manuscripta math. (2026) 177:26

Samuel Creedon · Volodymyr Mazorchuk

A stability phenomenon in Kazhdan-Lusztig combinatorics

Received: 18 April 2025 / Accepted: 5 March 2026 / Published online: 19 March 2026

Abstract. We prove that, when n goes to infinity, the expression, with respect to the dual Kazhdan-Lusztig basis, of the product $\underline{\hat{H}}_x \underline{H}_y$ of elements of the dual and the usual Kazhdan-Lusztig bases in the Hecke algebra of the symmetric group S_n stabilizes. As an application, we define the action of projective functors on the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_\infty$ and show that the subcategory of finite length objects is stable under this action. As a bonus, we also prove that this latter block is Koszul, answering, for this block, a question from Math. J. **19**(4), 655–693 (2019).

1. Introduction and description of the results

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra. To a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

of $\mathfrak{g}$, we can associate the corresponding Bernstein-Gelfand-Gelfand category $\mathcal{O}$, see [2,13]. The principal block $\mathcal{O}_0$ of $\mathcal{O}$ is equivalent to the category of finite dimensional modules over some finite dimensional associative algebra. This block is equipped with an action of the monoidal category $\mathscr{P}$ of projective endofunctors, see [1].

Both $\mathcal{O}_0$ and $\mathscr{P}$ admit $\mathbb{Z}$-graded lifts, ${}^{\mathbb{Z}}\mathcal{O}_0$ and ${}^{\mathbb{Z}}\mathscr{P}$, respectively. Taking the Grothendieck group, resp. ring, of the two latter categories, one obtains the right regular representation of the Hecke algebra $\mathbf{H}$ associated to the Weyl group W of $\mathfrak{g}$. In this picture, the Verma modules in $\mathcal{O}_0$ correspond to the standard basis $\{H_w : w \in W\}$ of the Hecke algebra, the indecomposable projective modules (as well as the indecomposable projective functors) correspond to the Kazhdan-Lusztig basis $\{\underline{H}_w : w \in W\}$, and the simple modules correspond to the dual Kazhdan-Lusztig basis $\{\underline{\hat{H}}_w : w \in W\}$. Consequently, the Kazhdan-Lusztig combinatorics of $\mathbf{H}$ becomes a very useful tool to study $\mathcal{O}_0$.

S. Creedon (✉) · V. Mazorchuk Department of Mathematics, Uppsala University, Box. 480, SE-75106 Uppsala, Sweden.
e-mail: samuel.creedon@math.uu.se

V. Mazorchuk
e-mail: mazor@math.uu.se

Mathematics Subject Classification 17B01 · 05E15 · 20F55.

https://doi.org/10.1007/s00229-026-01710-w

One particular representation-theoretic question for which Kazhdan-Lusztig combinatorics has shown to be very useful is the classical Kostant's problem for simple highest weight modules, see [14]. The original question asks whether the universal enveloping algebra surjects onto the algebra of adjointly locally finite endomorphisms of a module. In [16], a significant part of this question was reduced to the study of the properties of the elements in $\mathbf{H}$ that have the form $\underline{\hat{H}}_x\underline{H}_y$. At the level of $\mathcal{O}_0$, these elements of $\mathbf{H}$ correspond to the outcome of applying the indecomposable projective functor θ_y to the simple highest weight module L_x.

During our recent work on various aspects of Kostant's problem for simple highest weight $\mathfrak{sl}_n$-modules in [7,8], we computed, using SageMath and GAP3, a lot of products of the form $\underline{\hat{H}}_x\underline{H}_y$ in small ranks in the case of the Hecke algebra of the symmetric group S_n, with the outcome expressed as linear combinations of the elements of the dual KL basis. Not directly related to the questions we studied in [7,8], we made a bonus observation that, for fixed x and y, the expression of $\underline{\hat{H}}_x\underline{H}_y$ seems to stabilize when the rank increases. Let us directly point out that this is not an obvious phenomenon. The transformation matrix between the two bases changes very drastically with the rank due to the fact that the duality between the two bases swaps the identity e of W with the longest element w_0. And the element w_0 changes significantly with the rank. In particular, the expression $\underline{\hat{H}}_x\underline{H}_y$ itself does depend on the rank, but this dependence seems to stabilize. For comparison, the expression $\underline{H}_x\underline{H}_y$, when written in the KL basis, is independent of the rank.

The main result of the present paper, Theorem 1, confirms this phenomenon: given two elements $x, y \in S_n \subset S_k$, for $k \geq n$, we show that the expression $\underline{\hat{H}}_x\underline{H}_y$ with respect to the dual KL basis of S_k stabilizes for $k \gg 0$. As an application, in Theorem 7, we show that one can define an action of projective functors on the category of finite length modules in the principal block of the BGG category $\mathcal{O}$ for the algebra $\mathfrak{sl}_\infty$.

To prove the main result, we employ the Robinson-Schensted correspondence for the symmetric group, [23], and the properties of the Kazhdan-Lusztig μ-function listed in [28]. For our application, we heavily use the properties of the restricted inverse limits of categories, described in [9].

The paper is organized as follows: In Section 2 we collected all preliminaries necessary for the formulation of our main result, Theorem 1, which can be found in Subsection 2.4. This main theorem is proved in Section 3. Finally, Section 4 details the application to projective functors on category $\mathcal{O}$ for $\mathfrak{sl}_\infty$. As a bonus, we prove an extension vanishing property for simple modules in the principal block of $\mathcal{O}$ for $\mathfrak{sl}_\infty$, answering (in this particular case) a question from [6].

2. Hecke algebra of the symmetric group

2.1. Hecke algebra

For $n \in \mathbb{Z}_{>0}$, consider the *symmetric group* S_n and its *Hecke algebra* $\mathbf{H}_n$. The latter is a $\mathbb{Z}[v, v^{-1}]$-algebra with *standard basis* $\{H_w^{(n)} : w \in S_n\}$ and multiplication uniquely determined by

$$H_s^{(n)} H_w^{(n)} = \begin{cases} H_{sw}^{(n)}, & \text{if } \ell(sw) > \ell(w); \\ H_{sw}^{(n)} + (v^{-1} - v) H_w^{(n)}, & \text{if } \ell(sw) < \ell(w); \end{cases} \tag{1}$$

where s is an elementary transposition, $w \in S_n$ and ℓ denotes the length function. Note that $H_e^{(n)}$ is the unit element of this algebra and that $(H_s^{(n)})^2 = H_e^{(n)} + (v^{-1} - v)H_s^{(n)}$, for any elementary transposition s. Consequently, all $H_s^{(n)}$, and hence all $H_w^{(n)}$ as well, are invertible elements of $\mathbf{H}_n$.

We can view the algebra $\mathbf{H}_n$ as a subalgebra of $\mathbf{H}_{n+1}$ by sending $H_w^{(n)}$ to $H_w^{(n+1)}$, for $w \in S_n$.

2.2. *Kazhdan-Lusztig basis*

The *bar involution* on $\mathbf{H}_n$, denoted $h \mapsto \overline{h}$, is defined by sending v to v^{-1} and $H_s^{(n)}$ to $(H_s^{(n)})^{-1}$. The algebra $\mathbf{H}_n$ has, see [15,26], the *Kazhdan-Lusztig basis* $\{\underline{H}_w^{(n)} : w \in S_n\}$ which is uniquely determined by the properties

$$\overline{\underline{H}_w^{(n)}} = \underline{H}_w^{(n)} \qquad \text{and} \qquad \underline{H}_w^{(n)} = H_w^{(n)} + \sum_{x \in S_n} v\mathbb{Z}[v] H_x^{(n)}.$$

The entries of the transformation matrix between the two bases are known as the *Kazhdan-Lusztig polynomials* $\{p_{x,y}^{(n)} : x, y \in S_n\}$, that is, for $x, y \in S_n$, we have:

$$\underline{H}_y^{(n)} = \sum_{x \in S_n} p_{x,y}^{(n)} H_x^{(n)}. \tag{2}$$

Note that these polynomials do not really depend on n in the sense that, for $x, y \in S_n$, we have $p_{x,y}^{(n)} = p_{x,y}^{(n+1)}$. Also, note that $p_{x,y}^{(n)} \neq 0$ implies that $x \leq y$ with respect to the Bruhat order on S_n. In particular, this implies that our realization of $\mathbf{H}_n$ as a subalgebra of $\mathbf{H}_{n+1}$ is compatible with the KL basis, i.e. $\underline{H}_w^{(n)} = \underline{H}_w^{(n+1)}$, for $w \in S_n$.

