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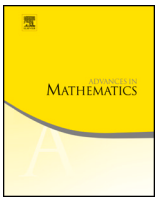

Dynamical covering sets in self-similar sets

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ABSTRACT

We study the size of *dynamical covering sets* on a self-similar set. Dynamical covering sets are limsup sets generated by placing shrinking target sets around points along an orbit in a dynamical system. In the case when the target sets are balls with sizes depending on the centre, we determine the size of the dynamical covering set as a function of the shrinking rate. In particular, we find sharp conditions guaranteeing when full Dvoretzky-type covering, and full measure occur. We also compute the Hausdorff dimension in the remaining cases. The proofs apply in the cases of targets centred at typical points of the self-similar set, with respect to any Bernoulli measure on it. Unlike in existing work on dynamical coverings, and despite the dimension value featuring phase transitions, we demonstrate that the behaviour can be characterised by a single pressure function over the full range of parameters. The techniques are a combination of classical dimension theoretical estimates and intricate martingale arguments.

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1. Introduction

An *iterated function system (IFS)* is a collection of contractive maps $\{f_1, \dots, f_N\}$, and is associated with a unique, non-empty, compact attractor, or invariant set $\Lambda = \cup_{i=1}^{N} f_i(\Lambda)$. Iterated function systems are one of the main methods for generating sets with fractal features. The invariance of the set Λ under the iterated function system implies an amount of scale-invariance that guarantees fine structure on all levels of magnification. The modern theory of iterated function systems began with Hutchinson [20], who computed the Hausdorff dimension of Λ in the case of similarity maps $\{f_1, \dots, f_N\}$, satisfying a separation condition. In this case, the set Λ is said to be *self-similar*.

There is a natural expanding dynamical system $T : \Lambda \to \Lambda$ given locally by the inverses of f_i. More precisely, for any point $x \in f_i(\Lambda)$, one can define $T(x)$ by $f_i^{-1}(x)$ for an appropriate choice of i. Such i must exist, but is not necessarily unique. Under sufficient separation conditions, the choice of f_i is always unique and the dynamical system is topologically conjugate to the left shift map on N symbols, via a coding map of points of Λ.

In the context of dynamical systems, a classical question is that of *recurrence*. A particular example is the *shrinking target problem*: How large is the set of points that hit a sequence of targets A_i infinitely many times? That is, how large is the set

$$R_{st} = \{x \in X : T^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

The term shrinking target was introduced by Hill and Velani, see [16,17]. Questions on such shrinking targets for iterated function systems have received an increasing amount of interest over the past decade, see [1,2,4–6,8,10,12,11,24,25,29] and the references therein. A particular feature of such sets is that they can be interpreted as lim sup sets, since

$$R_{st} = \limsup_{n\in\mathbb{N}} T^{-n}(A_n) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} T^{-k}(A_k).$$

Instead of a sequence of targets, one can consider the orbit of a (typical) point $x_0 \in X$ under the dynamics $T : X \to X$ and study the set of points that lie in $T^n(x_0) + A_n$ infinitely often, where A_n is a sequence of nested sets. For example, when $A_n = B(0, r_n)$ is a sequence of balls, we set

$$R(x_0, A_n) = \limsup_{n\in\mathbb{N}} T^n(x_0) + A_n = \limsup_{n\in\mathbb{N}} B(T^n(x_0), r_n) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} B(T^k(x_0), r_k).$$

The question on whether $R(x_0, A_n) = X$ is known as the *dynamical covering problem* and is the dynamical analogue of the classical Dvoretzky covering problem from probability theory (see e.g. [14] and, more recently, [21,22,3]). When $R(x_0, A_n) \subsetneq X$ one can ask the finer question of the size of $R(x_0, A_n)$ in terms of measure or dimension. This problem has been studied in some restricted set-ups for interval maps, see [15,18,26] and also [27,19]

for "typical" systems. (In these works, the dynamical covering problem is referred to as dynamical Diophantine approximation, or shrinking targets problem). In all of these works it is observed that the size of the dynamical covering goes through distinct regimes with several phase transitions. Namely, when $r_n \approx n^{-1/\alpha}$ and α is small, the dimension is linear in α, but the system goes through a phase transition when α equals the entropy of the measure, after which it follows a multifractal spectrum. One of the key techniques in these works is the mass transference principle, that gives a straightforward method to lower dimension bound estimates.

In the present article, we study the dynamical covering set in the context of self-similar sets satisfying the open set condition. For definitions see Section 1.1. For a typical centre (with respect to a Bernoulli measure on Λ), and for target sets given by symbolic balls shrinking at a rate associated with the centre point, we find sharp conditions under which the dynamical covering set is all of Λ. This is the analogue of the Dvoretzky covering problem in the study of random covering sets. Furthermore, we find sharp conditions under which the dynamical covering set is of full measure with respect to the Hausdorff measure on Λ (the Hausdorff measure $\mathcal{H}^{\dim \Lambda}$ is positive and finite due to the open set condition, see Hutchinson [20]). The bulk of the work in this article, however, is to compute the Hausdorff dimension of the dynamical covering set in all the remaining cases. We also compute the dimension of the complement of the dynamical covering set, for those parameter values when the dynamical covering set is of full measure.

To complement the past results on the topic, we present a single pressure formula governing the dimension value across the full range of shrinking rates. The phase transitions show as alternative forms to this pressure formula, see Proposition 2.2. This article is the first to study self-similar sets with inhomogeneous contraction rates, which reveals novel phenomena: We observe that for small values of α, the dimension value is no longer linear in α, but strictly concave. Furthermore, the dimension behaviour for larger values of α is not given by the multifractal spectrum of the associated measure, but through a different variational principle that we investigate. Further, in case of symbolic balls, the inhomogeneity of the contraction ratios makes it impossible to use the method of mass transference principle that underlays all previous results. Our set-up necessitates the use of intricate martingale methods to determine the dimension. The situation is analogous to a result from the shrinking target context [2], which also highlights that the mass transference principle is ineffectual when the contractions are inhomogeneous and the balls are symbolic.

1.1. Main result

Let us present more precise definitions before giving the full statements of the main theorems. Let $\Phi = \{f_1, \dots, f_N\}$ be an iterated function system of similarities on $\mathbb{R}^d$, with contraction ratios $\lambda_i \in (0,1)$. Denote by Λ the corresponding self-similar set, satisfying $\Lambda = \bigcup_{i=1}^N f_i(\Lambda)$. We assume throughout that the iterated function system satisfies the Open Set Condition (OSC), that is, there exists an open and bounded set $U \subset \mathbb{R}^d$ such

that $f_i(U) \subset U$ for every $i = 1, \dots, N$ and $f_i(U) \cap f_j(U) = \emptyset$ for every $i \neq j$. We assume without loss of generality that $\Lambda \cap U \neq \emptyset$, see Bandt and Graf [7] and Schief [28]. Denote by s_0 the Hausdorff dimension of Λ, which is the unique solution of

$$\sum_{i=1}^{N} \lambda_i^{s_0} = 1,$$

see Hutchinson [20]. For simplicity, let $\lambda_{\max} = \max\{\lambda_i : i \in \{1, \dots, N\}\} < 1$ and $\lambda_{\min} = \min\{\lambda_i : i \in \{1, \dots, N\}\} > 0$.

Denote the symbolic space by $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$, with the left shift map σ. We write Σ_n for the collection of words of length n, and $\Sigma_* = \cup_{n=0}^{\infty} \Sigma_n$. For $\overline{\imath} = (i_1, \dots, i_n) \in \Sigma_n$, let $f_{\overline{\imath}} = f_{i_1} \circ \cdots \circ f_{i_n}$ and $\lambda_{\overline{\imath}} = \lambda_{i_1} \cdots \lambda_{i_n}$. Let $\overline{\imath}|_n$ denote the first n digits of a word $\overline{\imath} \in \Sigma \cup \Sigma_*$, and $|\overline{\imath}|$ the length of a finite word. Denote by $\overline{\imath} < \overline{\jmath}$ when a finite word $\overline{\imath}$ appears as the initial segment of a finite or infinite $\overline{\jmath}$. Define cylinders for any $\overline{\imath} \in \Sigma_*$ by $[\overline{\imath}] = \{\overline{\jmath} \in \Sigma \mid \overline{\imath} < \overline{\jmath}\}$. The symbolic space gives a coding of points of Λ via the map $\pi : \Sigma \to \Lambda$,

$$\pi(\overline{\imath}) = \lim_{n \to \infty} f_{\overline{\imath}|_n}(0).$$

Note that the map $\pi \colon \Sigma \to \Lambda$ is "almost" one-to-one, namely, the points with multiple preimages in Σ are negligible in some proper sense.

Definition 1.1. Let $\overline{o} \in \Sigma$ and let $\ell \colon \mathbb{N} \to \mathbb{N}$ be any increasing function diverging to ∞. The **dynamical covering set** $R(\overline{o}, \ell) \subseteq \Lambda$ is

$$R(\overline{o}, \ell) = \pi\left(\left\{\overline{\imath} \in \Sigma \mid \overline{\imath} \in \left[(\sigma^n \overline{o})|_{\ell(n)}\right] \text{ infinitely often}\right\}\right) = \pi\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left[(\sigma^n \overline{o})|_{\ell(n)}\right]\right).$$

We can also consider the map $T \colon \Lambda \to \Lambda$ that takes any $x \in f_i(\Lambda)$ to $f_i^{-1}(x) \in \Lambda$. This is an expanding map on the points of unique coding, which conjugates to σ by $\pi \circ \sigma = T \circ \pi$. At points with multiple codings, a choice of the map f_i^{-1} has to be made. Notice that the set $R(\overline{o}, \ell)$ is analogous to the dynamical covering set on Λ following the orbit of the point $\pi(\overline{o})$ under T, namely, to the set

$$\widehat{R}(\overline{o}, \ell) = \left\{x \in \Lambda \mid x \in B\left(T^n(\pi(\overline{o})), \lambda_{(\sigma^n \overline{o})|_{\ell(n)}}\right) \text{ infinitely often}\right\}.$$

However, in what follows, we will focus our analysis on the projected symbolic dynamical covering set $R(\overline{o}, \ell)$.

Let $p = (p_1, \dots, p_N)$ be a probability vector, and $\mathbb{P}_p$ the corresponding Bernoulli measure on Σ. Without loss of generality, we may assume that $p_i > 0$ for every $i = 1, \dots, N$. For simplicity, we denote the push-forward measure $\pi_* \mathbb{P}_p$ on Λ also by $\mathbb{P}_p$. Let us denote the largest and the smallest probabilities by $p_{\max} = \max\{p_i : i \in \{1, \dots, N\}\}$

and $p_{\min} = \min\{p_i : i \in \{1, \dots, N\}\}$. Our main theorem on the dynamical covering set states:

Theorem 1.2. *Let $\{f_1, \dots, f_N\}$ be an IFS of similarities satisfying the open set condition, and with contraction ratios $\lambda_1, \dots, \lambda_N$. Let $\alpha > 0$ and assume that $\ell : \mathbb{N} \to \mathbb{N}$ is monotone increasing and it satisfies*

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \tfrac{1}{\alpha}. \tag{1}$$

Let $\mathbb{P}_p$ be a Bernoulli measure corresponding to a probability vector $p = (p_1, \dots, p_N)$. Then, for $\mathbb{P}_p$-almost every $\overline{o} \in \Sigma$, the Hausdorff dimension of $R(\overline{o}, \ell)$ is given by the solution $s = s(\alpha)$ to

$$s(\alpha) = \inf_{q\in[0,1]} \left\{ P_\alpha(q) : \sum_{i=1}^{N} \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q = 1 \right\}. \tag{2}$$

Furthermore, we have for $\mathbb{P}_p$-almost every $\overline{o}$ that if $-\log p_{\min} > \alpha > -\sum_{i=1}^{N} \lambda_i^{s_0} \log p_i$ then $\mathcal{H}^{s_0}(R(\overline{o}, \ell)) = \mathcal{H}^{s_0}(\Lambda)$ but $R(\overline{o}, \ell) \neq \Lambda$, and if $\alpha > -\log p_{\min}$ then $R(\overline{o}, \ell) = \Lambda$.

We remark that the statement of Theorem 1.2 remains valid for the set $\widehat{R}(\overline{o}, \ell)$. Indeed, we may assume without loss of generality that the diameter $\operatorname{diam}(\Lambda) = 1$, and so, $R(\overline{o}, \ell) \subseteq \widehat{R}(\overline{o}, \ell)$ for every $\overline{o} \in \Sigma$, which gives the lower bound. On the other hand, a similar argument to the one in Section 3 with only minor modification gives the upper bound. We leave the proof to the interested reader.

We further remark that, for $\alpha_0 < \alpha < \alpha_1$, our argument in fact gives the stronger exact-approximation statement $\dim_{\mathrm{H}} \left(R(\overline{o}, \ell) \setminus \bigcup_{0<\beta<\alpha} R(\overline{o}, \frac{\alpha}{\beta}\ell) \right) = s(\alpha)$. Indeed, the lower bound is witnessed by the maximising Bernoulli measure $\mathbb{Q}_{q^*}$ of Lemma 2.3, which has exact dimension $s(\alpha)$ and is null on every strictly lower-dimensional set $R(\overline{o}, \frac{\alpha}{\beta}\ell)$ with $\beta < \alpha$, hence gives full mass to the difference. We strongly suspect the same to hold for $0 < \alpha \leq \alpha_0$, by thinning the martingale construction of Section 5 across levels, but do not pursue this so as not to complicate the construction further.

Our second main result is on the size of the complement of the dynamical covering set:

Theorem 1.3. *Let $\{f_1, \dots, f_N\}$ be an IFS of similarities satisfying the open set condition, and with contraction ratios $\lambda_1, \dots, \lambda_N$. Let $\alpha > 0$ and assume that $\ell : \mathbb{N} \to \mathbb{N}$ is monotone increasing and it satisfies*

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \tfrac{1}{\alpha}.$$

Let $\mathbb{P}_p$ be a Bernoulli measure corresponding to a probability vector $p = (p_1, \dots, p_N)$. Then, for $\mathbb{P}_p$-almost every $\overline{o} \in \Sigma$, the Hausdorff dimension of $R(\overline{o}, \ell)^c$ is given by the solution $t = t(\alpha)$ to

$$t(\alpha) = \inf_{q<0} \left\{ P_\alpha(q) : \sum_{i=1}^{N} \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q = 1 \right\}. \tag{3}$$

Furthermore, if $\alpha < -\sum_{i=1}^{N} \lambda_i^{s_0} \log p_i$ then $\mathcal{H}^{s_0}(R(\overline{o}, \ell)^c) = \mathcal{H}^{s_0}(\Lambda)$ and if $\alpha > -\log p_{\min}$ then $R(\overline{o}, \ell)^c = \emptyset$ for $\mathbb{P}_p$-almost every $\overline{o}$.

The proofs fall into, broadly, two separate categories. For large α, as described in Sections 3 and 4, we rely on covering arguments and the Borel-Cantelli lemma, as well as the description of the multifractal spectrum, provided to us by a probabilistic pressure formula (see Eq. (13) underneath). For small α, in Section 5, the proof techniques are much more involved, and require sophisticated arguments involving martingale and large deviation estimates.

1.2. Discussion for different values of α

We now discuss the results of Theorem 1.2 and Theorem 1.3 in more detail. In particular, the purpose of this section is to make sense of the dimension formula and present the phase transitions that the size of the dynamical covering set goes through as α varies. We identify the four distinct regions of α for the different sizes of $R(\overline{o}, \ell)$, and the corresponding behaviour of $R(\overline{o}, \ell)^c$. Furthermore, at the end of this subsection we present figures showing examples of the possible behaviours of $R(\overline{o}, \ell)$ as a function of α.

Let $P_\alpha(q)$ be the unique solution of

$$\sum_{i=1}^{N} \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q = 1.$$

We refer to Section 2 for a proof of the existence of $P_\alpha(q)$. We will show that as α increases, the behaviour of the sizes of the sets $R(\overline{o}, \ell)$ and $R(\overline{o}, \ell)^c$ goes through phase transition at the values

$$\alpha_0 = -\sum_{i=1}^{N} \lambda_i^{P_{\alpha_0}(1)} p_i e^{\alpha_0} \log p_i, \quad \alpha_1 = -\sum_i \lambda_i^{s_0} \log p_i, \quad \text{and} \quad \alpha_2 = -\log p_{\min}.$$

The first region corresponds to "small" values of $\alpha \in (0, \alpha_0]$ and corresponds to the "pressure part". Namely, there is a unique value $\alpha_0 \in (0, \infty)$ such that $\alpha_0 = -\sum_{i=1}^{N} \lambda_i^{P_{\alpha_0}(1)} p_i e^{\alpha_0} \log p_i$ and $s(\alpha) = P_\alpha(1)$ holds for every $\alpha \in (0, \alpha_0]$. In particular, if

$$\alpha < -\sum_{i=1}^{N} \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i \quad \text{then} \quad s(\alpha) = P_\alpha(1).$$

In the special case when $\lambda_i \equiv \lambda$, we have that $\alpha_0 = -\sum_i p_i \log p_i$ and $s(\alpha) = \frac{\alpha}{-\log \lambda}$ for $\alpha \in (0, \alpha_0]$.

The second region corresponds to the "spectral behaviour part". More precisely, for every $\alpha \in [\alpha_0, \alpha_1]$, there exists a unique $q = q(\alpha) \in [0,1]$ such that almost surely,

$$\dim_{\mathrm{H}} R(\overline{o}, \ell) = P_\alpha(q(\alpha)) = \inf_{q \in \mathbb{R}} P_\alpha(q) = \sup \left\{ \frac{-\sum_i q_i \log q_i}{-\sum_i q_i \log \lambda_i} : -\sum_i q_i \log p_i \leq \alpha \right\}.$$

This optimisation resembles the multifractal spectrum, and it indeed coincides with the Legendre transform of the L^q-dimension of $\mathbb{P}_p$ if $\lambda_i \equiv \lambda$, which is the special dynamical case considered by Liao and Seuret [26]. For $\alpha \in (0, \alpha_1]$, the dimension satisfies $\dim_{\mathrm{H}} R(\overline{o}, \ell) < s_0$. We note here that for all values $\alpha < \alpha_1$, the complementary set $R(\overline{o}, \ell)^c$ has full $\mathcal{H}^{s_0}$-measure in Λ.

The next two regions are the "large" α cases when typical orbits cover $\mathcal{H}^{s_0}$-almost every point in Λ. However, in the region $\alpha \in (\alpha_1, \alpha_2)$, almost surely the whole set Λ is not covered, whereas if $\alpha > \alpha_2$ then every point in Λ is covered $\mathbb{P}_p$-almost surely infinitely often. This solves the analogue of Dvoretzky's covering problem in the self-similar setting. In the region $\alpha_2 > \alpha > \alpha_1$, the Hausdorff measure of the complementary set $R(\overline{o}, \ell)^c$ is zero and, in particular, its Hausdorff dimension is given by the "decreasing" part of the spectrum corresponding to negative q values

$$\dim_{\mathrm{H}} R(\overline{o}, \ell)^c = \inf_{q<0} P_\alpha(q) = \inf_{q \in \mathbb{R}} P_\alpha(q) = \sup \left\{ \frac{-\sum_i q_i \log q_i}{-\sum_i q_i \log \lambda_i} : -\sum_i q_i \log p_i \geq \alpha \right\}$$

almost surely. It is strictly smaller than s_0, see Sections 3.2 and 4.2. Once $\alpha_2 < \alpha$, the set $R(\overline{o}, \ell)^c$ is clearly empty almost surely.

Surprisingly, the $s(\alpha)$ defined in Eq. (2) has an equivalent characterisation as the unique root of

$$\limsup_{n \to \infty} \tfrac{1}{n} \log \sum_{|\bar{\jmath}|=n} \lambda_{\bar{\jmath}}^s \left(1 - (1 - p_{\bar{\jmath}})^{e^{\alpha n}}\right) = 0, \tag{4}$$

see Section 2. This alternative form of $s(\alpha)$ plays an important role in the proof of Theorem 1.2: It is more geometric in nature and gives a natural description of coverings by taking into account the multiplicity of the visits of cylinders by typical $\mathbb{P}_p$-orbits.

In the figures underneath we depict the possible behaviours of the sizes of $R(\overline{o}, \ell)$ and $R(\overline{o}, \ell)^c$ as α varies, for different types of iterated function systems. In the first figure, Fig. 1, we present a generic image describing the size of the dynamical covering set $R(\overline{o}, \ell)$. The dimension value passes from convex (green) to concave (yellow) at α_0, eventually ending in a plateau (red) at the dimension of the whole self-similar set Λ after α_1. The figures also show the final phase transition where $R(\overline{o}, \ell)$ becomes all of Λ at α_2 (dotted to solid line). The dimension value of its complement $R(\overline{o}, \ell)^c$ is shown in black

and behaves dually to the dimension of $R(\overline{o}, \ell)$, being maximal until α_1 after which it follows the concave region.

