{u}\|_{W^{1,p}(\Omega)} \leqslant C\big(\|\boldsymbol{f}\|_{L^p(\Omega)} + \|\boldsymbol{h}\|_{L^p(\Gamma)}\big), \tag{37}$$

where C depends on λ. Let us show uniqueness; clearly, it suffices to consider $\boldsymbol{u}$ a solution to the homogeneous problem. We set $\widetilde{\boldsymbol{f}} = |\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}}$, $\widetilde{\boldsymbol{h}} = |\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}}$ and observe that $\widetilde{\boldsymbol{f}} \in L^{p'}(\Omega)$, $\widetilde{\boldsymbol{h}} \in L^{p'}(\Gamma)$. By the above, there is some $\widetilde{\boldsymbol{u}} \in W^{1,p'}_{\sigma,n}(\Omega)$, a solution corresponding to these data. Testing the weak formulation for $\boldsymbol{u}$ by $\widetilde{\boldsymbol{u}}$ and vice versa, one deduces that

$$0 = \int_\Omega \widetilde{\boldsymbol{f}} \cdot \boldsymbol{u} \,\mathrm{d}x + \int_\Gamma \widetilde{\boldsymbol{h}} \cdot \boldsymbol{u} \,\mathrm{d}S = \|\boldsymbol{u}\|^p_{L^p(\Omega)} + \|\boldsymbol{u}\|^p_{L^p(\Gamma)} .$$

Hence $\boldsymbol{u} = 0$. □

Proof of Theorem 2.2 We split the solution $\boldsymbol{u} = \boldsymbol{u}^1 + \boldsymbol{u}^2$ as above, allowing moreover $\operatorname{div} \boldsymbol{u}^1 = g$ in (33). Hence, part (iii) of the theorem is taken into account.

Let us now consider the strong estimates, i.e., part (ii). Regularity of $\boldsymbol{u}^1$ follows as in [9], see also [10, Theorem IV.3.2]. Note that $\boldsymbol{\Phi} \in B^{1-1/p,p}_p(\Gamma)$ by the trace theorem [3, Theorem 7.39].

Concerning $\boldsymbol{u}^2$, we note that for a flat boundary, we have the so-called Robin boundary condition, whence the $W^{2,p}$-regularity follows by [19] or [14].

Nevertheless, we will sketch the proof that $\nabla\pi \in L^p(\Omega)$. In particular, we can adapt the argument from [9], which is here simplified in view of Lemma 4.1. (The estimate $\nabla\boldsymbol{u}^2$ is done similarly, or just follows by the regularity for the Laplacian again.)

Introduce an auxiliary function $\boldsymbol{v} = (\boldsymbol{v}', 0)$ as the solution to

$$\begin{aligned} (-\delta^2\Delta' - \partial_d^2)\,\boldsymbol{v}' &= 0 \quad \text{in } \Omega,\\ -\partial_d\,\boldsymbol{v}' &= \boldsymbol{\phi} \quad \text{on } \Gamma, \end{aligned}$$

where $\delta > 0$ is a suitably small number. We know (see e.g., [4, Theorem 14.1]) that

$$\|\nabla^2 \boldsymbol{v}\|_{L^p(\Omega)} \leqslant c\,\|\boldsymbol{\phi}\|_{B^{1-1/p,p}_p(\Gamma)}. \tag{38}$$

We proceed by noting that

$$\widehat{\nabla'\boldsymbol{v}}(\xi, x_d) = \frac{i\xi}{\delta|\xi|}\, e^{-\delta|\xi|x_d}\,\widehat{\boldsymbol{\phi}}(\xi), \quad \text{or} \quad \frac{i\xi}{|\xi|}\cdot\widehat{\boldsymbol{\phi}}(\xi) = \delta\, e^{\delta|\xi|x_d}\,\widehat{\nabla'\boldsymbol{v}}(\xi, x_d),$$

and hence, in view of (20), we can write

$$\widehat{\pi}(\xi, y) = f(z)\, e^{-\eta z y}\,\widehat{\nabla'\boldsymbol{v}}(\xi, y).$$