2.3. *Dual Kazhdan-Lusztig basis*

We have a symmetric $\mathbb{Z}[v, v^{-1}]$-bilinear form on $\mathbf{H}_n$ defined, for $x, y \in S_n$, via

$$(H_x^{(n)}, H_y^{(n)}) = \delta_{x,y}.$$

With respect to this form, we have the *dual Kazhdan-Lusztig basis* $\{\underline{\hat{H}}_w^{(n)} : w \in S_n\}$ (see e.g. [18, Subsection 7.1]), that is, for $x, y \in S_n$, we have

$$(\underline{H}_x^{(n)}, \underline{\hat{H}}_y^{(n)}) = \delta_{x,y}. \tag{3}$$

2.4. *Main result*

For $h \in \mathbf{H}_n$ and $w \in S_n$, we define $[h : \underline{\hat{H}}_w^{(n)}] \in \mathbb{Z}[v, v^{-1}]$ to be the coefficient of $\underline{\hat{H}}_w^{(n)}$ in h when expressed in the dual Kazhdan-Lusztig basis, i.e.

$$h = \sum_{w \in S_n} [h : \hat{\underline{H}}_w^{(n)}] \hat{\underline{H}}_w^{(n)}.$$

Theorem 1. (a) *For all $n \geq 1$, all $x, y, w \in S_n$ and all $k, m > n$, we have*

$$[\hat{\underline{H}}_x^{(k)} \underline{H}_y^{(k)} : \hat{\underline{H}}_w^{(k)}] = [\hat{\underline{H}}_x^{(m)} \underline{H}_y^{(m)} : \hat{\underline{H}}_w^{(m)}].$$

(b) *For any $n \geq 1$, all $x, y \in S_n$, all $k > 2^{\ell(y)} n$ and all $w \in S_k$, we have that the inequality $[\hat{\underline{H}}_x^{(k)} \underline{H}_y^{(k)} : \hat{\underline{H}}_w^{(k)}] \neq 0$ implies $w \in S_{2^{\ell(y)} n}$.*

2.5. Example

Denote the transposition $(i, i+1)$ by s_i. Then we have:

$$\hat{\underline{H}}^{(4)}_{s_1 s_2 s_3 s_1 s_2 s_1} \underline{H}^{(4)}_{s_3} = \hat{\underline{H}}^{(4)}_{s_2 s_1 s_3 s_2 s_1} + (v + v^{-1}) \hat{\underline{H}}^{(4)}_{s_1 s_2 s_3 s_1 s_2 s_1}$$

and, for all $m > 4$, we have

$$\hat{\underline{H}}^{(m)}_{s_1 s_2 s_3 s_1 s_2 s_1} \underline{H}^{(m)}_{s_3} = \hat{\underline{H}}^{(m)}_{s_2 s_1 s_3 s_2 s_1} + \hat{\underline{H}}^{(m)}_{s_1 s_2 s_3 s_1 s_2 s_1 s_4} + (v + v^{-1}) \hat{\underline{H}}^{(m)}_{s_1 s_2 s_3 s_1 s_2 s_1}.$$

The latter is easy to check, using GAP3, for $m \leq 8$.

3. Proof of Theorem 1

3.1. Proof of Claim (1)

Using (3), we can interpret the value $[\hat{\underline{H}}_x^{(k)} \underline{H}_y^{(k)} : \hat{\underline{H}}_w^{(k)}]$ as $(\hat{\underline{H}}_x^{(k)} \underline{H}_y^{(k)}, \underline{H}_w^{(k)})$. The latter equals $(\hat{\underline{H}}_x^{(k)}, \underline{H}_{y^{-1}}^{(k)} \underline{H}_w^{(k)})$ and hence coincides with the multiplicity of $\underline{H}_x^{(k)}$ in $\underline{H}_{y^{-1}}^{(k)} \underline{H}_w^{(k)}$. Since the canonical embeddings of the Hecke algebras are compatible with the KL basis, this multiplicity does not depend on k.

It remains to prove Claim (1).

3.2. Reduction to simple reflections

As a first step, we claim that it is enough to prove Theorem 1 under the assumption that $y = s$ is a simple reflection.

For this, consider the structure constants of $\mathbf{H}_n$ with respect to the Kazhdan-Lusztig basis:

$$\underline{H}_a^{(n)} \underline{H}_b^{(n)} = \sum_{c \in S_n} \gamma_{a,b}^c \underline{H}_c^{(n)}.$$

Note that $\gamma_{a,b}^c \in \mathbb{Z}[v, v^{-1}]$ does not depend on n. From [25, Theorem 2] it follows that each $\gamma_{a,b}^c$ has non-negative integer coefficients. Furthermore, if $y = s_1 s_2 \ldots s_r$

is a reduced expression, then $\underline{H}_y^{(n)}$ appears with a non-zero coefficient in the decomposition of the product

$$\underline{H}_{s_1}^{(n)}\underline{H}_{s_2}^{(n)}\dots\underline{H}_{s_r}^{(n)}. \tag{4}$$

If Theorem 1 is proved under the assumption that $y = s$ is a simple reflection, then, using distributivity of algebraic operations and induction on r, Claim (1) of Theorem 1 follows in the general case.

Therefore, in the remainder of this section, we will prove Claim (1) of Theorem 1 under the assumption that $y = s$ is a simple reflection.

3.3. The μ-function

For $x \in S_n$ and a simple reflection s, recall that

$$\underline{\hat{H}}_x^{(n)}\underline{H}_s^{(n)} = \begin{cases} (v+v^{-1})\underline{\hat{H}}_x^{(n)} + \sum\limits_{u\in S_n\,:\,\ell(us)>\ell(u)} \mu^{(n)}(x,u)\underline{\hat{H}}_u^{(n)}, & \text{if } \ell(xs) < \ell(x); \\ 0, & \text{if } \ell(xs) > \ell(x). \end{cases} \tag{5}$$

Here $\mu^{(n)}(x, u) \in \mathbb{Z}_{\geq 0}$ is the value of the *Kazhdan-Lusztig μ-function*, see [15]. Directly from the definition of μ it follows that, for $x, u \in S_n \subset S_{n+1}$, we have $\mu^{(n)}(x, u) = \mu^{(n+1)}(x, u)$. Therefore, from now on, we will drop the superscript (n) for μ.

We need to recall some basic properties of the μ-function. Denote by $\prec$ the Bruhat order on S_n. For $w \in S_n$, denote by $\mathbf{lds}(w)$ and $\mathbf{rds}(w)$ the *left and right descent sets* of w, respectively. This means that $\mathbf{lds}(w)$ consists of all simple reflections $t \in S_n$ such that $tw \prec w$. Also, $\mathbf{rds}(w)$ consists of all simple reflections $t \in S_n$ such that $wt \prec w$. The function μ has the following properties, see [3,15,28]:

- $\mu(x, u) \neq 0$ only if $x \prec u$ or $u \prec x$;
- $\mu(x, u) = \mu(u, x)$;
- $\mu(x, u) = 1$ if $x \prec u$ and $\ell(x) = \ell(u) - 1$;
- if $x \prec u$ and $\ell(x) < \ell(u) - 1$, then $\mu(x, u) = 0$ unless $\mathbf{lds}(u) \subset \mathbf{lds}(x)$ and $\mathbf{rds}(u) \subset \mathbf{rds}(x)$.