In Fig. 2, we present the Hausdorff dimensions of $R(\overline{o}, \ell)$ and $R(\overline{o}, \ell)^c$ in the case when the distribution of $\overline{o}$ follows the dimension maximising measure. Note that the second phase transition α_1 is the smallest by using the dimension maximising measure. In other words, it is the case where the dynamical covering set covers the full Hausdorff measure set with the fastest speed.

In the example presented in Fig. 3, unlike in the other examples depicted, the shape of the dimension spectrum is the same as in the work of [26]. This is due to the fact that in this example, the IFS is homogeneous, which implies that the radii of the balls in the dynamical covering do not depend on the orbit $\overline{o}$. In particular, the dimension of $R(\overline{o}, \ell)$ is linear in α for $\alpha < \alpha_0$.

The last figure, Fig. 4, shows that it is possible for the three critical values of α to coincide, *i.e.* $\alpha_0 = \alpha_1 = \alpha_2$. In this example, the dimension value is given by $P_\alpha(1)$ and furthermore, as soon as $\dim_{\mathrm{H}} R(\overline{o}, \ell)$ is equal to the dimension of Λ, also $R(\overline{o}, \ell) = \Lambda$. This is due to the fact that the Bernoulli measure has equal probabilities. Clearly, the phase transition α_2 is the smallest for the uniform distribution, *i.e.* the dynamical covering set covers the whole set with the fastest speed. Conversely, the dimension of $R(\overline{o}, \ell)^c$ is maximal until the phase transition after which $R(\overline{o}, \ell)^c = \varnothing$.

Finally, we briefly comment on the nature of the phase transitions α_0 and α_1. Since the functions $g_1 : \alpha \mapsto P_\alpha(1)$ and $g_2 : \alpha \mapsto \inf_{q \in \mathbb{R}} P_\alpha(q)$ are real analytic functions which are convex and concave, respectively, we have $g_1(\alpha) \geq g_2(\alpha)$ whenever defined, and the phase transition occurs at α_0, when both functions coincide, we must necessarily have that their derivatives agree at α_0. Again by convexity and concavity, the second derivatives are strictly positive and negative, respectively, and the phase transition is of second order. The only exception is when the Legendre transform is trivial and a singleton (and $\alpha_0 = \alpha_1 = \alpha_2$). In this case the derivative of the first function is strictly positive, whereas the second region is of derivative zero. Here we obtain a first order phase transition.

1.3. Structure of the paper

In Section 2 we prove that the dimension formula for $R(\overline{o}, \ell)$ can be computed from the probabilistic pressure Eq. (13), for all values of α. This probabilistic pressure is more closely related to the geometry of the set $R(\overline{o}, \ell)$ than the function $P_\alpha(q)$ and as such, is an important tool for obtaining the upper bound results in Section 3. They are also key to the lower bound arguments in the low α regime, $\alpha < \alpha_0$. In Section 3, we compute the dimension upper bounds for the dynamical covering set and its complement, proving the upper bounds in Theorems 1.2 and 1.3. These are probabilistic covering arguments relying on the Borel-Cantelli lemma.

In Section 4, we prove that $R(\overline{o}, \ell) = \Lambda$ for almost all $\overline{o}$ when $\alpha > \alpha_2$, see Theorem 4.1, that $\mathcal{H}^{s_0}(R(\overline{o}, \ell)) = \mathcal{H}^{s_0}(\Lambda)$ for almost all $\overline{o}$ when $\alpha_1 < \alpha < \alpha_2$, and that $\mathcal{H}^{s_0}(R(\overline{o}, \ell)^c) =$

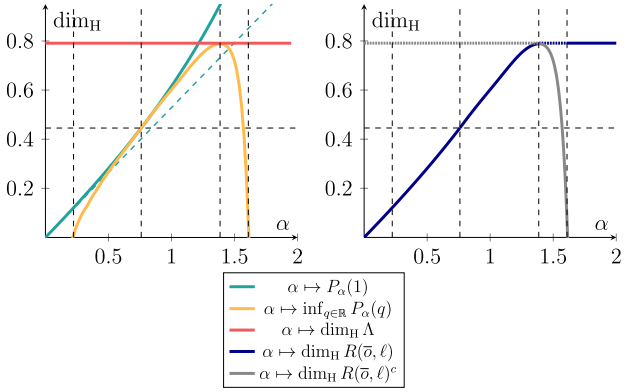

Fig. 1. Example with IFS consisting of two maps, $f_1 : x \mapsto 0.8x$ and $f_2 : x \mapsto 0.1x + 0.9$. The probabilities are $p_1 = 0.2$ and $p_2 = 0.8$. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

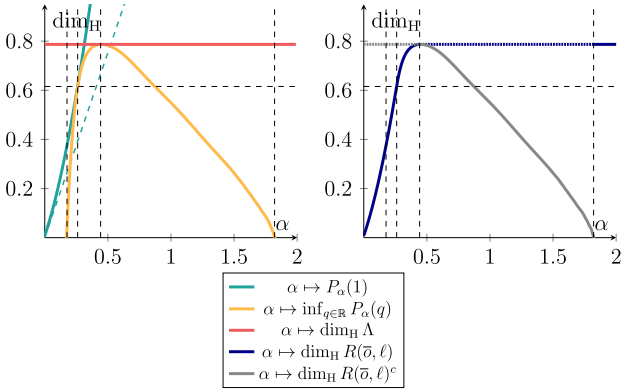

Fig. 2. Example with IFS $f_1 : x \mapsto 0.8x$ and $f_2 : x \mapsto 0.1x + 0.9$. The probabilities are $p_1 = 0.8^s \approx 0.838$ and $p_2 = 0.1^s \approx 0.162$, where $s = \dim_{\mathrm{H}} \Lambda$ (the dimension maximising measure).

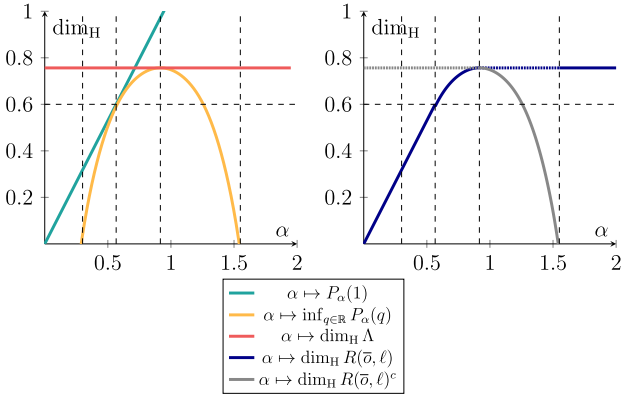

Fig. 3. Example with IFS $f_1 : x \mapsto 0.4x$ and $f_2 : x \mapsto 0.4x + 0.6$. The probabilities are $p_1 = 0.2$ and $p_2 = 0.8$. The case coincides with the result of Liao and Seuret [26].

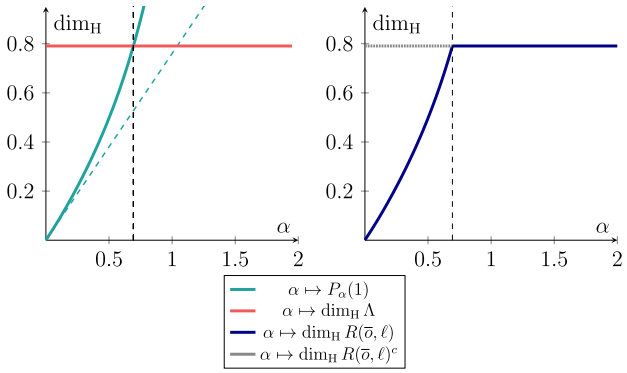

Fig. 4. Example with IFS $f_1 : x \mapsto 0.8x$ and $f_2 : x \mapsto 0.1x + 0.9$. The probabilities are equal $p_1 = 0.5 = p_2$.

$\mathcal{H}^{s_0}(\Lambda)$ for almost all $\overline{o}$ when $\alpha_1 > \alpha$. This section also contains the proofs of the lower bounds to $\dim_{\mathrm{H}} R(\overline{o}, \ell)$ for large $\alpha > \alpha_0$ and for $\dim_{\mathrm{H}} R(\overline{o}, \ell)^c$ for $\alpha > \alpha_1$. The arguments in Section 4 are similar to each other and to those in Section 3, with the additional analysis of certain Bernoulli measures on $R(\overline{o}, \ell)$ and its complement.

The remainder of the paper is dedicated to the intricate proof of the lower bound to $\dim_{\mathrm{H}} R(\overline{o}, \ell)$ in the case of small $\alpha \leq \alpha_0$. This proof is contained in Section 5. This section begins with a construction of a Cantor set that is utilised for the lower bound proof, see Section 5.1. On this Cantor set, we construct a random measure, using martingale methods, see Section 5.2. We go on to prove that the random measure has some nice distribution properties, see Section 5.3. Finally, in Section 5.4 we complete the lower bound proof through an energy estimate. The final short section (Section 6) proves the main theorems by combining the results from the previous sections.

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2. Preliminaries and the study of the probabilistic pressure

This section focuses on analysis of the dimension value in Theorems 1.2 and 1.3. In particular, we show in Lemma 2.3 that the number $P_\alpha(q)$ satisfies a variational principle,

and in Lemma 2.4 that it connects to the probabilistic pressure Eq. (13) defined underneath. These connections are crucial in the dimension estimates of the later sections.

We will repeatedly make use of the following well-known approximations throughout the whole paper. For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(1-x)^n \geq 1 - nx \quad \text{and} \quad (1-x)^n \leq e^{-xn}. \tag{5}$$

Further, from a second order Taylor approximation, for every $x > 0$ and $n \in \mathbb{N}$, we also always have

$$(1-x)^n \leq 1 - nx + \frac{n^2x^2}{2} = 1 - nx(1 - \frac{nx}{2}). \tag{6}$$

First we prove the following proposition, which describes the regions discussed in Section 1.2. Let $p = (p_1, \ldots, p_N)$ be a probability vector such that $p_i > 0$ for all $i = 1, \ldots, N$.

Lemma 2.1. *Let $P_\alpha(q)$ be the unique solution of $\sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q = 1$. Then the map $q \mapsto P_\alpha(q)$ is continuous and convex.*

Moreover, if $p_i \neq p_j$ for some $i \neq j$ then the map

$$\alpha \in [-\log p_{\max}, -\log p_{\min}] \mapsto \inf_{q \in \mathbb{R}} P_\alpha(q)$$

is also continuous, concave and its unique maximum is at $\alpha = -\sum_{i=1}^N \lambda_i^{s_0} \log p_i$.

Proof. It is easy to see that the solution $P_\alpha(q)$ is indeed unique since the map

$$x \mapsto \sum_{i=1}^N \lambda_i^x (p_i e^\alpha)^q$$

is strictly monotone decreasing, continuous, goes to 0 as $x \to \infty$ and goes to ∞ as $x \to -\infty$.

It is clear from the definition that $q \mapsto P_\alpha(q)$ is continuous, so, we show that the map $q \mapsto P_\alpha(q)$ is convex (but not necessarily strictly convex) on $\mathbb{R}$. Simple algebraic calculations show that for every $q \in \mathbb{R}$

$$0 = \sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q \left(\log p_i + \alpha + P'_\alpha(q) \log \lambda_i\right) \tag{7}$$

and

$$0 = \sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q \left(\log p_i + \alpha + P'_\alpha(q) \log \lambda_i\right)^2 + P''_\alpha(q) \sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q \log \lambda_i,$$

thus,

$$P''_\alpha(q) = \frac{\sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q \left(\log p_i + \alpha + P'_\alpha(q)\log\lambda_i\right)^2}{-\sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q \log\lambda_i} \geq 0,$$

hence, the convexity follows. Clearly, $P''_\alpha(q) = 0$ for some $q \in \mathbb{R}$ if and only if $P'_\alpha(q) = \frac{\log(p_i e^\alpha)}{-\log\lambda_i}$ for every i and every $q \in \mathbb{R}$. In particular, $P''_\alpha \equiv 0$ and $\frac{\log(p_i e^\alpha)}{-\log\lambda_i} = \frac{\log(p_j e^\alpha)}{-\log\lambda_j}$ for every $i \neq j$.

Now, let us study the continuity of $\alpha \mapsto \inf_{q\in\mathbb{R}} P_\alpha(q)$. It is easy to see that if $p_{\min}e^\alpha > 1$ then

$$P'_\alpha(q) = \frac{\sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q \log(p_i e^\alpha)}{-\sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q \log\lambda_i} \geq \frac{\log(p_{\min}e^\alpha)}{-\log\min_i\{\lambda_i\}} > 0 \quad \text{for every } q \in \mathbb{R},$$

and so $\inf_{q\in\mathbb{R}} P_\alpha(q) = \lim_{q\to-\infty} P_\alpha(q) = -\infty$. Similarly, if $p_{\max}e^\alpha < 1$ then $P'_\alpha(q) < 0$ for every $q \in \mathbb{R}$, and hence, $\inf_{q\in\mathbb{R}} P_\alpha(q) = \lim_{q\to\infty} P_\alpha(q) = -\infty$. Hence, $\alpha \mapsto \inf_{q\in\mathbb{R}} P_\alpha(q)$ defined only on $[-\log p_{\max}, -\log p_{\min}]$.

Let $\mathcal{I}_{\max} = \{i : p_i = p_{\max}\}$ and $\mathcal{I}_{\min} = \{i : p_i = p_{\min}\}$. Since α is an element in $[-\log p_{\max}, -\log p_{\min}]$, for $q \geq 0$ we get

$$1 = \sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q \geq \sum_{i\in\mathcal{I}_{\max}} \lambda_i^{P_\alpha(q)}.$$

Similarly, $1 \geq \sum_{i\in\mathcal{I}_{\min}} \lambda_i^{P_\alpha(q)}$ for $q \leq 0$. Thus, $P_\alpha(q) \geq 0$ for every $q \in \mathbb{R}$. Let $s_{\min}$ and $s_{\max}$ be the unique solutions of $1 = \sum_{i\in\mathcal{I}_{\min}} \lambda_i^{s_{\min}}$ and $1 = \sum_{i\in\mathcal{I}_{\max}} \lambda_i^{s_{\max}}$, respectively.

Under the assumption that $p_i \neq p_j$ for some $i \neq j$, a simple algebraic manipulation shows that if $\lambda_i = \lambda_j$ for every $i \neq j$ then there is no α such that $P''_\alpha \equiv 0$. Moreover, if $\lambda_i \neq \lambda_j$ for some $i \neq j$ then the only possible value of α for which $P''_\alpha \equiv 0$ is

$$\widehat{\alpha} = \frac{\frac{\log p_i}{-\log\lambda_i} - \frac{\log p_j}{-\log\lambda_j}}{\frac{1}{-\log\lambda_j} - \frac{1}{-\log\lambda_i}}.$$

However, $\widehat{\alpha} \notin [-\log p_{\max}, -\log p_{\min}]$. Indeed, if $\widehat{\alpha} \in [-\log p_{\max}, -\log p_{\min}]$ then $P'_{\widehat{\alpha}} \equiv c$ for some $c \in \mathbb{R}$. If $P'_{\widehat{\alpha}} \equiv c \neq 0$, then $P_{\widehat{\alpha}}(q) = s_0 + cq$ but then $\inf_{q\in\mathbb{R}} P_{\widehat{\alpha}}(q) = -\infty$, which contradicts to $P_{\widehat{\alpha}}(q) \geq 0$ for every q, and if $c = 0$ then $p_i e^{\widehat{\alpha}} = 1$ for every i but this contradicts to the assumption $p_i \neq p_j$ for some $i \neq j$. This implies that

$$P''_\alpha(q) > 0$$

for every $\alpha \in [-\log p_{\max}, -\log p_{\min}]$.

A simple algebraic manipulation shows that

$$\lim_{q\to\infty} P'_\alpha(q) = \frac{\sum_{i\in\mathcal{I}_{\max}} \lambda_i^{s_{\max}} \log(p_i e^\alpha)}{-\sum_{i\in\mathcal{I}_{\max}} \lambda_i^{s_{\max}} \log \lambda_i} \geq 0$$

and

$$\lim_{q\to-\infty} P'_\alpha(q) = \frac{\sum_{i\in\mathcal{I}_{\min}} \lambda_i^{s_{\min}} \log(p_i e^\alpha)}{-\sum_{i\in\mathcal{I}_{\min}} \lambda_i^{s_{\min}} \log \lambda_i} \leq 0.$$

Hence, if $p_i \neq p_j$ for some $i \neq j$ then for every $\alpha \in (-\log p_{\max}, -\log p_{\min})$ there exists a continuous function $\alpha \mapsto q(\alpha)$ such that $P'_\alpha(q(\alpha)) = 0$, and $\inf_{q\in\mathbb{R}} P_\alpha(q) = P_\alpha(q(\alpha))$. Moreover, the map $\alpha \mapsto P_\alpha(q(\alpha))$ is strictly concave, $\lim_{\alpha\to-\log p_{\max}} P_\alpha(q(\alpha)) = s_{\max}$ and $\lim_{\alpha\to-\log p_{\min}} P_\alpha(q(\alpha)) = s_{\min}$.

It remains to show that the unique maximum occurs at $\alpha = -\sum_{i=1}^N \lambda_i^{s_0} \log p_i$. Since $\alpha \mapsto \inf_{q\in\mathbb{R}} P_\alpha(q) = P_\alpha(q(\alpha))$ is strictly concave, it has a unique maximum, characterised by the vanishing of its derivative. By the multivariate chain rule,

$$\tfrac{d}{d\alpha} P_\alpha(q(\alpha)) = \left(\tfrac{d}{d\alpha} P_\alpha\right)(q(\alpha)) + P'_\alpha(q(\alpha)) \cdot q'(\alpha) = \left(\tfrac{d}{d\alpha} P_\alpha\right)(q(\alpha)), \tag{8}$$

where the last equality follows as $P'_\alpha(q(\alpha)) = 0$ by definition of $q(\alpha)$. Now differentiating

$$\sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q = 1$$

with respect to α gives

$$\sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q \left(\left(\tfrac{d}{d\alpha} P_\alpha\right)(q) \log \lambda_i + q\right) = 0,$$

and since $\sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q = 1$, solving for the parameter derivative yields

$$\left(\tfrac{d}{d\alpha} P_\alpha\right)(q) = \frac{-q}{\sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q \log \lambda_i}.$$

The denominator is strictly negative as $\log \lambda_i < 0$, so by Eq. (8) we have $\frac{d}{d\alpha} P_\alpha(q(\alpha)) = \left(\frac{d}{d\alpha} P_\alpha\right)(q(\alpha)) = 0$ if and only if $q(\alpha) = 0$. By definition of $q(\alpha)$ this means $P'_\alpha(0) = 0$, and evaluating Eq. (7) at $q = 0$, where $P_\alpha(0) = s_0$, gives

$$0 = \sum_{i=1}^N \lambda_i^{s_0} (\log p_i + \alpha) = \alpha + \sum_{i=1}^N \lambda_i^{s_0} \log p_i,$$

using $\sum_{i=1}^N \lambda_i^{s_0} = 1$. Hence the unique maximum of $\alpha \mapsto \inf_{q\in\mathbb{R}} P_\alpha(q)$ occurs at $\alpha = -\sum_{i=1}^N \lambda_i^{s_0} \log p_i$, which completes the proof. □

Lemma 2.1 shows that $\inf_q P_\alpha(q)$ has similar properties to the standard Legendre transform of convex functions.

Proposition 2.2. *Let $P_\alpha(q)$ be the unique solution of $\sum_{i=1}^N \lambda_i^{P_\alpha(q)}(p_i e^\alpha)^q = 1$. Then*

$$\inf_{q\in[0,1]}\{P_\alpha(q)\} = \begin{cases} P_\alpha(1), & \text{if } \alpha \le -\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i \\ \inf_{q\in\mathbb{R}}\{P_\alpha(q)\}, & \text{if } -\sum_{i=1}^N \lambda_i^{P_\alpha(0)} \log p_i > \alpha > -\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i \\ P_\alpha(0) = s_0, & \text{otherwise.} \end{cases} \quad (9)$$

Moreover, the map $\alpha \mapsto \inf_{q\in[0,1]} P_\alpha(q)$ is also continuous, and strictly smaller than $s_0 = \dim_{\mathrm{H}} \Lambda$ whenever $-\sum_{i=1}^N \lambda_i^{s_0} \log p_i > \alpha$.

Proof. Let us first prove Eq. (9). Suppose that $P_\alpha'' \equiv 0$, and so there exists $c \in \mathbb{R}$ such that $\frac{\log(p_i e^\alpha)}{-\log \lambda_i} = c$ for every i. In particular, $P_\alpha(q) = cq + s_0$. Then $\alpha \le -\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i = -\sum_{i=1}^N \lambda_i^{s_0} \log p_i$ holds if and only if $c \le 0$ and so $\inf_{q\in[0,1]} P_\alpha(q) = P_\alpha(1)$, which proves the claim in this case.