Here $\eta = 1 - \delta$, $z = |\xi|$, $y = x_d$ and

$$f(z) = \frac{\omega(\lambda, z) + z}{\alpha + \lambda + \omega(\lambda, z) + z}$$

is bounded and holomorphic in $\Sigma_{\pi/2}$. Hence

$$\widehat{\nabla\pi}(\xi, y) = (i\xi, \partial_y)\big[f(z)\,e^{-\eta z y}\,\widehat{\nabla' \boldsymbol{v}}(\xi, y)\big]$$
$$= \big[f(z)e^{-\eta z y}\big]\big(\widehat{(\nabla')^2\,\boldsymbol{v}}(\xi, y), (-\eta z + \partial_y)\,\widehat{\nabla' \boldsymbol{v}}(\xi, y)\big).$$

The trick is now to rewrite the last term as

$$(-\eta z + \partial_y)\,\widehat{\nabla' \boldsymbol{v}}(\xi, y) = -\frac{i\eta\xi}{|\xi|}\cdot\widehat{(\nabla')^2\,\boldsymbol{v}}(\xi, y) + \widehat{\partial_y \nabla' \boldsymbol{v}}(\xi, y).$$

We conclude that $\widehat{\nabla\pi} \simeq m_5(z, y)\widehat{\nabla^2\,\boldsymbol{v}}$, where $m_5(z, y)$ is holomorphic and bounded in z uniformly with respect to $y \geqslant 0$. By the second part of Lemma 4.1, cf. (22), we deduce

$$\|\nabla\pi\|_{L^p(\Omega)} \leqslant c\,\|\nabla^2\,\boldsymbol{v}\|_{L^p(\Omega)}.$$

In view of (38), the conclusion follows.

It remains to show part (i). The existence and regularity of $\boldsymbol{u}^1$ follows from [10, Theorem IV.3.3]. The key step is to bound $\mathrm{D}\boldsymbol{u}^1 \in B_p^{-1/p,p}(\Gamma)$, to which we need the following auxiliary result.

Lemma 5.1 *Let* $\boldsymbol{U} \in L^p(\Omega)$ *and* $\operatorname{div}\boldsymbol{U} \in W^{-1,p}(\Omega)$. *Then* $(\boldsymbol{U}\cdot\boldsymbol{n})_\tau \in B_{p;n}^{-1/p,p}(\Gamma)$.

Proof Integrating by parts, we write

$$\int_\Gamma (\boldsymbol{U}\cdot\boldsymbol{n})\,\boldsymbol{\varphi}\,\mathrm{d}S = \int_\Omega \boldsymbol{U}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega (\operatorname{div}\boldsymbol{U})\,\boldsymbol{\varphi}\,\mathrm{d}x$$

and note that the right-hand side can be bounded in terms of $\boldsymbol{\varphi} \in W_n^{1,p'}(\Omega)$. Denoting formally by γ the trace operator, it remains to observe that

$$\gamma(W_n^{1,p'}(\Omega)) = B_{p';n}^{1/p,p'}(\Gamma) = (B_{p;n}^{-1/p,p}(\Gamma))^*. \qquad \square$$

Applying now the above lemma to $\boldsymbol{U} = \nabla\boldsymbol{u}^1$, we see that $\boldsymbol{u}^2$ solves (34)–(35) where $\boldsymbol{\Phi} \in B_{p;n}^{-1/p,p}(\Gamma)$.

Set $\boldsymbol{v} = \mathfrak{J}_1^{x'}\boldsymbol{u}^2$, where $\mathfrak{J}_1^{x'}$ is the Bessel potential applied in the tangential directions x'. Obviously, $\boldsymbol{v}$ solves (34) with the boundary data $\mathfrak{J}_1^{x'}\boldsymbol{\phi} \in B_p^{1-1/p,p}$. Hence $\boldsymbol{v} \in W^{2,p}(\Omega)$, and thus $\boldsymbol{u}^2 \in W^{1,p}(\Omega)$ as required. $\square$

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Declarations

Conflict of interest The authors declare no conflict of interest.

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