3.4. Cell combinatorics

For two elements $a, b \in S_n$, we write $a \geq_L b$ provided that there exists $c \in S_n$ such that $\underline{H}_a^{(n)}$ appears with a non-zero coefficient when expressing $\underline{H}_c^{(n)}\underline{H}_b^{(n)}$ in the KL-basis. This defines the so-called *left pre-order* on S_n and the corresponding equivalence classes ($a \sim_L b$ if and only if $a \geq_L b$ and $b \geq_L a$) are called *left cells*. The *right pre-order* $\geq_R$ and the *right cells* ($\sim_R$) are defined similarly using $\underline{H}_b^{(n)}\underline{H}_c^{(n)}$. The *two-sided pre-order* $\geq_J$ and the *two-sided cells* ($\sim_J$) are defined similarly using $\underline{H}_{c_1}^{(n)}\underline{H}_b^{(n)}\underline{H}_{c_2}^{(n)}$. Note that the identity $e \in S_n$ is the minimum element with respect to all three pre-orders while the longest element $w_0 \in S_n$ is the maximum element with respect to all three pre-orders.

In [15, Section 5], one can find the following combinatorial description of cells. Recall the classical *Robinson-Schensted correspondence*

$$S_n \overset{1:1}{\leftrightarrow} \coprod_{\lambda \vdash n} \mathbf{SYT}_\lambda \times \mathbf{SYT}_\lambda$$

that provides a bijection between S_n and pairs of *standard Young tableaux* of the same shape (the shape itself is a partition of n). We denote the map from the left hand side to the right hand side of the above by **RS** and write $\mathbf{RS}(w) = (P_w, Q_w)$, where P_w is the *insertion tableau* and Q_w is the *recording tableau*, see [22,23]. Then, for $a, b \in S_n$, we have

- $a \sim_L b$ if and only if $Q_a = Q_b$;
- $a \sim_R b$ if and only if $P_a = P_b$;
- $a \sim_J b$ if and only if the shape of P_a is the same as the shape of P_b.

Consequently, for $a, b \in S_n \subset S_{n+1}$, we have $a \sim_L b$ in S_n if and only if $a \sim_L b$ in S_{n+1} and similarly for the right and two-sided cells. Hence there is no confusion that we do not use any superscripts in the notation.

One important property of the cell combinatorics is given by Lemma 13(a) of [17]: for $a, b, c \in S_n$, the fact that $[\underline{\hat{H}}_a^{(n)} \underline{H}_b^{(n)} : \underline{\hat{H}}_c^{(n)}] \neq 0$ implies $c \leq_R a$.

The rows of Young tableaux are numbered from top to bottom and the columns of Young tableaux are numbered from left to right. It is important that the descent sets of w can be directly read off from the tableaux P_w and Q_w. We have, see [22]:

- $s_i = (i, i+1) \in \mathbf{lds}(w)$ if and only if $i+1$ appears in P_w in a row with strictly higher number than the number of the row of P_w containing i;
- $s_i = (i, i+1) \in \mathbf{rds}(w)$ if and only if $i+1$ appears in Q_w in a row with strictly higher number than the number of the row of Q_w containing i.

Finally, we note that, for $w \in S_n \subset S_{n+1}$, the S_{n+1}-insertion tableau for w is obtained from the S_n-insertion tableau for w by adding a new box with $n+1$ in the first row. Similarly for the recording tableau.

In [10], it is shown that the order $\leq_J$ on the set of all partitions of n coincides with the (opposite of the) dominance order. We note that no combinatorial interpretations of $\leq_L$ (or $\leq_R$) are known. This is a major obstacle for many arguments. For some partial results, see e.g. [12, Corollary 3.26].

3.5. Stability for the μ-function

For $w \in S_n$, denote by $\mathbf{m}(w)$ the maximal positive integer k such that $w(k) \neq k$ (we set $\mathbf{m}(e) = 0$). The main observation that we need for our proof of Theorem 1 is the following:

Lemma 2. *Let $a, b \in S_n$ be such that $\mu(a, b) \neq 0$ and $\mathbf{rds}(a) \setminus \mathbf{rds}(b) \neq \varnothing$. Then $\mathbf{m}(b) \leq 2\mathbf{m}(a)$.*

Proof. We know that $\mu(a,b)\neq 0$ implies $a\prec b$ or $b\prec a$. If $b\prec a$, then the claim of the lemma is obvious. Therefore it remains to consider the case $a\prec b$.

Assume $\ell(a)=\ell(b)-1$ and let $t\in \mathbf{rds}(a)\setminus\mathbf{rds}(b)$. Assume, for contradiction, that $\mathbf{m}(b)>2\mathbf{m}(a)$ (note that $\mathbf{m}(a)\neq 0$ as we assume that $\mathbf{rds}(a)\neq\varnothing$ and hence, in particular, $a\neq e$). Then $b=as_{\mathbf{m}(b)-1}$ and $s_{\mathbf{m}(b)-1}$ commutes with all simple reflections involved in a, in particular, with t. Consequently, $t\in\mathbf{rds}(b)$, a contradiction.

Finally, assume $\ell(a)<\ell(b)-1$. Then we must have both $\mathbf{rds}(b)\subset\mathbf{rds}(a)$ and $\mathbf{lds}(b)\subset\mathbf{lds}(a)$. We assume $\mathbf{rds}(a)\setminus\mathbf{rds}(b)\neq\varnothing$. Let $t\in\mathbf{rds}(a)\setminus\mathbf{rds}(b)$. Then, from Formula (5), we have that $\underline{\hat{H}}_b^{(k)}$ appears with a non-zero coefficient in the decomposition of $\underline{\hat{H}}_a^{(k)}\underline{H}_t^{(k)}$ with respect to the dual KL basis. In particular, $b\leq_J a$.

Let λ be the shape of P_a and μ be the shape of P_b. Then we have $\lambda=(\lambda_1,\lambda_2,\dots)$ and $\mu=(\mu_1,\mu_2,\dots)$. Since $b\leq_J a$, the partition μ dominates λ, in particular, $n-\mu_1\leq n-\lambda_1$.

In order to prove that $\mathbf{m}(b)\leq 2\mathbf{m}(a)$, we need to show that, for any $i>2\mathbf{m}(a)$, the box i appears in the first row of both P_b and Q_b. We do it for P_b and for Q_b the argument is similar. Assume that i does not appear in the first row of P_b. Since $\mathbf{lds}(b)\subset\mathbf{lds}(a)$ and $s_{i-1}\notin\mathbf{lds}(a)$, we have $s_{i-1}\notin\mathbf{lds}(b)$. Hence $i-1$ must appear in P_b in the same row as i or in a row with higher number. Similarly for $i-2$ and so on all the way down to $\mathbf{m}(a)$. However, λ has at most $\mathbf{m}(a)-1$ boxes combined in rows 2, 3 and so on. Since μ dominates λ, we have that μ has at most $\mathbf{m}(a)-1$ boxes combined in rows 2, 3 and so on as well. This means that we will eventually find an element in $\mathbf{lds}(b)$ which is not in $\mathbf{lds}(a)$, a contradiction. The claim of the lemma follows. □

3.6. *Completing the proof of Theorem 1(1)*

Now we can prove Claim (1) of Theorem 1 under the assumption that $y=s$ is a simple reflection. If $x=e$, then $\underline{\hat{H}}_e^{(k)}\underline{H}_s^{(k)}=0$, so it remains to consider the case $x\neq e$. If we look at Formula (5) and take Lemma 2 into account, we see that, if we fix x, and hence also $\mathbf{m}(x)$, and let n increase, then all non-trivial summands in Formula (5) will correspond to u such that $\mathbf{m}(u)\leq 2\mathbf{m}(x)$. In other words, all these summands will be the same, for n big enough. This completes the proof.