We may suppose now that $P_\alpha''(q) > 0$ for all $q \in \mathbb{R}$. Hence, there exists at most one $q' \in \mathbb{R}$ such that $P_\alpha'(q') = 0$. Thus, if $\alpha \le -\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i$ then $P_\alpha'(1) \le 0$ by Eq. (7), and by convexity $1 \le q'$, and so, $P_\alpha'(q) \le 0$ for every $q \in [0,1]$, which clearly implies that $\inf_{q\in[0,1]}\{P_\alpha(q)\} = P_\alpha(1)$. The case $\alpha \ge -\sum_{i=1}^N \lambda_i^{P_\alpha(0)} \log p_i$ is similar, and the remaining case follows by the uniqueness of the infimal point of convex functions. □

Note that if $p_i = 1/N$ for every i, then the middle region in Eq. (9) is empty. Now, we show the following variational principle result:

Lemma 2.3. *For every $\alpha > 0$,*

$$\sup_{\substack{(q_1,\dots,q_N):\\ \sum_i q_i = 1,\, q_i>0}} \left\{ \frac{-\sum_{i=1}^N q_i \log q_i}{-\sum_{i=1}^N q_i \log \lambda_i} : \sum_{i=1}^N q_i \log p_i + \alpha \ge 0 \right\} = \inf_{q>0} P_\alpha(q),$$

and the supremum is attained for the probability vector $q_{i,\alpha}^* = \lambda_i^{P_\alpha(q^*)}(p_i e^\alpha)^{q^*}$, where q^* is where the infimum of $\inf_{q>0} P_\alpha(q)$ is attained.

Moreover, for every $\alpha > 0$,

$$\sup_{\substack{(q_1,\dots,q_N):\\ \sum_i q_i = 1,\, q_i>0}} \left\{ \frac{-\sum_{i=1}^N q_i \log q_i}{-\sum_{i=1}^N q_i \log \lambda_i} : \sum_{i=1}^N q_i \log p_i + \alpha \le 0 \right\} = \inf_{q<0} P_\alpha(q), \quad (10)$$

and the supremum is attained for the probability vector $q_{i,\alpha}^* = \lambda_i^{P_\alpha(q^*)}(p_i e^\alpha)^{q^*}$, where q^* is where the infimum of $\inf_{q<0} P_\alpha(q)$ is attained.

Proof. First, let us observe that by the concavity of the logarithm for every $q > 0$ and for every probability vector $\{q_i\}_{i=1}^N$ such that $\sum_{i=1}^N q_i \log p_i + \alpha \geq 0$ we get

$$\begin{aligned} 0 &= \log \sum_{i=1}^N \lambda_i^{P_\alpha(q)} (p_i e^\alpha)^q \geq \sum_{i=1}^N q_i \left(\log \left((p_i e^\alpha)^q \lambda_i^{P_\alpha(q)} \right) - \log q_i \right) \\ &= q \left(\sum_{i=1}^N q_i \log p_i + \alpha \right) + P_\alpha(q) \sum_{i=1}^N q_i \log \lambda_i - \sum_{i=1}^N q_i \log q_i \\ &\geq P_\alpha(q) \sum_{i=1}^N q_i \log \lambda_i - \sum_{i=1}^N q_i \log q_i, \end{aligned}$$

and so,

$$P_\alpha(q) \geq \frac{-\sum_{i=1}^N q_i \log q_i}{-\sum_{i=1}^N q_i \log \lambda_i}. \tag{11}$$

Now, suppose that $P_\alpha''(q) = 0$ for every $q \in \mathbb{R}$, and in particular, $P_\alpha'(q) = \frac{\log p_i + \alpha}{-\log \lambda_i} = c$ for every i and q. Then $c \geq 0$ if and only if $\alpha \geq -\sum_i \lambda_i^{s_0} \log p_i$ and in this case

$$\inf_{q>0} P_\alpha(q) = P_\alpha(0) = s_0 = \sup \frac{-\sum_{i=1}^N q_i \log q_i}{-\sum_{i=1}^N q_i \log \lambda_i}.$$

If $c < 0$ then $\inf_{q>0} P_\alpha(q) = -\infty$, and $\sum_{i=1}^N q_i \log p_i + \alpha = -c \sum_{i=1}^N q_i \log \lambda_i < 0$, thus, the claim of the lemma is trivial.

If $P_\alpha''(q) > 0$ for every $q \in \mathbb{R}$ then by Proposition 2.2, there exists a unique $q^* \in \mathbb{R}$ such that $\inf_{q \in \mathbb{R}} P_\alpha(q) = P_\alpha(q^*)$. If $q^* \leq 0$ then $\inf_{q>0} P_\alpha(q) = P_\alpha(0)$, and $\sum_{i=1}^N \lambda_i^{P_\alpha(0)} \log p_i + \alpha \geq 0$ by Proposition 2.2 and $\frac{-\sum_{i=1}^N \lambda_i^{P_\alpha(0)} \log \lambda_i^{P_\alpha(0)}}{-\sum_{i=1}^N \lambda_i^{P_\alpha(0)} \log \lambda_i} = P_\alpha(0) = s_0$. If $q^* > 0$ then $P_\alpha'(q^*) = 0$ and by Eq. (7)

$$0 = \sum_{i=1}^N \lambda_i^{P_\alpha(q^*)} (p_i e^\alpha)^{q^*} \log p_i + \alpha,$$

and then by Eq. (11)

$$P_\alpha(q^*) \geq \frac{-\sum_{i=1}^N \lambda_i^{P_\alpha(q^*)} (p_i e^\alpha)^{q^*} \log \lambda_i^{P_\alpha(q^*)} (p_i e^\alpha)^{q^*}}{-\sum_{i=1}^N \lambda_i^{P_\alpha(q^*)} (p_i e^\alpha)^{q^*} \log \lambda_i} = P_\alpha(q^*) - q^* P_\alpha'(q^*) = P_\alpha(q^*).$$

The proof of the second claim is similar and left for the reader. □

The three cases in Proposition 2.2 naturally define measures on Σ. Namely, in the first case let us denote by $\mathbb{Q}_1$ the Bernoulli measure on Σ associated to the probability vector

$\{\lambda_i^{P_\alpha(1)} p_i e^\alpha\}_{i=1}^N$, let us denote by $\mathbb{Q}_0$ the Bernoulli measure on Σ associated to the probability vector $\{\lambda_i^{s_0}\}_{i=1}^N$, and let $\mathbb{Q}_{q^*}$ be associated to $\{q^*_{i,\alpha} = \lambda_i^{P_\alpha(q^*(\alpha))}(p_i e^\alpha)^{q^*(\alpha)}\}_{i=1}^N$, where $q^*(\alpha)$ is the unique solution for $P_\alpha(q^*(\alpha)) = \inf_q P_\alpha(q)$.

We will turn now to study the relation of Eq. (4) and $P_\alpha(q)$. Let $\ell\colon \mathbb{N} \to \mathbb{N}$ be a monotone increasing function as in Theorem 1.2. Let us write

$$m(n) = \left\lfloor \frac{\#\ell^{-1}(n)}{n} \right\rfloor, \tag{12}$$

whenever $n \in \ell(\mathbb{N})$ and $m(n) = 0$ otherwise, as ℓ may not be surjective. Note further that $\limsup_{n\to\infty} \frac{\log m(n)}{n} = \alpha$ since $m(n)$ is non-zero for infinitely many values.

For any $s \geq 0$, we define the **probabilistic pressure** as

$$L\colon s \mapsto \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{|\overline{\jmath}|=n} \lambda_{\overline{\jmath}}^s \left(1 - (1 - p_{\overline{\jmath}})^{m(n)}\right). \tag{13}$$

It is clear that the map above is continuous and strictly monotone decreasing. Moreover, it tends to $-\infty$ as $s \to \infty$ and at $s = 0$ it is non-negative. Indeed,

$$\sum_{|\overline{\jmath}|=n} \left(1 - (1 - p_{\overline{\jmath}})^{m(n)}\right) \geq \sum_{|\overline{\jmath}|=n} p_{\overline{\jmath}} = 1.$$

Hence, there is a unique solution of $L(s) = 0$, which we denote by $s(\alpha)$ with a slight abuse of notation.

Next, we show that the two formulas of the dimension of the dynamical covering set are equivalent. Before doing so, we define two sets, which play an important role in this argument and further in the paper. Let $\epsilon > 0$ and

$$\mathcal{G}_n^\epsilon = \{\overline{\jmath} \in \Sigma_n \mid 1-(1-p_{\overline{\jmath}})^{m(n)} \leq (1-\epsilon)^n\} \text{ and } \mathcal{V}_n^\epsilon = \{\overline{\jmath} \in \Sigma_n \mid 1-(1-p_{\overline{\jmath}})^{m(n)} > (1-\epsilon)^n\}. \tag{14}$$

By the approximations Eq. (5) we immediately obtain

$$\{\overline{\jmath} \in \Sigma_n \mid p_{\overline{\jmath}} m(n) \leq (1-\epsilon)^n\} \subset \mathcal{G}_n^\epsilon \subset \{\overline{\jmath} \in \Sigma_n \mid p_{\overline{\jmath}} m(n) \leq \frac{1}{\epsilon}(1-\epsilon)^n\}, \tag{15}$$

and so,

$$\{\overline{\jmath} \in \Sigma_n \mid p_{\overline{\jmath}} m(n) > \frac{1}{\epsilon}(1-\epsilon)^n\} \subset \mathcal{V}_n^\epsilon \subset \{\overline{\jmath} \in \Sigma_n \mid p_{\overline{\jmath}} m(n) > (1-\epsilon)^n\}. \tag{16}$$

Lemma 2.4. *Let $s(\alpha)$ be the unique solution of $L(s(\alpha)) = 0$. Then $\inf_{q\in[0,1]} P_\alpha(q) = s(\alpha)$.*

Proof. Let us first suppose that $0 < \alpha < -\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i$. By Eq. (5), for every $s \geq 0$

$$\begin{aligned} L(s) = \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{|\overline{\jmath}|=n} \lambda_{\overline{\jmath}}^s \left(1-(1-p_{\overline{\jmath}})^{m(n)}\right) &\le \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{|\overline{\jmath}|=n} \lambda_{\overline{\jmath}}^s m(n) p_{\overline{\jmath}} \\ &= \log \sum_{i=1}^N \lambda_i^s p_i + \limsup_{n\to\infty} \tfrac{1}{n} \log m(n) \\ &= \log \sum_{i=1}^N \lambda_i^s p_i + \alpha, \end{aligned}$$

where in the last equality we have used the monotonicity of ℓ and Eq. (1). Thus, in particular, $L(P_\alpha(1)) \le 0$.

Choose $\epsilon > 0$ such that $\alpha + \sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i < 2\log(1-\epsilon) \le 0$. For sufficiently large $n \ge 1$, $m(n) \le e^{\alpha n}(1-\epsilon)^{-n}$. Then by Eq. (6) and the definition of $\mathcal{G}_n^\epsilon$.

$$\begin{aligned} L(P_\alpha(1)) &\ge \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{\overline{\jmath}\in\mathcal{G}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(1)} \left(1-(1-p_{\overline{\jmath}})^{m(n)}\right) \\ &\ge \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{\overline{\jmath}\in\mathcal{G}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(1)} m(n) p_{\overline{\jmath}} \left(1-\frac{m(n)p_{\overline{\jmath}}}{2}\right) \\ &\ge \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{\overline{\jmath}\in\mathcal{G}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(1)} m(n) p_{\overline{\jmath}} \left(1-\frac{(1-\epsilon)^n}{2\epsilon}\right) \\ &\ge \limsup_{n\to\infty} \left(\tfrac{1}{n} \log \left(\sum_{\overline{\jmath}\in\mathcal{G}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(1)} e^{\alpha n} p_{\overline{\jmath}} \right) + \tfrac{1}{n} \log(m(n)/e^{\alpha n}) \right) \\ &\ge \limsup_{n\to\infty} \left(\tfrac{1}{n} \log \mathbb{Q}_1 \left(\left\{ \overline{\imath} \in \Sigma : \tfrac{1}{n} \log p_{\overline{\imath}|_n} + \frac{\log m(n)}{n} \le \log(1-\epsilon) \right\} \right) \right. \\ &\qquad \left. + \tfrac{1}{n} \log(m(n)/e^{\alpha n}) \right) \\ &\ge \limsup_{n\to\infty} \left(\tfrac{1}{n} \log \mathbb{Q}_1 \left(\left\{ \overline{\imath} \in \Sigma : \tfrac{1}{n} \log p_{\overline{\imath}|_n} + \alpha \le 2\log(1-\epsilon) \right\} \right) + \tfrac{1}{n} \log(m(n)/e^{\alpha n}) \right) \\ &= 0, \end{aligned}$$

where the last equality follows by the fact that the expected value of $\log p_{i_1}$ w.r.t. $\mathbb{Q}_1$ is $\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i$ and the choice of ϵ.

Now suppose that $-\sum_{i=1}^N \lambda_i^{P_\alpha(1)} p_i e^\alpha \log p_i \le \alpha \le -\sum_{i=1}^N \lambda_i^{P_\alpha(0)} \log p_i$. Let $\epsilon > 0$ be arbitrary and write $h(\mathbb{Q}_q)$ for the entropy of the measure $\mathbb{Q}_q$. Then $m(n) \le e^{\alpha n}(1-\epsilon)^{-n}$ for every sufficiently large $n \ge 1$ and by Lemma 2.3 and Eq. (16)

$$\begin{aligned} &L(P_\alpha(q^*(\alpha))) \\ &\le \limsup_{n\to\infty} \tfrac{1}{n} \log \left(\sum_{\overline{\jmath}\in\mathcal{G}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(q^*(\alpha))} m(n) p_{\overline{\jmath}} + \sum_{\overline{\jmath}\in\mathcal{V}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(q^*(\alpha))} \right) \end{aligned}$$

$$\leq \limsup_{n\to\infty} \tfrac{1}{n} \log \left(\sum_{\overline{\jmath}\in\mathcal{G}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(q^*(\alpha))} (m(n)p_{\overline{\jmath}})^{q^*(\alpha)} (1-\epsilon)^{n(1-q^*(\alpha))} + \sum_{\overline{\jmath}\in\mathcal{V}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(q^*(\alpha))} \right)$$

$$\leq \limsup_{n\to\infty} \tfrac{1}{n} \log \Bigg((1-\epsilon)^{n(1-2q^*(\alpha))} \mathbb{Q}_{q^*}(\{\overline{\imath} : \overline{\imath}|_n \in \mathcal{G}_n^\epsilon\}) + \sum_{\substack{k_1,\dots,k_N\geq 0 \\ k_1+\cdots+k_N=n \\ n\alpha+\sum_i k_i \log p_i > n2\log(1-\epsilon)}} \binom{n}{k_1,\dots,k_N} \left(\lambda_1^{k_1}\cdots\lambda_N^{k_N}\right)^{P_\alpha(q^*(\alpha))} \Bigg)$$

$$= \limsup_{n\to\infty} \tfrac{1}{n} \log \Bigg(e^{o(n)} (1-\epsilon)^{n(1-2q^*)} \mathbb{Q}_{q^*}(\{\overline{\imath} : \overline{\imath}|_n \in \mathcal{G}_n^\epsilon\}) + \binom{n+N-1}{N-1} e^{n\left(h_{\mathbb{Q}_{q^*(\alpha-2\log(1-\epsilon))}} + P_\alpha(q^*(\alpha)) \sum_i q^*_{i,\alpha-2\log(1-\epsilon)} \log\lambda_i\right)} \Bigg)$$

$$\leq \max\left\{(1-2q^*(\alpha))\log(1-\epsilon), \left|\left(P_\alpha(q^*(\alpha)) - P_{\alpha-2\log(1-\epsilon)}(q^*(\alpha-2\log(1-\epsilon)))\right)\log\lambda_{\min}\right|\right\},$$

where in the second to last equality we have used Stirling's approximation $\binom{n}{k_1,\dots,k_N} = e^{nh(k_1/n,\dots,k_N/n)+o(n)}$ and that the number of compositions $k_1+\cdots+k_N = n$ is at most $\binom{n+N-1}{N-1}$, so that the sum is comparable to its largest term, whose exponent is identified via Lemma 2.3 with the measure $\mathbb{Q}_{q^*}$. To conclude this part, note that $\epsilon > 0$ was arbitrary, we get that $L(P_\alpha(q^*(\alpha))) \leq 0$.

For the other inequality, observe that for every $\epsilon > 0$ there exists a subsequence n_k such that

$$\frac{(1-\epsilon)^{n_k}}{\epsilon} < p_{\min} e^{-\alpha n_k} m(n_k) < p_1^{q^*_{1,\alpha} n_k} \cdots p_N^{q^*_{N,\alpha} n_k} m(n_k), \tag{17}$$

and so, every finite word $\overline{\imath}$ with $|\overline{\imath}| = n_k$ and $\#_i\overline{\imath} = q^*_{i,\alpha} n_k$ belongs to $\mathcal{V}_{n_k}^\epsilon$, where $\#_i\overline{\imath}$ denotes the number of occurrences of the symbol i in $\overline{\imath}$. Hence,

$$\begin{aligned} L(P_\alpha(q^*(\alpha))) &\geq \limsup_{n\to\infty} \tfrac{1}{n} \log \left(\sum_{\overline{\jmath}\in\mathcal{V}_n^\epsilon} \lambda_{\overline{\jmath}}^{P_\alpha(q^*(\alpha))} (1-\epsilon)^n \right) \\ &\geq \limsup_{k\to\infty} \tfrac{1}{n_k} \log \left(\binom{n_k}{q^*_{1,\alpha} n_k, \dots, q^*_{N,\alpha} n_k} \left(\lambda_1^{q^*_{1,\alpha} n_k} \cdots \lambda_N^{q^*_{N,\alpha} n_k}\right)^{P_\alpha(q^*(\alpha))} \times (1-\epsilon)^{n_k} \right) \\ &\geq \log(1-\epsilon), \end{aligned}$$

and since $\epsilon > 0$ was arbitrary $L(P_\alpha(q^*(\alpha))) \geq 0$.

Finally, suppose that $\alpha > -\sum_{i=1}^{N} \lambda_i^{P_\alpha(0)} \log p_i$. Since $L(s) \leq \log \sum_{i=1}^{N} \lambda_i^s$, we can further conclude that $L(P_\alpha(0)) \leq 0$. On the other hand, similarly to Eq. (17) one can show that $(1-\epsilon)^{n_k}/\epsilon < p_1^{\lambda_1^{s_0} n_k} \cdots p_N^{\lambda_N^{s_0} n_k} m(n_k)$ over a subsequence n_k, and so, similar argument leads us to the inequality $L(P_\alpha(0)) \geq 0$. □

Finally, let us note that a simple corollary of Lemma 2.4 is that the probabilistic pressures defined in Eq. (4) and Eq. (13) have the same root.

3. Upper bounds

This section is devoted to proving the almost sure upper bounds in the main theorems.

3.1. Upper bound for the dynamical covering set

First, we will prove the upper bound to the Hausdorff dimension in Theorem 1.2. Notice that the value $s(\alpha)$ given as the root of the probabilistic pressure from Eq. (13), by Lemma 2.4 is equal to the dimension value in Eq. (2) from Theorem 1.2.

Proposition 3.1. *Let $\{f_i(x) = \lambda_i O_i + t_i\}_{i=1}^N$ be an IFS of similarities. Let $\alpha > 0$ and assume that $\ell : \mathbb{N} \to \mathbb{N}$ is monotone increasing and it satisfies*

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \tfrac{1}{\alpha}.$$

Let $\mathbb{P}_p$ be a Bernoulli measure corresponding to a probability vector $p = (p_1, \ldots, p_N)$. Then, for $\mathbb{P}_p$-almost every $\overline{o} \in \Sigma$

$$\dim_{\mathrm{H}} R(\overline{o}, \ell) \leq s(\alpha),$$

where $s(\alpha)$ is the unique root of the probabilistic pressure defined in Eq. (13).

Proof. Notice that, for all $m \in \mathbb{N}$

$$R(\overline{o}, \ell) = \limsup_{n\to\infty} \pi[\sigma^n(\overline{o})|_{\ell(n)}] \subset \bigcup_{n\geq m} \bigcup_{\substack{|\overline{\imath}|=n \\ \sigma^k(\overline{o})|_{\ell(k)}=\overline{\imath} \text{ for some } k \text{ with } \ell(k)=n}} \pi[\overline{\imath}],$$

and (up to an absolute constant) $\operatorname{diam}(\pi[\overline{\imath}]) = \lambda_{\overline{\imath}}$. Hence, the Hausdorff dimension of $R(\overline{o}, \ell)$ is bounded from above by all those t for which

$$\sum_{n=1}^{\infty} \sum_{|\overline{\imath}|=n} \lambda_{\overline{\imath}}^t \cdot \mathbb{1}\{\overline{\imath} = \sigma^k(\overline{o})|_n \text{ for some } k \text{ for which } n = \ell(k)\} < \infty.$$

It is difficult to bound this series for a fixed $\overline{o}$, but as we are only looking for bounds that hold for $\mathbb{P}_p$-almost every $\overline{o}$, we aim to prove, instead, that

$$\begin{aligned}\mathbb{E}\left(\sum_{n=1}^{\infty}\sum_{|\overline{\imath}|=n}\lambda_{\overline{\imath}}^{t}\cdot\mathbb{1}\{\overline{\imath}=\sigma^{k}(\overline{o})|_{n}\text{ for some }k\text{ for which }n=\ell(k)\}\right)\\ =\sum_{n=1}^{\infty}\sum_{|\overline{\imath}|=n}\lambda_{\overline{\imath}}^{t}\mathbb{P}_{p}(\overline{\imath}=\sigma^{k}(\overline{o})|_{n}\text{ for some }k\text{ for which }n=\ell(k))<\infty.\end{aligned}\tag{18}$$

Here the expectation is with respect to the Bernoulli measure $\mathbb{P}_p$. For any fixed n such that there are values k for which $\ell(k)=n$, there are

$$\left\lfloor\frac{\#\ell^{-1}(n)}{n}\right\rfloor=m(n)\tag{19}$$

many k that correspond to disjoint stretches of $\overline{o}$. This is due to the fact that, denoting by $\min\ell^{-1}(n)=:M$ for any fixed n, there are approximately $m(n)$ elements p in the collection

$$P_k(n)=\{M+k,M+k+n,\dots,M+k+qn\mid q\in\mathbb{N}\text{ maximal s.t. }M+k+qn\in\ell^{-1}(n)\},$$

where $k\in\{0,\dots,n-1\}$. These correspond to disjoint stretches $\sigma^p(\overline{o})$. Hence we have, for any fixed n, $\overline{\imath}$

$$\begin{aligned}\mathbb{P}(\overline{\imath}=\sigma^{p}(\overline{o})|_{n}\text{ for some }p\text{ for which }n=\ell(p))&\leq\sum_{k=0}^{n-1}\mathbb{P}(\overline{\imath}=\sigma^{p}(\overline{o})|_{n}\text{ for some }p\in P_{k}(n))\\&\leq n\left(1-(1-p_{\overline{\imath}})^{m(n)}\right).\end{aligned}\tag{20}$$

Hence, when $t>s(\alpha)$, as L is strictly decreasing,

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\sum_{|\overline{\jmath}|=n}\lambda_{\overline{\jmath}}^{t}(1-(1-p_{\overline{\jmath}})^{m(n)})<0,$$

then

$$\sum_{n=1}^{\infty}n\sum_{|\overline{\imath}|=n}\lambda_{\overline{\imath}}^{t}(1-(1-p_{\overline{\imath}})^{m(n)})<\infty,$$

and in particular (18) and hence the upper bound to the Hausdorff dimension of $R(\overline{o},\ell)$ holds. □

3.2. Upper bound for the complement

Now, we verify the upper bound of Theorem 1.3.

Proposition 3.2. *Let $\{f_i(x) = \lambda_i O_i + t_i\}_{i=1}^N$ be an IFS of similarities. Let $\alpha > 0$ and assume that $\ell : \mathbb{N} \to \mathbb{N}$ is monotone increasing and it satisfies*

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \tfrac{1}{\alpha}.$$

Let $\mathbb{P}_p$ be a Bernoulli measure corresponding to a probability vector $p = (p_1, \dots, p_N)$. Then, for $\mathbb{P}_p$-almost every $\overline{o} \in \Sigma$

$$\dim_{\mathrm{H}} R(\overline{o}, \ell)^c \le t(\alpha),$$

where $t(\alpha)$ is defined in Eq. (3).

Proof. Notice that,

$$R(\overline{o}, \ell)^c = \bigcup_m \bigcap_{n\ge m} \bigcup_{\substack{|\overline{\imath}|=n \\ \sigma^k(\overline{o})|_{\ell(k)} \neq \overline{\imath} \text{ for every } k \text{ with } \ell(k)=n}} \pi[\overline{\imath}],$$

Let $\epsilon > 0$ be arbitrary but fixed. Let (n_p) be the subsequence for which

$$m(n_p) e^{-\alpha n_p} \ge (1+\epsilon)^{-n_p},$$

where $m(n)$ is as in Eq. (19). So, for every $m \ge 1$

$$R(\overline{o}, \ell)^c \subset \bigcup_{p=m}^{\infty} \bigcup_{\substack{|\overline{\imath}|=n_p \\ \sigma^k(\overline{o})|_{\ell(k)} \neq \overline{\imath} \text{ for every } k \text{ with } \ell(k)=n_p}} \pi[\overline{\imath}].$$

So, similarly to the previous case, the Hausdorff dimension of $R(\overline{o}, \ell)^c$ is bounded from above by t if

$$\sum_{p=1}^{\infty} \sum_{|\overline{\imath}|=n_p} \lambda_{\overline{\imath}}^t \cdot \mathbb{1}\{\overline{\imath} \neq \sigma^k(\overline{o})|_{n_p} \text{ for every } k \text{ for which } n_p = \ell(k)\} < \infty.$$

To show that $\dim_{\mathrm{H}} R(\overline{o}, \ell)^c \le t$ for $\mathbb{P}_p$-almost every $\overline{o}$, it is enough to show that

$$\mathbb{E}\left(\sum_{p=1}^{\infty} \sum_{|\overline{\imath}|=n_p} \lambda_{\overline{\imath}}^t \cdot \mathbb{1}\{\overline{\imath} \neq \sigma^k(\overline{o})|_{n_p} \text{ for every } k \text{ for which } n_p = \ell(k)\} \right)$$

$$\leq \sum_{p=1}^{\infty} \sum_{|\overline{\imath}|=n_p} \lambda_{\overline{\imath}}^t \cdot (1-p_{\overline{\imath}})^{m(n_p)} < \infty.$$

For $\epsilon > 0$, let us now define

$$\mathcal{G}'^{\epsilon}_n = \{\overline{\imath} \in \Sigma_n : p_{\overline{\imath}} m(n) \geq (1+\epsilon)^n\} \quad \text{and} \quad \mathcal{V}'^{\epsilon}_n = \{\overline{\imath} \in \Sigma_n : p_{\overline{\imath}} m(n) < (1+\epsilon)^n\}. \tag{21}$$

Then by choosing

$$t = (1+\epsilon) P_{\alpha - 2\log(1+\epsilon)}(q^*(\alpha - 2\log(1+\epsilon))) = (1+\epsilon) \inf_{q<0} P_{\alpha-2\log(1+\epsilon)}(q)$$

and Eq. (5)

$$\begin{aligned}
&\sum_{|\overline{\imath}|=n_p} \lambda_{\overline{\imath}}^t \cdot (1-p_{\overline{\imath}})^{m(n_p)} \\
&\quad \leq \sum_{\overline{\imath} \in \mathcal{G}'^{\epsilon}_{n_p}} \lambda_{\overline{\imath}}^t e^{-p_{\overline{\imath}} m(n_p)} + \sum_{\overline{\imath} \in \mathcal{V}'^{\epsilon}_{n_p}} \lambda_{\overline{\imath}}^t \\
&\quad \leq e^{-(1+\epsilon)^{n_p}} N^{n_p} + \sum_{\substack{k_1,\ldots,k_N \geq 0 \\ k_1+\cdots+k_N = n_p \\ n_p\alpha + \sum_i k_i \log p_i < 2n_p \log(1+\epsilon)}} \binom{n_p}{k_1,\ldots,k_N} \left(\lambda_1^{k_1} \cdots \lambda_N^{k_N}\right)^t \\
&\quad \leq e^{-(1+\epsilon)^{n_p}} N^{n_p} + \binom{n_p+N-1}{N-1} e^{n_p\left(h_{\mathbb{Q}_{q^*(\alpha-2\log(1+\epsilon))}} + t\sum_i q^*_{i,\alpha-2\log(1+\epsilon)} \log \lambda_i\right)} \\
&\quad \leq e^{-(1+\epsilon)^{n_p}} N^{n_p} + \binom{n_p+N-1}{N-1} \lambda_{\max}^{n_p \epsilon P_{\alpha-2\log(1+\epsilon)}(q^*(\alpha-2\log(1+\epsilon)))},
\end{aligned}$$

where the penultimate inequality uses Stirling's approximation $\binom{n_p}{k_1,\ldots,k_N} = e^{n_p h(k_1/n_p,\ldots,k_N/n_p)+o(n_p)}$ together with the bound $\binom{n_p+N-1}{N-1}$ on the number of compositions $k_1+\cdots+k_N = n_p$, so that the sum is comparable to its largest term, and the last inequality uses Lemma 2.3 to identify the exponential rate with $\inf_{q<0} P_{\alpha-2\log(1+\epsilon)}(q)$. The right-hand side forms a convergent series, and since $\epsilon > 0$ was arbitrary and the map $\alpha \to \inf_q P_\alpha(q)$ is continuous, the upper bound follows. □

4. Lower bounds for "large" α

4.1. The region $\alpha > \alpha_2$

We begin this section with a proof that for high α, the dynamical covering set turns out to be the entire attractor Λ. This is captured by the following theorem. Recall the notation $\alpha_2 = -\log p_{\min}$.

Theorem 4.1. *Let $\{f_i(x) = \lambda_i O_i + t_i\}_{i=1}^N$ be an IFS of similarities with attractor Λ and let $\mathbb{P}_p$ be a Bernoulli measure corresponding to a probability vector $p = (p_1, \dots, p_N)$. Let $\alpha > 0$ be such that $\alpha > \alpha_2$ and assume that $\ell : \mathbb{N} \to \mathbb{N}$ is monotone increasing and it satisfies*

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \tfrac{1}{\alpha}.$$

Then, for $\mathbb{P}_p$-almost all $\overline{o} \in \Sigma$,

$$R(\overline{o}, \ell) = \Lambda.$$

Proof. Let $\epsilon > 0$ be such that $p_{\min} e^{\alpha}(1-\epsilon) > 1$. Let $m(n)$ be as in Eq. (12), and let $\{n_p\}$ be the subsequence such that $\limsup_{n\to\infty} \frac{\log m(n)}{n} = \lim_{p\to\infty} \frac{\log m(n_p)}{n_p} = \alpha$. Without loss of generality we may assume that $m(n_p) \geq e^{\alpha n_p}(1-\epsilon)^{n_p}$.

We use a Borel-Cantelli argument to show that for typical $\overline{o}$, for sufficiently large p and for every cylinder $[\overline{\jmath}]$, where $\overline{\jmath} \in \Sigma_{n_p}$, there exists a k such that $\ell(k) = n_p$ and $\sigma^k(\overline{o})|_{\ell(k)} = \overline{\jmath}$. In particular, we show that the set

$$\bigcap_{k=1}^{\infty} \bigcup_{p=k}^{\infty} \bigcup_{\overline{\jmath}\in\Sigma_{n_p}} \{\overline{o} : (\sigma^q \overline{o})|_{n_p} \neq \overline{\jmath} \text{ for every } q \in \mathbb{N} \text{ with } \ell(q) = n_p\}$$

has zero $\mathbb{P}$ measure. This then shows that $R(\overline{o}, \ell) = \Lambda$.

First write M_p for the least integer and M_p' for the last integer such that $\ell(M_p) = \ell(M_p') = n_p$. We want to estimate the probability of the event that there exists at least one cylinder $\overline{\jmath} \in \Sigma_{n_p}$ that does not appear in $\overline{o}$ between M_p and M_p'. Denote this probability by P_p. Clearly, it is bounded by

$$\begin{aligned}
P_p &\leq \sum_{\overline{\jmath}\in\Sigma_{n_p}} \mathbb{P}\{\overline{o} \in \Sigma : (\sigma^k \overline{o})|_{n_p} \neq \overline{\jmath}, M_p \leq k < M_p'\} \\
&\leq \sum_{\overline{\jmath}\in\Sigma_{n_p}} \mathbb{P}\left\{\overline{o} \in \Sigma : \sigma^{M_p + n_p k}\overline{o}|_{n_p} \neq \overline{\jmath}, 0 \leq k < \frac{M_p' - M_p}{n_p}\right\} \\
&= \sum_{\overline{\jmath}\in\Sigma_{n_p}} \left(\prod_{k=0}^{(M_p'-M_p)/n_p - 1} \left(1 - \mathbb{P}\left\{\overline{o} \in \Sigma : \sigma^{M_p+n_p k}\overline{o}|_{n_p} = \overline{\jmath}\right\}\right)\right) \\
&= \sum_{\overline{\jmath}\in\Sigma_{n_p}} \left(1 - \mathbb{P}([\overline{\jmath}])\right)^{(M_p'-M_p)/n_p} \\
&\leq N^{n_p}\left(1 - p_{\min}^{n_p}\right)^{m(n_p)} = N^{n_p} \exp\log\left(1 - p_{\min}^{n_p}\right)^{m(n_p)}
\end{aligned}$$

and since $\log(1-x) \leq -x$ for all $x < 1$,

$$\leq N^{n_p} \exp\left(-m(n_p)p_{\min}^{n_p}\right) \leq N^{n_p} \exp\left(-e^{\alpha n_p}(1-\epsilon)^{n_p} p_{\min}^{n_p}\right).$$

By assumption, $e^{\alpha} p_{\min}(1-\epsilon) > 1$, and so, this bound decreases super-exponentially and $\sum_p P_p < \infty$. By the Borel-Cantelli lemmas we see that for almost every $\overline{o}$ there exists a level $K_{\overline{o}}$ such that for every cylinder $[\overline{\jmath}]$ where $\overline{\jmath} \in \Sigma_{n_p}$, $p > K_{\overline{o}}$ there exists a k such that $\ell(k) = n_p$ and $\sigma^k(\overline{o})|_{\ell(k)} = \overline{\jmath}$. It follows that $R(\overline{o}, \ell) = \Lambda$, as required. □

4.2. The region $\alpha_1 < \alpha \leq \alpha_2$

In this region, we will study the Hausdorff dimension of $R(\overline{o}, \ell)^c$ via the Bernoulli measures of the set $R(\overline{o}, \ell)$ with respect to measures for which $\sum_i q_i \log p_i + \alpha < 0$. Recall that

$$\alpha_1 = -\sum_{i=1}^{N} \lambda_i^{s_0} \log p_i,$$

where $s_0 = \dim_{\mathrm{H}} \Lambda$. Also recall $\alpha_2 = -\log p_{\min}$.

Theorem 4.2. *Let $\alpha_1 < \alpha$ and further assume that $\ell(n)$ satisfies*

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \frac{1}{\alpha}.$$

Let ν be a Bernoulli measure with probability vector $q = (q_1, \ldots, q_N)$ such that $\sum_i q_i \log p_i + \alpha < 0$. Then, $\nu(R(\overline{o}, \ell)) = 0$ for $\mathbb{P}$-almost every $\overline{o}$.

First, we prove the corollary of this result.

Corollary 4.3. *Let $\alpha_1 < \alpha \leq \alpha_2$, and assume that $\ell(n)$ satisfies $\liminf_{n\to\infty} \ell(n)/\log n = 1/\alpha$. Then, $\dim_{\mathrm{H}}(R(\overline{o}, \ell)^c) = \inf_{q<0} P_\alpha(q) < s_0$ and $\mathcal{H}^{s_0}(R(\overline{o}, \ell)) = \mathcal{H}^{s_0}(\Lambda)$, where $s_0 = \dim_{\mathrm{H}} \Lambda$ for $\mathbb{P}$-almost every $\overline{o}$.*

Proof. For α in this range, by Lemma 2.3 and Proposition 3.2,

$$\dim_{\mathrm{H}}(R(\overline{o}, \ell)^c) \leq \inf_{q<0} P_\alpha(q) < s_0$$

and so, the claim for the Hausdorff measure of $R(\overline{o}, \ell)$ follows.

Let $q = (q_1, \ldots, q_N)$ be the probability vector, where the supremum in Eq. (10) is attained. Then by Lemma 2.3, $\sum_i q_i \log p_i + \alpha < 0$, and so, by Theorem 4.2,

$$\dim_{\mathrm{H}}(R(\overline{o}, \ell)^c) \geq \sup\left\{ \frac{-\sum_{i=1}^N q_i \log q_i}{-\sum_{i=1}^N q_i \log \lambda_i} : \sum_{i=1}^{N} q_i \log p_i + \alpha < 0 \right\} = \inf_{q<0} P_\alpha(q). \quad \square$$

Proof of Theorem 4.2. The proof is similar to Section 3. That is, by Borel-Cantelli's lemma, it is enough to show that for $\mathbb{P}$-almost every $\overline{o}$

$$\sum_{n=1}^{\infty} \sum_{|\overline{\imath}|=n} q_{\overline{\imath}} \cdot \mathbb{1}\{\overline{\imath} = \sigma^k(\overline{o})|_n \text{ for some } k \text{ for which } n = \ell(k)\} < \infty.$$

Similarly to Eq. (18) and Eq. (20), it is enough to show that

$$\sum_{n=1}^{\infty} \sum_{|\overline{\imath}|=n} q_{\overline{\imath}} \cdot n(1-(1-p_{\overline{\imath}})^{m(n)}) < \infty.$$

Let $\epsilon > 0$ be such that $\sum_{i=1}^{N} q_i \log p_i + \alpha < 2\log(1-\epsilon) < 0$. Recalling the definition of $\mathcal{G}_n^\epsilon$ and $\mathcal{V}_n^\epsilon$ from Eq. (14) together with Eq. (15) and Eq. (16), we get for sufficiently large n that

$$\begin{aligned}\sum_{|\overline{\imath}|=n} q_{\overline{\imath}} \cdot n(1-(1-p_{\overline{\imath}})^{m(n)}) &\le \sum_{\overline{\imath}\in\mathcal{G}_n^\epsilon} q_{\overline{\imath}} m(n) p_{\overline{\imath}} + \sum_{\overline{\imath}\in\mathcal{V}_n^\epsilon} q_{\overline{\imath}} \\ &\le \tfrac{1}{\epsilon}(1-\epsilon)^n + \nu\left(\left\{\overline{\imath} : p_{\overline{\imath}|_n} m(n) > (1-\epsilon)^n\right\}\right).\end{aligned}$$

By $m(n) \le (1-\epsilon)^{-n} e^{\alpha n}$ for n sufficiently large

$$\begin{aligned}&\le \tfrac{1}{\epsilon}(1-\epsilon)^n + \nu\left(\left\{\overline{\imath} : p_{\overline{\imath}|_n} e^{n\alpha} > (1-\epsilon)^{2n}\right\}\right) \\ &\le \tfrac{1}{\epsilon}(1-\epsilon)^n + e^{-\delta n},\end{aligned}$$

where the last inequality follows by the large deviation principle for some $\delta = \delta(q,\epsilon)$. Then $\sum_{n=1}^{\infty}\sum_{|\overline{\imath}|=n} q_{\overline{\imath}} \cdot n(1-(1-p_{\overline{\imath}})^{m(n)}) < \infty$ clearly follows. □

4.3. The region $\alpha_0 < \alpha < \alpha_1$

Similarly to Section 4.2, we will study the Hausdorff dimension of $R(\overline{o},\ell)$ via the Bernoulli measures of the set $R(\overline{o},\ell)^c$ with respect to measures for which $\sum_i q_i \log p_i + \alpha > 0$.

Theorem 4.4. *Let $\alpha > 0$ be such that $\alpha < \alpha_1$ and further assume that $\ell(n)$ satisfies $\liminf_{n\to\infty} \ell(n)/\log n = 1/\alpha$. Let ν be a Bernoulli measure with probability vector $q = (q_1,\dots,q_N)$ such that $\sum_i q_i \log p_i + \alpha > 0$. Then, $\nu(R(\overline{o},\ell)^c) = 0$ for $\mathbb{P}$-almost every $\overline{o}$.*

This result has the immediate corollary:

Corollary 4.5. *Let $\alpha_0 < \alpha < \alpha_1$ and assume that $\ell(n)$ satisfies $\liminf_{n\to\infty} \ell(n)/\log n = 1/\alpha$. Then, $\dim_{\mathrm{H}}(R(\overline{o},\ell)) = \inf_{q\in[0,1]} P_\alpha(q)$ and also $\mathcal{H}^{s_0}(R(\overline{o},\ell)^c) = \mathcal{H}^{s_0}(\Lambda)$, where $s_0 = \dim_{\mathrm{H}}\Lambda$ for $\mathbb{P}$-almost every $\overline{o}$.*

Proof. By Section 3 and Lemma 2.4, $\dim_{\mathrm{H}}(R(\overline{o},\ell)) \le \inf_{q\in[0,1]} P_\alpha(q) < s_0$ and so, the claim for the Hausdorff measure of $R(\overline{o},\ell)^c$ follows.