4. Application: category $\mathcal{O}$ for $\mathfrak{sl}_\infty$

4.1. *Classical category $\mathcal{O}$ for $\mathfrak{sl}_n$*

For a Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ its universal enveloping algebra.

For $n\geq 2$, consider the Lie algebra $\mathfrak{sl}_n$ over the complex numbers and consider the standard triangular decomposition

$$\mathfrak{sl}_n=\mathfrak{n}_-^{(n)}\oplus\mathfrak{h}^{(n)}\oplus\mathfrak{n}_+^{(n)}$$

of this Lie algebra (here $\mathfrak{n}_-^{(n)}$ is the subalgebra of all strictly lower triangular matrices, $\mathfrak{n}_+^{(n)}$ is the subalgebra of all strictly upper triangular matrices, and $\mathfrak{h}^{(n)}$ is the Cartan subalgebra of all traceless diagonal matrices). Associated to this triangular decomposition, we have the BGG category $\mathcal{O}$, see [2,13], defined as the full subcategory of the category of all finitely generated $\mathfrak{sl}_n$-modules consisting of all $\mathfrak{h}$-semi-simple and $U(\mathfrak{n}_+)$-locally finite modules.

Consider the *principal block* $\mathcal{O}_0$ of $\mathcal{O}$, defined as the direct summand containing the trivial $\mathfrak{sl}_n$-module. To emphasize dependence on n (which we will need later), we will write $\mathcal{O}_0^{(n)}$. Simple objects in $\mathcal{O}_0^{(n)}$ are indexed naturally by the elements of the symmetric group S_n, the latter being the Weyl group of $(\mathfrak{sl}_n, \mathfrak{h}^{(n)})$. We denote by $L_w^{(n)}$ the simple object corresponding to $w \in S_n$. As an $\mathfrak{sl}_n$-module, $L_w^{(n)}$ is the simple highest weight module with highest weight $w \cdot 0$, where $\cdot$ is the dot-action of S_n on $(\mathfrak{h}^{(n)})^*$.

The category $\mathcal{O}_0^{(n)}$ carries the natural action of the monoidal category $\mathscr{P}^{(n)}$ of *projective endofunctors*, see [1]. A projective functor is defined as a summand of the functor of tensoring with a finite dimensional $\mathfrak{sl}_n$-module. Indecomposable projective functors are in bijection with the elements of S_n, we write $w \mapsto \theta_w^{(n)}$, where $\theta_w^{(n)}$ is uniquely determined by the property that it maps the indecomposable projective cover of $L_e^{(n)}$ to the indecomposable projective cover of $L_w^{(n)}$.

The split Grothendieck ring $[\mathscr{P}^{(n)}]_\oplus$ acts on the Grothendieck group of $\mathcal{O}_0^{(n)}$ and this gives rise to a right regular representation of the integral group algebra $\mathbb{Z}[S_n]$.

The category $\mathcal{O}_0^{(n)}$ is equivalent to the module category of a finite dimensional associative algebra $A^{(n)}$ which is known to be Koszul, see [24]. With respect to the corresponding $\mathbb{Z}$-grading, both $\mathcal{O}_0^{(n)}$ and $\mathscr{P}^{(n)}$ admit (unique up to equivalence) graded lifts, ${}^{\mathbb{Z}}\mathcal{O}_0^{(n)}$ and ${}^{\mathbb{Z}}\mathscr{P}^{(n)}$, respectively. Interpreting the corresponding Grothendieck groups as $\mathbb{Z}[v, v^{-1}]$-modules, where v acts as the shift of grading, the decategorification of the action of ${}^{\mathbb{Z}}\mathscr{P}^{(n)}$ on ${}^{\mathbb{Z}}\mathcal{O}_0^{(n)}$ gives the right regular $\mathbf{H}$-module, see [27].

Under these identifications, $\theta_w^{(n)}$ corresponds to $\underline{H}_w^{(n)}$ and $L_w^{(n)}$ to $\underline{\hat{H}}_w^{(n)}$, so that we can directly see the relevance of the formula in Theorem 1 in this context: this formula combinatorially describes the outcome of the action of $\theta_y^{(k)}$ on $L_x^{(k)}$.

4.2. Projective limit

Fix now $n \in \mathbb{Z}_{>0}$ such that $n > 2$ and consider the standard embedding $\mathfrak{sl}_{n-1} \subset \mathfrak{sl}_n$ with respect to the left upper corner of a matrix. Note that we have $\mathfrak{sl}_{n-1} \cap \mathfrak{n}_-^{(n)} = \mathfrak{n}_-^{(n-1)}$, $\mathfrak{sl}_{n-1} \cap \mathfrak{n}_+^{(n)} = \mathfrak{n}_+^{(n-1)}$ and $\mathfrak{sl}_{n-1} \cap \mathfrak{h}^{(n)} = \mathfrak{h}^{(n-1)}$.

Let $\mathbf{R}^{(n)} \subset (\mathfrak{h}^{(n)})^*$ be the root system of $(\mathfrak{sl}_n, \mathfrak{h}^{(n)})$. Let $\mathbb{Z}\mathbf{R}^{(n)} \subset (\mathfrak{h}^{(n)})^*$ be the additive subgroup generated by $\mathbf{R}^{(n)}$. Then, for each $M \in \mathcal{O}_0^{(n)}$, we have

$$M \cong \bigoplus_{\lambda \in \mathbb{Z}\mathbf{R}^{(n)}} M_\lambda, \quad \text{where} \quad M_\lambda = \{v \in M : hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}^{(n)}\}.$$

Let $\mathbf{Q}^{(n-1)} \subset \mathbf{R}^{(n)}$ be the subset of all roots corresponding to the elements of $\mathfrak{sl}_{n-1}$ and $\mathbb{Z}\mathbf{Q}^{(n-1)} \subset (\mathfrak{h}^{(n)})^*$ be the corresponding additive subgroup. Then, for any $M \in \mathcal{O}_0^{(n)}$, the space

$$\mathbf{F}_{n-1}^{n} M := \bigoplus_{\lambda \in \mathbb{Z}\mathbf{Q}^{(n-1)}} M_\lambda$$

is invariant under the $\mathfrak{sl}_{n-1}$-action and is, in fact, an object of $\mathcal{O}_0^{(n-1)}$. By restriction, this defines a functor

$$\mathbf{F}_{n-1}^{n} : \mathcal{O}_0^{(n)} \to \mathcal{O}_0^{(n-1)}$$

and hence gives rise to an inverse system of categories and functors

$$\begin{tikzcd} \mathcal{O}_0^{(2)} & \mathcal{O}_0^{(3)} \arrow[l, "\mathbf{F}_2^3"'] & \mathcal{O}_0^{(4)} \arrow[l, "\mathbf{F}_3^4"'] & \dots \arrow[l, "\mathbf{F}_4^5"'] \end{tikzcd} \tag{6}$$

We note that there is an alternative way to think of $\mathbf{F}_{n-1}^{n}$, see [5]. Consider the usual embedding $S_{n-1} \subset S_n$ and recall that the simple objects of $\mathcal{O}_0^{(n)}$ are $L_w^{(n)}$, where $w \in S_n$. Let $\mathcal{X}_n$ be the Serre subcategory of $\mathcal{O}_0^{(n)}$ generated by all simples L_w, where $w \in S_n \setminus S_{n-1}$. Then $\mathbf{F}_{n-1}^{n}$ induces and equivalence between $\mathcal{O}_0^{(n)}/\mathcal{X}_n$ and $\mathcal{O}_0^{(n-1)}$ and this equivalence sends $L_w^{(n)}$ to $L_w^{(n-1)}$, for $w \in S_{n-1}$. This alternative description, in particular, implies that (6) admits a graded lift

$$\begin{tikzcd} {}^{\mathbb{Z}}\mathcal{O}_0^{(2)} & {}^{\mathbb{Z}}\mathcal{O}_0^{(3)} \arrow[l, "{}^{\mathbb{Z}}\mathbf{F}_2^3"'] & {}^{\mathbb{Z}}\mathcal{O}_0^{(4)} \arrow[l, "{}^{\mathbb{Z}}\mathbf{F}_3^4"'] & \dots \arrow[l, "{}^{\mathbb{Z}}\mathbf{F}_4^5"'] \end{tikzcd} \tag{7}$$

We define the category $\mathcal{C}$ as the limit of (6) and the category ${}^{\mathbb{Z}}\mathcal{C}$ as the limit of (7), see e.g. [9, Section 3].

Note that all categories $\mathcal{O}_0^{(n)}$ are length categories and the functors $\mathbf{F}_{n-1}^{n}$ are exact and send simple objects to either simple objects or zero. Therefore they are "shortening" in the terminology of [9, Section 4], so that we can consider the *restricted limits* of the above inverse systems, which we denote by $\mathcal{C}^{\mathrm{res}}$ and ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$, respectively. We will provide more details on these later, in Subsection 4.4. An important property of both $\mathcal{C}^{\mathrm{res}}$ and ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$ is that they are length categories, see [9, Lemma 4.1.3].

4.3. Simple highest weight modules for $\mathfrak{sl}_\infty$

This subsection is inspired by [6, 19–21].

Consider the directed system $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3 \hookrightarrow \dots$ given by inclusions with respect to the left upper corner and denote by $\mathfrak{sl}_\infty$ the Lie algebra that is the limit of this system. Taking the component-wise limit, gives a triangular decomposition

$$\mathfrak{sl}_\infty = \mathfrak{n}_-^{(\infty)} \oplus \mathfrak{h}^{(\infty)} \oplus \mathfrak{n}_+^{(\infty)}.$$

In this setup, one can define a few variations of the classical category $\mathcal{O}$, see e.g. [6, 19–21]. We will restrict our attention to the following category.

For $\lambda \in (\mathfrak{h}^{(\infty)})^*$, consider the *Verma module* $M(\lambda)^{(\infty)}$ defined as the quotient of $U(\mathfrak{sl}_\infty)$ by the left ideal generated by $\mathfrak{n}_+^{(\infty)}$ and all $h - \lambda(h)$, where $h \in \mathfrak{h}^{(\infty)}$ (note that this ideal is not finitely generated). The module $M(\lambda)^{(\infty)}$ is an $\mathfrak{h}^{(\infty)}$-weight module with finite dimensional weight spaces. As usual, the Verma module $M(\lambda)^{(\infty)}$ has simple top, denoted $L(\lambda)^{(\infty)}$. The module $L(\lambda)^{(\infty)}$ is the simple highest weight module with highest weight λ.

Consider the full subcategory $\mathcal{O}^{(\infty,\mathrm{fl})}$ of the category of all weight $\mathfrak{h}^{(\infty)}$-weight $\mathfrak{sl}_\infty$-modules which consists of all objects that have a finite composition series with simple highest weight subquotients. Note that the analogue of $\mathcal{O}^{(\infty,\mathrm{fl})}$ for each $\mathfrak{sl}_n$, with $n < \infty$, is the usual BGG category $\mathcal{O}$. We denote by $\mathcal{O}_0^{(\infty,\mathrm{fl})}$ the indecomposable direct summand (i.e. the block) of $\mathcal{O}^{(\infty,\mathrm{fl})}$ which contains the trivial $\mathfrak{sl}_\infty$-module.

We emphasize that, in general, the module $M(\lambda)^{(\infty)}$ has infinite length and hence does not belong to $\mathcal{O}^{(\infty,\mathrm{fl})}$. For example, $M(0)^{(\infty)}$ does not belong to $\mathcal{O}^{(\infty,\mathrm{fl})}$.

Denote by S_∞ the limit of the directed system $S_1 \hookrightarrow S_2 \hookrightarrow \dots$ defined via the obvious inclusions. The group S_∞ acts on $(\mathfrak{h}^{(\infty)})^*$ both in the obvious way and via the dot-action.

For $n \geq 2$, let $\mathbf{Q}^{(n)} \subset (\mathfrak{h}^{(\infty)})^*$ be the set of all roots of the $\mathfrak{sl}_n$-part of $\mathfrak{sl}_\infty$. Let $\mathbb{Z}\mathbf{Q}^{(n)}$ be the additive subgroup of $(\mathfrak{h}^{(\infty)})^*$ spanned by $\mathbf{Q}^{(n)}$. For $M \in \mathcal{O}_0^{(\infty,\mathrm{fl})}$ consider the $\mathfrak{sl}_n$-module

$$\mathbf{F}_n^\infty M := \bigoplus_{\mu \in \mathbb{Z}\mathbf{Q}^{(n)}} M_\mu. \tag{8}$$

Then $\mathbf{F}_n^\infty$ gives rise to a functor from $\mathcal{O}_0^{(\infty,\mathrm{fl})}$ to $\mathcal{O}_0^{(n)}$ which sends $L(w \cdot 0)^{(\infty)}$ to $L_w^{(n)}$, if $w \in S_n$, and to zero otherwise.

Proposition 3. *For $\lambda \in (\mathfrak{h}^{(\infty)})^*$, we have $L(\lambda)^{(\infty)} \in \mathcal{O}_0^{(\infty,\mathrm{fl})}$ if and only if $\lambda \in S_\infty \cdot 0$.*

Proof. The "only if" part follows from [20, Proposition 3.2].

Let $w \in S_\infty$ and s be a simple reflection such that $sw \succ w$. To prove the "if" part, it is enough to show that $\mathrm{Ext}^1_{\mathcal{O}_0^{(\infty)}}(L(w \cdot 0)^{(\infty)}, L(sw \cdot 0)^{(\infty)}) \neq 0$. For this, consider the Verma module $M(w \cdot 0)^{(\infty)}$. Since s is a simple reflection and $sw \succ w$, the dimension of $M(w \cdot 0)^{(\infty)}_{sw \cdot 0}$ equals 1.

Let N be the biggest submodule of $M(w \cdot 0)^{(\infty)}$ with the property $N_{w \cdot 0} = N_{sw \cdot 0} = 0$. Then $M(w \cdot 0)^{(\infty)}/N$ is an indecomposable module with simple top $L(sw \cdot 0)^{(\infty)}$ and simple socle $L(ws \cdot 0)^{(\infty)}$. If $M(w \cdot 0)^{(\infty)}/N$ has length 2, we are done. If not, $M(w \cdot 0)^{(\infty)}/N$ has some simple subquotient of the form $L(x \cdot 0)^{(\infty)}$, for some $x \in S_\infty$ such that $x \neq w$ and $x \neq sw$.