Let $q = (q_1,\dots,q_N)$ be the probability vector, where the supremum in Eq. (10) is attained. Then by Lemma 2.3, $\sum_i q_i \log p_i + \alpha > 0$. Thus, by Theorem 4.4,

$$\dim_{\mathrm{H}}(R(\overline{o},\ell)) \ge \sup\left\{ \frac{-\sum_{i=1}^N q_i \log q_i}{-\sum_{i=1}^N q_i \log \lambda_i} : \sum_{i=1}^N q_i \log p_i + \alpha > 0 \right\} = \inf_{q>0} P_\alpha(q),$$

where in the last equality, we have used again Lemma 2.3. □

Proof of Theorem 4.4. The proof will be similar to Section 3.2. Let $\epsilon > 0$ and let (n_p) be the subsequence for which

$$m(n_p)e^{-\alpha n_p} \ge (1+\epsilon)^{-n_p}.$$

By Borel-Cantelli's lemma, it is enough to show that for $\mathbb{P}$-almost every $\overline{o}$

$$\sum_{p=1}^{\infty} \sum_{|\overline{\imath}|=n_p} q_{\overline{\imath}} \cdot \mathbb{1}\{\overline{\imath} \ne \sigma^k(\overline{o})|_{n_p} \text{ for every } k \text{ for which } n_p = \ell(k)\} < \infty.$$

To show this for $\mathbb{P}_p$-almost every $\overline{o}$, it is enough to show that

$$\mathbb{E}\left(\sum_{p=1}^{\infty} \sum_{|\overline{\imath}|=n_p} q_{\overline{\imath}} \cdot \mathbb{1}\{\overline{\imath} \ne \sigma^k(\overline{o})|_{n_p} \text{ for every } k \text{ for which } n_p = \ell(k)\}\right) \le \sum_{p=1}^{\infty} \sum_{|\overline{\imath}|=n_p} q_{\overline{\imath}} \cdot (1-p_{\overline{\imath}})^{m(n_p)} < \infty. \tag{22}$$

Choose $\epsilon > 0$ such that $\sum_i q_i \log p_i + \alpha > 2\log(1+\epsilon) > 0$, and by recall the definition of $\mathcal{G}'^{\epsilon}_n$ and $\mathcal{V}'^{\epsilon}_n$ from Eq. (21), we get

$$\begin{aligned}
\sum_{|\overline{\imath}|=n_p} q_{\overline{\imath}} \cdot (1-p_{\overline{\imath}})^{m(n_p)} &\le \sum_{\overline{\imath}\in\mathcal{G}'^{\epsilon}_{n_p}} q_{\overline{\imath}} e^{-p_{\overline{\imath}} m(n_p)} + \sum_{\overline{\imath}\in\mathcal{V}'^{\epsilon}_{n_p}} q_{\overline{\imath}} \\
&\le e^{-(1+\epsilon)^{n_p}} + \nu\left(\left\{\overline{\imath}\in\Sigma : \alpha + \tfrac{1}{n_p}\log p_{\overline{\imath}} < 2\log(1+\epsilon)\right\}\right) \\
&\le e^{-(1+\epsilon)^{n_p}} + e^{-n_p\delta}
\end{aligned}$$

where the last inequality follows by the large deviation principle. Then Eq. (22) clearly follows. □

5. Lower bound for $\alpha \leq \alpha_0$

At its heart, the strategy of proof of the lower bound from Theorem 1.2 in this case is standard: We construct a Cantor subset to $R(\overline{o}, \ell)$ in Section 5.1, and then a measure on this Cantor set in Section 5.2. We prove well-distribution properties of the measure in Section 5.3, and show that $\dim_{\mathrm{H}} R(\overline{o}, \ell) \geq s(\alpha)$ via an energy estimate in Section 5.4. By far the most difficult and technical part of this proof is the definition of the measure and handling its intricacies in order to compute its energy.

Let $\alpha_0 \geq \alpha > 0$ and $\ell \colon \mathbb{N} \to \mathbb{N}$ be an increasing function such that

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \frac{1}{\alpha}.$$

In this section, our standing assumption is that $\alpha \leq -\sum_{i=1}^{N} \lambda_i^{P_\alpha(1)} p_i e^{\alpha} \log p_i$. Let $\gamma < \alpha$ be arbitrary. Then $\gamma < -\sum_{i=1}^{N} \lambda_i^{P_\gamma(1)} p_i e^{\gamma} \log p_i$ holds.

For short, we use the notation $s(\gamma) = P_\gamma(1)$ (recall Proposition 2.2 and Lemma 2.4). So, let us fix $\gamma \in \mathbb{R}$ such that $\gamma < -\sum_{i=1}^{N} \lambda_i^{s(\gamma)} p_i e^{\gamma} \log p_i$. Then there exists $\epsilon_\gamma > 0$ such that $\gamma + \sum_{i=1}^{N} \lambda_i^{s(\gamma)} p_i e^{\gamma} \log p_i < \log(1-\epsilon_\gamma)$ holds. Then by Cramér's bound, there exists $\beta = \beta_\gamma > 0$ such that for every $n \in \mathbb{N}$

$$\mathbb{Q}_1 \left(\left\{ \overline{\jmath} \in \Sigma_n : p_{\overline{\jmath}} e^{\gamma |\overline{\jmath}|} > (1-\epsilon_\gamma)^{|\overline{\jmath}|} \right\} \right) \leq e^{-\beta n}, \tag{23}$$

where we recall that $\mathbb{Q}_1$ is the Bernoulli measure induced by the probabilities $(\lambda_i^{s(\gamma)} p_i e^{\gamma})_{i=1}^{N}$.

5.1. Cantor subset for the lower bound

Denote, for each $n \in \mathbb{N}$, $M_n := \min \ell^{-1}(n)$. Let q be the maximal integer such that $q \leq 2e^{\gamma n}/n$ and $M_n + qn \in \ell^{-1}(n)$. Let

$$P(n) = \{M_n, M_n + n, \ldots, M_n + (q-1)n\}.$$

Let $\{n_p\}_{p\geq 1}$ be a sequence such that $\limsup_{n\to\infty} \frac{\log m(n)}{n} = \lim_{p\to\infty} \frac{\log m(n_p)}{n_p} = \alpha > \gamma$. Without loss of generality, we may assume that

$$\frac{e^{\gamma n_p}}{2n_p} \leq \#P(n_p) \leq \frac{2e^{\gamma n_p}}{n_p} \quad \text{for every } p \in \mathbb{N}. \tag{24}$$

Let us now define a class of functions $n : \Sigma_* \to \mathbb{N}$, which plays a central role in our construction.

Definition 5.1. We call a function $n : \Sigma_* \to \mathbb{N}$ a *constructor function* if it satisfies the following properties: For every $\overline{\imath}, \overline{\jmath} \in \Sigma_*$

(C1) $n(\overline{\imath}) \in \{n_p\}_{p\geq 1}$,
(C2) if $\overline{\imath} \neq \overline{\jmath}$, then $n(\overline{\imath}) \neq n(\overline{\jmath})$,
(C3) if $|\overline{\jmath}| > |\overline{\imath}|$, then $n(\overline{\jmath}) > n(\overline{\imath})$,
(C4) $\frac{n(\overline{\imath})}{p_{\overline{\imath}} e^{\gamma|\overline{\imath}|}} > 2$,
(C5) $\frac{|\overline{\imath}|}{n(\overline{\imath})} \leq \frac{1}{n(\emptyset)} \leq \frac{1}{2}$.

It is easy to construct constructor functions. Arrange the elements of Σ_* such that their lengths are increasing and the same length words are in alphabetic order. Let this be $(\emptyset, 1, \ldots, N, 11, \ldots) = (\overline{\imath}_0, \overline{\imath}_1, \ldots)$. Fix $2 \leq n(\emptyset) \in \{n_p\}_{p\geq 1}$ being arbitrary and then define $n(\overline{\imath}_n)$ by induction such that let $n(\overline{\imath}_n) \in \{n_p\}_{p\geq 1} \cap (n(\overline{\imath}_{n-1}), \infty)$ be the smallest element such that (C4) and (C5) are satisfied.

Denote, for $\overline{\imath} \in \Sigma_*$

$$m(\overline{\imath}) := \#P(n(\overline{\imath})).$$

Set inductively, for $\overline{\imath} \in \Sigma_*, n \in \mathbb{N}$,

$$\Xi_0^{\overline{\imath}} = \{\overline{\imath}\} \quad \text{and} \quad \Xi_n^{\overline{\imath}} = \{\overline{\jmath} \in \Sigma_* \mid n(\overline{\imath}') = |\overline{\jmath}| \text{ and } \overline{\jmath}|_{|\overline{\imath}'|} = \overline{\imath}' \text{ for some } \overline{\imath}' \in \Xi_{n-1}^{\overline{\imath}}\},$$

the set of children of $\overline{\imath}$ at the nth construction level. Observe that $[\overline{\imath}] = \bigcup\{[\overline{\jmath}] : \overline{\jmath} \in \Xi_n^{\overline{\imath}}\}$ for every $n \geq 0$, hence $\{\Xi_n^{\overline{\imath}}\}_{n\in\mathbb{N}}$ form a very sparse subtree of Σ_*, which still covers the whole set Σ. Now, we will choose the random subset which are actually visited by the random cover process.

Let $\mathcal{C}_1$ be the collection of finite words, which have been visited by the "children of the empty set". That is,

$$\mathcal{C}_1 := \{(\sigma^p \overline{o})|_{n(\emptyset)} : p \in P(n(\emptyset))\} \subseteq \Xi_1^{\emptyset}$$

Now, let us define the nth generation by induction, namely,

$$\mathcal{C}_n := \bigcup_{\overline{\imath}\in\mathcal{C}_{n-1}} \{(\sigma^p \overline{o})|_{n(\overline{\imath})} : p \in P(n(\overline{\imath})) \ \&\ \overline{\imath} < \sigma^p \overline{o}\} \subseteq \Xi_n^{\emptyset}.$$

Observe that

$$\{(\sigma^p \overline{o})|_{n(\overline{\imath})} : p \in P(n(\overline{\imath})) \ \&\ \overline{\imath} < \sigma^p \overline{o}\} \subseteq \Xi_1^{\overline{\imath}}.$$

The random Cantor subset of $R(\overline{o}, \ell)$ we consider is $\pi(C)$, where $C \subset \Sigma$ is defined as the intersection of the nested sequence of compact sets $C_n \subset \Sigma$,

$$C_n = \bigcup_{\overline{\imath}\in\mathcal{C}_n} [\overline{\imath}] \qquad \text{and} \qquad C = \bigcap_{n=1}^{\infty} C_n.$$

Let us observe that the random Cantor subset C depends on the constructor function defined in Definition 5.1. Clearly, $C = \emptyset$ with positive probability. In particular, we will construct a random measure $\mu = \mu_{\overline{o}}$ for $\mathbb{P}$-almost every $\overline{o}$ such that $0 < \mu(C) < \infty$ with positive probability. Our goal is to show that for every $\epsilon > 0$ there is a properly chosen constructor function such that the $(s(\gamma) - \epsilon)$-energy of μ is finite for every $\epsilon > 0$ almost surely. In the remaining part of the section, we suppress the notation $\mu_{\overline{o}}$, but the reader shall keep in mind that the measure actually depends on $\overline{o}$.

5.2. Construction of a random measure on the Cantor set

For every $\overline{\imath} \in \Sigma_*$ we will define an L^2-martingale that will be used to determine the measure of $[\overline{\imath}]$. First we will need some notation. Let $s_{\overline{\imath}}$ be such that

$$\sum_{|\overline{\jmath}|=n(\overline{\imath}),\ \overline{\imath}<\overline{\jmath}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}}(1-(1-p_{\overline{\jmath}})^{m(\overline{\imath})}) = 1.$$

Then recall the definition of $P(n)$ from Section 5.1 and let $\mathbb{1}_{\overline{\jmath}}^{\overline{\imath}}$ denote the characteristic function of the set

$$\{\overline{o} \in \Sigma \mid \exists p \in P(n(\overline{\imath})) \text{ s.t. } (\sigma^p \overline{o})|_{\ell(p)} = \overline{\jmath} \text{ and } \overline{\imath} < \overline{\jmath}\}.$$

Note that since $p \in P(n(\overline{\imath}))$, $\ell(p) = n(\overline{\imath})$.

We set $X_{\overline{\imath}}^{(0)} = 1$ for every $\overline{\imath} \in \Sigma_*$. Then, for $n \in \mathbb{N}$, we define $X_{\overline{\imath}}^{(n+1)}$ inductively for every $\overline{\imath} \in \Sigma_*$ by

$$X_{\overline{\imath}}^{(n+1)} = \sum_{|\overline{\jmath}|=n(\overline{\imath})} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \cdot \mathbb{1}_{\overline{\jmath}}^{\overline{\imath}} \cdot X_{\overline{\jmath}}^{(n)}.$$

In particular,

$$X_{\overline{\imath}}^{(1)} = \sum_{|\overline{\jmath}|=n(\overline{\imath})} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \cdot \mathbb{1}_{\overline{\jmath}}^{\overline{\imath}}.$$

Note that by choice of $s_{\overline{\imath}}$, we obtain $\mathbb{E}(X_{\overline{\imath}}^{(1)}) = 1$.

Let us now set up a filtration of sigma-algebras. Let

$$a_n^{\overline{\imath}} = \max\{\max \ell^{-1}(n(\overline{\jmath})) + n(\overline{\jmath}) \mid \overline{\jmath} \in \Xi_{n-1}^{\overline{\imath}}\}, \qquad \text{with } a_0^{\overline{\imath}} = 0.$$

Then let $\mathcal{F}_n^{\overline{\imath}}$ be the sigma-algebra generated by the cylinders of words with length at most $a_n^{\overline{\imath}}$ on Σ, with $\mathcal{F}_0^{\overline{\imath}}$ the trivial sigma-algebra.

Proposition 5.2. *For each $\overline{\imath} \in \Sigma_*$ the collection $(\mathcal{F}_n^{\overline{\imath}})$ is a filtration, i.e. $\mathcal{F}_n^{\overline{\imath}} \supset \mathcal{F}_{n-1}^{\overline{\imath}}$. Further,*

$$\mathcal{F}_n^{\overline{\imath}} = \bigcup_{|\overline{\jmath}|=n(\overline{\imath}),\overline{\imath}<\overline{\jmath}} \mathcal{F}_{n-1}^{\overline{\jmath}}.$$

Proof. The first claim follows immediately from $a_n^{\overline{\imath}} > a_{n-1}^{\overline{\imath}}$.

For the second claim, we prove by induction that

$$\Xi_n^{\overline{\imath}} = \bigcup_{|\overline{\jmath}|=n(\overline{\imath}),\overline{\imath}<\overline{\jmath}} \Xi_{n-1}^{\overline{\jmath}}.$$

The case $n = 1$ is immediate from definitions. Assume the claim holds up to n. Then

$$\begin{aligned}\Xi_{n+1}^{\overline{\imath}} &= \{\overline{\jmath} \mid |\overline{\jmath}| = n(\overline{\imath}'), \overline{\imath}' < \overline{\jmath} \text{ for some } \overline{\imath}' \in \Xi_n^{\overline{\imath}}\} \\ &= \bigcup_{|\overline{\jmath}|=n(\overline{\imath}),\overline{\imath}<\overline{\jmath}} \{\overline{\jmath}' \mid n(\overline{\imath}') = |\overline{\jmath}'|, \overline{\imath}' < \overline{\jmath}' \text{ for some } \overline{\imath}' \in \Xi_{n-1}^{\overline{\jmath}}\} \\ &= \bigcup_{|\overline{\jmath}|=n(\overline{\imath}),\overline{\imath}<\overline{\jmath}} \Xi_n^{\overline{\jmath}},\end{aligned}$$

by the induction hypothesis and the definition of $\Xi_n^{\overline{\jmath}}$. Hence,

$$a_n^{\overline{\imath}} = \max\{a_{n-1}^{\overline{\jmath}} \mid |\overline{\jmath}| = n(\overline{\imath}), \overline{\imath} < \overline{\jmath}\}.$$

Finally, we note that the fact that the finite union of increasing σ-algebras is itself a σ-algebra from which our claim follows. □

Proposition 5.3. *For every $\epsilon_0 > 0$ there exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$*

$$0 \leq s(\gamma) - s_{\overline{\imath}} < \epsilon_0 \quad \text{for all } \overline{\imath} \in \Sigma_*.$$

Moreover, there exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$ we get that for all $\overline{\imath} \in \Sigma_*$

$$\left(\frac{n(\overline{\imath})}{2p_{\overline{\imath}}e^{\gamma|\overline{\imath}|}}\right)^{\frac{1}{n(\overline{\imath})-|\overline{\imath}|}} \leq \sum_{j=1}^{N} \lambda_j^{s_{\overline{\imath}}} p_j e^{\gamma} \leq \left(\frac{5n(\overline{\imath})}{p_{\overline{\imath}}e^{\gamma|\overline{\imath}|}}\right)^{\frac{1}{n(\overline{\imath})-|\overline{\imath}|}}. \tag{25}$$

Proof. Let $\overline{\imath} \in \Sigma_*$. By the definition of $s_{\overline{\imath}}$ and using the approximations (24) and (5), we obtain

$$1 = \sum_{|\overline{a}|=n(\overline{\imath})-|\overline{\imath}|} \lambda_{\overline{a}}^{s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}}p_{\overline{a}})^{m(\overline{\imath})}) \leq \frac{2p_{\overline{\imath}}e^{\gamma|\overline{\imath}|}}{n(\overline{\imath})} \left(\sum_{i=1}^{N} \lambda_i^{s_{\overline{\imath}}} p_i e^{\gamma}\right)^{n(\overline{\imath})-|\overline{\imath}|},$$

which rearranges to

$$\left(\frac{n(\overline{\imath})}{2p_{\overline{\imath}}e^{\gamma|\overline{\imath}|}}\right)^{\frac{1}{n(\overline{\imath})-|\overline{\imath}|}} \leq \sum_{i=1}^{N} \lambda_i^{s_{\overline{\imath}}} p_i e^{\gamma}.$$

By (C4),

$$\left(\frac{n(\overline{\imath})}{2p_{\overline{\imath}}e^{\gamma|\overline{\imath}|}}\right)^{\frac{1}{n(\overline{\imath})-|\overline{\imath}|}} > 1$$

so that $s(\gamma) \geq s_{\overline{\imath}}$.

Let ϵ_γ be as in Eq. (23). Then denote

$$\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma} := \{\overline{\jmath} \in \Sigma_{n(\overline{\imath})-|\overline{\imath}|} \mid p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|} \leq (1-\epsilon_\gamma)^{n(\overline{\imath})-|\overline{\imath}|}\}.$$

We estimate,

$$\begin{aligned} 1 &= \sum_{|\overline{\jmath}|=n(\overline{\imath})-|\overline{\imath}|} \lambda_{\overline{\jmath}}^{s_{\overline{\imath}}} (1-(1-p_{\overline{\imath}}p_{\overline{\jmath}})^{m(\overline{\imath})}) \\ &\geq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)(s_{\overline{\imath}}-s(\gamma))} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s(\gamma)} (1-(1-p_{\overline{\imath}}p_{\overline{\jmath}})^{m(\overline{\imath})}) \qquad (26) \\ &\geq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)(s_{\overline{\imath}}-s(\gamma))} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s(\gamma)} m(\overline{\imath}) p_{\overline{\imath}}p_{\overline{\jmath}} \left(1-\frac{m(\overline{\imath})p_{\overline{\imath}}p_{\overline{\jmath}}}{2}\right) \end{aligned}$$

by Eq. (6), and

$$\geq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)(s_{\overline{\imath}}-s(\gamma))} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s(\gamma)} \frac{e^{\gamma n(\overline{\imath})}}{2n(\overline{\imath})} p_{\overline{\imath}}p_{\overline{\jmath}} \left(1-\frac{e^{\gamma n(\overline{\imath})}p_{\overline{\imath}}p_{\overline{\jmath}}}{n(\overline{\imath})}\right)$$

by the definition of $\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}$. So,

$$\geq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)(s_{\overline{\imath}}-s(\gamma))} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s(\gamma)} \frac{e^{\gamma n(\overline{\imath})}}{2n(\overline{\imath})} p_{\overline{\imath}}p_{\overline{\jmath}} \left(1-\frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}(1-\epsilon_\gamma)^{n(\overline{\imath})-|\overline{\imath}|}}{n(\overline{\imath})}\right)$$

and by (C4),

$$\begin{aligned} &\geq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)(s_{\overline{\imath}}-s(\gamma))} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s(\gamma)} \frac{e^{\gamma n(\overline{\imath})}}{4n(\overline{\imath})} p_{\overline{\imath}}p_{\overline{\jmath}} \\ &\geq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)(s_{\overline{\imath}}-s(\gamma))} \frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}}{4n(\overline{\imath})} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s(\gamma)} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}}. \qquad (27) \end{aligned}$$

The sum in Eq. (27) is the $\mathbb{Q}_1$ measure of $\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}$. Then by Cramér's theorem (Eq. (23))

$$\sum_{\overline{\jmath}\in\left(\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}\right)^c} \lambda_{\overline{\jmath}}^{s(\gamma)} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}} \leq e^{-\beta(n(\overline{\imath})-|\overline{\imath}|)}.$$

Now, taking logarithms in Eq. (27) and dividing by $(n(\overline{\imath})-|\overline{\imath}|)\log\lambda_{\max}$, we obtain

$$(s_{\overline{\imath}} - s(\gamma)) + \frac{\gamma|\overline{\imath}| + \log p_{\overline{\imath}} - \log 4n(\overline{\imath}) + \log(1-e^{-\beta(n(\overline{\imath})-|\overline{\imath}|)})}{(n(\overline{\imath})-|\overline{\imath}|)\log\lambda_{\max}} \geq 0.$$

The second term is bounded above by

$$\begin{aligned}
&\frac{\gamma|\overline{\imath}| + \log p_{\overline{\imath}} - \log 4n(\overline{\imath}) + \log(1-e^{-\beta(n(\overline{\imath})-|\overline{\imath}|)})}{(n(\overline{\imath})-|\overline{\imath}|)\log\lambda_{\max}} \\
&\leq \frac{|\overline{\imath}|\log p_{\min} - \log 4n(\overline{\imath}) + \log(1-e^{-\beta n(\overline{\imath})/2})}{\frac{1}{2}n(\overline{\imath})\log\lambda_{\max}} \\
&\leq 2\cdot\frac{\log p_{\min} - \log 4n(\emptyset) + \log(1-e^{-\beta n(\emptyset)/2})}{n(\emptyset)\log\lambda_{\max}},
\end{aligned}$$

where we have used (C3), (C5) and the monotonicity of the above maps. Hence, we may set K large enough such that $s(\gamma) - s_{\overline{\imath}} < \epsilon_0$ holds for our chosen ϵ_0.

We finish the proof by showing the upper bound in Eq. (25). Choose $\epsilon_0 > 0$ such that $\lambda_{\min}^{-\epsilon_0} e^{-\beta} < 1$, where β is the exponent from Eq. (23). Then choose $K \geq 1$ corresponding to $\epsilon_0 > 0$. A computation similar to that from Eq. (26) to Eq. (27) gives

$$\begin{aligned}
1 &\geq \frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}}{4n(\overline{\imath})} \sum_{\overline{\jmath}\in\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma}} \lambda_{\overline{\jmath}}^{s_{\overline{\imath}}} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}} \\
&= \frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}}{4n(\overline{\imath})} \left(\sum_{\overline{\jmath}\in\Sigma_{n(\overline{\imath})-|\overline{\imath}|}} \lambda_{\overline{\jmath}}^{s_{\overline{\imath}}} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}} - \sum_{\overline{\jmath}\in(\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma})^c} \lambda_{\overline{\jmath}}^{s_{\overline{\imath}}} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}} \right) \\
&= \frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}}{4n(\overline{\imath})} \left(\left(\sum_{j=1}^{N} \lambda_j^{s_{\overline{\imath}}} e^{\gamma} p_j \right)^{n(\overline{\imath})-|\overline{\imath}|} - \sum_{\overline{\jmath}\in(\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma})^c} \lambda_{\overline{\jmath}}^{s_{\overline{\imath}}} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}} \right) \\
&\geq \frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}}{4n(\overline{\imath})} \left(\left(\sum_{j=1}^{N} \lambda_j^{s_{\overline{\imath}}} e^{\gamma} p_j \right)^{n(\overline{\imath})-|\overline{\imath}|} - \lambda_{\min}^{(s_{\overline{\imath}}-s(\gamma))(n(\overline{\imath})-|\overline{\imath}|)} \sum_{\overline{\jmath}\in(\mathcal{G}_{\overline{\imath}}^{\epsilon_\gamma})^c} \lambda_{\overline{\jmath}}^{s(\gamma)} e^{\gamma(n(\overline{\imath})-|\overline{\imath}|)} p_{\overline{\jmath}} \right) \\
&\geq \frac{e^{\gamma|\overline{\imath}|}p_{\overline{\imath}}}{4n(\overline{\imath})} \left(\left(\sum_{j=1}^{N} \lambda_j^{s_{\overline{\imath}}} e^{\gamma} p_j \right)^{n(\overline{\imath})-|\overline{\imath}|} - \lambda_{\min}^{-(s(\gamma)-s_{\overline{\imath}})(n(\overline{\imath})-|\overline{\imath}|)} e^{-\beta(n(\overline{\imath})-|\overline{\imath}|)} \right)
\end{aligned}$$

by Cramér's theorem (Eq. (23)), as above. Now, rearranging gives

$$\sum_{j=1}^{N} \lambda_j^{s_{\overline{\imath}}} e^{\gamma} p_j \le \left(\frac{4n(\overline{\imath})}{e^{\gamma|\overline{\imath}|} p_{\overline{\imath}}} + \lambda_{\min}^{-(s(\gamma)-s_{\overline{\imath}})(n(\overline{\imath})-|\overline{\imath}|)} e^{-\beta(n(\overline{\imath})-|\overline{\imath}|)} \right)^{1/(n(\overline{\imath})-|\overline{\imath}|)}$$

$$\le \left(\frac{5n(\overline{\imath})}{e^{\gamma|\overline{\imath}|} p_{\overline{\imath}}} \right)^{1/(n(\overline{\imath})-|\overline{\imath}|)} . \quad \square$$

Proposition 5.4. *For all $\overline{\imath} \in \Sigma_*$, the sequence of random variables $(X_{\overline{\imath}}^{(n)})$ is a martingale with respect to the sequence of sigma-algebras $\mathcal{F}_n^{\overline{\imath}}$.*

Proof. We will argue by induction. Let $n = 1$ and $\overline{\imath} \in \Sigma_*$. Then, since $|\overline{\imath}| \le \min \ell^{-1}(n(\overline{\imath}))$,

$$\mathbb{E}(X_{\overline{\imath}}^{(1)} \mid \mathcal{F}_0^{\overline{\imath}}) = \mathbb{E}(X_{\overline{\imath}}^{(1)}) = 1 = X_{\overline{\imath}}^{(0)}.$$

Assume, inductively, that for all $\overline{\imath} \in \Sigma_*$ and for n the claim holds, that is,

$$\mathbb{E}(X_{\overline{\imath}}^{(n)} \mid \mathcal{F}_{n-1}^{\overline{\imath}}) = X_{\overline{\imath}}^{(n-1)}.$$

Since $a_1^{\overline{\imath}} = \max \ell^{-1}(n(\overline{\imath})) + n(\overline{\imath}) \le a_n^{\overline{\imath}}$, the random variable $\mathbb{1}_{\overline{\jmath}}^{\overline{\imath}}$ is measurable with respect to the sigma-algebra $\mathcal{F}_n^{\overline{\imath}}$, and hence

$$\begin{aligned} \mathbb{E}(X_{\overline{\imath}}^{(n+1)} \mid \mathcal{F}_n^{\overline{\imath}}) &= \sum_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \lambda_{\sigma_{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \mathbb{E}(\mathbb{1}_{\overline{\jmath}}^{\overline{\imath}} X_{\overline{\jmath}}^{(n)} \mid \mathcal{F}_n^{\overline{\imath}}) \\ &= \sum_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \lambda_{\sigma_{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\jmath}}^{\overline{\imath}} \mathbb{E}(X_{\overline{\jmath}}^{(n)} \mid \mathcal{F}_n^{\overline{\imath}}). \end{aligned} \tag{28}$$

Notice now that the random variable $X_{\overline{\jmath}}^{(n)}$ is independent of digits of $\overline{o}$ beyond index $a_{n-1}^{\overline{\jmath}}$. Hence conditioning on $\mathcal{F}_n^{\overline{\imath}}$ is equivalent to conditioning on $\mathcal{F}_{n-1}^{\overline{\imath}}$, and

$$\mathbb{E}(X_{\overline{\jmath}}^{(n)} \mid \mathcal{F}_n^{\overline{\imath}}) = \mathbb{E}(X_{\overline{\jmath}}^{(n)} \mid \mathcal{F}_{n-1}^{\overline{\imath}}) = X_{\overline{\jmath}}^{(n-1)},$$

where we have made use of the induction hypothesis. Continuing with this from (28),

$$\sum_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \lambda_{\sigma_{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\jmath}}^{\overline{\imath}} \mathbb{E}(X_{\overline{\jmath}}^{(n)} \mid \mathcal{F}_n^{\overline{\imath}}) = \sum_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \lambda_{\sigma_{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\jmath}}^{\overline{\imath}} X_{\overline{\jmath}}^{(n-1)} = X_{\overline{\imath}}^{(n)}.$$

This proves the martingale property. $\square$

Proposition 5.5. *There exists a $K \ge 1$ such that for every constructor function with $n(\emptyset) \ge K$ there is $c > 0$ such that for all $n \in \mathbb{N}, \overline{\imath} \in \Sigma_*$, we have*

$$\mathbb{E}((X_{\overline{\imath}}^{(n)})^2) < c.$$

Proof. We will find $\tilde{\lambda} < 1$ such that for all $n \in \mathbb{N}$,

$$\max_{\overline{\imath} \in \Sigma_*} \mathbb{E}((X_{\overline{\imath}}^{(n)})^2) \leq 1 + \max_{\overline{\imath} \in \Sigma_*} \mathbb{E}((X_{\overline{\imath}}^{(n-1)})^2)\tilde{\lambda},$$

from which it follows by induction that for all $n \in \mathbb{N}$,

$$\max_{\overline{\imath} \in \Sigma_*} \mathbb{E}((X_{\overline{\imath}}^{(n)})^2) \leq \frac{1}{1-\tilde{\lambda}}.$$

Fix $\overline{\imath} \in \Sigma_*$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}\mathbb{E}((X_{\overline{\imath}}^{(n+1)})^2) = &\sum_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{2s_{\overline{\imath}}} \mathbb{E}(\mathbb{1}_{\overline{\jmath}}^{\overline{\imath}}(X_{\overline{\jmath}}^{(n)})^2) \\ &+ \sum_{\substack{\overline{\jmath}_1 \neq \overline{\jmath}_2,\ k=1,2 \\ |\overline{\jmath}_k|=n(\overline{\imath}),\ \overline{\imath}<\overline{\jmath}_k}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath}_1)}^{s_{\overline{\imath}}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath}_2)}^{s_{\overline{\imath}}} \mathbb{E}(\mathbb{1}_{\overline{\jmath}_1}^{\overline{\imath}} \mathbb{1}_{\overline{\jmath}_2}^{\overline{\imath}} X_{\overline{\jmath}_1}^{(n)} X_{\overline{\jmath}_2}^{(n)}) \qquad (29) \\ =: &\, S_1 + S_2.\end{aligned}$$

Note that since $n(\overline{\imath}) \neq n(\overline{\jmath})$, $\mathbb{1}_{\overline{\jmath}}^{\overline{\imath}}$ is independent of $X_{\overline{\jmath}}^{(n)}$. Hence, S_1 in the sum above is equal to

$$S_1 = \sum_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{2s_{\overline{\imath}}} (1-(1-p_{\overline{\jmath}})^{m(\overline{\imath})}) \mathbb{E}((X_{\overline{\jmath}}^{(n)})^2) \leq \lambda_{\max}^{(n(\overline{\imath})-|\overline{\imath}|)s_{\overline{\imath}}} \max_{|\overline{\jmath}|=n(\overline{\imath}), \overline{\imath}<\overline{\jmath}} \mathbb{E}((X_{\overline{\jmath}}^{(n)})^2),$$

where we have used the definition of $s_{\overline{\imath}}$. Let us choose $\epsilon_0 > 0$ such that $s(\gamma) - \epsilon_0 > 0$. By Proposition 5.3, there exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$, $s_{\overline{\imath}} \geq s(\gamma) - \epsilon_0$ for every $\overline{\imath} \in \Sigma_*$. Thus, $\lambda_{\max}^{s_{\overline{\imath}}(n(\overline{\imath})-|\overline{\imath}|)} \leq \lambda_{\max}^{(s(\gamma)-\epsilon_0)} =: \tilde{\lambda} < 1$, where we have used the fact that $n(\overline{\imath}) - |\overline{\imath}| \geq \frac{1}{2} n(\overline{\imath}) \geq \frac{1}{2} n(\emptyset) \geq 1$.

For the second term in the sum (29) we note first of all that since $n(\overline{\imath}) \neq n(\overline{\jmath}_k)$, $\mathbb{1}_{\overline{\jmath}_1}^{\overline{\imath}} \mathbb{1}_{\overline{\jmath}_2}^{\overline{\imath}}$ is independent of $X_{\overline{\jmath}_k}^{(n)}$. Since, furthermore, $\mathbb{E}(X_{\overline{\jmath}_k}^{(n)}) = 1$ by definition, we obtain for S_2

$$\begin{aligned}&\sum_{\substack{\overline{\jmath}_1 \neq \overline{\jmath}_2,\ k=1,2 \\ |\overline{\jmath}_k|=n(\overline{\imath}),\ \overline{\imath}<\overline{\jmath}_k}} \lambda_{\sigma^{|\overline{\imath}|}}^{s_{\overline{\imath}}}(\overline{\jmath}_1) \lambda_{\sigma^{|\overline{\imath}|}}^{s_{\overline{\imath}}}(\overline{\jmath}_2) \mathbb{E}(\mathbb{1}_{\overline{\jmath}_1}^{\overline{\imath}} \mathbb{1}_{\overline{\jmath}_2}^{\overline{\imath}} X_{\overline{\jmath}_1}^{(n)} X_{\overline{\jmath}_2}^{(n)}) \\ &= \sum_{\substack{\overline{\jmath}_1 \neq \overline{\jmath}_2,\ k=1,2 \\ |\overline{\jmath}_k|=n(\overline{\imath}),\ \overline{\imath}<\overline{\jmath}_k}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath}_1)}^{s_{\overline{\imath}}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath}_2)}^{s_{\overline{\imath}}} (1-(1-p_{\overline{\jmath}_1})^{m(\overline{\imath})} - (1-p_{\overline{\jmath}_2})^{m(\overline{\imath})} + (1-p_{\overline{\jmath}_1}-p_{\overline{\jmath}_2})^{m(\overline{\imath})}).\end{aligned}$$

Here, from the elementary computation

$$\begin{aligned}1 - p_{\overline{\jmath}_1} - p_{\overline{\jmath}_2} \leq 1 - p_{\overline{\jmath}_1} - p_{\overline{\jmath}_2} + p_{\overline{\jmath}_1} p_{\overline{\jmath}_2} \Leftrightarrow \\ (1-p_{\overline{\jmath}_1}-p_{\overline{\jmath}_2})^{m(\overline{\imath})} \leq (1-p_{\overline{\jmath}_1})^{m(\overline{\imath})} (1-p_{\overline{\jmath}_2})^{m(\overline{\imath})} \Leftrightarrow\end{aligned}$$

$$1-(1-p_{\overline{\jmath}_1})^{m(\overline{\imath})}-(1-p_{\overline{\jmath}_2})^{m(\overline{\imath})}+(1-p_{\overline{\jmath}_1}-p_{\overline{\jmath}_2})^{m(\overline{\imath})}$$
$$\leq (1-(1-p_{\overline{\jmath}_1})^{m(\overline{\imath})})(1-(1-p_{\overline{\jmath}_2})^{m(\overline{\imath})}) \tag{30}$$

we can deduce that

$$S_2 \leq \left(\sum_{|\overline{\jmath}|=n(\overline{\imath}),\overline{\imath}<\overline{\jmath}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} (1-(1-p_{\overline{\jmath}})^{m(\overline{\imath})}) \right)^2,$$

which equals 1 by definition of $s_{\overline{\imath}}$. Thus, $S_1 + S_2 \leq 1 + \max \mathbb{E}((X_{\overline{\jmath}}^{(n)})^2)\tilde{\lambda}$, as in the claim. □

The upshot of the content of this section is the following: For each $\overline{\imath} \in \Sigma_*$ there is a limiting value $X_{\overline{\imath}}$, which we will be able to use for the definition of the mass distribution on the Cantor subset of $R(\overline{o}, \ell)$ from Section 5.1.

Proposition 5.6. *For all $\overline{\imath} \in \Sigma_*$, there is $X_{\overline{\imath}}$ such that $X_{\overline{\imath}}^{(n)} \to X_{\overline{\imath}}$ as $n \to \infty$ almost surely and in L^2. Furthermore, there exists a $C > 0$ such that $\mathbb{E}(X_{\overline{\imath}}^2) \leq C$, $\mathbb{E}(X_{\overline{\imath}}) = 1$ and*

$$X_{\overline{\imath}} = \sum_{|\overline{\jmath}|=n(\overline{\imath}),\overline{\imath}<\overline{\jmath}} \lambda_{\sigma^{|\overline{\imath}|}(\overline{\jmath})}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\jmath}}^{\overline{\imath}} X_{\overline{\jmath}}$$

for all $\overline{\imath} \in \Sigma_*$.

Proof. This is immediate from Propositions 5.4 and 5.5 and Doob's L^2-martingale convergence theorem [13, Theorem VII.4.1]. □

We will define a random measure $\mu = \mu_{\overline{o}}$ on Σ using the weights $X_{\overline{\imath}}$ constructed above (see Proposition 5.6), for cylinders of the following form: Consider $[\overline{\imath}_1\overline{\imath}_2\ldots\overline{\imath}_n]$ where $n(\emptyset) = |\overline{\imath}_1|$, $n(\overline{\imath}_1) = |\overline{\imath}_1\overline{\imath}_2|$, and so on, up to $n(\overline{\imath}_1\ldots\overline{\imath}_{n-1}) = |\overline{\imath}_1\ldots\overline{\imath}_n|$. For any cylinder of this form, set

$$\mu([\overline{\imath}_1\ldots\overline{\imath}_n]) := \lambda_{\overline{\imath}_1}^{s_\emptyset}\lambda_{\overline{\imath}_2}^{s_{\overline{\imath}_1}}\cdots\lambda_{\overline{\imath}_n}^{s_{\overline{\imath}_1\ldots\overline{\imath}_{n-1}}}\mathbb{1}_{\overline{\imath}_1}^{\emptyset}\ldots\mathbb{1}_{\overline{\imath}_1\ldots\overline{\imath}_n}^{\overline{\imath}_1\ldots\overline{\imath}_{n-1}}X_{\overline{\imath}_1\ldots\overline{\imath}_n}.$$

By Proposition 5.6 and Kolmogorov's extension theorem [30, Theorem 1.7.8], for almost every $\overline{o}$ the measure μ is a well-defined Borel measure on Σ. By the construction, it is supported on the random Cantor set defined in Section 5.1 and satisfies $\mu(\Sigma) = \mu(R(\overline{o}, \ell)) = X_\emptyset$. Note that μ can be a zero measure, but using the martingale property, $\mathbb{E}(\mu(\Sigma)) = 1$. For the remainder of this section we study properties of μ.

5.3. Properties of the random measure on the Cantor set

Lemma 5.7. *There exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$ we have that for every $\overline{\imath}, \overline{\jmath} \in \Sigma_*$ with $|\overline{\jmath}| \leq n(\overline{\imath}) - |\overline{\imath}|$,*

$$\sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}}} \left(1 - (1 - p_{\overline{\imath}\overline{\jmath}\hbar})^{m(\overline{\imath})}\right) \leq 10 p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|}.$$

Proof. By Eq. (5) and Eq. (24),

$$\begin{aligned}\sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}}} \left(1 - (1 - p_{\overline{\imath}\overline{\jmath}\hbar})^{m(\overline{\imath})}\right) &\leq 2\frac{p_{\overline{\imath}} e^{\gamma|\overline{\imath}|}}{n(\overline{\imath})} p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|} \sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}}} p_{\hbar} e^{\gamma|\hbar|} \\ &= 2\frac{p_{\overline{\imath}} e^{\gamma|\overline{\imath}|}}{n(\overline{\imath})} p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|} \left(\sum_{j=1}^{N} \lambda_j^{s_{\overline{\imath}}} p_j e^{\gamma}\right)^{n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|}.\end{aligned}$$

By Proposition 5.3, one can choose $K \geq 1$ such that for every constructor function we get

$$\begin{aligned}&\leq 2\frac{p_{\overline{\imath}} e^{\gamma|\overline{\imath}|}}{n(\overline{\imath})} p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|} \left(\frac{5n(\overline{\imath})}{p_{\overline{\imath}} e^{\gamma|\overline{\imath}|}}\right)^{\frac{n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|}{n(\overline{\imath})-|\overline{\imath}|}} \\ &\leq 10 p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|} \left(\frac{p_{\overline{\imath}} e^{\gamma|\overline{\imath}|}}{n(\overline{\imath})}\right)^{\frac{|\overline{\jmath}|}{n(\overline{\imath})-|\overline{\imath}|}}\end{aligned}$$

By (C4)

$$\leq 10 p_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|}. \quad \square$$

Lemma 5.8. *For all $\epsilon_0 > 0$ there exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$ the following holds: For all $\overline{\jmath} \in \Sigma_*$,*

$$\mathbb{E}(\mu([\overline{\jmath}])) \leq 10 \lambda_{\min}^{-\epsilon_0|\overline{\jmath}|} \mathbb{Q}_1([\overline{\jmath}]),$$

where $\mathbb{Q}_1$ is the Bernoulli measure with the probability vector $q_i = \lambda_i^{s(\gamma)} p_i e^{\gamma}$.