Choose n such that $w, sw, x \in S_n$ and consider $K := \mathbf{F}_n^\infty(M(w \cdot 0)^{(\infty)}/N)$. This module is, by construction, a quotient of the Verma module $M(w \cdot 0)^{(n)}$. As the space $\mathrm{Ext}^1_{\mathcal{O}_0^{(n)}}(L(w \cdot 0)^{(n)}, L(sw \cdot 0)^{(n)})$ is one-dimensional, the module $M(w \cdot 0)^{(n)}$ has a length 2 quotient with simple top $L(w \cdot 0)^{(n)}$ and simple socle $L(sw \cdot 0)^{(n)}$. Consequently, any $\mathfrak{sl}_n$-submodule of K generated by any $L(x \cdot 0)^{(\infty)}$ as above does not contain any subquotient of the form $L(sw \cdot 0)^{(n)}$. Since this is true for any x as above and any $n \gg 0$, we get a contradiction with the fact that $L(sw \cdot 0)^{(\infty)}$ is the simple socle of $M(w \cdot 0)^{(\infty)}/N$. Therefore our assumption that $M(w \cdot 0)^{(\infty)}/N$ has length different from 2 is false. The claim of the proposition follows. □

4.4. Action of projective functors on restricted limits

For each $w \in S_\infty$, we want to define an endofunctor $\theta_w^{(\infty)}$ of the category $\mathcal{C}$ (and then of the category $\mathcal{C}^{\text{res}}$ as well). For this, recall from [9, Definition 3.1.1] that objects of $\mathcal{C}$ are pairs $(\{C_i : i \geq 2\}, \{\phi_i : i \geq 3\})$, where $C_i \in \mathcal{O}_0^{(i)}$ and $\phi_i : \mathbf{F}_{i-1}^i C_i \xrightarrow{\cong} C_{i-1}$ is an isomorphism. A morphism f from an object $(\{C_i : i \geq 2\}, \{\phi_i : i \geq 3\})$ to $(\{C_i' : i \geq 2\}, \{\phi_i' : i \geq 3\})$ consists of $f_i : C_i \to C_i'$, for all $i \geq 2$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{F}_{i-1}^i C_i \arrow[r, "\phi_i"] \arrow[d, "\mathbf{F}_{i-1}^i f_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \\
\mathbf{F}_{i-1}^i C_i' \arrow[r, "\phi_i'"] & C_{i-1}'
\end{tikzcd}$$

Composition of morphisms is component-wise and the identity morphisms are given by the identities.

If w is the identity, we just define θ_e^∞ as the identity endofunctor of $\mathcal{C}$. If $w \neq e$, let $k = \mathbf{m}(w)$. By [5, Theorem 37], for all $i > k$, we can fix isomorphisms $\alpha_i : \mathbf{F}_{i-1}^i \circ \theta_w^{(i)} \cong \theta_w^{(i-1)} \circ \mathbf{F}_{i-1}^i$.

We define the action of $\theta_w^{(\infty)}$ on the object $(\{C_i : i \geq 2\}, \{\phi_i : i \geq 3\})$ as the object $(\{D_i : i \geq 2\}, \{\psi_i : i \geq 3\})$, where

$$D_i = \begin{cases} \theta_w^{(i)} C_i, & i \geq k; \\ \mathbf{F}_i^{i+1} \circ \cdots \circ \mathbf{F}_{k-2}^{k-1} \circ \mathbf{F}_{k-1}^k \theta_w^{(k)} C_k, & i < k; \end{cases}$$

and

$$\psi_i = \begin{cases} \theta_w^{(i-1)}(\psi_{i-1}) \circ (\alpha_i)_{C_i}, & i > k; \\ \text{id}, & i \leq k. \end{cases}$$

The action of $\theta_w^{(\infty)}$ on a morphism $f = \{f_i\}$ is defined as $g = \{g_i\}$, where

$$g_i = \begin{cases} \theta_w^{(i-1)}(f_i) & i > k; \\ \mathbf{F}_i^{i+1} \circ \cdots \circ \mathbf{F}_{k-2}^{k-1} \circ \mathbf{F}_{k-1}^k \theta_w^{(k)}(f_{k+1}), & i \leq k. \end{cases}$$

Lemma 4. *The above gives a well-defined endofunctor of $\mathcal{C}$. The functor $\theta_w^{(\infty)}$ is exact and biadjoint to $\theta_{w^{-1}}^{(\infty)}$.*

Proof. This follows directly from the construction and definitions since each $\theta_w^{(i)}$ is exact and biadjoint to $\theta_{w^{-1}}^{(i)}$, see also [9, Proposition 3.2.4]. □

Proposition 5. *For any $w \in S_\infty$, the functor $\theta_w^{(\infty)}$ restricts to an endofunctor of the category $\mathcal{C}^{\mathrm{res}}$.*

Proof. Since the functor $\theta_w^{(\infty)}$ is exact, we just need to prove that, for a fixed $u \in S_\infty$, the length of the module $\theta_w^{(i)} L_u^{(i)}$ stabilizes, for all $i \gg 0$. This follows directly from Theorem 1 and the categorification remarks in Subsection 4.1. □

4.5. The Hecke algebra for S_∞

Consider the *Hecke algebra* $\mathbf{H}_\infty$, defined as the $\mathbb{Z}[v, v^{-1}]$-algebra with *standard basis* $\{H_w^{(\infty)} : w \in S_\infty\}$ and multiplication uniquely defined by (1) (with the superscript (n) replaced by (∞)). We have the *Kazhdan-Lusztig basis* $\{\underline{H}_w^{(\infty)} : w \in S_\infty\}$ of $\mathbf{H}_\infty$, defined by (2) (with the superscript (n) replaced by (∞), here it is important that the KL polynomials do not depend on n at all).

Consider the free $\mathbb{Z}[v, v^{-1}]$-module $\hat{\mathbf{H}}_\infty$ with the basis $\{\hat{\underline{H}}_w^{(\infty)} : w \in S_\infty\}$. For $x, y, w \in S_\infty$, define $[\hat{\underline{H}}_x^{(\infty)} \underline{H}_y^{(\infty)} : \hat{\underline{H}}_w^{(\infty)}]$ as the stabilized value of the expression $[\hat{\underline{H}}_x^{(k)} \underline{H}_y^{(k)} : \hat{\underline{H}}_w^{(k)}]$ in Theorem 1, where $k \gg 0$. From Theorem 1, it follows that, setting, for arbitrary $x, y \in S_\infty$,

$$\hat{\underline{H}}_x^{(\infty)} \underline{H}_y^{(\infty)} := \sum_{w \in S_\infty} [\hat{\underline{H}}_x^{(\infty)} \underline{H}_y^{(\infty)} : \hat{\underline{H}}_w^{(\infty)}] \hat{\underline{H}}_w^{(\infty)},$$

turns $\hat{\mathbf{H}}_\infty$ into a (right) $\mathbf{H}_\infty$-module.

Remark 6. Unlike the finite case, $\hat{\mathbf{H}}_\infty$ does not coincide with the right regular $\mathbf{H}_\infty$-module. However, one can identify $\hat{\mathbf{H}}_\infty$ with a submodule of a certain enlargement of the right regular $\mathbf{H}_\infty$-module. For this, we need to allow formal Laurent series as coefficients in front of the elements in the standard (or Kazhdan-Lusztig) basis. Then one can take the limit, where n goes to infinity, of the expressions of the elements of the dual Kazhdan Lusztig basis in terms of the standard (or Kazhdan-Lusztig) basis. This will output an infinite sum with formal Laurent series as coefficients.

4.6. Projective functors on $\mathcal{O}_0^{(\infty,\mathrm{fl})}$

Denote by $\mathcal{O}_0^{(\infty)}$ the full subcategory of the category of all $\mathfrak{sl}_\infty$-modules that consists of all M such that $\mathbf{F}_i^\infty M \in \mathcal{O}_0^{(i)}$, for all i. Our next observation is the following.