Proof. Let $\overline{\jmath} \in \Sigma_*$ be arbitrary. Since $\{\Xi_\ell^\emptyset\}_{\ell \in \mathbb{N}}$ is a monotone refining sequence of partitions of Σ, there exists a maximal $n \in \mathbb{N}$ such that for every $k \leq n$ and every $\overline{\imath} \in \Xi_k^\emptyset$, $[\overline{\jmath}] \cap [\overline{\imath}] = \emptyset$ or $[\overline{\jmath}] \subsetneq [\overline{\imath}]$. Thus, there exist $\overline{\imath}_1, \ldots, \overline{\imath}_n, \overline{\jmath}' \in \Sigma_*$ such that $\overline{\jmath} = \overline{\imath}_1 \ldots \overline{\imath}_n \overline{\jmath}'$ and $\overline{\imath}_1 \ldots \overline{\imath}_k \in \Xi_k^\emptyset$ for every k, moreover, for every $\hbar \in \Sigma_*$ with $|\hbar| = n(\overline{\imath}_1 \ldots \overline{\imath}_n) - |\overline{\jmath}|$, $\overline{\jmath}\hbar \in \Xi_{n+1}^\emptyset$. Hence,

$$\mu([\overline{\jmath}]) = \lambda_{\overline{\imath}_1}^{s_\emptyset} \lambda_{\overline{\imath}_2}^{s_{\overline{\imath}_1}} \cdots \lambda_{\overline{\imath}_n}^{s_{\overline{\imath}_1\ldots\overline{\imath}_{n-1}}} \mathbb{1}_{\overline{\imath}_1}^{\emptyset} \ldots \mathbb{1}_{\overline{\imath}_1\ldots\overline{\imath}_n}^{\overline{\imath}_1\ldots\overline{\imath}_{n-1}} \lambda_{\overline{\jmath}'}^{s_{\overline{\imath}_1\ldots\overline{\imath}_n}} \sum_{|\hbar|=n(\overline{\imath}_1\ldots\overline{\imath}_n)-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}_1\ldots\overline{\imath}_n}} \mathbb{1}_{\overline{\jmath}\hbar}^{\overline{\imath}_1\ldots\overline{\imath}_n} X_{\overline{\jmath}\hbar}. \tag{31}$$

Then by the independence of the indicators and $X_{\overline{\jmath}\hbar}$,

$$\mathbb{E}(\mu([\overline{\jmath}])) = \prod_{k=1}^{n} \lambda_{\overline{\imath}_k}^{s_{\overline{\imath}_1\cdots\overline{\imath}_{k-1}}} \left(1 - (1 - p_{\overline{\imath}_1\ldots\overline{\imath}_k})^{m(\overline{\imath}_1\ldots\overline{\imath}_{k-1})}\right) \cdot \lambda_{\overline{\jmath}'}^{s_{\overline{\imath}_1\ldots\overline{\imath}_n}} \sum_{|\hbar|=n(\overline{\imath}_1\ldots\overline{\imath}_n)-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}_1\ldots\overline{\imath}_n}} \left(1 - (1 - p_{\overline{\jmath}\hbar})^{m(\overline{\imath}_1\ldots\overline{\imath}_n)}\right).$$

Choose $K \geq 1$ sufficiently large such that Proposition 5.3 and Lemma 5.7 hold. Then by Eq. (5), Eq. (24) and Lemma 5.7

$$\begin{aligned}\mathbb{E}(\mu([\overline{\jmath}])) &\leq 10\lambda_{\overline{\imath}_1}^{s_\emptyset} \cdots \lambda_{\overline{\imath}_n}^{s_{\overline{\imath}_1\ldots\overline{\imath}_{n-1}}} p_{\overline{\imath}_1\ldots\overline{\imath}_n} e^{\gamma|\overline{\imath}_1\ldots\overline{\imath}_n|} \prod_{k=0}^{n-1} \frac{2p_{\overline{\imath}_1\ldots\overline{\imath}_k} e^{\gamma|\overline{\imath}_1\ldots\overline{\imath}_k|}}{n(\overline{\imath}_1\ldots\overline{\imath}_k)} \lambda_{\overline{\jmath}'}^{s_{\overline{\imath}_1\ldots\overline{\imath}_n}} p_{\overline{\jmath}'} e^{\gamma|\overline{\jmath}'|} \\ &\leq 10\lambda_{\min}^{-\epsilon_0|\overline{\jmath}|} \mathbb{Q}_1([\overline{\jmath}]),\end{aligned}$$

where in the last inequality we have used (C4) and Proposition 5.3. □

Lemma 5.9. *There exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$ we have that for every $\overline{\imath}, \overline{\jmath} \in \Sigma_*$ with $|\overline{\jmath}| \leq n(\overline{\imath}) - |\overline{\imath}|$*

$$\mathbb{E}\left(\left(\sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar}^{\overline{\imath}} X_{\overline{\imath}\overline{\jmath}\hbar}\right)^2\right) \leq 10Cp_{\overline{\jmath}} e^{\gamma|\overline{\jmath}|} + 100p_{\overline{\jmath}}^2 e^{2\gamma|\overline{\jmath}|},$$

where $C > 0$ is the universal constant such that $\mathbb{E}(X_{\overline{\imath}}^2) \leq C$ for all $\overline{\imath} \in \Sigma_*$.

Proof. Let $\overline{\imath}, \overline{\jmath} \in \Sigma_*$ be fixed such that $|\overline{\jmath}| \leq n(\overline{\imath}) - |\overline{\imath}|$. Then

$$\mathbb{E}\left(\left(\sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar}^{\overline{\imath}} X_{\overline{\imath}\overline{\jmath}\hbar}\right)^2\right) = \sum_{\substack{|\hbar_1|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ |\hbar_2|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|}} \mathbb{E}\left(\lambda_{\hbar_1}^{s_{\overline{\imath}}} \lambda_{\hbar_2}^{s_{\overline{\imath}}} \mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar_1}^{\overline{\imath}} \mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar_2}^{\overline{\imath}} X_{\overline{\imath}\overline{\jmath}\hbar_1} X_{\overline{\imath}\overline{\jmath}\hbar_2}\right).$$

By the independence of the indicators and $X_{\overline{\imath}\overline{\jmath}\hbar}$'s

$$\begin{aligned}&= \sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{2s_{\overline{\imath}}} \mathbb{E}(\mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar}^{\overline{\imath}}) \mathbb{E}(X_{\overline{\imath}\overline{\jmath}\hbar}^2) \\ &\quad + \sum_{\substack{|\hbar_1|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ |\hbar_2|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ \hbar_1\neq\hbar_2}} \lambda_{\hbar_1}^{s_{\overline{\imath}}} \lambda_{\hbar_2}^{s_{\overline{\imath}}} \mathbb{E}\left(\mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar_1}^{\overline{\imath}} \mathbb{1}_{\overline{\imath}\overline{\jmath}\hbar_2}^{\overline{\imath}}\right) \mathbb{E}(X_{\overline{\imath}\overline{\jmath}\hbar_1}) \mathbb{E}(X_{\overline{\imath}\overline{\jmath}\hbar_2}),\end{aligned}$$

and by Proposition 5.6

$$\leq C \sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{2s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar})^{m(\overline{\imath})})$$
$$+ \sum_{\substack{|\hbar_1|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ |\hbar_2|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ \hbar_1\neq\hbar_2}} \lambda_{\hbar_1}^{s_{\overline{\imath}}}\lambda_{\hbar_2}^{s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar_1})^{m(\overline{\imath})}-(1-p_{\overline{\imath}\overline{\jmath}\hbar_2})^{m(\overline{\imath})}+(1-p_{\overline{\imath}\overline{\jmath}\hbar_1}-p_{\overline{\imath}\overline{\jmath}\hbar_2})^{m(\overline{\imath})}).$$

By Eq. (30)

$$\leq C \sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{2s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar})^{m(\overline{\imath})})$$
$$+ \sum_{\substack{|\hbar_1|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ |\hbar_2|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}| \\ \hbar_1\neq\hbar_2}} \lambda_{\hbar_1}^{s_{\overline{\imath}}}\lambda_{\hbar_2}^{s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar_1})^{m(\overline{\imath})})(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar_2})^{m(\overline{\imath})})$$

$$\leq C \sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{2s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar})^{m(\overline{\imath})}) + \left(\sum_{|\hbar|=n(\overline{\imath})-|\overline{\imath}|-|\overline{\jmath}|} \lambda_{\hbar}^{s_{\overline{\imath}}}(1-(1-p_{\overline{\imath}\overline{\jmath}\hbar})^{m(\overline{\imath})})\right)^2.$$

By Lemma 5.7

$$\leq C10p_{\overline{\jmath}}e^{\gamma|\overline{\jmath}|} + \left(10p_{\overline{\jmath}}e^{\gamma|\overline{\jmath}|}\right)^2$$

for every constructor function with sufficiently large $n(\emptyset)$. □

Proposition 5.10. *For every $\delta > 0$ there exists a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$ we have*

$$\mu_{\overline{o}}\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{\ell=\lceil\delta n\rceil}^{\infty}\{\overline{\jmath}\in\Sigma \mid p_{(\sigma^n\overline{\jmath})|_\ell}e^{\gamma\ell}\geq(1-\epsilon_\gamma)^\ell\}\right)=0 \quad \text{for } \mathbb{P}\text{-almost every } \overline{o},$$

where $\epsilon_\gamma > 0$ is defined in Eq. (23).

Proof. Let $\delta > 0$ be arbitrary but fixed. By Borel-Cantelli's Lemma, it is enough to show that

$$\mathbb{E}\left(\sum_{n=1}^{\infty}\sum_{\ell=\lceil\delta n\rceil}^{\infty}\mu\left(\{\overline{\jmath}\in\Sigma \mid p_{(\sigma^n\overline{\jmath})|_\ell}e^{\gamma\ell}\geq(1-\epsilon_\gamma)^\ell\}\right)\right)<\infty.$$

Clearly,

$$\mu\left(\{\overline{\jmath} \in \Sigma \mid p_{(\sigma^n\overline{\jmath})|_\ell} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\}\right) = \sum_{|\overline{\imath}|=n} \sum_{|\overline{\jmath}|=\ell} \mu([\overline{\imath\jmath}]) \mathbb{1}\{p_{\overline{\jmath}} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\}.$$

Let $\epsilon_0 > 0$ be such that $\lambda_{\min}^{-\epsilon_0(1+\delta)} e^{-\beta\delta} < 1$, where β is defined in Eq. (23). Note that $\lambda_{\min}^{-\epsilon_0} e^{-\beta} < 1$. Let $K \geq 1$ be such that Lemma 5.8 holds with ϵ_0. Hence, for every $n \geq K$

$$\begin{aligned}
&\mathbb{E}\left(\mu\left(\{\overline{\jmath} \in \Sigma \mid p_{(\sigma^n\overline{\jmath})|_\ell} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\}\right)\right) \\
&\quad = \sum_{|\overline{\imath}|=n} \sum_{|\overline{\jmath}|=\ell} \mathbb{E}(\mu([\overline{\imath\jmath}])) \mathbb{1}\{p_{\overline{\jmath}} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\} \\
&\quad \leq 10\lambda_{\min}^{-\epsilon_0(n+\ell)} \sum_{|\overline{\imath}|=n} \sum_{|\overline{\jmath}|=\ell} \mathbb{Q}_1([\overline{\imath\jmath}]) \mathbb{1}\{p_{\overline{\jmath}} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\} \\
&\quad = 10\lambda_{\min}^{-\epsilon_0(n+\ell)} \sum_{|\overline{\jmath}|=\ell} \mathbb{Q}_1([\overline{\jmath}]) \mathbb{1}\{p_{\overline{\jmath}} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\} \\
&\quad = 10\lambda_{\min}^{-\epsilon_0(n+\ell)} \mathbb{Q}_1\left(\{\overline{\jmath} \in \Sigma \mid p_{\overline{\jmath}|_\ell} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\}\right) \\
&\quad \leq 10\lambda_{\min}^{-\epsilon_0(n+\ell)} e^{-\beta\ell}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{\ell=\lceil\delta n\rceil}^{\infty} \mathbb{E}\left(\mu\left(\{\overline{\jmath} \in \Sigma \mid p_{(\sigma^n\overline{\jmath})|_\ell} e^{\gamma\ell} \geq (1-\epsilon_\gamma)^\ell\}\right)\right) &\leq 10 \sum_{\ell=\lceil\delta n\rceil}^{\infty} \lambda_{\min}^{-\epsilon_0(n+\ell)} e^{-\beta\ell} \\
&\leq 10 \frac{\lambda_{\min}^{-\epsilon_0(1+\delta)n} e^{-\beta\delta n}}{1 - \lambda_{\min}^{-\epsilon_0} e^{-\beta}},
\end{aligned}$$

which forms a convergent series. □

For $\delta > 0$, let

$$E_k^\delta := \bigcap_{n=k}^{\infty} \bigcap_{\ell=\lceil n\delta\rceil}^{\infty} \{\overline{\jmath} \in \Sigma \mid p_{(\sigma^n\overline{\jmath})|_\ell} e^{\gamma\ell} < (1-\epsilon_\gamma)^\ell\}.$$

It is an immediate consequence of Proposition 5.10, $0 < \mathbb{E}(\mu(\Sigma)) < \infty$ and the monotone convergence theorem that for a well-chosen constructor function

$$\mathbb{E}(\mu(E_k^\delta)) > 0$$

for all large enough k.

Proposition 5.11. *For all $\epsilon_0 > 0$ there exist a $\delta > 0$ and a $K \geq 1$ such that for every constructor function with $n(\emptyset) \geq K$ and every $q \in \mathbb{N}$:*

$$\sum_{\overline{\jmath}\in\Sigma_*} \lambda_{\overline{\jmath}}^{-s(\gamma)+3\epsilon_0}\mathbb{E}(\mu([\overline{\jmath}])^2)\mathbb{1}\{[\overline{\jmath}]\cap E_q^\delta\neq\emptyset\}<\infty.$$

Proof. Let $\overline{\jmath}\in\Sigma_*$ be arbitrary but fixed. Similarly to the proof of Lemma 5.8, there exist $\overline{\imath}_1,\dots,\overline{\imath}_n,\overline{\jmath}'\in\Sigma_*$ such that $\overline{\jmath}=\overline{\imath}_1\dots\overline{\imath}_n\overline{\jmath}'$, $\overline{\imath}_1\dots\overline{\imath}_k\in\Xi_k^\emptyset$ for every $1\le k\le n$, and $\overline{\jmath}\hbar\in\Xi_{n+1}^\emptyset$ for every $\hbar\in\Sigma_*$ with $|\hbar|=n(\overline{\imath}_1\dots\overline{\imath}_n)-|\overline{\jmath}|$. By Eq. (31)

$$\mu([\overline{\jmath}])^2=\prod_{k=1}^n \lambda_{\overline{\imath}_k}^{2s_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}\mathbb{1}_{\overline{\imath}_1\dots\overline{\imath}_k}^{\overline{\imath}_1\dots\overline{\imath}_{k-1}}\lambda_{\overline{\jmath}'}^{2s_{\overline{\imath}_1\dots\overline{\imath}_n}}\left(\sum_{|\hbar|=n(\overline{\imath}_1\dots\overline{\imath}_n)-|\overline{\jmath}|}\lambda_\hbar^{s_{\overline{\imath}_1\dots\overline{\imath}_n}}\mathbb{1}_{\overline{\jmath}\hbar}^{\overline{\imath}_1\dots\overline{\imath}_n}X_{\overline{\jmath}\hbar}\right)^2.$$

By the independence of the indicators and $X_{\overline{\jmath}\hbar}$'s

$$\mathbb{E}(\mu([\overline{\jmath}])^2)=\prod_{k=1}^n \lambda_{\overline{\imath}_k}^{2s_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}\left(1-(1-p_{\overline{\imath}_1\dots\overline{\imath}_k})^{m(\overline{\imath}_1\dots\overline{\imath}_{k-1})}\right)$$
$$\cdot\lambda_{\overline{\jmath}'}^{2s_{\overline{\imath}_1\dots\overline{\imath}_n}}\mathbb{E}\left(\left(\sum_{|\hbar|=n(\overline{\imath}_1\dots\overline{\imath}_n)-|\overline{\jmath}|}\lambda_\hbar^{s_{\overline{\imath}_1\dots\overline{\imath}_n}}\mathbb{1}_{\overline{\jmath}\hbar}^{\overline{\imath}_1\dots\overline{\imath}_n}X_{\overline{\jmath}\hbar}\right)^2\right).$$

By Lemma 5.9

$$\le\prod_{k=1}^n \lambda_{\overline{\imath}_k}^{2s_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}\left(1-(1-p_{\overline{\imath}_1\dots\overline{\imath}_k})^{m(\overline{\imath}_1\dots\overline{\imath}_{k-1})}\right)\cdot\lambda_{\overline{\jmath}'}^{2s_{\overline{\imath}_1\dots\overline{\imath}_n}}\left(10Cp_{\overline{\jmath}'}e^{\gamma|\overline{\jmath}'|}+100p_{\overline{\jmath}'}^2e^{2\gamma|\overline{\jmath}'|}\right).$$

By Eq. (5), Eq. (24) and (C4) and Proposition 5.3

$$\le\prod_{k=1}^n \lambda_{\overline{\imath}_k}^{2s_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}m(\overline{\imath}_1\dots\overline{\imath}_{k-1})p_{\overline{\imath}_1\dots\overline{\imath}_k}\cdot\lambda_{\overline{\jmath}'}^{2s_{\overline{\imath}_1\dots\overline{\imath}_n}}\left(10Cp_{\overline{\jmath}'}e^{\gamma|\overline{\jmath}'|}+100p_{\overline{\jmath}'}^2e^{2\gamma|\overline{\jmath}'|}\right)$$
$$\le\prod_{k=1}^n \lambda_{\overline{\imath}_k}^{2s_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}\frac{2e^{\gamma n(\overline{\imath}_1\dots\overline{\imath}_{k-1})}p_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}{n(\overline{\imath}_1\dots\overline{\imath}_{k-1})}p_{\overline{\imath}_k}\cdot\lambda_{\overline{\jmath}'}^{2s_{\overline{\imath}_1\dots\overline{\imath}_n}}\left(10Cp_{\overline{\jmath}'}e^{\gamma|\overline{\jmath}'|}+100p_{\overline{\jmath}'}^2e^{2\gamma|\overline{\jmath}'|}\right)$$
$$\le\prod_{k=1}^n \lambda_{\overline{\imath}_k}^{2s_{\overline{\imath}_1\dots\overline{\imath}_{k-1}}}e^{\gamma|\overline{\imath}_k|}p_{\overline{\imath}_k}\cdot\lambda_{\overline{\jmath}'}^{2s_{\overline{\imath}_1\dots\overline{\imath}_n}}\left(10Cp_{\overline{\jmath}'}e^{\gamma|\overline{\jmath}'|}+100p_{\overline{\jmath}'}^2e^{2\gamma|\overline{\jmath}'|}\right)$$
$$\le 10C\lambda_{\overline{\jmath}}^{s(\gamma)-2\epsilon_0}\mathbb{Q}_1([\overline{\jmath}])+100\lambda_{\overline{\jmath}}^{s(\gamma)-2\epsilon_0}\mathbb{Q}_1([\overline{\imath}_1\dots\overline{\imath}_n])\lambda_{\overline{\jmath}'}^{s(\gamma)}p_{\overline{\jmath}'}^2e^{2\gamma|\overline{\jmath}'|}.$$

Let $A:=\max\{1,\sum_{j=1}^N\lambda_j^{s(\gamma)}p_j^2e^{2\gamma}\}$. Now, choose $\delta>0$ such that $\lambda_{\max}^{\epsilon_0/2}A^\delta<1$. By the construction of the sets $\Xi_n^\emptyset$,

$$\sum_{\overline{\jmath}\in\Sigma_*}\lambda_{\overline{\jmath}}^{-s(\gamma)+3\epsilon_0}\mathbb{E}(\mu([\overline{\jmath}])^2)\mathbb{1}\{[\overline{\jmath}]\cap E_q^\delta\neq\emptyset\}$$
$$=\sum_{n=0}^\infty\sum_{\overline{\imath}\in\Xi_n^\emptyset}\sum_{\ell=0}^{n(\overline{\imath})-|\overline{\imath}|}\sum_{|\hbar|=\ell}\lambda_{\overline{\imath}\hbar}^{-s(\gamma)+3\epsilon_0}\mathbb{E}(\mu([\overline{\imath}\hbar])^2)\mathbb{1}\{[\overline{\imath}\hbar]\cap E_q^\delta\neq\emptyset\}$$

$$\leq 10C \sum_{\overline{\jmath}\in\Sigma_*} \lambda_{\overline{\jmath}}^{\epsilon_0} \mathbb{Q}_1([\overline{\jmath}])$$

$$+ 100 \sum_{n=0}^{\infty} \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \sum_{\ell=0}^{n(\overline{\imath})-|\overline{\imath}|} \sum_{|\hbar|=\ell} \lambda_{\overline{\imath}\hbar}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \lambda_{\hbar}^{s(\gamma)} p_{\hbar}^2 e^{2\gamma|\hbar|} \mathbb{1}\{[\overline{\imath}\hbar] \cap E_q^{\delta} \neq \emptyset\}.$$

Since the first sum is clearly convergent, we only need to check the summability of the second part. For n sufficiently large, $|\overline{\imath}| \geq q$ for every $\overline{\imath} \in \bigcup_{k=n}^{\infty} \Xi_k^{\emptyset}$ and $|\overline{\imath}| \leq \lambda_{\max}^{-\epsilon_0/2|\overline{\imath}|}$. So, for every sufficiently large n

$$\sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \sum_{\ell=0}^{n(\overline{\imath})-|\overline{\imath}|} \sum_{|\hbar|=\ell} \lambda_{\overline{\imath}\hbar}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \lambda_{\hbar}^{s(\gamma)} p_{\hbar}^2 e^{2\gamma|\hbar|} \mathbb{1}\{[\overline{\imath}\hbar] \cap E_q^{\delta} \neq \emptyset\}$$

$$= \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \sum_{\ell=0}^{\lfloor \delta|\overline{\imath}| \rfloor} \sum_{|\hbar|=\ell} \lambda_{\overline{\imath}\hbar}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \lambda_{\hbar}^{s(\gamma)} p_{\hbar}^2 e^{2\gamma|\hbar|} \mathbb{1}\{[\overline{\imath}\hbar] \cap E_q^{\delta} \neq \emptyset\}$$

$$+ \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \sum_{\ell=\lceil \delta|\overline{\imath}| \rceil}^{n(\overline{\imath})-|\overline{\imath}|} \sum_{|\hbar|=\ell} \lambda_{\overline{\imath}\hbar}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \lambda_{\hbar}^{s(\gamma)} p_{\hbar}^2 e^{2\gamma|\hbar|} \mathbb{1}\{[\overline{\imath}\hbar] \cap E_q^{\delta} \neq \emptyset\}$$

$$\leq \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \lambda_{\overline{\imath}}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \sum_{\ell=0}^{\lfloor \delta|\overline{\imath}| \rfloor} \left(\sum_{i=1}^{N} \lambda_i^{s(\gamma)} p_i^2 e^{2\gamma} \right)^{\ell}$$

$$+ \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \sum_{\ell=\lceil \delta|\overline{\imath}| \rceil}^{n(\overline{\imath})-|\overline{\imath}|} \sum_{|\hbar|=\ell} \lambda_{\overline{\imath}\hbar}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \lambda_{\hbar}^{s(\gamma)} p_{\hbar} e^{\gamma|\hbar|} (1-\epsilon_{\gamma})^{|\hbar|}$$

$$\leq \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \lambda_{\overline{\imath}}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \delta|\overline{\imath}| A^{\delta|\overline{\imath}|} + \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \sum_{\ell=\lceil \delta|\overline{\imath}| \rceil}^{n(\overline{\imath})-|\overline{\imath}|} \lambda_{\overline{\imath}}^{\epsilon_0} \mathbb{Q}_1([\overline{\imath}]) \lambda_{\max}^{\epsilon_0 \ell} (1-\epsilon_{\gamma})^{\ell}$$

$$\leq \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \lambda_{\max}^{\epsilon_0/2|\overline{\imath}|} \mathbb{Q}_1([\overline{\imath}]) A^{\delta|\overline{\imath}|} + \sum_{\overline{\imath}\in\Xi_n^{\emptyset}} \lambda_{\max}^{\epsilon_0|\overline{\imath}|} \mathbb{Q}_1([\overline{\imath}]) \left(1 - \lambda_{\max}^{\epsilon_0}(1-\epsilon_{\gamma})\right)^{-1}$$

$$\leq \lambda_{\max}^{\epsilon_0/2n} A^{\delta n} + \lambda_{\max}^{\epsilon_0 n} \left(1 - \lambda_{\max}^{\epsilon_0}(1-\epsilon_{\gamma})\right)^{-1},$$

which completes the proof. □

5.4. Energy estimates of the random Cantor measure

Lemma 5.12. *Let $\Phi = \{f_i(x) = \lambda_i O_i x + t_i\}_{i=1}^N$ be an IFS of similarities of $\mathbb{R}^d$ such that Φ satisfies the open set condition. For every Borel measure ν on $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ with $0 < \nu(\Sigma) < \infty$, if $\sum_{\overline{\imath}\in\Sigma_*} \lambda_{\overline{\imath}}^{-t} \nu([\overline{\imath}])^2 < \infty$ then $\dim_{\mathrm{H}} \pi_* \nu \geq t$, where π denotes the natural projection.*

The proof of Lemma 5.12 is standard, however, we could not find any proper reference for it.

Proof. Without loss of generality, we may assume that $|U| = 1$, where U is the open and bounded subset of $\mathbb{R}^d$ with respect to the OSC holds. Let $\lambda_{\min}$ be the smallest contraction ratio as usual. For simplicity, let $\mu := \pi_*\nu$. By Frostman's lemma [9, Lemma 1.9.9], it is enough to show that $\iint \|x-y\|^{-t} d\mu(x)d\mu(y) < \infty$. Then

$$\iint \frac{d\mu(x)d\mu(y)}{\|x-y\|^t} = \sum_{n=0}^{\infty} \iint \|x-y\|^{-t} \mathbb{1}\{\lambda_{\min}^{n+1} < \|x-y\| \le \lambda_{\min}^n\} d\mu(x)d\mu(y)$$
$$\le \sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \int \mu\left(B(x, \lambda_{\min}^n)\right) d\mu(x).$$

Let

$$\mathcal{Q}_n = \{\prod_{j=1}^{d} [m_j \lambda_{\min}^n, (m_j+1)\lambda_{\min}^n) : m_j \in \mathbb{Z} \text{ for } j = 1, \dots, d\}$$

be the partition of $\mathbb{R}^d$ into axis parallel squares of side length $\lambda_{\min}^n$ such that the origin is a vertex of a partition element. Let $\mathcal{Q}_n(x)$ be the unique partition element in $\mathcal{Q}_n$ containing x. Furthermore, for a cube $Q \in \mathcal{Q}_n(x)$ of the form $\prod_{j=1}^d [m_j\lambda_{\min}^n, (m_j+1)\lambda_{\min}^n)$, let $\widehat{Q} = \prod_{j=1}^d [(m_j-1)\lambda_{\min}^n, (m_j+2)\lambda_{\min}^n)$ be the union of the neighbouring cubes of Q in $\mathcal{Q}_n$. Then

$$\sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \int \mu\left(B(x, \lambda_{\min}^n)\right) d\mu(x) \le \sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \sum_{Q \in \mathcal{Q}_n} \mu\left(\widehat{Q}\right)^2$$

Let

$$\Gamma_n := \{\overline{\imath} \in \Sigma_* : \lambda_{\overline{\imath}} \le \lambda_{\min}^n < \lambda_{\overline{\imath}_-}\},$$

where $\overline{\imath}_-$ denotes the parent of $\overline{\imath}$, i.e. the finite word obtained from $\overline{\imath}$ by deleting its last symbol (with the convention $\lambda_{\emptyset_-} = +\infty$). Clearly, $\Gamma_n \cap \Gamma_m = \emptyset$ for every $n \neq m$. By the open set condition, there exists $C > 0$ such that $\#\{\overline{\imath} \in \Gamma_n : f_{\overline{\imath}}(\overline{U}) \cap \widehat{Q} \neq \emptyset\} \le C$ for every $Q \in \mathcal{Q}_n$. Thus,

$$\sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \sum_{Q \in \mathcal{Q}_n} \mu\left(\widehat{Q}\right)^2 \le \sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \sum_{Q \in \mathcal{Q}_n} \left(\sum_{\substack{\overline{\imath} \in \Gamma_n \\ f_{\overline{\imath}}(\overline{U}) \cap \widehat{Q} \neq \emptyset}} \nu([\overline{\imath}]) \right)^2$$
$$\le C^2 \sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \sum_{Q \in \mathcal{Q}_n} \sum_{\substack{\overline{\imath} \in \Gamma_n \\ f_{\overline{\imath}}(\overline{U}) \cap \widehat{Q} \neq \emptyset}} \nu([\overline{\imath}])^2$$

$$= C^2 \sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \sum_{\overline{\imath}\in\Gamma_n} \#\{Q \in \mathcal{Q}_n : f_{\overline{\imath}}(\overline{U}) \cap \widehat{Q} \neq \emptyset\} \nu([\overline{\imath}])^2.$$

Since for every $\overline{\imath} \in \Gamma_n$, there are at most 5^d-many $Q \in \mathcal{Q}_n$ such that $f_{\overline{\imath}}(\overline{U}) \cap \widehat{Q} \neq \emptyset$

$$\leq C^2 5^d \sum_{n=0}^{\infty} \lambda_{\min}^{-t(n+1)} \sum_{\overline{\imath}\in\Gamma_n} \nu([\overline{\imath}])^2$$

$$\leq C^2 5^d \lambda_{\min}^{-t} \sum_{n=0}^{\infty} \sum_{\overline{\imath}\in\Gamma_n} \lambda_{\overline{\imath}}^{-t} \nu([\overline{\imath}])^2$$

$$\leq C^2 5^d \lambda_{\min}^{-t} \sum_{n=0}^{\infty} \sum_{\overline{\imath}\in\Sigma_n} \lambda_{\overline{\imath}}^{-t} \nu([\overline{\imath}])^2. \quad \square$$

Proposition 5.13. *Let $\alpha > 0$ be such that*

$$\alpha \leq -\sum_{i=1}^{N} \lambda_i^{s(\alpha)} p_i e^{\alpha} \log p_i,$$

where $\sum_{i=1}^{N} \lambda_i^{s(\alpha)} p_i e^{\alpha} = 1$. Suppose that $\ell(n)$ satisfies $\liminf_{n\to\infty} \ell(n)/\log n = 1/\alpha$. Then, $\dim_{\mathrm{H}}(R(\overline{o}, \ell)) \geq s(\alpha)$ for $\mathbb{P}$-almost every $\overline{o}$.

Proof. Clearly, the map $\alpha \mapsto s(\alpha)$ is continuous and monotone increasing. Thus, it is enough to show that $\mathbb{P}(\{\overline{o} \in \Sigma : \dim_{\mathrm{H}}(R(\overline{o}, \ell)) \geq s(\alpha - 1/n) - 1/n\}) = 1$ for every $n \in \mathbb{N}$. It is also clear that $R(\overline{o}', \ell) = R(\overline{o}, \ell)$ for every $\overline{o}', \overline{o} \in \Sigma$ such that there exists $n \geq 1$ such that $o'_k = o_k$ for every $k \geq n$. Thus, $\{\overline{o} \in \Sigma : \dim_{\mathrm{H}}(R(\overline{o}, \ell)) \geq s(\alpha - 1/n) - 1/n\}$ is a tail event and by Kolmogorov's zero-one law [23, Corollary 1.7.1], it is enough to show that $\mathbb{P}(\{\overline{o} \in \Sigma : \dim_{\mathrm{H}}(R(\overline{o}, \ell)) \geq s(\alpha - 1/n) - 1/n\}) > 0$ for every $n \in \mathbb{N}$.

Let $\mu = \mu_{\overline{o}}$ be the random measure constructed in Section 5.2 for $\alpha - 1/n$. By Proposition 5.11, choose $\delta > 0$ and $K \geq 1$ such that for $\epsilon_0 = (3n)^{-1}$ we have

$$\sum_{\overline{\jmath}\in\Sigma_*} \lambda_{\overline{\jmath}}^{-s(\alpha-1/n)+1/n} \mathbb{E}(\mu([\overline{\jmath}])^2) \mathbb{1}\{[\overline{\jmath}] \cap E_q^{\delta} \neq \emptyset\} < \infty, \tag{32}$$

for every $q \in \mathbb{N}$. By Proposition 5.6 and Proposition 5.10, there exists $q \in \mathbb{N}$ such that $\mathbb{P}(\{\overline{o} \in \Sigma : \mu(E_q^{\delta} \cap R(\overline{o}, \ell)) > 0\}) > 0$. Hence, it is enough to show that $\mathbb{P}(\{\overline{o} \in \Sigma : \dim_{\mathrm{H}} \mu|_{E_q^{\delta}} \geq s(\alpha - 1/n) - 1/n\}) > 0$. By Lemma 5.12, it is enough to show that

$$\mathbb{E}\left(\sum_{\overline{\imath}\in\Sigma_*} \lambda_{\overline{\imath}}^{-s(\alpha-1/n)+1/n} \mu|_{E_q^{\delta}}([\overline{\imath}])^2\right) < \infty.$$

But $\mu|_{E_q^{\delta}}([\overline{\imath}])^2 = \mu([\overline{\imath}] \cap E_q^{\delta})^2 \leq \left(\mu([\overline{\imath}]) \mathbb{1}\{[\overline{\imath}] \cap E_q^{\delta} \neq \emptyset\}\right)^2 = \mu([\overline{\imath}])^2 \mathbb{1}\{[\overline{\imath}] \cap E_q^{\delta} \neq \emptyset\}$, and so, the claim follows by Eq. (32). $\square$

6. Completing the proofs

Finally, we summarise the proofs of the main theorems.

Proof of Theorem 1.2. Let $\alpha > 0$ and $\ell\colon \mathbb{N} \to \mathbb{N}$ be an increasing function such that

$$\liminf_{n\to\infty} \frac{\ell(n)}{\log n} = \frac{1}{\alpha},$$

and let $\mathbb{P}_p$ be the Bernoulli measure defined by the probability vector $p = (p_1, \dots, p_N)$. If $\alpha > \alpha_2$ then by Theorem 4.1, $R(\overline{o}, \ell) = \Lambda$ almost surely. If $\alpha_1 < \alpha \le \alpha_2$ then $\mathcal{H}^{s_0}(R(\overline{o}, \ell)) = \mathcal{H}^{s_0}(\Lambda)$ $\mathbb{P}_p$-almost surely by Corollary 4.3. So we may assume that $\alpha \le \alpha_1$.

Let $s(\alpha)$ be as in Eq. (2). By Lemma 2.4, $s(\alpha)$ is the unique root of the probabilistic pressure defined in Eq. (13), and so, by Proposition 3.1, $\dim_{\mathrm{H}} R(\overline{o}, \ell) \le s(\alpha)$ $\mathbb{P}_p$-almost surely. If $\alpha_0 < \alpha < \alpha_1$ then $\dim_{\mathrm{H}} R(\overline{o}, \ell) \ge s(\alpha)$ by Corollary 4.5 $\mathbb{P}$-almost surely. If $\alpha = \alpha_1$ then by taking $c > 1$ arbitrary and having $\ell'(n) := c\ell(n)$, we get that $\dim_{\mathrm{H}} R(\overline{o}, \ell) \ge \dim_{\mathrm{H}} R(\overline{o}, \ell') = s(\alpha/c)$ $\mathbb{P}_p$-almost surely. By taking the limit $c \to 1$ on a countable sequence and using the continuity of $\alpha \mapsto s(\alpha)$ (see Proposition 2.2) we get that $\dim_{\mathrm{H}} R(\overline{o}, \ell) \ge \lim_{c\to 1} s(\alpha/c) = s(\alpha) = s_0$ $\mathbb{P}$-almost surely.

Finally, suppose that $\alpha \le \alpha_0$. Then, the almost sure lower bound $\dim_{\mathrm{H}} R(\overline{o}, \ell) \ge s(\alpha)$ follows by Proposition 5.13. □

Proof of Theorem 1.3. The proof is the combination of Proposition 3.2, Theorem 4.1, Corollary 4.3 and Corollary 4.5 for every $\alpha \ne \alpha_1$. One can verify the case $\alpha = \alpha_1$ by relying on the continuity of $\alpha \mapsto t(\alpha)$ and on the monotonicity of the set $R(\overline{o}, \ell)$ similarly to the previous proof. □

References

[1] D. Allen, S. Baker, B. Bárány, Recurrence rates for shifts of finite type, Adv. Math. 460 (2025) 110039.
[2] D. Allen, B. Bárány, On the Hausdorff measure of shrinking target sets on self-conformal sets, Mathematika 67 (4) (2021) 807–839.
[3] R. Anttila, M. Myllyoja, Dvoretzky covering problem for general measures, Preprint, available at arXiv:2601.11470, 2026.
[4] S. Baker, Overlapping iterated function systems from the perspective of metric number theory, Mem. Am. Math. Soc. 287 (1428) (2023) v+95.
[5] S. Baker, Intrinsic Diophantine approximation for overlapping iterated function systems, Math. Ann. 388 (3) (2024) 3259–3297.
[6] S. Baker, H. Koivusalo, Quantitative recurrence and the shrinking target problem for overlapping iterated function systems, Adv. Math. 442 (2024) 109538.
[7] C. Bandt, S. Graf, Self-similar sets. VII. A characterization of self-similar fractals with positive Hausdorff measure, Proc. Am. Math. Soc. 114 (4) (1992) 995–1001.
[8] B. Bárány, M. Rams, Shrinking targets on Bedford–McMullen carpets, Proc. Lond. Math. Soc. (3) 117 (5) (2018) 951–995.
[9] B. Bárány, K. Simon, B. Solomyak, Self-Similar and Self-Affine Sets and Measures, Mathematical Surveys and Monographs, vol. 276, American Mathematical Society, Providence, RI, 2023.
[10] B. Bárány, S. Troscheit, Dynamically defined subsets of generic self-affine sets, Nonlinearity 35 (10) (2022) 4986–5013.

[11] E. Daviaud, Hausdorff dimension of dynamical Diophantine approximation associated with ergodic mixing systems, Adv. Math. 482 (2025) 110621.
[12] E. Daviaud, Dynamical Diophantine approximation and shrinking targets for C^1 weakly conformal IFSs with overlaps, Ergod. Theory Dyn. Syst. 45 (6) (2025) 1777–1826.
[13] J.L. Doob, Stochastic Processes, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990, Reprint of the 1953 original, A Wiley-Interscience Publication.
[14] A. Dvoretzky, On covering a circle by randomly placed arcs, Proc. Natl. Acad. Sci. USA 42 (1956) 199–203.
[15] A.-H. Fan, J. Schmeling, S. Troubetzkoy, A multifractal mass transference principle for Gibbs measures with applications to dynamical Diophantine approximation, Proc. Lond. Math. Soc. (3) 107 (5) (2013) 1173–1219.
[16] R. Hill, S.L. Velani, The ergodic theory of shrinking targets, Invent. Math. 119 (1) (1995) 175–198.
[17] R. Hill, S.L. Velani, The shrinking target problem for matrix transformations of tori, J. Lond. Math. Soc. (2) 60 (2) (1999) 381–398.
[18] Z. Hu, B. Li, Y. Xiao, On the intersection of dynamical covering sets with fractals, Math. Z. 301 (1) (2022) 485–513.
[19] Z. Hu, B. Li, L. Yang, Large intersection property for limsup sets in metric space, Potential Anal. 60 (1) (2024) 341–363.
[20] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (5) (1981) 713–747.
[21] E. Järvenpää, M. Järvenpää, M. Myllyoja, Ö. Stenflo, The Ekström-Persson conjecture regarding random covering sets, J. Lond. Math. Soc. (2) 111 (1) (2025) e70058.
[22] E. Järvenpää, M. Myllyoja, S. Seuret, Hitting probabilities and the Ekström-Persson conjecture, Preprint, available at arXiv:2506.10448, 2025.
[23] D. Khoshnevisan, Multiparameter Processes: An Introduction to Random Fields, Springer Monographs in Mathematics, Springer-Verlag, New York, 2002.
[24] M. Kirsebom, P. Kunde, T. Persson, On shrinking targets and self-returning points, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 24 (3) (2023) 1499–1535.
[25] H. Koivusalo, F.A. Ramírez, Recurrence to shrinking targets on typical self-affine fractals, Proc. Edinb. Math. Soc. (2) 61 (2) (2018) 387–400.
[26] L. Liao, S. Seuret, Diophantine approximation by orbits of expanding Markov maps, Ergod. Theory Dyn. Syst. 33 (2) (2013) 585–608.
[27] T. Persson, M. Rams, On shrinking targets for piecewise expanding interval maps, Ergod. Theory Dyn. Syst. 37 (2) (2017) 646–663.
[28] A. Schief, Separation properties for self-similar sets, Proc. Am. Math. Soc. 122 (1) (1994) 111–115.
[29] S. Seuret, Inhomogeneous random coverings of topological Markov shifts, Math. Proc. Camb. Philos. Soc. 165 (2) (2018) 341–357.
[30] T. Tao, An Introduction to Measure Theory, Graduate Studies in Mathematics, vol. 126, American Mathematical Society, Providence, RI, 2011.