Theorem 7. *The categories $\mathcal{C}^{\mathrm{res}}$ and $\mathcal{O}_0^{(\infty,\mathrm{fl})}$ are equivalent.*

Proof. Directly from the construction, for every $2 \leq m < n$, we have $\mathbf{F}_m^n \circ \mathbf{F}_n^\infty = \mathbf{F}_m^\infty$. Therefore, by the universal properties of limits, we have a faithful functor Φ from $\mathcal{O}_0^{(\infty)}$ to $\mathcal{C}$ sending M to $(\{\mathbf{F}_i^\infty M\}, \{\mathrm{id}_{\mathbf{F}_i^\infty M}\})$ and with the obvious restriction action on morphisms. This restricts to a faithful functor Φ^{res} from $\mathcal{O}_0^{(\infty,\mathrm{fl})}$ to $\mathcal{C}^{\mathrm{res}}$ and we claim that this latter functor is, in fact, an equivalence.

Given $(\{C_i : i \geq 2\}, \{\phi_i : i \geq 3\})$ in $\mathcal{C}^{\mathrm{res}}$, let M be the vector space consisting of the equivalence classes of all sequences $(v_k, v_{k+1}, \dots)$, where $k \geq 2$, satisfying the condition $v_i = \phi_i^{-1}(v_{i-1})$ (which makes sense as, by definition, $\mathbf{F}^i_{i-1}C_i$ is a subset of C_i), for all $i > k$. Two such sequences $(v_k, v_{k+1}, \dots)$ and $(w_m, w_{m+1}, \dots)$ are equivalent if there is $r > \max(k, m)$ such that $v_i = w_i$, for all $i > r$. The vector space structure is defined component-wise on those components for which it makes sense (this means that, when adding $(v_k, v_{k+1}, \dots)$ and $(w_m, w_{m+1}, \dots)$, the indexing of the outcome will start from $\max(k, m)$). Furthermore, the vector space M carries the natural structure of an $\mathfrak{sl}_\infty$-module defined again by acting on those components for which the action makes sense (that is, $u \in U(\mathfrak{sl}_m)$ acts on $(v_k, v_{k+1}, \dots)$ outputting a vector whose indexing starts with $\max(k, m)$). In fact, with the obvious action on morphisms, this defines a functor Ψ from $\mathcal{C}^{\mathrm{res}}$ to $\mathcal{O}_0^{(\infty,\mathrm{fl})}$.

It follows directly from the definitions that $\Phi^{\mathrm{res}} \circ \Psi$ is isomorphic to the identity endofunctor of $\mathcal{C}^{\mathrm{res}}$, implying that Φ^{res} is both dense and full. The claim of the theorem follows. □

Combining Proposition 5 with Theorem 7, for every $w \in S_\infty$, we have an endofunctor $\theta_w^{(\infty)}$ of $\mathcal{O}_0^{(\infty,\mathrm{fl})}$. Putting all $\theta_w^{(\infty)}$ together, we have a monoidal category $\mathscr{P}^{(\infty)}$ of projective endofunctors of $\mathcal{O}_0^{(\infty,\mathrm{fl})}$. The objects are the functors from the additive closure of all $\theta_w^{(\infty)}$ and the morphisms are natural transformations of functors. The monoidal product is composition of functors. The split Grothendieck ring of $\mathscr{P}^{(\infty)}$ is isomorphic to $\mathbb{Z}[S_\infty]$ (resp. to the Hecke algebra of S_∞ in the graded version), where $\theta_w^{(\infty)}$ corresponds to $\underline{H}_w$.

The Grothendieck group of $\mathcal{O}_0^{(\infty,\mathrm{fl})}$ is a free abelian group (resp. a free $\mathbb{Z}[v, v^{-1}]$-module in the graded version) with basis $[L_w^\infty]$, where $w \in S_n$.

Corollary 8. *The action of projective functors on the graded version* ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$ *of* $\mathcal{O}_0^{(\infty,\mathrm{fl})}$ *decategorifies to the* $\mathbf{H}_\infty$*-module* $\hat{\mathbf{H}}_\infty$*, where* $[L_w^\infty]$ *corresponds to* $\hat{\underline{H}}_w$.

Proof. This follows from the definitions and Theorem 1. □

4.7. *Koszulity of* ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$

The following answers [6, Question 5.3.3] in the case of the category ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$. As usual, ext denotes homogeneous extensions of degere zero and $\langle 1\rangle$ is the degree shift that sends degree 0 to degree -1.

Theorem 9. *For* $x, y \in S_\infty$ *and* $i, j \in \mathbb{Z}_{\geq 0}$, *the inequality* $i \neq j$ *implies*

$$\mathrm{ext}^i_{\mathcal{C}^{\mathrm{res}}}(L(x \cdot 0)^{(\infty)}, L(y \cdot 0)^{(\infty)}\langle -j\rangle) = 0.$$

Proof. For brevity, we write L_x and L_y for $L(x \cdot 0)^{(\infty)}$ and $L(y \cdot 0)^{(\infty)}$, respectively. We are going to prove the necessary extension vanishing using the interpretation of extensions as equivalence classes of exact sequences.

Consider two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L_y\langle -j\rangle \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \dots \arrow[r] & X_i \arrow[r] & L_x \arrow[r] & 0 \end{tikzcd} \tag{9}$$

and

$$\begin{tikzcd} 0 \arrow[r] & L_y\langle -j\rangle \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \dots \arrow[r] & Y_i \arrow[r] & L_x \arrow[r] & 0 \end{tikzcd} \tag{10}$$

in ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$. Choose k such that the indices of all simple subquotients of all modules involved in these two sequences belong to S_k. Such k exists as all X_s and all Y_t involved have finite length. Then, applying the exact functor $\mathbf{F}_k^\infty$ to (9) and (10) gives two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] & \mathbf{F}_k^\infty X_1 \arrow[r] & \dots \arrow[r] & \mathbf{F}_k^\infty X_i \arrow[r] & \mathbf{F}_k^\infty L_x \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] & \mathbf{F}_k^\infty Y_1 \arrow[r] & \dots \arrow[r] & \mathbf{F}_k^\infty Y_i \arrow[r] & \mathbf{F}_k^\infty L_x \arrow[r] & 0. \end{tikzcd}$$

As $\mathcal{O}_0^{(k)}$ is Koszul, see [24], we know that $\mathrm{ext}^i_{\mathcal{O}_0^{(k)}}(L_x^{(k)}, L_y^{(k)}\langle -j\rangle) = 0$. Therefore, by [4, Theoreme 1, A X.121], there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] \arrow[d, equal] & \mathbf{F}_k^\infty Y_1 \arrow[r] \arrow[d] & \dots \arrow[r] & \mathbf{F}_k^\infty Y_i \arrow[r] \arrow[d] & \mathbf{F}_k^\infty L_x \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] & Z_1 \arrow[r] & \dots \arrow[r] & Z_i \arrow[r] & \mathbf{F}_k^\infty L_x \arrow[r] & 0 \\ 0 \arrow[r] & \mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] \arrow[u, equal] & \mathbf{F}_k^\infty X_1 \arrow[r] \arrow[u] & \dots \arrow[r] & \mathbf{F}_k^\infty X_i \arrow[r] \arrow[u] & \mathbf{F}_k^\infty L_x \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \tag{11}$$

over ${}^{\mathbb{Z}}\mathcal{O}_0^{(k)}$ with exact rows. Applying to (11) the exact parabolic induction $\mathbf{I}_k^\infty$ from $\mathfrak{sl}_k$ to $\mathfrak{sl}_\infty$ (this induction is, as usual, left adjoint to $\mathbf{F}_k^\infty$) outputs a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] \arrow[d, equal] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty Y_1 \arrow[r] \arrow[d] & \dots \arrow[r] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty Y_i \arrow[r] \arrow[d] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty L_x \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] & \mathbf{I}_k^\infty Z_1 \arrow[r] & \dots \arrow[r] & \mathbf{I}_k^\infty Z_i \arrow[r] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty L_x \arrow[r] & 0 \\ 0 \arrow[r] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty L_y\langle -j\rangle \arrow[r] \arrow[u, equal] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty X_1 \arrow[r] \arrow[u] & \dots \arrow[r] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty X_i \arrow[r] \arrow[u] & \mathbf{I}_k^\infty\mathbf{F}_k^\infty L_x \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \tag{12}$$

over ${}^{\mathbb{Z}}\mathcal{O}_0^{(\infty)}$ with exact rows.

Set $\Lambda := S_\infty \setminus S_k$. Let $\mathbf{G}$ be the functor of taking the trace of all projectives $P(w\cdot 0)^{(\infty)}$, where $w \in \Lambda$, in a given module. Note that, in general, this functor is neither left nor right exact, despite of the fact that it preserves both monomorphisms and epimorphisms, see [11].

Lemma 10. *Applying* $\mathbf{G}$ *to* (12)*, outputs a commutative diagram with exact rows.*

Proof. We only need to prove that $\mathbf{G}$, applied to a row of (12), outputs an exact sequence. Splitting longer exact sequences into shorter, it is enough to prove the claim in the special case $i = 1$, that is for short exact sequences. If the original short exact sequence splits, the claim is clear as each $\mathbf{I}_k^\infty L_w^{(k)}$ has simple top $L(w \cdot 0)^\infty$ and all other subquotients are indexed by elements of Λ. Hence the application of $\mathbf{G}$ sends $\mathbf{I}_k^\infty L_w^{(k)}$ to its radical. In particular, $\mathbf{G}$ acts on the whole short exact sequence as taking the radical.

The case when the original short exact sequence does not split corresponds to non-vanishing of some $\mu(u, w)$. Since the value of $\mu(u, w)$ does not depend on k, for all k for which the value makes sense, the non-split extension of some $L_w^{(k)}$ and $L_u^{(k)}\langle -1\rangle$ gives a non-split extension of $L(w \cdot 0)^\infty$ and $L(u \cdot 0)^\infty\langle -1\rangle$ which is realized in the middle term of our short exact sequence with end terms $\mathbf{I}_k^\infty L_w^{(k)}$ and $\mathbf{I}_k^\infty L_u^{(k)}\langle -1\rangle$ by the same arguments as in the proof of Proposition 3. This implies that the application of $\mathbf{G}$ to the non-split extension of $\mathbf{I}_k^\infty L_w^{(k)}$ and $\mathbf{I}_k^\infty L_u^{(k)}\langle -1\rangle$ outputs the extensions of the radicals of these two modules. The claim of the lemma follows. □

Given Lemma 10, we can take the component-wise quotient of (12) by the subdiagram obtained by applying $\mathbf{G}$ to (12). This will now be a commutative diagram with exact rows over ${}^{\mathbb{Z}}\mathcal{C}^{\mathrm{res}}$. Evaluating the adjunction morphism for the adjoint pair $(\mathbf{I}_k^\infty, \mathbf{F}_k^\infty)$ and comparing characters, we obtain that the first row of the resulting diagram is isomorphic to (9) while the third row is isomorphic to (10). By [4, Theoreme 1, A X.121], we thus obtain that the images of (9) and (10) in $\mathrm{ext}^i_{\mathcal{C}^{\mathrm{res}}}(L_x, L_y\langle -j\rangle)$ must coincide. Since (9) and (10) were arbitrary, we obtain that this extension space is zero. This completes the proof. □

Acknowledgements The first author is partially supported by Vergstiftelsen. The second author is partially supported by the Swedish Research Council. We thank the referee for helpful comments.

Declarations

Data Availibility No data was used for the creation of this paper

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Funding Open access funding provided by Uppsala University.

References

[1] Bernstein, I., Gelfand, S.: Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41**(2), 245–285 (1980)
[2] Bernstein, I., Gelfand, I., Gelfand, S.: A certain category of $\mathfrak{g}$-modules. Funkcional. Anal. i Prilozhen. **10**(2), 1–8 (1976)
[3] Björner, A., Brenti, F.: Combinatorics of Coxeter groups. Grad. Texts in Math., **231** Springer, New York, xiv+363 pp (2005)
[4] Bourbaki, N.: Elements of mathematics. Commutative algebra. Chapter 10. Springer-Verlag, Berlin, ii+187 pp (2007)
[5] Coulembier, K., Mazorchuk, V., Zhang, X.: Indecomposable manipulations with simple modules in category $\mathcal{O}$. Math. Res. Lett. **26**(2), 447–499 (2019)
[6] Coulembier, K., Penkov, I.: On an infinite limit of BGG categories $\mathcal{O}$. Mosc. Math. J. **19**(4), 655–693 (2019)
[7] Creedon, S., Mazorchuk, V.: Almost all permutations and involutions are Kostant negative. Math. Pannon. (N. S.) **31**(1), 54–59 (2025)
[8] Creedon, S., Mazorchuk, V.: Consecutive Patterns, Kostant's Problem and Type A_6. (2026) arXiv:2503.07809
[9] Entova-Aizenbud, I.: Notes on Restricted Inverse Limits of Categories. arXiv:1504.01121. (2015)
[10] Geck, M.: Kazhdan-Lusztig cells and the Murphy basis. Proc. London Math. Soc. (3) **93**(3), 635–665 (2006)
[11] Grensing, A.-L., Mazorchuk, V.: Finitary 2-categories associated with dual projection functors. Commun. Contemp. Math. **19**(3), 1650016 (2017). (**40 pp**)
[12] He, Z., Hu, J., Sun, Y.: Relations between Kazhdan-Lusztig bases, Murphy bases and seminormal bases. Commun. Algebra (2025). https://doi.org/10.1080/00927872.2024.2448244
[13] Humphreys, J.: Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$. Grad. Stud. Math., **94** American Mathematical Society, Providence, RI, (2008), xvi+289 pp
[14] Joseph, A.: Kostant's problem, Goldie rank and the Gelfand-Kirillov conjecture. Invent. Math. **56**(3), 191–213 (1980)
[15] Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math. **53**(2), 165–184 (1979)
[16] Ko, H., Mazorchuk, V., MrĐen, R.: Some homological properties of category $\mathcal{O}$. V. Int. Math. Res. Not. IMRN **2023**(4), 3329–373
[17] Mazorchuk, V., Miemietz, V.: Cell 2-representations of finitary 2-categories. Compos. Math. **147**(5), 1519–1545 (2011)
[18] Mazorchuk, V., Stroppel, C.: Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219**(4), 1363–1426 (2008)
[19] Nampaisarn, T.: Categories $\mathcal{O}$ for Dynkin Borel Subalgebras of Root-Reductive Lie Algebras. arXiv:1706.05950
[20] Nampaisarn, T.: On Categories $\mathcal{O}$ for Root-Reductive Lie Algebras. arXiv:1711.11234. (2017)
[21] Nampaisarn, T.: Categories $\mathcal{O}$ for Root-Reductive Lie Algebras: II. Translation Functors and Tilting Modules. arXiv:2012.01003 (2020)

[22] Sagan, B.: The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition. Graduate Texts in Mathematics, **203**. Springer-Verlag, New York, (2001). xvi+238 pp

[23] Schensted, C.: Longest increasing and decreasing subsequences. Canadian J. Math. **13**, 179–191 (1961)

[24] Soergel, W.: Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3**(2), 421–445 (1990)

[25] Soergel, W.: The combinatorics of Harish-Chandra bimodules. J. Reine Angew. Math. **429**, 49–74 (1992)

[26] Soergel, W.: Kazhdan-Lusztig polynomials and a combinatoric for tilting modules. Represent. Theory **1**, 83–114 (1997)

[27] Stroppel, C.: Category $\mathcal{O}$: gradings and translation functors. J. Algebra **268**(1), 301–326 (2003)

[28] Warrington, G.: Equivalence classes for the μ-coefficient of Kazhdan-Lusztig polynomials in S_n. Exp. Math. **20**(4), 457–466 (2011)